Lecture Notes

PET504E
Advanced Well Test Analysis

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1 Introduction

The term "Well Testing" [c1]as it is used in Petroleum Industry means the measuring of a formation's (or reservoir's) pressure (and/or rate) response to flow from a well. The term "Well Testing" is generally used with the term "Pressure Transient Analysis", interchangeably. It is an indirect measurement technique as opposed to direct methods such as fluid sampling or coring. Well testing provides dynamic information on the reservoir whereas direct measurements only provide static information, which is not sufficient for predicting the behavior of the reservoir.

[c1] *Murat Çınar: Text added.*

Simply, the objective of well testing is to deduce quantitative information about the well/reservoir system under consideration from its response to a given input. Input (or input signal) is used for perturbing one or more wells so that the output (signal) exhibiting the response of the reservoir is obtained at the perturbated well and/or adjacent wells. In practice, the input is equivalent to controlling the well behavior [c2] created by changing the flow rate or the pressure at the well (Mathematically specifying the well behavior is equivalent to specifying a boundary condition). A common example for creating an input signal is [c3]a build up test [c4]where we change the rate to zero by shutting-in the well. Reservoir response, [c5] also called output signal, to a given input is monitored by measuring the pressure change (or rate change) at the [c6] well. This process is illustrated as,

[c2] *Murat Çınar:* ~~and~~

[c3] *Murat Çınar: Text added.*

[c4] *Murat Çınar:* ~~in which~~

[c5] *Murat Çınar:* ~~which is~~

[c6] *Murat Çınar:* ~~same~~

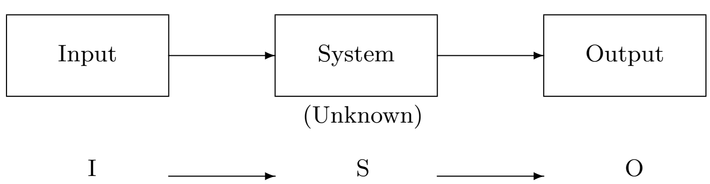


Fig. 1.1. Block diagram ?????

Typical examples for input and output signals as used in petroleum industry are shown in Fig. 1.2.

From reservoir response as monitored by the "output signal", we would like to determine information related to the followings:

- Fluid in place; pore volume, $\phi h A$.

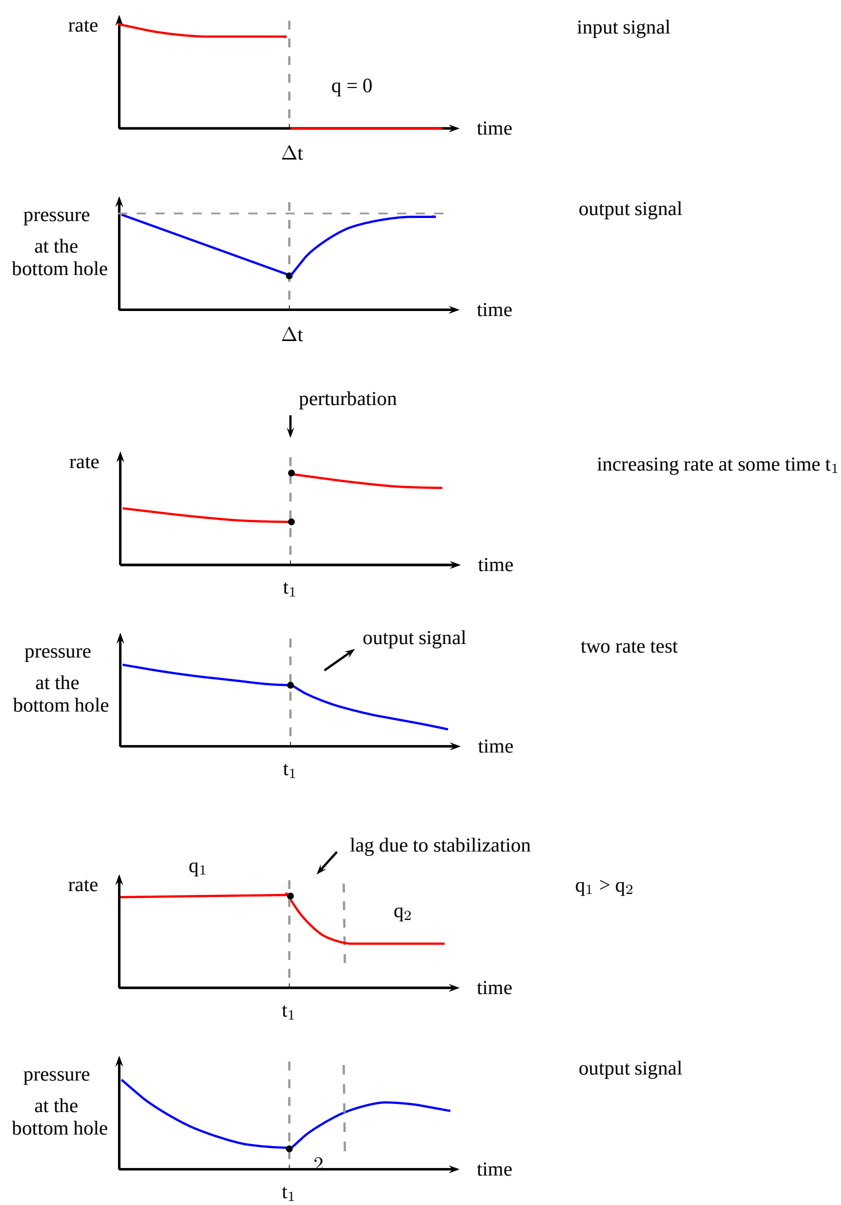


Fig. 1.2. Typical input and output signals - Transient phenomena.

- Ability of reservoir to transfer fluid, kh (or transmissibility, $\frac{kh}{\mu}$).
- Determination of average reservoir [c1]pressure, $\overline{P}$, which is the driving force in the reservoir [c2]
- Prediction of rate versus time data.
- Initial recovery, is the reservoir worth producing.
- Is there any damage around the wellbore impeding the flow? skin factor, s.
- Reservoir description (type of reservoir, flow boundaries (faults)).
- Distance to fluid interface [c3]that is important determining swept zone for secondary and tertiary methods.

Interpretation of well test data consist of basically three steps:

(i) Determination of the one most appropriate reservoir / wellbore (mathematical) model [c4]of the actual system. We also call such a model as the interpretation model. [c5]Here our intention is to find a representative mathematical model that reproduces, as close as possible, the output of the actual system for a given input. This is known as the inverse problem. [c6]We are trying to obtain information about the physical system by using observed measurements. Unfortunately, the solution of inverse problem often yields non-unique results. [c7]By non-unique results, we mean that several different interpretation models [c8]may generate an output signal (response) to a given input [c9]that is similar (or identical) to that of the actual system. The inverse problem can be represented by the following equation.

$$\Sigma = O/I \approx S \tag{1.1}$$

where Σ denotes the interpretation model, S denotes the actual system. In inverse problem, as can be seen from Eq.1.1, it may be possible to obtain the same outputs to a given I for different Σ_i's[c10],however, the number of alternative models (solutions) can be reduced as the number and the range of output signal measurements.

(ii) Once the appropriate model is determined, estimate the parameters of the actual system S. [c11]These parameters are kh, s, ϕ, C, λ, w etc. This is known as [c12]parameter estimation[c13][c14] and achieved by adjusting the parameters of the model by different [c15]mathematical methods to obtain an output signal, Ω, that is always qualitatively identical (within some tolerance) to that of [c16]the actual system, O. The computation of Ω is known as the "[c17]forward problem" in mathematics. Contrary to the inverse problem, the solution of the [c18]forward problem is always unique for a given system; that is,

$$I \times \Sigma = \Omega \approx 0 \tag{1.2}$$

The adjusted parameters of the interpretation model are assumed to represent the parameters of the real system S. [c19]

(iii) Validate the results of the interpretation. This can be achieved by using the parameters determined from part (ii) in the model to generate output

[c1] *Murat Çınar*: ~~average~~

[c2] *Murat Çınar*: Based upon the explanation you gave about the pressure decline recently, I am not sure if this statement is correct.

