Chapter 1 Direct Current Circuits

* Ohm's law (ch. 25)

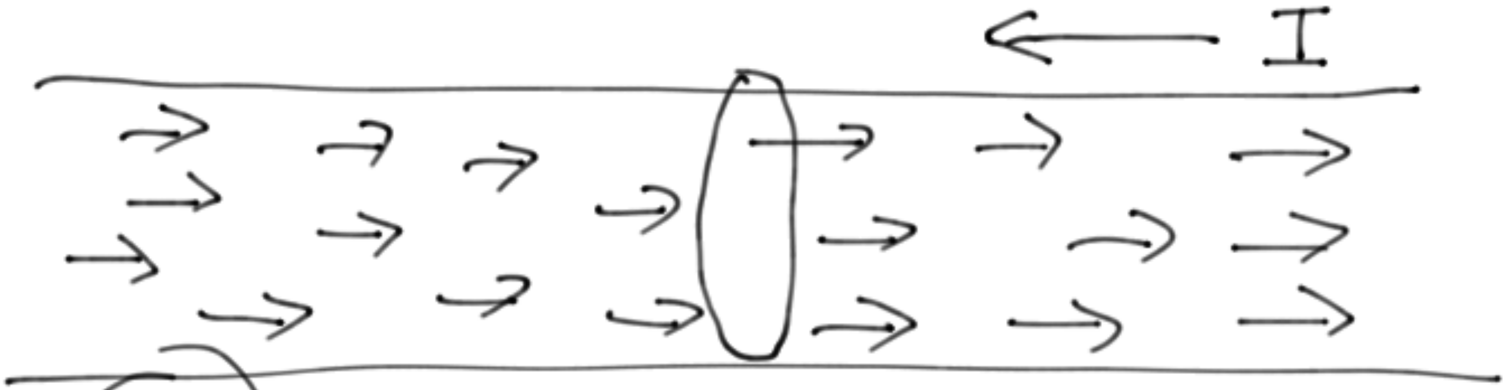
Basic Concepts

Electric charge Q: fundamental quantity of electricity

Smallest charge, electron, -1.6×10^{-19} Coulomb (C)

Electric current: $I = \frac{\Delta Q}{\Delta t}$, unit: Ampere (A)
Coulomb/sec.

continuous flow of charges gives rise to electric current. Electric current (I) is the rate at which charges cross a given cross-section in a circuit.



$-e$ electrons moving to the right

Electric current flows to the left.

Direction of current is opposite to direction of motion of electrons.

During time period Δt, amount of charge ΔQ passes through cross-section.

$$I = \frac{\Delta Q}{\Delta t}$$

Next question: what drives electric current?

Electric field $\xleftrightarrow{\text{in analogy}}$ gravitational field.

Electric potential (voltage) $\longleftrightarrow$ gravitational potential energy

electric charge $\longleftrightarrow$ object.

Voltage: electric potential difference.

Total work per unit charge done by electric field associated with the motion of charge between two points.

unit: volt, 1 V = 1 Joule/Coulomb

potential energy difference per unit charge.

$q\underline{V}$: electric potential energy

$mg\underline{h}$: gravitational potential energy

V is in analogy to gh.

see voltage as a "driving force" for electric currents.

Just like water currents flow from a higher point to a lower point, electric current flows from a higher potential to a lower potential.

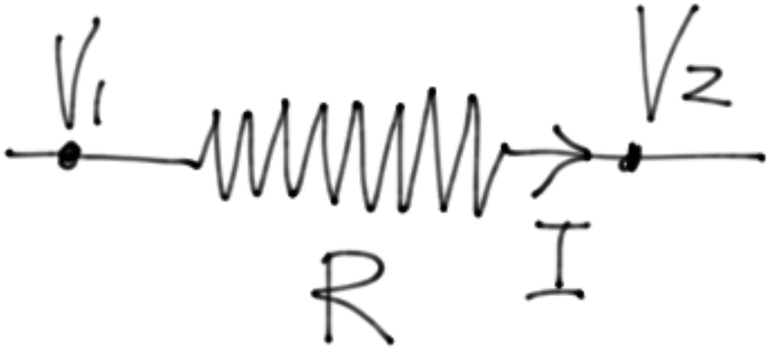
Ohm's law

$$R=\frac{\Delta V}{I}$$

ΔV → voltage difference

I → electric current

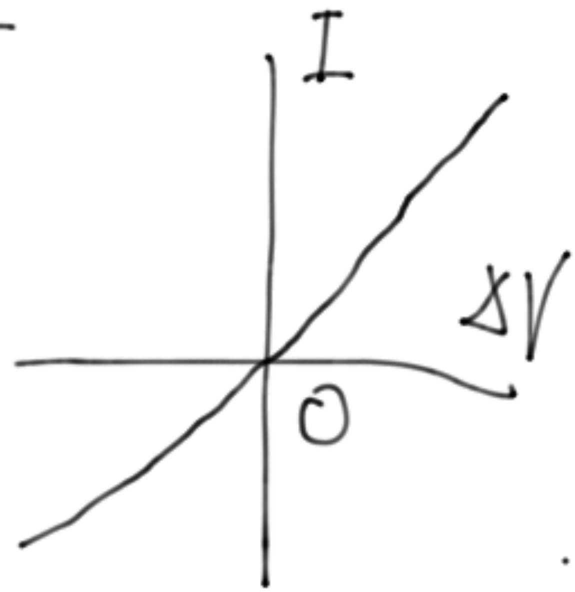
Electric current is proportional to the voltage difference between two ends of a conductor. The proportionality constant is defined as resistance R.



$V_1 > V_2$

$\Delta V = V_1 - V_2$

$I = \frac{\Delta V}{R}$



slope of curve : $\frac{1}{R}$

Recap:

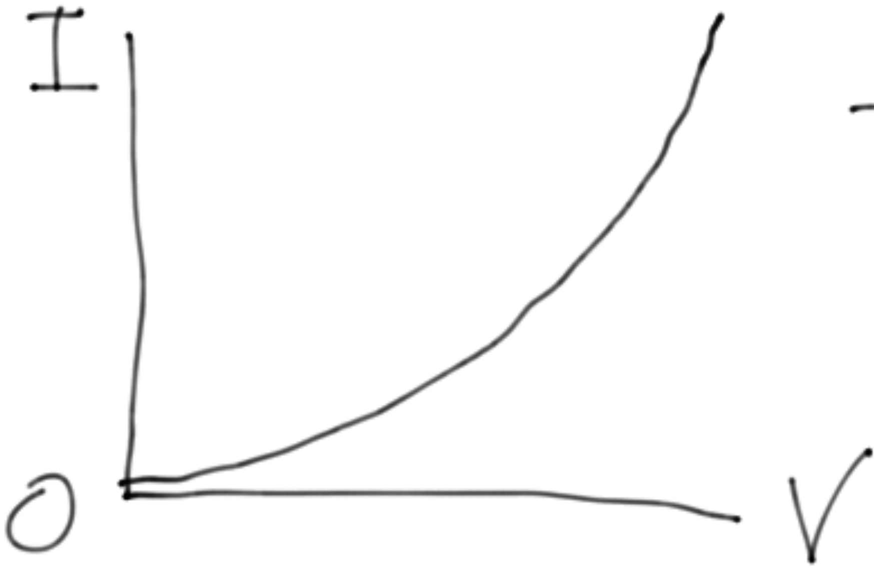
* Current flows from higher voltage to lower voltage.
* R is independent of ΔV or I for conductors.
* voltage difference causes current in conductors.

