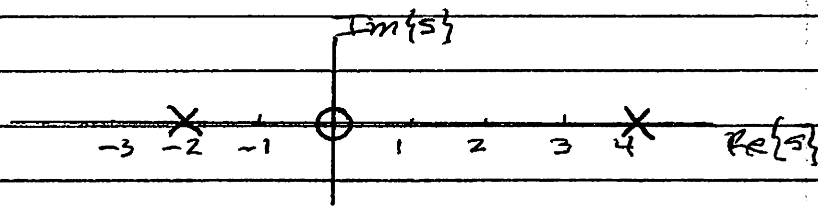
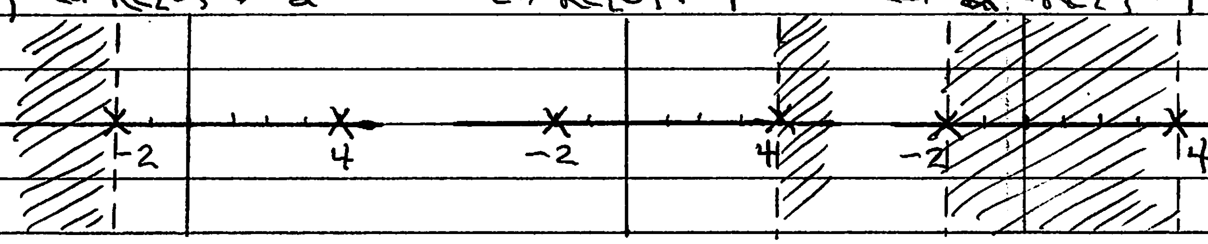
SOLUTIONS TO EXAM #3 (5/10/18)

#1. $H(s) = \dfrac{6s}{s^2-2s-8} = \dfrac{6s}{(s-4)(s+2)}$

a.) poles $(H(s) \to \infty)$: $s = 4,\ s = -2$
zeros $(H(s) \to 0)$: $s = 0,\ s = \infty$



b.) (i) $Re\{s\} < -2$ (ii) $Re\{s\} > 4$ (iii) $-2 < Re\{s\} < 4$



left of leftmost pole — LS signal

right of rightmost pole — RS signal (causal)

bounded by poles — 2S signal (stable)

c.) $H(s) = \dfrac{6s}{(s-4)(s+2)} = \dfrac{A}{s-4} + \dfrac{B}{s+2}$

$A = H(s)(s-4)\Big|_{s=4} = \dfrac{6(4)}{(4+2)} = 4$

$B = H(s)(s+2)\Big|_{s=-2} = \dfrac{6(-2)}{(-2-4)} = 2$

$H(s) = \dfrac{4}{s-4} + \dfrac{2}{s+2}$

4.c. cont'd) (i) $Re\{s\} < -2 \Rightarrow$ both signals are left sided

$$h(t) = -4e^{4t}u(-t) - 2e^{-2t}u(-t)$$

(ii) $Re\{s\} > 4 \Rightarrow$ both signals are right sided

$$h(t) = 4e^{4t}u(t) + 2e^{-2t}u(t)$$

(iii) $-2 < Re\{s\} < 4 \Rightarrow$ two sided signal

$$h(t) = -4e^{4t}u(-t) + 2e^{-2t}u(t)$$

d.) $$H(s) = \frac{Y(s)}{X(s)} = \frac{6s}{s^2 - 2s - 8}$$

$$Y(s)(s^2 - 2s - 8) = 6s\,X(s)$$

$$\Downarrow \mathcal{L}^{-1}$$

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 8y(t) = 6\frac{dx(t)}{dt}$$

e.) No.

