SOLUTION TO MATH PRACTICE PROBLEMS

1. Find the roots of

(a) $x^2 + 2x - 15 = (x-3)(x+5) = 0$

So, the roots are 3 and -5.

(b) $x^3 + 12x^2 + 27x = x(x+3)(x+9) = 0$

So, the roots are 0, -3, and -9.

2. Evaluate

(a) $\tan 30^\circ = \dfrac{\sin 30^\circ}{\cos 30^\circ} = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$

(b) If $\sin\theta = \dfrac{1}{3}$, what is $\cos\theta$?

$\sin^2\theta + \cos^2\theta = 1$, so $\cos^2\theta = 1 - \sin^2\theta = 1 - \dfrac{1}{9} = \dfrac{8}{9}$

Therefore, $\cos\theta = \pm\dfrac{2\sqrt{2}}{3}$

(c) $\cos 45^\circ = \dfrac{\sqrt{2}}{2}$

3. Multiply the following (with $j = \sqrt{-1}$):

(a) $(3+4j)(1-2j) = 3 - 6j + 4j - 8(j)^2 = 3 - 6j + 4j - 8(-1) = 11 - 2j$

(b) $(3+4j)(1-2j)^* = (3+4j)(1+2j) = 3 + 6j + 4j + 8(j)^2 = 3 + 6j + 4j - 8 = -5 + 10j$

(c) $\dfrac{1}{2+j}(\dfrac{3+4j}{1+j}) = \dfrac{3+4j}{2+2j+j-1} = \dfrac{3+4j}{1+3j} = \dfrac{(3+4j)(1-3j)}{(1+3j)(1-3j)} = \dfrac{3+4j-9j+12}{1+9}$
$= \dfrac{15-5j}{10} = \dfrac{3-j}{2}$

4. Convert to polar coordinates (i.e., $re^{j\theta}$)

(a) $1 - j$

$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}(\dfrac{-1}{1}) = -45^\circ$

(b) $-2 + j$

$r = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

$\theta = \cos^{-1}(\frac{-2}{\sqrt{5}}) = 153.4°$

(c) $\frac{4+3j}{1-j} = \frac{(4+3j)(1+j)}{(1-j)(1+j)} = \frac{4+3j+4j-3}{1+1} = \frac{1+7j}{2} = \frac{1}{2} + \frac{7}{2}j$

$r = \sqrt{(\frac{1}{2})^2 + (\frac{7}{2})^2} = \sqrt{\frac{50}{4}} = \frac{5\sqrt{2}}{2}$

$\theta = \tan^{-1}(\frac{7/2}{1/2}) = \tan^{-1}(7) = 81.9°$

5. Convert to rectangular coordinates (use Euler's Relation)

(a) $e^{j\frac{\pi}{3}} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$

(b) $3e^{j\frac{\pi}{6}} = 3(\cos\frac{\pi}{6} + j\sin\frac{\pi}{6}) = 3(\frac{\sqrt{3}}{2} + \frac{1}{2}j) = \frac{3}{2}(\sqrt{3}+j)$

(c) $2e^{j\frac{\pi}{2}} + 3e^{-j\frac{\pi}{2}} = 2j + 3(-j) = -j$

6. Compute the absolute value of

(a) $e^{j\frac{\pi}{3}}$

$e^{j\frac{\pi}{3}} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$

$|e^{j\frac{\pi}{3}}| = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1$

(b) $3+4j$

$|3+4j| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

(c) $2e^{j\frac{\pi}{2}}$

$|2e^{j\frac{\pi}{2}}| = 2|e^{j\frac{\pi}{2}}| = 2$

7. Compute the real part of

(a) Using Euler's Relation, $2e^{j\frac{\pi}{2}} + 3e^{-j\frac{\pi}{2}} = 2(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}) + 3(\cos(-\frac{\pi}{2}) + j\sin(-\frac{\pi}{2}))$

$= 2j - 3j = -j$

$\therefore Re\{2e^{j\frac{\pi}{2}} + 3e^{-j\frac{\pi}{2}}\} = Re\{-j\} = 0$

(b) $(3+4j)(1-2j) = 3+4j-6j+8 = 11-2j$

$\therefore Re\{11-2j\} = 11$

(c) $5e^{j\frac{\pi}{3}} = 5(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}) = 5(\frac{1}{2} + j\frac{\sqrt{3}}{2})$

$\therefore Re\{5e^{j\frac{\pi}{3}}\} = 5/2$

8. Differentiate

(a) $(\sin^2\theta\cos\theta)' = (\sin^2\theta)'(\cos\theta) + (\sin^2\theta)(\cos\theta)' = 2\sin\theta(\sin\theta)'(\cos\theta) - (\sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta - \sin^3\theta = \sin\theta(2\cos^2\theta - \sin^2\theta) = \sin\theta(2 - 3\sin^2\theta)$

(b) $(x^2\sqrt{1-x^2} + \dfrac{3}{1-x})' = (x^2\sqrt{1-x^2})' + (\dfrac{3}{1-x})'$