Also note, ΔV can be simplified to V

so, $$R = \frac{V}{I}$$

Ohm's law is not universal.
Some materials (semiconductor or polymers) do not obey Ohm's law.

semiconductor diode :



→ not "linear"

non-ohmic.

Resistivity :

Consider a conducting wire with uniform cross section.

$$R = \rho \frac{L}{A}$$

L : length

A : cross-sectional area.

ρ : resistivity

Resistance (R) depends on both materials and geometries.

Resistivity (ρ) only depends on materials.

Joule heating power.

Eleceric power

electric field moving charge $\longrightarrow$ doing work.

$$\text{Power}(P) = \frac{\text{Work}}{\text{time}} = \underbrace{\frac{\text{Work}}{\text{charge}}}_{V} \cdot \underbrace{\frac{\text{charge}}{\text{time}}}_{I}$$

$$= V \cdot I$$

unit of P: Joule/time = Watt

$P = IV$ $\quad\quad I = \frac{V}{R}$

$P = \left(\frac{V}{R}\right)V = \frac{V^2}{R}$

$P = I(IR) = I^2R$

only for conductors that obey ohm's law.

Example

1000 W hair dryer, $V = 120V$.

$I = ?$ $R = ?$

$P = IV$, $I = \frac{P}{V} = \frac{1000W}{120V} = 8.33(A)$

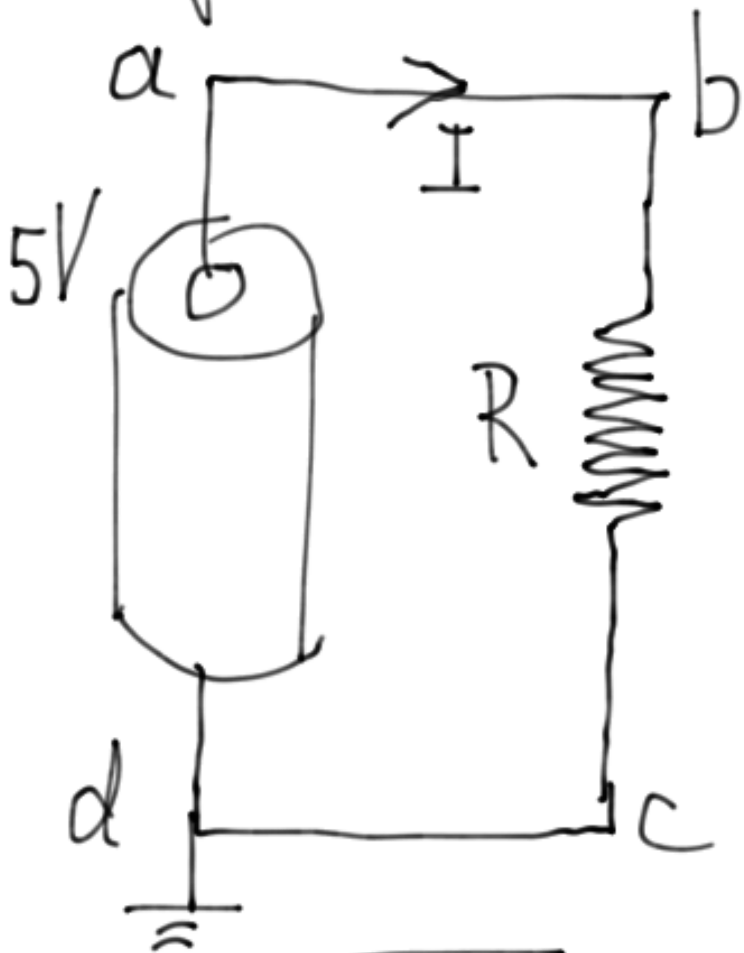
$P = \frac{V^2}{R}$ $R = \frac{V^2}{P} = \frac{(120)^2}{1000} = 14.4(\Omega)$

what happens if you use it in 220V lines.

$P = \frac{V^2}{R} = \frac{(220)^2}{14.4} = 3361(W) > 1000W$

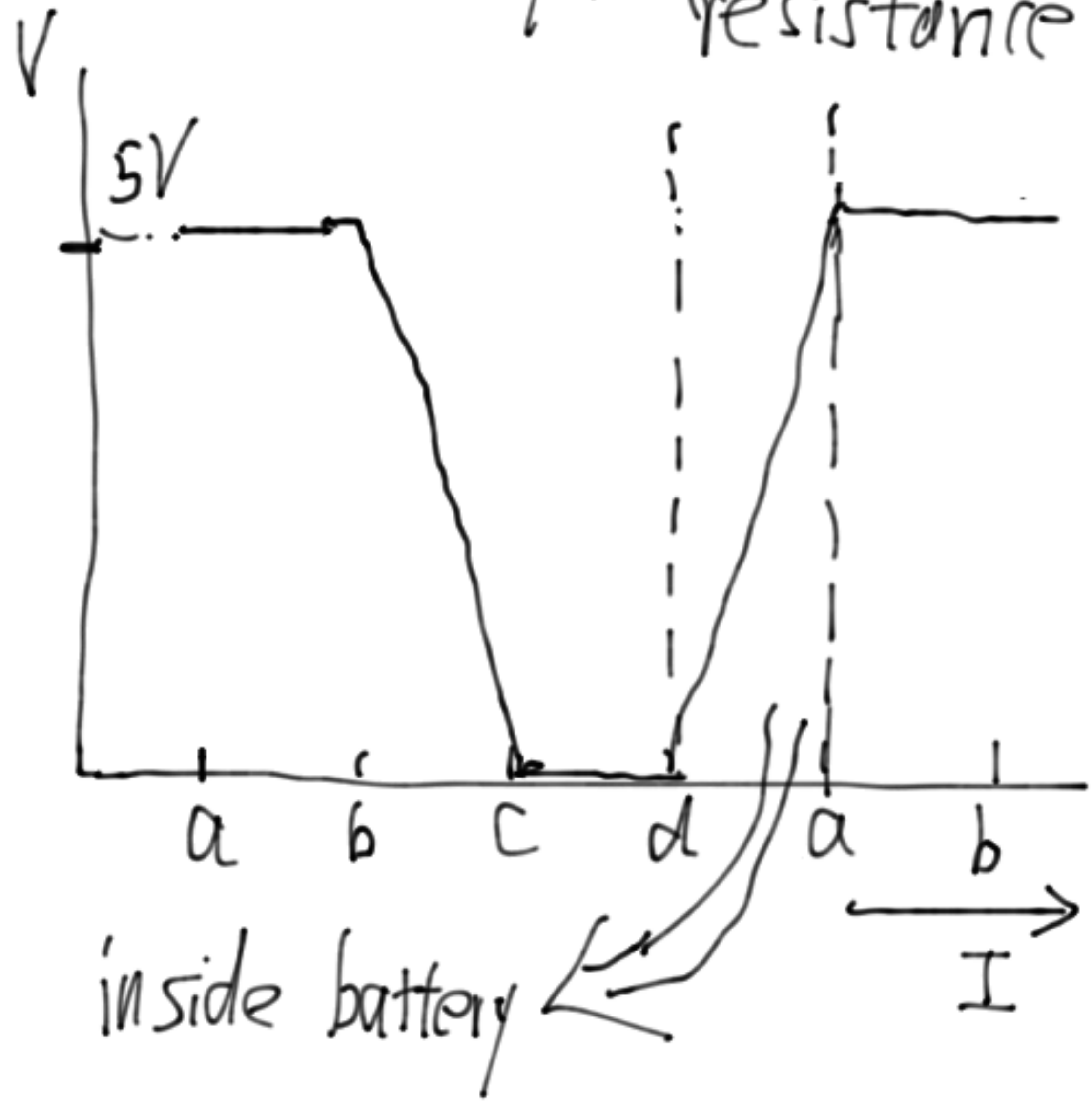
way above rated power. may damage the filament (resistor inside).