causal $\longrightarrow$ ROC is right-half plane

stable $\longrightarrow$ ROC includes the unit circle

(i) is not causal and not stable

(ii) is causal but not stable

(iii) is stable but not causal

#2. LTI System

input $x[n] = (-3)^n u[n]$

output $y[n] = 4(2)^n u[n] - (\frac{1}{2})^n u[n]$

a.) transfer function $H(z) = \dfrac{Y(z)}{X(z)}$

$$Y(z) = \mathcal{Z}\{y[n]\} = \sum_{n=0}^{\infty} 4(2)^n z^{-n} - \sum_{n=0}^{\infty} (\tfrac{1}{2})^n z^{-n}$$

$$= 4\sum_{n=0}^{\infty}(2z^{-1})^n - \sum_{n=0}^{\infty}(\tfrac{1}{2}z^{-1})^n$$

$$= \frac{4}{1-2z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$|z| > 2 \ \cap \ |z| > \tfrac{1}{2}$$

$$Y(z) = \frac{4(1-\cancel{\frac{1}{2}z^{-1}}) - (1-\cancel{2z^{-1}})}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{3}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}, \ |z| > 2$$

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty}(-3)^n z^{-n}$$

$$= \sum_{n=0}^{\infty}(-3z^{-1})^n = \frac{1}{1+3z^{-1}}, \ |z| > 3$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{3(1+3z^{-1})}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

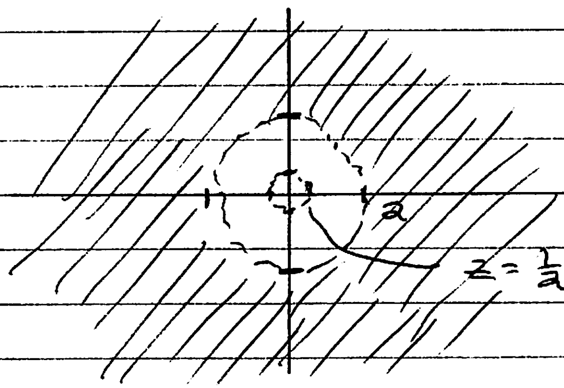
ROC: $|z| > 2$ outside outermost pole

#2 cont'd)

$$H(z) = \frac{3(1+3z^{-1})}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{3z(z+3)}{(z-2)(z-\frac{1}{2})}$$

poles: $z = 2,\ z = 1/2$

zeros: $z = 0,\ z = -3$ $\Rightarrow$ didn't ask for these



c) impulse response $h[n] = \mathcal{Z}^{-1}\{H(z)\}$

$$\frac{3(1+3z^{-1})}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \quad |z|>2$$

RS sequence

$$H(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = H(z)(1-2z^{-1})\Big|_{z=2} = \frac{3(1+\frac{3}{2})}{1-\frac{1}{4}} = \frac{\cancel{3}(\frac{5}{2})}{\frac{\cancel{3}}{4}} = 10$$

$$B = H(z)(1-\tfrac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}} = \frac{3(1+6)}{1-4} = -7$$

$$H(z) = \frac{10}{1-2z^{-1}} - \frac{7}{1-\frac{1}{2}z^{-1}}$$

$$\Big\Downarrow \mathcal{Z}^{-1}$$

$$h[n] = 10(2^n)u[n] - 7\left(\tfrac{1}{2}\right)^n u[n]$$

#2. cont'd) d.) $$H(z) = \frac{Y(z)}{X(z)} = \frac{3(1+3z^{-1})}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{3+9z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}$$

$$Y(z)\left(1-\frac{5}{2}z^{-1}+z^{-2}\right) = X(z)(3+9z^{-1})$$

$$\Big\Downarrow \mathcal{Z}^{-1}$$

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = 3x[n] + 9x[n-1]$$

e.) The system is <u>not</u> <u>stable</u> because the ROC does not contain the unit circle. (Or, you can simply compute $\sum_n |h[n]|^2 < \infty$)

