SOLUTION TO HOMEWORK #4

Problem #1

(a) The signal $x(t) = 3\cos(\frac{\pi t}{2} + \frac{\pi}{4})$ has fundamental radian frequency $\omega_0 = \frac{\pi}{2}$ and hence its fundamental period is $T = \frac{2\pi}{\omega_0} = 4$. Using Euler's Relation, $x(t)$ can be expressed in terms of complex exponentials as

$$\begin{aligned} x(t) &= 3\cos(\frac{\pi t}{2} + \frac{\pi}{4}) \\ &= \frac{3}{2}\left(e^{\frac{j\pi t}{2}} e^{\frac{j\pi}{4}} + e^{-\frac{j\pi t}{2}} e^{-\frac{j\pi}{4}}\right) \\ &= \left(\frac{3}{2} e^{\frac{j\pi}{4}}\right) e^{\frac{j\pi t}{2}} + \left(\frac{3}{2} e^{-\frac{j\pi}{4}}\right) e^{-\frac{j\pi t}{2}} \end{aligned}$$

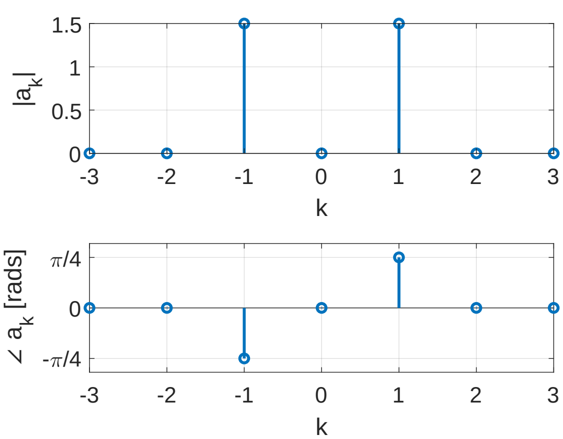
Note that $x(t)$ has a Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{2}t} = \ldots + a_{-1}e^{-j\frac{\pi}{2}t} + a_0 + a_1 e^{j\frac{\pi}{2}t} \ldots$$

So, if we equate terms, by inspection, the Fourier coefficients are

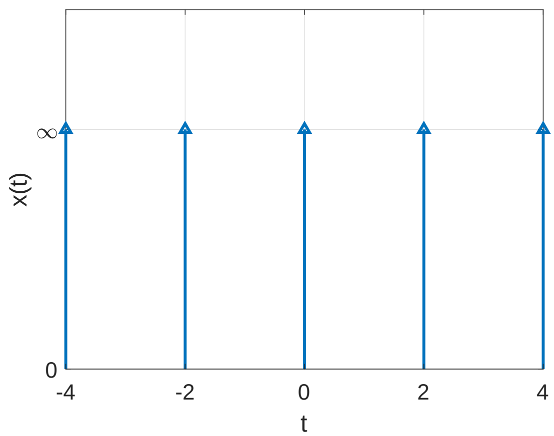
$$a_k = \begin{cases} \frac{3}{2}e^{\frac{j\pi}{4}}, & k = 1 \\ \frac{3}{2}e^{-\frac{j\pi}{4}}, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

The magnitude of the coefficients $|a_k|$ and the phase $\angle a_k$ are



where the symbol $\angle$ means angle.

(b) (i) The signal $x(t) = \sum_{m=\infty}^{\infty} \delta(t - 2m)$, shown below,



This signal has fundamental period $T = 2$, so $\omega_0 = \frac{2\pi}{2} = \pi$. The Fourier coefficients are computed as

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

where the integration is over a period T. For this signal, it is convenient to integrate from $-\frac{T}{2}$ to $\frac{T}{2}$. Therefore,

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{-1}^{1}\delta(t)e^{-jk\pi t}dt \\
&= \frac{1}{2}e^{-jk\pi 0}\int_{-1}^{1}\delta(t)dt = \frac{1}{2}\cancelto{1}{\int_{-1}^{1}\delta(t)dt} = \frac{1}{2}, \text{ for all } k
\end{aligned}$$

(ii) The signal $x(t) = e^{-2t}, 0 \le t \le 2$, has period $T = 2$, and hence $\omega_0 = \pi$. The Fourier coefficients can be calculated as

$$\begin{aligned}
a_k &= \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\omega_0 t}dt = \frac{1}{2}\int_0^2 \delta(t)e^{-jk\pi t}dt \\
&= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}dt = \frac{1}{2}\left[\frac{1}{-(2+jk\pi)}\right]e^{-(2+jk\pi)t}\Big|_0^2 \\
&= \frac{1}{2}\frac{(e^{-(2+jk\pi)2}-1)}{-(2+jk\pi)} \\
&= \frac{1}{2}\frac{(1-e^{-4}\cancelto{1}{e^{-jk2\pi}})}{(2+jk\pi)} = \frac{1-e^{-4}}{2(2+jk\pi)}
\end{aligned}$$