[c3] *Murat Çınar*: ~~which~~

[c4] *Murat Çınar*: ~~to~~

[c5] *Murat Çınar*: ~~Here our hope is that the model chosen will produce an output signal to a given input which is as close as possible to that of the actual system.~~

[c6] *Murat Çınar*: *Text added.*

[c7] *Murat Çınar*: ~~With~~

[c8] *Murat Çınar*: ~~can~~

[c9] *Murat Çınar*: ~~which~~

[c10] *Murat Çınar*: ~~. However~~

[c11] *Murat Çınar*: ~~Such parameters can be~~

[c12] *Murat Çınar*: ~~the~~

[c13] *Murat Çınar*: ~~problem~~

[c14] *Murat Çınar*: ~~. It is~~

[c15] *Murat Çınar*: *Text added.*

[c16] *Murat Çınar*: *Text added.*

[c17] *Murat Çınar*: ~~direct~~

[c18] *Murat Çınar*: ~~direct~~

[c19] *Murat Çınar*: I think we should discuss this phrase. Does the real system have any parameters? I do not think so... This is something conceptual we need to think about

signals for the entire range of [c20]the test and by comparing these outputs with the [c21]physical measurements.

[c20]*Murat Çınar*: *Text added.*

[c21]*Murat Çınar*: ~~measured ones~~

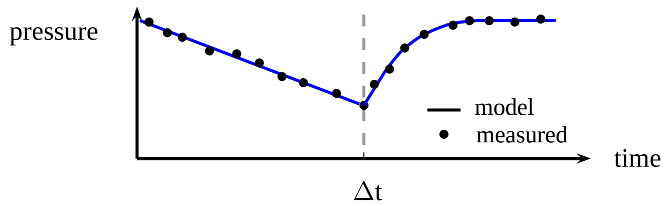


Fig. 1.3. Parameters estimated based on the analysis of buildup data.

[c1]Now we consider single phase flow in a cylindrical reservoir produced by a well at the center. The[c2]partial differential equation (p.d.e.) describing the flow is given by,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{kr}{\mu}\frac{\partial p}{\partial r}\right) = \phi c_t \frac{\partial p}{\partial t} \tag{1.3}$$

or if k, μ are constant,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\phi c_t \mu}{k}\frac{\partial p}{\partial t} = \frac{1}{\eta}\frac{\partial p}{\partial t} \tag{1.4}$$

[c3]$\eta = \frac{k}{\phi c_t \mu}$ is the hydraulic diffusivity[c4]; a measure of the [c5]speed at which a pressure disturbance [c6]propagates through the formation. If we specify, k, ϕ, c_t, μ, and the flow rate, then $p(r,t)$ is uniquely determined. This is an example for the [c7]forward problem.

Inverse problem, given q and p, [c8]helps us to

1. determine the p.d.e. that describes the reservoir best
2. find k, ϕ, etc.

[c9]

[c10]Analytical solutions have been presented in the literature for a variety of different well and reservoir settings for single phase flow. A summary of these responses are given by Bourdet [2].

2 Flow Equations

In this chapter, we will derive the equations which describe the fluid flow in porous media. Such equations are derived from the conservation of mass and the momentum equation as given by Darcy's semi-empirical equation. With the exception of thermal recovery schemes, all well-testing models assume isothermal conditions in the reservoir and thus the energy conservation is not needed.

[c1]*Murat Çınar*: ~~To illustrate an example of direct and inverse problems, let's consider single phase flow in a closed cylindrical reservoir produced by a single well at the center.~~

[c2]*Murat Çınar*: ~~pd.E~~

[c3]*Murat Çınar*: ~~$\eta = \frac{\phi c_t \mu}{k}$~~

[c4]*Murat Çınar*: ~~which is~~

[c5]*Murat Çınar*: ~~rapidity with~~

[c6]*Murat Çınar*: *Text added.*

[c7]*Murat Çınar*: ~~direct~~

[c8]*Murat Çınar*: *Text added.*

[c9]*Murat Çınar*: ~~During the past ten years, a lot of work has been done to develop models to a wide variety of reservoir / well configurations such as fractures, layered reservoirs, multiple porosities (composite zones), fractured wells, slanted and horizontal wells, etc. Well testing literature is almost complete for single phase problems, but some work needs to be done for multi-phase problems and heterogeneous reservoir systems.~~

~~Recently, people are much focused on the model recognition problem and the computer-aided parameter estimation by using non-linear regression techniques. Pressure derivatives and integrals are proven to be very useful in identifying the appropriate~~

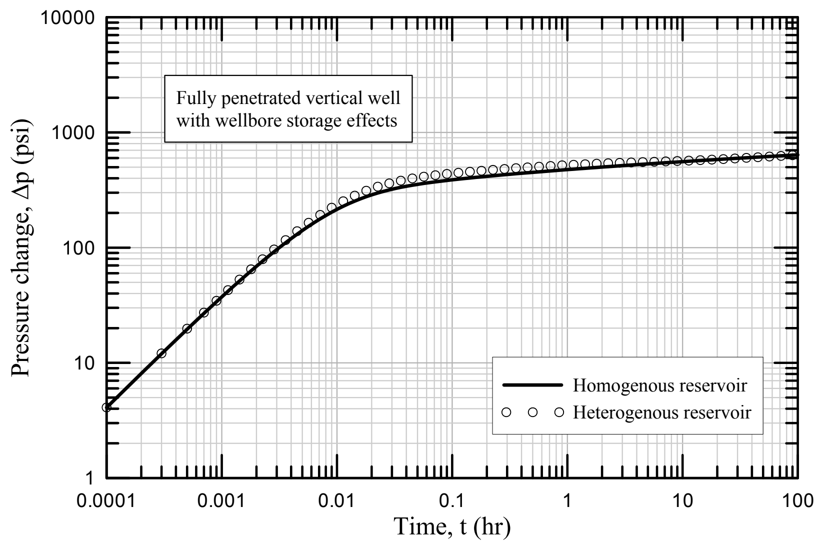


Fig. 1.4. Homogeneous vs heterogeneous reservoir, pressure difference .

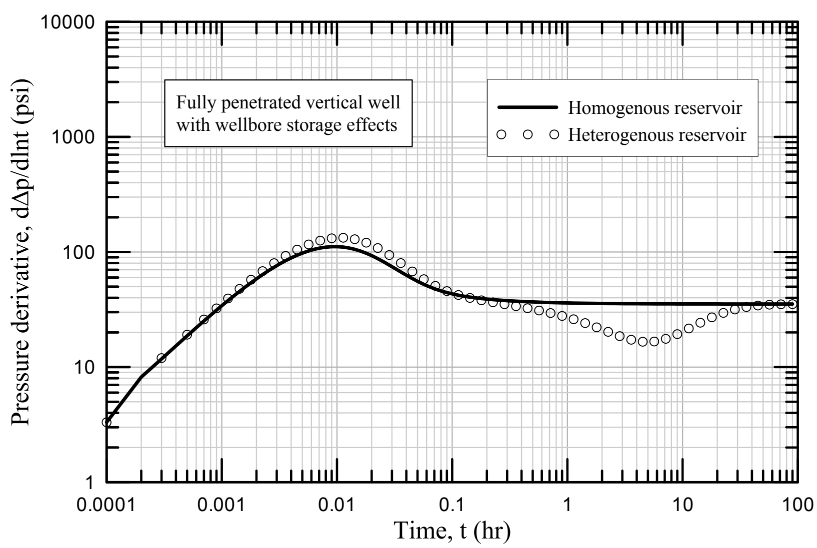


Fig. 1.5. Homogeneous vs heterogeneous reservoir - logarithmic derivative.

