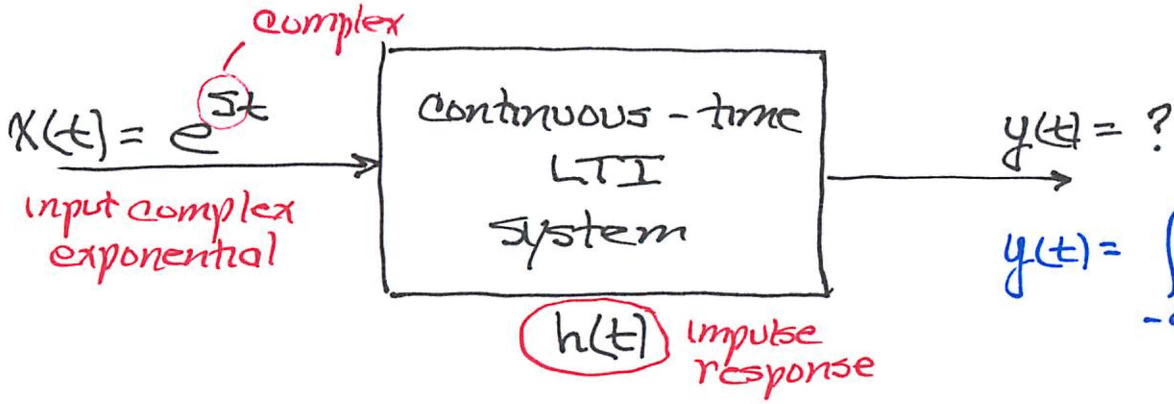
Section 3.2
The Response of LTI Systems to Complex Exponentials

Why is Fourier analysis important?

i. Almost all signals can be represented as linear combinations of complex exponentials.

ii. Complex exponentials are eigenfunctions of LTI systems.

Response of Continuous-Time LTI System to e^{st}



$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

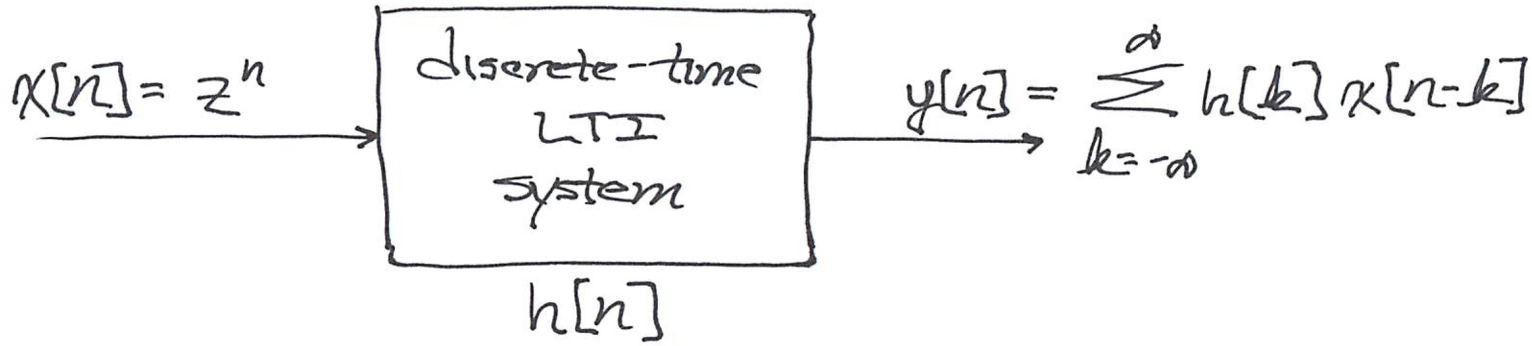
"complex exponentials are eigenfunctions of linear time-invariant systems"

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \underbrace{e^{st}}_{\text{complex exp}} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{\text{not a function of } t\text{, just a multiplier}}$$

$$= e^{st} \underbrace{H(s)}_{\text{Laplace Transform}} \quad \xRightarrow{s=j\omega} \quad e^{j\omega t} \underbrace{H(j\omega)}_{\text{Fourier Transform (CT)}}$$

Response of Discrete-Time LTI System to z^n



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$= \underbrace{z^n}_{\text{comp. exp.}} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{\text{not a function of } n\text{, just a multiplier}}$$

$$= z^n \underbrace{H(z)}_{\text{Z Transform}} \xRightarrow{z = e^{j\omega}} e^{j\omega n} \underbrace{H(e^{j\omega})}_{\text{Fourier Transform (DT)}}$$

LTI System Response

"almost all signals ..."

$$x(t) = \sum_k a_k e^{s_k t} \xrightarrow{LTI} y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

$$x[n] = \sum_k a_k z_k^n \xrightarrow{LTI} y[n] = \sum_k a_k H(z_k) z_k^n$$

Ch. 3 periodic signals – Fourier Series

Chs 4 & 5 aperiodic signals – Fourier Transform

$s = j\omega$ $z = e^{j\omega}$

More general,

$s = \sigma + j\omega$ ⟶ Laplace Transform Ch. 9

$z = re^{j\omega}$ ⟶ Z Transform Ch. 10

Ch. 7 Sampling

Sections 3.3, 3.5
Fourier Series

Continuous-Time Periodic Signals

$$x(t) = x(t+T)$$

fundamental period

$$\omega_0 = \frac{2\pi}{T}$$

Representation as Linear Comb. of Complex Exponentials

$\boxed{\text{harmonically related set}}$ $\quad \phi_k(t) = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \quad k = 0, \pm 1, \pm 2, \ldots$

$$x(t) = \text{linear combination of } \{\phi_k(t)\}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier Series representation

$k = 0$ constant

$k = 1$ ω_0 , first harmonic

$\vdots$

How do we determine the coefficients a_k?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier Series Representation

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^{T=\frac{2\pi}{\omega_0}} x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

?

- $k = n \quad \int_0^T 1 \, dt = T$
- $k \neq n \quad \int_0^T e^{j(k-n)\omega_0 t} dt = \frac{1}{j(k-n)\omega_0} e^{j(k-n)\omega_0 t} \Big|_0^T$

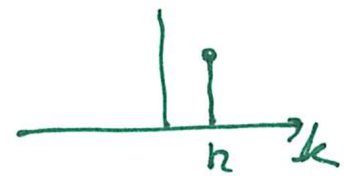
$$= \frac{1}{j(k-n)\omega_0} \left[e^{j(k-n)\frac{2\pi}{\cancel{T}}\cancel{T}} - 1 \right]$$

$$= 0$$

$\omega_0 = \frac{2\pi}{T}$

How do we determine the coefficients a_k?

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \underbrace{\begin{pmatrix} T\,, & k=n \\ 0\,, & k\neq n \end{pmatrix}}_{T\,\delta[k-n]}$$



$$= T \sum_{k=-\infty}^{\infty} a_k\, \delta[k-n]$$

$$= T\, a_n$$

$$\therefore \quad \boxed{a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt}$$

Fourier Series coefficients

Spectrum

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$a_k = \frac{1}{T}\int_{T} x(t)\, e^{-jk\omega_0 t}\, dt$$

Fourier series coefficients

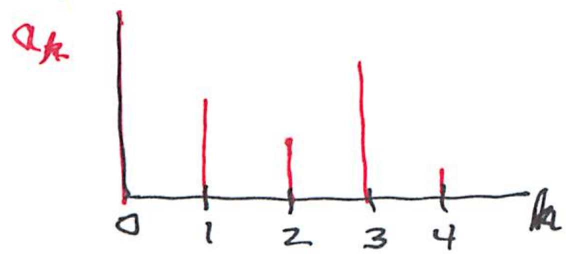
Integrate over a period of length T

$$\int_0^T \qquad \int_{-T/2}^{T/2}$$

- $a_0 = \frac{1}{T}\int_T x(t)\,dt$

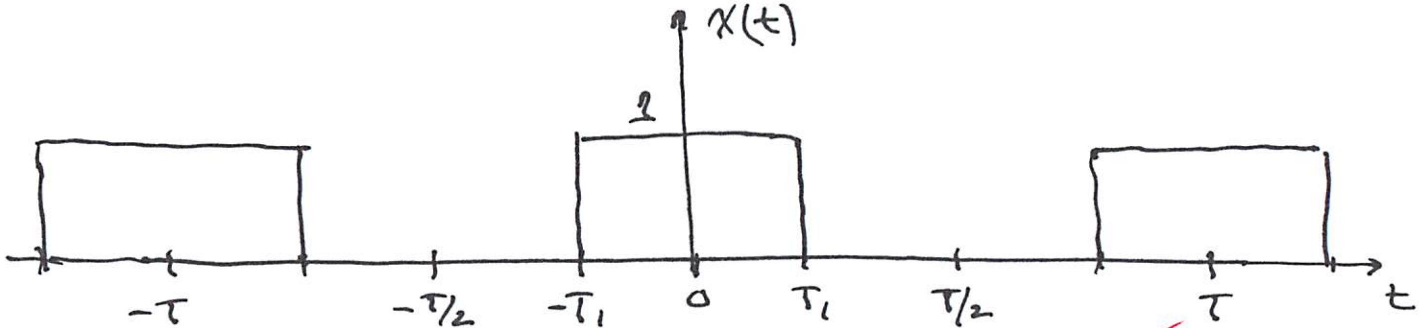
  = average value

- line spectrum



$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

CT Fourier Series
Example (#3.5, pp. 193-195)



periodic square wave

fundamental period

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$ (then it repeats)

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\,dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\frac{2\pi}{T}t}\,dt$$

$$= \frac{1}{\cancel{T}} \cdot \frac{1}{-jk\frac{2\pi}{\cancel{T}}}\, e^{-jk\frac{2\pi}{T}t}\Big|_{-T_1}^{T_1} = \frac{e^{-jk2\pi\frac{T_1}{T}} - e^{jk\frac{2\pi}{T}T_1}}{-jk2\pi}$$

$$= \frac{1}{k\pi}\left(\frac{e^{jk2\pi\frac{T_1}{T}} - e^{-jk2\pi\frac{T_1}{T}}}{2j}\right) \Rightarrow \frac{\sin 2\pi k\frac{T_1}{T}}{k\pi}$$

CT Fourier Series Example (#3.5, pp. 193-195)

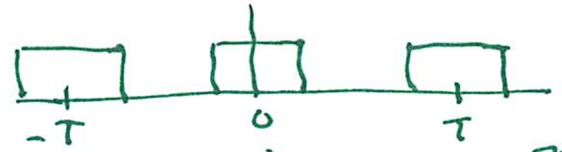
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{jk\frac{2\pi}{T}t}\,dt$$

$$= \frac{\sin 2\pi k(T_1/T)}{k\pi}$$

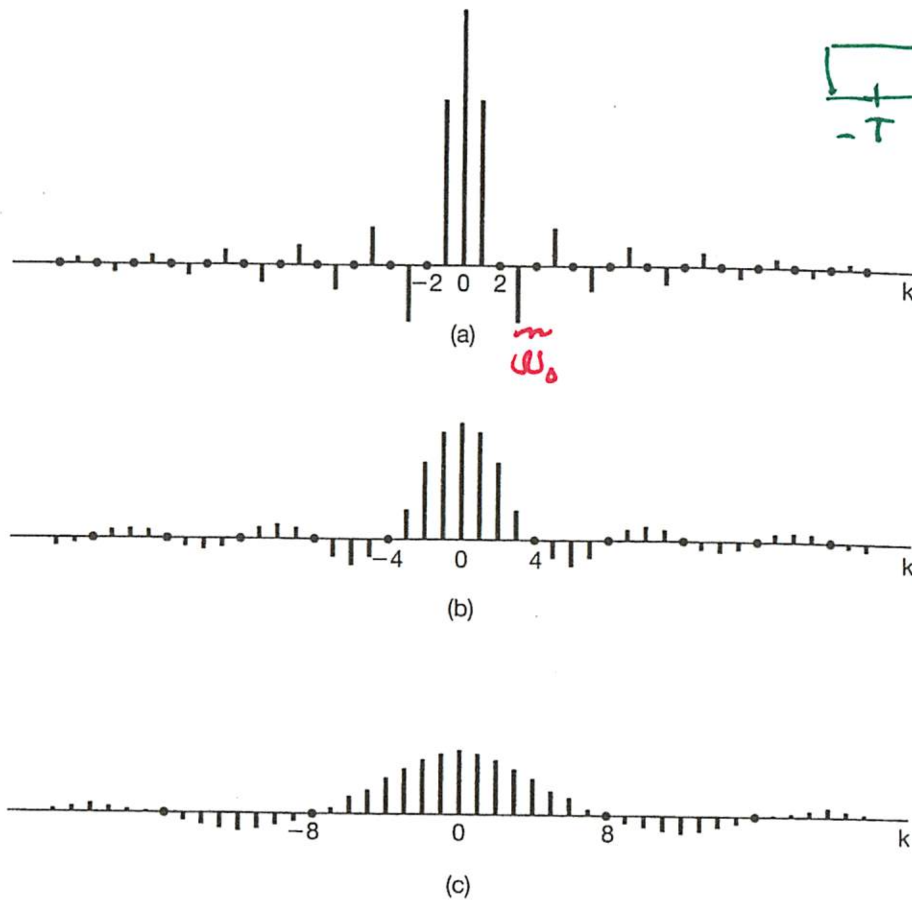
$$a_0 = \frac{2T_1}{T}$$

L'Hopital's Rule

$$\frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}$$



duty cycle $= \frac{2T_1}{T}$



line spectrum

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

duty cycle 1/2, 1/4, 1/8

CT Fourier Series
Example

$$x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$$

periodic with period $T=4$
$\left(\frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/2} = 4\right)$

- $x(t) = \sum_k a_k e^{jk\omega_0 t} = \sum_k a_k e^{jk\frac{\pi}{2}t} = \cdots a_{-2}e^{-j\pi t} + a_{-1}e^{-j\frac{\pi}{2}t} + a_0 + a_1 e^{j\frac{\pi}{2}t} + a_2 e^{j\pi t} + \cdots$

- $x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$
  $= \frac{3}{2}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{2}t} + \frac{3}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{2}t}$

equate terms

$a_1 = \frac{3}{2}e^{j\frac{\pi}{4}}$

$a_{-1} = \frac{3}{2}e^{-j\frac{\pi}{4}}$

all other $a_k = 0$

Properties of Continuous-Time Fourier Series (Table 3.1)

$$x(t) \overset{\mathcal{F}_s}{\longleftrightarrow} a_k$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

- Linearity
- Time shift
- Time Reversal
- Time Scaling
- Multiplication
- Conjugation
- Parseval's Relation

Properties of CT Fourier Series

Parseval's Relation

Let $\overset{h}{z}(t) = \overset{a_\ell}{x}(t)\, y(t)$

and $y(t) = x^*(t)$

$b_\ell = a^*_{-\ell}$

$$\Rightarrow \quad z(t) = |x(t)|^2 = x(t)\, x^*(t)$$

$$\text{average power} = \frac{1}{T}\int_T |x(t)|^2\, dt \;\Rightarrow\; ?$$

$$h_0 = \sum_{\ell=-\infty}^{\infty} a_\ell\, b_{-\ell}$$

But $b_{-\ell} = a^*_\ell$,

$$h_0 = \sum_{\ell=-\infty}^{\infty} |a_\ell|^2$$

$$\therefore \quad \text{average power} = \frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{\ell=-\infty}^{\infty} |a_\ell|^2$$

Sections 3.6-3.7
Fourier Series
Discrete-Time Periodic Signals

$$x[n] = x[n+N]$$

fundamental period, $\omega_0 = \frac{2\pi}{N}$

Representation as Linear Comb. of Complex Exponentials

harmonically related set $\phi_k[n] = e^{jk\omega_0 n} = e^{jk(\frac{2\pi}{N})n}, \quad k = 0, \pm 1, \pm 2, \cdots$

There are only N distinct values because $\phi_k[n] = \phi_{k+N}[n]$

only N complex exponentials

only N values for Fourier coefficients

$x[n]$ = linear combination of $\phi_k[n]$

$= \sum_k a_k e^{jk\omega_0 n}$

only N

$= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$

$k = 0, 1, 2, \ldots, N-1$

$k = 3, 4, 5, \ldots, N+2$

How do we determine the coefficients a_k?

Linear Equation Solution

N unknowns

N equations

$$x[0] = \sum_{k=\langle N \rangle} a_k$$

$$x[1] = \sum_{k=\langle N \rangle} a_k e^{j\frac{2\pi k}{N}}$$

$$x[2]$$

$$\vdots$$

$$x[N-1] = \sum_{k=\langle N \rangle} a_k e^{j\frac{2\pi k}{N}(N-1)}$$

$$= \sum_{k=\langle N \rangle} a_k e^{-j\frac{2\pi k}{N}}$$

How do we determine the coefficients a_k?

