# SOLUTION TO HOMEWORK #2

## #1

To solve these problems, we use the sifting property of impulse functions:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$$

For example, for continuous-time, this is easily derived

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$\therefore \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

**(a)** $x_1(t) = \int_{-3}^{-1} t^4 \underbrace{\delta(t+2)}_{\text{impulse at } t=-2} dt = t^4\Big|_{t=-2} = 16$

**(b)** $x_2(t) = \int_{-2}^{2} (1+t)^2 \underbrace{\delta(t-1)}_{\text{impulse at } t=1} dt = (1+t)^2\Big|_{t=1} = 4$

**(c)** $x_3(t) = \int_{-2}^{2} (1-t) \underbrace{\delta(t+3)}_{\substack{\text{impulse at t=-3} \\ \text{but not in range} \\ \text{of integral}}} dt = 0$

**(d)** $x_4[n] = \sum_{n=-3}^{\infty} (1/2)^n \underbrace{\delta[n-1]}_{\text{impulse at } n=1} = (1/2)^n\Big|_{n=1} = 1/2$

**(e)** $x_5[n] = \sum_{n=0}^{\infty} (e^{j2\pi/3})^n \underbrace{\delta[n-k]}_{\text{impulse at } n=k} = (e^{j2\pi/3})^n\Big|_{n=k} = e^{j2\pi k/3},\ k \geq 0$

**(f)** $x_6[n] = \sum_{n=-3}^{3} (-3)^n \underbrace{\delta[n+4]}_{\substack{\text{impulse at n=4} \\ \text{but not in range} \\ \text{of sum}}} = 0$

## #2

**(a)** $y(t) = x(2-t)$

(1) Since $y(0) = x(2-0) = x(2)$, the output at time $t = 0$ depends on the input at a different time ($t = 2$). Therefore, the system is **not memoryless**.

(2)
$$y_1(t) = x_1(2-t)$$

$$\begin{aligned} y_1(t-t_0) = &\text{ shifted output} \\ &= x_1(2-(t-t_0)) \\ &= x_1(2-t+t_0). \end{aligned}$$

Consider a shifted input $x_2(t) = x_1(t-t_0)$. Then, the corresponding output is

$$\begin{aligned} y_2(t) &= x_2(2-t) \\ &= x_1(2-t-t_0). \end{aligned}$$

But, $y_2(t) =$ output for the shifted input $\neq$ shifted output $y_1(t-t_0)$. Therefore, the system is **not time invariant**.

(3) Consider $x(t) = ax_1(t) + bx_2(t)$, where $a$ and $b$ are constants. Then,

$$\begin{aligned} y(t) &= x(2-t) \\ &= ax_1(2-t) + bx_2(2-t) \\ &= ay_1(t) + by_2(t). \end{aligned}$$

Therefore, since the linear combination of the input results in the same linear combination of the outputs, the system is **linear**.

(4) Since $y(0) = x(2)$, the output at the present time $t = 0$ may depend on the input at a future time. Therefore, the system is **not causal**.

(5) The output $y(t)$ is simply the flipped and shifted version of $x(t)$. So, if $x(t)$ is bounded, then $y(t)$ is bounded. Therefore, the system is **stable**.

**(b)** $y(t) = [\sin 2t]x(t-2)$

(1) The system is **not memoryless** since the output at a given time (for example, $t = 1$) depends on the input at a different time ($t = -1$).

(2)
$$y_1(t) = [\sin 2t]x_1(t-2)$$

$$\begin{aligned} y_1(t-t_0) = &\text{ shifted output} \\ &= [\sin 2(t-t_0)]x_1(t-t_0-2). \end{aligned}$$

Consider a shifted input $x_2(t) = x_1(t-t_0)$. Then,

$$\begin{aligned} y_2(t) &= [\sin 2t]x_2(t-2) \\ &= [\sin 2t]x_1(t-t_0-2). \end{aligned}$$

But, $y_2(t) =$ output for the shifted input $\neq$ shifted output $y_1(t-t_0)$. Therefore, the system is **not time invariant**.