Problem #2

(a) The signal $x[n] = 1 + \sin(\frac{n\pi}{12} + \frac{3\pi}{8})$ has fundamental radian frequency $\omega_0 = \frac{\pi}{12}$ and hence its fundamental period is $N = \frac{2\pi}{\omega_0} = 24$. By using Euler's Relation, $x[n]$

can be expressed in terms of complex exponentials as

$$\begin{aligned} x[n] &= 1 + \frac{1}{2j}\left(e^{\frac{jn\pi}{12}}e^{\frac{j3\pi}{8}} - e^{\frac{-jn\pi}{12}}e^{-\frac{j3\pi}{8}}\right) \\ &= 1 + \left(\frac{1}{2j}e^{\frac{j3\pi}{8}}\right)e^{\frac{j\pi n}{12}} - \left(\frac{1}{2j}e^{-\frac{j3\pi}{8}}\right)e^{-\frac{j\pi n}{12}} \end{aligned}$$

Note that $x[n]$ has Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} = \sum_{k=-11}^{12} a_k e^{jk\frac{2\pi}{24}n}$$

So, if we equate terms, by inspection, the Fourier coefficients are

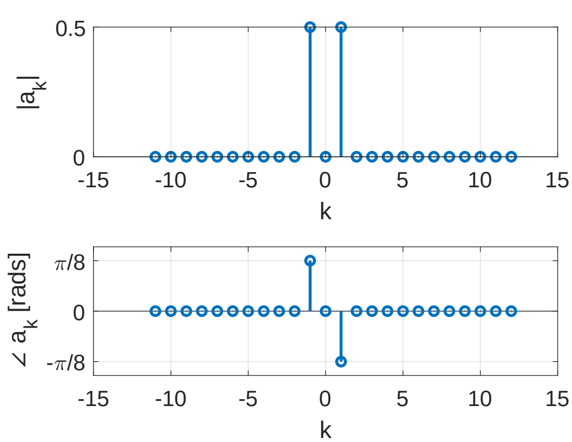
$$a_k = \begin{cases} \frac{1}{2j}e^{\frac{j3\pi}{8}} = \frac{1}{2}e^{-\frac{j\pi}{8}}, & k = 1 \\ 1, & k = 0 \\ -\frac{1}{2j}e^{-\frac{j3\pi}{8}} = \frac{1}{2}e^{\frac{j\pi}{8}}, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

Recall that the Fourier coefficients of a discrete-time signal are periodic; therefore, the a_k's repeat with period $N = 24$. The magnitude and phase of the coefficients a_k are

$$|a_{-1}| = 1/2,\ |a_0| = 1,\ |a_1| = 1/2,\ \text{and 0 otherwise}$$

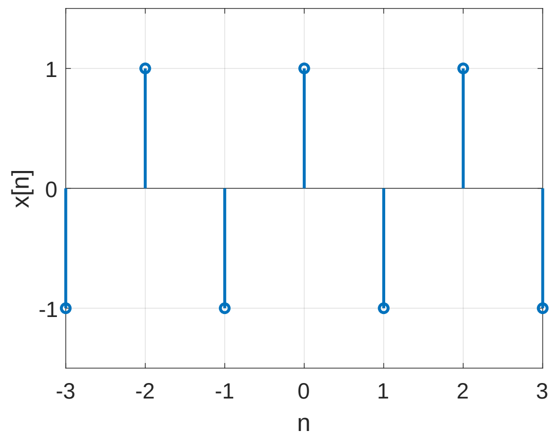
$$\theta_k = \begin{cases} \theta_{-1} = \frac{\pi}{8} \text{ since } a_{-1} = -\frac{1}{2j}e^{-\frac{j3\pi}{8}} = \frac{1}{2}e^{j\frac{\pi}{2}}e^{-\frac{j3\pi}{8}} = \frac{1}{2}e^{\frac{j\pi}{8}} \\ \theta_0 = 0 \text{ since } a_1 = 1 \text{ is purely real} \\ \theta_1 = -\frac{\pi}{8} \text{ since } a_1 = \frac{1}{2j}e^{\frac{j3\pi}{8}} = \frac{1}{2}e^{-\frac{j\pi}{2}}e^{\frac{j3\pi}{8}} = \frac{1}{2}e^{-\frac{j\pi}{8}} \end{cases}$$

and plotted here



Again, note that both the magnitude and phase repeat with period $N = 24$; the above figure displays a single period.

