ELEG 810

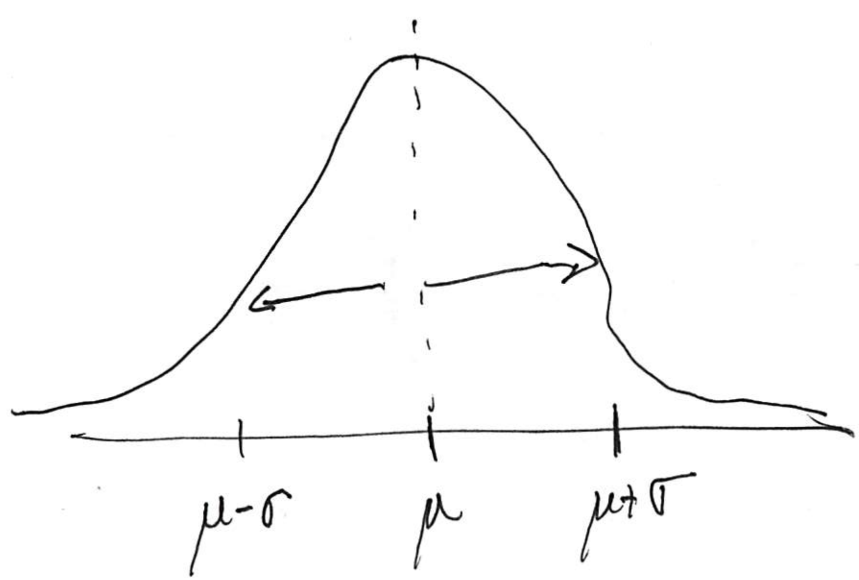
4/19/2018

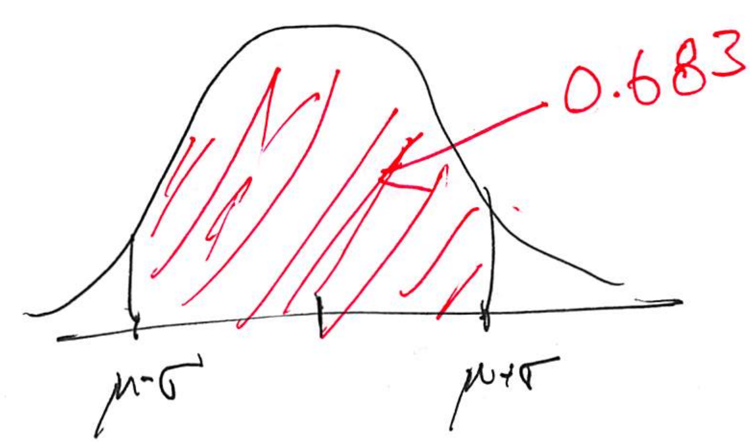
Midterm 2 Thursday May 3

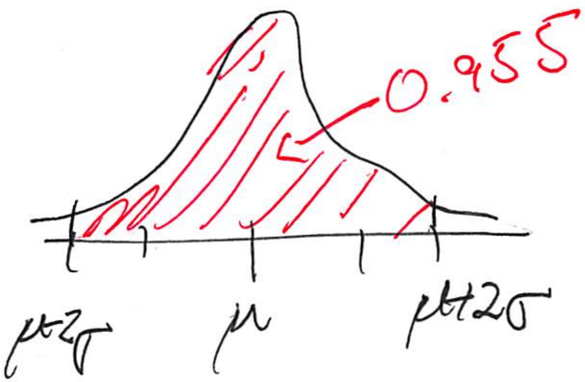
Covers Chapters 1-9

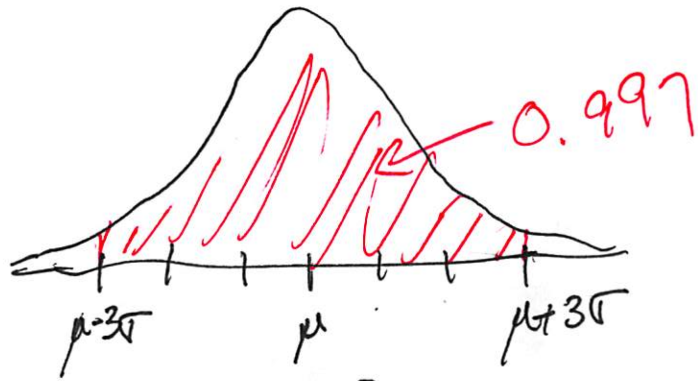
Gaussian Distribution - also Normal Distribution

$X \sim N(\mu, \sigma^2)$ X is Normal with mean μ and variance σ^2









$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$X \sim N(\mu, \sigma^2)$

$Z \sim N(0,1)$ "Standard Normal"