emf: electro-motive force.



ideal wires: zero resistance

ideal battery: no internal resistance

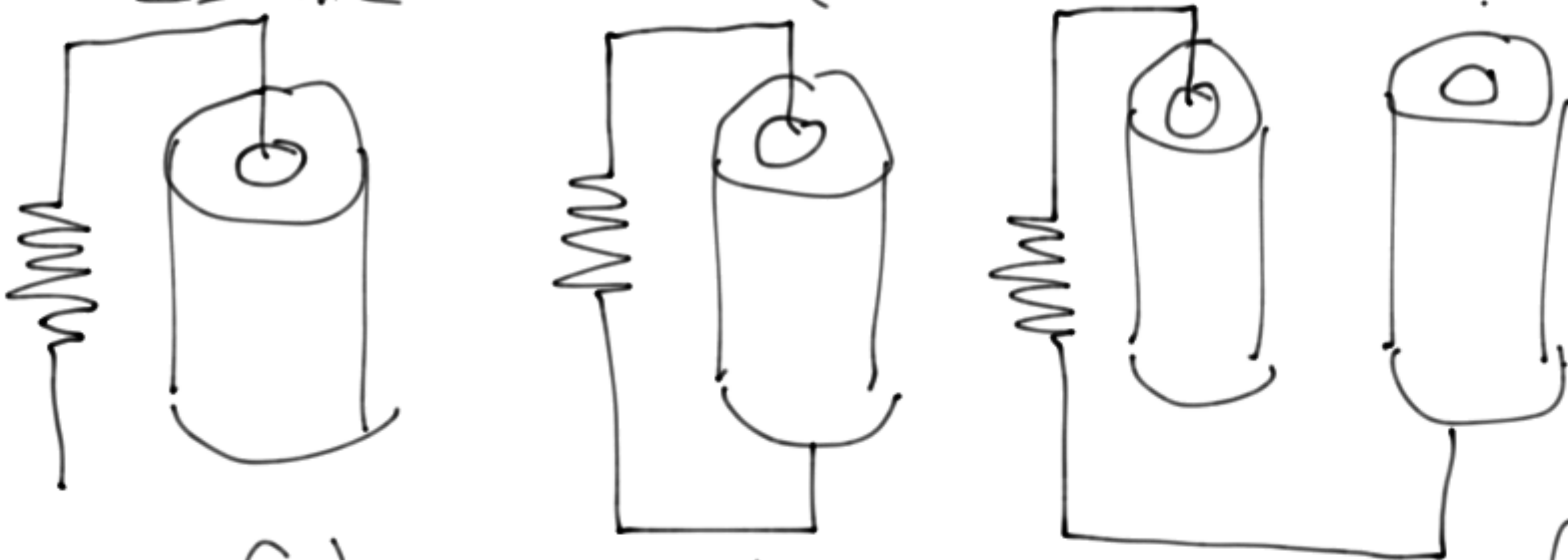


battery: charge pump

emf: the current-causing energy sources are called electromotive forces (emf).
batteries and generators are emfs.
emf (ε) is in unit of volt (V)
Work per unit charge done by energy sources other than electricity for some specific path.

Question

Is the a current in each case?



(a) (b) (c)

Have close circuits for current flow.

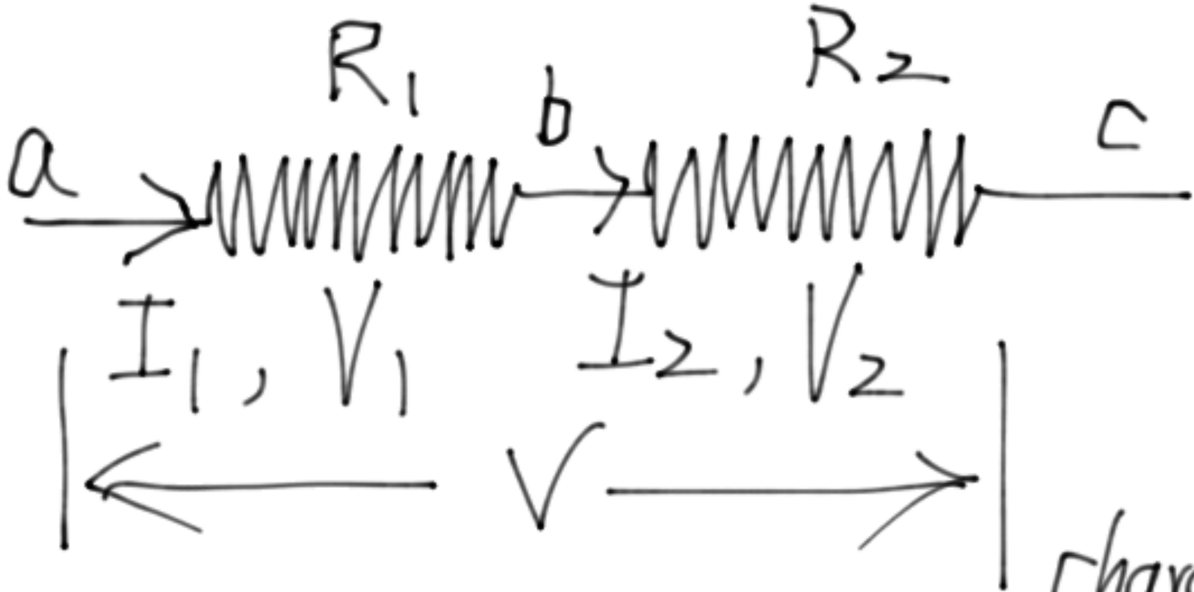
Also avoid a short circuit:

$$I = \frac{V}{R} = \frac{\mathcal{E}}{R=0} = \infty$$

$$P = \frac{V^2}{R=0} \rightarrow \infty$$

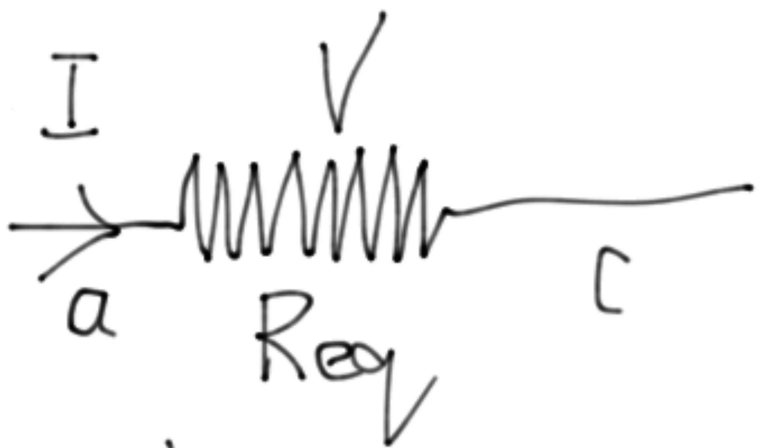
* Resistors in series and in parallel

Resistors in series



$$I_1 = I_2 = I$$

charge conservation



equivalent resistor

$$V_1 = V_a - V_b$$

$$V_2 = V_b - V_c$$

$$V_1 + V_2 = (V_a - V_b) + (V_b - V_c)$$

$$= V_a - V_c = V$$

$$V_1 = I_1 R_1 = I R_1 \quad V_2 = I_2 R_2 = I R_2$$

$$V = I R_{eq}$$

$$V = V_1 + V_2, \quad I R_{eq} = I R_1 + I R_2$$

$$R_{eq} = R_1 + R_2$$

Voltage divider rule:

