HW4 Solution

Problem 4.1

X has the probabilities listed in the table below. What are $E[X]$ and $\mathrm{Var}[X]$?

k	1	2	3	4
$\Pr[X = k]$	0.5	0.2	0.1	0.2

Solution to Problem 4.1

$$E[X] = 1 \times 0.5 + 2 \times 0.2 + 3 \times 0.1 + 4 \times 0.2 = 2.0$$
$$E[X^2] = 1^2 \times 0.5 + 2^2 \times 0.2 + 3^2 \times 0.1 + 4^2 \times 0.2 = 5.4$$
$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = 1.4$$

4.3 $F(1) = P(X \leq 1) = P(X=1) = 0.5$

$F(2) = P(X \leq 2) = P(X \leq 1) + P(X=2) = 0.5 + 0.2 = 0.7$

$F(3) = P(X \leq 3) = P(X \leq 2) + P(X=3) = 0.7 + 0.1 = 0.8$

$F(4) = P(X \leq 4) = P(X \leq 3) + P(X=4) = 0.8 + 0.2 = 1$

Problem 4.10

Let N be Geometric with parameter p. What is $\Pr[N \geq k]$ for arbitrary integer $k > 0$. Give a simple interpretation of your answer.

Solution to Problem 4.10

$$\Pr[N \geq k] = \sum_{l=k}^{\infty} p(1-p)^{l-1} = \sum_{m=1}^{\infty} p(1-p)^{m+k-2} \tag{4.28}$$
$$= (1-p)^{k-1} \sum_{m=1}^{\infty} p(1-p)^{m-1} = (1-p)^{k-1} \tag{4.29}$$

The event $\{N \geq k\}$ is the same as the event the first $k-1$ flips are tails. Therefore, $\Pr[N \geq k] = (1-p)^{k-1}$.

4.11 $P(N=l \mid N \geq k) = \dfrac{P(N=l \cap N \geq k)}{P(N \geq k)} = \dfrac{P(N=l)}{P(N \geq k)}$ $\underline{l \geq k}$

$= \dfrac{P(1-P)^{l-1}}{(1-P)^{k-1}} = P(1-P)^{l-k}$

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$$= \frac{P(1-P)^{l-1}}{(1-P)^{k-1}} = P(1-P)^{l-k}$$

4.13 $P(N \leq 2) = 1 - P(N > 2) = 1 - (1-P)^2 = 1 - \frac{4}{9} = \frac{5}{9}$

$P(N=2) = P(1-P) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

$P(N \geq 2) = (1-P) = \frac{2}{3}$

4.20 $E(y^2) = E[(ax+b)^2] = E(a^2x^2 + 2abx + b^2) = a^2E(x^2) + 2abE(x) + b^2$

$E(y) = E(ax+b) = aE(x) + b$

$\sigma_y = Var(y) \qquad u_y = E(y) \qquad u_x = E(x) \qquad \sigma_x = Var(x)$

$\sigma_y^2 = Var^2(y) = E(y^2) - E^2(y) = a^2E(x^2) + 2abE(x) + b^2 - (aE(x) + b)^2$

$= a^2E(x^2) + 2abE(x) + b^2 - a^2E^2(x) - 2abE(x) - b^2$

$= a^2E(x^2) - a^2E^2(x) = a^2(E(x^2) - E^2(x))$

$= a^2\sigma_x^2$

$\therefore \sigma_y^2 = a^2\sigma_x^2$

4.21

a. $Var(X) = \sum_k (x_x - u_x)^2 P(k)$

$\because P(k) \geq 0 \quad (x_x - u_x)^2 \geq 0$

$\therefore Var(x) \geq 0$

b. $E(X^{2k}) = \sum_l x_l^{2k} P(l) = \sum_l (x_l^k)^2 P(l)$

$\because P(l) \geq 0 \quad (x_l^k)^2 \geq 0$

$\therefore E(X^{2k}) \geq 0$

c.

x	2	2	-3
$P(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$E(X) = \frac{2+2-3}{3} = \frac{1}{3}$

$E(X^3) = \frac{8+8-27}{3} = -\frac{11}{3}$

Problem 4.22

What value of a minimizes $E\left[(\mathbf{X}-a)^2\right]$? Show this two ways.

a) Write $E\left[(\mathbf{X}-a)^2\right]$ in terms of σ^2, μ, and a (no expected values at this point) and find the value of a that minimizes the expression.

b) Use calculus and (4.14) to find the minimizing value of a.

Solution to Problem 4.22

Let $Q(a) = E\left[(\mathbf{X}-a)^2\right]$.

$$\begin{aligned} Q(a) &= E\left[\left((\mathbf{X}-\mu)+(\mu-a)\right)^2\right] \\ &= E\left[(\mathbf{X}-\mu)^2\right] + 2(\mu-a)E\left[\mathbf{X}-\mu\right] + (\mu-a)^2 \\ &= \sigma^2 + (\mu-a)^2 \end{aligned}$$

The first term does not depend on a. The second term is minimized when $a=\mu$.

Here's the calculus solution:

$$\frac{d}{da}Q(a) = E\left[\frac{d}{da}(\mathbf{X}-a)^2\right] = -2E\left[\mathbf{X}-a\right] = -2(\mu-a) = 0$$

Therefore, $a=\mu$.

Problem 4.28

An American roulette wheel has 18 red numbers, 18 black numbers, and 2 green numbers. Assume all numbers are equally likely.

a) What is the probability the number is red? black? green?

b) The player can make simple bets on the color. A dollar bet on red or black returns a dollar profit if red or black comes up. What is the expected value of this bet? What is the variance of this bet?

c) A bet on a single number pays 35 to 1 (if the number comes up the bettor profits $35 for each dollar bet). What is the expected value of this bet? What is the variance of this bet?

Solution to Problem 4.28

The American roulette wheel has $18+18+2=38$ slots.

a) $\Pr[\text{red}] = \Pr[\text{black}] = 18/38 = 0.474$, $\Pr[\text{green}] = 2/38 = 0.052$.

b)

$$\begin{aligned} E[\text{red}] &= +1 \times 0.474 - 1 \times 0.526 = -0.052 \\ E[\text{red}^2] &= 1^2 \times 0.474 + (-1)^2 \times 0.526 = 1 \\ \text{Var}[\text{red}] &= 1 - (-0.052)^2 = 0.997 \end{aligned}$$

c)

$$\begin{aligned} E[\text{single number}] &= +35 \times 1/38 - 1 \times 37/38 = -2/38 = -0.052 \\ E[\text{single number}^2] &= 35^2 \times 1/38 + (-1)^2 \times 37/38 = 33.21 \\ \text{Var}[\text{single number}] &= 33.21 - (-0.052)^2 = 33.21 \end{aligned}$$

4.31 a. $P(\text{winning}) = \frac{1}{10^5}$

b. $P(\text{winning}) = \frac{5!}{10^5}$

c. $P(\text{winning}) = \frac{5}{10^5}$