ELEG 305
SOLUTIONS TO EXAM #2 (4/18/17)

#1. $x(t) = \sin\frac{\pi}{4}t + \cos\frac{5\pi}{4}t$

a) The fundamental period, T, is the time interval over which $x(t)$ repeats, i.e.,

$$x(t) = x(t+T)$$

or, we can determine the fundamental radian frequency ω_0, and $T = 2\pi/\omega_0$

In this case, it is clear that $\omega_0 = \pi/4$, and

$$T = \frac{2\pi}{\omega_0} = 8$$

b) $x(t) = \sin\omega_0 t + \cos 5\omega_0 t$

The Fourier Series coefficients for this signal can easily be found by inspection.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$= \frac{1}{2j}e^{j\frac{\pi}{4}t} - \frac{1}{2j}e^{-j\frac{\pi}{4}t} + \frac{1}{2}e^{j\frac{5\pi}{4}t} + \frac{1}{2}e^{-j\frac{5\pi}{4}t}$$

$\therefore$
$a_1 = \frac{1}{2j}$
$a_{-1} = -\frac{1}{2j}$ $\qquad \omega_0 = \frac{\pi}{4}$
$a_5 = \frac{1}{2}$
$a_{-5} = \frac{1}{2}$
and all other $a_k = 0$

#1 (cont'd) c.) b_k = Fourier Series coefficients of $y(t)$
$= a_k H(jk\omega_0)$

The filter cuts off anything with frequency lower than π. So, a_1 and a_{-1} will be zeroed - the $\sin\frac{\pi}{4}t$ term will not pass through the filter.

$$b_5 = a_5 H\left(j\frac{5\pi}{4}\right)$$
$$= \frac{1}{2}\left(\frac{5\pi}{4} - \frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{8}$$

$$b_{-5} = a_{-5} H\left(-j\frac{5\pi}{4}\right)$$
$$= \frac{1}{2}\left(\frac{5\pi}{4} - \frac{\pi}{2}\right) = \frac{3\pi}{8}$$

all other b_k's are zero.

d.) $$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{\pi}{4}t}$$
$$= \frac{3\pi}{8} e^{j\frac{5\pi}{4}t} + \frac{3\pi}{8} e^{-j\frac{5\pi}{4}t}$$
$$= \frac{3\pi}{4}\cos\frac{5\pi}{4}t$$

#2 a.) $$y(t) = \overbrace{\beta x(t-2t_0)}^{x_1(t)} + \overbrace{2\beta x(t) * x(t) * x(t)}^{x_2(t)} + \underbrace{\delta(t+2t_0)}_{x_3(t)} + \underbrace{d^2x(t)/dt^2}_{x_4(t)}$$

$$\mathcal{F}[y(t)] = Y(j\omega) = X_1(j\omega) + X_2(j\omega) + X_3(j\omega) + X_4(j\omega)$$

2a. cont'd)

$$X_1(j\omega) = \beta X(j\omega) e^{-j\omega t_0}$$

$$X_2(j\omega) = 2\rho X^3(j\omega)$$

$$X_3(j\omega) = e^{j\omega 2 t_0}$$

$$X_4(j\omega) = (j\omega)^2 X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\,dt, \quad a>0$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt = \frac{1}{a+j\omega}$$

$$\therefore\ Y(j\omega) = \beta \frac{e^{-j\omega t_0}}{a+j\omega} + 2\rho \frac{1}{(a+j\omega)^3} + e^{j\omega 2 t_0} + \frac{(j\omega)^2}{a+j\omega}$$

b.) $$x[n] = \underbrace{\left(\left(-\tfrac{1}{2}\right)^n u[n]\right)}_{x_1[n]} * \underbrace{\left(\tfrac{1}{4}\right)^{-n} u[n]}_{x_2[n]}$$

$$X(e^{j\omega}) = X_1(e^{j\omega}) X_2(e^{j\omega})$$

- $$X_1(e^{j\omega}) = \mathcal{F}\left\{\left(-\tfrac{1}{2}\right)^n u[n]\right\} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n e^{-j\omega n} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

- $$X_2(e^{j\omega}) = \mathcal{F}\left\{\left(\tfrac{1}{4}\right)^{-n} u[-n]\right\} = \sum_{n=-\infty}^{0} \left(\frac{1}{4}\right)^{-n} e^{-j\omega n} \quad \text{let } m=-n$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{4}e^{j\omega}\right)^m = \frac{1}{1-\frac{1}{4}e^{j\omega}}$$

#2b. cont'd $\therefore X(e^{j\omega}) = \dfrac{1}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{j\omega})}$

#3. a.) (i) $X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\therefore X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n](-1)^n$$

$$= \cancel{(1)(-1)} + \cancel{(-2)(1)} + \cancel{(2)(-1)} + \cancel{(1)(1)}$$

$$+ \cancel{(-2)(-1)} + \cancel{(2)(4)} + (2)(-1)$$

$$+ (-1)(1) + (2)(-1)$$

$$= -5$$

(ii) $x[n] = \dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$

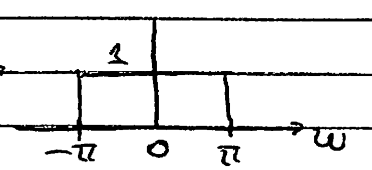
$$\therefore \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}\,d\omega = 2\pi\, x[n]\Big|_{n=1}$$

$$= 2\pi(2) = 4\pi$$

b.) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 \pi t}{\pi t^2}\,dt = ?$