$$V_1 = I R_1 = \frac{V}{R_{eq}} \cdot R_1 = \frac{R_1}{R_1 + R_2} V$$

$$V_2 = I R_2 = \frac{V}{R_{eq}} R_2 = \frac{R_2}{R_1 + R_2} V$$

$$\frac{V_1}{V_2} = \frac{R_1}{R_2}, \quad \frac{V_1}{V} = \frac{R_1}{R_1 + R_2}, \quad \frac{V_2}{V} = \frac{R_2}{R_1 + R_2}$$

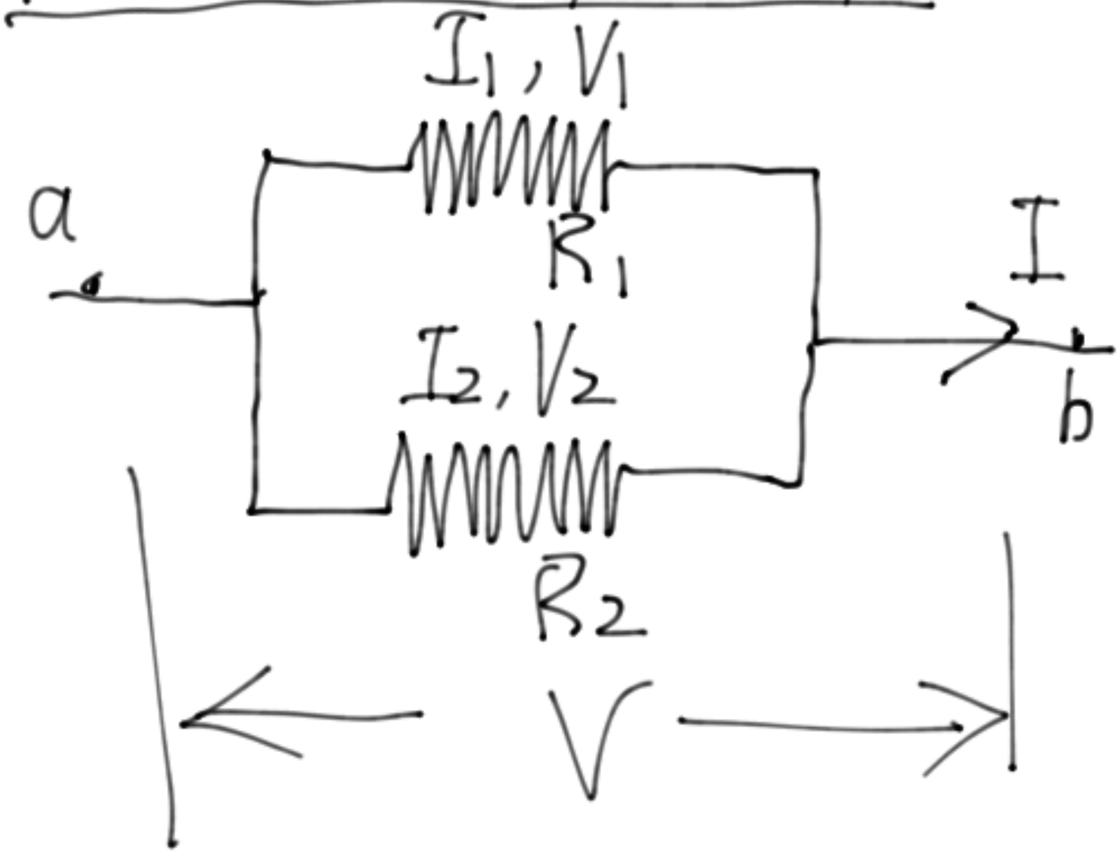
Extend to N resistors in series

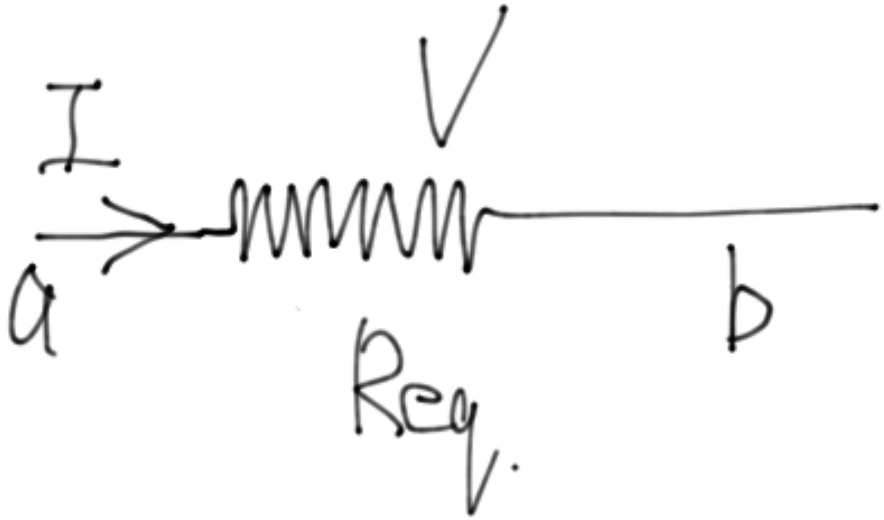
$$V_1 : V_2 : V_3 : \cdots\cdots V_N = R_1 : R_2 : R_3 : \cdots\cdots R_N$$

$$R_{eq} = R_1 + R_2 + \cdots\cdots + R_N$$

$$R_{eq} > \max(R_1, R_2, \cdots, R_N)$$

Resistors in parallel





$$I = I_1 + I_2$$

charge conservation

$$V = V_1 = V_2 = V_a - V_b$$

$$I = \frac{V}{R_{eq}}, \quad I_1 = \frac{V}{R_1}$$

$$I_2 = \frac{V}{R_2}$$

$$\frac{V}{R_{eq}} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

Define $\frac{1}{R} = G$ conductance

$$G_{eq} = G_1 + G_2$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2}$$

$$R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2} = \frac{R_1}{\left(\frac{R_1}{R_2} + 1\right)} < R_1$$

$$= \frac{R_2}{\left(1 + \frac{R_2}{R_1}\right)} < R_2$$

$$R_{eq} < \min(R_1, R_2)$$

Current divider rule

$$I_1 = \frac{V}{R_1} = \frac{I R_{eq}}{R_1} = I\left(\frac{1}{1/R_1 + 1/R_2}\right)\frac{1}{R_1}$$