Closed-Form Solution

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$$\sum_{n=\langle N\rangle} x[n] e^{-jr(\frac{2\pi}{N})n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n} e^{-jr(\frac{2\pi}{N})n}$$

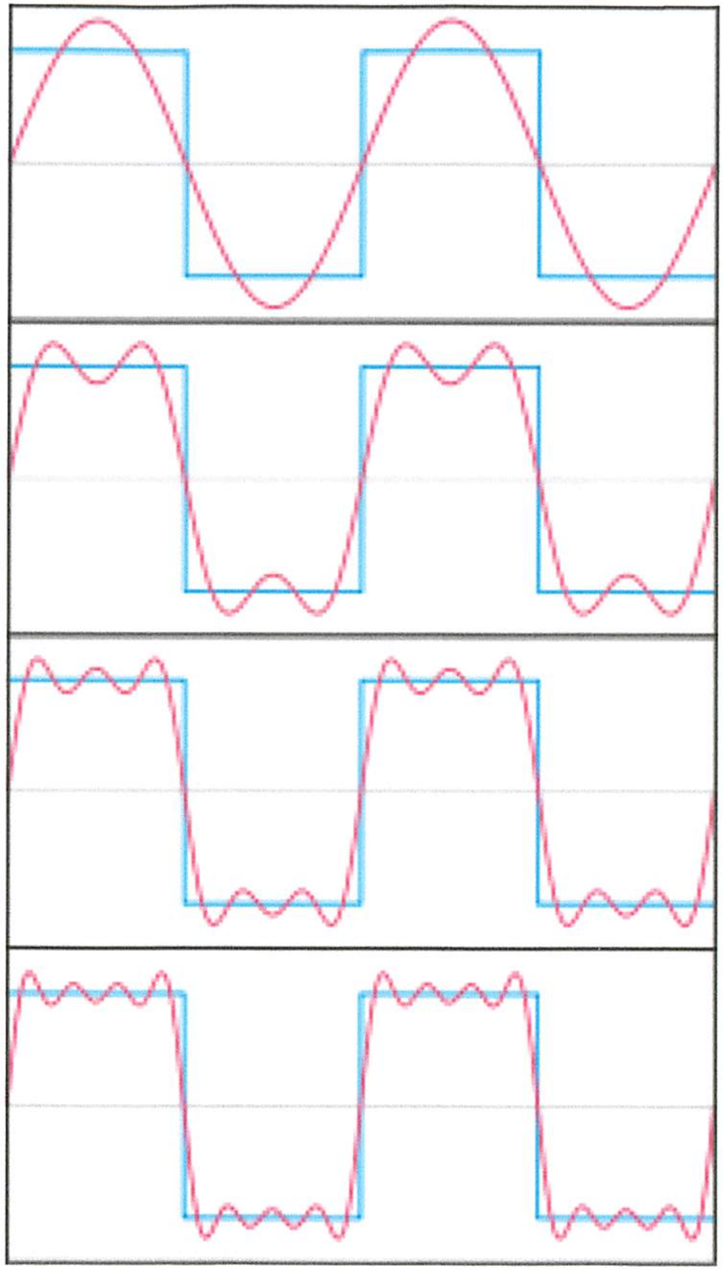
$$= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} (\underbrace{e^{j(k-r)\frac{2\pi}{N}}}_{\alpha})^n}$$

$$\frac{1-\alpha^N}{1-\alpha} = \frac{1-e^{j(k-r)\frac{2\pi}{\cancel{N}}\cancel{N}}}{1-e^{j(k-r)\frac{2\pi}{N}}}$$

$$\left.\begin{array}{ll} k \neq r & \Rightarrow 0 \\ k = r & \sum_{n=\langle N\rangle} 1 = N \end{array}\right\} N\delta[k-r]$$

$$\therefore \quad a_r = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jr\frac{2\pi}{N}n} \quad *$$

Periodic Square-Wave



$x(t) = \sum a_k e^{jk\omega_0 t}$

$k=1$

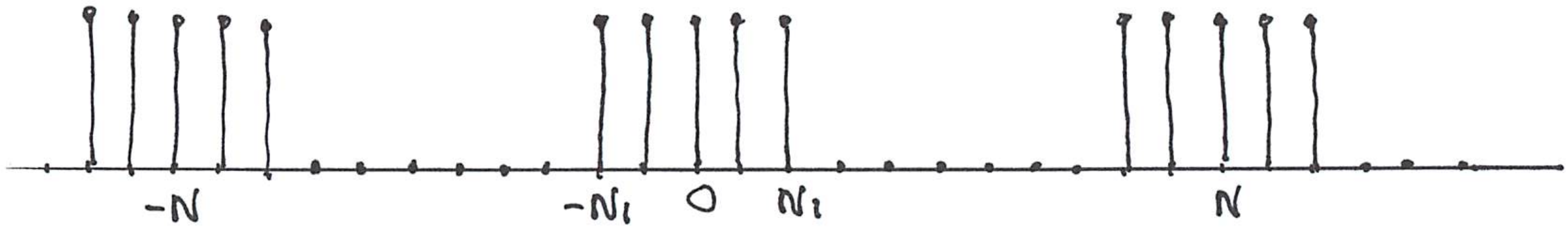
$k=1$
$k=3$

$a_1 \cos \omega_0 t + a_3 \cos 3\omega_0 t$

$k=1, 3, 5$

DT Fourier Series
Example (#3.12, pp. 218-219)

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$



periodic square wave

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

geometric series

$2N_1+1$ terms

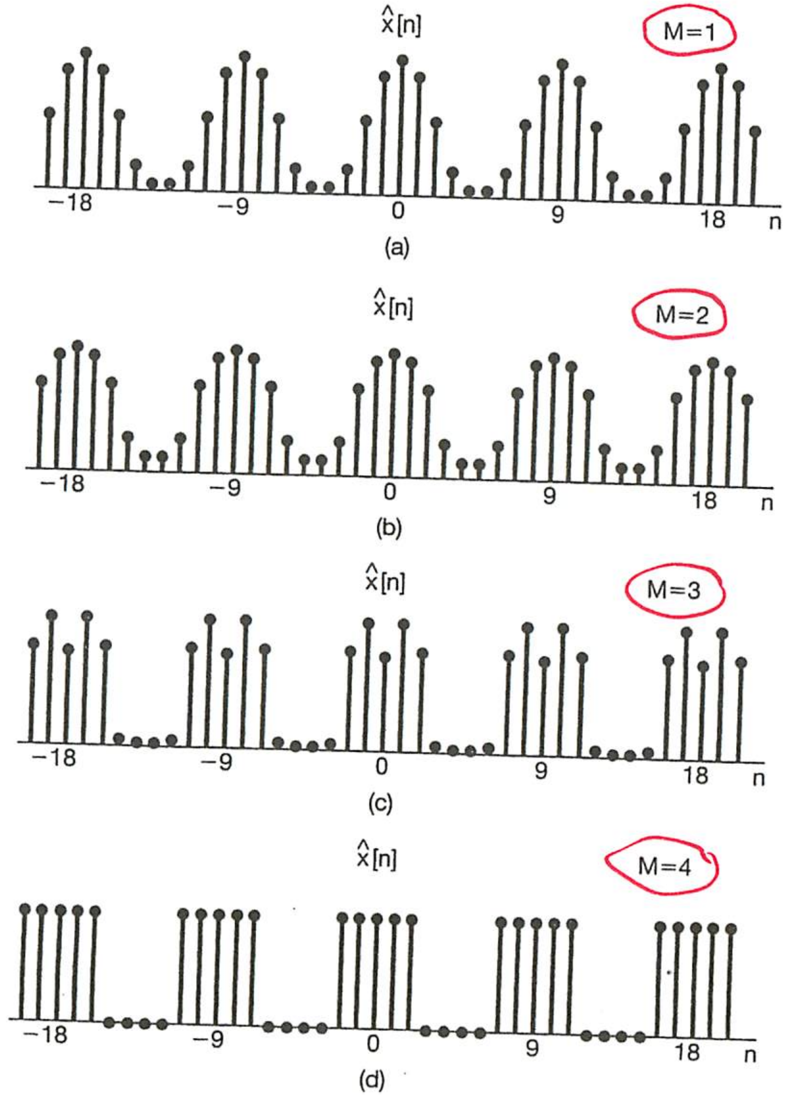
$$= \frac{1}{N} e^{-jk\left(\frac{2\pi}{N}\right)(-N_1)}\left[\frac{1-\left(e^{-jk\frac{2\pi}{N}}\right)^{2N_1+1}}{1-e^{-jk\frac{2\pi}{N}}}\right]$$

$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 = \frac{2N_1+1}{N}$$

note $$a_N = \frac{1}{N}\sum_{n=-N_1}^{N_1} \underbrace{e^{-jn2\pi}}_{1} = \frac{2N_1+1}{N}$$ repeats

$$a_k = \frac{1}{N}\,\frac{\sin\left[2\pi k\left(N_1+\frac{1}{2}\right)/N\right]}{\sin\frac{\pi k}{N}}$$

DT Fourier Series Example



$$\hat{x}[n] = \sum_k a_k e^{jk\omega_0 n}$$

perfect

DT Fourier Series Example

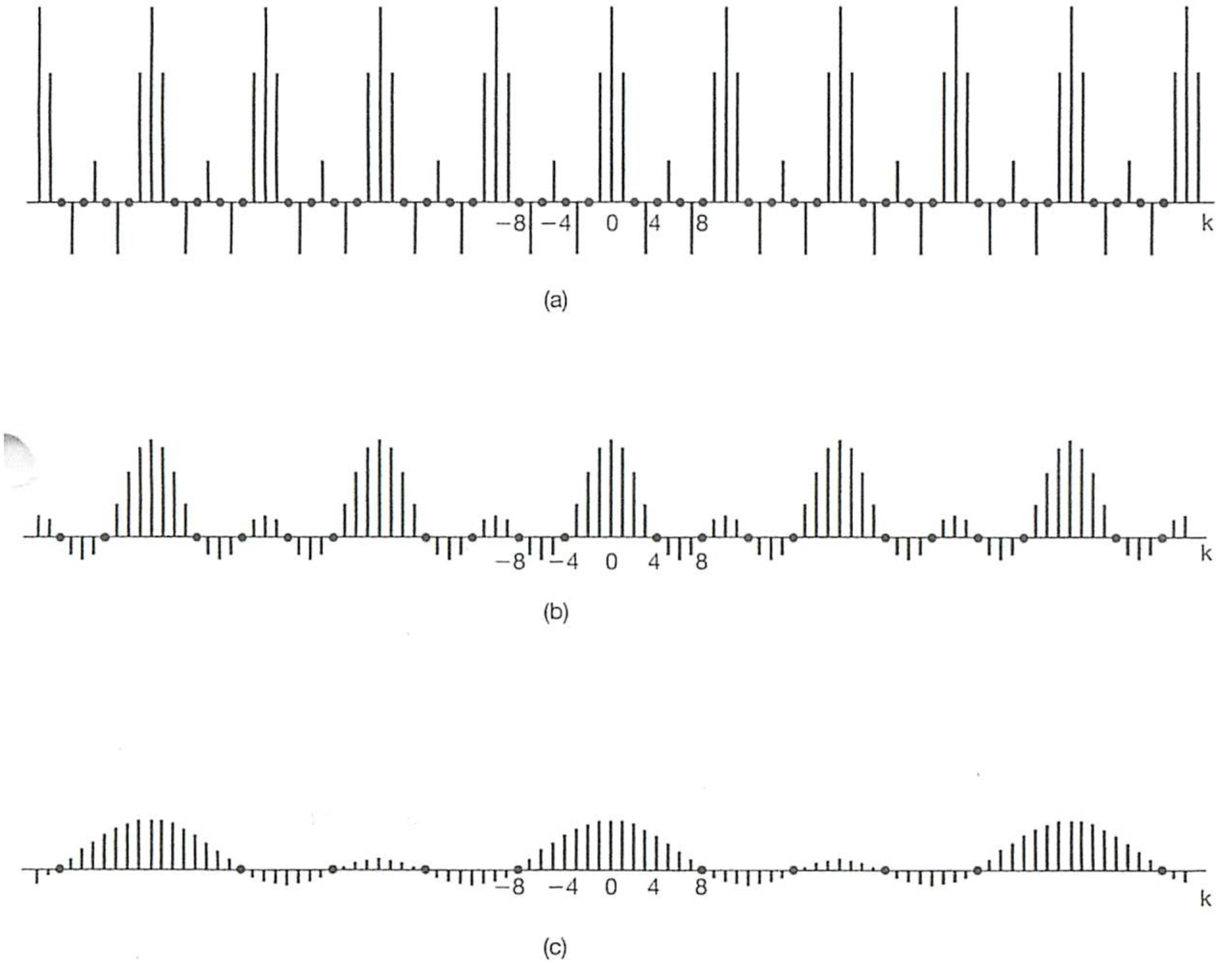


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

Properties of DT Fourier Series
Parseval's Relation

average power $= \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2$

$$= \sum_{k=\langle N \rangle} |a_k|^2$$

Useful Math Facts

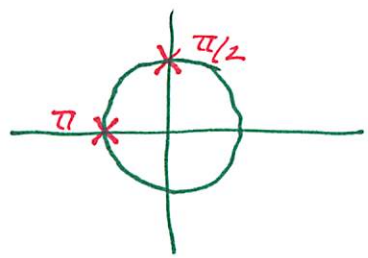
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\frac{\pi}{2}} = j$$

$$e^{j\pi} = -1$$



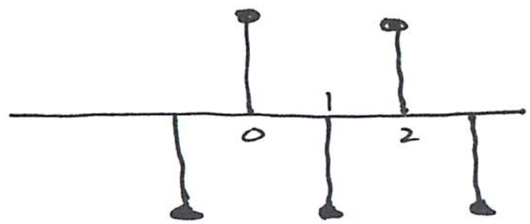
$$\therefore (j)^k = e^{j\frac{\pi}{2}k}$$

$$(-1)^k = e^{j\pi k}$$

$$\sum_n (-1)^n = \sum_n (e^{j\pi})^n \rightarrow \text{geometric series}$$

DT Fourier Series
Examples

$x[n] = (-1)^n$



periodic

$N = 2$

$\omega_0 = \frac{2\pi}{N} = \pi$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{2} \sum_{n=0}^{1} (-1)^n e^{-jk\pi n} = \frac{1}{2}\left[1 - e^{-jk\pi}\right]$$

$$= \begin{cases} 0 \ , & k \text{ even} \\ 1 \ , & k \text{ odd} \end{cases}$$

Section 3.8
Fourier Series and LTI Systems

Frequency Response

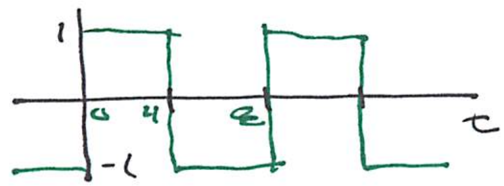
Discrete-Time Periodic Input Signals

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$y[n] = \sum_{k=\langle N \rangle} a_k H\left(e^{j2\pi k/N}\right) e^{jk\left(\frac{2\pi}{N}\right)n}$$

Fourier Series and LTI Systems
Example (Problem #3.13, p. 253)

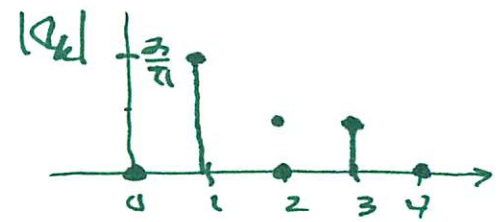
$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

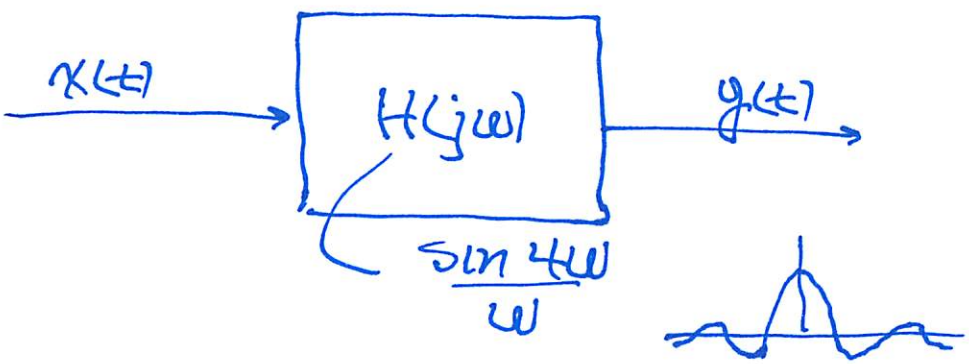


$T = 8$

$\omega_0 = \frac{\pi}{4}$

$$a_k = \frac{1}{j\pi k}\left(1 - e^{-jk\pi}\right) = \begin{cases} 0, & k \text{ even} \\ \frac{2}{j\pi k}, & k \text{ odd} \end{cases}$$





$$b_k = a_k H(jk\omega_0) = a_k \left.\frac{\sin 4\omega}{\omega}\right|_{\omega = \frac{k\pi}{4}} = a_k \frac{\sin k\pi}{\frac{k\pi}{4}} \begin{cases} k \ne 0, & 0 \\ k = 0, & 4 \end{cases}$$

