ELEG 310

5/1/2018

X_1, X_2, X_3 independent

$X_1 \sim N(0, 1) \quad X_2 \sim N(-2, 4) \quad X_3 \sim N(3, 16)$

$Y_1 = X_1 + X_3 \quad Y_2 = X_2 - X_3$

$E(Y_1) = E(X_1 + X_3) = E(X_1) + E(X_3) = 0 + 3 = 3$

$E(Y_2) = E(X_2 - X_3) = E(X_2) - E(X_3) = -2 - 3 = -5$

$Var(Y_1) = Var(X_1 + X_3) = Var(X_1) + 2\,Cov(X_1, X_3) + Var(X_3)$

$= 1 + 0 + 16 = 17$

$Var(Y_2) = Var(X_2) - 2\,Cov(X_2, X_3) + Var(X_3)$

$= 4 - 2 \times 0 + 16 = 20$

$$\operatorname{Cov}(Y_1, Y_2) = E\big((Y_1 - E(Y_1))(Y_2 - E(Y_2))\big)$$

$$= E\big((X_1 + X_3 - 0 - 3)(X_2 - X_3 - (-2) + 3)\big)$$

$$= E\Big(\big((X_1 - 0) + (X_3 - 3)\big)\big((X_2 + 2) - (X_3 - 3)\big)\Big)$$

$$= E\big((X_1 - 0)(X_2 + 2)\big) - E\big((X_1 - 0)(X_3 - 3)\big)$$

$$+ E\big((X_3 - 3)(X_2 + 2)\big) - E\big((X_3 - 3)^2\big)$$

$$= \operatorname{Cov}(X_1, X_2) - \operatorname{Cov}(X_1, X_3) + \operatorname{Cov}(X_3, X_2) - \operatorname{Var}(X_3)$$

$$= 0 - 0 + 0 - 16 = -16$$

$\Rightarrow E(Y_1) = 3 \qquad \operatorname{Var}(Y_1) = 17 \qquad Y_1 \sim N(3, 17)$

$E(Y_2) = -5 \qquad \operatorname{Var}(Y_2) = 20 \qquad Y_2 \sim N(-5, 20)$

$f(Y_1, Y_2) = ?$

$$u = \frac{Y_1 - E(Y_1)}{\sigma_1} \qquad v = \frac{Y_2 - E(Y_2)}{\sigma_2}$$

$$f(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u^2 - 2\rho uv + v^2}{1-\rho^2}\right)}$$

$$\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

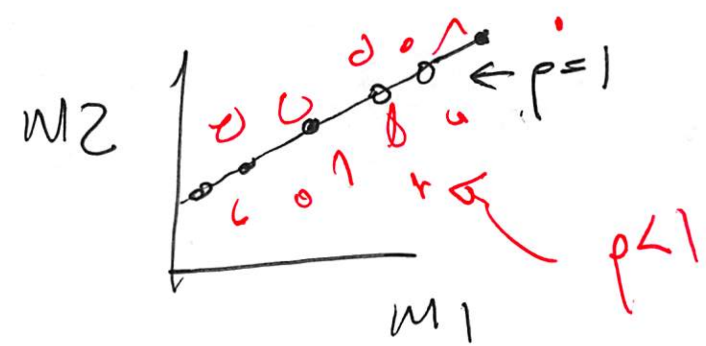
= correlation coefficient

$$\sigma_{12} = \mathrm{Cov}(Y_1, Y_2)$$

$$\sigma_1 = \sqrt{\mathrm{Var}(Y_1)}$$

$$\sigma_2 = \sqrt{\mathrm{Var}(Y_2)}$$

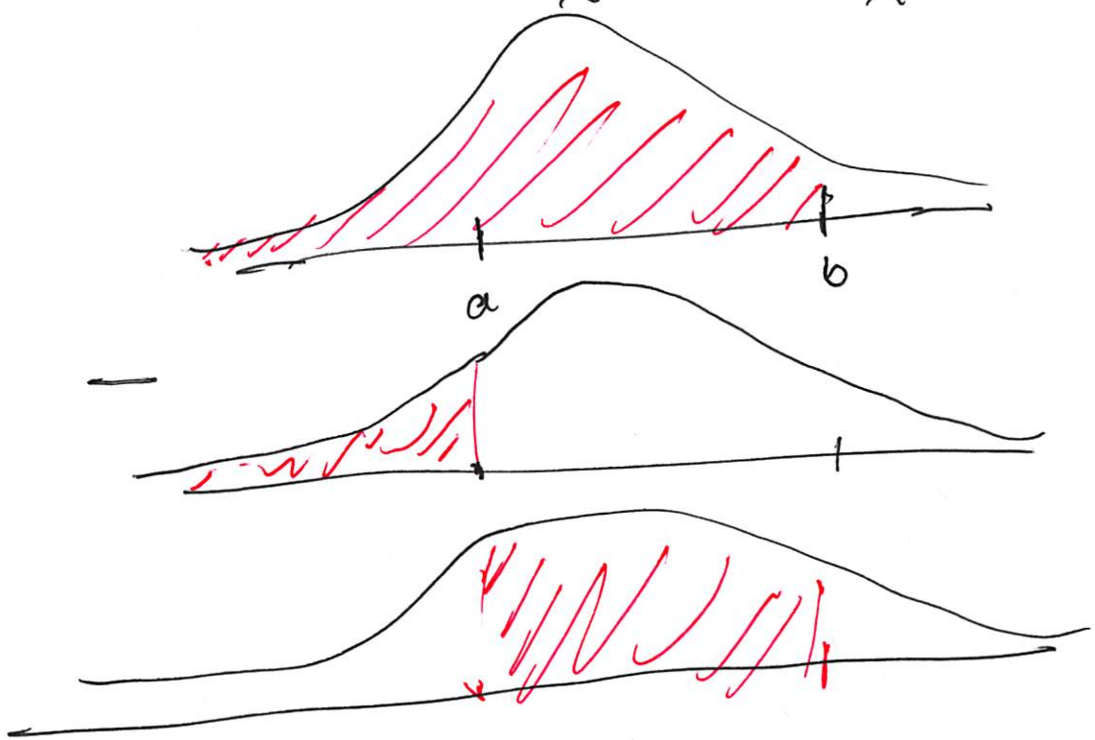
$$-1 \le \rho \le 1$$



$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

$$F_X(x) = P(X \le x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f_X(w)\,dw$$

$$P(a \le X \le b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f(w)\,dw - \int_{-\infty}^{a} f(w)\,dw$$



$X \sim N(\mu, \sigma^2)$

$$P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$$

$$= F_Z\left(\frac{b-\mu}{\sigma}\right) - F_Z\left(\frac{a-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$X \sim N(-1, 4)$ $\quad P(-2 \leq X \leq 2) = \Phi\left(\frac{2-(-1)}{2}\right) - \Phi\left(\frac{-1}{2}\right)$

$$= \Phi\left(\frac{3}{2}\right) - \Phi\left(-\frac{1}{2}\right)$$

$$= \Phi\left(\frac{3}{2}\right) + \Phi\left(\frac{1}{2}\right) - 1 = 0.9332 + 0.6915 - 1$$

$$f_{X|Y=y}(x \mid Y=y) = \frac{f_{XY}(x,y)}{f(y)}$$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Bayes Theorem

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$

$$E(X \mid Y=y) = \int_{-\infty}^{\infty} x\, f(x \mid Y=y)\, dx$$

$$f_{Y|X=x}(y \mid X=x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{f(x,y)\, f(y)}{f(x)\, f(y)}$$

$$= \frac{f(x \mid Y=y)\, f(y)}{f(x)}$$