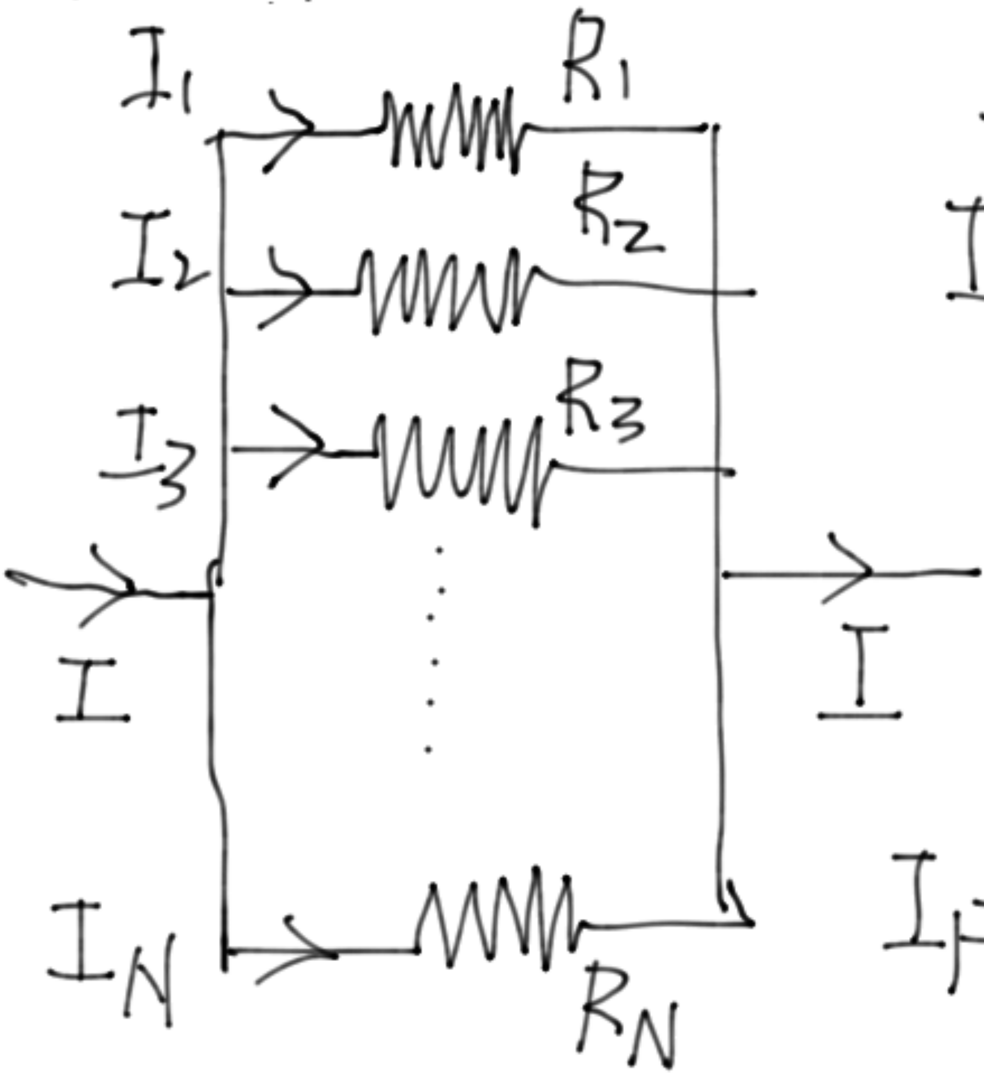
$$= I\,\frac{1/R_1}{1/R_1 + 1/R_2} = I\,\frac{G_1}{G_1 + G_2}$$

Similarly $I_2 = I\,\dfrac{1/R_2}{1/R_1 + 1/R_2} = I\,\dfrac{G_2}{G_1 + G_2}$

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} = \frac{G_1}{G_2}$$

branch current inversely proportional to branch resistance.

Generalize to N resistors



same V for each R

$$I = I_1 + I_2 + \cdots + I_N$$

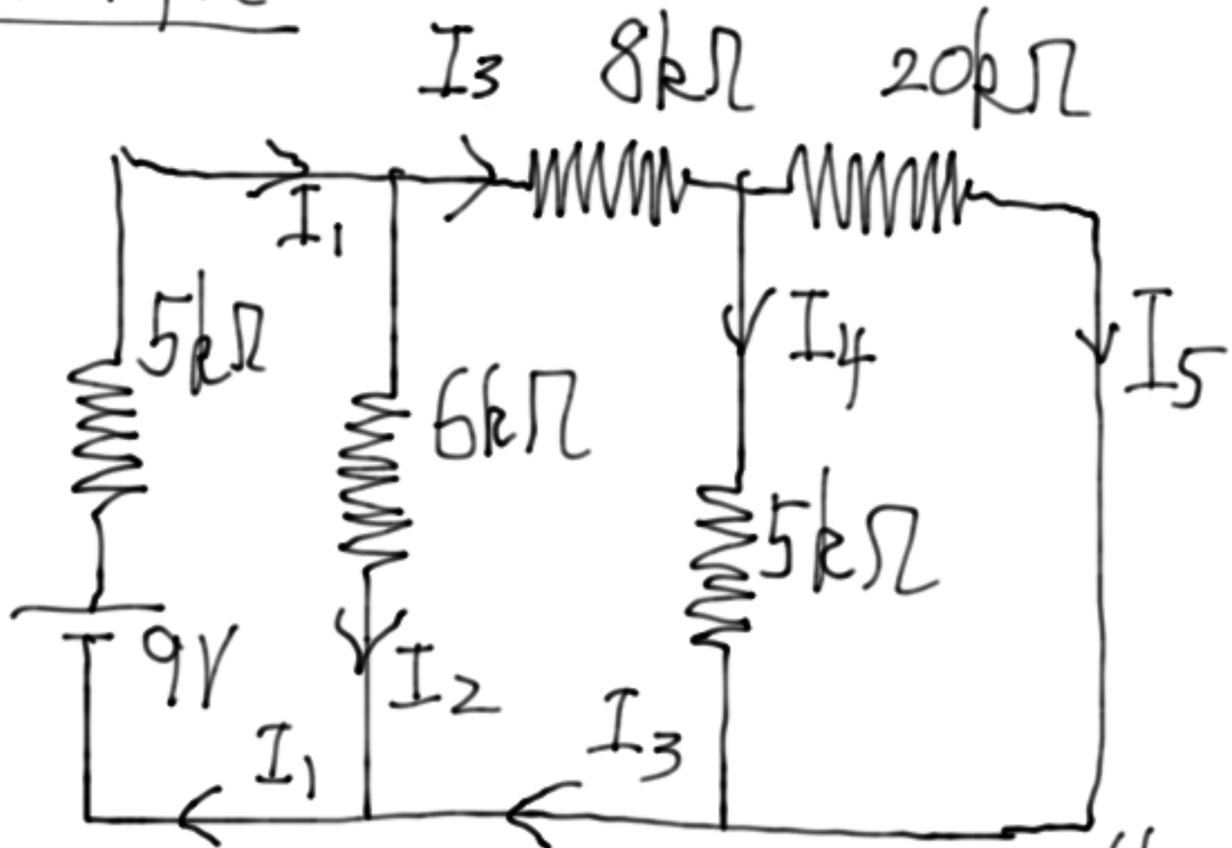
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

