EXAM #3
ELEG 305 SIGNALS AND SYSTEMS
SPRING 2019

Name: _______________________________ *(Family Name, Given Name)*
Major:_______________________________

Read each problem carefully before you start.

There are five problems; they are NOT equally weighted. There are also two Extra Credit questions worth a maximum of ten points total.

Closed book.
No calculators, no cellphones.

Sheets with the sum of a geometric series, the defining equations, and the properties of the Discrete-Time Fourier transform and the Laplace transform are attached after the grading rubric page. Feel free to detach those pages and discard them after the exam. (Please press down on the staple if you do detach the pages.) Some other useful equations:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\alpha^n u[n] \leftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & elsewhere \end{cases} \leftrightarrow \frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\frac{\omega}{2})}$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a} \quad, Re\{s\} > -a$$
$$-e^{-at}u(-t) \leftrightarrow \frac{1}{s+a} \quad, Re\{s\} < -a$$

Problem #1 (15 points)

Consider a discrete-time LTI system described by the following difference equation:

$$y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = 5x[n] - \frac{5}{6}x[n-1]$$

a.) (5 pts) Derive the frequency response, $H\left(e^{j\omega}\right)$, for this system.

b.) (10 pts) What is the corresponding impulse response, $h[n]$?

Problem #2 (25 points) (Part c) is on the next page.)

a.) (10 pts) Derive the Fourier transform of the following discrete-time signal:

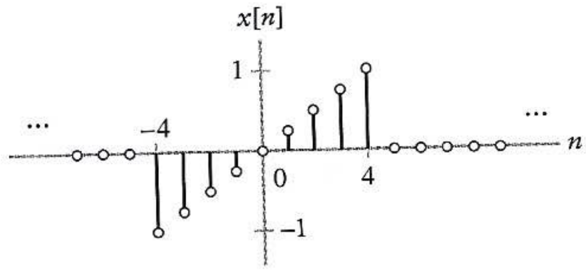
$$x[n] = e^{j\frac{\pi}{4}n}\left(\frac{1}{2}\right)^n u[n-2]$$

b.) (10 pts) Assume the discrete-time signal $x[n] = \left(\frac{3}{4}\right)^{|n|}$ has Fourier transform $X(e^{j\omega})$. Without evaluating $X(e^{j\omega})$, find $y[n]$ if

$$Y(e^{j\omega}) = \frac{d}{d\omega}\left\{e^{-4j\omega}\left[X\left(e^{j\left(\omega+\frac{\pi}{4}\right)}\right) - X\left(e^{j\left(\omega-\frac{\pi}{4}\right)}\right)\right]\right\}$$

Problem #2 (25 points) (cont'd)

c.) (5 pts) Consider the discrete-time signal $x[n]$ given below. Evaluate the following without computing $X(e^{j\omega})$.

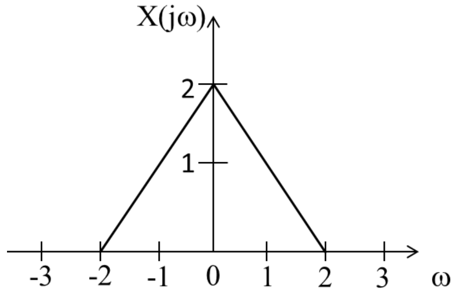


(i) $X(e^{j0})$

(ii) $\int_{-\pi}^{\pi} X(e^{j\omega})e^{4j\omega}\, d\omega$

Problem #3 (15 points)

Consider a continuous-time signal, $x(t)$, with the frequency characteristic shown below.



a.) (5 pts) What is the Nyquist rate (in rad/sec)?

b.) (5 pts) Suppose $x(t)$ is sampled at ***two times*** the Nyquist rate, using impulse-train sampling. Sketch the **spectrum** of the resulting sampled signal.

c.) (5 pts) Suppose $x(t)$ is sampled at ***half*** the Nyquist rate, using impulse-train sampling. Sketch the **spectrum** of the resulting sampled signal.

Problem #4 (25 points)

Consider a continuous-time LTI system, with transfer function

$$H(s) = \frac{5s + 16}{s^2 + 6s + 8}$$

a.) (5 pts) Derive the differential equation relating the input and output for this system.
b.) (5 pts) Determine the locations of the poles and zeros, and plot them in the s-plane.
c.) (6 pts) Draw the three possibilities for the regions of convergence (ROC).
d.) (9 pts) Derive the impulse response, h(t), for a causal system.

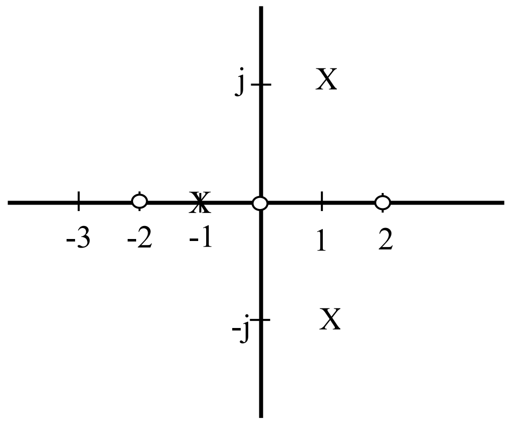
Problem #5 (20 points) (Parts b) and c) are on the next two pages.)

a.) (10 pts) Determine the Laplace transform of

$$x(t) = \left(e^t \frac{d}{dt}[e^{-2t}u(-t)]\right) * \left(\frac{d^2}{dt^2}[-e^{-t}u(-t)]\right)$$

Problem #5 (20 points) (cont'd)

b.) (6 pts) Consider the following pole-zero plot for a rational Laplace transform, $H(s)$, corresponding to an LTI system with impulse response $h(t)$.



There are three zeros $(-2, 0, 1)$ and three poles $(-1, 1 \pm j)$ in the finite s-plane.

(i) Can this system be causal? If it can, provide the ROC (drawing or equation).

(ii) Can this system be stable? If it can, provide the ROC.

(iii)Can this system be both causal and stable? If it can, provide the ROC.

Problem #5 (20 points) (cont'd)

c.) (4 pts) Consider the following transfer function for a causal LTI system

$$H(s) = \frac{2s + 3}{s(s + 5)}$$

Compute (i) $h(t)$ at $t = 0^+$ and (ii) $h(t)$ as $t \rightarrow \infty$.

Extra Credit #1 (5 points) (*from Sample Exam #3B*)

Consider an impulse response $h(t) = e^t u(t)$; this system is clearly not stable. How can you make this system stable by using feedback?

Extra Credit #2 (5 points) (*from HW #8 Conceptual Question*)

Describe, in words, not equations, the possible reactions that a second-order system might have to an impulse input.

Problem #1 (out of 15)	
Problem #2 (out of 25)	
Problem #3 (out of 15)	
Problem #4 (out of 25)	
Problem #5 (out of 20)	
Extra Credit #1 (out of 5)	
Extra Credit #2 (out of 5)	
TOTAL (out of 100)	

Geometric series

$$\sum_{n=0}^{M-1} \beta^n = \begin{cases} \dfrac{1-\beta^M}{1-\beta}, & \beta \neq 1 \\ M, & \beta = 1 \end{cases}$$

Discrete-Time Fourier Transform

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \end{aligned}}$$

Laplace Transform

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds.}$$

$$\boxed{X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt,}$$

Properties of Discrete-Time Fourier Transform

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

Properties of Laplace Transform

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$