Use Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$$

$x(t) = \dfrac{\sin \pi t}{\pi t} \xleftrightarrow{\mathcal{F}} X(j\omega) =$ 

$$\therefore \pi\int_{-\infty}^{\infty} \frac{\sin^2 \pi t}{\pi^2 t^2}\,dt = \frac{\pi}{2\pi}\int_{-\pi}^{\pi} 1\,d\omega = \pi\,\frac{\cancel{2\pi}}{\cancel{2\pi}} = \pi$$

#3. cont'd) d.) $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{-e^{-j\omega}+3}{1+\frac{1}{3}e^{-j\omega}-\frac{1}{3}e^{-2j\omega}}$

$$Y(e^{j\omega})\left(1+\frac{1}{3}e^{-j\omega}-\frac{1}{3}e^{-2j\omega}\right)$$

$$= X(e^{j\omega})(-e^{-j\omega}+3)$$

$$\Downarrow \mathcal{F}^{-1}$$

$$y[n]+\frac{1}{3}y[n-1]-\frac{1}{3}y[n-2] = 3x[n]-x[n-1]$$

#4. $$H(j\omega) = \frac{-(j\omega+1)}{-\omega^2+7j\omega+12}$$

a.) $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$

$$H(j\omega) = \frac{-j\omega-1}{(j\omega)^2+7j\omega+12} = \frac{-j\omega-1}{(j\omega+3)(j\omega+4)}$$

$$= \frac{A}{j\omega+3} + \frac{B}{j\omega+4}$$

$$A = (j\omega+3)H(j\omega)\Big|_{j\omega=-3} = \frac{-j\omega-1}{j\omega+4}\Big|_{j\omega=-3} = \frac{2}{1} = 2$$

$$B = (j\omega+4)H(j\omega)\Big|_{j\omega=-4} = \frac{-j\omega-1}{j\omega+3}\Big|_{j\omega=-4} = \frac{3}{-1} = -3$$

$$\therefore H(j\omega) = \frac{2}{j\omega+3} - \frac{3}{j\omega+4}$$

$$\Downarrow \mathcal{F}^{-1}$$

$$h(t) = 2e^{-3t}u(t) - 3e^{-4t}u(t)$$

#4. cont'd) b.) $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{-j\omega - 1}{(j\omega)^2 + 7j\omega + 12}$

$$Y(j\omega)\left((j\omega)^2 + 7j\omega + 12\right) = X(j\omega)(-j\omega - 1)$$

$$\Downarrow \mathcal{F}^{-1}$$

$$\frac{d^2 y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = -\frac{dx(t)}{dt} - x(t)$$

#5. $x(t) \rightarrow \boxed{\text{Earth}} \rightarrow y(t)$

- $x(t) = \delta(t)$

  $y(t) =$ impulse response, $h(t)$

  $= \frac{3}{2}e^{-4t}u(t) + \frac{3}{2}e^{-2t}u(t)$

a.) frequency response $H(j\omega) = \mathcal{F}\{h(t)\}$

$$= \frac{3}{2}\frac{1}{j\omega+4} + \frac{3}{2}\frac{1}{j\omega+2}$$

$$= \frac{3}{2}\left(\frac{j\omega+2+j\omega+4}{(j\omega+2)(j\omega+4)}\right)$$

$$= \frac{3}{\cancel{2}} \cdot \frac{\cancel{2}(j\omega+3)}{(j\omega+2)(j\omega+4)}$$

$$H(j\omega) = \frac{3(j\omega+3)}{(j\omega+2)(j\omega+4)}$$

#5. cont'd) b.) $y(t) = 2e^{-t}u(t) - 2e^{-4t}u(t)$

$x(t) = ?$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$\therefore X(j\omega) = \frac{Y(j\omega)}{H(j\omega)}$$

- $$H(j\omega) = \frac{3(j\omega+3)}{(j\omega+2)(j\omega+4)}$$

- $$Y(j\omega) = \mathcal{F}\{y(t)\} = \frac{2}{j\omega+1} - \frac{2}{j\omega+4}$$

$$= 2\,\frac{j\omega+4-j\omega-1}{(j\omega+1)(j\omega+4)}$$

$$= \frac{6}{(j\omega+1)(j\omega+4)}$$

- $$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = \frac{\cancel{6}^{\,2}}{(j\omega+1)\cancel{(j\omega+4)}} \cdot \frac{\cancel{(j\omega+4)}(j\omega+2)}{\cancel{3}(j\omega+3)}$$

$$= \frac{2(j\omega+2)}{(j\omega+1)(j\omega+3)}$$

$$= \frac{A}{j\omega+1} + \frac{B}{j\omega+3}$$

$$A = X(j\omega)(j\omega+1)\Big|_{j\omega=-1} = \frac{2(j\omega+2)}{j\omega+3}\Big|_{j\omega=-1} = \frac{2}{2} = 1$$

$$B = X(j\omega)(j\omega+3)\Big|_{j\omega=-3} = \frac{2(j\omega+2)}{j\omega+1}\Big|_{j\omega=-3} = \frac{-2}{-2} = 1$$

$$\therefore X(j\omega) = \frac{1}{j\omega+1} + \frac{1}{j\omega+3}$$

#5.b cont'd

$$\Downarrow \mathcal{F}^{-1}$$

$$x(t) = e^{-t}u(t) + e^{-3t}u(t)$$

Extra Credit

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \qquad Y(j\omega) = H(j\omega)X(j\omega)$$

$$\longrightarrow \boxed{h^{inv}(t)} \longrightarrow x(t)$$

if we want to recover $x(t)$ simply multiply $Y(j\omega)$ by $1/H(j\omega)$

$$\therefore H^{inv}(j\omega) = \frac{1}{H(j\omega)} = \frac{(j\omega+4)(j\omega+2)}{3(j\omega+3)}$$