ELEG 310

Midterm 2

May 2, 2013

Name: ______________________________

1. (25 points) Let $\boldsymbol{X} \sim N(3,4)$ and $\boldsymbol{Y} = 2\boldsymbol{X} - 1$.
   (a) What is $\Pr[0 \le \boldsymbol{X} \le 4]$?
   (b) What is $E[\boldsymbol{X}^2]$?
   (c) What is $\Pr[0 \le \boldsymbol{Y} \le 4]$?
   (d) What is $E[\boldsymbol{Y}]$?
   (e) What is $\mathrm{Var}[\boldsymbol{Y}]$?

a) $P[0 \le X \le 4] = P\left[\frac{0-3}{2} \le \frac{X-3}{2} \le \frac{4-3}{2}\right]$

$= P[-1.5 \le Z \le 0.5] = \Phi(0.5) - \Phi(-1.5)$

$= \Phi(0.5) + \Phi(1.5) - 1 = 0.6915 + 0.9332 - 1$

$= \boxed{0.6247}$

b) $E[X^2] = \sigma^2 + \mu^2 = 4 + 3^2 = \boxed{13}$

c) $P[0 \le Y \le 4] = P\left[\frac{0-5}{4} \le \frac{Y-5}{4} \le \frac{4-5}{4}\right]$ $\quad 4 = \sqrt{16}$

$= \Phi(-0.25) - \Phi(-1.25) = \Phi(1.25) - \Phi(0.25)$

$= 0.8944 - 0.5987 = \boxed{0.2957}$

d) $E[Y] = 2E[X] - 1 = \boxed{5}$

e) $\mathrm{Var}[Y] = 2^2\,\mathrm{Var}[X] = \boxed{16}$

2. (25 points) Let $X_1, X_2, \ldots, X_n$ be a sequence of IID Bernoulli random variables ($\Pr[X=1]=p$ and $\Pr[X=0]=1-p$). Let $S_n = X_1 + X_2 + \cdots + X_n$.

(a) What is $\Pr[3 \le S_4 \le 4]$?

(b) Give a good approximation for $\Pr[40 \le S_{64} \le 64]$?

a) $P[3 \le S_4 \le 4] = P[S_4 = 3] + P[S_4 = 4]$

$$\boxed{= \binom{4}{3} p^3 (1-p) + \binom{4}{4} p^4}$$

$$\boxed{= 4p^3 - 3p^4}$$

b) $P[40 \le S_{64} \le 64]$

$E S_{64} = 64p$

$\mathrm{Var}\, S_{64} = 64p(1-p)$

$$= P\left[\frac{40-64p}{8\sqrt{p(1-p)}} \le \frac{S_{64} - 64p}{8\sqrt{p(1-p)}} \le \frac{64-64p}{8\sqrt{p(1-p)}}\right]$$

$$\approx \Phi\left(\frac{64-64p+0.5}{8\sqrt{p(1-p)}}\right) - \Phi\left(\frac{40-64p-0.5}{8\sqrt{p(1-p)}}\right)$$

Note, even better is to recognize $P[40 \le S_{64} \le 64]$
$= P[40 \le S_{64}]$

$$\boxed{\approx 1 - \Phi\left(\frac{40-64p-0.5}{8\sqrt{p(1-p)}}\right)}$$

3. (25 points) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ have joint density $f_{XY}(x,y) = c$ in the shaded region below and $f_{XY}(x,y) = 0$ otherwise.

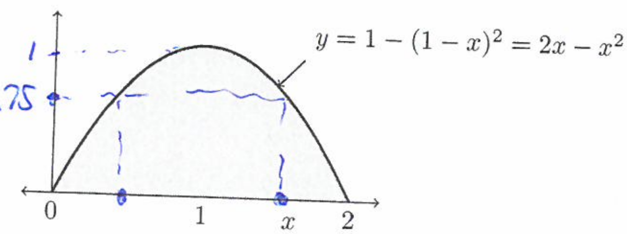
   (a) What value does c have?

   (b) What is $f_X(x)$?

   (c) What is $E[\boldsymbol{X}]$?

   (d) What is $f_{X|Y}(x|y)$?

   (e) How would you calculate $\Pr[\boldsymbol{Y} \geq 0.75]$. Be precise, but you do not need to do the actual computation.



$\Leftrightarrow (1-x)^2 = 1-y \Rightarrow x = 1 \pm \sqrt{1-y}$

e.g. $y = 0.75 \Rightarrow x = \frac{1}{2}, \frac{3}{2}$

a) $$1 = \int_0^2 \int_0^{2x-x^2} c \, dy \, dx = \int_0^2 c(2x-x^2)\,dx = c\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^2$$

$$= c\left(4 - \frac{8}{3}\right) = c\,\frac{4}{3} \Rightarrow \boxed{c = \frac{3}{4}}$$

b) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{2x-x^2} c\,dy = \boxed{\begin{cases} c(2x-x^2) & 0<x<2 \\ 0 & \text{o.w.} \end{cases}}$$

c) $E[X] = 1$ by symmetry $\left(\text{OR, } E[X] = \int_0^2 x \cdot c(2x-x^2)\,dx = c\left(\frac{16}{3} - \frac{16}{4}\right) = 1\right)$

d) $$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \boxed{\begin{cases} \dfrac{c}{2c\sqrt{1-y}} & 1-\sqrt{1-y} < x < 1+\sqrt{1-y},\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}}$$

e) $$P[Y \geq 0.75] = \boxed{\int_{0.75}^{1} \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} c\,dx\,dy} = \boxed{\int_{1/2}^{3/2} \int_{0.75}^{2x-x^2} c\,dy\,dx}$$ ← Either of these.

4. (25 points) In World War II, the US tested all soldiers for syphilis. The test was expensive and a method was devised to reduce the number of test needed. Assume the soldiers have syphilis with probability p independently of all other soldiers (typically p is a small number). A group of n ($n \geq 2$) blood samples are mixed together and tested. A single test can indicate if none of the group has syphilis. However, if any of the group has syphilis, each soldier in the group is then tested individually. Let $\boldsymbol{T}_n$ be the total number of tests required to test n soldiers.

   (a) What is the PMF of $\boldsymbol{T}_n$?

   (b) What is the mean of $\boldsymbol{T}_n$?

   (c) Verify that the mean of $\boldsymbol{T}_n$ has the right value for $p = 0$ and $p = 1$.

$T_n = 1$ if first test is negative (none of group have syphilis)

$T_n = n+1$ if first test is positive (at least one soldier in group has syphilis)

a) $\boxed{P[T_n = 1] = (1-p)^n \qquad P[T_n = n+1] = 1-(1-p)^n}$

b) $E[T_n] = (1-p)^n + (n+1)(1-(1-p)^n)$

$\boxed{= n+1 - n(1-p)^n}$

c) if $p=0$, no one has syphilis $\Rightarrow P[T_n = 1] = 1$

$E[T_n] = n+1 - n(1-0)^n = n+1-n = \boxed{1}$

if $p=1$, everyone has syphilis $P[T_n = n+1] = 1$

$E[T_n] = n+1 - n(1-1)^n = \boxed{n+1}$