CISC260 Machine Organization and Assembly Language

Data representation & arithmetic in binary

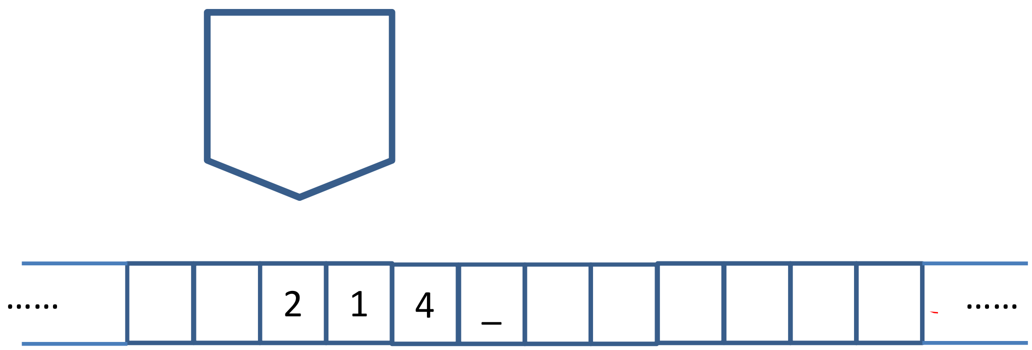
What is computing?

Y = f(x)

output   input

Turing Machine

States: Q, E, O, F
Symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, _
moves: →, ←, *



tape

input

Quintuples:

Q 0 : _ → E
Q 1 : _ → O
Q 2 : _ → E
Q 3 : _ → O
…

E 0 : _ → E
E 1 : _ → O
E 2 : _ → E
E 3 : _ → O
…

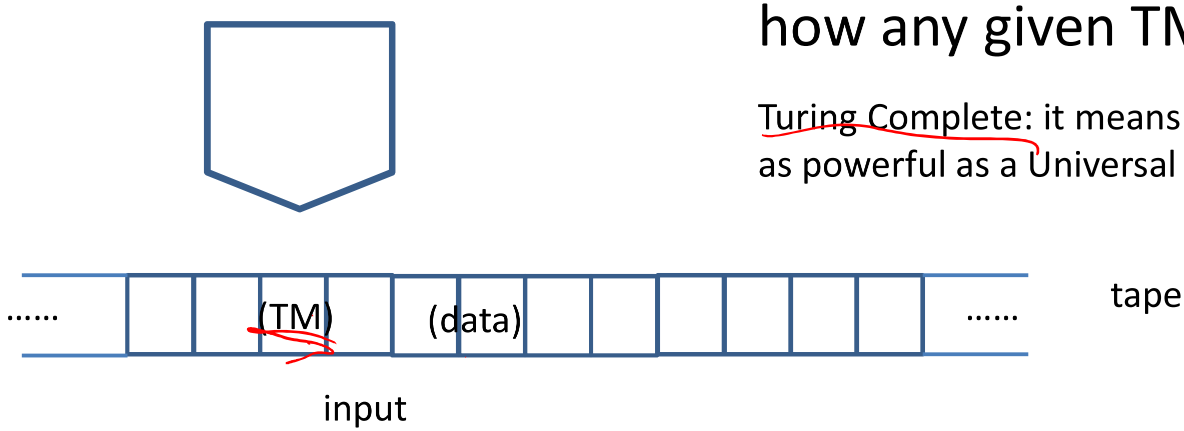
O 0 : _ → E
O 1 : _ → O
…

E _ : 0 * F
O _ : 1 * F

Universal Turing Machine

UTM is essentially an interpreter/simulator, that can simulate how any given TM runs on any data.

Turing Complete: it means a machine (programming language) is as powerful as a Universal Turing machine.



Big Idea: the description of any TM (whose quintuples are encoded in certain way) can be treated as data stored on the tape for UTM to interpret. This leads to the Stored-Program architecture (aka von Neumann architecture) used for all current digital computers.

Halting Problem: Can there be an algorithm (i.e., Turing machine) that is capable of predicting any TM halts with any given data?

How do we count?

Once we have developed the notion of numbers, how do we represent them, in an *economical* way?

To keep the size small (or rather constant), we can try to invent a unique symbol for each number. But there are infinite many numbers, we will run out of symbols.

Let's use compound symbols made from a finite set of atom symbols.

	hex	decimal	binary
	0	0	0
*	1	1	1
**	2	2	10
***	3	3	11
****	4	4	100
*****	5	5	101
******	6	6	110
*******	7	7	111
********	8	8	1000
*********	9	9	1001
**********	A	10	1010
***********	B	11	1011
************	C	12	1100
*************	D	13	1101
**************	E	14	1110
***************	F	15	1111

How economical?