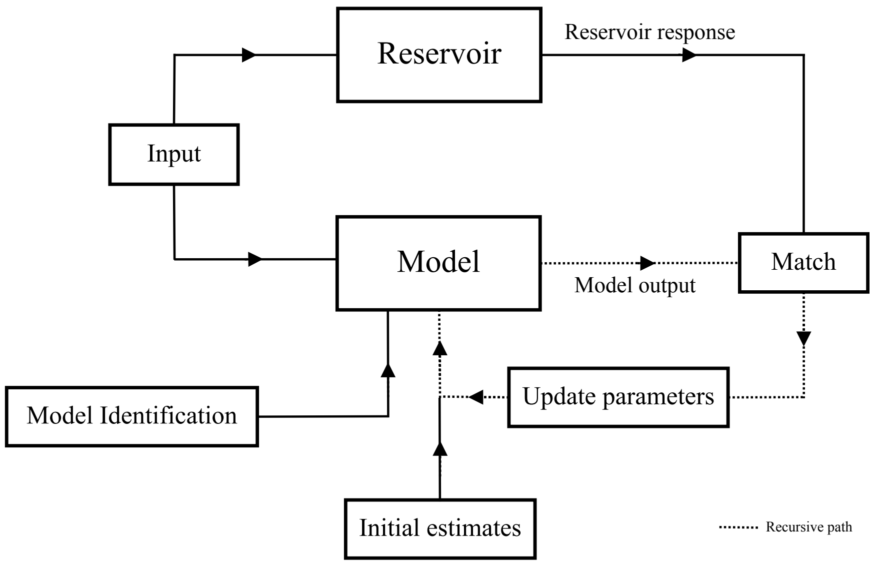


Fig. 1.6. Flow diagram of computer aided parameter estimation.

2.1 Conservation of mass

For a single phase fluid [c11], the mathematical form of the mass balance in porous media is given by

$$-\nabla \cdot (\rho \mathbf{v}) = \frac{\partial (\rho \phi)}{\partial t} \tag{2.1}$$

[c1] where ρ is the density of the fluid in [c2]$\mathbf{M/L^3}$ and $\mathbf{v}$ is the fluid velocity vector in [c3]$\mathbf{L/T}$. Note that the units of Eq. 2.1 is [c4]$\mathbf{M/L^3T}$ It is also important to note that Eq. 2.1 applies for any coordinate system and can be derived either from a mass balance done on a control volume for a coordinate system under consideration or from [c5] divergence theorem (or Gauss Theorem see Supplement II).

In[c6]Cartesian coordinate system,

$$\nabla \cdot (\rho \mathbf{v}) = \frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) \tag{2.2}$$

In[c7]cylindrical coordinate system,

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) \tag{2.3}$$

[c8]

[c11] *Murat Çınar*: ~~(or a fluid composed of one hydrocarbon component)~~

[c1] *Murat Çınar*: I removed the constant -5.615 from the equation

[c2] *Murat Çınar*: ~~lbm/ft^3~~

[c3] *Murat Çınar*: ~~$RB/ft^2 day$~~

[c4] *Murat Çınar*: ~~$lbm/ft^3 day$~~

[c5] *Murat Çınar*: ~~a~~

[c6] *Murat Çınar*: ~~x-y-x~~

[c7] *Murat Çınar*: ~~r-θ-z~~

[c8] *Murat Çınar*: typo in the equation corrected

2.2 Conservation of momentum in porous media

The principle of momentum conservation is described by the equation of motion. For most hydrocarbon fluids, the shear stress - shear rate behavior [c9]is described by the Newton's law of friction, combined with the equation of motion, results in the well known Navier-Stokes equation. Solution of the Navier-Stokes equation with the appropriate boundary conditions yields the velocity distribution of a given problem. Although, it is possible to solve Navier-Stokes in pipe flow, it is almost impossible to solve due to complexity of the pore geometry and its distribution. This hinders the formation of the boundary conditions for flow through a porous medium. Therefore, a different approach [c10]is taken. In 1856, [c11]Darcy discovered that [c12]for a single phase viscous flow in porous media, the velocity is proportional to the pressure gradient with a proportionality constant k. The general form of Darcy's Law including gravity effects is given by Eq.2.4.

[c9] *Murat Çınar*: ~~can be~~

[c10] *Murat Çınar*: ~~must be~~

[c11] *Murat Çınar*: ~~a French hydraulic engineer Henry~~

[c12] *Murat Çınar*: ~~the velocity vector and the pressure gradient for a single phase viscous flow~~

$$\mathbf{v} = -\frac{\mathbf{k}}{\mu}\left(\nabla p - \rho g \nabla z'\right) \tag{2.4}$$

[c1] where $\mathbf{v}$ is defined as a volumetric flow rate across a unit cross-section area (solid+fluid) averaged over a small region of space. The unit of $\mathbf{v}$ is in [c2]**L/T**. Eq. 2.4 yield[c3]s a velocity vector [c4]that replaces the solution of Navier-Stokes equation. In Eq. 2.4 $\mathbf{k}$ is [c5]the permeability tensor. Operationally, $\mathbf{k}$ acts like a matrix in [c6]coordinate system, we usually [c7]assume

[c1] *Murat Çınar*: I removed the filed unit constants

[c2] *Murat Çınar*: ~~*RB/ft²day*~~

[c3] *Murat Çınar*: *Text added.*

[c4] *Murat Çınar*: ~~which~~

[c5] *Murat Çınar*: ~~a~~

[c6] *Murat Çınar*: ~~x-y-z~~

[c7] *Murat Çınar*: ~~use~~

$$\mathbf{k}\,\nabla p = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} = \begin{bmatrix} k_x \frac{\partial p}{\partial x} \\ k_y \frac{\partial p}{\partial y} \\ k_z \frac{\partial p}{\partial z} \end{bmatrix}$$

$$\nabla z' = \begin{bmatrix} \frac{\partial z'}{\partial x} \\ \frac{\partial z'}{\partial y} \\ \frac{\partial z'}{\partial z} \end{bmatrix}$$

where, z' is the direction in which gravity acts, i.e., the direction towards the center of the earth. In [c8]Cartesian coordinate system, [c9] for each velocity component,

[c8] *Murat Çınar*: ~~x-y-z~~

[c9] *Murat Çınar*: ~~Eq. 2.5~~

$$v_\xi = -\frac{k_\xi}{\mu}\left(\frac{\partial p}{\partial \xi} - \rho g \frac{\partial z'}{\partial \xi}\right), \quad \xi = x, y, z \tag{2.5}$$

We generally denote γ as the specific weight of fluid and define as,

$$\gamma = \rho g$$

It follows from Eq. 2.5, 2.2, and 2.1 that the p.d.e. describing conservation of mass in [c10]Cartesian coordinate system is

[c10] *Murat Çınar*: ~~x-y-z~~

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\rho\frac{k_x}{\mu}\left(\frac{\partial p}{\partial x}-\gamma\frac{\partial z'}{\partial x}\right)\right]\\&+\frac{\partial}{\partial y}\left[\rho\frac{k_y}{\mu}\left(\frac{\partial p}{\partial y}-\gamma\frac{\partial z'}{\partial y}\right)\right]\\&+\frac{\partial}{\partial z}\left[\rho\frac{k_z}{\mu}\left(\frac{\partial p}{\partial z}-\gamma\frac{\partial z'}{\partial z}\right)\right]=\frac{\partial}{\partial t}(\rho\phi)\end{aligned} \tag{2.6}$$

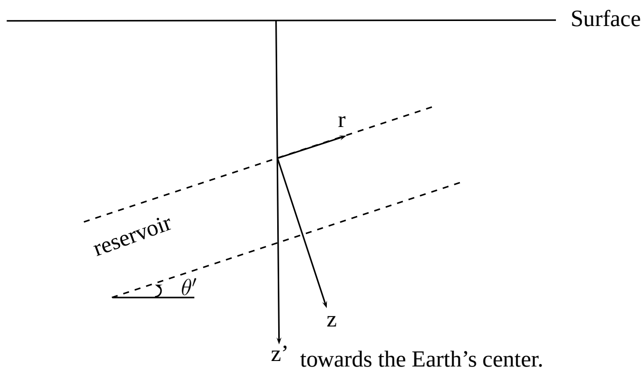
In [c1]Cylindrical coordinates (neglecting flow in θ direction)