$$G_{eq} = G_1 + G_2 + \cdots + G_N$$

$$I_1 = I \cdot \frac{1/R_1}{1/R_1 + \cdots + 1/R_N}$$

$$I_n = I \cdot \frac{1/R_n}{1/R_1 + \cdots + 1/R_N}$$

Example



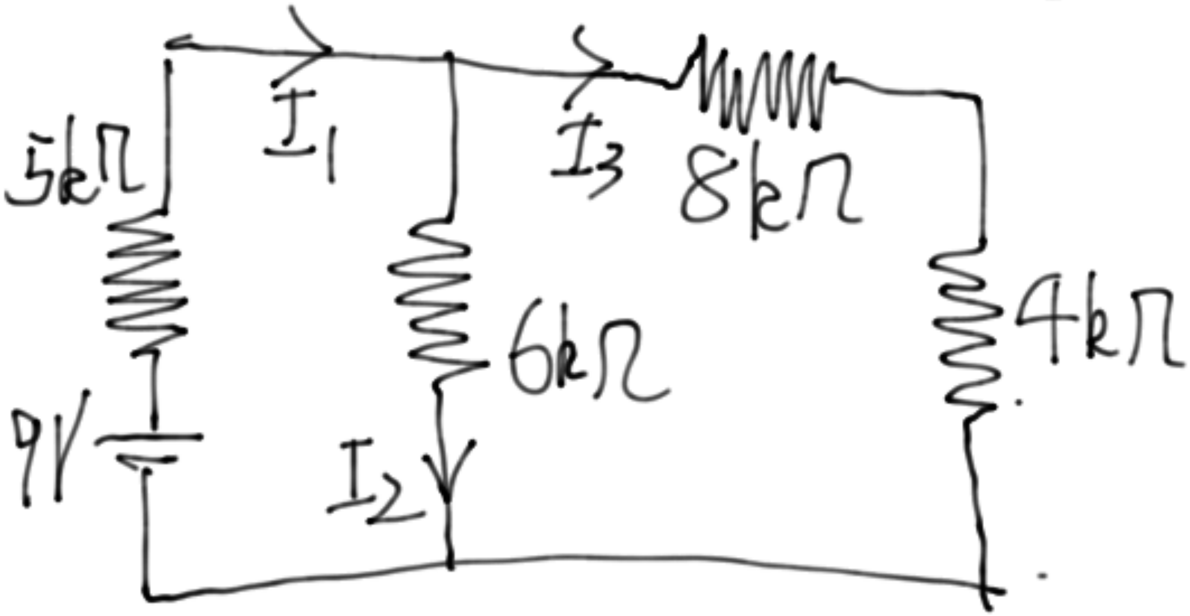
(a) What is the current through battery?

(b) What is the current through each resistor?

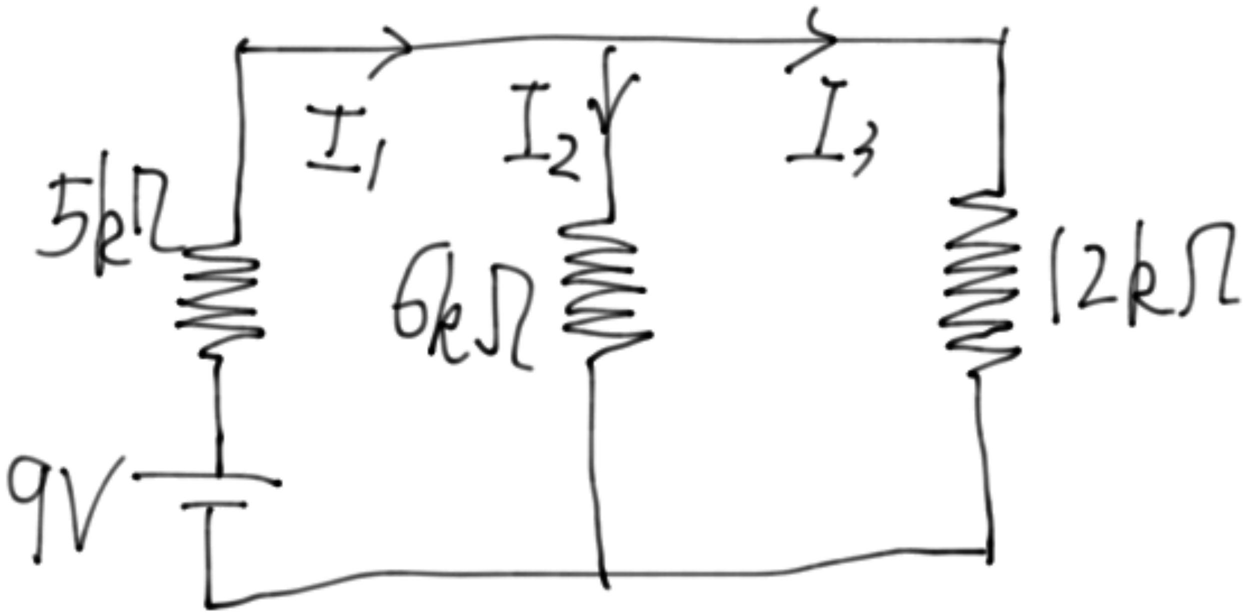
First reduce circuit by finding R_{eq}.

Then restore circuit step by step.

