Final, May 24, 2011

NAME:

1. You are dealt three cards from a well shuffled deck of 52. What are the following:

   a) Pr[three Aces]?

   b) Pr[any three of a kind]?

   c) Pr[two Aces and a King]?

   d) Pr[any pair and any other card]?

a) $P(3\text{ Aces}) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{4}{\frac{52\cdot 51\cdot 50}{3\cdot 2}} = \frac{4}{26\cdot 17\cdot 50} = \frac{1}{13\cdot 17\cdot 25} = \frac{1}{\cancel{44725}\ 5525}$

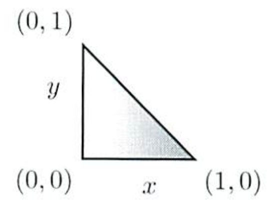
b) $P(\text{any 3 of a kind}) = \frac{13\cdot\binom{4}{3}}{\binom{52}{3}} = \frac{13\cdot 4\cdot 3\cdot 2}{52\cdot 51\cdot 50} = \frac{1}{17\cdot 25} = \frac{1}{425}$

c) $P[2\text{ Aces and }1\text{ K}] = \frac{\binom{4}{2}\binom{4}{1}}{\binom{52}{3}} = \frac{6\cdot 4}{\frac{52\cdot 51\cdot 50}{3\cdot 2}} = \frac{6}{5525}$

d) P[any pair and any other card]

$$= \frac{\binom{4}{2}\binom{4}{1}\cdot 13\cdot 12}{\binom{52}{3}} = \frac{6\cdot 4\cdot 13\cdot 12}{22100} = 0.169$$

2. $\boldsymbol{X}$ and $\boldsymbol{Y}$ have density $f_{\boldsymbol{XY}}(x,y) = cx^3$ in the triangle below and $f_{\boldsymbol{XY}}(x,y) = 0$ elsewhere.



a) What is $f_{\boldsymbol{X}}(x)$?

b) What is $f_{\boldsymbol{Y}}(y)$?

c) What is $f_{\boldsymbol{X}|\boldsymbol{Y}}(x|\boldsymbol{Y}=y)$? Show the integral of $f_{\boldsymbol{X}|\boldsymbol{Y}}(x|\boldsymbol{Y}=y)$ over the appropriate range of x is 1.

d) What is $E[\boldsymbol{X}]$?

a) $f(x,y) = cx^3 \quad f_x(x) = \int_0^{1-x} cx^3\,dy = cx^3(1-x) \quad 0<x<1$

$$1 = \int_0^1 c x^3(1-x)\,dx = c\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = c\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{c}{20}$$

$$\Rightarrow c = 20 \Rightarrow \boxed{f_x(x) = \begin{cases} 20x^3(1-x) & 0<x<1 \\ 0 & \text{o.w.} \end{cases}}$$

b) $$f_y(y) = \int_0^{1-y} cx^3\,dx = c\,\frac{x^4}{4}\Big|_0^{1-y} = \frac{c}{4}(1-y)^4 = \begin{cases} 5(1-y)^4 & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

c) $$f_{x|y}(x|y) = \frac{cx^3}{\frac{c}{4}(1-y)^4} = \frac{4x^3}{(1-y)^4} \quad 0<x<1-y$$

$$\int_0^{1-y} \frac{4x^3}{(1-y)^4}\,dx = \frac{4x^4}{4(1-y)^4}\Big|_0^{1-y} = \frac{(1-y)^4}{(1-y)^4} = 1$$

d) $$EX = c\int_0^1 \cancel{x}(x^4 - x^5)\,dx = c\left(\frac{x^5}{5} - \frac{x^6}{6}\right) = \frac{20}{30} = \boxed{\frac{2}{3}}$$

3. Let X_1, X_2, ..., X_5 be IID exponential random variables with density $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.

   a) What is $\Pr[X \geq 1]$?

   b) What is $\Pr[\text{at least 3 of the 5 } X\text{'s are} \geq 1]$?

   c) What is $\Pr[\max(X_1, X_2, \ldots, X_5) \geq 1]$? (where $\max(2,5) = 5$ is the maximum function)

a) $P(X \geq 1) = \int_1^\infty \lambda e^{-\lambda x}\,dx = \int_\lambda^\infty e^{-s}\,ds = \boxed{e^{-\lambda}}$ $\quad s = \lambda x$, $ds = \lambda\,dx$

b) $P(\text{at least 3 of 5 } X\text{'s are} \geq 1) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + \binom{5}{5}p^5$

where $p = e^{-\lambda}$

c) $P(\max(X_1,\ldots,X_5) \geq 1) = 1 - P(\max(X_1,\ldots,X_5) \leq 1)$

$= 1 - P(\text{all } X_i \leq 1)$

$= 1 - (1-p)^5$ where $p = e^{-\lambda}$

4. A simple joint Probability Mass Function for $\boldsymbol{X}$ and $\boldsymbol{Y}$ is shown below:

y \ x	0	1	2	3
-1	0.1	0.0	0.1	0.1
0	0.0	0.1	0.4	0.2

a) What are $E[\boldsymbol{X}]$ and $\mathrm{Var}[\boldsymbol{X}]$?