(3) Consider $x(t) = ax_1(t) + bx_2(t)$, where $a$ and $b$ are constants. Then,

$$\begin{aligned} y(t) &= [\sin 2t](ax_1(t-2) + bx_2(t-2)) \\ &= ay_1(t) + by_2(t), \end{aligned}$$

where $y_1(t) = [\sin 2t]x_1(t-2)$ and $y_2(t) = [\sin 2t]x_2(t-2)$. Therefore, the system is **linear**.

(4) The system is **causal** because the output at a given time does not depend on a future value of the input.

(5) Assume the input is bounded, i.e., $|x(t)| \leq B < \infty$. In this case, $|y(t)| = |[\sin 2t]x(t-2)| < |x(t-2)| < \infty$. Therefore, a bounded input produces a bounded output, and the system is **stable**.

**(c)** $y[n] = |x[n-3]|$

(1) For this system, we have $y[1] = |x[-2]|$. The output at a given time (for example, $n = 1$) depends on the input at a different time ($n = -2$), which means the system is **not memoryless**.

(2)

$$y_1[n] = |x_1[n-3]|$$

$$\begin{aligned} y_1[n - N_0] &= \text{ shifted output} \\ &= |x_1[(n - N_0) - 3]| \\ &= |x_1[n - N_0 - 3]|. \end{aligned}$$

Consider a shifted input $x_2[n] = x_1[n - N_0]$. Then,

$$\begin{aligned} y_2[n] &= |x_2[n-3]| \\ &= |x_1[n - N_0 - 3]|. \end{aligned}$$

But, $y_2[n]$ = output for the shifted input = shifted output $y_1[n - N_0]$. Therefore, the system is **time invariant**.

(3) Consider $x[n] = ax_1[n] + bx_2[n]$, where $a$ and $b$ are constants. Then,

$$y[n] = |x[n-3]| = |ax_1[n-3] + bx_2[n-3]|$$

However,

$$ay_1[n] + by_2[n] = a|x_1[n-3]| + b|x_2[n-3]| \neq |ax_1[n-3] + bx_2[n-3]|$$

Therefore, the system is **not linear**.

(4) The system is **causal** because the output at a given time does not depend on a future value of the input.

(5) Assume the input is bounded, i.e., $|x[n]| \leq B < \infty$. In this case, $|y[n]| = |x[n-3]| < \infty$. Therefore, a bounded input produces a bounded output, and the system is **stable**.

**(d)** $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$

(1) For this system, we have $y[1] = \frac{1}{3}(x[2]+x[1]+x[0])$. The output at a given time depends on the input at a different time, which means the system is **not memoryless**.

(2)
$$y_1[n] = \frac{1}{3}(x_1[n+1] + x_1[n] + x_1[n-1])$$

$$\begin{aligned} y_1[n-N_0] &= \text{ shifted output} \\ &= \frac{1}{3}(x_1[n-N_0+1] + x_1[n-N_0] + x_1[n-N_0-1]). \end{aligned}$$

Consider a shifted input $x_2[n] = x_1[n-N_0]$. Then,

$$\begin{aligned} y_2[n] &= \frac{1}{3}(x_2[n+1] + x_2[n] + x_2[n-1]) \\ &= \frac{1}{3}(x_1[n-N_0+1] + x_1[n-N_0] + x_1[n-N_0-1]). \end{aligned}$$

But, $y_2[n]$ = output for the shifted input = shifted output $y_1[n-N_0]$. Therefore, the system is **time invariant**.

(3) Consider $x[n] = ax_1[n] + bx_2[n]$, where $a$ and $b$ are constants. Then,

$$\begin{aligned} y[n] &= \frac{1}{3}(x[n-N_0+1] + x[n-N_0] + x[n-N_0-1]) \\ &= \frac{1}{3}(ax_1[n-N_0+1] + bx_2[n-N_0+1] + ax_1[n-N_0] + bx_2[n-N_0] \\ &\quad + ax_1[n-N_0-1] + bx_2[n-N_0-1]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

where

$$\begin{aligned} y_1[n] &= \frac{1}{3}(x_1[n+1] + x_1[n] + x_1[n-1]) \\ y_2[n] &= \frac{1}{3}(x_2[n+1] + x_2[n] + x_2[n-1]) \end{aligned}$$

Therefore, the system is **linear**.