Size of number 1000 in unary = 1000

Size of number 1000 in decimal = 4

Size of number 1000 in binary = 10

Size of number N in b-nary = $\log_b(N)$

Remember the big O you learned in CISC220!

So, both decimal and binary has "complexity" O(log N).

$$(9038)_{ten} = 9 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0 = 9038$$

$$(10011)_{two} = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 19$$

$$(x_n x_{n-1} \ldots x_0)_b = \sum_{i=0}^{n} x_i \cdot b^i$$

16's column
8's column
4's column
2's column
1's column

$$10110_2 = \underbrace{1 \times 2^4}_{\text{one sixteen}} + \underbrace{0 \times 2^3}_{\text{no eight}} + \underbrace{1 \times 2^2}_{\text{one four}} + \underbrace{1 \times 2^1}_{\text{one two}} + \underbrace{0 \times 2^0}_{\text{no one}} = 22_{10}$$

Positional notation (or place-value) numbering system (to support arithmetic operations)

Two bits can encode 4 numbers, why we have to map
00 ->0, 01-> 1, 10 -> 2, 11 -> 3
can we map
00->0, 10 -> 1, 01 ->3, 11 -> 2 ?
Probably yes. But when you mess up the meaning, it will require more twisted rules in order to make the arithmetic operations right (i.e., for input, operators, and output to be consistent with what they mean, as dictated by the mapping)

Use addition as example

Addition can be done by following a set of simple rules position by position.
Addition table

A	B	Sum	carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

```
    1011
+)  0101
--------
   10000
```

Addition table in decimal will be 10x10 = 100 rows
Tradeoff between efficiency and simplicity
Decimal: more efficient (requires fewer digits) but also more complicated (more rules)
Binary: less efficient (requires more digits) but also less complicated (fewer rules)

Table 1.2 **Hexadecimal number system**

Hexadecimal Digit	Decimal Equivalent	Binary Equivalent
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

256's column
16's column
1's column

$$2ED_{16} = \underbrace{2 \times 16^2}_{\text{two two hundred fifty six's}} + \underbrace{E \times 16^1}_{\text{fourteen sixteens}} + \underbrace{D \times 16^0}_{\text{thirteen ones}} = 749_{10}$$

Converting between hex and binary

- Hex to binary

  e.g., 0x8F7A93 = 100011110111101010010011

- Binary to Hex

  e.g., binary 1011011110011100 = 0xB79C

hex	binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

Conversion between decimal and binary

- Binary to decimal
  - Trivial. Use the formula: $$(x_n x_{n-1} \ldots x_0)_b = \sum_{i=0}^{n} x_i \cdot b^i$$

  example:

$$(10011)_{two} = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 19$$

-Decimal to binary

(A decimal number, dividing successively by 2 and prefixing the remainder to the result until a quotient of zero is obtained)

e.g., decimal number 19

19 / 2	=	9 remains 1	→ 1	(LSB)
9/2	=	4 remains 1	→ 1	
4/2	=	2 remains 0	→ 0	
2/2	=	1 remains 0	→ 0	
½	= 0	0 remains 1	→ 1	(MSB)

So the binary number is 10011

udel.edu

ASCII value	Character	Control character	ASCII value	Character	ASCII value	Character	ASCII value	Character
000	(null)	NUL	032	(space)	064	@	096	
001	☺	SOH	033	!	065	A	097	a
002	☻	STX	034	"	066	B	098	b
003	♥	ETX	035	#	067	C	099	c
004	♦	EOT	036	$	068	D	100	d
005	♣	ENQ	037	%	069	E	101	e
006	♠	ACK	038	&	070	F	102	f
007	(beep)	BEL	039	'	071	G	103	g
008	◘	BS	040	(	072	H	104	h
009	(tab)	HT	041	)	073	I	105	i
010	(line feed)	LF	042	*	074	J	106	j
011	(home)	VT	043	+	075	K	107	k
012	(form feed)	FF	044	,	076	L	108	l
013	(carriage return)	CR	045	-	077	M	109	m
014	♫	SO	046	.	078	N	110	n
015	☼	SI	047	/	079	O	111	o
016	►	DLE	048	0	080	P	112	p
017	◄	DC1	049	1	081	Q	113	q
018	↕	DC2	050	2	082	R	114	r
019	‼	DC3	051	3	083	S	115	s
020	¶	DC4	052	4	084	T	116	t
021	§	NAK	053	5	085	U	117	u
022	▬	SYN	054	6	086	V	118	v
023	↨	ETB	055	7	087	W	119	w
024	↑	CAN	056	8	088	X	120	x
025	↓	EM	057	9	089	Y	121	y
026	→	SUB	058	:	090	Z	122	z
027	←	ESC	059	;	091	[	123	{
028	(cursor right)	FS	060	<	092	\	124	¦
029	(cursor left)	GS	061	=	093	]	125	}
030	(cursor up)	RS	062	>	094	^	126	~
031	(cursor down)	US	063	?	095	_	127	⌂