[c1] *Murat Çınar*: ~~r-z~~

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{k_r}{\mu}\rho\left(\frac{\partial p}{\partial r}-\gamma\frac{\partial z'}{\partial r}\right)\right]+\frac{\partial}{\partial z}\left[\frac{k_z}{\mu}\rho\left(\frac{\partial p}{\partial z}-\gamma\frac{\partial z'}{\partial z}\right)\right]=\frac{\partial}{\partial t}(\rho\phi) \tag{2.7}$$

Remark on gravity term:

In [c2]Cylindrical coordinates,[c3]assume z and z' are in the same direc-



tion then,

[c2] *Murat Çınar*: ~~r-θ-z~~

[c3] *Murat Çınar*: ~~if reservoir is horizontal; i.e. $\theta'=0$~~

$$\frac{\partial z'}{\partial r}=\frac{\partial z'}{\partial \theta}=0 \quad and \quad \frac{\partial z'}{\partial z}=1$$

[c4]if the reservoir is dipping with an angle of θ',

[c4] *Murat Çınar*: ~~If we had a dip angle θ'~~

$$\frac{\partial z'}{\partial r}=\sin\theta' \quad \frac{\partial z'}{\partial \theta}=\cos\theta' \quad \frac{\partial z'}{\partial z}=\cos\theta'$$

[c5]Now consider a single well [c6]at the center of a cylindrical reser-

[c5] *Murat Çınar*: ~~If we~~

[c6] *Murat Çınar*: ~~in~~

voir, then Eq. 2.7 correctly describes the fluid flow for both partially penetrating and fully penetrating cases, see Fig 2.1.
c7 c8 Now define formation volume factor B as,

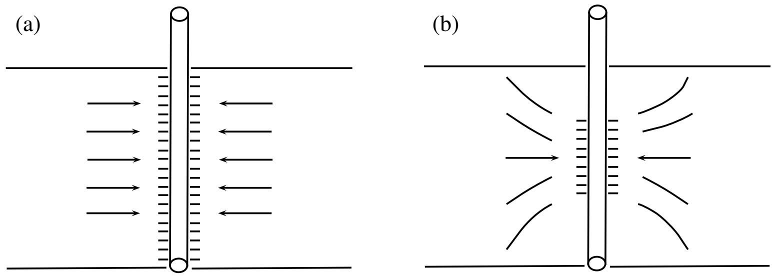


Fig. 2.1. (a) Fully penetrated vertical well. (b) Partially penetrated vertical well.

c7 *Murat Çınar*: ~~Recalling our general continuity equation for the single phase flow gives,~~

c8 *Murat Çınar*: *Text added.*

$$B = \frac{V_{reservoir}}{V_{SC}} = \frac{m/\rho}{m/\rho_{\ SC}} = \frac{\rho_{SC}}{\rho} \quad ; \quad \rho_{SC}\,\text{is constant} \tag{2.8}$$

c1 *Murat Çınar*: ~~which we call it as Formation volume factor, then we can write Equation (2.8) as~~

c2 *Murat Çınar*: *Text added.*

c1 c2 Recall general continuity equation, Eq. 2.1 and insert Eq. 2.8, then we have,

$$-\nabla \cdot \left(\frac{\mathbf{v}}{B}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B}\right) \tag{2.9}$$

$B = B(p)$, $\rho = \rho(p)$, and $\phi = \phi(p) \rightarrow$ single valued functions of p

c3 *Murat Çınar*: *Text added.*

c3 Expanding right hand side (RHS) of Eq. 2.9,

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\phi}{B}\right) &= \phi\frac{\partial}{\partial t}\left(\frac{1}{B}\right) + \frac{1}{B}\frac{\partial \phi}{\partial t} \\ &= \phi\left(-\frac{1}{B^2}\frac{dB}{dp}\right)\frac{\partial p}{\partial t} + \frac{1}{B}\frac{d\phi}{dt}\frac{\partial p}{\partial t} \\ &= \frac{\phi}{B}\left(-\frac{1}{B}\frac{dB}{dp} + \frac{1}{\phi}\frac{d\phi}{dt}\right)\frac{\partial p}{\partial t} \end{aligned} \tag{2.10}$$

c4 *Murat Çınar*: Typo in Eq corrected.

c5 *Murat Çınar*: ~~but~~

c4 c5 Fluid and rock compressibilities are defined, respectively, by,

$$c_{fluid} = -\frac{1}{V}\frac{dV}{dp} = -\frac{1}{B}\frac{dB}{d\rho} = \frac{1}{\rho}\frac{d\rho}{dp} \tag{2.11}$$

$$c_r = c_f = \frac{1}{\phi}\frac{d\phi}{dp} \tag{2.12}$$

(here p is the fluid pressure in the pore, therefore, $\frac{d\phi}{dp} > 0$.)
Using Eqs. 2.11 and 2.12 in Eq. 2.10 and the resulting equation in Eq. 2.9 gives,

$$-\nabla \cdot \left(\frac{\mathbf{v}}{B}\right) = \frac{\phi c_t}{B}\frac{\partial p}{\partial t} \tag{2.13}$$

c1 Note that here the total compressibility c_t is defined as $c_t = c_{fluid} + c_f$. Under the assumptions of Darcy's Law we have,

c1 *Murat Çınar*: *Text added.*

$$\nabla \cdot \left(\frac{\mathbf{k}}{\mu B}\left(\nabla p - \gamma \nabla z'\right)\right) = \frac{\phi c_t}{B}\frac{\partial p}{\partial t} \tag{2.14}$$

Now assuming negligible gravity effects and k/μ is constant then,

$$\frac{k}{\mu}\nabla \cdot (\rho \nabla p) = \phi c_t \rho \frac{\partial p}{\partial t} \tag{2.15}$$

c2 Assuming $c(\nabla p)^2$ is small here $c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$, Eq. 2.15 is well approximated by

c2 *Murat Çınar*: ~~If we assume~~

$$\frac{k}{\mu}\nabla^2 p = \phi c_t \frac{\partial p}{\partial t} \tag{2.16}$$

Remark on coordinate systems:

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \quad \text{in Cartesian coordinates}$$

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} \quad \text{in cylindrical coordinates}$$

$$\begin{aligned}\nabla^2 p = &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial p}{\partial \theta}\right) \\ &+ \frac{1}{r^2 sin^2\theta}\frac{\partial^2 p}{\partial \phi^2}\end{aligned} \quad \text{in spherical coordinates}$$

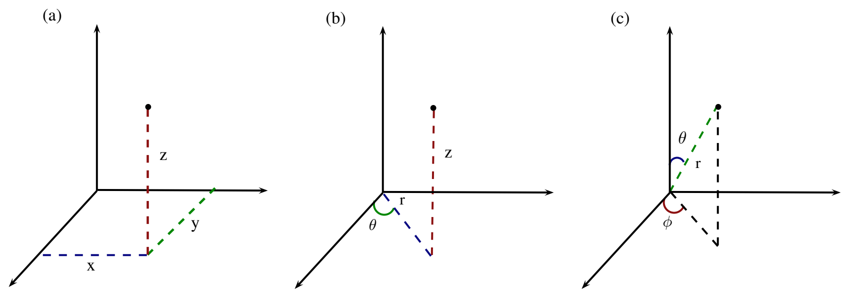


Fig. 2.2. (a) Cartesian coordinates. (b) Cylindrical coordinates. (c) Spherical coordinates.

