SOLUTION TO HOMEWORK #1

Problem #1

(a) Given the continuous-time signal

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

the total energy of the signal can be computed as

$$\begin{aligned} E_\infty &= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt \\ &= \int_0^1 t^2 \, dt + \int_1^2 (2-t)^2 \, dt \\ &= \frac{t^3}{3}\bigg|_0^1 + \int_1^2 (4 - 4t + t^2) \, dt \\ &= \frac{1}{3} + (4t - 4\frac{t^2}{2} + \frac{t^3}{3})\bigg|_1^2 \\ &= \frac{1}{3} + ((8 - 8 + \frac{8}{3}) - (4 - 2 + \frac{1}{3})) = \frac{2}{3} \end{aligned}$$

Since $E_\infty < \infty$, $x(t)$ is an **energy** signal with average power $P_\infty = 0$.

(b) Given the discrete-time signal

$$\begin{aligned} x[n] &= 3(\frac{1}{3})^n u[n] \\ &= \begin{cases} 3(\frac{1}{3})^n, & n \ge 0 \\ 0, & n < 0 \end{cases} \end{aligned}$$

$$E_\infty = \sum_{n=-\infty}^{\infty} |x_[n]|^2 = \sum_{n=0}^{\infty} |3(\frac{1}{3})^n|^2 = \sum_{n=0}^{\infty} 9(\frac{1}{9})^n$$

Notice that $(\frac{1}{9})^n$ is an infinite geometric series with ratio $|\alpha| < 1$; so, from Problem 1.54c

$$E_\infty = 9\sum_{n=0}^{\infty} (\frac{1}{9})^n = 9\frac{1}{1 - 1/9} = \frac{81}{8}$$

Because $E_\infty < \infty$, $x[n]$ is an **energy** signal with average power $P_\infty = 0$.

(c) Given the periodic signal

$$x(t) = je^{j(\pi t+10)}$$

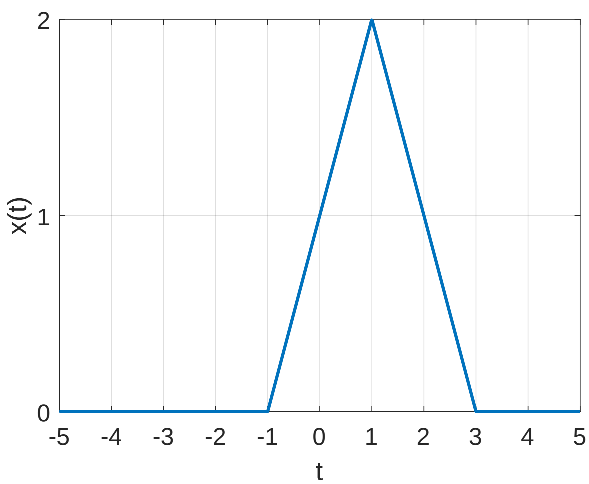
the total energy and average power of the signal can be calculated as

$$\begin{aligned}
E_\infty &= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2(t)\,dt \\
&= \lim_{T\to\infty} \int_{-T}^{T} x(t)x^*(t)\,dt = \lim_{T\to\infty} \int_{-T}^{T} \underbrace{(je^{j(\pi t+10)})(-je^{-j(\pi t+10)})}_{1}\,dt \\
&= \lim_{T\to\infty} \int_{-T}^{T} 1\,dt \Rightarrow \infty \\
P_\infty &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1\,dt \\
&= \lim_{T\to\infty} \frac{2T}{2T} = \lim_{T\to\infty} 1 = 1
\end{aligned}$$

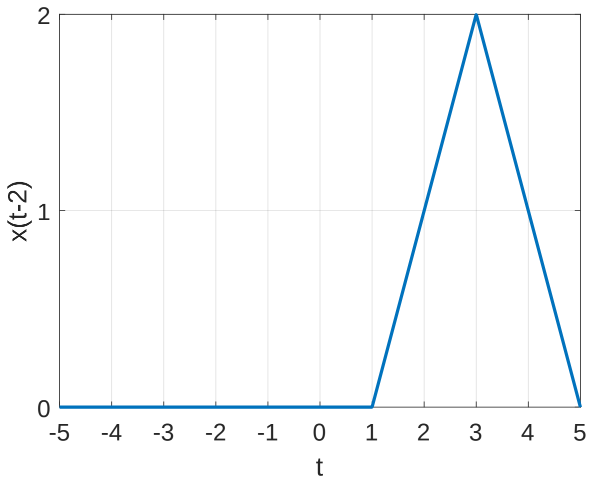
Since $P_\infty < \infty$, $x(t)$ is a **power** signal.