01000001

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0		☺	☻	♥	♦	♣	♠					♂	♀		♫	☼
1	►	◄	↕	‼	¶	§	▬	↨	↑	↓	→	←	∟	↔	▲	▼
2		!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	¦	}	~	⌂
8	Ç	ü	é	â	ä	à	å	ç	ê	ë	è	ï	î	ì	Ä	Å
9	É	æ	Æ	ô	ö	ò	û	ù	ÿ	Ö	Ü	¢	£	¥	₧	ƒ
A	á	í	ó	ú	ñ	Ñ	ª	º	¿	⌐	¬	½	¼	¡	«	»
B	░	▒	▓	│	┤	╡	╢	╖	╕	╣	║	╗	╝	╜	╛	┐
C	└	┴	┬	├	─	┼	╞	╟	╚	╔	╩	╦	╠	═	╬	╧
D	╨	╤	╥	╙	╘	╒	╓	╫	╪	┘	┌	█	▄	▌	▐	▀
E	α	ß	Γ	π	Σ	σ	µ	τ	Φ	Θ	Ω	δ	∞	φ	ε	∩
F	≡	±	≥	≤	⌠	⌡	÷	≈	°	∙	·	√	ⁿ	²	■	

- Viewing a binary file as text file

>more

- Bits are just bits (with no inherent meaning)
  conventions define relationship between bits and numbers
- Binary numbers (base 2)
  0000 0001 0010 0011 0100 0101 0110 0111 1000 1001...
  decimal: $0,...,2^n-1$
- Of course it gets more complicated:
  numbers are finite (overflow)
  fractions and real numbers
  negative numbers
- How do we represent negative numbers?
  i.e., which bit patterns will represent which numbers?

Possible Representations

Sign Magnitude:	One's Complement	Two's Complement
000 = +0	000 = +0	000 = +0
001 = +1	001 = +1	001 = +1
010 = +2	010 = +2	010 = +2
011 = +3	011 = +3	011 = +3
100 = -0	100 = -3	100 = -4
101 = -1	101 = -2	101 = -3
110 = -2	110 = -1	110 = -2
111 = -3	111 = -0	111 = -1

- Issues: balance, number of zeros, ease of operations

Issues with sign-magnitude.

- Two zero's
- Arithmetic operations are cumbersome to implement.
- E.g. when adding two numbers, if the second operand is negative, ordinary addition will give wrong answer:

```
   2 + (-3)              5 + (-5) = 0

   010                    1101
   111                    0101
-----------            ------------
 1|001                 10010
```

- There is an overflow;
- The answer is wrong

Therefore, depending on the signs of the operands, different rules are needed to ensure the correct arithmetic operations.

Issues with one's complement.

- Arithmetic operations are cumbersome to implement.
- E.g. when adding two numbers, if the second operand is negative, ordinary addition may give wrong answer:

```
    2 + (-1)

    010
    110
-----------
  1|000
```

1), there is an overflow; 2) the answer is wrong
Therefore, depending on the signs of the operands, different rules are needed to ensure the correct arithmetic operations.

```
   subtraction:
      000     (0)
-)   001     (1)
---------------------
      111     (-0)
```

Motivation to two’s complement.

What has led to negative numbers? Subtraction. Let’s see the effect of borrowing.

```
    0000
 -  0001
------------
    1111
```

If we can borrow “out of the range”, this gives us negative one.
So 1111 here actually represents -1, as the result of 0 subtract by 1. This is two’s complement.

So when we add two integers, a + b, if b is negative, it is equivalent to

$$a - |b| = a + 0 - |b| = a + (0 - |b|) = a + b'',$$

where b” stands for b in two’s complement.

This shows:

1. addition can be done as usual for integers, both positive and negative, when represented in two’s complement.
2. subtraction can be done by negation (which is a simpler operation) and addition.