[c1] *Murat Çınar*: ~~if we assume~~

Further [c1]assuming,

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p} = \text{constant} \tag{2.17}$$

$$\int_{p_i}^{p} c dp = \int_{\rho_i}^{\rho} \frac{1}{\rho} d\rho$$

$$c\,(p - p_i) = \ln\left(\frac{\rho}{\rho_i}\right) \quad ; \quad p_i \text{ initial pressure}$$

or

$$\rho = \rho_i \exp\left[-c\,(p_i - p)\right] \tag{2.18}$$

Using Taylor series representation of $\exp\left[-c\,(p_i - p)\right]$ gives,

$$\begin{aligned}\rho &= \rho_i \left[1 - c\,(p_i - p) + \frac{c^2}{2}(p_i - p)^2 + \cdots\right] \\ &= \rho_i \left[1 - c\,(p_i - p) + \frac{c^2}{2}(p_i - \widetilde{p})^2\right] \quad p < \widetilde{p} < p_i\end{aligned}$$

$$\rho = \rho_i \left[1 - c\,(p_i - p)\right] \tag{2.19}$$

[c2] *Murat Çınar*: *Text added.*

[c2]Note that c is very small for oil (or liquids); $c \approx 10^{-5} \sim 10^{-6}$. Using Eq. 2.19 in Eq. 2.16.

$$\frac{k}{\mu}\nabla^2 p = \phi c_t \frac{\partial p}{\partial t} \tag{2.20}$$

2.3 Multiphase flow

Three distinct phases, gas, oil, and water occur i a petroleum reservoir. Varying pressure conditions (isothermal system assumed) cause a mass exchange between [c3]two hydrocarbon phases [c4]- oil-gas (water-oil and gas-water systems [c5]is assumed immiscible). The [c6]mass transfer between oil and gas is described by solution gas-oil ration, R_s, which gives the amount of gas dissolved in oil as a function of pressure, i.e. $[V_{dissolved\,gas}/V_o]_{STC}$.

[c3] *Murat Çınar*: ~~the~~

[c4] *Murat Çınar*: *Text added.*

[c5] *Murat Çınar*: ~~can be~~

[c6] *Murat Çınar*: ~~material~~

The fluid flow equations (based on β-model) with the introduction of phase saturations for oil-water-gas system [c1]is written as;

[c1] *Murat Çınar*: ~~can be~~

$$-\nabla\cdot\left(\frac{\mathbf{v}_o}{B_o}\right)=\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right)\qquad for\ oil \tag{2.21}$$

$$-\nabla\cdot\left(\frac{\mathbf{v}_w}{B_w}\right)=\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right)\qquad for\ water \tag{2.22}$$

$$-\nabla\cdot\left(\frac{R_s\mathbf{v}_o}{B_o}+\frac{\mathbf{v}_g}{B_g}\right)=\frac{\partial}{\partial t}\left[\phi\left(\frac{R_s}{B_o}S_o+\frac{S_g}{B_g}\right)\right]\qquad for\ gas \tag{2.23}$$

With introducing relative permeability [c2], the velocity vector for each phase is given by,

[c2] *Murat Çınar*: ~~concept~~

$$\mathbf{v}_\varphi=-\frac{\mathbf{k}\,k_{r\varphi}}{\mu_\varphi}\left(\nabla p_\varphi-\gamma_\varphi\nabla z'\right) \tag{2.24}$$

where $\varphi=o,w,or g$.

Using Eq. 2.24 in Eqs. 2.21, 2.22, and 2.23 for corresponding [c3]phase,

[c3] *Murat Çınar*: ~~Hydro Carbon component~~

$$\nabla\cdot\left(\frac{\mathbf{k}\,k_{ro}}{B_o\mu_o}\left(\nabla p_o-\gamma_o\nabla z'\right)\right)=\frac{\partial}{\partial t}\left(\frac{\phi S_o}{B_o}\right) \tag{2.25}$$

$$\nabla\cdot\left(\frac{\mathbf{k}\,k_{rw}}{B_w\mu_w}\left(\nabla p_w-\gamma_w\nabla z'\right)\right)=\frac{\partial}{\partial t}\left(\frac{\phi S_w}{B_w}\right) \tag{2.26}$$

$$\begin{aligned}\nabla\cdot\left(\frac{R_s\mathbf{k}\,k_{ro}}{B_o\mu_o}\left(\nabla p_o-\gamma_o\nabla z'\right)+\frac{\mathbf{k}\,k_{rg}}{B_g\mu_g}\left(\nabla p_g-\gamma_g\nabla z'\right)\right)\\=\frac{\partial}{\partial t}\left[\phi\left(\frac{R_s}{B_o}S_o+\frac{S_g}{B_g}\right)\right]\end{aligned} \tag{2.27}$$

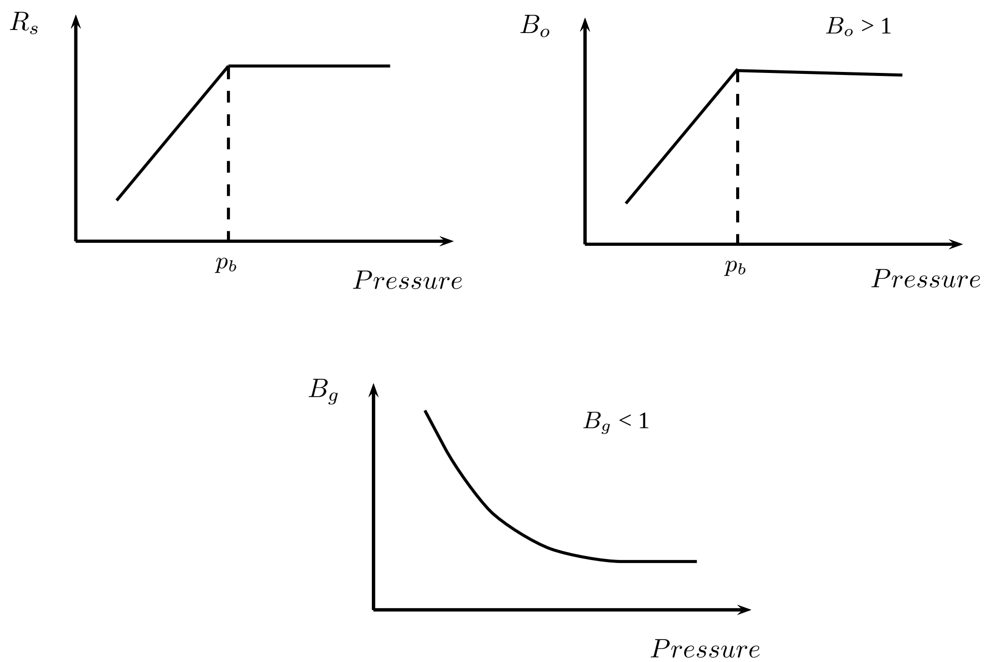


Fig. 2.3. R_s, B_o, B_g behavior.

It is important to note that B defined by Eq. 2.8[c4], considering oil, [c5] is [c6]valid if [c7]oil is single component or [c8] "black-oil" with no dissolved gas ($R_s = 0$). [c9]On the other hand, Eq. 2.14 is [c10]valid for black-oil problems with $R_s \neq 0$ provided that [c11]the pressure is above bubble-point. [c12]Eq. 2.25 and Eq. 2.27 [c13]becomes identical[c14] if the pressure is above bubble point.

[c1]In some cases, the flow equation is written in terms of pseudo potential.

$$\psi = \int_{p_0}^{p} \frac{1}{\gamma} dp - (z' - {z_0}') \tag{2.28}$$

where z' is measured positive in the direction of gravity and ${z_0}'$ is the datum where [c2]p_0 [c3]is measured. Also $\gamma = \mathbf{M}/\mathbf{L}^2\mathbf{T}^2$) and $\gamma = \gamma(p)$.

$$\begin{aligned}\nabla\psi &= \nabla \int_{p_0}^{p} \frac{1}{\gamma} dp - \nabla (z' - {z_0}') \\ &= \frac{d}{dp}\left\{\int_{p_0}^{p} \frac{1}{\gamma} dp\right\} \nabla p - \nabla z' \\ &= \frac{1}{\gamma}\nabla p - \nabla z'\end{aligned}$$

Then,

$$\gamma\, \nabla\psi = \nabla p - \gamma\, \nabla z' \tag{2.29}$$

and

$$\frac{\partial\psi}{\partial t} = \frac{1}{\gamma}\frac{\partial p}{\partial t} \tag{2.30}$$

