ELEG 305

SOLUTIONS TO EXAM #1 (3/8/18)

#1. i.) Almost all signals can be represented as linear combinations of complex exponentials.

ii.) Complex exponentials are eigenfunctions of linear, time-invariant systems.

#2. a.)

$$x[n] = e^{-(1+j\pi)n}$$
$$= \underbrace{e^{-n}}_{\text{decaying exponential}} \underbrace{e^{-j\pi n}}_{\text{periodic}}$$

$\therefore$ $x[n+N] \neq x[n]$ for all n

NOT PERIODIC

b.)

$$\int_0^\infty e^{-2t}\,\delta(t-2)\,dt = ?$$

In general, $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

So,

$$e^{-2t}\,\delta(t-2) = e^{-4}\,\delta(t-2)$$

$$\therefore \int_0^\infty e^{-2t}\,\delta(t-2)\,dt = e^{-4}\underbrace{\int_0^\infty \delta(t-2)\,dt}_{1}$$
$$= e^{-4}$$

c.)

$$\sum_{k=0}^{\infty}(1+k)^3\,\underbrace{\delta[k-2]}_{\substack{1 \text{ for } k=2 \\ 0 \text{ for } k\neq 2}}$$

$$= (1+k)^3\Big|_{k=2} = 3^3 = 27$$

#2 cont'd) d.)

$$y(t) = \int_{t-1}^{t} x(\tau)\,d\tau$$

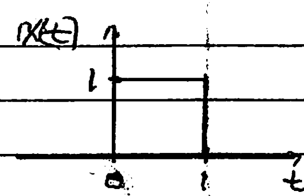
i.) impulse response $\quad h(t) = y(t)\Big|_{x(t)=\delta(t)}$

$$h(t) = \int_{t-1}^{t} \delta(\tau)\,d\tau$$

$\delta(\tau) = \begin{cases} 1 \text{ when } \tau = 0 \\ 0 \text{ when } \tau \neq 0 \end{cases}$

- As long as integration interval $(t-1, t)$ includes 0, the integral will equal 1, otherwise it is 0.
- When $t<0$, the limits are both negative, and $h(t) = 0$.
- When $t>1$, the limits are both greater than 0, and $h(t)=0$.
- When $0<t<1$, the limits include 0, and $h(t) = 1$

$$\therefore \; h(t) = \begin{cases} 1, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$$



ii) causal? YES
The output $y(t)$ depends only on present and past values of $x(t)$ (integration limits mean we only "add up" values of $x(\tau)$ from $t-1$ up to t).

Using the impulse response: To be causal, $h(t) = 0$ for $t<0$, which is true.

2d. cont'd) iii) stable? YES

Assume $x(t)$ is bounded, for example, $|x(t)| \leq B < \infty$. Then,

$$|y(t)| = \left| \int_{t-1}^{t} x(\tau)\,d\tau \right| \leq B \left| \int_{t-1}^{t} d\tau \right| = B < \infty$$

Therefore, a bounded input results in a bounded output, and the system is stable.

Using the impulse response: To be stable,

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

In our case,

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{1} 1\,dt = 1 < \infty$$

#3. $x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$

$x(t) = u(t) - u(t-1) \longrightarrow y(t) = 2x(t-1) + x(t-2)$

$g(t) = x(t-1) + x(t-3) \longrightarrow r(t) = ?$

- <u>Analytical Solution</u>: The system is linear and time-invariant. So, the sum of the inputs is the sum of the outputs, and the response to a shifted input is just the shifted output.

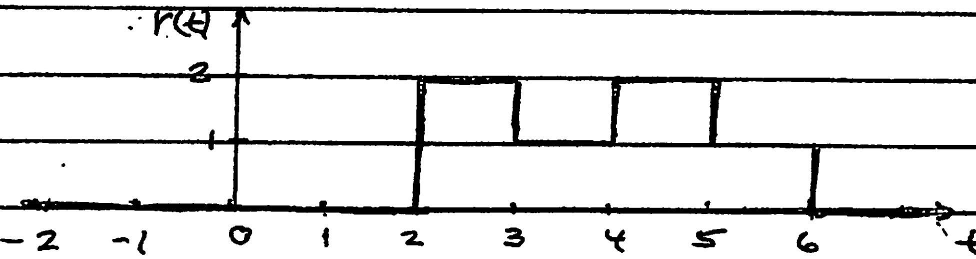
$\therefore$ $r(t)$ = response of the system to $g(t)$