(4) Since $y[1] = \frac{1}{3}(x[2] + x[1] + x[0])$, the output at time $n = 1$ depends on the future input at $n = 2$, so, the system is **not causal**.

(5) For a bounded input $|x[n]| = K < \infty$, the system produces an output $|y[n]| = |\frac{1}{3}(x[n+1] + x[n] + x[n-1])| < |\frac{1}{3}(K + K + K)| = K < \infty$. Thus, the system is **stable**.

**#3** $h(t) = \text{Impulse response}$

$$= \text{Output of continuous-time system when the input is a unit impulse} = y(t)\Big|_{x(t)=\delta(t)}$$

$h[n] = \text{Impulse response}$

$$= \text{Output of discrete-time system when the input is a unit impulse} = y[n]\Big|_{x[n]=\delta[n]}$$

**(a)**

$$y[n] = 2x[n] + x[n-\alpha] + x[n+\beta]$$

$$h[n] = y[n]\Big|_{x[n]=\delta[n]} = 2\delta[n] + \delta[n-\alpha] + \delta[n+\beta]$$

**(b)**

$$y(t) = \int_0^\infty e^{-\tau} x(t-\tau)d\tau$$

$$h(t) = y(t)\Big|_{x(t)=\delta(t)} = \int_0^\infty e^{-\tau}\delta(t-\tau)d\tau$$

- If $t < 0$, $h(t) = 0$, because the impulse is located where $\tau = t$, but $\tau$ is never negative.
- If $t > 0$, $h(t) = e^{-t}$ (sifting property).

$$\therefore\ h(t) = e^{-t}u(t)$$

**#4**

**(a)** Causal?

If the system is causal, there can be no output before the input is applied. However, the input $x_1(t)$ starts at $t = 0$ and gives an output, $y_1(t)$ that starts at $t = -1 \Rightarrow$ **not causal**.

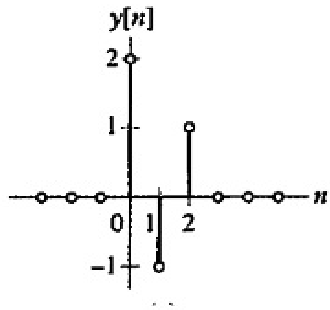
**(b)** Time-invariant?

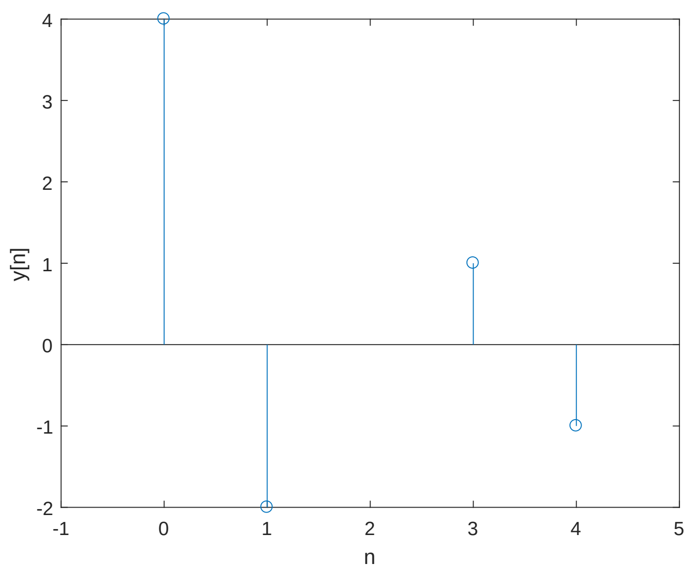
If the system is time-invariant, a shifted input will give the same output, shifted by the same amount. Notice that input $x_4(t)$ is simply $x_2(t-1)$. However, $y_4(t) \neq y_2(t-1) \Rightarrow$ **not time invariant**.