Using Eqs. 2.29 and 2.30 in Eq. 2.14 we obtain,

$$\nabla \cdot \left(\frac{\mathbf{k}\gamma}{B\mu}\nabla\psi\right) = \frac{\phi c_t \gamma}{B}\frac{\partial\psi}{\partial t} \tag{2.31}$$

If we consider that we have stress dependent reservoir, that is permeability decreases as the fluid pressure in the pores decreases, then

$$\mathbf{k} = k\,(p)\,\widetilde{\mathbf{k}}$$

where the entries of $\widetilde{\mathbf{k}}$ are independent of pressure. It has been observed that tight (and geothermal) reservoirs are [c4] examples of stress

[c4] *Murat Çınar*: *Text added.*
[c5] *Murat Çınar*: ~~this~~
[c9] *Murat Çınar*: ~~However~~
[c6] *Murat Çınar*: ~~correct~~
[c10] *Murat Çınar*: ~~correct~~
[c7] *Murat Çınar*: ~~we have a~~
[c11] *Murat Çınar*: ~~we are~~
[c8] *Murat Çınar*: ~~if we are talking about~~
[c12] *Murat Çınar*: ~~If we are above bubble-point,~~
[c13] *Murat Çınar*: ~~will be~~
[c14] *Murat Çınar*: *Text added.*
[c1] *Murat Çınar*: ~~Sometimes we write the flow equation~~
[c2] *Murat Çınar*: ~~we take as reference~~
[c3] *Murat Çınar*: *Text added.*
[c4] *Murat Çınar*: ~~good~~

dependent reservoirs. Then Eq. 2.31 [c5]is expressed as

[c5] *Murat Çınar*: ~~can be~~

$$\nabla \cdot \left(\frac{k(p)\,\widetilde{\mathbf{k}}\gamma}{B\mu} \nabla \psi \right) = \frac{\phi c_t \gamma}{B} \frac{\partial \psi}{\partial t} \tag{2.32}$$

[c1] [c2]A pseudo pressure function is defined to partially linearize Eq. 2.32 (or Eq. 2.14) under the assumption that the gravity effects are not important.

[c1] *Murat Çınar*: ~~It is important to note that Eq. 2.38 is non-linear because ϕ, c_t, γ, μ, B, and k are all pressure dependent (or potential). To partially linearize the Eq. 2.38 (or Eq. 2.14), we normally define a pseudo pressure function if gravity effects are not important.~~

[c2] *Murat Çınar*: Following paragraph is added.

$$m(p) = \int_{p_0}^{p} \frac{k(p)}{\mu(p)\,B(p)} dp \tag{2.33}$$

$$\nabla m(p) = \frac{d}{dp} m(p) \nabla p = \frac{k(p)}{\mu(p)\,B(p)} \nabla p \tag{2.34}$$

$$\frac{\partial m(p)}{\partial t} = \frac{k(p)}{\mu(p)\,B(p)} \frac{\partial p}{\partial t} \tag{2.35}$$

If [c3]gravity effects are ignored, Eq. 2.14 reduces to

[c3] *Murat Çınar*: ~~we don't have gravity effects note that~~

$$\nabla \cdot \left(\frac{k(p)\,\widetilde{\mathbf{k}}}{B(p)\,\mu(p)} \nabla p \right) = \frac{\phi c_t}{B} \frac{\partial p}{\partial t} \tag{2.36}$$

Using Eqs. 2.33, 2.34, and 2.35 in 2.36 gives

$$\nabla \cdot \left(\widetilde{\mathbf{k}} \nabla m(p) \right) = \frac{\phi(p)\, c_t(p)\, \mu(p)}{k(p)} \frac{\partial m(p)}{\partial t} \tag{2.37}$$

Note Eq. 2.37 is still non-linear. [c4] Eq. 2.32 [c5]is the expression of basic flow equations in terms of the potential ψ. Therefore, to partially linearize Eq. 2.32, [c6]a "pseudo pressure" is defined as

[c4] *Murat Çınar*: ~~In~~

[c5] *Murat Çınar*: ~~we have written our basic flow equations~~

[c6] *Murat Çınar*: ~~we can define a "pseudo pressure" by~~

$$m(\psi) = \int_{\psi_0}^{\psi} \frac{k(\psi)\,\gamma(\psi)}{\mu(\psi)\,B(\psi)} d\psi \tag{2.38}$$

2.4 Diffusivity equation for single phase gas flow - real gas flow

For [c7]gases μ, ρ are strong functions of pressure. Permeability typically is independent of pressure[c8], however, at low pressures Klinkenberg effect may cause some pressure dependence in permeability and/or [c9]tight reservoirs are considered as discussed earlier. To account for the dependence of $k/\mu B_g$ on pressure,[c10] Eq.2.33 or 2.38 [c11]is used. Note that Eq. 2.37 is also valid for [c12]flow of real gases in

[c7] *Murat Çınar*: ~~gas~~

[c8] *Murat Çınar*: ~~although~~

[c9] *Murat Çınar*: ~~if we have tight reservoirs~~

[c10] *Murat Çınar*: ~~we can use~~

[c11] *Murat Çınar*: *Text added.*

[c12] *Murat Çınar*: ~~real flow of~~

porous media.

With some modifications, the above procedure is the current approach used to derive the p.d.e. for gas flow. The method was first introduced in the literature by Al-Hussainy, Ramey and Crawford [1]. Below [c13]the p.d.e. [c14]is derived using Al-Hussainy et. al. [1] approach. [c15c16] Note that Eq. 2.36 holds for real gases. [c17]Assuming $k(p)\mathbf{k}$ is independent of pressure and same in all directions, then Eq. 2.36 [c18]becomes

[c13] *Murat Çınar*: ~~we will derive~~
[c14] *Murat Çınar*: *Text added.*
[c15] *Murat Çınar*: ~~Since Eq. 2.32 is also valid for real gases, we have~~
[c16] *Murat Çınar*: The following sentence is added.
[c17] *Murat Çınar*: ~~Further, assuming~~
[c18] *Murat Çınar*: ~~can be written as~~

$$\nabla \cdot \left(\frac{1}{B\left(p\right)\mu\left(p\right)} \nabla p \right) = \frac{\phi c_t}{kB} \frac{\partial p}{\partial t} \tag{2.39}$$

Since $B = \frac{(\rho_g)_{SC}}{\rho_g}$ and $(\rho_g)_{SC}$ is constant Eq. 2.39 is equivalent to

$$\nabla \cdot \left(\frac{\rho}{\mu} \nabla p \right) = \frac{\phi c_t \rho}{k} \frac{\partial p}{\partial t} \tag{2.40}$$

Recall that ρ for real gases is given by the following equation of state (EOS),

$$\rho = \frac{pM}{zRT} \tag{2.41}$$

Using Eq. 2.41 in 2.40 gives

$$\nabla \cdot \left(\frac{pM}{zRT\mu} \nabla p \right) = \frac{pM}{zRT} \frac{\phi c_t}{k} \frac{\partial p}{\partial t} \tag{2.42}$$

Since M/RT is constant, then Eq. 2.42 reduces to

$$\nabla \cdot \left(\frac{p}{z\mu} \nabla p \right) = \frac{\phi c_t p}{zk} \frac{\partial p}{\partial t} \tag{2.43}$$

Al-Hussainy et. al. [1] defined the integral transform $m'(p)$ to be

$$m'\left(p\right) = 2 \int_{p_0}^{p} \frac{p'}{\mu z} dp' \tag{2.44}$$

$$\nabla m'\left(p\right) = \frac{2p}{\mu z} \nabla p \tag{2.45}$$

$$\frac{\partial m'}{\partial t} = 2 \frac{p}{\mu z} \frac{\partial p}{\partial t} \tag{2.46}$$

Using Eqs. 2.44, 2.45, and 2.46 in 2.43 gives,

$$\nabla \cdot \left[\nabla m'\left(p\right) \right] = \frac{\phi c_t \mu\left(p\right)}{k} \frac{\partial m'\left(p\right)}{\partial t} \tag{2.47}$$

