Final, May 18, 2012

NAME:

1. Two players, you and Doyle Brunson (former world champion), play a Texas Hold'em poker game. Each player is dealt two cards (from a well shuffled deck). If your two cards are $(A\clubsuit, K\clubsuit)$. What are the following probabilities for Doyle's hand?

a) Pr[two aces]

b) Pr[an ace and a king]

c) Pr[a pair of tens]

d) Pr[any pair]

e) Pr[an ace or a king and another card (not an ace or king)]

a) $\frac{\binom{3}{2}}{\binom{50}{2}} = \frac{3 \cdot 2}{50 \cdot 49} = \frac{3}{25 \cdot 49}$

b) $P(AK) = \frac{\binom{3}{1}\binom{3}{1}}{\binom{50}{2}} = \frac{3 \times 3 \times 2}{50 \times 49} = \frac{9}{25 \times 49}$

c) $P(10,10) = \frac{\binom{4}{2}}{\binom{50}{2}} = \frac{6 \times 2}{50 \times 49} = \frac{6}{25 \times 49}$

d) $P(\text{any pair}) = \frac{\underbrace{\binom{3}{2}}_{\text{Aces}} + \underbrace{\binom{3}{2}}_{\text{Kings}} + 11 \times \underbrace{\binom{4}{2}}_{\text{all others}}}{\binom{50}{2}} = \frac{3+3+11 \times 6}{\binom{50}{2}} = \frac{72 \times 2}{50 \times 49}$

e) $P(A \text{ or } K, \text{other}) = \frac{\binom{6}{1}\binom{44}{1}}{\binom{50}{2}} = \frac{6 \times 44 \times 2}{50 \times 49}$

6 = # Aces or Kings

44 = # others

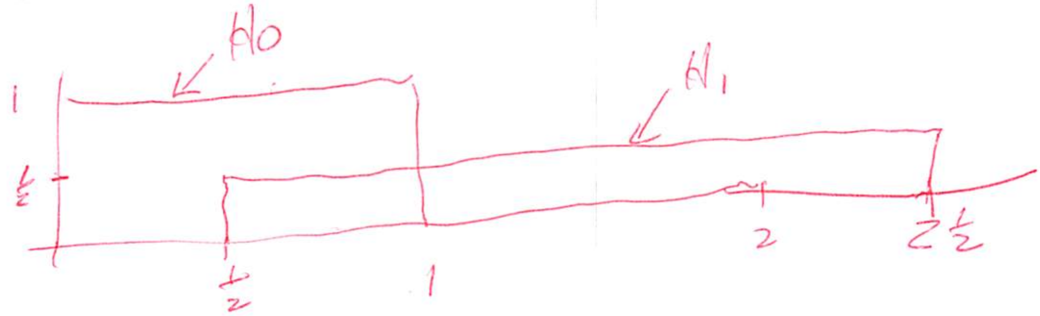
2. Consider a simple hypothesis testing problem: Under hypothesis H_0, $\boldsymbol{X} \sim U(0,1)$ and under hypothesis H_1, $\boldsymbol{X} \sim U(0.5, 2.5)$.

a) Plot both densities on the same graph, labeling each hypothesis.

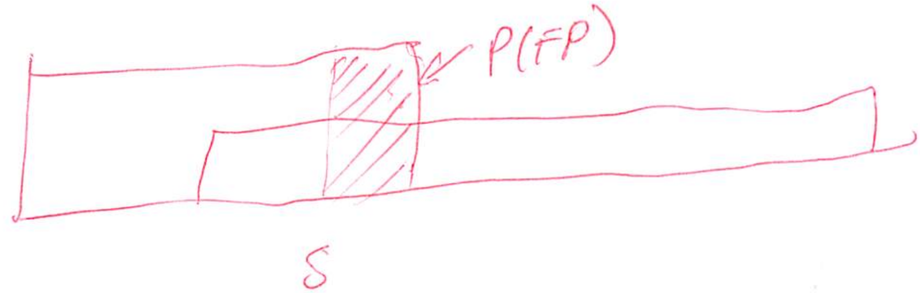
b) In a simple test comparing $\boldsymbol{X}$ to a threshold s, what value of s results in a false positive probability equal to 0.2?

c) For the same value of s as above, what is the false negative probability?

a)