**#5**

Consider the discrete-time LTI system with impulse response shown below. So, $h[n] = 2\delta[n] - \delta[n-1] + \delta[n-2]$. Now, the new input $x[n] = 2\delta[n] - \delta[n-2]$. Because the system is linear, the output is the same linear combination of the inputs, and because the system is time invariant, the output to $\delta[n-2]$ is simply $h[n-2]$. Therefore,

$$y[n] = 2h[n] - h[n-2] = 4\delta[n] - 2\delta[n-1] + \delta[n-3] - \delta[n-4]$$



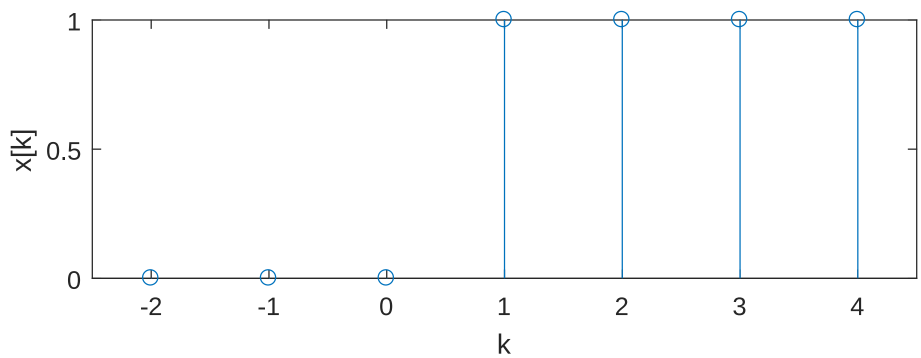



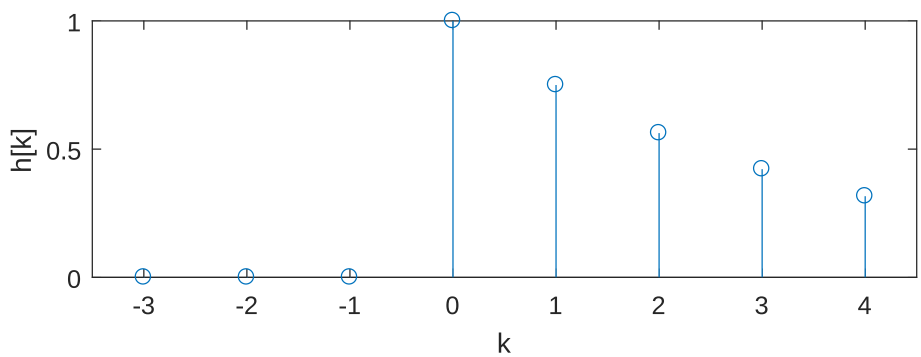

# #6

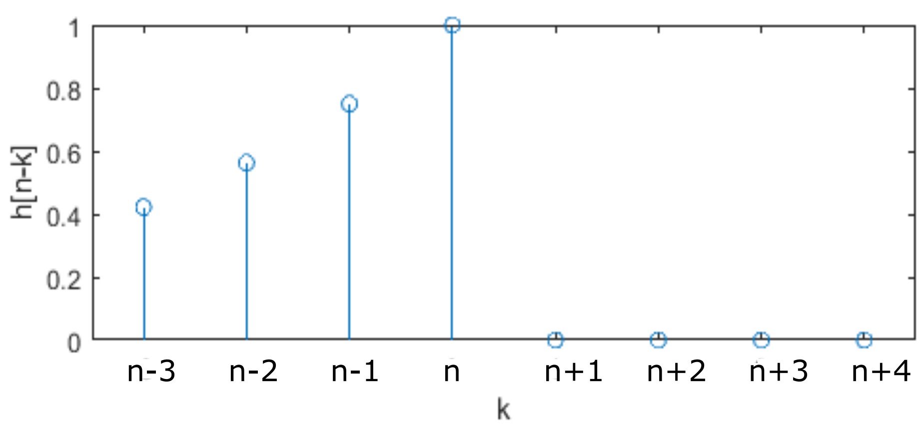
**(a)** The convolution of $x[n] = u[n-1]$ and $h[n] = (3/4)^n u[n]$ is given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We will compute this graphically. First draw $x[k]$ and $h[k]$.