Now look at two's complement

Two's Complement

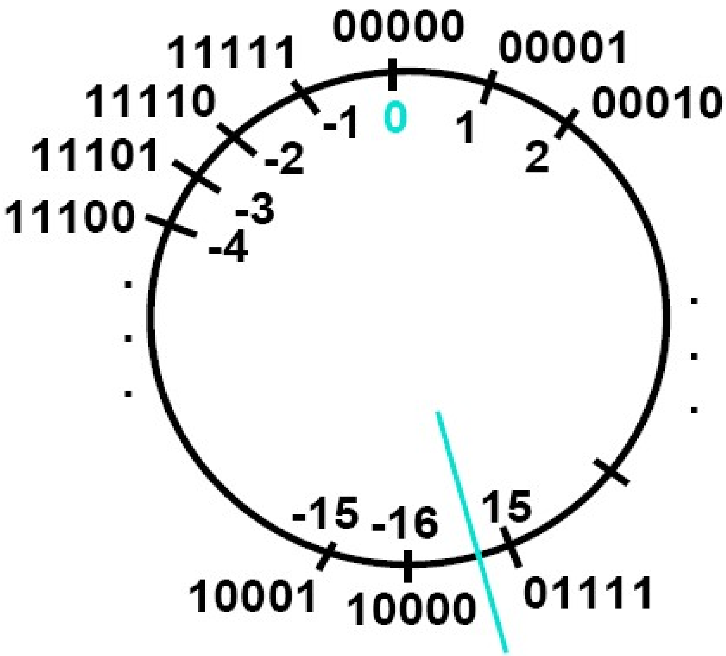
011 = +3
010 = +2
001 = +1
000 = 0
111 = -1
110 = -2
101 = -3
100 = -4 => left-most bit is sign bit but also contributes a value

$$A = d_2 (-2^2) + d_1 2^1 + d_0 2^0$$

Starting at 000, keep subtracting 1 to get negative numbers, and keep adding 1 to get positive numbers.

- Sign bit contribution: (-2^2) in this example, and -2^{N-1} for N-bit integers.
- There is one zero.
- Unbalanced: the most negative integer has no positive counterpart.
- Look at pairs (1, -1), (2, -2) … to discover patterns for negation.
- Check the correctness of arithmetic operations.

2's Complement Number "line": N = 5



- 2^{N-1} non-negatives
- 2^{N-1} negatives
- one zero
- how many positives?

CS 61C L02 Number Representation (16)

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Generalized to 32 bit (used in ARM)

$[-2^{N-1} \quad 0, \quad 2^{N-1}-1]$

32 bit signed numbers:

$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_{two} = 0_{ten}$
$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0001_{two} = +\ 1_{ten}$
$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two} = +\ 2_{ten}$
...
$0111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1110_{two} = +\ 2{,}147{,}483{,}646_{ten}$
$0111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111_{two} = +\ 2{,}147{,}483{,}647_{ten}$ — ***maxint***
$1000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_{two} = -\ 2{,}147{,}483{,}648_{ten}$ — ***minint***
$1000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0001_{two} = -\ 2{,}147{,}483{,}647_{ten}$
$1000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two} = -\ 2{,}147{,}483{,}646_{ten}$
...
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two} = -\ 3_{ten}$
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1110_{two} = -\ 2_{ten}$
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111_{two} = -\ 1_{ten}$

Two's Complement Formula

- Can represent positive and negative numbers in terms of the bit value times a power of 2:

  $d_{31} \times -2^{31} + d_{30} \times 2^{30} + \ldots + d_2 \times 2^2 + d_1 \times 2^1 + d_0 \times 2^0$

- Example: $1111\ 1100_{two}$

  $= 1 \times -2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$

  $= -2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 0 + 0$

  $= -128 + 64 + 32 + 16 + 8 + 4$

  $= -128 + 124$

  $= -4_{ten}$

Two's Complement Operations

- **Negating a two's complement number: invert all bits and add 1.**

Why?

For a number X, let X’ be its invert, then X + X’ = 11…1 = -1_{ten}

Therefore, X’ + 1 = - X

E.g., X = 5 in 8-bit

X	=	0000 0101
X’	=	1111 1010
X’ +1	=	1111 1011

  – Special case1: negate 0 should still give you 0 (by ignoring carry out at the leftmost bit)
  – Special case2: negate the most negative number, say 1111 in 4-bit, you still get the same number since the most negative number does not have a positive counterpart in two’s complement – its range is asymmetric!
  – remember: for numbers in two’s complement, **“negate” and “invert” are quite different!**

Addition & Subtraction

- Just like in grade school (carry/borrow 1s)

```
  0111 (7)       0111 (7)       0110 (6)
+ 0110 (6)     - 0110 (6)     - 0111 (7)
----------------------------------------
  1101           0001 (1)       1111 (-1)
```

- Two's complement operations easy
  - subtraction using addition of negative numbers, and unlike the above example, here the sign is automatically taken care of.

```
  0111 (7)           0110     (6)
+ 1010 (-6)        + 1001     (-7)
-----------        -------------
   0001               1111
```

More examples

Overflow is not indicated by spilling over a bit out of the range; rather it is indicated by sign bit flipping: both operands have the same sign bit, but the result gets an opposite sign bit.

