Name: ________________________________

1. (25 points) Let $\boldsymbol{X} \sim N(3,4)$. What are the following?
   (a) $\Pr[-2 \le \boldsymbol{X} \le 2]$
   (b) $\Pr[\boldsymbol{X} \ge 3]$
   (c) $\Pr[0 \le \boldsymbol{X} \le 5]$
   (d) $\Pr[|\boldsymbol{X}| \ge 2]$
   (e) $\Pr[\boldsymbol{X}^2 \le 1]$

a) $P[-2 \le X \le 2] = P\left[\frac{-2-3}{2} \le \frac{X-3}{2} \le \frac{2-3}{2}\right] = \Phi(-\frac{1}{2}) - \Phi(-\frac{5}{2})$

$= \Phi(\frac{5}{2}) - \Phi(\frac{1}{2}) = 0.9938 - 0.6915 = \boxed{0.3023}$

b) $P[X \ge 3] = P\left[\frac{X-3}{2} \ge \frac{3-3}{2}\right] = 1 - \Phi(0) = \boxed{0.5}$

c) $P[0 \le X \le 5] = P\left[\frac{0-3}{2} \le \frac{X-3}{2} \le \frac{5-3}{2}\right] = \Phi(1) - \Phi(-\frac{3}{2})$

$= \Phi(1) + \Phi(\frac{3}{2}) - 1 = 0.8413 + \cancel{0.6915}\ 0.9332 - 1 = \boxed{\cancel{0.5328}}\ \boxed{0.7745}$

d) $P[|X| \ge 2] = P[X \ge 2] + P[X \le -2] = 1 - P[-2 \le X \le 2]$

$= 1 - 0.3023 = \boxed{0.6977}$

e) $P[X^2 \le 1] = P[-1 \le X \le 1] = P\left[\frac{-1-3}{2} \le \frac{X-3}{2} \le \frac{1-3}{2}\right]$

$= \Phi(-1) - \Phi(-2) = \Phi(2) - \Phi(1) = 0.9772 - 0.8413$

$= \boxed{0.1359}$

2. (25 points) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be IID Gaussian with mean μ and variance σ^2. What are the following?
   (a) $\Pr\left[-1 \le \boldsymbol{X} \le 1 \cap 0 \le \boldsymbol{Y} \le 2\right]$
   (b) $\Pr\left[\boldsymbol{X} > \boldsymbol{Y}\right]$
   (c) $\Pr\left[\boldsymbol{X} + \boldsymbol{Y} \le 2\right]$
   (d) $\mathrm{E}\left[2\boldsymbol{X} + 3\boldsymbol{Y}\right]$
   (e) $\mathrm{Var}\left[2\boldsymbol{X} + 3\boldsymbol{Y}\right]$

a) $P[-1 \le X \le 1 \cap 0 \le Y \le 2] = P\left[\frac{-1-\mu}{\sigma} \le Z_1 \le \frac{1-\mu}{\sigma}\right] P\left[\frac{0-\mu}{\sigma} \le Z_2 \le \frac{2-\mu}{\sigma}\right]$

$= \boxed{\left(\Phi\left(\frac{1-\mu}{\sigma}\right) - \Phi\left(\frac{-1-\mu}{\sigma}\right)\right)\left(\Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{-\mu}{\sigma}\right)\right)}$

b) $P[X > Y] = P[X - Y \ge 0]$ $\quad Z = X - Y \sim N(0, 2\sigma^2)$

$= \boxed{\frac{1}{2}}$

c) $P[X + Y \le 2] = P[Z \le 2]$ $\quad Z \sim N(2\mu, 2\sigma^2)$

$= P\left(\frac{Z - 2\mu}{\sqrt{2}\sigma} \le \frac{2 - 2\mu}{\sqrt{2}\sigma}\right) = \boxed{\Phi\left(\frac{2-2\mu}{\sqrt{2}\sigma}\right)}$

d) $E[2X + 3Y] = 2E[X] + 3E[Y] = 2\mu + 3\mu = \boxed{5\mu}$

e) $Var[2X + 3Y] = 2^2 Var[X] + 3^2 Var[Y]$

$= (4 + 9)\sigma^2 = \boxed{13\sigma^2}$

3. (25 points) Let $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\ldots$, $\boldsymbol{X}_n$ be a sequence of IID Bernoulli random variables with parameter p. Let $\boldsymbol{S} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_n$.