$\therefore\ k \ne 0,\ b_k = 0$

$k = 0,\ b_k = 0$

$\Rightarrow\ y(t) = 0$

Section 3.9
Filtering

Decibels

$$\text{dB} \triangleq 20\log_{10}(\text{amplitude}) = 10\log_{10}(\text{power})$$

relative to some reference value

$$= 20\log_{10}|H(j\omega)| = 10\log_{10}|H(j\omega)|^2$$

Math Facts

$$10\log_{10}10^x = 10x\,\cancel{\log_{10}10}^{\,1} = 10x \text{ dB}$$

(factors of 10 in power)

$10 \rightarrow 10$ dB
$100 \rightarrow 20$ dB
$1000 \rightarrow 30$ dB

$$\log ab = \log a + \log b$$

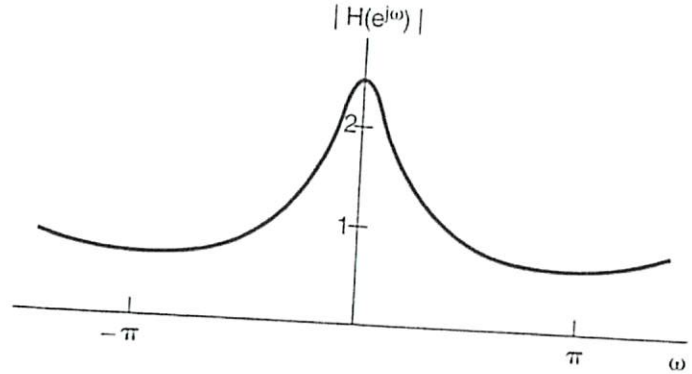
$$\log \frac{a}{b} = \log a - \log b$$

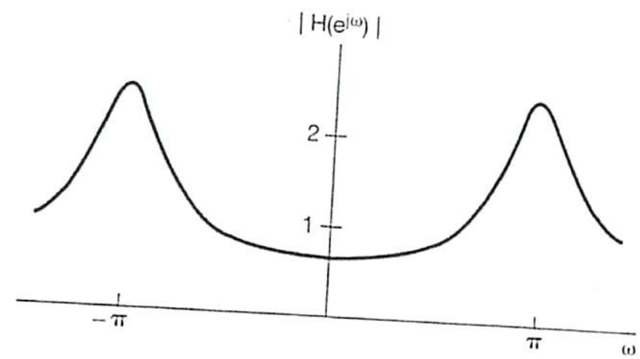
factor of 2 $\longrightarrow$ $10\log_{10}2 \sim 3$ dB

Discrete-Time Filtering (Section 3.11)

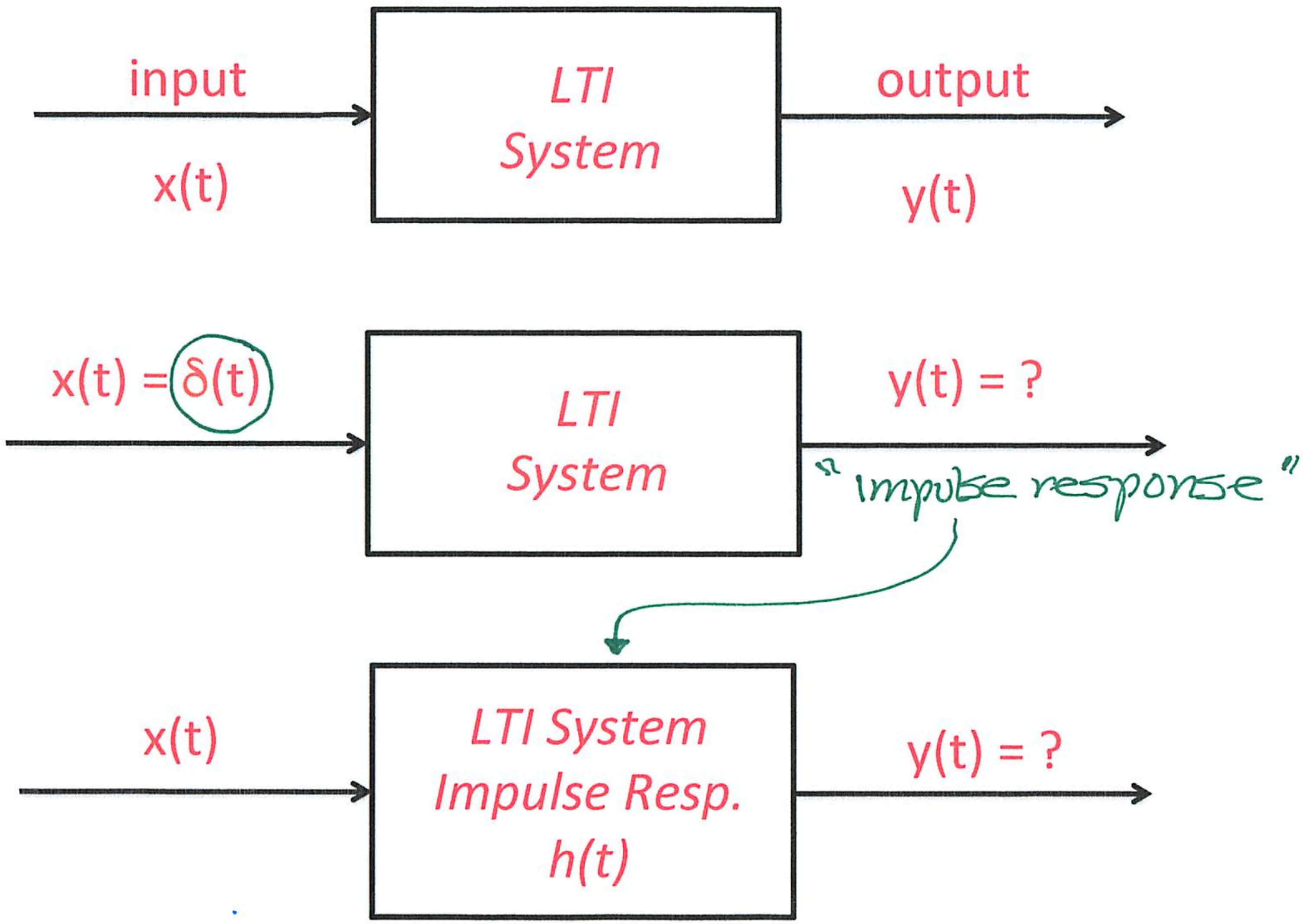
First-Order Recursive

$y[n] - a\,y[n-1] = x[n]$



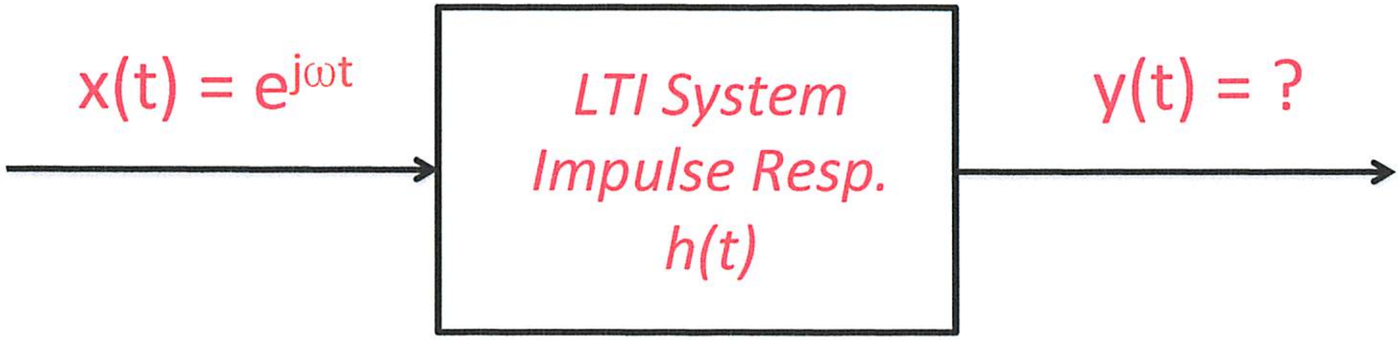


Review: LTI Systems



$y(t) = x(t) * h(t)$

Review: LTI Systems



"Complex exponentials are eigenfunctions of LTI systems."

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

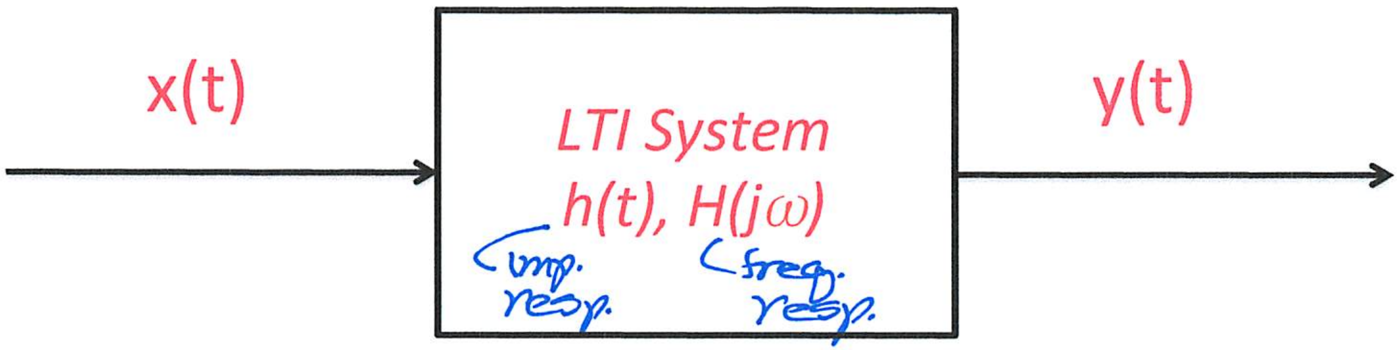
$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega t} e^{-j\omega\tau}\, d\tau$$

$$= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}$$

- not a function of time
- just a complex number that depends on frequency of input tone

$\triangleq H(j\omega)$ frequency response

Review: LTI Systems



"Almost all signals can be represented as linear combinations of complex exponentials."

periodic signals

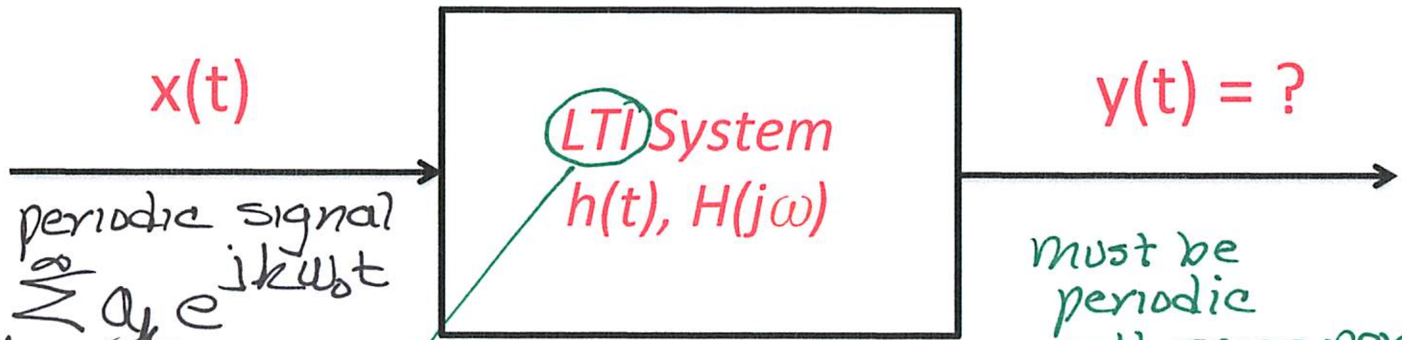
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\frac{2\pi}{T}$

Fourier Series coefficients

frequency content of signal

Review: LTI Systems



periodic signal

$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

must be periodic with same period

linear $y(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{\left[\text{response to } e^{jk\omega_0 t}\right]}_{e^{jk\omega_0 t} H(j\omega)\big|_{\omega = k\omega_0}}$

$= \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)\, e^{jk\omega_0 t}$

must be Fourier series coefficients of output

$\boxed{b_k = a_k H(jk\omega_0)}$

conv. in time is mult. in frequency

system frequency response modifies each spectral component

Filtering

"The process of changing the relative amplitudes and phases of the frequency components of a signal."

(i) change shape → frequency shaping

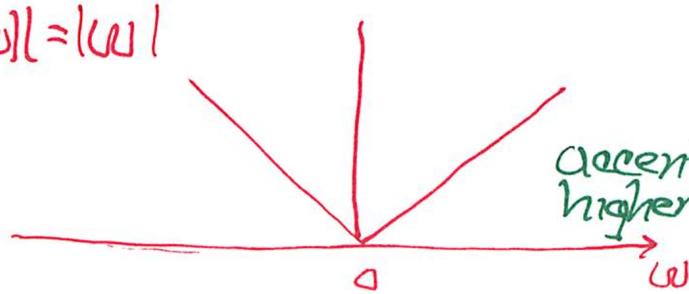
(ii) pass some frequencies and eliminate others → frequency selective

e.g. differentiator

$$y(t) = \frac{dx(t)}{dt}$$

$$x(t) = e^{j\omega t} \longrightarrow y(t) = \underbrace{j\omega}_{H(j\omega)} \underbrace{e^{j\omega t}}_{x(t)}$$

$|H(j\omega)| = |\omega|$



accentuates higher frequencies

$$\measuredangle H(j\omega) = \begin{cases} \frac{\pi}{2}, & \omega > 0 \\ -\frac{\pi}{2}, & \omega < 0 \end{cases}$$

Frequency Shaping Filters

Example: Differentiator

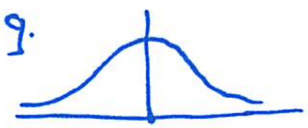
$$y(t) = \frac{dx(t)}{dt}$$

Frequency Selective Filters

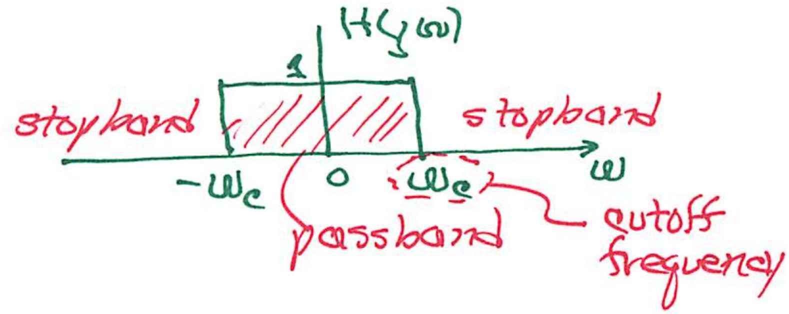
< selects some frequency bands and rejects others

lowpass $\Rightarrow$ passes low frequencies (around $\omega = 0$) and attenuates or rejects higher frequencies

e.g.