Problem #2

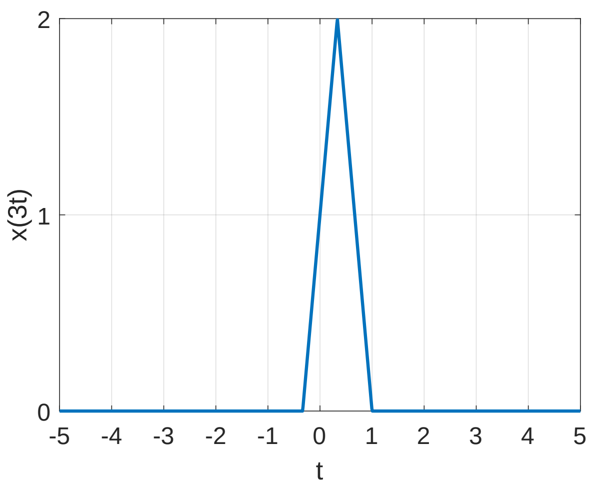
Consider the continuous-time signal $x(t)$



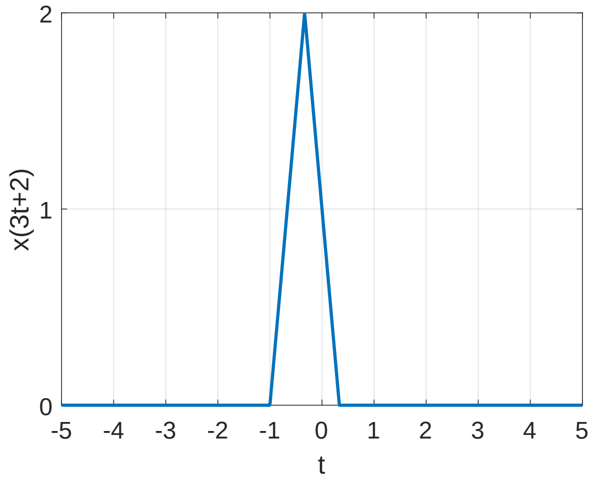
(a) $x(t-2) \Rightarrow$ shift to the right by two units



(b) $x(3t) \Rightarrow$ compression of the t-axis by a factor of three.

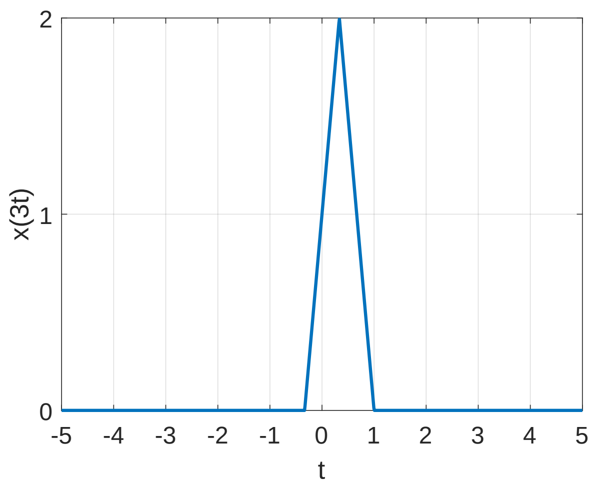


(c) $x(3t+2) \Rightarrow$ compression of the t-axis by a factor of three, and shift to left, but be careful about the shift; $x(3t+2) \Leftrightarrow x(3(t+2/3)))$.

