Math 342

Homework#3 solutions

Sec. 7.1 (P):

2) With $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$, we get

(a)

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^\top A \boldsymbol{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}^\top \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = -4$$

(b)

$$\|\boldsymbol{u}\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} = \sqrt{\begin{pmatrix} 1 \\ -2 \end{pmatrix}^\top \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}} = \sqrt{10}$$

(c)

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{\begin{pmatrix} -3 \\ -5 \end{pmatrix}^\top \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix}} = \sqrt{189}$$

6)

(a)

$$\langle p(x), q(x) \rangle = \int_0^1 (3-2x)(1+x+x^2)\,dx = \int_0^1 (-2x^3+x^2+x+3)\,dx = \left(-\frac{1}{2}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x\right)_0^1 = \frac{10}{3}$$

(b)

$$\|p(x)\| = \sqrt{\langle p(x), p(x) \rangle} = \sqrt{\int_0^1 (3-2x)^2 dx} = \sqrt{\left(9x - 6x^2 + \frac{4}{3}x^3\right)_0^1} = \sqrt{\frac{13}{3}}$$

(c)

$$\begin{aligned} d(p(x), q(x)) &= \|p(x) - q(x)\| = \|2 - 3x - x^2\| = \sqrt{\int_0^1 (2 - 3x - x^2)^2 dx} \\ &= \sqrt{\left(\frac{1}{5}x^5 + \frac{3}{2}x^4 + \frac{5}{3}x^3 - 6x^2 + 4x\right)_0^1} = \sqrt{\frac{41}{30}} \end{aligned}$$

34) We have

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + 2\langle \boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{v}\|^2 \ , \quad \|\boldsymbol{u} - \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 - 2\langle \boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{v}\|^2$$

therefore, by subtracting these two equations,

$$2\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{2}\Big(\|\boldsymbol{u} + \boldsymbol{v}\|^2 - \|\boldsymbol{u} - \boldsymbol{v}\|^2\Big)$$

or equivalently

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$$

36) We have

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{\|\boldsymbol{u}\|^2 - 2\langle \boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{v}\|^2}$$

which is equal to $\sqrt{\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2}$ if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$, meaning if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

Sec. 5.1 (Z):

1) The ratio test requires

$$\lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{n+1}x^{n+1}\frac{n}{2^n x^n}\right| = 2|x|\lim_{n\to\infty}\frac{n}{n+1} = 2|x| < 1$$

for convergence, hence $|x| < 1/2$. Therefore the radius of convergence is $R = 1/2$ and the interval of convergence is $-1/2 < x < 1/2$.

2) The ratio test requires

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x+7)^{n+1}}{c_n(x+7)^n}\right| = \lim_{n\to\infty}\left|\frac{100^{n+1}}{(n+1)!}(x+7)^{n+1}\frac{n!}{100^n(x+7)^n}\right| = 100\,|x+7|\lim_{n\to\infty}\frac{1}{n+1} < 1$$

for convergence, hence $|x+7| < \infty$. Therefore the radius of convergence is $R = \infty$ and the interval of convergence is $-\infty < x < \infty$.

3) The ratio test requires

$$\lim_{k\to\infty}\left|\frac{c_{k+1}(x-5)^{k+1}}{c_k(x-5)^k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{10^{k+1}}(x-5)^{k+1}\frac{10^k}{(-1)^k(x-5)^k}\right| = \frac{1}{10}|x-5| < 1$$

for convergence, hence $|x-5| < 10$. Therefore the radius of convergence is $R = 10$ and the interval of convergence is $-5 < x < 15$.

4) The ratio test requires

$$\lim_{k\to\infty}\left|\frac{c_{k+1}(x-1)^{k+1}}{c_k(x-1)^k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!(x-1)^{k+1}}{k!(x-1)^k}\right| = |x-1|\lim_{k\to\infty}(k+1) < 1$$

for convergence, hence $|x-1| < 0$. Therefore the radius of convergence is $R = 0$ and the interval of convergence is $x = 1$.