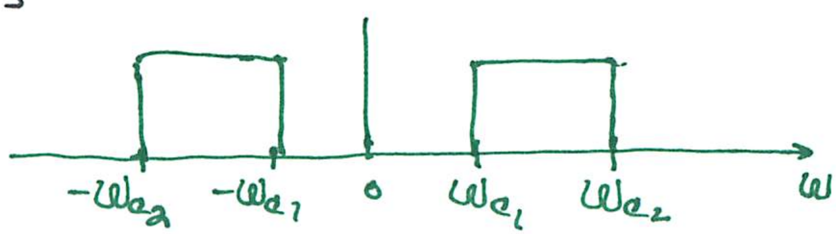


Ideal $H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$



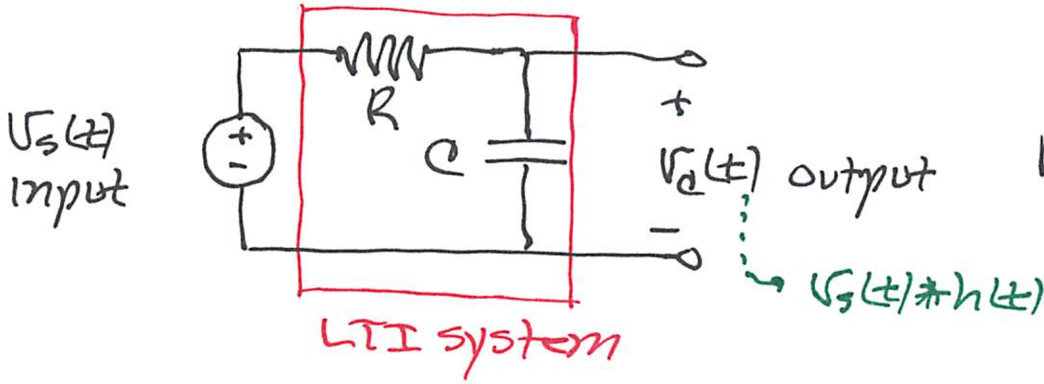
highpass

bandpass



Continuous-Time Filter (Section 3.10)

RC Circuit



$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

Let $v_s(t) = e^{j\omega t}$, then $v_c(t) = H(j\omega)e^{j\omega t}$

$$RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

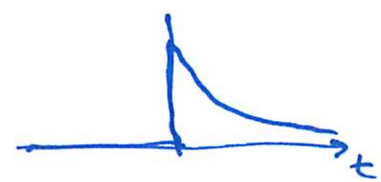
$$RC\,H(j\omega)\,j\omega\,\cancel{e^{j\omega t}} + H(j\omega)\cancel{e^{j\omega t}} = \cancel{e^{j\omega t}}$$

$$j\omega RC\,H(j\omega) + H(j\omega) = 1$$

$$\boxed{H(j\omega) = \frac{1}{1 + j\omega RC}}$$

frequency response

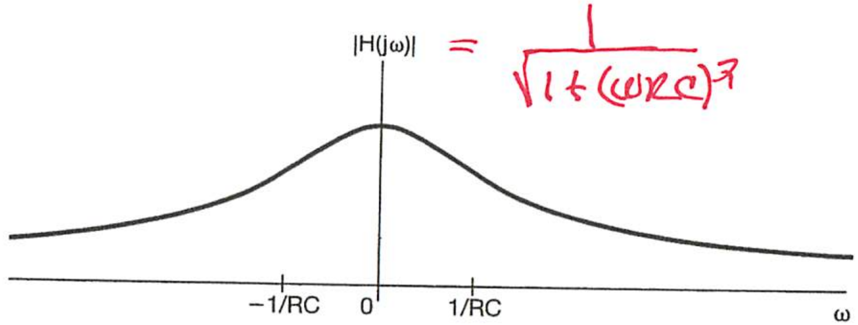
$h(t) =$ impulse response $= \frac{1}{RC}e^{-t/RC}u(t)$



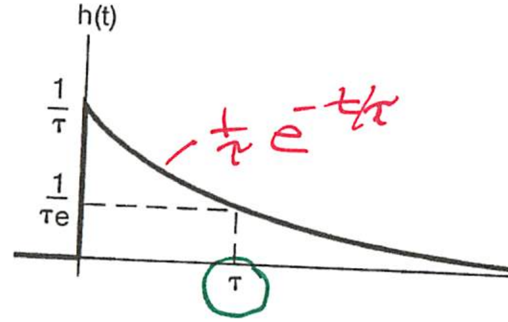
Continuous-Time Filter (Section 3.10)

RC Circuit (cont'd)

$\tau = RC$

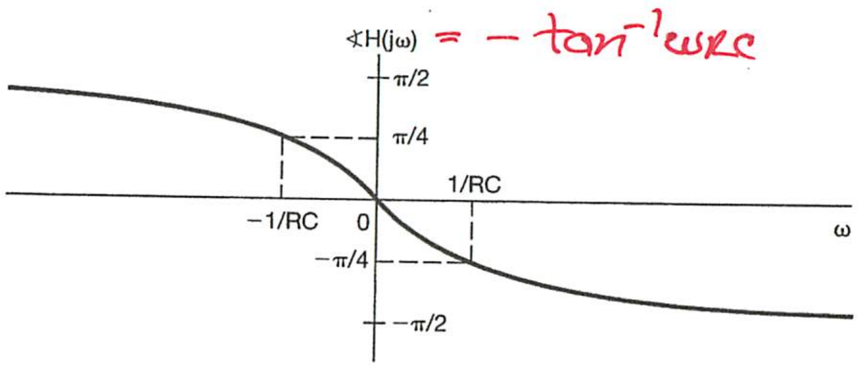


(a)

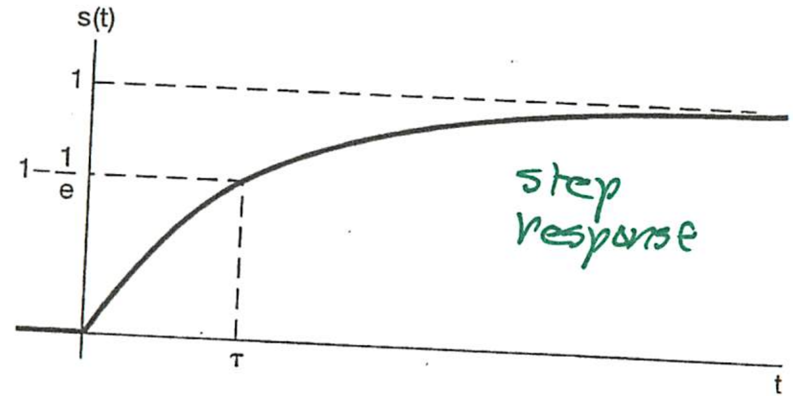


impulse response

(a)



(b)



step response

(b)

Chapter 4
Continuous-Time Fourier Transforms $e^{j\omega t}$

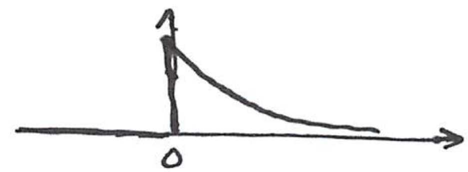
Ch. 5 Discrete-Time Fourier Transforms $e^{j\omega n}$

Ch. 9 Laplace Transform e^{st} $s = \sigma + j\omega$
Ch. 10 Z-Transform z^n $z = re^{j\omega}$

Ch. 7 Sampling

Continuous-Time Fourier Transform Example (#4.1, pp. 290-291)

$$x(t) = e^{-at}u(t), \quad a>0$$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = -\frac{1}{a+j\omega}\Big[\underbrace{e^{-j\omega\infty}e^{-a\infty}}_{a>0 \quad =0} - 1\Big]$$

$$\therefore X(j\omega) = \frac{1}{a+j\omega}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

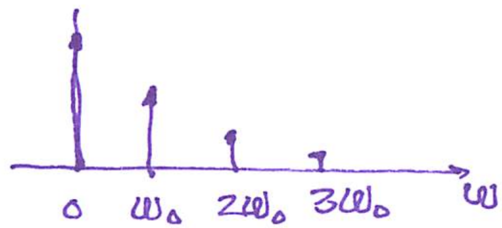
$$\measuredangle X(j\omega) = 0 - \tan^{-1}\frac{\omega}{a} = -\tan^{-1}\frac{\omega}{a}$$

Continuous-Time Fourier Transforms

- periodic signal $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ Fourier series

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

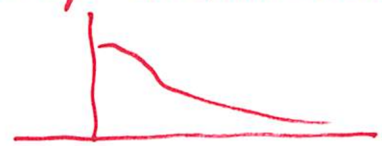
line spectrum



spacing $\omega_0 = \frac{2\pi}{T}$

- aperiodic signal = periodic signal with $T \to \infty$

  - spacing $\to 0$
  - line spectrum becomes continuous



Fourier Transform pair ($\omega = 2\pi f$)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \text{inverse FT}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{FT}$$

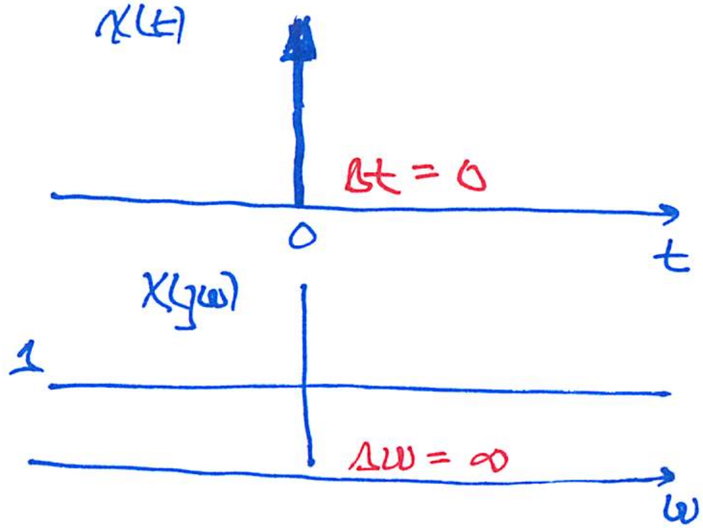
Continuous-Time Fourier Transform
Example (#4.3, p. 292)

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \qquad x(t_0) = \int x(t)\,\delta(t-t_0)\,dt$$

$$= e^{j\omega t}\Big|_{t=0}$$

$$= 1$$

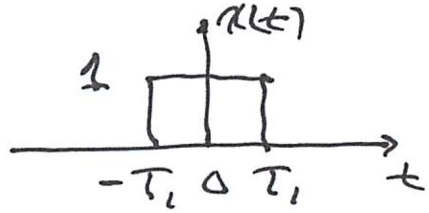


$$x(t) = \delta(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = e^{-j\omega t_0}$$

Continuous-Time Fourier Transform Example (#4.4, pp. 293-294)

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



rectangular pulse

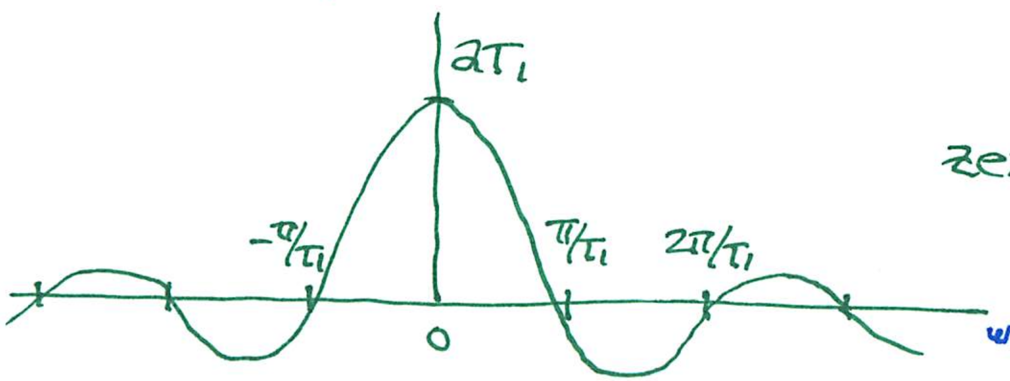
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right) = \frac{2}{2j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right)$$

$$= \frac{2}{\omega}\sin\omega T_1 = \frac{2T_1}{\omega T_1}\sin\omega T_1$$

$$X(j\omega) = 2T_1 \frac{\sin\omega T_1}{\omega T_1}$$

"sinc function" $\left(\mathrm{sinc}\,\theta = \frac{\sin\pi\theta}{\pi\theta}\right)$



zeros $\omega T_1 = m\pi$

$\omega = \dfrac{m\pi}{T_1}$

ELEG 305 March 22, 2016

Review: Continuous-Time Fourier Transform

aperiodic signals ← periodic $T \to \infty$, $\omega_0 = 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Inv. FT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

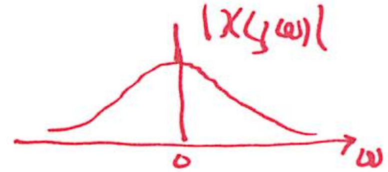
FT

Fourier Transform pair

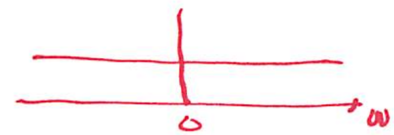
$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

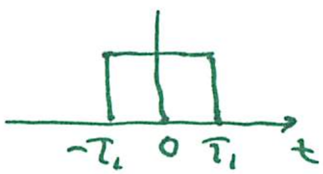
Useful pairs:

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

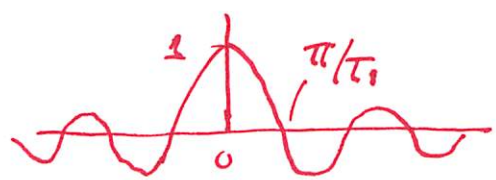


$$\delta(t) \longleftrightarrow 1$$





$$\longleftrightarrow 2T_1\frac{\sin\omega T_1}{\omega T_1}$$



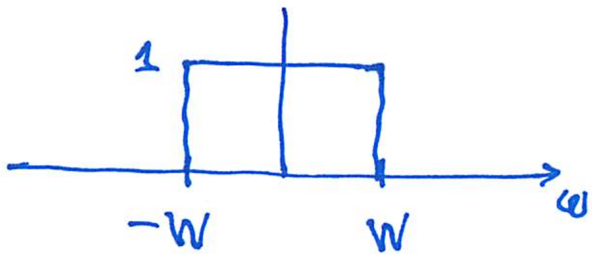
$$\boxed{\operatorname{sinc}\theta \triangleq \frac{\sin\pi\theta}{\pi\theta}}$$

Continuous-Time Fourier Transform Example (#4.5, pp. 294-296)

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

rectangular frequency characteristic

"dual"

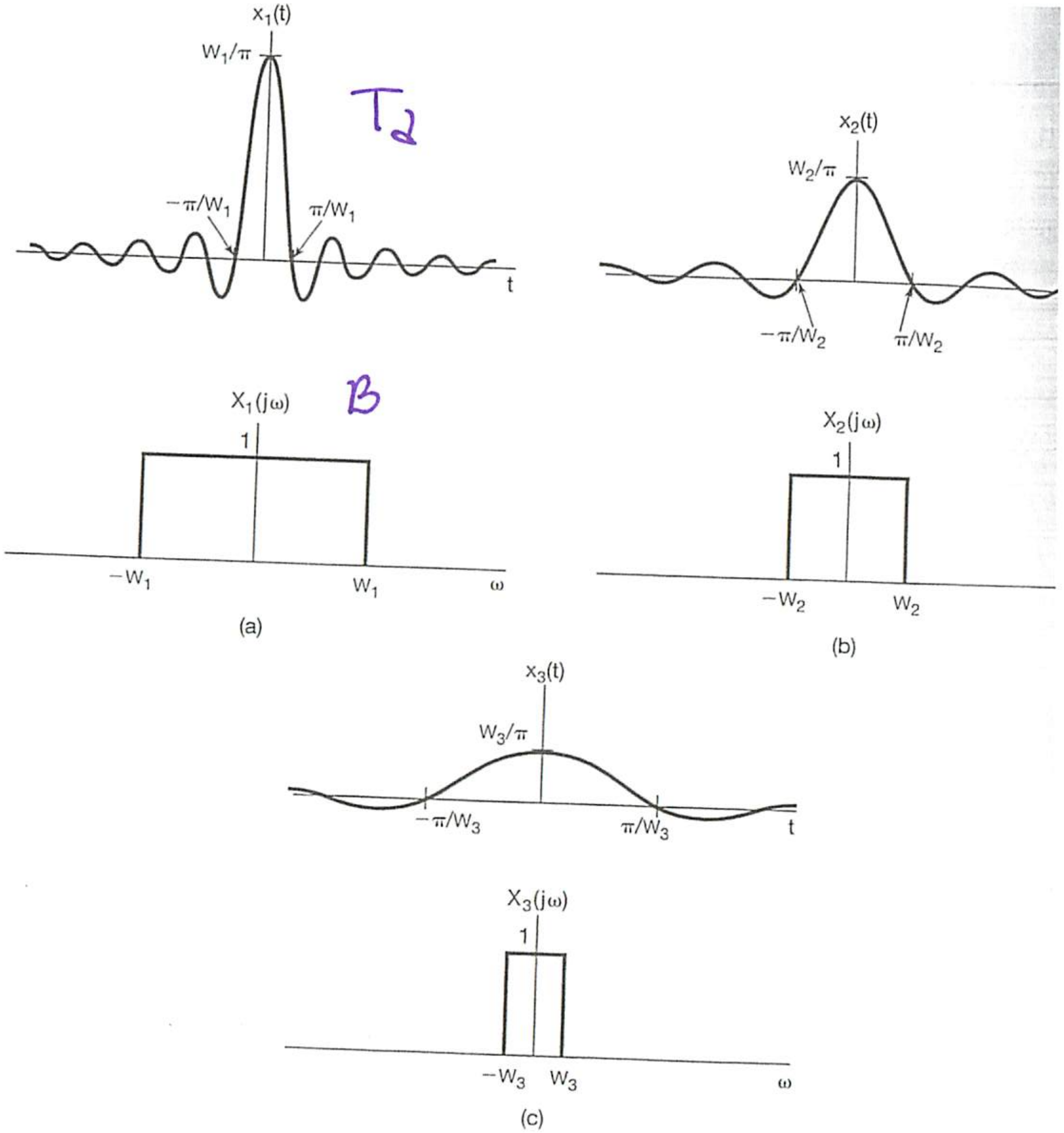


ideal lowpass filter (LPF)

$$x(t) = \frac{W}{\pi}\,\frac{\sin Wt}{Wt}$$

$$= \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

Example (#4.5, pp. 294-296, cont'd)



inverse-time-frequency relationship

$T_d B = c$

Section 4.2
Fourier Transforms of Periodic Signals

Fourier Transform of Periodic Signals

Fourier transform of periodic signal = ?

$$X(j\omega) = 2\pi \sum a_k \, \delta(\omega - k\omega_0)$$

FS coefficients

- line spectrum
- "tones" separated by ω_0

Example (#4.7, pp. 298-299)

$$x(t) = \sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$= \sum a_k e^{jk\omega_0 t}$$