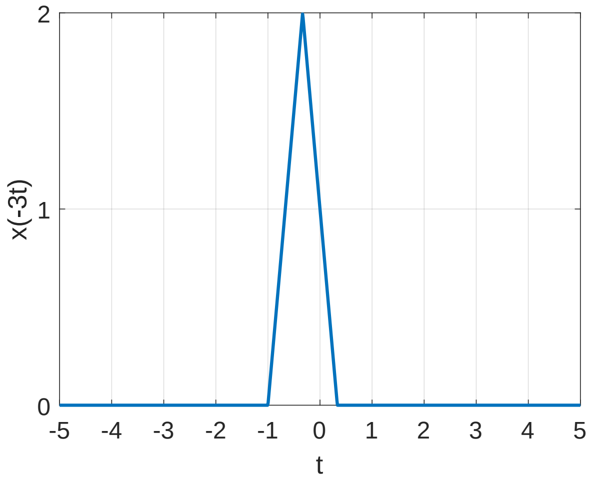


(d) $x(-3t-1) \Rightarrow$ the first part flips and compresses, and the second part shifts but be careful about the shift; $x(-3t-1) \Leftrightarrow x(-3(t+1/3))$. Accordingly, we next show the transformations in three steps.

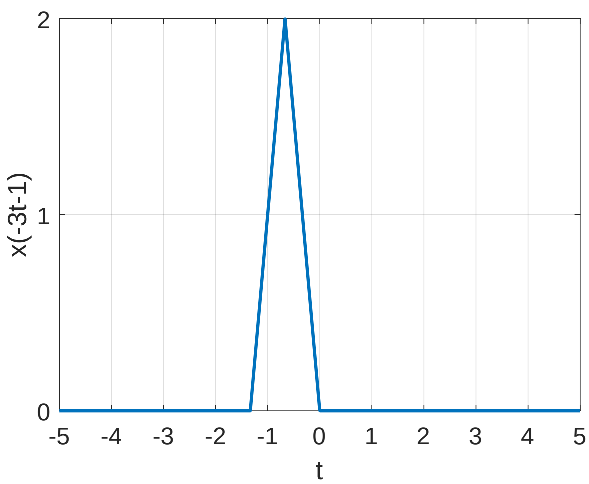
(1) Compress by a factor of 3



(2) Flip the direction of the time axis



(3) Shift $\frac{1}{3}$ units to the left.

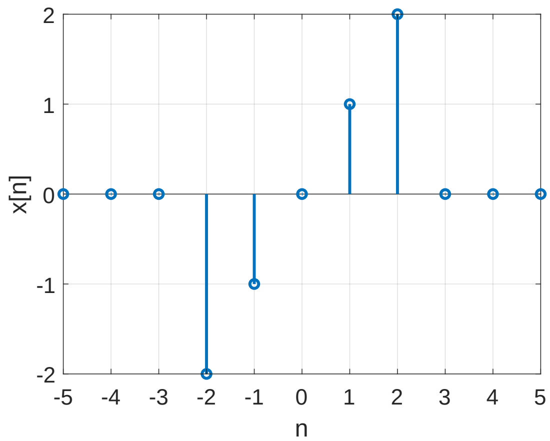


Problem #3

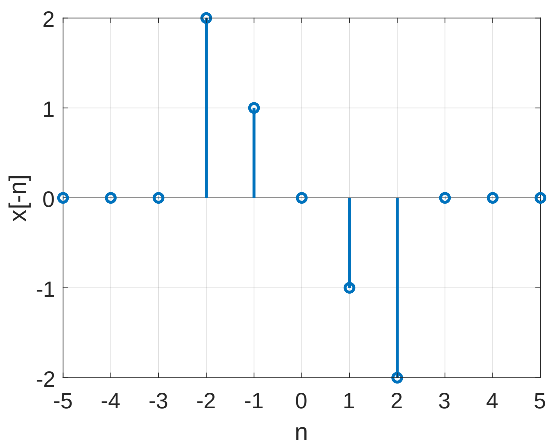
Consider the discrete-time signal given by

$$x[n] = \begin{cases} n, & -2 \leq n \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