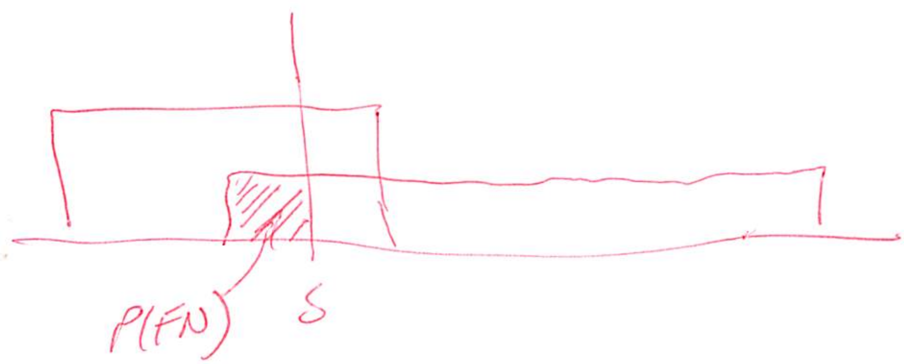


b)



$$P(FP) = P(X > s \mid H_0) = 1 - s = 0.2 \Rightarrow \boxed{s = 0.8}$$

c) $$P(FN) = P(X < s \mid H_1) = \frac{s - \frac{1}{2}}{2} = \frac{0.8-0.5}{2} = \boxed{0.15}$$

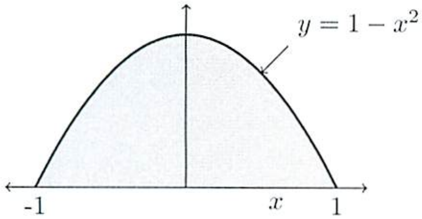


3. Two random variables, $\boldsymbol{X}$ and $\boldsymbol{Y}$, have density $f_{\boldsymbol{XY}}(x, y) = ky$ in the shaded region below.

   a) What is k?

   b) What is $f_{\boldsymbol{X}}(x)$?

   c) What is $f_{\boldsymbol{Y}|\boldsymbol{X}}(y|x)$?



a) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{-1}^{1}\int_{0}^{1-x^2} ky\,dy\,dx$

$= \int_{-1}^{1}\left(\frac{ky^2}{2}\Big|_0^{1-x^2}\right)dx = \int_{-1}^{1}\frac{k}{2}(1-x^2)^2\,dx = \int_{-1}^{1}\frac{k}{2}(1-2x^2+x^4)\,dx$

$= \frac{k}{2}\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-1}^{1} = \frac{k}{2}\cdot 2\cdot\left(1-\frac{2}{3}+\frac{1}{5}\right) = k\,\frac{8}{15}$

$\Rightarrow \boxed{k = \frac{15}{8}}$

b) $f_X(x) = k\int_0^{1-x^2} y\,dy = \boxed{\frac{k}{2}(1-x^2)^2}$ $\quad -1 < x < 1$

c) $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{ky}{\frac{k}{2}(1-x^2)^2} = \boxed{\frac{2y}{(1-x^2)^2} \quad \begin{matrix} 0 < y < 1-x^2 \\ -1 < x < 1 \end{matrix}}$

4. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be IID exponential random variables with density $f_{\boldsymbol{X}}(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f_{\boldsymbol{X}}(x) = 0$ for $x < 0$ (and same for $\boldsymbol{Y}$). Let $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y}$.

a) What is the density of $\boldsymbol{S}$, i.e., $f_{\boldsymbol{S}}(s)$?

b) What is the conditional density of $\boldsymbol{X}$ given $\boldsymbol{S} = s$, i.e., $f_{\boldsymbol{X}|\boldsymbol{S}}(x|\boldsymbol{S} = s)$?

a) $f_S = f_X * f_Y \qquad f_S(s) = \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$

$$\boxed{f_S(s) = \lambda^2 e^{-\lambda s} s \qquad 0 < s < \infty}$$

Check: $1 = \int_0^\infty f(s)\,ds = \lambda^2 \int_0^\infty s e^{-\lambda s}\,ds = -\frac{\lambda^2 s}{\lambda} e^{-\lambda s}\Big|_0^\infty - \underbrace{\int_0^\infty \frac{-\lambda^2}{\lambda} e^{-\lambda s}\,ds}_{1}$

(using $u = s$, $du = ds$, $dv = e^{-\lambda s}ds$, $v = -\frac{1}{\lambda}e^{-\lambda s}$)

b) $f_{X|S}(x|s) = \frac{f_{X,S}(x,s)}{f_S(s)} \approx P(X=x|S=s) = \frac{P(X=x \cap X+Y=s)}{P(S=s)}$

Can be solved for with the Jacobian. Simpler argument

$= \frac{P(X=x)P(Y=s-x)}{P(S=s)}$

$= \frac{\lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}}{\lambda^2 s e^{-\lambda s}} = \frac{1}{s}$

for $0 < X < S$

i.e. Given $S=s$, X is uniform between 0 and s.

5.