   (a) What is $\Pr[\boldsymbol{S} = k]$?

   (b) What is the probability at least one $\boldsymbol{X}_i$ equals 1?

   (c) What is $\mathrm{E}[\boldsymbol{S}]$?

   (d) What is $\mathrm{Var}[\boldsymbol{S}]$?

   (e) What is a good Gaussian approximation to $\Pr[k_0 \le \boldsymbol{S} \le k_1]$?

$q = 1-p$

a) $P[S=k] = \binom{n}{k} p^k q^{n-k}$

b) $P[S \geq 1] = 1 - P[S=0] = 1-(1-p)^n$

c) $E[S] = np$

d) $Var[S] = npq$

e) $P[k_0 \leq S \leq k_1] = P\left[\frac{k_0 - np}{\sqrt{npq}} \leq \frac{S-np}{\sqrt{npq}} \leq \frac{k_1 - np}{\sqrt{npq}}\right]$

$\approx \Phi\left(\frac{k_1 + \frac{1}{2} - np}{\sqrt{npq}}\right) - \Phi\left(\frac{k_0 - \frac{1}{2} - np}{\sqrt{npq}}\right)$

4. (25 points) A test for a disease counts antibodies in a blood sample. Let $\boldsymbol{N}$ be the count. Under the null hypothesis that there is no disease, $\boldsymbol{N}$ is Poisson with parameter λ_0; under the alternative that there is a disease, $\boldsymbol{N}$ is Poisson with parameter $\lambda_1 > \lambda_0$.
   (a) Describe the hypothesis test for the two hypotheses. What is the decision rule?
   (b) What is the false alarm probability?
   (c) What threshold value results in a false alarm probability equal to α?

a) $H_0: P(N=k) = \dfrac{\lambda_0^k e^{-\lambda_0}}{k!}$

$H_1: P(N=k) = \dfrac{\lambda_1^k e^{-\lambda_1}}{k!}$

$$L(k) = \frac{\dfrac{\lambda_1^k e^{-\lambda_1}}{k!}}{\dfrac{\lambda_0^k e^{-\lambda_0}}{k!}} = \left(\frac{\lambda_1}{\lambda_0}\right)^k e^{-(\lambda_1-\lambda_0)}$$

Choose H_1 if $L(k) \geq L_0$; else choose H_1

Simplifies to Choose H_1 if $k \geq k_0$; else choose H_0

b) P(false alarm) = P(choose H_1 | H_0 true)

$$= \sum_{k=k_0}^{\infty} \frac{\lambda_0^k e^{-\lambda_0}}{k!}$$

c) Solve this equation for k_0 (as close as possible)

$$P(FA) = \alpha = \sum_{k=k_0}^{\infty} \frac{\lambda_0^k e^{-\lambda_0}}{k!}$$

5. (25 points) Let a sequence of observations be 1, 2, -3, 2, 5, 2.

   (a) What is the sample mean?

   (b) What is the (unbiased) sample variance?

a) $\overline{X}_n = \dfrac{1+2+(-3)+2+5+2}{6} = \dfrac{9}{6} = \boxed{1.5}$

$$\begin{array}{r} 4.5 \\ \underline{4.5} \\ 225 \\ \underline{180} \\ 20.25 \end{array}$$

b) $\hat{\sigma}^2 = \dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 =$

$= \dfrac{1}{5}\left((1-1.5)^2 + (2-1.5)^2 + (-3-1.5)^2 + (2-1.5)^2 + (5-1.5)^2 + (2-1.5)^2 \right)$

$= \dfrac{0.25 + 0.25 + 20.25 + 0.25 + 12.25 + 0.25}{5}$

$= \dfrac{33.5}{5} = \boxed{6.7}$

$$\begin{array}{r} 3.5 \\ \underline{3.5} \\ 175 \\ \underline{105} \\ 12.25 \end{array}$$

6. (25 points) Let $\boldsymbol{X}(t) = \cos(\omega_C t + \theta)$ with ω_0 a known constant and $\theta \sim U(0, 2\pi)$.
   (a) What is $\mathrm{E}\big[\boldsymbol{X}(t)\big]$?
   (b) What is $\mathrm{Var}\big[\boldsymbol{X}(t)\big]$?
   (c) What is the autocorrelation of $\boldsymbol{X}(t)$?
   (d) What is the power spectral density of $\boldsymbol{X}(t)$?