1001 = −7 +0101 = 5 1110 = −2 (a) (−7) + (+5)	1100 = −4 +0100 = 4 10000 = 0 (b) (−4) + (+4)
0011 = 3 +0100 = 4 0111 = 7 (c) (+3) + (+4)	1100 = −4 +1111 = −1 11011 = −5 (d) (−4) + (−1)
0101 = 5 +0100 = 4 1001 = Overflow (e) (+5) + (+4)	1001 = −7 +1010 = −6 10011 = Overflow (f) (−7) + (−6)

Overflow for addition and subtraction of signed integers in two's complement

Operation	Operand A	Operand B	Result indicating overflow
$A + B$	≥ 0	≥ 0	< 0
$A + B$	< 0	< 0	≥ 0
$A - B$	≥ 0	< 0	< 0
$A - B$	< 0	≥ 0	≥ 0

Sign extension: If you get larger size to represent numbers, say from 16bit to 32 bit, how do you migrate? It is simple, you just pad these new bits with zeros or with ones as decided by the sign bit. This is sign extension.

Proof of the sign extension for negative numbers:

In b+1 bit representation

$$Z = -1x\,(2)^b + \overbrace{z_{b-1}\,x\,(2)^{b-1} + \ldots + z_0}^{y}$$

$$= -\,(2)^b + y$$

Extended to B+1 bit with B > b:

$Z' = -1x\,(2)^B + (2)^{B-1} + \ldots\ (2)^{b+1} + (2)^b + y.$

It is easy to see that

$$Z' - Z = -1x\,(2)^B + (2)^{B-1} + \ldots\ (2)^{b+1} + (2)^b + y + (2)^b - y$$

$$= -1x\,(2)^B + (2)^{B-1} + \ldots\ (2)^{b+1} + \underbrace{(2)^b + (2)^b}_{(2)^{b+1}}$$

Starting from right end, the two neighboring terms are identical and summing them will give a new term identical the third term. Repeating this process till the n

Multiplication

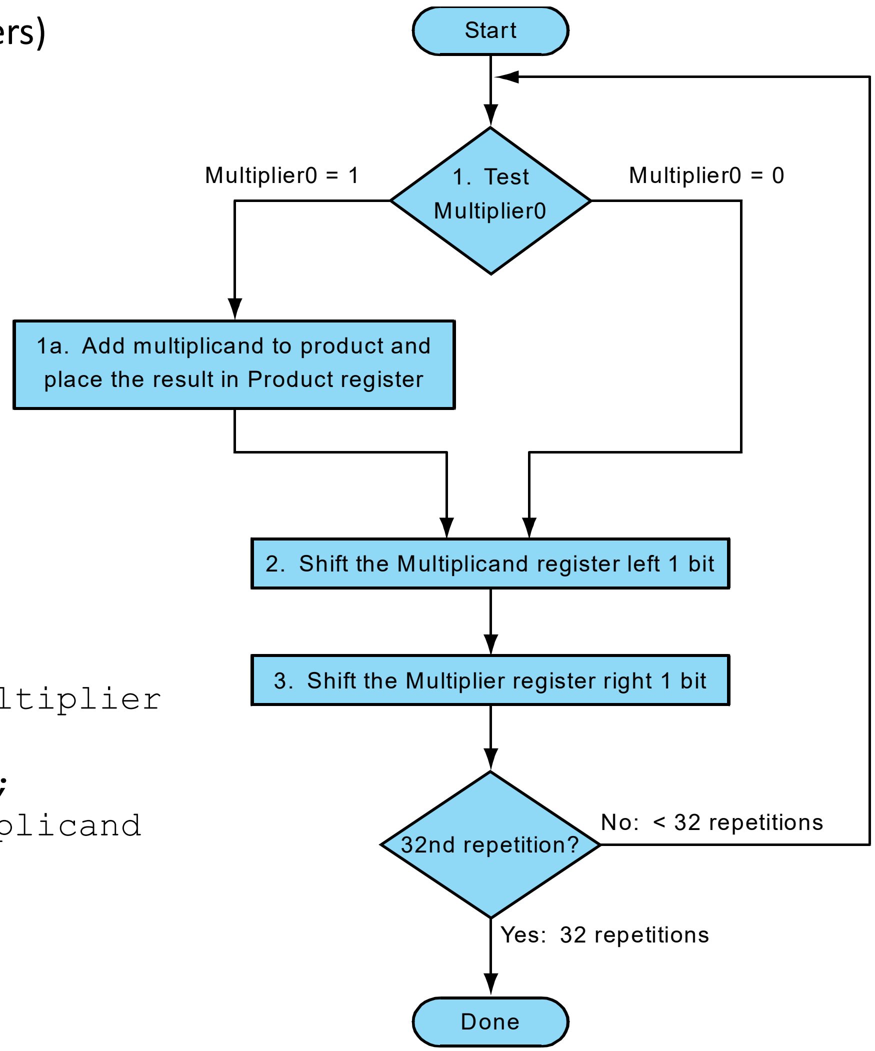
- More complicated than addition
  - accomplished via shifting and addition
- More time
- Let's look at a grade school (bit-shifting) algorithm

```
      0010   (multiplicand)
    x 1011   (multiplier)
  ________
```

Multiplication (for **unsigned** integers)

```
       0010    (multiplicand)
     x 1011    (multiplier)
     ------
       0010
      0010
     0000
    0010
-----------
    0010110    (product)
```

```
product  = 0
while (b != 0) { // b is multiplier
   if(b & 0x01) == 1
      product = product + a;
              // a is multiplicand
   a = a << 1;
   b = b >> 1;
}
return product;
```



What if either operant is negative?