Let $\boldsymbol{X} \sim N(-1, 4)$, use the table on the right to find,

a) $\Pr[\boldsymbol{X} \leq 0]$.

b) $\Pr[-1 \leq \boldsymbol{X} \leq 1]$.

c) $\Pr[\boldsymbol{X} \geq 1]$.

d) $\Pr[2\boldsymbol{X} + 1 \leq 3]$.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	1.00	0.8413	2.00	0.9772	3.00	0.9987
0.05	0.5199	1.05	0.8531	2.05	0.9798	3.05	0.9989
0.10	0.5398	1.10	0.8643	2.10	0.9821	3.10	0.9990
0.15	0.5596	1.15	0.8749	2.15	0.9842	3.15	0.9992
0.20	0.5793	1.20	0.8849	2.20	0.9861	3.20	0.9993
0.25	0.5987	1.25	0.8944	2.25	0.9878	3.25	0.9994
0.30	0.6179	1.30	0.9032	2.30	0.9893	3.30	0.9995
0.35	0.6368	1.35	0.9115	2.35	0.9906	3.35	0.9996
0.40	0.6554	1.40	0.9192	2.40	0.9918	3.40	0.9997
0.45	0.6736	1.45	0.9265	2.45	0.9929	3.45	0.9997
0.50	0.6915	1.50	0.9332	2.50	0.9938	3.50	0.9998
0.55	0.7088	1.55	0.9394	2.55	0.9946	3.55	0.9998
0.60	0.7257	1.60	0.9452	2.60	0.9953	3.60	0.9998
0.65	0.7422	1.65	0.9505	2.65	0.9960	3.65	0.9999
0.70	0.7580	1.70	0.9554	2.70	0.9965	3.70	0.9999
0.75	0.7734	1.75	0.9599	2.75	0.9970	3.75	0.9999
0.80	0.7881	1.80	0.9641	2.80	0.9974	3.80	0.9999
0.85	0.8023	1.85	0.9678	2.85	0.9978	3.85	0.9999
0.90	0.8159	1.90	0.9713	2.90	0.9981	3.90	1.0000
0.95	0.8289	1.95	0.9744	2.95	0.9984	3.95	1.0000

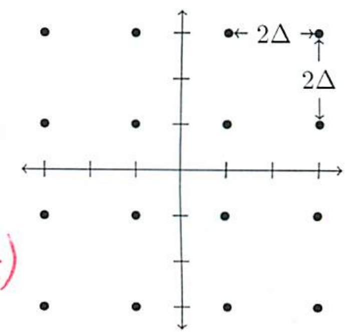
a) $P[X < 0] = P\left[\frac{X+1}{2} \leq \frac{0+1}{2}\right]$
$= \Phi\left(\frac{1}{2}\right) = \boxed{0.6915}$

b) $P[-1 \leq X \leq 1] = P\left[\frac{-1+1}{2} \leq \frac{X+1}{2} \leq \frac{1+1}{2}\right] = \Phi(1) - \Phi(0)$
$= 0.8413 - 0.5 = \boxed{0.3413}$

c) $P[X \geq 1] = P\left[\frac{X+1}{2} \geq \frac{1+1}{2}\right] = 1 - \Phi(1)$
$= 1 - 0.8413 = \boxed{0.1587}$

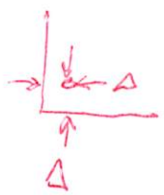
d) $P[2X+1 \leq 3] = P\left[X \leq \frac{3-1}{2}\right] = P[X \leq 1]$
$= P\left[\frac{X+1}{2} \leq \frac{1+1}{2}\right] = \Phi(1) = \boxed{0.8413}$

6. For 16 QAM (same assumptions as usual: all 16 points are equally likely to be transmitted, the noises N_X and N_Y are IID $N(0, \sigma^2)$), derive the probability of error.



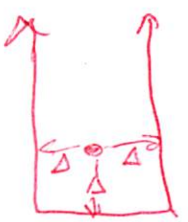
3 regions:

a) Corner



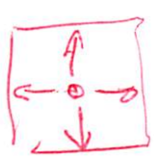
$$P(\text{correct}) = P(N_X > -\Delta \cap N_Y > -\Delta)$$
$$= P\left(\frac{N_X - 0}{\sigma} > \frac{-\Delta - 0}{\sigma}\right) P\left(\frac{N_Y}{\sigma} > \frac{-\Delta}{\sigma}\right)$$
$$= \left(1 - \Phi\left(-\frac{\Delta}{\sigma}\right)\right)\left(1 - \Phi\left(-\frac{\Delta}{\sigma}\right)\right)$$
$$= \Phi\left(\frac{\Delta}{\sigma}\right)^2$$