- When $n < 1$, there is no overlap $\Rightarrow x[k]h[n-k] = 0 \Rightarrow y[n] = 0$
- When $n \geq 1$, the overlap will be from $k = 1$ to $k = n$ (and $x[k]h[n-k] \neq 0$ over the range) $\Rightarrow y[n] = \sum_{k=1}^{n} (3/4)^{n-k}$

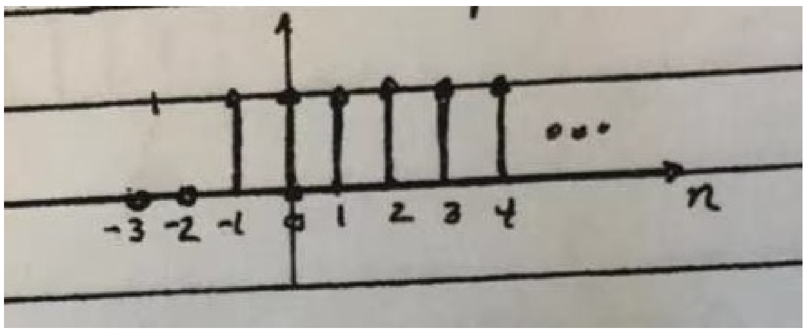
Let $r = n - k$,

$$y[n] = \sum_{r=n-1}^{0} (3/4)^r = \sum_{r=0}^{n-1} (3/4)^r = \frac{1-(3/4)^n}{1-3/4} = 4(1-(3/4)^n),\ n \geq 1$$

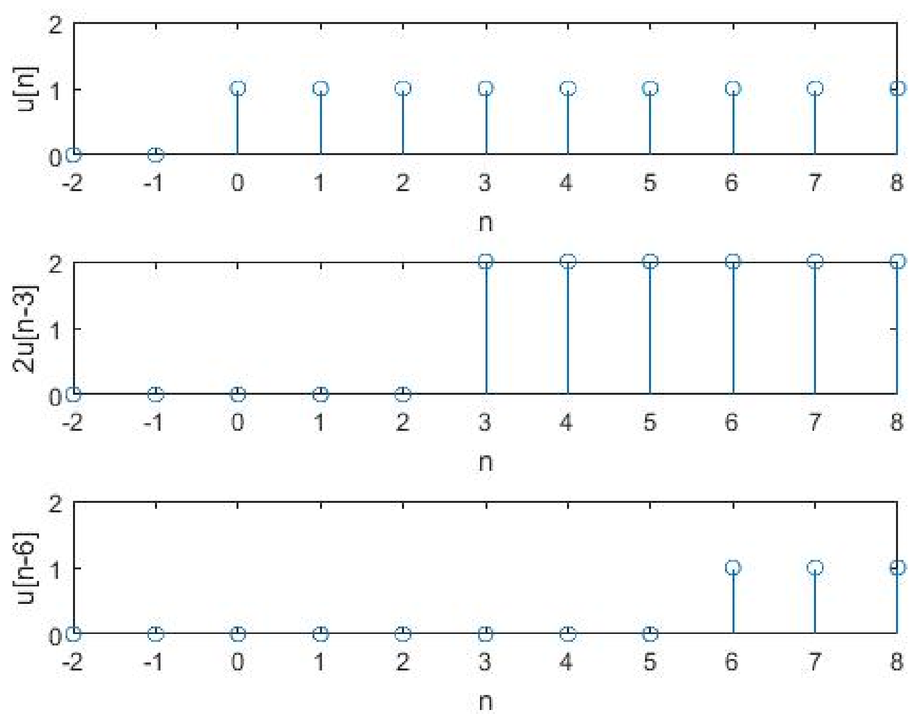
$$\text{and } y[n] = 0,\ n < 1$$
$$\therefore y[n] = 4[1 - (3/4)^n]u[n-1]$$