$5k\Omega // 20k\Omega \quad R_{eq} = \frac{5 \times 20}{5+20} = 4k\Omega$

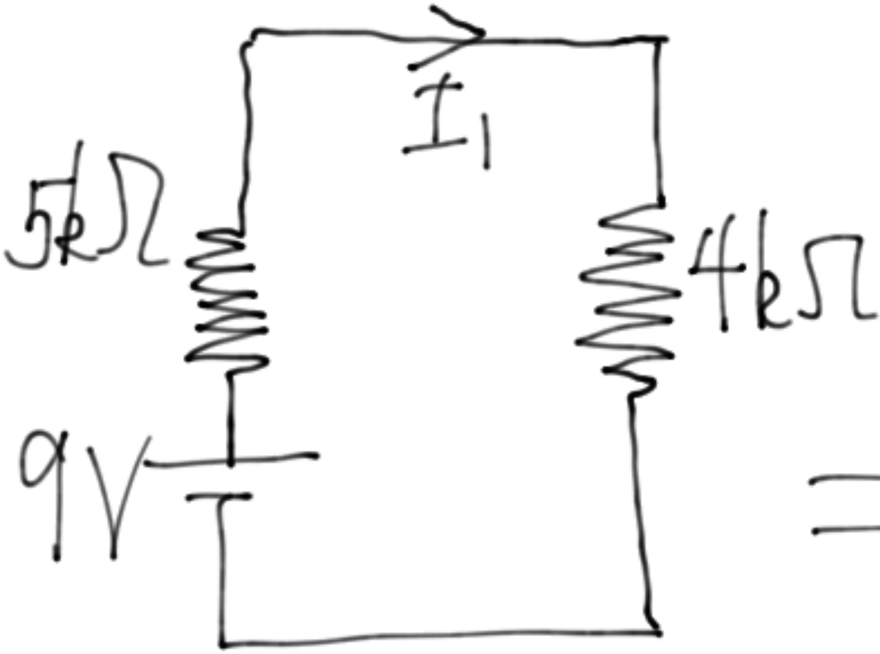


$$R_{eq} = 8 + 4 = 12\,k\Omega$$

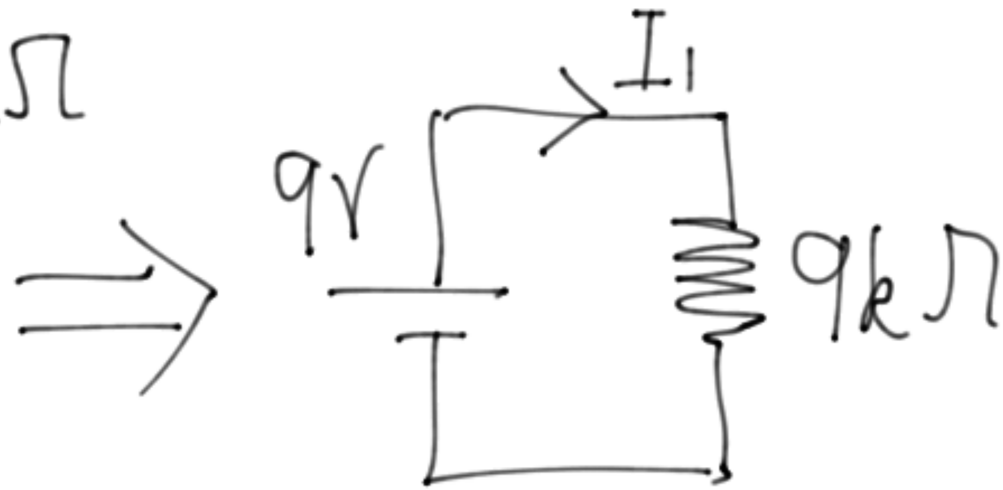


$6k\Omega // 12k\Omega$

$$\frac{12 \times 6}{12+6} = 4k\Omega$$



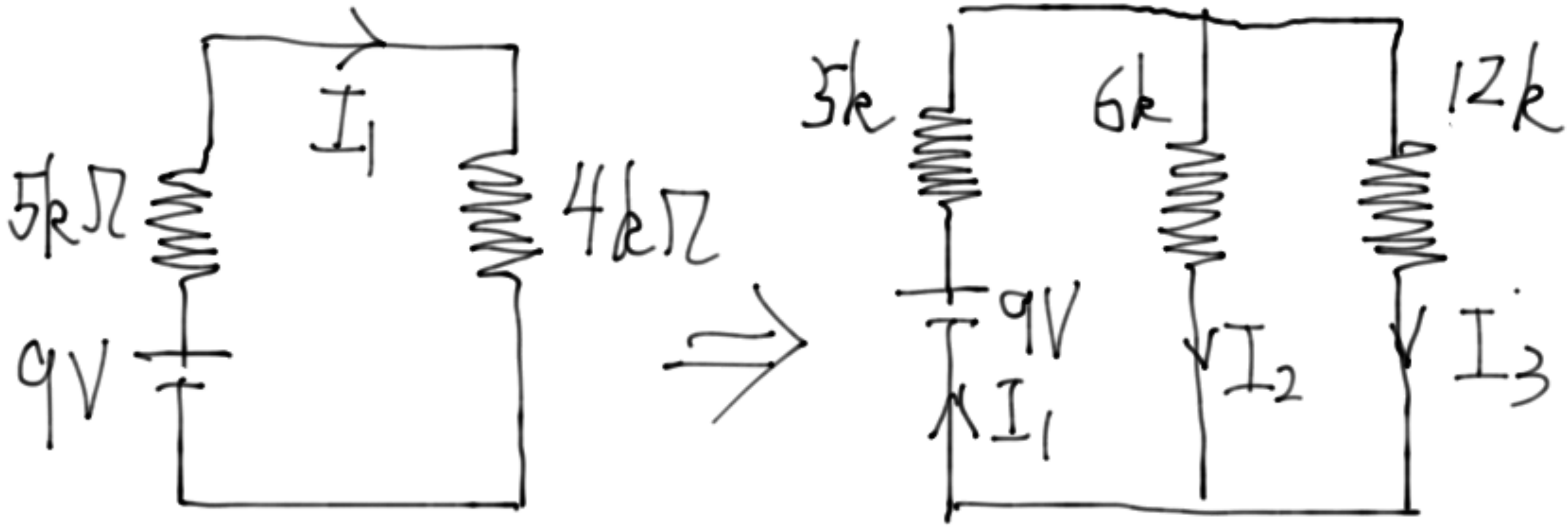
$5+4=9k\Omega.$



$$I_1 = \frac{9V}{9k\Omega} = 1mA$$

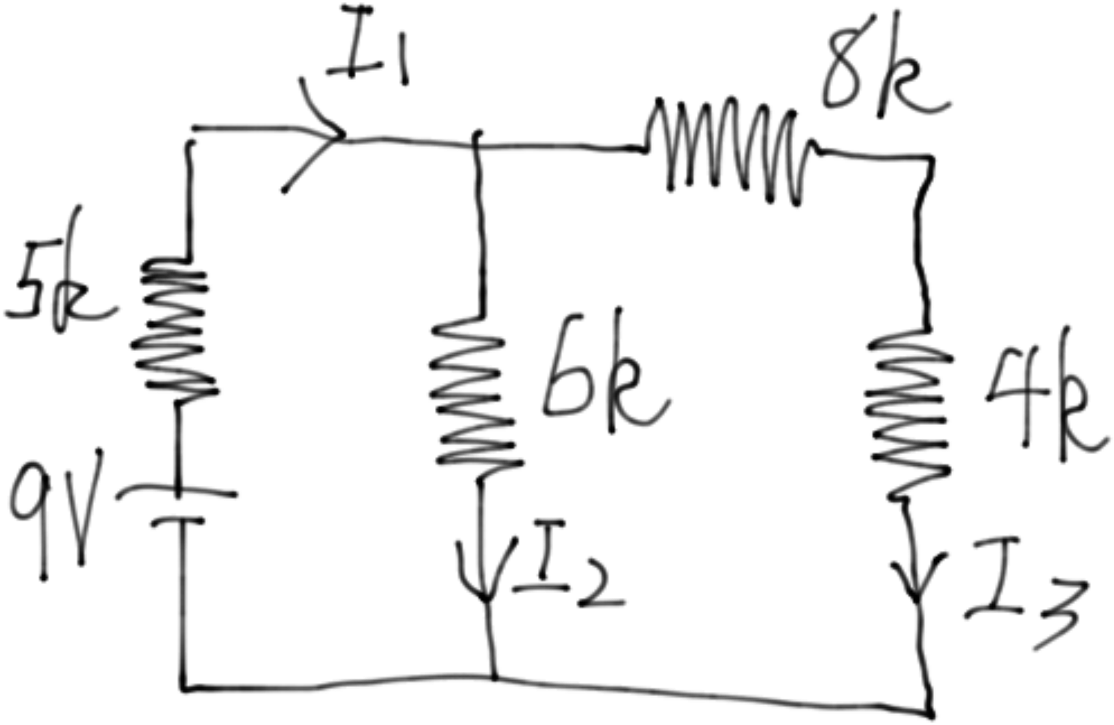
Current through battery

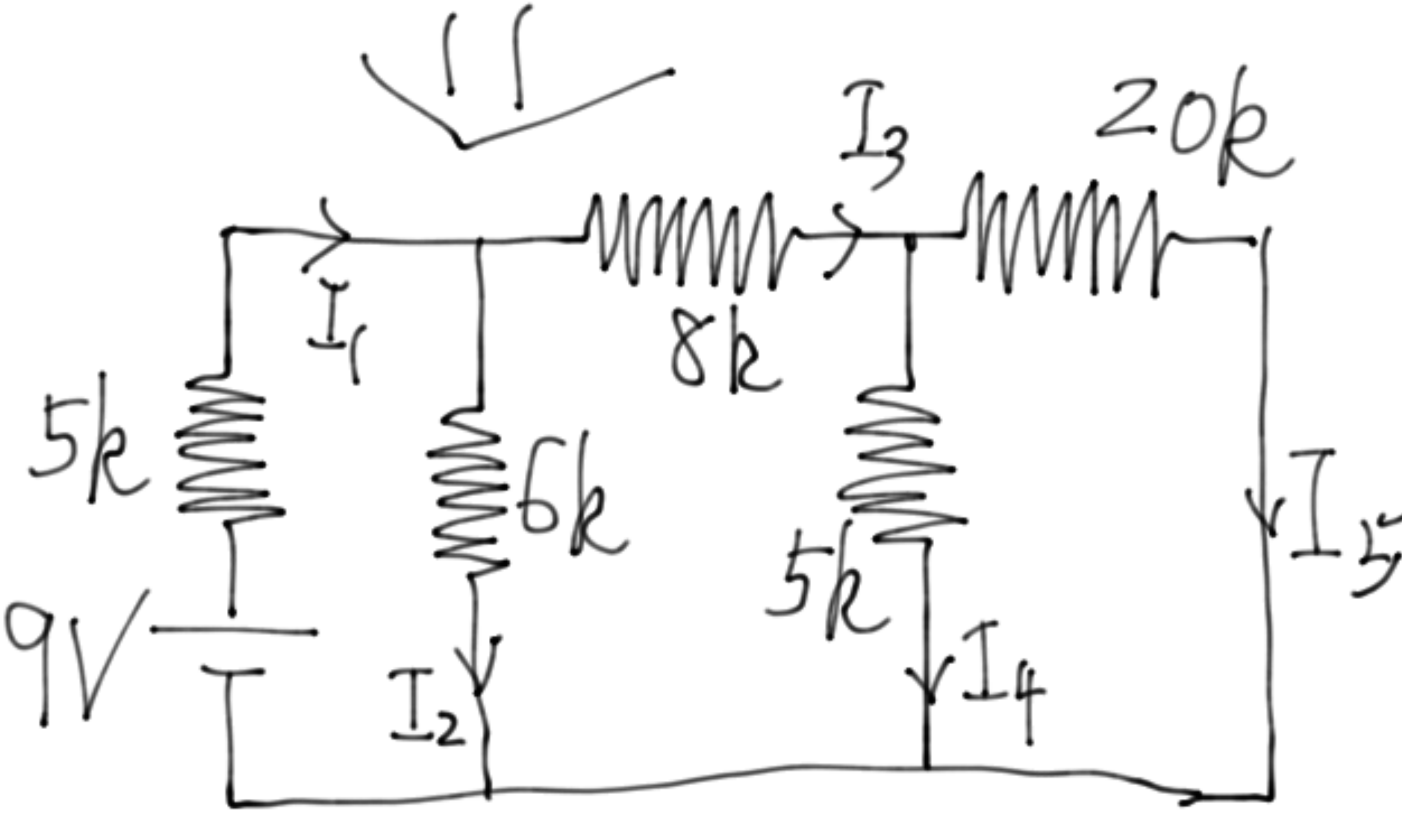
Now restore circuit to find current in each resistor.



$$I_2 = I_1 \cdot \frac{1/6}{1/6 + 1/12} = 0.67\,mA$$

$$I_3 = I_1 \cdot \frac{1/12}{1/6 + 1/12} = 0.33\,mA$$





$$I_4 = I_3 \cdot \frac{1/5}{1/5 + 1/20}$$