$= 2x\sqrt{1-x^2} + x^2\dfrac{1}{2\sqrt{1-x^2}}(-2x) + \dfrac{-3(-1)}{(1-x)^2}$

$= 2x\sqrt{1-x^2} - \dfrac{x^3}{\sqrt{1-x^2}} + \dfrac{3}{(1-x)^2}$

(c) $(\dfrac{x\sin x}{\sqrt{1+x}})' = \dfrac{(x\sin x)'\sqrt{1+x} - x\sin x(\sqrt{1+x})'}{1+x}$

$$= \frac{(\sin x + x\cos x)\sqrt{1+x} - x\sin x\dfrac{-1}{2\sqrt{1+x}}}{1+x}$$

$$= \frac{(\sin x + x\cos x)(1+x) + \dfrac{1}{2}x\sin x}{(1+x)\sqrt{1+x}}$$

$$= \frac{\sin x + x\cos x + \dfrac{3}{2}x\sin x + x^2\cos x}{(1+x)\sqrt{1+x}}$$

9. Integrate (please do not use tables)

(a) $\int_0^\pi \cos 2\theta\, d\theta \Longrightarrow$ Use substitution.

Let $u = 2\theta$, $du = 2d\theta$.

$\int_0^\pi \cos 2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}\cos u\, du = \frac{1}{2}\sin u|_0^{2\pi} = \frac{1}{2}(\sin 2\pi - \sin 0) = 0$

(b) $\int_0^4 e^{-4x}\, dx \Longrightarrow$ Use substitution.

Let $u = 4x, du = 4dx$.

$\int_0^4 e^{-4x}\, dx = \frac{1}{4}\int_0^{16} e^{-u}\, du = -\frac{1}{4}e^{-u}|_0^{16} = -\frac{1}{4}(e^{-16} - 1) = \frac{1}{4}(1 - e^{-16})$

(c) $\int_0^\infty xe^{-x}\, dx \Longrightarrow$ Use integration by parts.

Let $u = x$ and $dv = e^{-x}dx$. Then, $du = dx$ and $v = -e^{-x}$.

$\int_0^\infty xe^{-x}\, dx = [-xe^{-x}]_{x=0}^\infty + \int_0^\infty e^{-x}\, dx = [0-0] + \int_0^\infty e^{-x}\, dx = \int_0^\infty e^{-x}\, dx$

$= -e^{-x}|_0^\infty = 1$

(Note: When $x = 0$, $xe^{-x} = 0$. When $x \to \infty$, $\frac{x}{e^x} \to 0$)

(d) $\int_1^2 \frac{1}{(x+a)(x+b)}\, dx, a \neq -1$ or -2, $b \neq -1$ or -2. Use partial fraction expansion.

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} = \frac{A(x+b) + B(x+a)}{(x+a)(x+b)} = \frac{(A+B)x + (Ab+Ba)}{(x+a)(x+b)}$$

Therefore $A = \dfrac{1}{b-a}$ and $B = \dfrac{1}{a-b}$.

$\int_1^2 \dfrac{1}{(x+a)(x+b)}\, dx = \int_1^2 [\dfrac{A}{x+a} + \dfrac{B}{x+b}]\, dx = [Aln(x+a) + Bln(x+b)]_{x=1}^2$

$= \dfrac{1}{b-a} ln(\dfrac{a+2}{a+1}) + \dfrac{1}{a-b} ln(\dfrac{b+2}{b+1}) = \dfrac{1}{a-b}[ln(\dfrac{b+2}{b+1}) - ln(\dfrac{a+2}{a+1})]$

$= \dfrac{1}{a-b} ln[\dfrac{(b+2)(a+1)}{(b+1)(a+2)}]$

Note that a and b cannot be equal to -1 or -2, or the argument of the natural logarithm will be 0 or ∞.

10. Compute the sum

(a) $\sum_{k=0}^{\infty} (\dfrac{1}{3})^k$

This is an infinite geometric series with $r = \dfrac{1}{3}$.

Therefore, since $|r| < 1$, we can use the formula $\sum_{k=0}^{\infty} r^k = \dfrac{1}{1-r}$

$\sum_{k=0}^{\infty} (\dfrac{1}{3})^k = \dfrac{1}{1-1/3} = \dfrac{3}{2}$.

(b) $\sum_{k=0}^{\infty} z^k, |z| < 1$

$\sum_{k=0}^{\infty} z^k = \dfrac{1}{1-z}$

(c) $\sum_{k=0}^{4} (-3)^k$

Using the formula $\sum_{k=0}^{n-1} r^k = \dfrac{1-r^n}{1-r}$, with $r = -3$ and $n = 5$ (number of terms), we get

$\sum_{k=0}^{4} (-3)^k = \dfrac{1-(-3)^5}{4} = \dfrac{244}{4} = 61$.

(d) $\sum_{n=0}^{5} (-2)^{n-1} = -\dfrac{1}{2} \sum_{n=0}^{5} (-2)^n = -\dfrac{1}{2}(\dfrac{1-(-2)^6}{1-(-2)}) = -\dfrac{1}{2}(-\dfrac{63}{3}) = \dfrac{21}{2}$