a) $E[X(t)] = E[\cos(\omega_c t + \theta)] = 0$

b) $Var[X(t)] = E[(X(t) - 0)^2] = E[\cos^2(\omega_c t + \theta)]$

$= \frac{1}{2}$ (over full period, average of $\cos^2\theta = \frac{1}{2}$)

c) $R_{xx}(t, t+\tau) = E[X(t)X(t+\tau)]$

$= E[\cos(\omega_c t + \theta)\cos(\omega_c(t+\tau) + \theta)]$

$= \frac{1}{2}\cos(\omega_c \tau)$

d) $S(\omega) = \mathcal{F}(R(\tau)) = \mathcal{F}\left(\frac{1}{2}\cos(\omega_c \tau)\right)$

$= \frac{2\pi}{2}\left(\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right)$

7. (10 points) (Extra credit) Let $\boldsymbol{X}$ be geometric with parameter p. What is the entropy of $\boldsymbol{X}$?

$$H(X) = -\sum_{k=1}^{\infty} p(k) \log p(k) \qquad p(k) = (1-p)^{k-1} p$$

$$= -\sum_{k=1}^{\infty} p(1-p)^{k-1} \log\left((1-p)^{k-1} p\right) = -p \log p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$-\sum_{k=1}^{\infty} (k-1)(1-p)^{k-1} p \log(1-p)$$

$$\boxed{\log\left((1-p)^{k-1} p\right) = (k-1)\log(1-p) + \log p}$$

$$\text{now } \sum_{k=1}^{\infty} (1-p)^{k-1} = \sum_{l=0}^{\infty} (1-p)^{l} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$\sum_{k=1}^{\infty} (k-1)(1-p)^{k-1} p = \underbrace{\sum_{k=1}^{\infty} k(1-p)^{k-1} p}_{\text{expected value of geometric} = \frac{1}{p}} - \underbrace{\sum_{k=1}^{\infty} (1-p)^{k-1} p}_{\frac{p}{p} = 1 = \text{sum of geometric probs}}$$

$$\Rightarrow H(X) = -\frac{p \log p}{p} - \frac{1}{p}\log(1-p) + \log(1-p) \qquad \frac{1}{p} - 1 = \frac{1-p}{p}$$

$$= \frac{-p\log p - (1-p)\log(1-p)}{p} = \boxed{\frac{h(p)}{p}}$$

Comment: n independent bits have $nh(p)$ bits of information. On average there are np 1's in the n bits. Therefore each geometric (sequence of 0's followed by a 1) has $\frac{nh(p)}{np} = \frac{h(p)}{p}$ bits of information.

Table 1: Values of the Standard Normal Distribution Function

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	1.00	0.8413	2.00	0.9772	3.00	0.9987
0.05	0.5199	1.05	0.8531	2.05	0.9798	3.05	0.9989
0.10	0.5398	1.10	0.8643	2.10	0.9821	3.10	0.9990
0.15	0.5596	1.15	0.8749	2.15	0.9842	3.15	0.9992
0.20	0.5793	1.20	0.8849	2.20	0.9861	3.20	0.9993
0.25	0.5987	1.25	0.8944	2.25	0.9878	3.25	0.9994
0.30	0.6179	1.30	0.9032	2.30	0.9893	3.30	0.9995
0.35	0.6368	1.35	0.9115	2.35	0.9906	3.35	0.9996
0.40	0.6554	1.40	0.9192	2.40	0.9918	3.40	0.9997
0.45	0.6736	1.45	0.9265	2.45	0.9929	3.45	0.9997
0.50	0.6915	1.50	0.9332	2.50	0.9938	3.50	0.9998
0.55	0.7088	1.55	0.9394	2.55	0.9946	3.55	0.9998
0.60	0.7257	1.60	0.9452	2.60	0.9953	3.60	0.9998
0.65	0.7422	1.65	0.9505	2.65	0.9960	3.65	0.9999
0.70	0.7580	1.70	0.9554	2.70	0.9965	3.70	0.9999
0.75	0.7734	1.75	0.9599	2.75	0.9970	3.75	0.9999
0.80	0.7881	1.80	0.9641	2.80	0.9974	3.80	0.9999
0.85	0.8023	1.85	0.9678	2.85	0.9978	3.85	0.9999
0.90	0.8159	1.90	0.9713	2.90	0.9981	3.90	1.0000
0.95	0.8289	1.95	0.9744	2.95	0.9984	3.95	1.0000