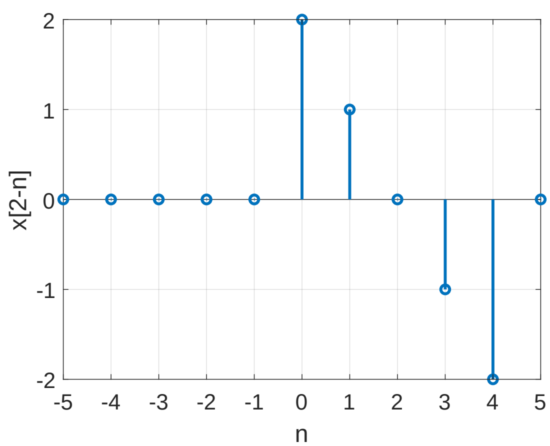
Then



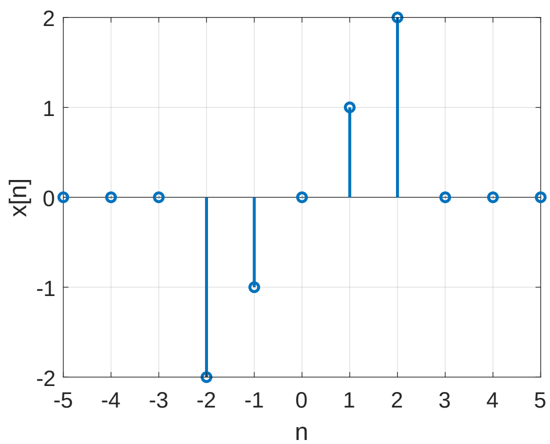
(a) $x[2-n] \Rightarrow$ flip and shift to the right. The transformations can be done as follows
(1) Flip the direction of the time axis