- Two's complement works well for addition and subtraction—how about multiplication?

  Example (numbers interpreted in two's complement):

$$
\begin{array}{rl}
1011 & (-5) \\
\times\ 0111 & (7) \\
\hline
1011 & \\
1011 & \\
1011 & \\
+\,0000 & \\
\hline
01001101 & (77?)
\end{array}
$$

  – This multiplication scheme fails if either operand is negative

Multiplication of signed integers

- Covert the multiplicand and multiplier to positive numbers and remember the original signs
- Run multiplication algorithm (for 31 iterations) leaving the signs out of the calculation
- Negate the product if the original signs disagree (sign extension may be involved)

Multiplication (for signed integers)

```
product  = 0
sign_a = 1;
sign_b = 1;
if(a < 0) sign_a = -1; a = -a;
if(b < 0) sign_b = -1; b = -b;
while (b != 0) { // b is multiplier
   if(b & 0x01) == 1
      product = product + a;
              // a is multiplicand
   a = a << 1;
   b = b >> 1;
}
If (sign_a != sign_b) product = -product;
return product;
```

Multiplication with two's complement?

- Why does multiplication fail?
- Remember: the multiplication of two n-bit numbers requires $2n$-bit additions to add the partial products
  - This is what actually happens:

$$
\begin{array}{rl}
1\,0\,1\,1 & (-5) \\
\times\,0\,1\,1\,1 & (7) \\
\hline
0\,0\,0\,0\,1\,0\,1\,1 & ☡ \\
0\,0\,0\,1\,0\,1\,1\,0 & ☡ \\
0\,0\,1\,0\,1\,1\,0\,0 & ☡ \\
+\,0\,0\,0\,0\,0\,0\,0\,0 & \\
\hline
0\,1\,0\,0\,1\,1\,0\,1 & (77\,☡)
\end{array}
$$

  - Note: interpreted as 8-bit numbers, the shifted numbers in the ☡ lines are *positive* (and not *negative* as expected)

Multiplication with two's complement?

- To **extend** a n-bit two's complement number to m bits ($m > n$), pad the number on the left; the new $m - n$ bits have the **same value as the sign bit**

  Example (extend 4-bit to 8-bit number, two's complement):

$$-5_{10} = \underset{\uparrow}{1}011_2 = \underbrace{1111}_{\text{new}}1011_2$$

- 4-bit multiplication (corrected 8-bit extension):

$$\begin{array}{rl}
1011 & (-5) \\
\times\, 0111 & (7) \\
\hline
11111011 & \\
11110110 & \\
11101100 & \\
+\,00000000 & \\
\hline
11011101 & (-35)
\end{array}$$

Do an example when the multiplier is also negative and show how the sign bit should be treated specially: the intermediate product from the sign bit should be subtracted from the total. With the consecutive-one approach, the sign-bit is taken care of automatically.

Multiplication (for signed integers in two's complement)
Example when multiplier is negative

```
        0010    (2)
    x   1011    (-5)
    --------
    00000010
    00000100
    00000000
+)  00010000
    --------
    00010110    (22)
```

```
        0010    (2)
    x   1011    (-5)
    --------
    00000010
    00000100
    00000000
+) -00010000
    --------
```

This last row is contributed by the leftmost bit (i.e., sign bit of the multiplier (-5), which is negative, and therefore should be subtracted rather added to the final product.

This subtraction can be implemented by negation first and addition.

-00010000 = 11110000

```
        0010    (2)
    x   1011    (-5)
    --------
    00000010
    00000100
    00000000
+)  11110000
    --------
    11110110    (-10)
```

Consecutive one

C = A x B

76543210
B = 00111100
= $2^5 + 2^4 + 2^3 + 2^2$

C = A x $(2^5 + 2^4 + 2^3 + 2^2)$

```
           aaaaaaaa   (A)
         x 00111100   (B)
         ----------
           00000000
          00000000
         aaaaaaaa
        aaaaaaaa
       aaaaaaaa
      aaaaaaaa
     00000000
+   00000000
-------------------------
```

A x $(2^5 + 2^4 + 2^3 + 2^2)$ (C)

Consecutive one

```
          aaaaaaaa   (A)
       x  00111100   (B)
          --------
          00000000
         00000000
        aaaaaaaa
       aaaaaaaa
      aaaaaaaa
     aaaaaaaa
    00000000
+) 00000000
```