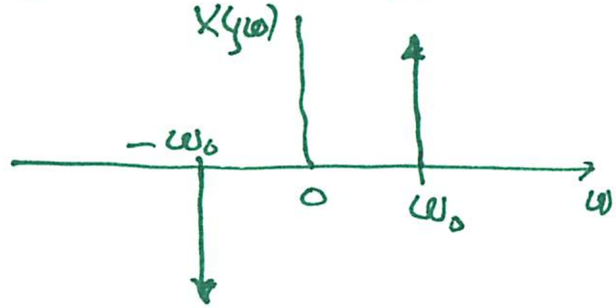
$a_0 = 0$

$a_1 = \frac{1}{2j}$

$a_{-1} = -\frac{1}{2j}$

$a_k = 0, \; k \neq 1 \text{ or } -1$

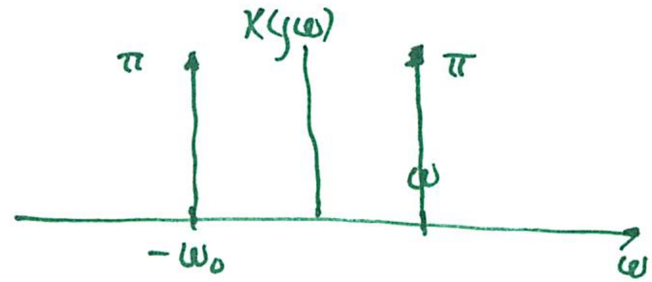
$$X(j\omega) = 2\pi \sum a_k \delta(\omega - k\omega_0)$$



$$x(t) = \cos \omega_0 t$$

$a_1 = \frac{1}{2} = a_{-1}$

$a_k = 0$ otherwise



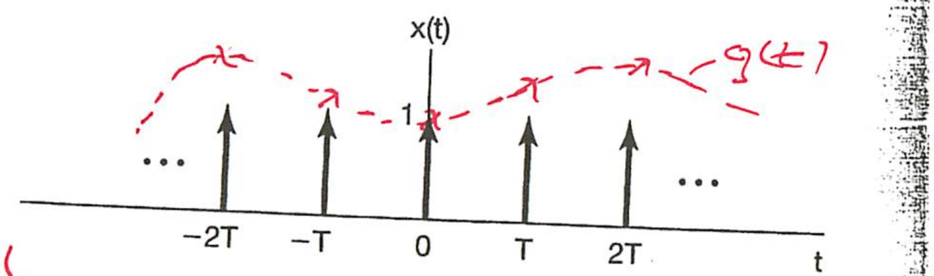
$x(t_0) = \int x(t)\,\delta(t-t_0)\,dt$

Example (#4.8, pp. 299-300)

impulse train $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

"sampling function"

$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}$

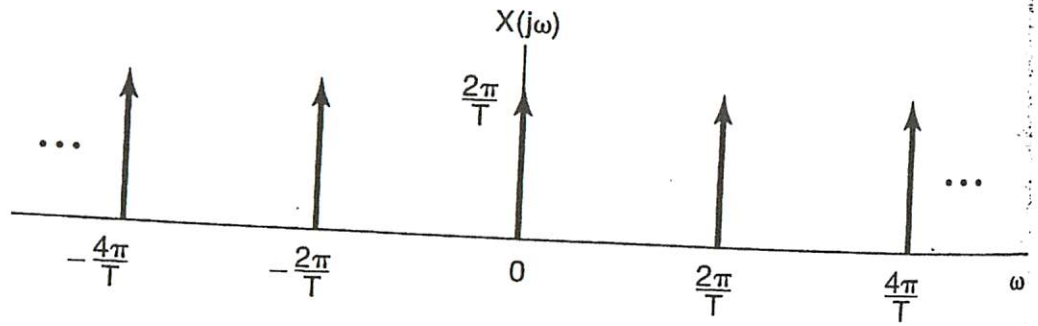


(a)

$X(j\omega) = 2\pi \sum a_k \delta(\omega - k\omega_0)$

$= \frac{2\pi}{T} \sum \delta(\omega - k\omega_0)$

(another impulse train



(b)

Properties of Continuous-Time Fourier Transform (Table 4.1)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- Linearity
- Time Shift
- Conjugation and Conjugate Symmetry
- Differentiation and Integration
- Time and Frequency Scaling
- Parseval's Relation
- Convolution
- Multiplication

Properties of CT Fourier Transform

Time Shift and Time Reversal

• Time Shift

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

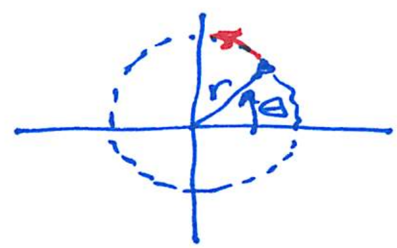
$$\mathcal{F}[x(t)] = X(j\omega) = \underbrace{|X(j\omega)|}_{r}\, e^{j \underbrace{\angle X(j\omega)}_{\theta}}$$

$$\mathcal{F}[x(t-t_0)] = X(j\omega)e^{-j\omega t_0}$$

$$= \underbrace{|X(j\omega)|}_{r}\, e^{j(\underbrace{\angle X(j\omega)}_{\theta} - \omega t_0)}$$

– magnitude unchanged

– phase shift introduced ($-\omega t_0$)



• Time Reversal

$$\mathcal{F}[x(-t)] = \int_{-\infty}^{\infty} x(-t)\,e^{-j\omega t}\,dt \;\overset{\tau=-t}{\underset{d\tau=-dt}{=}}\; -\int_{\infty}^{-\infty} x(\tau)\,e^{j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\underbrace{e^{+j\omega\tau}}_{e^{-j(-\omega)\tau}}\,d\tau = X(-j\omega)$$

$$x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

Example (Problem 4.6a, p. 335)

$$x_1(t) = \underbrace{x(1-t)}_{\substack{\text{time reversed} \\ \text{and shifted} \\ \text{or shifted and} \\ \text{time reversed}}} + x(-1-t)$$

linearity
time shift
time reversal

$$\mathcal{F}[x(-t)] = X(-j\omega)$$

$$\mathcal{F}[x(-t+1)] = X(-j\omega)e^{j\omega}$$

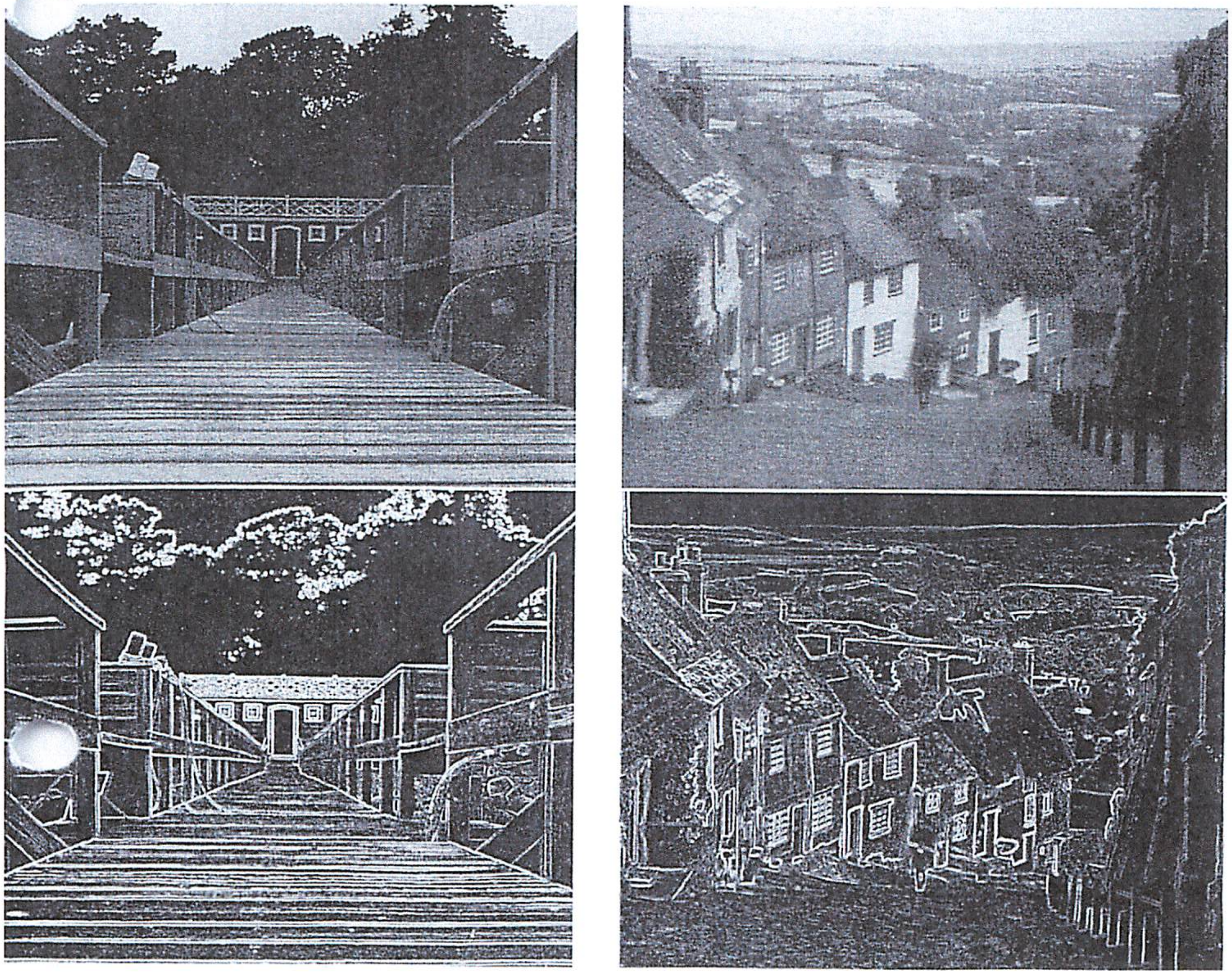
$$\mathcal{F}[x(-t-1)] = X(-j\omega)e^{-j\omega}$$

$$X_1(j\omega) = X(-j\omega)\underbrace{\left(e^{j\omega} + e^{-j\omega}\right)}_{2\cos\omega}$$

$$\therefore X_1(j\omega) = 2X(-j\omega)\cos\omega$$

Frequency Shaping Filters

Example: Differentiator



Properties of CT Fourier Transform
Differentiation and Integration

Differentiation

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

Integration

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Example (#4.11, p. 307)

$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1$$

$$x(t) = u(t) = \int_{-\infty}^{t} g(t)\,dt$$

$$X(j\omega) = \frac{\cancel{G(j\omega)}^{1}}{j\omega} + \pi\,\cancel{G(0)}^{1}\,\delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\,\delta(\omega)$$

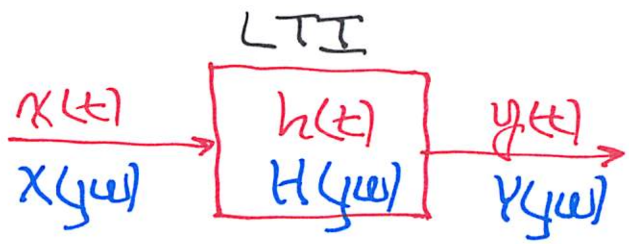
Properties of CT Fourier Transform
Convolution (Section 4.4)

convolution in time $\longleftrightarrow$ multiplication in frequency

$$x(t) * h(t) \longleftrightarrow X(j\omega)H(j\omega)$$

- often much easier to compute
- but still need to determine $y(t)$ from $Y(j\omega)$

NOTE:

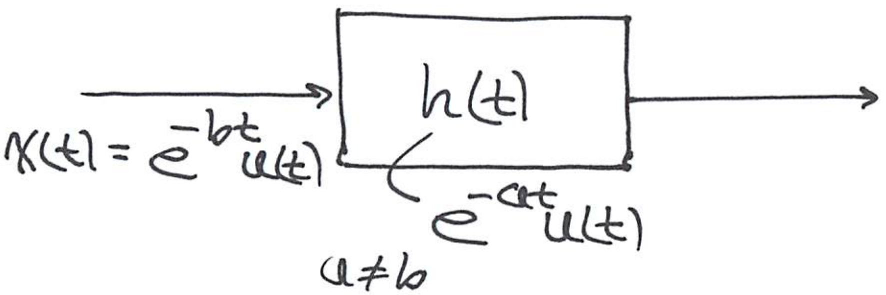


$$Y(j\omega) = H(j\omega)X(j\omega) \Longrightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$$

Example (#4.19, pp. 320-321)

$$\boxed{e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}}$$



HW from Ch. 2

$y(t) = x(t) * h(t)$

$$X(j\omega) = \frac{1}{b+j\omega} \qquad H(j\omega) = \frac{1}{a+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{b+j\omega}\cdot\frac{1}{a+j\omega}$$

$$y(t) = \mathcal{F}^{-1}[Y(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(a+j\omega)(b+j\omega)}e^{j\omega t}\,d\omega$$

- table look-up
- Matlab
- ⋮
- partial fraction expansion (PFE)

PFE

$$\frac{1}{(b+j\omega)(a+j\omega)} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} \Rightarrow y(t) = Ae^{-at}u(t) + Be^{-bt}u(t)$$

Example (#4.19, pp. 320-321, cont'd)

$$Y(j\omega) = \frac{1}{(b+j\omega)(a+j\omega)} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} \qquad a \neq b$$

$$y(t) = A e^{-at} u(t) + B e^{-bt} u(t)$$

- cross multiply

$$1 = A(b+j\omega) + B(a+j\omega)$$
$$= Ab + Ba + j\omega(A+B)$$

$$\Downarrow$$

$$\left.\begin{aligned} A+B &= 0 \\ Ab+Ba &= 1 \end{aligned}\right\} \Rightarrow \quad A = \frac{1}{b-a}, \quad B = \frac{1}{a-b}$$

Section 4.7
System Characterized by Linear, Constant-Coefficient, Differential Equations

① impulse & frequency response

$$x(t) \longrightarrow \boxed{\ h(t)\ } \longrightarrow y(t)$$

$$H(j\omega)$$

$$y(t) = x(t) * h(t)$$
$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

② differential equation

Linear, Constant-Coefficient, Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$ Nth order

$$\Downarrow \mathcal{F} \qquad \frac{d}{dt} \rightarrow j\omega$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$Y(j\omega) \sum_{k=0}^{N} a_k (j\omega)^k = X(j\omega) \sum_{k=0}^{M} b_k (j\omega)^k$$

frequency response $\quad H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$

$$\Downarrow \mathcal{F}^{-1}$$

impulse response $\quad h(t) = \mathcal{F}^{-1}[H(j\omega)]$

Example (#4.25, pp. 331-332)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

* Solve for $h(t)$

$$(j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 4j\omega + 3\right] = X(j\omega)(j\omega + 2)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega+3)(j\omega+1)}$$

- remember $e^{-at}u(t) \longleftrightarrow \dfrac{1}{j\omega + a}$

$$H(j\omega) = \frac{j\omega + 2}{(j\omega+3)(j\omega+1)} = \frac{A}{j\omega+3} + \frac{B}{j\omega+1}$$

$$= \frac{1}{2}\left[\frac{1}{j\omega+3} + \frac{1}{j\omega+1}\right]$$

$$h(t) = \mathcal{F}^{-1}\left[H(j\omega)\right] = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

Example (#4.26, pp. 332-333)

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$\boxed{x(t) = e^{-t}u(t)} \Rightarrow X(j\omega) = \frac{1}{j\omega+1}$$

* Solve for $y(t)$

$$H(j\omega) = \frac{j\omega+2}{(j\omega+3)(j\omega+1)}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega+2}{(j\omega+3)(j\omega+1)^2}$$

$$= \frac{A_{11}}{j\omega+3} + \frac{A_{21}}{j\omega+1} + \frac{A_{12}}{(j\omega+1)^2}$$ (Appendix p. 915)

$$y(t) = \underbrace{A_{11}}_{\frac{1}{4}} e^{-3t}u(t) + \underbrace{A_{21}}_{-\frac{1}{4}} e^{-t}u(t) + \underbrace{A_{12}}_{\frac{1}{2}} t e^{-t}u(t)$$