#3 a.) $$H(s) = \frac{s+2}{s+1}, \quad \underbrace{Re\{s\} > -1}_{\text{right-sided signal}}$$

$$= 1 + \frac{1}{s+1}$$

$$\Big\Downarrow \mathcal{L}^{-1}$$

$$h(t) = \delta(t) + e^{-t}u(t)$$

Alternative Approach:

$$H(s) = \frac{s}{s+1} + \frac{2}{s+1}$$

$$h(t) = \frac{d}{dt}\left(e^{-t}u(t)\right) + 2e^{-t}u(t)$$

$$= -e^{-t}u(t) + e^{-t}\delta(t) + 2e^{-t}u(t)$$

$$= \delta(t) + e^{-t}u(t)$$

3 cont'd) b.) Let $x_1(t) = e^{-t}u(t)$

$$X_1(s) = \frac{1}{s+1}, \quad Re\{s\} > -1$$

Let $x_2(t) = e^{-(t+1)}u(t+1)$. Then,
$x_2(t) = x_1(t+1)$ and

$$X_2(s) = e^{s} \cdot \frac{1}{s+1}, \quad Re\{s\} > -1 \quad \text{(Prop. 9.5.2)}$$

Let $x_3(t) = \frac{d}{dt}x_2(t)$. Then,

$$X_3(s) = \underset{\text{(Prop 9.5.7)}}{s X_2(s)} = e^{s}\frac{s}{s+1}, \quad Re\{s\} > -1$$

Finally, $x(t) = e^{-t}x_3(t)$. Using Prop. 9.5.3 with a shift $s_0 = -1$

$$X(s) = X_3(s+1) = \frac{e^{s+1}(s+1)}{s+2}, \quad Re\{s\} > -2$$

c.)
$$x[n] = x_1[n] * x_2[n]$$
where $x_1[n] = u[n-2]$
and $x_2[n] = (\frac{2}{3})^n u[n]$

Then, $X(z) = X_1(z) \cdot X_2(z)$

$$X_1(z) = \sum_{n=2}^{\infty} z^{-n} = \frac{z^{-2}}{1-z^{-1}}, \quad |z| > 1$$

$$X_2(z) = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}z^{-1}\right)^n$$

$$= \frac{1}{1-\frac{2}{3}z^{-1}}, \quad |z| > \frac{2}{3}$$

$$\therefore X(z) = \frac{z^{-2}}{(1-z^{-1})(1-\frac{2}{3}z^{-1})} = \frac{1}{(z-1)(z-\frac{2}{3})}, \quad |z| > 1$$

#3. cont'd) d.) (4) Initial Value Theorem

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} e^{-5s}\left(\frac{-2(s+3)}{s+2}\right)$$

$$= \underbrace{\lim_{s\to\infty} e^{-5s}}_{0} \; \underbrace{\lim_{s\to\infty}\left(\frac{-2(1+3/s)}{(1+2/s)}\right)}_{-2}$$

$$= 0$$

(4) Final Value Theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

$$= \lim_{s\to 0} e^{-5s}\left(\frac{-2(s+3)}{s+2}\right)$$

$$= \frac{1(-2)(3)}{2} = -3$$

e.) $$H(z) = \frac{2z(4z^2-1)}{8z^3-26z^2+5z+3}, \quad \frac{1}{2} < |z| < 3$$

Yes. This system is stable because the ROC includes the unit circle.

f.) Let $y(t) = e^{2t}\sin t\, \overset{\text{left sided}}{u(-t)}$. Then, $x(t) = \int_{-\infty}^{t} y(\tau)\,d\tau$ and $X(s) = \frac{1}{s}Y(s)$ with ROC, $R_y \cap \mathrm{Re}\{s\} > 0$

$$Y(s) = \int_{-\infty}^{0} e^{2t}\sin t\, e^{-st}\,dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{0} e^{(2-s+j)t}\,dt + \int_{-\infty}^{0} e^{(2-s-j)t}\,dt\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{2}-\cancel{s}-j-\cancel{2}+\cancel{s}-j}{(2-s+j)(2-s-j)}\right], \quad 2-s>0 \;\Downarrow\; \mathrm{Re}\{s\}<2$$