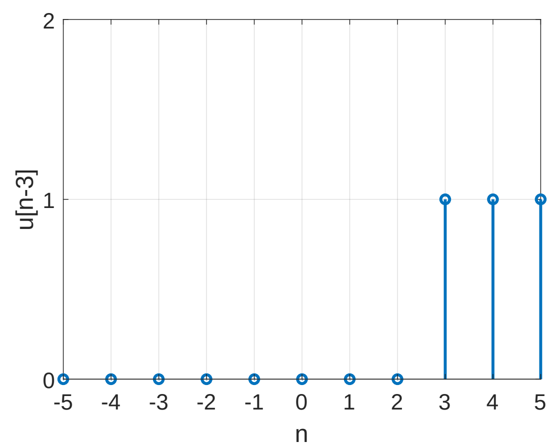


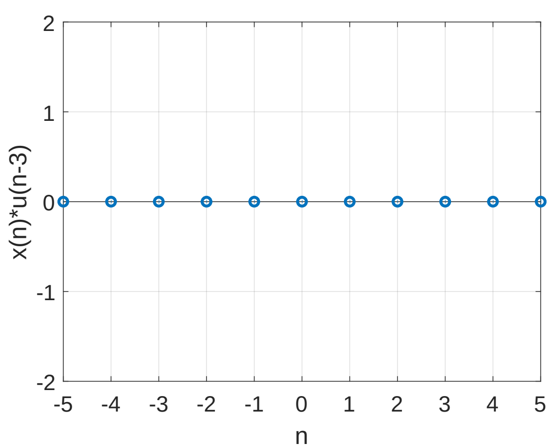
(2) Shift to the right by two units



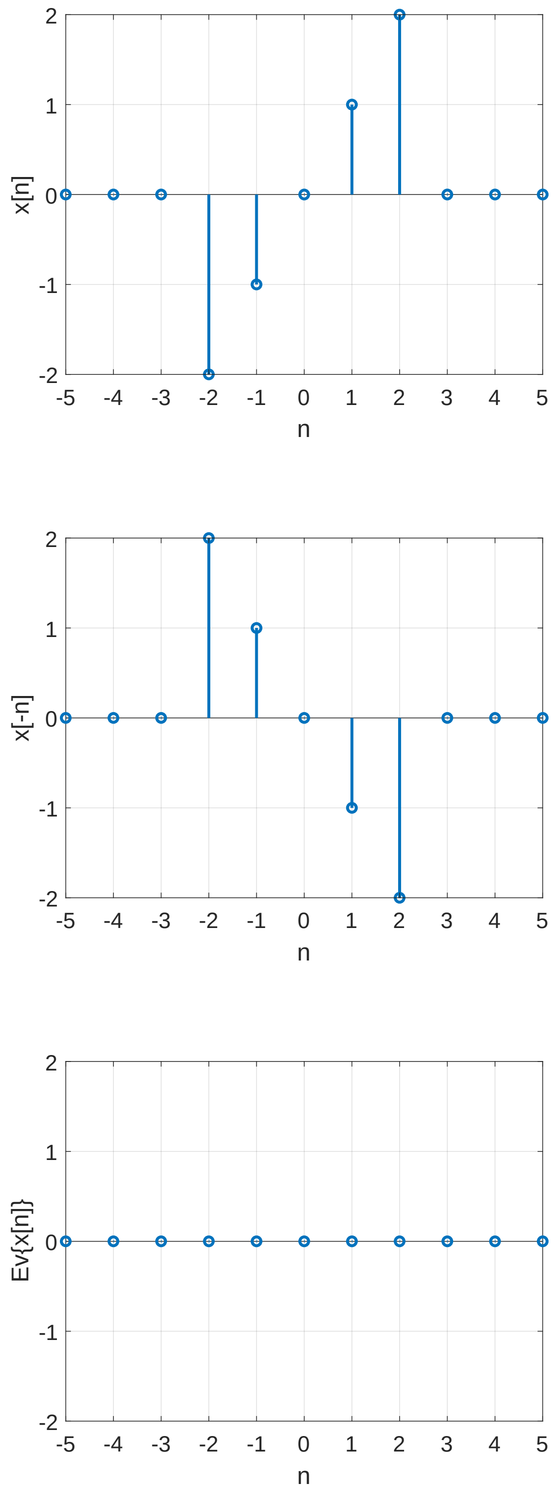
(b) $x[n]u[n-3] \Rightarrow$ component-wise multiplication between $x[n]$ and $u[n-3]$. Note $x[n]u[n-3] = 0$ when either $x[n] = 0, i.e., |n| > 2$, or $u[n] = 0, i.e., n < 3$.







(c) $\mathcal{E}v\{x[n]\} = (x[n] + x[-n])/2$ is the even part of $x[n]$. Since $x[n]$ is a purely odd function, the even part is all zeros. The next Figure illustrates the process of generating the even part of $x[n]$ by adding $x[n]$ and its time-reversed version $x[-n]$.

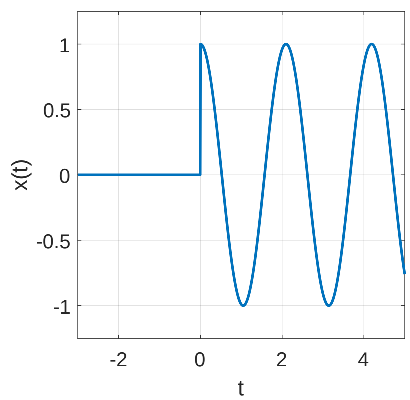


Problem #4

(a) $x(t) = \cos^2(2\pi t) = \frac{1}{2} + \frac{1}{2}\cos(4\pi t)$ using the trigonometric identity $\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$. Recall that a function is periodic if $x(t) = x(t+T)$ for all t for some value of T. This is just a continuous-time cosine and is always **periodic** with fundamental period $T_0 = \frac{2\pi}{w_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$.

(b) $x(t) = e^{-2t}\cos(2\pi t)$ is **not periodic**. Note that the first term is a decaying exponential and $e^{-2t} \neq e^{-2(t+T)}$ for any T.

(c) $x(t) = \cos(3t)u(t)$ is **not periodic**. Again, for it to be periodic $x(t) = x(t+T)$ for all t. In the Figure below, it is clearly seen that $x(t)$ does not repeat for $t < 0$, so it is not periodic. (Remember that $u(t) = 0$ for $t < 0$.)

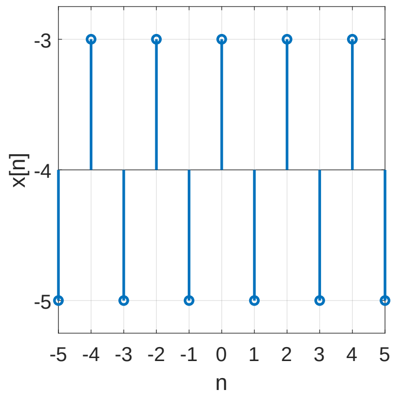


(d) $x(t) = e^{j(\pi t-2)} = \cos(\pi t - 2) + j\sin(\pi t - 2)$ is **periodic** since both cosine and sine are periodic functions. In general, for this example, $x(t) = x(t+T) \Leftrightarrow e^{j(\pi t-2)} = e^{j(\pi(t+T)-2)} \Rightarrow e^{j\pi T} = 1$, so $T = \frac{2\pi}{w_0} = \frac{2\pi}{\pi} = 2$.