2.5 1-D Radial flow equation

[c1] *Murat Çınar*: ~~we have axisymmetric flow~~

Consider a completely penetrating well in an infinite porous medium of uniform thickness filled with a single phase fluid. Further assume that [c1]flow is axisymmetric, i.e., no variation in θ-direction or in a plane z'=constant equipotential curves are circles - see Figure 2.48,

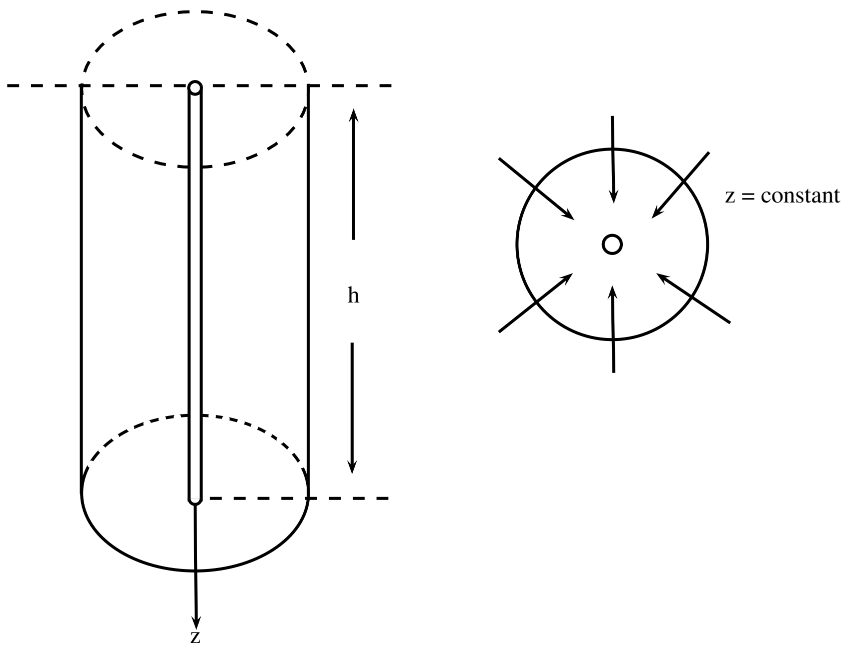


Fig. 2.4. Radial flow geometry.

[c1] *Murat Çınar*: ~~our general equation;~~

If reservoir is not horizontal, [c1] Eq. 2.14, applies with $v_\theta = 0$ and so Eq. 2.14 becomes in $r - z$ coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r k_r}{\mu B}\left(\frac{\partial p}{\partial r} - \gamma\frac{\partial z'}{\partial r}\right)\right] + \frac{\partial}{\partial z}\left[\frac{k_z}{\mu B}\left(\frac{\partial p}{\partial z} - \gamma\frac{\partial z'}{\partial z}\right)\right] = \frac{\phi c_t}{B}\frac{\partial p}{\partial t} \tag{2.48}$$

[c2] *Murat Çınar*: ~~If we assume~~

[c2]Now assume $z = z'$ and $\theta = 0$, then $\frac{\partial z'}{\partial r} = 0$

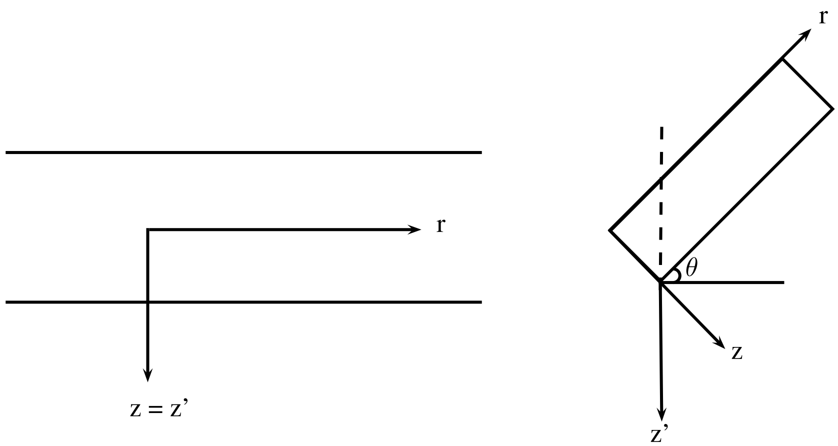


Fig. 2.5. r-z coordinates.

For a completely penetrating well, it is physically reasonable to assume $v_z \approx 0$ ($k_z \ll k_r$), i.e.,

$$
\begin{aligned}
&v_z = -\frac{k_z}{\mu}\left(\frac{\partial p}{\partial z} - \gamma \frac{\partial z'}{\partial z}\right) = 0 \quad ; \quad \frac{\partial z'}{\partial z} = 1 \\
&\frac{\partial p}{\partial z} - \gamma = 0 \quad ; \quad p\left(z_2\right) = p\left(z_1\right) + \gamma\left(z_2 - z_1\right) \quad z_z > z_1
\end{aligned}
\tag{2.49}
$$

Then [c1]the general radial flow problem [c2]becomes,

$$
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{k_r}{\mu B}\frac{\partial p}{\partial r}\right) = \frac{\phi c_t}{B}\frac{\partial p}{\partial t} \tag{2.50}
$$

where ϕ, c_t, B, k_r, and μ [c3]are functions of pressure.
[c4]Recall pseudo-pressure function defined as,

$$
m\left(p\right) = \int_{p_b}^{p} \frac{k_r\left(p\right)}{\mu\left(p\right)B\left(p\right)}dp \tag{2.51}
$$

Using Eq. 2.51, [c5] Eq. 2.50 [c6]is written as,

$$
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m\left(p\right)}{\partial r}\right) = \frac{\phi c_t \mu}{k_r}\frac{\partial m\left(p\right)}{\partial t} \tag{2.52}
$$

Dimensionless Variables In well testing [c7]dimensionless variables are used for two main reasons;

[c1] *Murat Çınar*: ~~we arrive at~~

[c2] *Murat Çınar*: *Text added.*

[c3] *Murat Çınar*: ~~can be~~

[c4] *Murat Çınar*: ~~Recalling our~~

[c5] *Murat Çınar*: ~~we write~~

[c6] *Murat Çınar*: *Text added.*

[c7] *Murat Çınar*: ~~for two reasons, we are using dimensionless variables to prevent our results,~~

(i) minimize number of variables [c8] (by grouping parameters)
(ii) provide general solutions

[c8] *Murat Çınar*: ~~(find group parameters)~~

[c1] *Murat Çınar*: ~~If we define a dimensionless time~~

[c1] Dimensionless time is defined as,

$$t_D = \frac{k_i t}{(\phi c_t \mu)_i r_w^2} \tag{2.53}$$

where subscript "i" refers to initial conditions, i.e.,

$$k_i = k(p_i) \;\; ; \;\; \mu_i = \mu(p_i) \;\; etc.$$

here p_i is the initial reservoir pressure (at some datum) and we assume p_i is independent of r, then

$$\frac{\partial m}{\partial t} = \frac{\partial m}{\partial t_D}\frac{\partial t_D}{\partial t} = \frac{\partial m}{\partial t_D}\left(\frac{k_i}{(\phi c_t \mu)_i r_w^2}\right) \tag{2.54}$$

[c2] *Murat Çınar*: Typo corrected in the following equation

Using Eq. 2.54 in Eq. 2.52, and simplifying gives,[c2]

$$r_w^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m\,(p)}{\partial r}\right)\right] = \left(\frac{\phi c_t \mu}{k}\right)\left[\frac{k_i}{(\phi c_t \mu)_i}\right]\frac{\partial m}{\partial t_D} \tag{2.55}$$

[c3] *Murat Çınar*: ~~If we~~

[c3] Now define,

$$r_D = \frac{r}{r_w} \tag{2.56}$$

[c4] *Murat Çınar*: *Text added.*

and [c4] dimensionless diffusivity,

$$\eta_D = \frac{k/(\phi c_t \mu)}{k_i/(\phi c_t \mu)_i} \tag{2.57}$$

[c5] *Murat Çınar*: ~~we can write~~

[c6] *Murat Çınar*: ~~as~~

then[c5] Eq. 2.55 [c6] becomes

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial m\,(p)}{\partial r_D}\right) = \frac{1}{\eta_D}\frac{\partial m\,(p)}{\partial t_D} \tag{2.58}$$

If $\frac{1}{\eta_D} = 0$, then Eq. 2.58 is a linear p.d.e. .