Properties of CT Fourier Transform

Time and Frequency Scaling

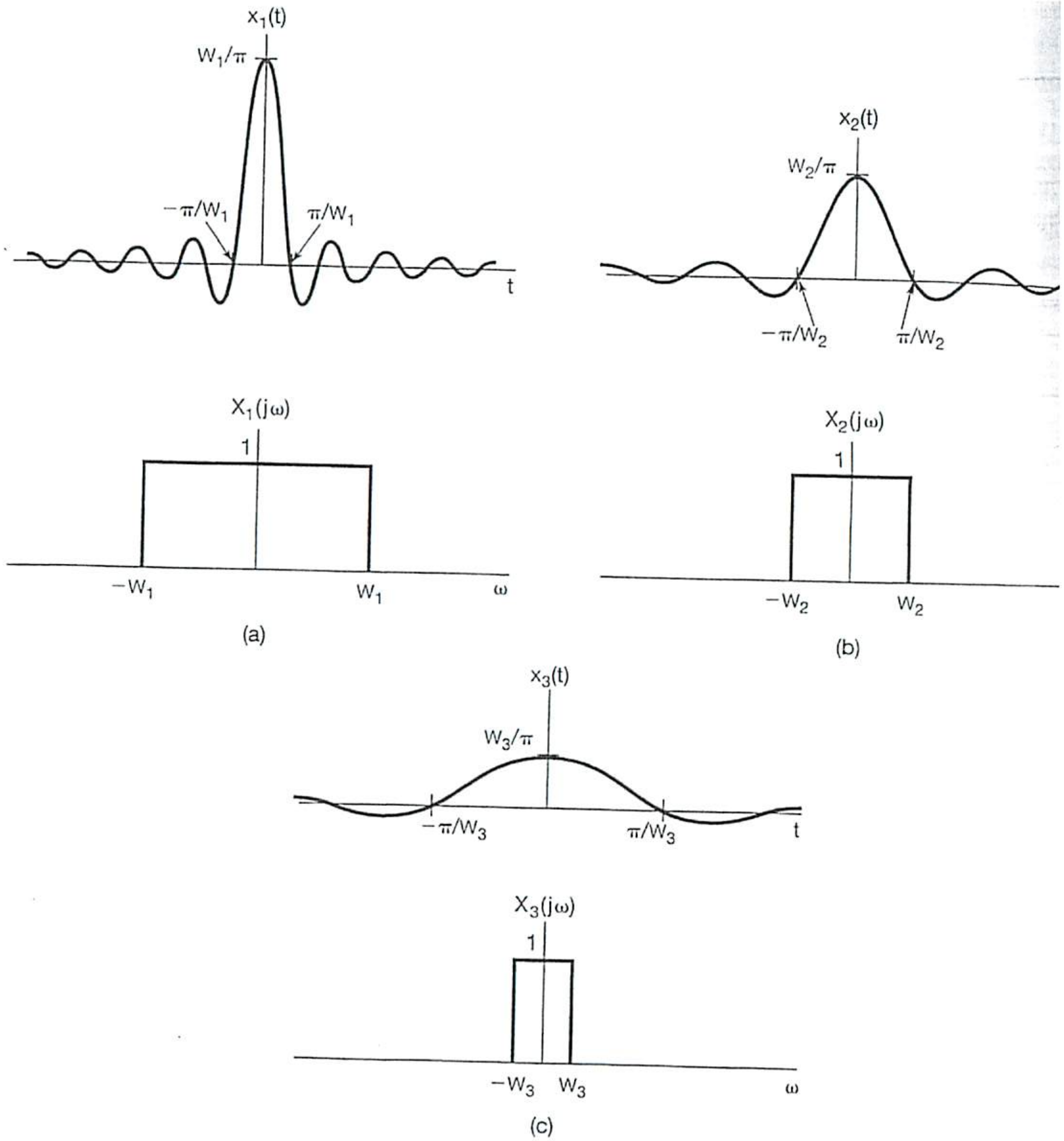
$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

Inverse-time-frequency relationship

("uncertainty principle")

"Uncertainty" Principle



Properties of CT Fourier Transform

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

energy computed in time domain

energy computed in frequency domain

$$\text{energy density spectrum} = \frac{|X(j\omega)|^2}{2\pi}$$

Properties of CT Fourier Transform

Conjugation and Conjugate Symmetry

If $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$, then $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$

special case: $\underline{x(t) \text{ real}} \Rightarrow x(t) = x^*(t)$

$$X(j\omega) = X^*(-j\omega)$$

$$\boxed{X(-j\omega) = X^*(j\omega)}$$ conjugate symmetry

$$X(j\omega) = Re[X(j\omega)] + j\,Im[X(j\omega)]$$

$$X^*(j\omega) = Re[X(j\omega)] - j\,Im[X(j\omega)]$$

$$= Re[X(-j\omega)] + j\,Im[X(-j\omega)]$$

$\therefore$ $Re[X(j\omega)] = Re[X(-j\omega)]$ $\Rightarrow$ even

$-Im[X(j\omega)] = Im[X(-j\omega)]$ $\Rightarrow$ odd

polar coordinates:
- magnitude even
- phase odd

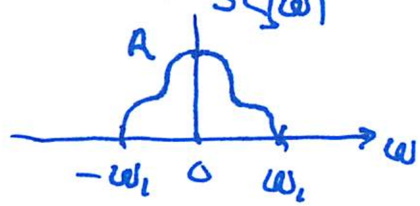
Amplitude Modulation $\Rightarrow$ application Ch. 8
Example (#4.21 & #4.22, pp. 323-325)

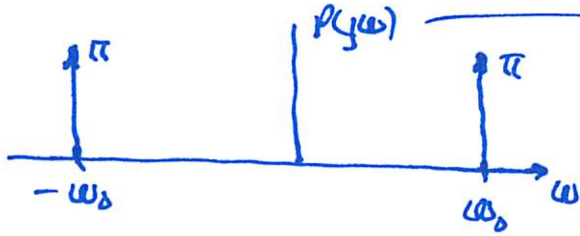
< "modulation" < "demodulation"

modulation signal $s(t)$

carrier $p(t) = \cos \omega_0 t$ — carrier frequency

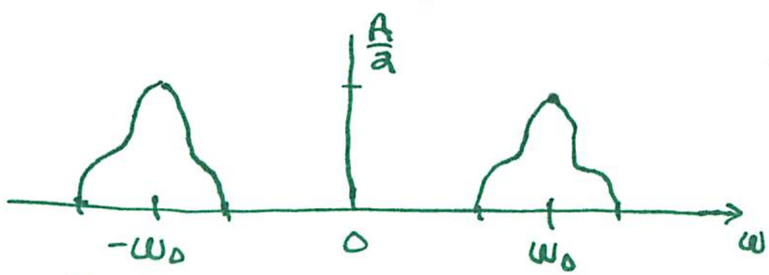
AM signal $r(t) = s(t)\,p(t) = s(t)\cos\omega_0 t$





$P(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$

$R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$



shift up and down in frequency

demodulation $\hat{s}(t) = r(t)\cos\omega_0 t$ then lowpass filter

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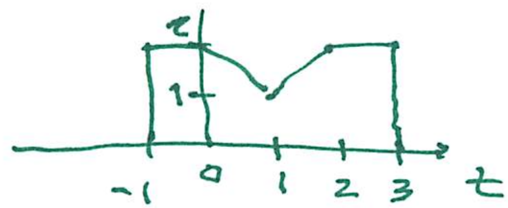
Examples

#4.25 e) $\displaystyle\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = ?$

Given $x(t)$



Use Parseval's Relation

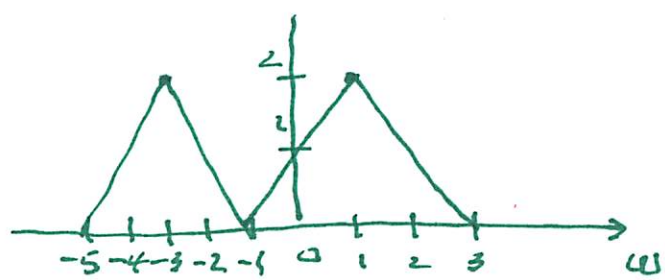
$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 \, dt = 2\pi \cdot \frac{38}{3}$$

Sample Exam

Given $X(j\omega)$

$$\int_{-\infty}^{\infty} x(t) e^{2jt} \, dt$$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$$\therefore \int_{-\infty}^{\infty} x(t) e^{2jt} \, dt = X(j\omega)\Big|_{\omega=-2} = 1$$

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Examples

#4.33 $$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

Impulse Response ?

Method #1: $x(t) = \delta(t)$ then solve differential equation for this input

Method #2: $\mathcal{F}\left\{\begin{matrix}\text{diff.}\\ \text{eq.}\end{matrix}\right\} \longrightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$

$$\Downarrow$$

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$$

$$(j\omega)^2 Y(j\omega) + 6j\omega Y(j\omega) + 8Y(j\omega) = 2X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 6j\omega + 8\right] = 2X(j\omega)$$

frequency response $$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{(j\omega)^2 + 6j\omega + 8} = \frac{2}{(j\omega+4)(j\omega+2)}$$

impulse response $$h(t) = \mathcal{F}^{-1}\{H(j\omega)\} = \mathcal{F}^{-1}\left\{\frac{-1}{j\omega+4} + \frac{1}{j\omega+2}\right\} = -e^{-4t}u(t) + e^{-2t}u(t)$$

Properties of Continuous-Time Fourier Transform (Table 4.1)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- Linearity
- Time Shift $\longrightarrow$ phase shift (rotation) in frequency
- Conjugation and Conjugate Symmetry
- Differentiation and Integration
- Time and Frequency Scaling
- Parseval's Relation
- Convolution
- Multiplication

— makes computation of $\mathcal{F}$ & $\mathcal{F}^{-1}$ easier

— how some operation/transformation in time affects the frequency content

Properties of CT Fourier Transform
Multiplication (Section 4.5)

multiplication in time $\longleftrightarrow$ convolution in frequency

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = S(j\omega) * P(j\omega)$$

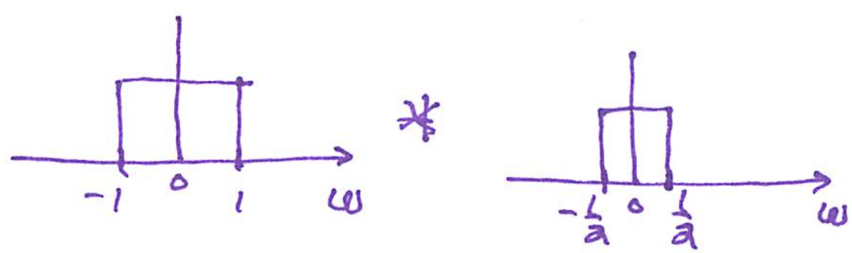
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)\, P(j(\omega-\theta))\, d\theta$$

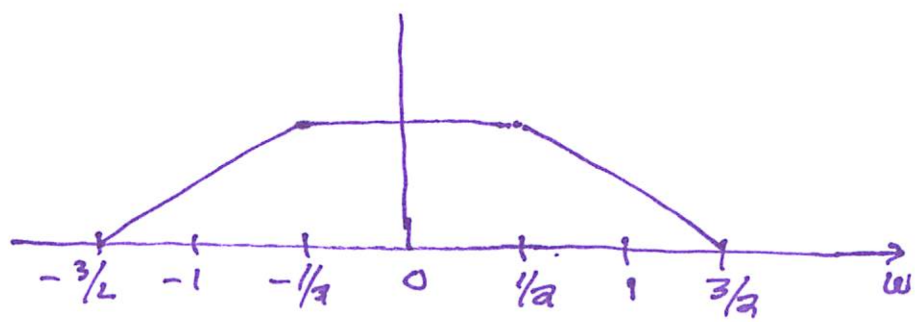
Example (#4.23, p. 325)

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}, \quad X(j\omega) = ? = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \pi \left[\frac{\sin t}{\pi t}\right]\left[\frac{\sin t/2}{\pi t}\right]$$

$$X(j\omega) = \frac{1}{2} \mathcal{F}\left[\frac{\sin t}{\pi t}\right] * \mathcal{F}\left[\frac{\sin t/2}{\pi t}\right]$$





Chapter 5
Discrete-Time Fourier Transforms

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transforms

periodic signal with period N $\xrightarrow{N \to \infty}$ aperiodic signal

Review - Discrete Time Fourier Series $x[n] = x[n+N]$ ← period

i.) complex exponentials $e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N} n}$

only N distinct complex exp.

ii.) $e^{j(\omega_0 + 2\pi)n} = e^{j\omega_0 n}$

periodic ⇒ only need to consider a frequency interval of 2π

iii.) $x[n] = \sum_{\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$

$a_k = \frac{1}{N} \sum_{\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$ Fourier series coefficients

- line spectrum
- $a_{k+N} = a_k$ ⇒ periodic
- spacing $\omega_0 = 2\pi/N$

Discrete-Time Fourier Transforms

— highlight differences with continuous-time

$$N \rightarrow \infty \Rightarrow \frac{1}{N} \rightarrow \frac{\omega_0}{2\pi} \rightarrow \frac{d\omega}{2\pi}$$

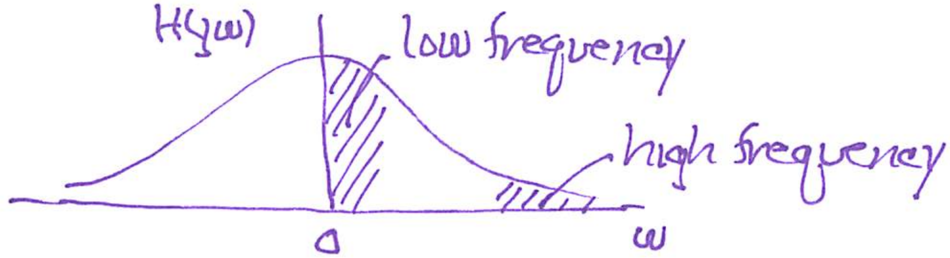
Fourier Transform pair

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \underbrace{X(e^{j\omega})}_{\text{periodic}} e^{j\omega n}\, d\omega$$

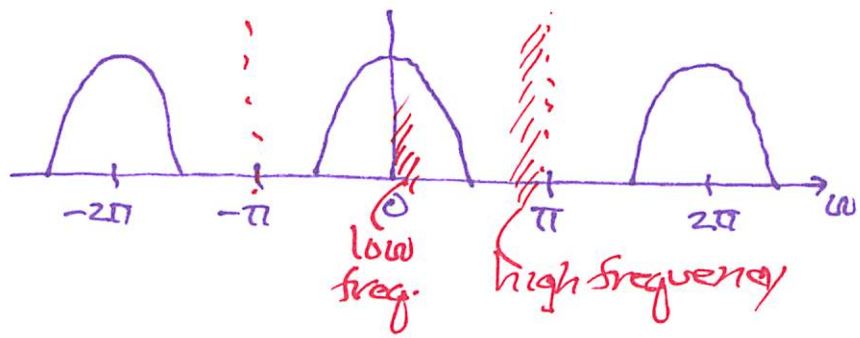
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

spectrum

continuous time



discrete time



Discrete-Time Fourier Transform Example (#5.1, pp. 362-363)

$x[n] = a^n u[n] \longrightarrow$ only converges if $|a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n \underbrace{u[n]}_{0 \text{ for } n<0} e^{-j\omega n}$$

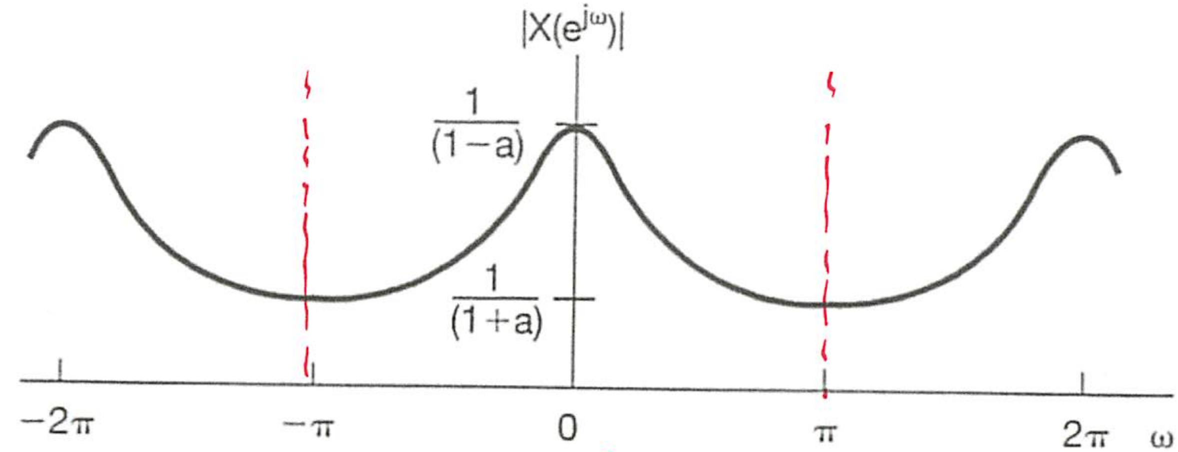
$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\underbrace{ae^{-j\omega}}_{\alpha,\ |\alpha|<1})^n \quad \text{geometric series}$$

$$= \frac{1}{1-\alpha} = \frac{1}{1-ae^{-j\omega}}$$

Discrete-Time Fourier Transform Example (#5.1, pp. 362-363)