Additional problems:

1)

(a) $\langle e^x, e^{-x}\rangle = \int_0^1 e^x e^{-x}\,dx = \int_0^1 dx = 1$

(b)

$$\begin{aligned}\langle x, \sin(\pi x)\rangle &= \int_0^1 x\sin(\pi x)\,dx = -\frac{x}{\pi}\cos(\pi x)\Big|_0^1 + \frac{1}{\pi}\int_0^1 \cos(\pi x)\,dx \\ &= -\frac{1}{\pi}\cos\pi + \frac{1}{\pi^2}\sin(\pi x)\Big|_0^1 = \frac{1}{\pi}\end{aligned}$$

(c) $\langle x^2, x^3\rangle = \int_0^1 x^5\,dx = \frac{1}{6}x^6\big|_0^1 = \frac{1}{6}$

2)

(a)

$$\cos\theta = \frac{\langle 1, x\rangle}{\|1\|\|x\|} = \frac{\int_0^1 x dx}{\sqrt{\int_0^1 dx}\sqrt{\int_0^1 x^2 dx}} = \frac{\sqrt{3}}{2} \quad\Longrightarrow\quad \theta = \frac{\pi}{6}\text{ rad (or } 30^\circ)$$

(b)

$$p = \frac{\langle 1, x\rangle}{\|x\|^2}x = \frac{3}{2}x$$

$$\langle 1-p, p\rangle = \int_0^1 \left(1 - \frac{3}{2}x\right)\frac{3}{2}x dx = \left(\frac{3}{4}x^2 - \frac{3}{4}x^3\right)\Big|_0^1 = 0$$

therefore $1-p$ and p are orthogonal

(c)

$$\|1\|^2 = \int_0^1 dx = 1\ , \quad \|p\|^2 = \int_0^1 \frac{9}{4}x^2 dx = \frac{3}{4}\ , \quad \|1-p\|^2 = \int_0^1 \left(1 - \frac{3}{2}x\right)^2 dx = \frac{1}{4}$$

Therefore $\|1\| = 1$, $\|p\| = \sqrt{3}/2$, $\|1-p\| = 1/2$ and $\|1-p\|^2 + \|p\|^2 = 1 = \|1\|^2$ (Pythagorean)

3) If $m \neq n$, we have

$$\begin{aligned}\langle\cos(mx), \sin(nx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\sin((n+m)x) + \sin((n-m)x)]\,dx \\ &= \frac{1}{2\pi}\left[-\frac{1}{n+m}\cos((n+m)x)\Big|_{-\pi}^{\pi} - \frac{1}{n-m}\cos((n-m)x)\Big|_{-\pi}^{\pi}\right] = 0 \\ &\quad\text{(orthogonal functions)}\end{aligned}$$

Note: saying that it is zero because the integrand $\cos(mx)\sin(nx)$ is an odd function over $[-\pi, \pi]$ is also OK.

$$\begin{aligned}\|\cos(mx)\|^2 = \langle\cos(mx), \cos(mx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(mx)\,dx = \frac{1}{\pi}\int_0^{\pi}(1 + \cos(2mx))\,dx \\ &= \frac{1}{\pi}\left[x|_0^{\pi} + \frac{1}{2m}\sin(2mx)\Big|_0^{\pi}\right] = 1 \\ &\quad\text{(unit function)}\end{aligned}$$

$$\begin{aligned}\|\sin(nx)\|^2 = \langle\sin(nx), \sin(nx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}(1 - \cos(2nx))\,dx \\ &= \frac{1}{\pi}\left[x|_0^{\pi} - \frac{1}{2n}\sin(2nx)\Big|_0^{\pi}\right] = 1 \\ &\quad\text{(unit function)}\end{aligned}$$

$$\begin{aligned}\|\cos(mx)-\sin(nx)\|^2 &= \|\cos(mx)\|^2 - 2\langle\cos(mx),\sin(nx)\rangle + \|\sin(nx)\|^2\\ &= \|\cos(mx)\|^2 + \|\sin(nx)\|^2 = 2\end{aligned}$$

Therefore the distance between $\cos(mx)$ and $\sin(nx)$ is $\sqrt{2}$.

