Short questions: Circle your answer so there is no ambiguity.

1. (5 points) If $X \sim N(3,4)$, what is $\Pr[X < 0]$?

$$P(X<0) = P\left(\frac{X-3}{2} < \frac{0-3}{2}\right) = \Phi\left(-\frac{3}{2}\right) = 1-\Phi\left(\frac{3}{2}\right)$$
$$= 1-0.9332 = \boxed{0.0668}$$

2. (5 points) If $X \sim U(-1,2)$, what is $E(X)$?

$$E(X) = \int_{-1}^{2} \frac{1}{3} x\, dx = \frac{1}{3}\frac{x^2}{2}\Big|_{-1}^{2} = \frac{4-1}{6} = \boxed{\frac{1}{2}}$$

3. (5 points) If X and Y are independent with $X \sim N(0,2)$ and $Y \sim N(-1,5)$, what is the variance of $Z = 2X + 3Y$?

$$\text{Var } Z = 2^2 \text{Var } X + 3^2 \text{Var } Y = 4 \times 2 + 9 \times 5 = \boxed{53}$$

4. (5 points) If $f(x) = cx$ for $0 < x < 2$ and $f(x) = 0$ elsewhere, what is c?

$$1 = \int_0^2 cx\, dx = c\frac{x^2}{2}\Big|_0^2 = c\frac{4}{2} \Rightarrow \boxed{c = \frac{1}{2}}$$

5. (5 points) If $f(x) = 3x^2/8$ for $0 < x < 2$ and $f(x) = 0$ elsewhere, what is $E(X)$?

$$EX = \int_0^2 \frac{3x^2}{8} x \, dx = \frac{3}{8} \frac{x^4}{4}\Big|_0^2 = \boxed{\frac{3}{2}}$$

6. (5 points) If $F(x) = 1 - e^{-x^2}$ for $x > 0$, what is $f(x)$?

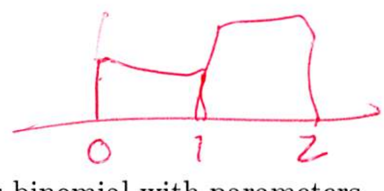
$$f(x) = \frac{d}{dx} F(x) = \boxed{2xe^{-x^2}}$$

7. (5 points) If X is Bernoulli with parameter p and Y is $U(0, 1)$ (uniform), what is the density of $Z = X + Y$?

$$F_Z(z) = P(Z \le z) = P(X+Y \le z) = P(X+Y \le z \mid X=0)P(X=0) + P(X+Y \le z \mid X=1) P(X=1$$

$$= F_Y(z)(1-p) + F_Y(z-1)p$$

$$f_Z(z) = f_Y(z)(1-p) + f_Y(z-1)p$$



8. (5 points) If X is binomial with parameters n and p and Y is binomial with parameters m and p, what is $Z = X + Y$?

$\boxed{Z \text{ is binomial with parameters } n+m \text{ and } p}$

9. (5 points) If X and Y are IID exponential with parameter λ, what is the density of ~~$X = Z + Y$~~? $Z = X + Y$

$f_Z = f_X * f_Y \quad f_Z(z) = \int_{-\infty}^{\infty} f_X(s) f_Y(z-s) = \int_0^z \lambda e^{-\lambda s} \lambda e^{-\lambda(z-s)} ds$

$$\boxed{= \lambda^2 \cancel{z} e^{-\lambda z} \quad z > 0}$$

10. (5 points) If an alphabet with probabilities $p = [0.3, 0.3, 0.2, 0.1, 0.1]$ is encoded with codes $[0, 10, 110, 1110, 1111]$, what is the average length (in bits) of the code?

$EL = 0.3 \times 1 + 0.3 \times 2 + 0.2 \times 3 + 0.1 \times 4 + 0.1 \times 4$

$= 0.3 + 0.6 + 0.6 + 0.4 + 0.4 = \boxed{2.3 \ \frac{\text{bits}}{\text{symbol}}}$

11. (5 points) If $X \sim N(2, 4)$, what is $\Pr[X < 0]$?