(b) (i) The signal $x[n] = \sum_{m=-\infty}^{\infty} (-1)^n \delta[n-m]$, shown below,



This signal has fundamental period $N = 2$, thus $\omega_0 = 2\pi/N = \pi$. The Fourier series coefficients are computed as

$$
\begin{aligned}
a_k &= \frac{1}{2}\sum_{n=0}^{1} x[n]e^{-jk\pi n} \\
&= \frac{1}{2}(x[0]e^{-jk\pi 0} + x[1]e^{-jk\pi}) \\
&= \frac{1}{2}(1 - e^{-jk\pi}) = \begin{cases} 0, k \text{ even} \\ 1, k \text{ odd} \end{cases}
\end{aligned}
$$

(ii) The signal $x[n]$ shown in the figure has fundamental period $N = 6$ and hence radian frequency $w_0 = \frac{2\pi}{N} = \frac{\pi}{3}$. The Fourier series coefficients can be computed by summing over a period from $n = -2$ to $n = 3$. Thus,

$$
\begin{aligned}
a_k &= \frac{1}{6}\sum_{n=-2}^{3} x[n]e^{-jk\omega_0 n} \\
&= \frac{1}{6}\sum_{n=-2}^{3} x[n]e^{-jk\frac{\pi}{3}n} = \frac{1}{6}(x[-1]e^{jk\frac{\pi}{3}} + x[0] + x[1]e^{-jk\frac{\pi}{3}}) \\
&= \frac{1}{6}(1 + x[1](e^{jk\frac{\pi}{3}} + e^{-jk\frac{\pi}{3}})) \text{ because } x[1] = x[-1] \\
&= \frac{1}{6}(1 + 4\cos(\frac{k\pi}{3})),\ k = -2, -1, 0, 1, 2, 3
\end{aligned}
$$

Problem #3

(a) The signal $x(t)$ (with $\omega_0 = \pi$) can be reconstructed from its Fourier series coefficients as

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \\
&= a_2 e^{j2\pi t} + a_{-2} e^{-j2\pi t} + a_3 e^{j3\pi t} + a_{-3} e^{-j3\pi t} \\
&= -je^{j2\pi t} + je^{-j2\pi t} + 2e^{j3\pi t} + 2e^{-j3\pi t} \\
&= -j(e^{j2\pi t} - e^{-j2\pi t}) + 2(e^{j3\pi t} + e^{-j3\pi t}) \\
&= 2\sin(2\pi t) + 4\cos(3\pi t)
\end{aligned}$$

(b) The signal $x[n]$ (with $N = 10$) can be reconstructed from its periodic, Fourier series coefficients as

$$\begin{aligned}
x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} = \sum_{k=0}^{9} (\frac{1}{2})^k e^{jk\frac{2\pi}{10}n} \\
&= \sum_{k=0}^{9} (\frac{1}{2} e^{j\frac{2\pi}{10}n})^k
\end{aligned}$$

Using the geometric series formula with $\alpha = \frac{1}{2}e^{j\frac{2\pi}{10}n}$,

$$\begin{aligned}
x[n] &= \frac{1 - (\frac{1}{2}e^{j\frac{2\pi}{10}n})^{10}}{1 - \frac{1}{2}e^{j\frac{2\pi}{10}n}} = \frac{1 - (\frac{1}{2})^{10} \cancelto{1}{(e^{j\frac{2\pi}{10}n})^{10}}}{1 - \frac{1}{2}e^{j\frac{2\pi}{10}n}} \\
&= \frac{1 - (\frac{1}{2})^{10}}{1 - \frac{1}{2}e^{j\frac{2\pi}{10}n}}, \quad n = 0, 1, 2, \ldots, 9
\end{aligned}$$

Problem #4

The signal $x(t) = \sin(\frac{\pi}{4}t) + \cos(\frac{5\pi}{4}t)$ has fundamental period $T = \frac{2\pi}{w_0} = \frac{2\pi}{\pi/4} = 8$ and hence its radian frequency $\omega_0 = \pi/4$.

(a) The Fourier series coefficients for this signal can be easily found by inspection

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{4}t} \\
&= \frac{1}{2j} e^{j\frac{\pi}{4}t} - \frac{1}{2j} e^{-j\frac{\pi}{4}t} + \frac{1}{2} e^{j\frac{5\pi}{4}t} + \frac{1}{2} e^{j\frac{5\pi}{4}t}
\end{aligned}$$

Therefore, $a_1 = \frac{1}{2j}$, $a_{-1} = -\frac{1}{2j}$, $a_5 = -\frac{1}{2}$, and $a_{-5} = \frac{1}{2}$ and all other $a_k = 0$.