b)



$$P(\text{correct}) = P(-\Delta \leq N_X \leq \Delta \cap N_Y \geq -\Delta)$$
$$= P\left(\frac{-\Delta}{\sigma} \leq \frac{N_X}{\sigma} \leq \frac{\Delta}{\sigma}\right) P\left(\frac{N_Y}{\sigma} \geq -\frac{\Delta}{\sigma}\right)$$
$$= \left[\Phi\left(\frac{\Delta}{\sigma}\right) - \Phi\left(\frac{-\Delta}{\sigma}\right)\right] \cdot \left[1 - \Phi\left(\frac{-\Delta}{\sigma}\right)\right]$$
$$= \left(2\Phi\left(\frac{\Delta}{\sigma}\right) - 1\right)\Phi\left(\frac{\Delta}{\sigma}\right)$$

c)



$$P(\text{correct}) = P(-\Delta < N_X < \Delta \cap -\Delta < N_Y < \Delta)$$
$$= \left(2\Phi\left(\frac{\Delta}{\sigma}\right) - 1\right)^2$$

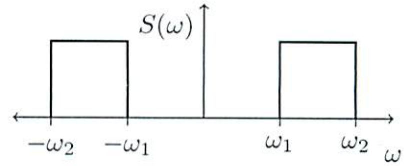
Combining gives
$$P(\text{correct}) = \frac{1}{4}\Phi\left(\frac{\Delta}{\sigma}\right)^2 + \frac{1}{2}\Phi\left(\frac{\Delta}{\sigma}\right)\left(2\Phi\left(\frac{\Delta}{\sigma}\right) - 1\right) + \frac{1}{4}\left(2\Phi\left(\frac{\Delta}{\sigma}\right) - 1\right)^2$$

$$P(\text{error}) = 1 - P(\text{correct}) = \frac{1}{4}\left(1 - \Phi\left(\frac{\Delta}{\sigma}\right)\right)^2 + \frac{1}{2}\left(1 - \Phi\left(\frac{\Delta}{\sigma}\right)\left(2\Phi\left(\frac{\Delta}{\sigma}\right) - 1\right)\right) + \frac{1}{4}\left(1 - \left(2\Phi\left(\frac{\Delta}{\sigma}\right) - 1\right)^2\right) = \frac{3}{4} + \frac{3}{2}\Phi\left(\frac{\Delta}{\sigma}\right) - \frac{9}{4}\Phi\left(\frac{\Delta}{\sigma}\right)^2$$

7. (Extra credit: credit only for correct work) Let $\boldsymbol{X}(t)$ be a WSS random process with autocorrelation function $R(\tau)$ and power spectral density $S(\omega)$.

a) (5 points) If $\boldsymbol{R}(\tau) = e^{-\alpha|\tau|}$ (for $\alpha > 0$ and all τ), what is $\boldsymbol{S}(\omega)$?

b) (5 points) If $\boldsymbol{S}(\omega) = 1$ for $\omega_1 \leq |\omega| \leq \omega_2$, what is $\boldsymbol{R}(\tau)$?



a) $S(\omega) = \mathcal{F}(R(\tau)) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau}\, d\tau$

$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|} e^{-j\omega\tau}\, d\tau$

$= \int_{-\infty}^{0} e^{\alpha\tau - j\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-\alpha\tau - j\omega\tau}\, d\tau$

$= \frac{1}{\alpha - j\omega} e^{\alpha\tau - j\omega\tau}\Big|_{-\infty}^{0} + \frac{-1}{\alpha + j\omega} e^{-\alpha\tau - j\omega\tau}\Big|_{0}^{\infty}$

$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega} = \boxed{\frac{2\alpha}{\alpha^2 + \omega^2}}$

b) $R(\tau) = \mathcal{F}^{-1}(S(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau}\, d\omega$

$= \frac{1}{2\pi} \int_{-\omega_2}^{-\omega_1} e^{j\omega\tau}\, d\omega + \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} e^{j\omega\tau}\, d\tau = \frac{1}{2\pi}\left[\frac{1}{j\tau} e^{j\omega\tau}\Big|_{-\omega_2}^{-\omega_1} + \frac{1}{j\tau} e^{j\omega\tau}\Big|_{\omega_1}^{\omega_2}\right]$

$= \frac{1}{2\pi}\frac{1}{j\tau}\left[e^{-j\omega_1\tau} - e^{-j\omega_2\tau} + e^{j\omega_2\tau} - e^{+j\omega_1\tau}\right]$

$= \boxed{\frac{1}{\pi\tau}\left[\sin \omega_2\tau - \sin \omega_1\tau\right]}$

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