$$= 2g(t-1) + g(t-2)$$

$$= 2x(t-2) + 2x(t-4) + x(t-3) + x(t-5)$$

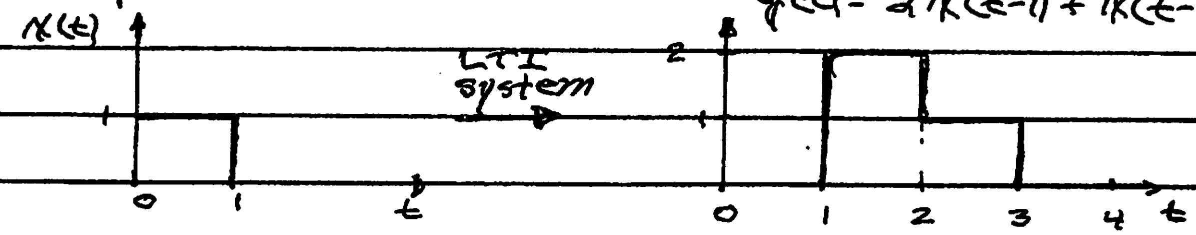
3. cont'd)

$$= 2(u(t-2) - u(t-3)) + 2(u(t-4) - u(t-5))$$
$$+ \; u(t-3) - u(t-4) + u(t-5) - u(t-6)$$
$$= 2u(t-2) - u(t-3) + u(t-4) - u(t-5) - u(t-6)$$

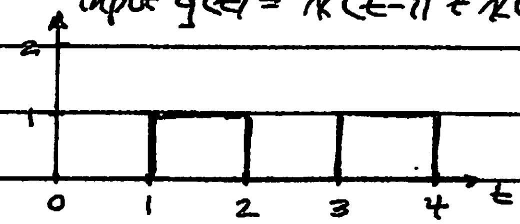


- Graphical Solution

$y(t) = 2x(t-1) + x(t-2)$

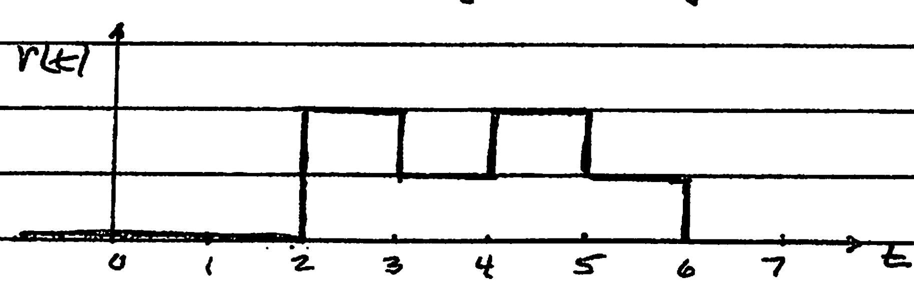


input $g(t) = x(t-1) + x(t-3)$



$\Downarrow$ LTI system

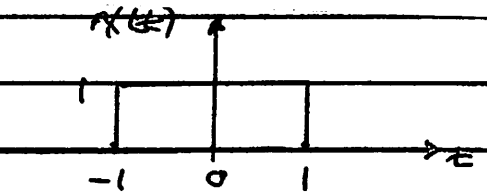
output $r(t) = y(t-1) + y(t-3)$

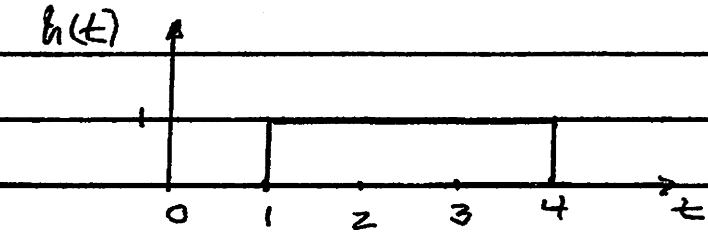


#4. $x(t) = u(t+1) - u(t-1)$

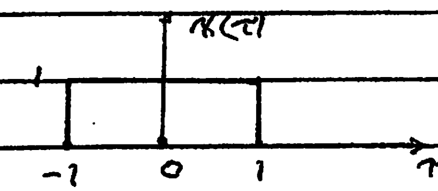
$h(t) = u(4-t) - u(1-t)$

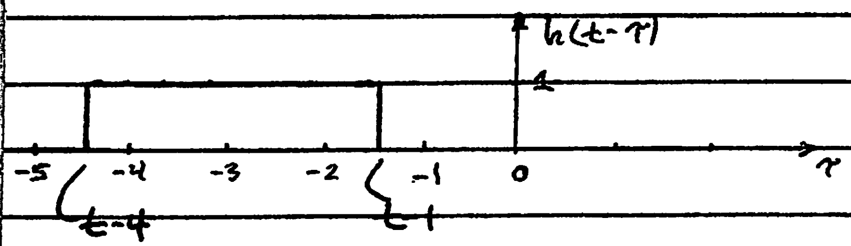
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$