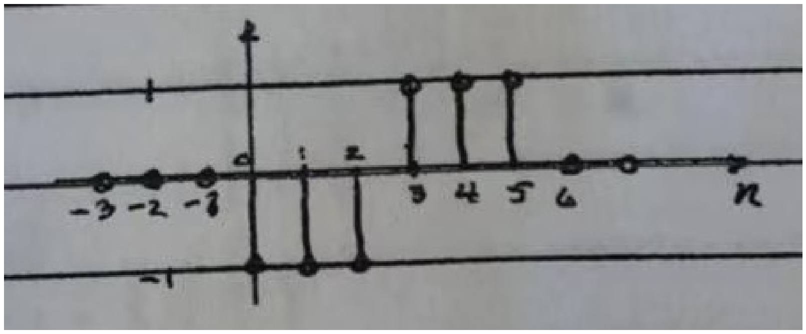
**(b)** Convolve $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$. Again, we perform this graphically. First draw these individual signals

$$x[n] = u[n+1] \Rightarrow \text{step that starts at } n = -1$$

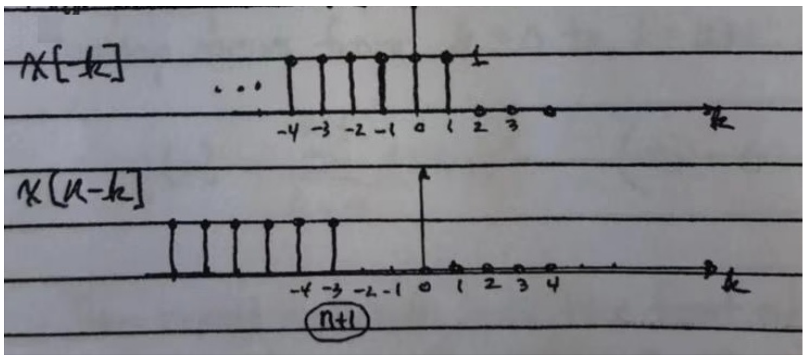



$$\begin{aligned} h[n] &= -u[n] + 2u[n-3] - u[n-6] \\ &= -\delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] \end{aligned}$$

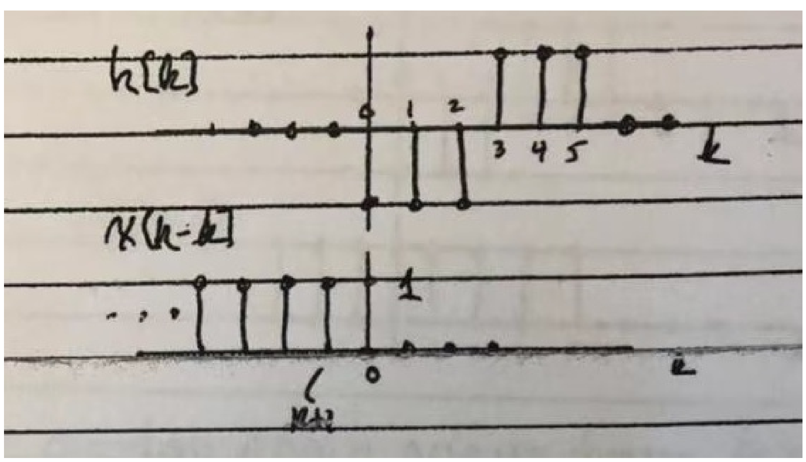





For this problem, it is easier to leave $h[k]$ alone and flip and shift $x[k]$.