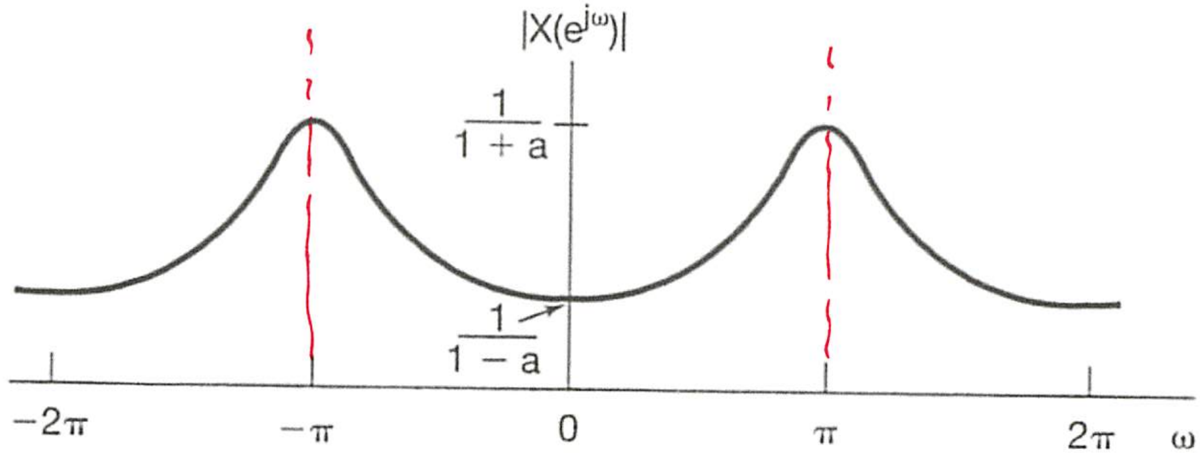
$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

periodic



$a > 0$

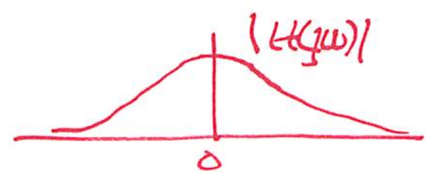
"lowpass"



$a < 0$

"highpass"

NOTE: continuous-time $e^{-at}u(t) \longleftrightarrow \frac{1}{j\omega + a}$



lowpass filter

Discrete-Time Fourier Transform
Example (#5.3, pp. 365-366)

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$ rectangular pulse

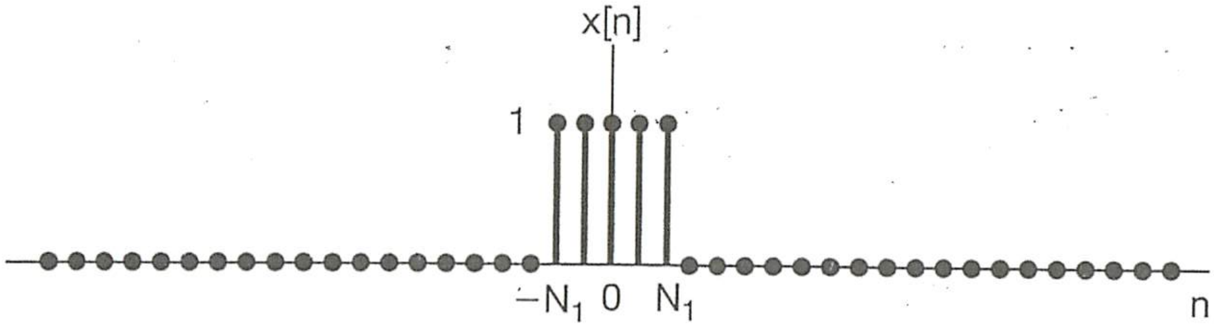
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n$$

geometric series

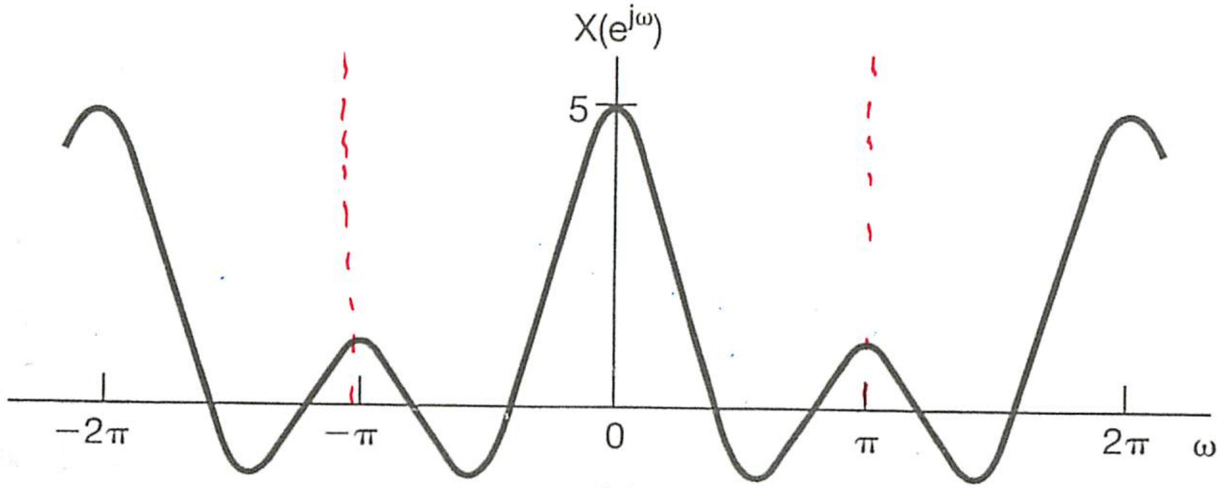
$$= e^{j\omega N_1} \left(\frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}} \right)$$

$$X(e^{j\omega}) = \frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\frac{\omega}{2})}$$

Discrete-Time Fourier Transform Example (#5.3, pp. 365-366, cont'd)



(a)



(b)

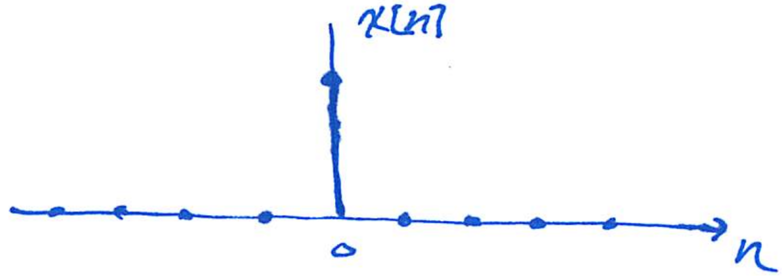
Discrete-Time Fourier Transform
Example (#5.4, p. 367)

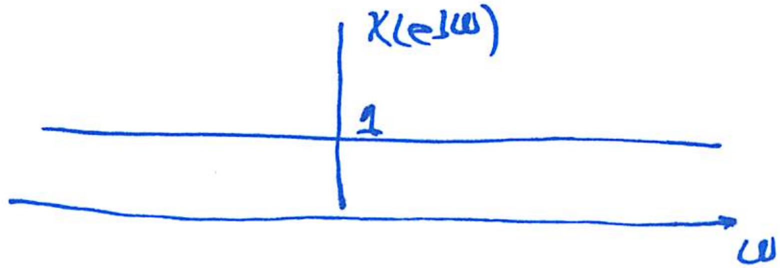
$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] \, e^{-j\omega n}$$

(only has value at $n=0$

$$= 1$$





Sections 5.3-5.6
Properties of DT Fourier Transforms

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega + a}$$

Problem #4.36

LTI system input $x(t) = [e^{-t} + e^{-3t}]u(t)$

output $y(t) = [2e^{-t} - 2e^{-4t}]u(t)$

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)}$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

a.) frequency response of the system $H(j\omega)$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$Y(j\omega) = \frac{2}{j\omega+1} - \frac{2}{j\omega+4} = \frac{6}{(j\omega+1)(j\omega+4)}$$

$$X(j\omega) = \frac{1}{j\omega+1} + \frac{1}{j\omega+3} = \frac{2j\omega+4}{(j\omega+1)(j\omega+3)} = \frac{2(j\omega+2)}{(j\omega+1)(j\omega+3)}$$

$$\therefore H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\cancel{6}^{\,3}}{\cancel{(j\omega+1)}(j\omega+4)} \cdot \frac{\cancel{(j\omega+1)}(j\omega+3)}{\cancel{2}(j\omega+2)} = \frac{3(j\omega+3)}{(j\omega+2)(j\omega+4)} \; *$$

b.) impulse response of the system $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$

Problem #4.36 (cont'd)

$$H(j\omega) = \frac{3(j\omega+3)}{(j\omega+2)(j\omega+4)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+4} \Rightarrow \begin{array}{l} A = \frac{3}{2} \\ B = \frac{3}{2} \end{array}$$

$$\therefore h(t) = \frac{3}{2}\left[e^{-2t} + e^{-4t}\right]u(t)$$

c.) differential equation?

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3(j\omega+3)}{(j\omega+2)(j\omega+4)} = \frac{3j\omega+9}{(j\omega)^2+6j\omega+8}$$

$$(3j\omega+9)X(j\omega) = Y(j\omega)\left[(j\omega)^2+6j\omega+8\right]$$

$$3j\omega X(j\omega) + 9X(j\omega) = (j\omega)^2 Y(j\omega) + 6j\omega Y(j\omega) + 8Y(j\omega)$$

$$\Downarrow \mathcal{F}^{-1}$$

$$3\frac{dx(t)}{dt} + 9x(t) = \frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t)$$

Section 5.2
Fourier Transforms of Periodic Signals

Fourier Transform of Periodic Signals

continuous time $\quad e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega-\omega_0)$

discrete time $\quad e^{j\omega_0 n} \longleftrightarrow ?$

- should be impulse function
- but must be periodic (2π)

$$\sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)$$

impulses separated by 2π

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \longleftrightarrow X(e^{j\omega})$$

$$= \sum_{k=-\infty}^{\infty} a_k\, 2\pi\,\delta\left(\omega-\frac{2\pi k}{N}\right)$$

- line spectrum
- periodic

Example (#5.5, p. 371)

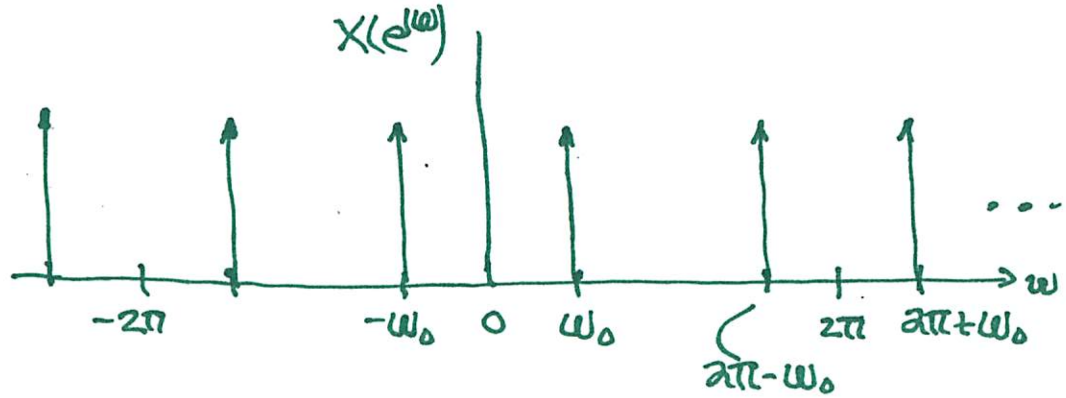
$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n} \quad \left(\omega_0 = \frac{2\pi}{5}\right)$$

$$X(e^{j\omega}) = \ ?$$

$$= \frac{1}{\cancel{2}} \sum_{l=-\infty}^{\infty} \cancel{2}\pi\, \delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right)$$

$$+ \frac{1}{\cancel{2}} \sum_{l=-\infty}^{\infty} \cancel{2}\pi\, \delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

$$\therefore \quad X(e^{j\omega}) = \pi\, \delta\left(\omega - \frac{2\pi}{5}\right) + \pi\, \delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi < \omega < \pi$$



Properties of DT Fourier Transform

Multiplication (Section 5.5)

$$y[n] = x_1[n]\, x_2[n] \longleftrightarrow$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

convolution

* periodic convolution

Example (Problem #5.21k, p. 403)

$$x[n] = \left(\frac{\sin \pi n/5}{\pi n}\right)\cdot \cos\left(\frac{7\pi}{2}n\right)$$

$$x_1[n] \cdot \underbrace{x_2[n]}_{\text{carrier}}$$

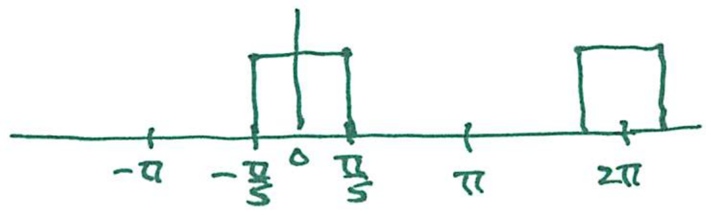
amplitude modulation

$$= \cos\frac{\pi}{2}n$$

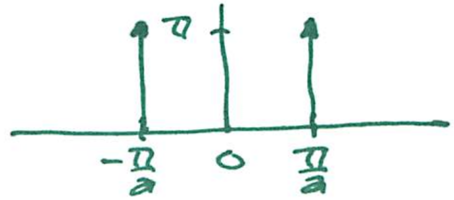
$$= \frac{e^{j\frac{\pi}{2}n}+e^{-j\frac{\pi}{2}n}}{2}$$

$X(e^{j\omega})$ = periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$

$$X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \le |\omega| < \pi \end{cases}$$



$$X_2(e^{j\omega}) = \pi\left(\delta(\omega-\tfrac{\pi}{2}) + \delta(\omega+\tfrac{\pi}{2})\right), \quad 0 \le |\omega| \le \pi$$



$$x(t) * \delta(t-t_0) = x(t-t_0) \implies X(e^{j\omega})$$

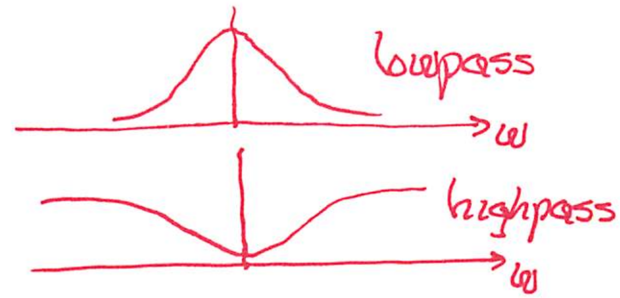
Properties of Discrete-Time Fourier Transform (Table 5.1)

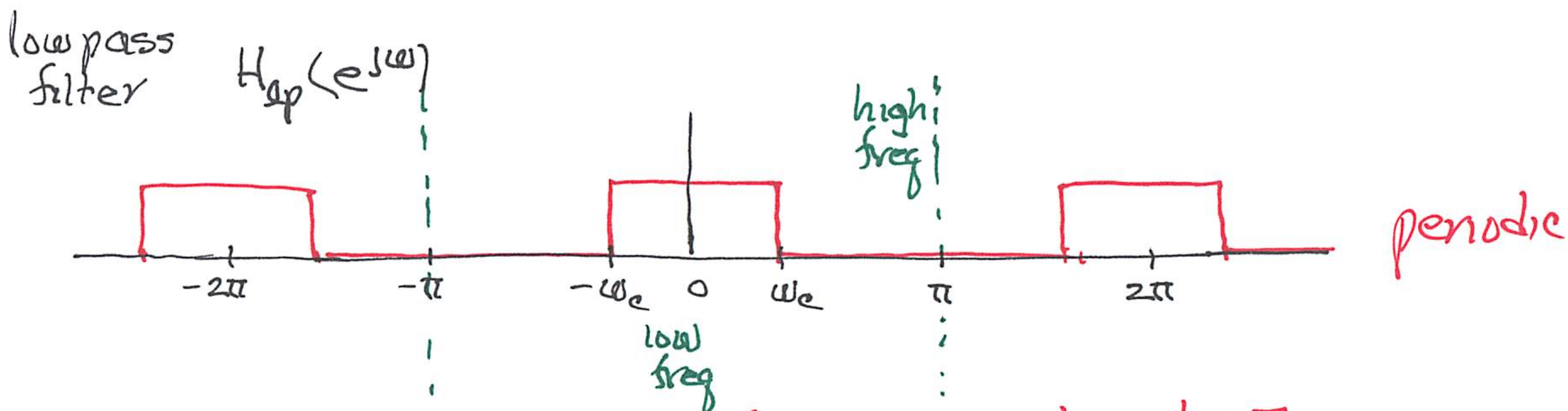
$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$
$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$
$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

- Periodicity $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
- Linearity
- Time Shift/Frequency Shift/Time Reversal
  - $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$
    time shift — phase shift
  - $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$
- Conjugation and Conjugate Symmetry
  - $x^*(n] \longleftrightarrow X^*(e^{-j\omega})$
    If $x(n]$ is real,
    $x^*[n] = x(n]$
    $X^*(e^{-j\omega}) = X(e^{j\omega})$
    "conjugate symmetry"
- Differencing and Accumulation
- Time Scaling
- Parseval's Relation
- Convolution
- Multiplication