$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

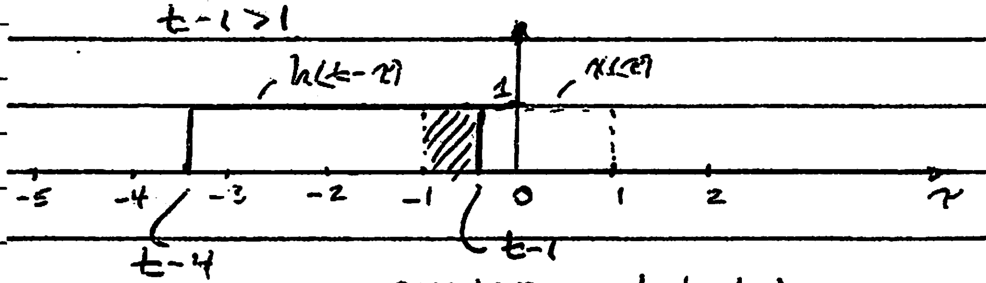




• $t-1 < -1 \Rightarrow t < 0$, $y(t) = 0$ no overlap

#4 cont'd)

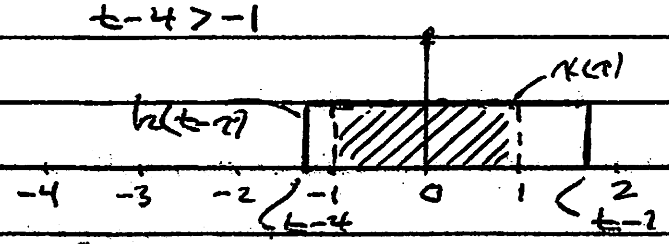
• $0 < \underbrace{t}_{t-1>1} < 2$



overlap -1 to $t-1$

$$y(t) = \int_{-1}^{t-1} 1 \, dt = t-1-(-1) = t.$$

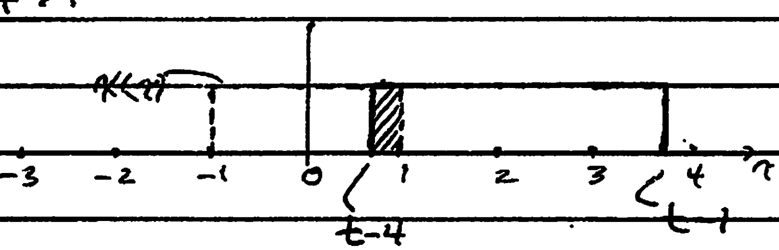
• $2 < \underbrace{t}_{t-4>-1} < 3$



overlap -1 to 1

$$y(t) = \int_{-1}^{1} 1 \, dt = 2$$

• $3 < \underbrace{t}_{t-4>1} < 5$



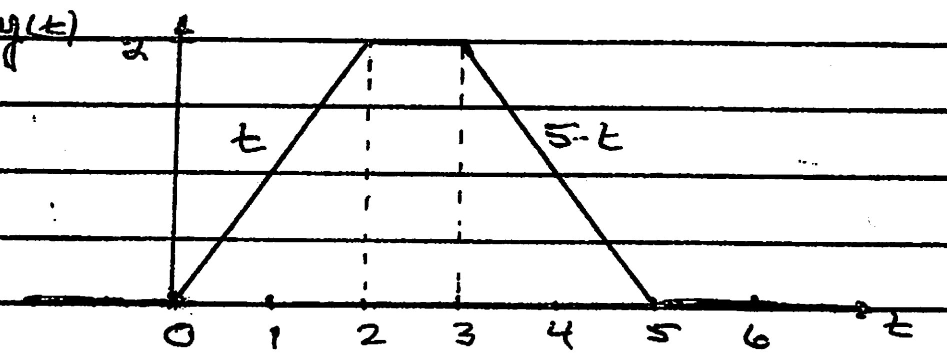
overlap $t-4$ to 1

$$y(t) = \int_{t-4}^{1} 1 \, dt = 1-(t-4) = 5-t$$

#4 cont'd)

- $t > 5$, $y(t) = 0$ no overlap

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 2 \\ 2, & 2 < t < 3 \\ 5-t, & 3 < t < 5 \\ 0, & t > 5 \end{cases}$$