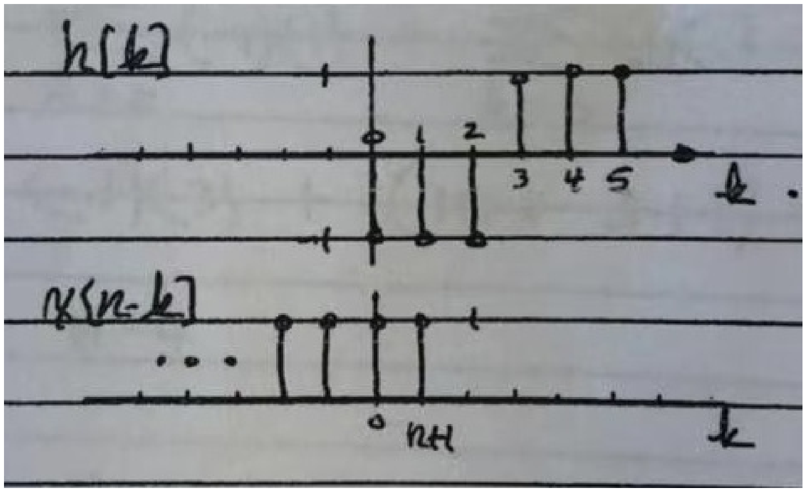



- $n + 1 < 0 \Rightarrow n \leq -1$




There is no overlap between $h[k]$ and $x[n-k]$. So, $y[n] = 0$

- $n + 1 > 0 \Rightarrow n > -1$



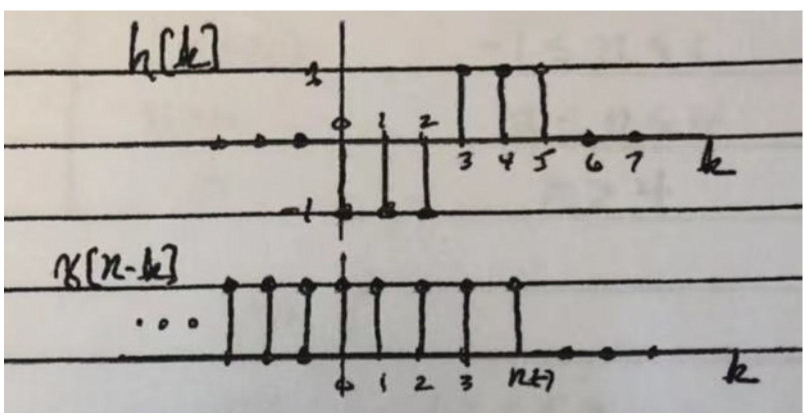

Now, the product $x[n-k]h[k]$ is no longer zero. The overlap occurs from $k = 0$ to $k = n + 1$.

$$\therefore y[n] = \sum_{k=0}^{n+1} (1)(-1) = -(n + 1 - 0 + 1) = -(n + 2)$$

This result continues until the front edge of $x[n-k]$ reaches $k = 2 \Rightarrow n+1 = 2 \Rightarrow n = 1$

$$\therefore y[n] = -(n + 2) \text{ for } -1 \le n \le 1$$

- $n + 1 > 2 \Rightarrow n > 1$




The overlap again occurs from $k = 0$ to $k = n + 1$.

$$y[n] = \sum_{k=0}^{n+1} h[k]x[n-k] = \sum_{k=0}^{2} (-1)(1) + \sum_{k=3}^{n+1} (1)(1) = (-1)(3) + (n + 1 - 3 + 1) = n - 4$$

This result continues until the front edge of $x[n-k]$ reaches $k = 5 \Rightarrow n+1 = 5 \Rightarrow n = 4$

$$\therefore y[n] = n - 4 \text{ for } 2 \le n \le 4$$

- $n+1>5 \Rightarrow n>4$, $x[n-k]$ covers all of $h[k]$ and

$$y[n] = \sum_{k=0}^{2}(-1)(1) + \sum_{k=3}^{5}(1)(1) = -3+3 = 0$$

So, the output is

$$y[n] = \begin{cases} 0, & n \leq -1 \\ -(n+2), & -1 \leq n \leq 1 \\ n-4, & 2 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$$