4)

(a) $\|x\| = \sqrt{\sum_{i=1}^5 x_i^2} = \sqrt{(-1)^2+(-\frac{1}{2})^2+(\frac{1}{2})^2+1^2} = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$

(b) $\|x^2\| = \sqrt{\sum_{i=1}^5 x_i^4} = \sqrt{(-1)^4+(-\frac{1}{2})^4+(\frac{1}{2})^4+1^4} = \sqrt{\frac{17}{8}} = \frac{\sqrt{34}}{4}$

(c) distance $\|x-x^2\| = \sqrt{\sum_{i=1}^5 (x_i-x_i^2)^2} = \sqrt{\sum_{i=1}^5 x_i^2 - 2x_i^3 + x_i^4} = \sqrt{\frac{37}{8}} = \frac{\sqrt{74}}{4}$

(d) $\langle x, x^2\rangle = \sqrt{\sum_{i=1}^5 x_i^3} = \sqrt{(-1)^3+(-\frac{1}{2})^3+(\frac{1}{2})^3+1^3} = 0$ hence x and x^2 are orthogonal.

5) We have

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + 2\langle\boldsymbol{u},\boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2 \ , \quad \|\boldsymbol{u}-\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 - 2\langle\boldsymbol{u},\boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2$$

therefore

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 + \|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$$

6)

(a)

$$\|\boldsymbol{x}_1\|^2 = \|\boldsymbol{x}_2\|^2 = \cos^2\theta + \sin^2\theta = 1 \quad \text{(unit functions)}$$
$$\langle\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0 \quad \text{(orthogonal)}$$

(b)

$$\boldsymbol{y} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \langle\boldsymbol{y},\boldsymbol{x}_1\rangle\boldsymbol{x}_1 + \langle\boldsymbol{y},\boldsymbol{x}_2\rangle\boldsymbol{x}_2 = (\alpha\cos\theta+\beta\sin\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + (-\alpha\sin\theta+\beta\cos\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

(c)

$$c_1^2 + c_2^2 = (\alpha\cos\theta+\beta\sin\theta)^2 + (-\alpha\sin\theta+\beta\cos\theta)^2 = (\alpha^2+\beta^2)(\cos^2\theta+\sin^2\theta) = \alpha^2+\beta^2$$

7)

(a)

$$\begin{aligned}\sin^4 x &= \sin^2 x\,\sin^2 x = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right) = \frac{1}{4}(1-\cos 2x)^2 = \frac{1}{4}(1-2\cos 2x+\cos^2 2x)\\ &= \frac{1}{4}(1-2\cos 2x + \frac{1+\cos 4x}{2}) = \frac{1}{4}\left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\end{aligned}$$

(b) By identification of the coefficients, we find

(i) $\int_{-\pi}^{\pi} \sin^4 x \, \cos x \, dx = \pi \, \langle \sin^4 x, \cos x \rangle = 0$

(ii) $\int_{-\pi}^{\pi} \sin^4 x \, \cos 2x \, dx = \pi \, \langle \sin^4 x, \cos 2x \rangle = -\frac{\pi}{2}$

(iii) $\int_{-\pi}^{\pi} \sin^4 x \, \cos 3x \, dx = \pi \, \langle \sin^4 x, \cos 3x \rangle = 0$

(iv) $\int_{-\pi}^{\pi} \sin^4 x \, \cos 4x \, dx = \pi \, \langle \sin^4 x, \cos 4x \rangle = \frac{\pi}{8}$