[c7] *Murat Çınar*: ~~If we assume~~

Slightly compressible fluid of constant compressibility [c7] Consider production at a specified rate q; i.e.,

Flow rate out $= qB = \int\limits_S \mathbf{v}\,\mathbf{n}\,dS$ and $\mathbf{n}$ is the unit outward normal to S and is equal to

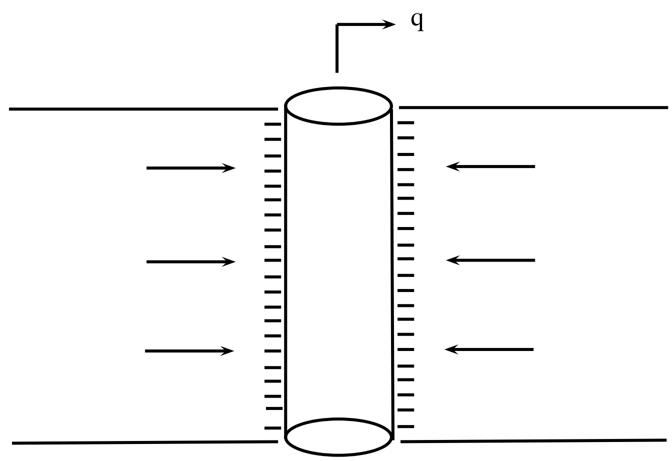


Fig. 2.6. Production from a vertical full penetrating well. Volumetric flux, q, into the wellbore (flow out of the reservoir the boundary represented by wellbore).

$$\mathbf{n} = -i_r + 0i_\theta + 0i_z = (1,0,0)$$
$$\mathbf{v} = (v_r + v_\theta + v_z)$$

$$qB = \int_S -v_r|_{r=r_w} dS \quad ; \quad ds = r_w d\theta dz$$

$$qB = \int_0^{2\pi} \int_0^h (-v_r)_{r_w} r_w d\theta dz$$

$$q = \int_0^{2\pi} \int_0^h \frac{k}{\mu B} \left(r\frac{\partial p}{\partial r} \right)_{r=r_w} d\theta dz$$

$$q = 2\pi \int_0^h \left(r\frac{\partial m}{\partial r} \right)_{r_w} dz$$

[c1]Now we assume [c2]that variation of $r\frac{\partial m(p)}{\partial r}$ in z-direction is insignificant or

[c1]*Murat Çınar*: ~~If~~

[c2]*Murat Çınar*: *Text added.*

$$\int_0^h \left(r\frac{\partial m\,(p)}{\partial r} \right)_{r_w} dz = \left(r\frac{\partial m\,(p)}{\partial r} \right)_{r_w,\widehat{z}} h$$

where $\widehat{z}$ is a mean value between $0 \leq \widehat{z} \leq h$. Then the boundary condition is

$$q = 2\pi h\left(r\frac{\partial m\left(p\right)}{\partial r}\right)_{r_w} \tag{2.59}$$

[c1] *Murat Çınar*: ~~If we define~~

[c1]Define,

$$\begin{aligned} m_D &= \frac{2\pi h\left[m\left(p_i\right)-m\left(p\right)\right]}{q} \\ &= \frac{h}{q}\left[m\left(p_i\right)-m\left(p\right)\right] \\ &= \frac{h}{q}\int_{p}^{p_i}\frac{k\left(p\right)}{\mu\left(p\right)B\left(p\right)}dp \end{aligned} \tag{2.60}$$

[c2] *Murat Çınar*: ~~one~~

[c3] *Murat Çınar*: ~~, respectively, can be~~

Then [c2]we can show that Eqs. 2.58 and 2.59[c3] is written as

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial m_D}{\partial r_D}\right) = \frac{1}{\eta_D}\frac{\partial m_D}{\partial t_D} \tag{2.61}$$

$$\left(r_D\frac{\partial m_D}{\partial r_D}\right)_{r_D=1} = -1 \tag{2.62}$$

Note that $r_D = 1$ corresponds to $r = r_w$. Initial condition, $p = p_i$ at values of r at $\widehat{z}$. $m_D = 0$ at $t_D = 0$ then,

$$p\left(r,t\right)|_{t=0} = p_i \tag{2.63}$$

[c4] *Murat Çınar*: ~~Since we consider an infinite reservoir, then~~

[c4]Infinite acting reservoir is considered implying,

$$\lim_{r\to\omega} p\left(r,t\right) = p_i$$

which corresponds to

$$\lim_{r_D\to\infty} m_D\left(r_D,t_D\right) = 0 \tag{2.64}$$

[c5] *Murat Çınar*: ~~we have the following IBVP,~~

In summary, [c5]the following initial boundary value problem (IBVP) is achieved with the appropriate boundary conditions.

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial m_D}{\partial r_D}\right) = \frac{1}{\eta_D}\frac{\partial m_D}{\partial t_D} \tag{2.65}$$

$$\left(r_D\frac{\partial m_D}{\partial r_D}\right)_{r_D=1} = -1 \tag{2.66}$$

$$\lim_{r_D \to \infty} m_D\left(r_D, t_D\right) = 0 \tag{2.67}$$

$$m_D\left(r_D, t_D = 0\right) = 0 \tag{2.68}$$

Eqs. 2.65-2.68 lead to give a complete mathematical description of [c6]the physical problem. Because of η_D term, it is a non-linear IBVP. It can also be solved analytically (see Kale and Mattar[3] or Peres et.al.[4])

[c6] *Murat Çınar*: ~~our~~

[c1]For simplicity, assume that variations in k, ϕ, c_t, and B are small ("negligible") for the pressure change considered. [c2]Then,

[c1] *Murat Çınar*: ~~Let's now, for~~

[c2] *Murat Çınar*: ~~wit this assumption~~

$$\eta_D = \frac{\left(k/\phi c_t \mu\ \right)}{\left(k/\phi c_t \mu\ \right)_{p_i}} \approx 1 \tag{2.69}$$

and

$$\eta_D = \frac{\left(k/\phi c_t \mu\ \right)}{\left(k/\phi c_t \mu\ \right)_{p_i}} \approx 1 \tag{2.70}$$

$$m\left(p_i\right) - m\left(p\right) = \int_p^{p_i} \frac{k\left(p\right)}{\mu\left(p\right) B\left(p\right)} dp \approx \frac{k_i}{\mu_i B_i}\left(p_i - p\right) \tag{2.71}$$

and then it follows from Eq. 2.60 that

$$m_D = \frac{k_i h\left(p_i - p\right)}{q B_i \mu_i} = p_D = \frac{kh\left(p_i - p\right)}{q B \mu} \tag{2.72}$$

[c3]that is the definition of dimensionless pressure p_D in well testing. Considering $\frac{1}{\eta_D} \approx 1$ in Eq. 2.65, we have

[c3] *Murat Çınar*: ~~which is the normal~~

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial m_D}{\partial r_D}\right) = \frac{\partial m_D}{\partial t_D} \tag{2.73}$$

$$\left(r_D \frac{\partial m_D}{\partial r_D}\right)_{r_D = 1} = -1 \tag{2.74}$$

$$\lim_{r_D \to \infty} m_D\left(r_D, t_D\right) = 0 \tag{2.75}$$

$$m_D\left(r_D, t_D = 0\right) = 0 \tag{2.76}$$

Note that Eq. 2.73 is a linear p.d.e. . We seek a solution to the IBVP given by Eqs. 2.73 - 2.76. To find a solution we assume that

$$m_D = m_D\left(\varepsilon_D\right)$$

where

$$\varepsilon_D = \frac{r_D^2}{4t_D} = \varepsilon_D\left(r_D, t_D\right)$$

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