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

$$F_Z(z) = \int_{-\infty}^{z} f_Z(v)\,dv = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\,dv$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} > 0 \quad \text{for all } z$$

f_Z very smooth

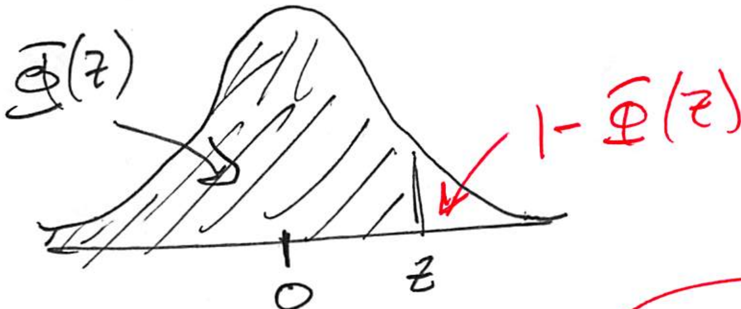
$$\int_{-\infty}^{\infty} f_Z(z)\,dz = 1$$

$\int_{-\infty}^{z} f_Z(v)\,dv$ cannot be computed in closed form

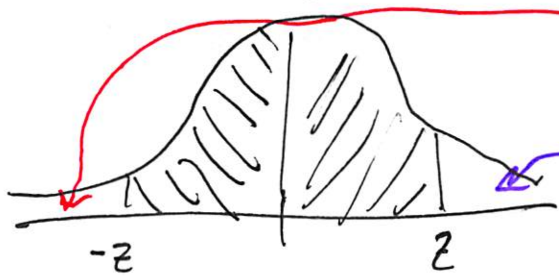
$$\Phi(z) = \int_{-\infty}^{z} f_Z(v)\,dv = P(Z \le z) = F_Z(z)$$

computed numerically

$$P(-2 \le Z \le 3) = \Phi(3) - \Phi(-2)$$



$$P(Z > z) = 1 - \Phi(z)$$



$$\Phi(-z) = 1 - \Phi(z)$$

$$P(Z \le -z) = P(Z \ge z)$$

Ex. $P(-2 \le Z \le 3) = \Phi(3) - \Phi(-2) = \Phi(3) + \Phi(2) - 1$

$X \sim N(\mu, \sigma^2)$ $\quad X = \sigma Z + \mu$

$$E(X) = E(\sigma Z + \mu) = \sigma E(Z) + \mu = \mu$$

$$Var(X) = Var(\sigma Z + \mu) = \sigma^2 \, Var\, Z = \sigma^2$$

$Z \sim N(0,1)$ $\quad X = \sigma Z + \mu \Rightarrow X \sim N(\mu, \sigma^2)$

$$P(X \le x) = F_X(x) = \int_{-\infty}^{x} f_X(v)\,dv = P(\sigma Z + \mu \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Moments: $Z \sim N(0,1)$

$0 = E(Z) = \cancel{1} = \int_{-\infty}^{\infty} z\,\phi(z)\,dz \qquad \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, z e^{-z^2/2}\,dz = \frac{-1}{\sqrt{2\pi}} e^{-z^2/2} \Big|_{-\infty}^{\infty} = 0 - 0 = 0$

$\int z e^{-z^2/2}\,dz = -e^{-z^2/2}$

$E(z^m) = 0 \quad m$ odd

$E(z^m) = (m-1)(m-3) \cdots 3 \cdot 1$

m even

$E(z^2) = 1$

$E(z^4) = 3$

$E(z^6) = 3 \times 5 = 15$

$Z_1 \; Z_2$ IID $N(0,1)$

$$f_{Z_1 Z_2}(v_1, v_2) = f_{Z_1}(v_1)\, f_{Z_2}(v_2) = \frac{1}{2\pi} e^{-\frac{(v_1^2+v_2^2)}{2}}$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Z_1 Z_2}(v_1, v_2)\, dv_2\, dv_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(v_1^2+v_2^2)}{2}}\, dv_2\, dv_1$$

$v_1 = r\sin\theta \qquad v_1^2 + v_2^2 = r^2$

$v_2 = r\cos\theta \qquad dv_2\, dv_1 = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2}\, r\,dr\,d\theta = 1$$

$X_1 \sim N(\mu_1, \sigma_1^2) \quad X_2 \sim N(\mu_2, \sigma_2^2) \quad Cov(X_1, X_2) = \sigma_{12}$

$$\cancel{f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right.}}$$

Let $U = \frac{X_1 - \mu_1}{\sigma_1}$ $\quad V = \frac{X_2 - \mu_2}{\sigma_2}$ $\quad \rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$

corellation coefficient

$-1 \leq \rho \leq 1$

$$f_{UV}(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right)}$$