(b) The Fourier series coefficients b_k of the output signal $y(t)$ can be computed by multiplying the coefficients of the input signal by the frequency response of the highpass filter, using

$$b_k = a_k H(jk\omega_0)$$

The filter cuts off any coefficient a_k with frequency lower than π. So, a_1 and a_{-1} will be zeroed i.e., the signal $\sin(\frac{pi}{4}t)$ will not pass though the filter. Thus,

$$b_5 = a_5 H(j5\omega_0) = \frac{1}{2}H(j\frac{5\pi}{4}) = \frac{1}{2}(\frac{5\pi}{4} - \frac{\pi}{2}) = \frac{1}{2}\frac{3\pi}{4} = \frac{3\pi}{8}$$

$$b_{-5} = a_{-5} H(-j5\omega_0) = \frac{1}{2}H(-j\frac{5\pi}{4}) = \frac{1}{2}(\frac{5\pi}{4} - \frac{\pi}{2}) = \frac{3\pi}{8}$$

All other b_k's are zero.

(c) The output signal $y(t)$ can be written in terms of its coefficients b_k as

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{\pi}{4}t} = \frac{3\pi}{8}e^{j\frac{5\pi}{4}t} + \frac{3\pi}{8}e^{-j\frac{5\pi}{4}t} = \frac{3\pi}{4}\cos(\frac{5\pi}{4}t)$$

Problem #5

(a) The Fourier series coefficients are periodic. Therefore, this frequency characteristic must represent a discrete-time signal.

(b) $w_0 = \dfrac{2\pi}{N} = \frac{2\pi}{9}$; so, $N = 9$. Then, $x[n]$ can be reconstructed from its Fourier series coefficients as

$$\begin{aligned} x[n] = \sum_{k=-4}^{4} a_k e^{jk\omega_0 n} &= 1e^{-j3\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n} + 1e^{j\omega_0 n} + \frac{1}{2j}e^{j3\omega_0 n} \\ &= 2\cos(3\omega_0 n) + \sin(\omega_0 n) \\ &= 2\cos(\frac{2\pi}{3}n) + \sin(\frac{2\pi}{9}n) \end{aligned} \tag{1}$$

(c) Since the lowpass filter only allows frequencies $|\omega| \leq 2\omega_0$ to pass, when $x[n]$ is passed through the filter, it cancels out the $\cos(\dfrac{2\pi}{3}n)$ term; so the resulting output $y[n] = \sin(\frac{2\pi}{9}n)$

Conceptual

Another analogy: In our inner ears, the cochlea enables us to hear differences in the sounds coming to our ears. The cochlea consists of a spiral of tissue filled with liquid and thousands of tiny hairs which gradually get smaller from the outside of the spiral to the inside. Each hair is connected to a nerve which feeds into the auditory nerve bundle going to the brain. The longer hairs resonate with lower frequency sounds, and the shorter hairs with higher frequencies. Thus, the cochlea serves to transform the air pressure signal experienced by the ear drum into frequency information which can be interpreted by the brain. The information our brains receive is not what the shape of the sound wave is, but how much of each component wave is presented. In effect, our ears act like a prism for sound waves.

Math Review

- $$\frac{4x-19}{x^2-9x+20} = \frac{A_1}{x-4} + \frac{A_2}{x-5}$$

  Method #1: Cross-multiplying gives

  $$A_1(x-5) + A_2(x-4) = 4x - 19$$

  $$(A_1 + A_2)x - (5A_1 + 4A_2) = 4x - 19$$

  which leads to solving the following system of equations

  $$\begin{aligned} A_1 + A_2 &= 4 \\ 5A_1 + 4A_2 &= 19 \end{aligned}$$

  Therefore, $A_1 = 3$ and $A_2 = 1$.

  Method #2:

  $$A_1 = \frac{4x-19}{x^2-9x+20}(x-4)\Big|_{x=4} = \frac{4x-19}{x-5}\Big|_{x=4} = 3$$

  $$A_2 = \frac{4x-19}{x^2-9x+20}(x-5)\Big|_{x=5} = \frac{4x-19}{x-4}\Big|_{x=5} = 1$$

- For this problem, you can simply use long division and divide $(x+1)$ into $(x^2 + x + 3)$, obtaining

  $$\frac{x^2+x+3}{x+1} = x + \frac{3}{x+1}$$