#5. input $x[n]$ → filter LTI → output $y[n]$

$h[n]$ = impulse response = $\beta^n u[n]$, $\beta > 0$

$x[n] = \alpha^n u[n-1]$, $\alpha > 0$, $\alpha \neq \beta$

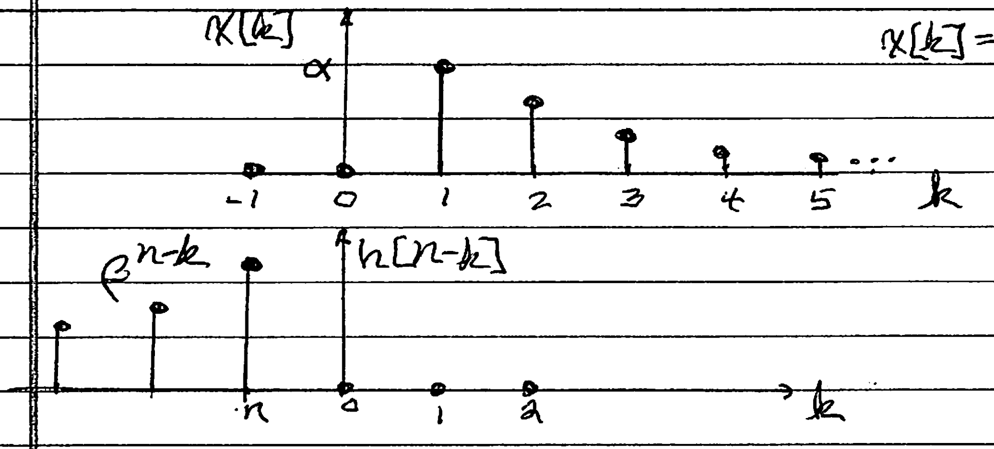
For a linear, time-invariant system, the output $y[n]$ can be computed using convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

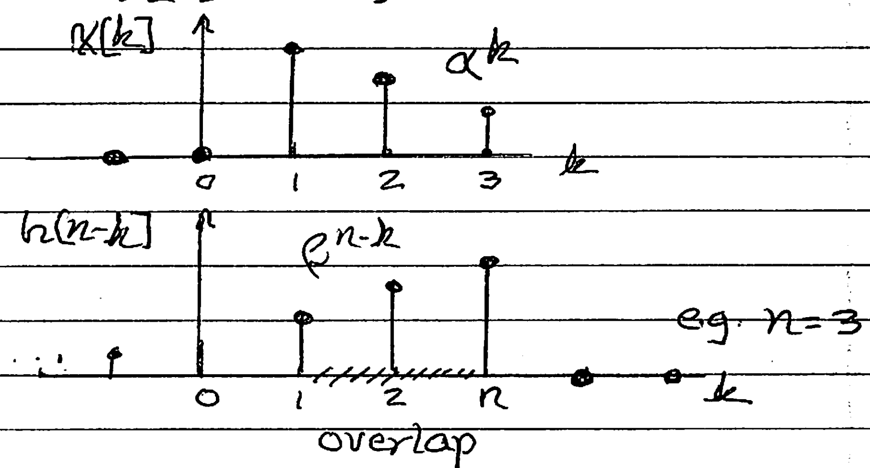
(NOTE: This is very similar to Prob. #2.21a in HW #2.)

#5 cont'd)

$x[k] = \alpha^k u[k-1]$



- $n < 1$ there is no overlap and the product is 0, so $y[n] = 0$
- $n \geq 1$ there is a non-zero product $x[k]\,h[n-k]$ from $k=1$ to $k=n$



$$y[n] = \sum_{k=1}^{n} \alpha^k \beta^{n-k} = \beta^n \underbrace{\sum_{k=1}^{n} \left(\frac{\alpha}{\beta}\right)^k}_{\text{finite geometric series in } (\alpha/\beta)}$$

Let $l = k-1$

$$y[n] = \beta^n \sum_{l=0}^{n-1} \left(\frac{\alpha}{\beta}\right)^{l+1}$$

$$= \beta^n \left(\frac{\alpha}{\beta}\right) \underbrace{\sum_{l=0}^{n-1} \left(\frac{\alpha}{\beta}\right)^{l}}_{n \text{ terms}} = \frac{1-(\alpha/\beta)^n}{1-\alpha/\beta}$$

#5 cont'd)

$$y[n] = \beta^n\left(\frac{\alpha}{\beta}\right)\left(\frac{1-(\alpha/\beta)^n}{1-\alpha/\beta}\right)$$

This answer is acceptable. But as for the problem in HW#2 it can be simplified using algebraic manipulations to

$$y[n] = \cancel{\beta^{n-1}}\alpha\left(\frac{\beta^n-\alpha^n}{\cancel{\beta^{n-1}}(\beta-\alpha)}\right)$$

$$= \alpha\,\frac{\beta^n-\alpha^n}{\beta-\alpha}\;\underbrace{u[n-1]}$$

(because it is 0 for $n<1$

Extra Credit a.) causal?

If the system is causal, there can be no output before the input is applied. However, the input $x_2(t)$ doesn't start until $t=1$ but it gives an output $y_2(t)$ at time $t=0$.

Therefore, this system is <u>not</u> <u>causal</u>

b.) time invariant?

If the system is time-invariant, a shifted input will give the same output, shifted by the same amount. Notice that the input $x_3(t)$ is simply $x_1(t-1)$. However, $y_3(t) \neq y_1(t-1)$.

Therefore, this system is <u>not</u> <u>time-invariant</u>