$A \times (2^5 + 2^4 + 2^3 + 2^2)$ (C)

$C = A \times B$

```
     76543210
B  = 00111100
```

$= 2^5 + 2^4 + 2^3 + 2^2$

$B + 2^2 = 2^5 + 2^4 + 2^3 + 2^2 + 2^2$
$= 2^5 + 2^4 + 2^3 + 2^3$
$= 2^5 + 2^4 + 2^4$
$= 2^5 + 2^5$
$= 2^6$

$B = 2^6 - 2^2$

$C = A \times (2^6 - 2^2)$

```
          aaaaaaaa   (A)
       x  00111100   (B)
          --------
          00000000
         00000000
       -aaaaaaaa
       00000000
      00000000
     00000000
    aaaaaaaa
+) 00000000
```

$A2^6 - A2^2$ (C)

Observation: A left shifted 6 positions and then added to C; A left shifted 2 positions and then subtracted from C

Rules: Enter a "consecutive-one" block, subtract the shifted A from product;
Within the block, just shift A (without adding to C)
Exit the block, add the shifted A to the product

“consecutive-one” taking care of the sign bit for multiplier automatically.

C = A x B

$$\begin{array}{ll} & 76543210 \\ B & = 10111100 \\ & = -2^7 + 2^5 + 2^4 + 2^3 + 2^2 \end{array}$$

$B = -2^7 + (2^6 - 2^2)$

$C = A \times (-2^7) + A \times (2^6 - 2^2)$

$B_0 \; B_{-1}$

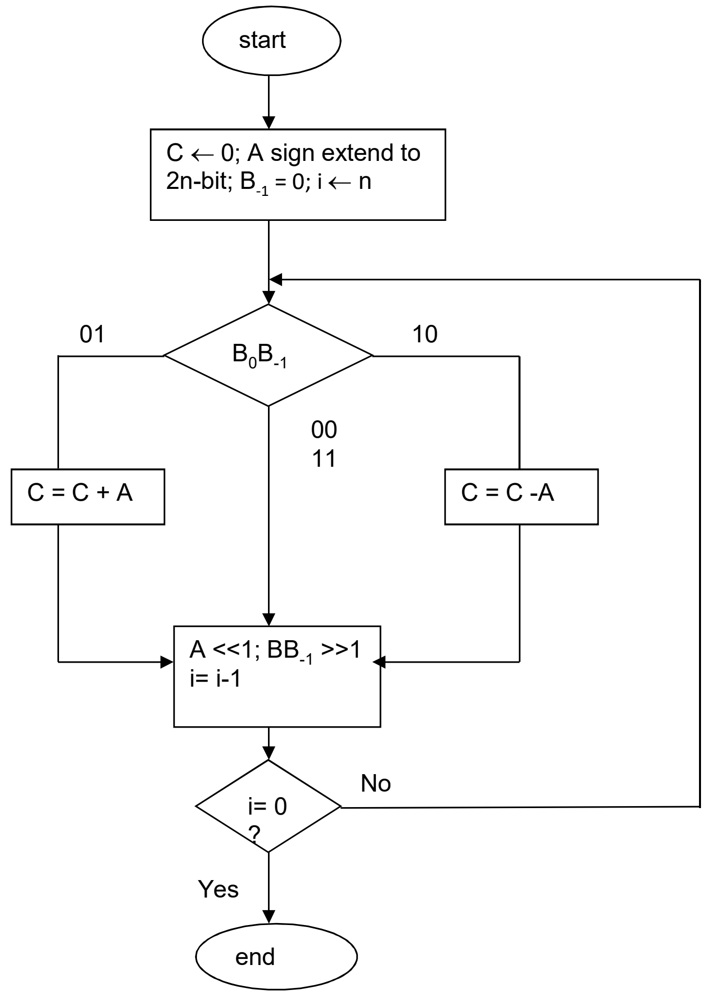
```
           aaaaaaaa    (A)
        x  101111000   (B)
        ____________
           00000000
          00000000
        -aaaaaaaa
        00000000
       00000000
      00000000
     aaaaaaaa
+  -aaaaaaaa
_______________________
   A(-2^7) + A2^6 - A2^2    (C)
```

Ask: What if this bit is also 1? Then the whole consecutive-block is just sign ext of a multiplier 100.

The sign bit for a negative multiplier is taken care of automatically in Booth algorithm.