b) What are $E[\boldsymbol{Y}]$ and $\mathrm{Var}[\boldsymbol{Y}]$?

c) What is $\mathrm{Cov}[\boldsymbol{X}, \boldsymbol{Y}]$?

d) What is $E[\boldsymbol{X}|\boldsymbol{Y}=0]$?

a) $E[X] = 0 \times 0.1 + 1 \times 0.1 + 2 \times 0.5 + 3 \times 0.3 = \boxed{2.0}$

$E[X^2] = 0^2 \times 0.1 + 1^2 \times 0.1 + 2^2 \times 0.5 + 3^2 \times 0.3 = 4.8$

$Var[X] = 4.8 - (2.0)^2 = \boxed{0.8}$

b) $E[Y] = 0 \times 0.7 + 1 \times 0.3 = \boxed{0.3}$

$E[Y^2] = 0^2 \times 0.7 + 1^2 \times 0.3 = 0.3$

$Var[Y] = 0.3 - (0.3)^2 = \boxed{0.21}$

c) $Cov[X, Y] = E[XY] - E[X]E[Y]$

$E[XY] = 2 \times 0.1 + 3 \times 0.1 = 0.5$ (all other terms are 0)

$Cov[X, Y] = 0.5 - 2.0 \times 0.3 = \boxed{\cancel{-1.1}}$ $\boxed{-0.1}$

d) $E[X | Y=0] = \dfrac{0 \times 0.0 + 1 \times 0.1 + 2 \times 0.4 + 3 \times 0.2}{0.0 + 0.1 + 0.4 + 0.2} = \dfrac{1.5}{0.7}$

$= \dfrac{15}{7} = \boxed{2\tfrac{1}{7}}$

5. Consider a hypothesis testing problem:

$$\boldsymbol{H}_0 : \boldsymbol{X} \sim N(0, \sigma^2)$$
$$\boldsymbol{H}_1 : \boldsymbol{X} \sim N(s, \sigma^2)$$

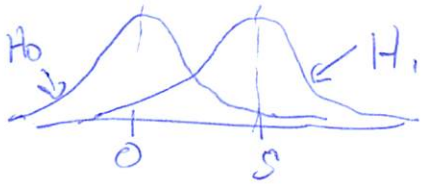
where $s > 0$ is a known parameter. Assume the test uses a threshold of $3s/4$.

a) Sketch the two densities on the same plot. Label each curve.

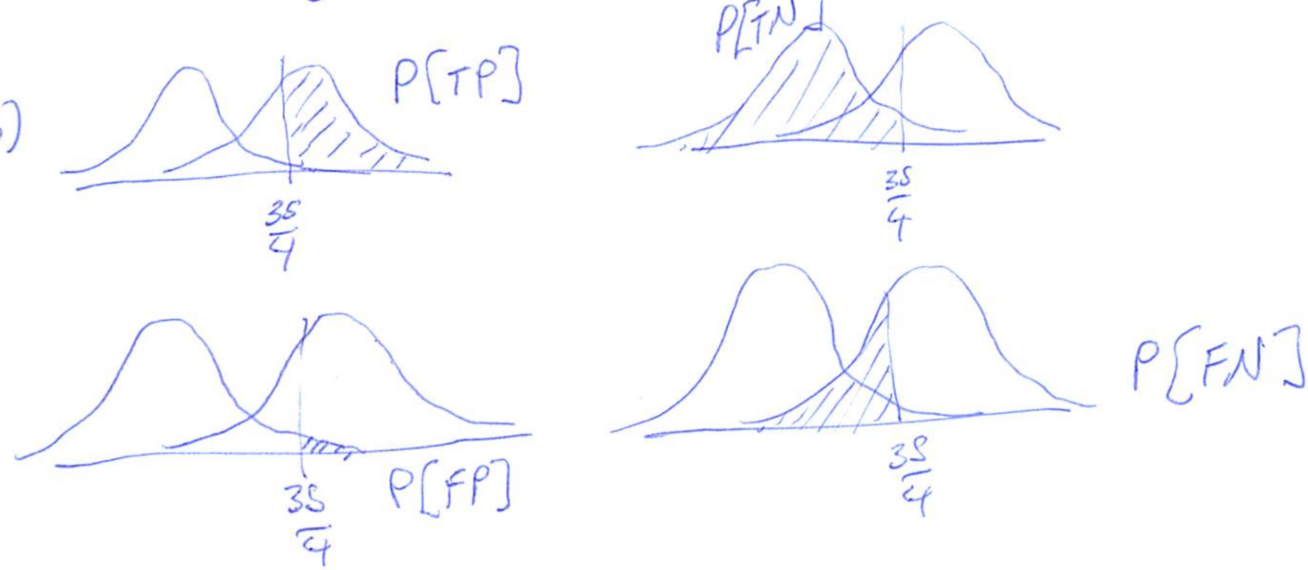
b) Shade and label on separate sketches the true positive probability $\Pr[\text{TP}]$, the true negative probabiity $\Pr[\text{TN}]$, the false positive probability $\Pr[\text{FP}]$, and the false negative probability $\Pr[\text{FN}]$.