$P(X < 0) = P\left(\frac{X-2}{2} < \frac{0-2}{2}\right) = \Phi(-1) = 1 - \Phi(1)$

$= 1 - 0.8413 = \boxed{0.1587}$

12. (5 points) If $X \sim U(-1, 3)$, what is $E(X)$?

$EX = \int_{-1}^{3} \frac{1}{4} x\, dx = \frac{1}{4} \frac{x^2}{2} \Big|_{-1}^{3} = \frac{9-1}{8} = \boxed{1}$

13. (20 points) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ are IID $N(0, \sigma^2)$ and let $\boldsymbol{R} = \sqrt{\boldsymbol{X}^2 + \boldsymbol{Y}^2}$ and $\Theta = \tan^{-1}(\boldsymbol{Y}/\boldsymbol{X})$ be a transformation to polar coordinates.What are the following?

   (a) $f_\Theta(\theta)$

   (b) $F_{\boldsymbol{R}}(r)$

   (c) $f_{\boldsymbol{R}}(r)$

   (d) What value of r causes $F_{\boldsymbol{R}}(r) = 0.5$? (This is the median.)

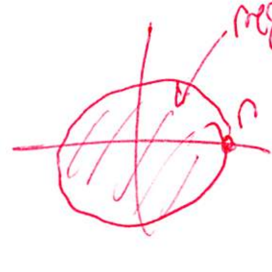
a). 2D Gaussian points are uniform in angle

$$f_{XY} = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$ depends on $r = \sqrt{x^2+y^2}$ and ~~not~~ not on Θ

$$\Rightarrow \Theta \sim U(0, 2\pi) \qquad f_\Theta(\theta) = \begin{cases} 1 & 0 < \theta < 2\pi \\ 0 & \text{o.w.} \end{cases}$$

b) $$F_R(r) = \int_0^{2\pi}\int_0^r \frac{1}{2\pi\sigma^2} e^{-s^2/2\sigma^2}\, s\, ds\, d\theta \qquad t = \frac{s}{\sigma}$$

$$= \int_0^{r/\sigma} t e^{-t^2/2}\, dt = \boxed{1 - e^{-r^2/2\sigma^2}} \qquad 0 < r < \infty$$

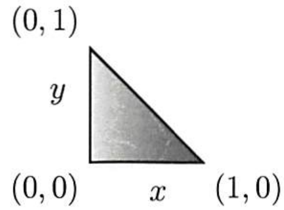


c) $$f_R(r) = \frac{d}{dr} F_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$$

d) $$F_R(r) = 0.5 = 1 - e^{-r^2/2\sigma^2} \Rightarrow e^{-r^2/2\sigma^2} = 0.5$$

$$-\frac{r^2}{2\sigma^2} = \log(0.5) = -\log 2 \Rightarrow \boxed{r = \sigma\sqrt{2 \log 2}}$$

13. (20 points) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ have density $f_{\boldsymbol{XY}}(x,y) = cx$ in the triangle below, and $f_{\boldsymbol{XY}}(x,y) = 0$ elsewhere, where c is an unknown constant.



What are the following?

(a) c

(b) $f_{\boldsymbol{X}}(x)$

(c) $E\boldsymbol{X}$

(d) $f_{\boldsymbol{Y}|\boldsymbol{X}=x}(y|\boldsymbol{X}=x)$

a) $1 = \int_0^1 \int_0^{1-x} cx\,dy\,dx = \int_0^1 cx(1-x)\,dx = c\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{c}{6}$

$\Rightarrow \boxed{c = 6}$

b) $f_X(x) = \int_0^{1-x} 6x\,dy = \boxed{6x(1-x)} \quad 0 < x < 1$

c) $E(X) = \int_0^1 x\, 6x(1-x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{6}{12} = \boxed{\frac{1}{2}}$

d) $f_{Y|X=x}(y|X=x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{6x}{6x(1-x)} = \boxed{\frac{1}{1-x} \quad 0 < y < 1-x}$