Booth Algorithm

A x B = C
A: multiplicand, n-bit
B: multiplier, n-bit
C: product, 2n-bit



A = 1011
B = 0111

C = A x B

i	**A**	**B**	**B_{-1}**	**C**	**actions**
4	1111 1011	0111	0	0000 0000	B_0B_{-1} = "10" entering c1 block
4	1111 1011	0111	0	0000 0101	C = C-A = 5
4	1111 0110	0011	1	0000 0101	A <<1, BB_{-1} >>1
3	1111 0110	0011	1	0000 0101	B_0B_{-1} = "11", inside c1 block
3	1110 1100	0001	1	0000 0101	A <<1, BB_{-1}>>1, i--
2	1110 1100	0001	1	0000 0101	B_0B_{-1} = "11", inside c1 block
2	1101 1000	0000	1	0000 0101	A>>1, BB_{-1} >>1, i--
1	1101 1000	0000	1	1101 1101	B_0B_{-1} = "01", exiting c1 block, C = C+A = -35
1	1011 0000	0000	0	1101 1101	A>>1, BB_{-1} >>1, i--
0	stop				

A more space efficient implementation

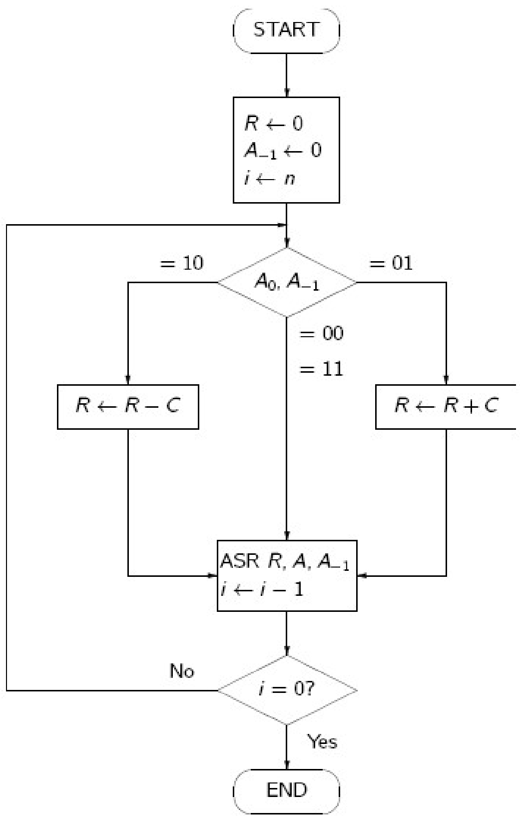
- **Booth's algorithm** is capable of multiplying two's complement numbers; multiplicand and/or multiplier may be negative
  - As before, two n-bit numbers are multiplied to yield a $2n$-bit result
  - Input: $A = A_{n-1}A_{n-2}\ldots A_1A_0, C$ (n-bit numbers)
  - Output: $\underbrace{RA}_{2n\text{-bit}} = A \times C$
  - Booth's algorithm uses an operation ASR (**arithmetic shift right**) which behaves *almost* like a right bit shift: the sign bit is *preseverd*
    Example (right shift and ASR of 4-bit number):

$$\underset{\rightarrow\rightarrow\rightarrow}{1\,0\,1\,1} \underset{\text{right shift}}{\rightarrow} \mathbf{0}101 \qquad \underset{\circlearrowleft\rightarrow\rightarrow}{\mathbf{1}\,0\,1\,1} \underset{\text{ASR}}{\rightarrow} \mathbf{1}101 \qquad \underset{\circlearrowleft\rightarrow\rightarrow}{\mathbf{0}\,0\,1\,1} \underset{\text{ASR}}{\rightarrow} \mathbf{0}001$$

  - In the algorithm, ASR is applied to 3 "connected" registers:

$$\underbrace{\mathbf{1}011}_{R} \curvearrowright \underbrace{0010}_{A} \curvearrowright \underbrace{1}_{A_{-1}} \quad \xrightarrow[\text{ASR } R, A, A_{-1}]{} \quad \underbrace{\mathbf{1}101}_{R}\ \underbrace{1001}_{A}\ \underbrace{0}_{A_{-1}}$$

Booth's algorithm



$A = 1011_2 = -5_{10} \qquad C = 0111_2 = 7_{10}$

i	R	A	A_{-1}	C	**Remark**
4	0000	1011	0	0111	$A_0A_{-1} = 10$
4	1001	1011	0	0111	$R \leftarrow R - C$
4	1100	1101	1	0111	ASR
3	1100	1101	1	0111	$A_0A_{-1} = 11$
3	1110	0110	1	0111	ASR
2	1110	0110	1	0111	$A_0A_{-1} = 01$
2	0101	0110	1	0111	$R \leftarrow R + C$
2	0010	1011	0	0111	ASR
1	0010	1011	0	0111	$A_0A_{-1} = 10$
1	1011	1011	0	0111	$R \leftarrow R - C$
1	1101	1101	1	0111	ASR
0	1101	1101	1	0111	END

$$RA \quad = \quad 1101\,1101_2 \quad = \quad -35_{10}$$