c) Give an expression for $\Pr[\text{FP}]$ and $\Pr[\text{FN}]$ in terms of $\Phi(\cdot)$.

a)



b)



c) $P[FP] = P\left[X > \frac{3s}{4} \middle| H_0\right] = P\left[\frac{X}{\sigma} \geq \frac{3s}{4\sigma} \middle| H_0\right]$

$= 1 - \Phi\left(\frac{3s}{4\sigma}\right)$

$P[FN] = P\left(X < \frac{3s}{4} \middle| H_1\right) = P\left(\frac{X-s}{\sigma} \leq \frac{\frac{3s}{4} - s}{\sigma} \middle| H_1\right)$

$= \Phi\left(-\frac{s}{4\sigma}\right) = 1 - \Phi\left(\frac{s}{4\sigma}\right)$

6. Imagine a coin with probability p is flipped independently until a head appears. Let $\boldsymbol{N}$ be a random variable denoting the number of flips needed.

a) What is $\Pr[\boldsymbol{N} = k]$ for $k = 1, 2, \ldots$?

b) What is the entropy of $\boldsymbol{N}$ (or, more precisely, of the PMF of $\boldsymbol{N}$)?

c) How would you encode $\boldsymbol{N}$ with a binary strings (1's and 0's)? (Your code must be decodable, so that given a sequence of bits, we can uniquely determine $\boldsymbol{N}$.)

d) What is the expected length of your code?

a) $P(N=k) = p\,q^{k-1}$ for $k = 1, 2, \ldots$ $\quad q = 1-p$

b) $H(N) = E[-\log_2 P(N=k)] = -\sum_{k=1}^{\infty} \left(\log p + (k-1)\log q\right) p\,q^{k-1}$

$$= -\log p + \log q - \underbrace{\left(\sum_{k=1}^{\infty} k\,p\,q^{k-1}\right)}_{EN = \frac{1}{p}} \log q$$

$$= -\log p + \left(1 - \frac{1}{p}\right)\log q$$

$$= \frac{-p\log p - q\log q}{p} = \boxed{\frac{h(p)}{p}}$$

c) Simplest method is to encode N just as it happened $N = \underbrace{000\cdots0}_{k-1}1$

d) $E[L] = 1 \cdot P[N=1] + 2\,P[N=2] + 3 \times P[N=3] + \cdots$

$$= E[N] = \boxed{\frac{1}{p}}$$

7. (Extra credit: credit only for correct work) Alice wears contact lenses. Every night she removes her lenses and puts the left lens in a container marked L and the right lens in a container marked R. However, sometimes she accidentally switches the lenses (left lens into R container and right lens into L container). If she switches the lenses each day with probability p, and the days are independent, what is the probability after n days that she has the correct lens in each container?

$P(k \text{ switches in } N \text{ days}) = \binom{n}{k} p^k (1-p)^{n-k}$

$P(\text{lenses in right containers}) = P(\text{even \# of switches})$

$$= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

can get a closed form solution. First n even

$$(a+b)^n = \sum_{k \text{ even}} \binom{n}{k} a^k b^{n-k} + \sum_{k \text{ odd}} \binom{n}{k} a^k b^{n-k}$$

$$(a-b)^n = \sum_{k \text{ even}} \binom{n}{k} a^k b^{n-k} - \sum_{k \text{ odd}} \binom{n}{k} a^k b^{n-k}$$

$$\Rightarrow (a+b)^n + (a-b)^n = 2 \sum_{k \text{ even}} \binom{n}{k} a^k b^{n-k}$$

$$\Rightarrow \frac{1+(2p-1)^n}{2} = P(\text{lenses in right container}) \quad \text{when } n = \text{even}$$

when n is odd, $P(\text{lenses in right container}) = \dfrac{1-(2p-1)^n}{2}$