#3 f. cont'd)

$$Y(s) = \frac{-1}{(a-s)^2+1}, \quad Re\{s\} < a$$

$$\therefore X(s) = \frac{-1}{s[(s-a)^2+1]}, \quad 0 < Re\{s\} < a$$

#4.

$$y[n] = x[n] + a\,x[n-1], \quad |a|<1$$

$$\Downarrow \mathcal{Z}$$

$$Y(z) = X(z) + a z^{-1} X(z)$$
$$= X(z)(1 + a z^{-1})$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = 1 + a z^{-1} \Rightarrow$$ ROC is the entire z-plane except for $z=0$

$$\Downarrow \mathcal{Z}^{-1}$$

$$h[n] = \delta[n] + a\,\delta[n-1]$$

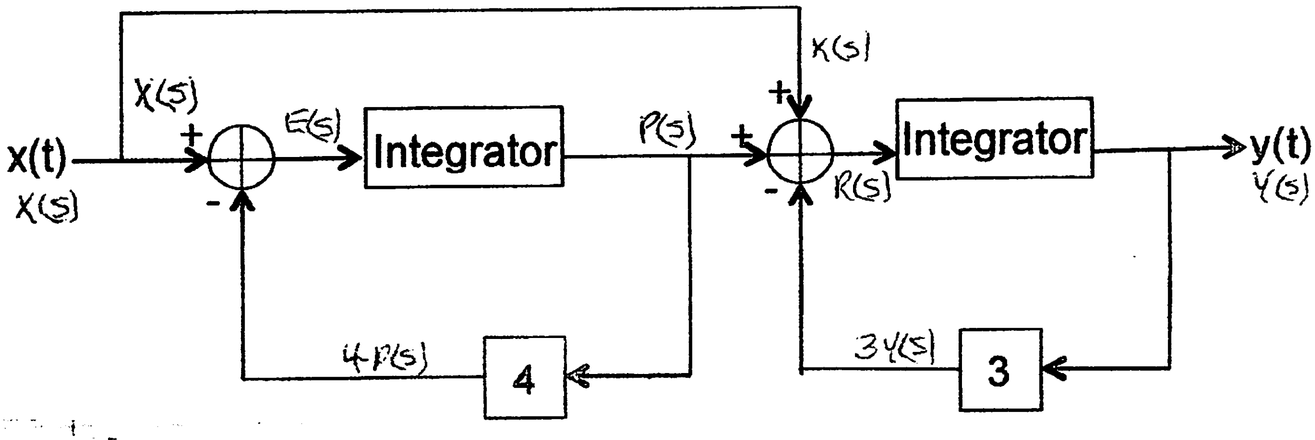
NOTE: You can get the impulse directly from the difference equation by replacing $x[n]$ by an impulse.

i.e. $$h[n] \triangleq y[n]\Big|_{x[n]=\delta[n]}$$

$$= \delta[n] + a\,\delta[n-1]$$

Extra credit

Label variables as shown on the figure



Then, we can write the following equations:

- $E(s) = X(s) - 4P(s)$

  But $P(s) = \frac{1}{s} E(s) \Rightarrow E(s) = sP(s)$

  $$\therefore \; sP(s) = X(s) - 4P(s)$$
  $$(s+4)P(s) = X(s)$$
  $$P(s) = \frac{X(s)}{s+4}$$

- $R(s) = X(s) + P(s) - 3Y(s)$

  But $Y(s) = \frac{1}{s} R(s) \Rightarrow R(s) = sY(s)$

  $$\therefore \; sY(s) = X(s) + P(s) - 3Y(s)$$
  $$= X(s) + \frac{X(s)}{s+4} - 3Y(s)$$
  $$(s+3)Y(s) = \frac{s+4+1}{s+4} X(s) = \frac{s+5}{s+4} X(s)$$
  $$H(s) = \frac{Y(s)}{X(s)} = \frac{s+5}{(s+3)(s+4)} = \frac{s+5}{s^2+7s+12}$$