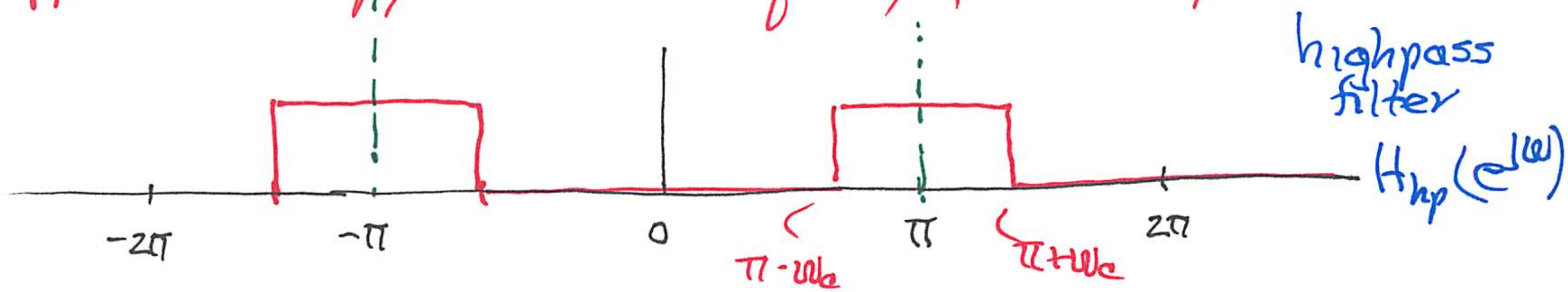
Example (#5.7, pp. 374-375)

(consequence of (1) periodicity
(2) frequency shifting





– suppose we simply shift the frequency spectrum by π



$$\therefore\ H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

impulse response

$$h_{hp}[n] = h_{lp}[n]\,e^{j\omega_0 n} = h_{lp}[n]\,e^{j\pi n} = h_{lp}[n](-1)^n$$

Properties of DT Fourier Transform

Differencing and Accumulation

derivative $\quad \frac{d}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega$

- first difference $\quad x[n] - x[n-1]$

$$\Big\updownarrow \mathcal{F}$$

$$X(e^{j\omega}) - X(e^{j\omega})e^{-j\omega}$$

$$X(e^{j\omega})\left(1 - e^{-j\omega}\right)$$

integral $\quad \int \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi X(j0)\delta(\omega)$

- accumulation $\quad y[n] = \sum_{m=-\infty}^{n} x[m]$

$$\Big\updownarrow \mathcal{F}$$

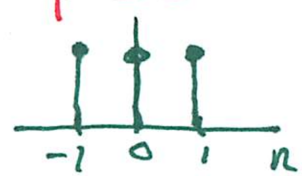
$$\frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Example (Problem #5.8, p. 401)

$$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}\left(\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right) + 5\pi\delta(\omega), \quad -\pi \le \omega \le \pi$$

$x[n] = ?$

$$\underbrace{\frac{1}{1-e^{-j\omega}}}_{\text{accumulator}}\ \underbrace{\left(\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right)}_{\text{rectangular pulse}}$$



$$x_1[n] = \begin{cases} 1, & |n| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

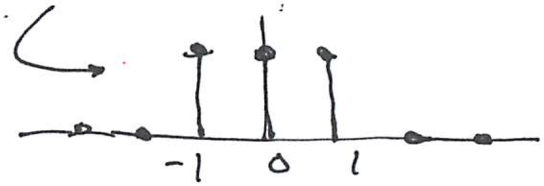
$$\sum_{k=-\infty}^{n} x_1[k] \longleftrightarrow \frac{1}{1-e^{-j\omega}}\left(\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right) + \underbrace{\pi \underbrace{X(e^{j0})}_{3} \cdot \delta(\omega)}_{\text{but need } 5\pi, \text{ so add } 2\pi\delta(\omega)}$$

$$\therefore\ x[n] = \sum_{k=-\infty}^{n} x_1[k] + \mathcal{F}^{-1}[2\pi\delta(\omega)]$$

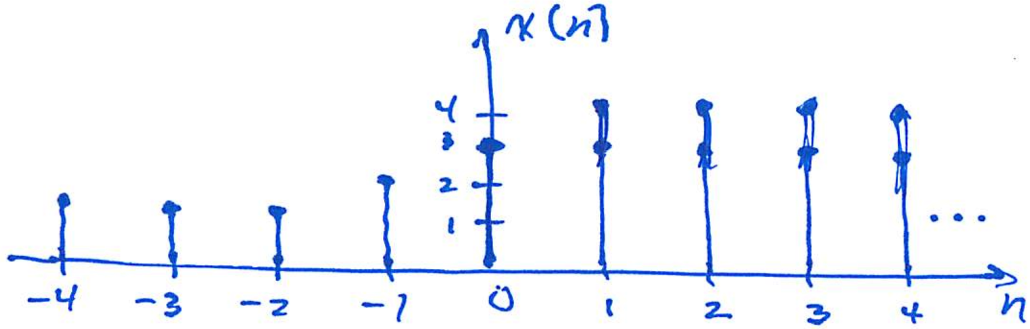
$$= \sum_{k=-\infty}^{n} x_1[k] + 1$$

Example (Problem #5.8, p. 401, cont'd)

$$x[n] = \sum_{k=-\infty}^{n} x_1[k] + 1$$



$$\therefore \quad x[n] = \begin{cases} 1, & n \le -2 \\ n+3, & -1 \le n \le 1 \\ 4, & n \ge 2 \end{cases}$$



Properties of DT Fourier Transform

Time "Scaling"

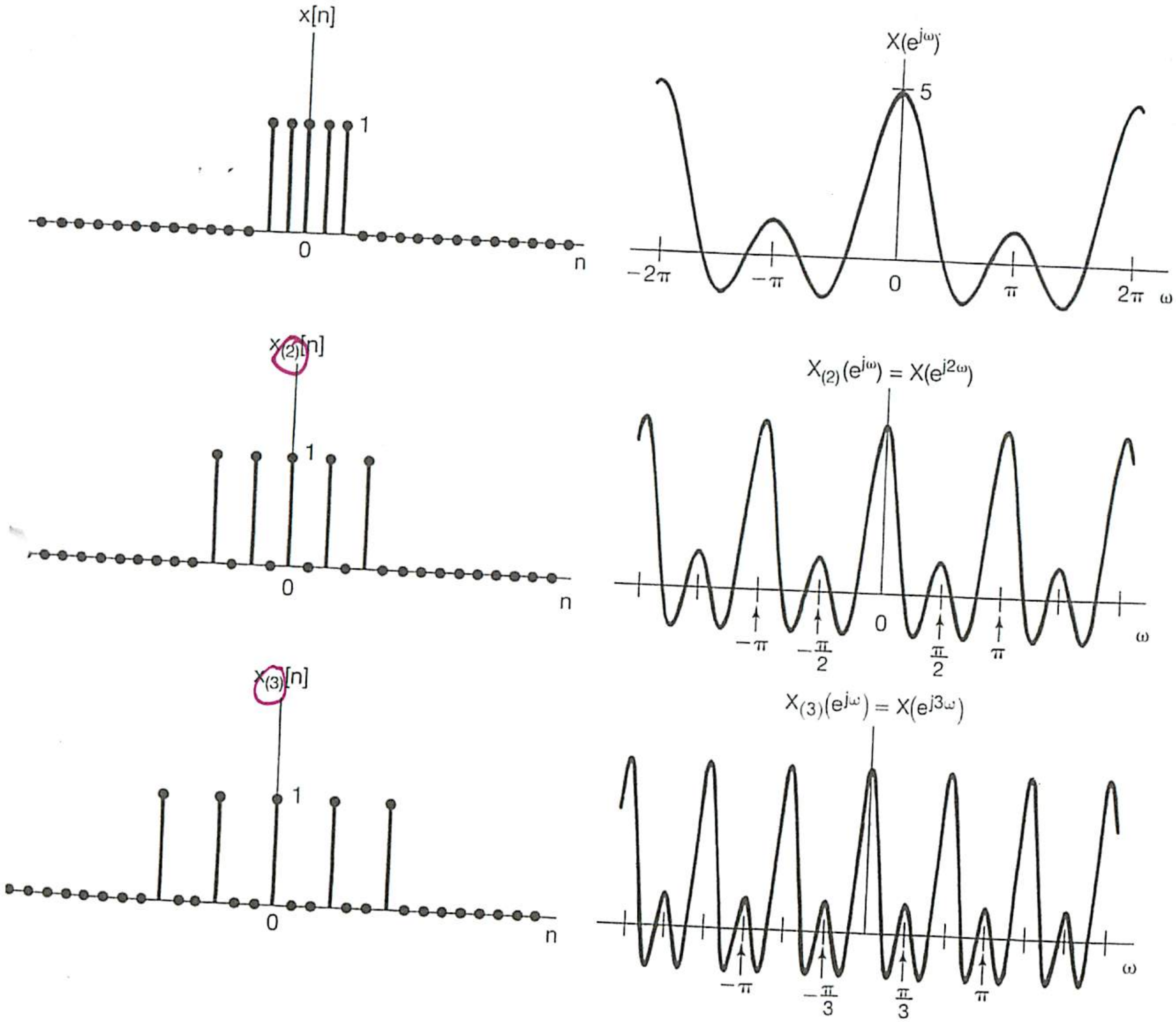
continuous time $\quad x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

discrete time ??

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

Properties of DT Fourier Transform

Time "Scaling" Example



Properties of DT Fourier Transform

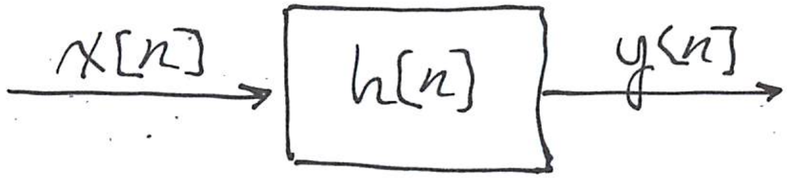
Parseval's Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\substack{\text{energy} \\ \text{computed} \\ \text{in time}}} = \underbrace{\frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega}_{\substack{\text{energy} \\ \text{computed} \\ \text{in frequency}}}$$

$$\text{energy density spectrum} = \frac{|X(e^{j\omega})|^2}{2\pi}$$

Properties of DT Fourier Transform

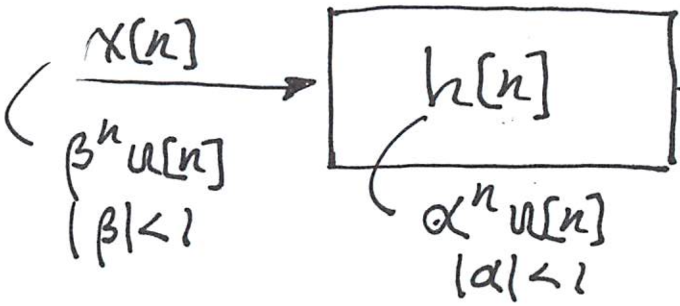
Convolution (Section 5.4)



$$\boxed{\begin{aligned} y[n] &= x[n] * h[n] \\ Y(e^{j\omega}) &= X(e^{j\omega})\, H(e^{j\omega}) \end{aligned}}$$

$$\alpha^n u[n] \longleftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

Example (#5.13, pp. 385-386)



$y[n] = h[n] * x[n]$ (HW in ch. 2)

$\alpha \neq \beta$

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}, \quad H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$\therefore \; Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1-\beta e^{-j\omega})(1-\alpha e^{-j\omega})}$$

$$= \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$y(t) = A\,\alpha^n u[n] + B\,\beta^n u[n] \quad \checkmark$$

Example (#5.13, pp. 385-386, cont'd)

$$Y(e^{j\omega}) = \frac{1}{(1-\beta e^{-j\omega})(1-\alpha e^{-j\omega})} = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$v = e^{-j\omega}$$

$$Y(v) = \frac{1}{(1-\beta v)(1-\alpha v)} = \frac{A}{1-\alpha v} + \frac{B}{1-\beta v}$$

$$\Downarrow$$

$$A = \frac{\alpha}{\alpha-\beta}$$

$$B = -\frac{\beta}{\alpha-\beta}$$

Section 5.8
System Characterized by Linear, Constant-Coefficient, Difference Equations

Linear, Constant-Coefficient, Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Nth-order difference equation

$$\Big\Downarrow \mathcal{F}$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

$$\Big\Downarrow \mathcal{F}^{-1}$$

$$h[n]$$

ELEG 305 April 5, 2016
Review: Fourier Transforms

continuous time

"almost all signals..."

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

+ properties

LTI System

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$

impulse response

$Y(j\omega) = H(j\omega)\cdot X(j\omega)$

frequency response

differential equations

discrete time

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ periodic

+ properties

LTI System

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$

impulse response

$Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$

frequency response

difference equations

+ geometric series, complex numbers, partial fraction expansion

Example (#5.19, p. 398)

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \qquad \text{2nd order}$$

- Find $h[n]$

$\downarrow \mathcal{F}$

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-2j\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}$$

polynomial in $v = e^{-j\omega}$

Remember that $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$

$$H(e^{j\omega}) = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}} = \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$\overset{PFE}{=} \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\therefore\ h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Example (#5.20, pp. 398-399)

$$y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n]$$

Find $y[n]$ for $x[n] = \left(\frac{1}{4}\right)^n u[n]$

from previous problem

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} \cdot \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})^2}$$

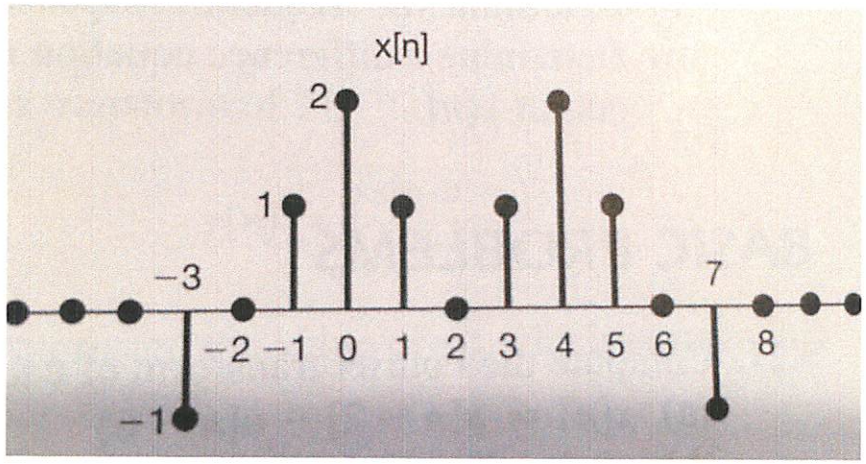
$$\overset{PFE}{=} \frac{-4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

$$\Downarrow \mathcal{F}^{-1}$$

$$y[n] = \left[-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right] u[n]$$

ELEG 305 April 11, 2017
Examples

Problem #5.23 (f) (ii)



$$\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = ?$$

Parseval's $\Rightarrow \displaystyle\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

Let $Y(e^{j\omega}) = \dfrac{dX(e^{j\omega})}{d\omega} \Rightarrow$ Prop. 5.3.8 $\quad y[n] = \dfrac{n}{j} x[n]$
$= -nj x[n]$

$$\int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |y[n]|^2 = 2\pi \sum_{n=-\infty}^{\infty} |-nj x[n]|^2 = 2\pi \sum_{n=-\infty}^{\infty} |n x[n]|^2$$

$= 316\pi$

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Examples

Problem # 5.29 (a)

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$x[n] = \left(\tfrac{3}{4}\right)^n u[n]$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

(i) $y[n] = ?$

- convolution $y[n] = x[n] * h[n]$
- frequency domain $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

$$\Big\Downarrow \mathcal{F}^{-1}$$

$$y[n]$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{(1-\frac{3}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} \overset{PFE}{=} \frac{A}{1-\frac{3}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}}$$

$$y[n] = A\left(\tfrac{3}{4}\right)^n u[n] + B\left(\tfrac{1}{2}\right)^n u[n]$$

ELEG 305 April 11, 2017
Examples

$$Y(e^{j\omega}) = \frac{1}{(1-\frac{3}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} = \frac{A}{1-\frac{3}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}}$$

$$\frac{1}{(1-\frac{3}{4}v)(1-\frac{1}{2}v)} = \frac{A}{1-\frac{3}{4}v} + \frac{B}{1-\frac{1}{2}v}$$

$$1 = A(1-\tfrac{1}{2}v) + B(1-\tfrac{3}{4}v)$$

$$\Downarrow$$

$$\left.\begin{aligned} A+B &= 1 \\ -\frac{A}{2} - \frac{3B}{4} &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= 3 \\ B &= -2 \end{aligned}$$

→ another question: given $H(e^{j\omega})$, find the difference equation relating the input & output

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega})(1-\tfrac{1}{2}e^{-j\omega}) = X(e^{j\omega})$$

$$\Downarrow \mathcal{F}^{-1}$$

$$y[n] - \tfrac{1}{2}y[n-1] = x[n]$$