(e) For $x[n] = \cos[2\pi n + \frac{\pi}{8}]$ to be periodic, there must exist an integer N such that $x[n] = x[n+N]$ for all n. As we showed in class, this means that $\frac{w_0}{2\pi}$ must be rational (the ratio of integers). In this case, it is **periodic** because $\frac{w_0}{2\pi} \Rightarrow N = \frac{2\pi}{w_0} = 1$ (or if < 1, some integer multiple).

(f) $x[n] = \cos[3n]$ is **not periodic**. Following the logic of the previous problem, $w_0 = 3 \Rightarrow \frac{w_0}{2\pi} = \frac{3}{2\pi}$ is not rational. Therefore, $x[n]$ is not periodic.

(g) $x[n] = (-1)^n$ is **periodic**. In the figure below, we observe that the signal repeats with period $N = 2$.



(h) $x[n] = e^{-(1+j\pi)n} = e^{-n}e^{-j\pi n}$ is **not periodic**. Although the second term is periodic, the first term is a decaying exponential. So, there is no N for which $x[n] = x[n+N]$ for all n.

Problem #5

(a) $|1+2j| = \sqrt{1+2^2} = \sqrt{5}$
$|(1+2j)^*| = |(1-2j)| = \sqrt{1+2^2} = \sqrt{5}$. Notice $|z^*| = |z^*|$
$|2e^{j\pi/4}| = |2||e^{j\pi/4}| = 2\sqrt{\cos^2(\pi/4) + \sin^2(\pi/4)} = 2$

(b) By using the geometric series formula $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$, $|\alpha| < 1$, and taking $\alpha = 0.5e^{j\pi/4} \Rightarrow |\alpha| = 0.5 < 1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}(\frac{1}{2}e^{j\pi/4})^n &= \frac{1}{1-(\frac{1}{2}e^{j\pi/4})} \\
&= \frac{1}{1-(\frac{1}{2}\cos(\pi/4) + \frac{j}{2}\sin(\pi/4))} \\
&= \frac{1}{1-(\frac{1}{2}\frac{\sqrt{2}}{2} + \frac{j}{2}\frac{\sqrt{2}}{2})} \\
&= \frac{1}{1-(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}j}{4})}\frac{1-\frac{\sqrt{2}}{4}+\frac{\sqrt{2}j}{4}}{1-\frac{\sqrt{2}}{4}+\frac{\sqrt{2}j}{4}} \\
&= \frac{1-\frac{\sqrt{2}}{4}+\frac{\sqrt{2}j}{4}}{(1-\frac{\sqrt{2}}{4})^2+\frac{2}{16}} \\
&= \frac{1-\frac{\sqrt{2}}{4}+\frac{\sqrt{2}j}{4}}{\frac{5}{4}-\frac{\sqrt{2}}{2}}
\end{aligned}$$

Similarly, for the second problem, we can express $\sum_{n=0}^{\infty}(\frac{1}{2})^n\cos(\frac{\pi}{4}n)$ as the sum of two geometric series by using Euler's formula on the cosine term; that is, $\cos(\pi n/4) = \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n})$. Consequently,

$$\begin{aligned}
\sum_{n=0}^{\infty}(\frac{1}{2})^n \cos(\frac{\pi}{4}n) &= \sum_{n=0}^{\infty}(\frac{1}{2}e^{j\pi/4})^n + \sum_{n=0}^{\infty}(\frac{1}{2}e^{-j\pi/4})^n \\
&= \frac{1}{1-(\frac{1}{2}e^{j\pi/4})} + \frac{1}{1-(\frac{1}{2}e^{-j\pi/4})} \\
&= \frac{1-(\frac{1}{2}e^{-j\pi/4})+1-(\frac{1}{2}e^{j\pi/4})}{1+\frac{1}{4}-\frac{1}{2}(e^{j\pi/4}+e^{-j\pi/4})} \\
&= \frac{2-\cos(\frac{\pi}{4})}{\frac{5}{4}-\cos(\frac{\pi}{4})}
\end{aligned}$$