$$= 0.26\ mA$$

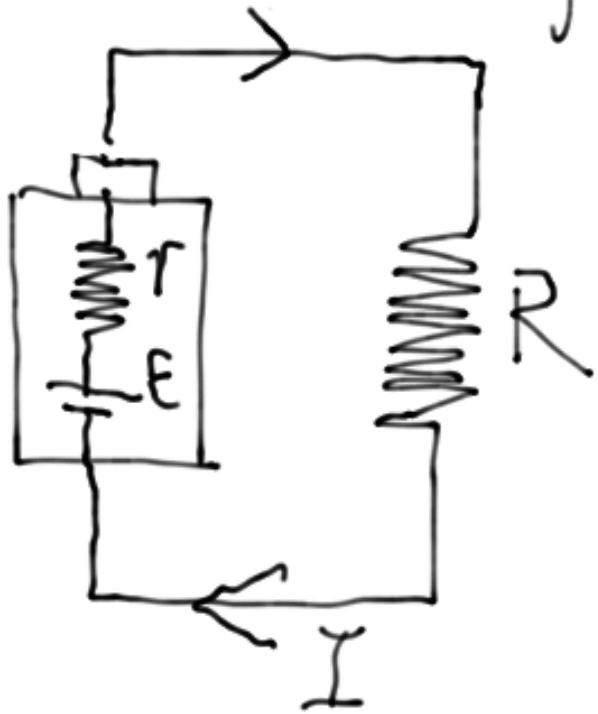
$$I_5 = I_3 - I_4$$

$$= 0.33 - 0.26$$

$$= 0.07$$

(mA)

* Practical emf, ammeters, and voltmeters



ideal battery: $r = 0$

and $I = \frac{\varepsilon}{r}$

practical battery:

$r > 0$ internal resistance

Load resistor R and internal resistor r are in series. $I = \frac{\varepsilon}{R+r}$

voltage on load resistor (terminal voltage):

$$\Delta V_R = I \cdot R = \left(\frac{\varepsilon}{R+r}\right) R$$

$$= \varepsilon \cdot \frac{R}{R+r} < \varepsilon$$

Load voltage < emf because of r.

Same result can be obtained by voltage divider rule:

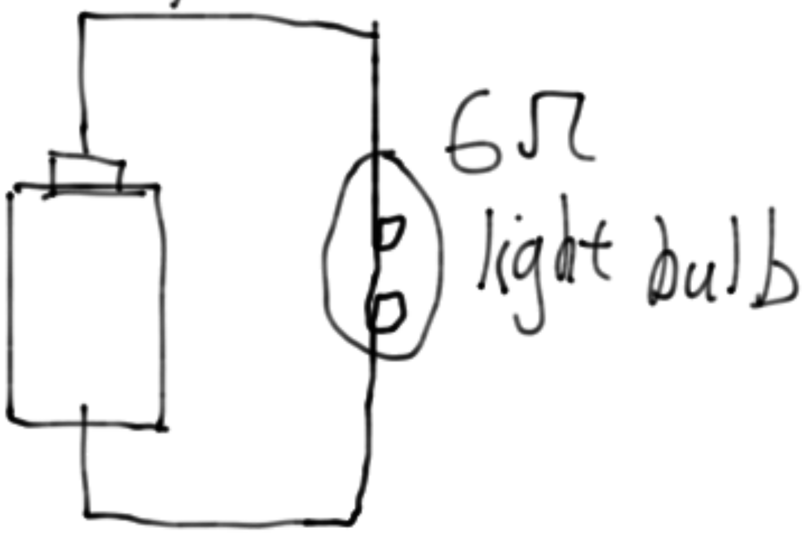
$$\Delta V_R = \varepsilon \cdot \frac{R}{R+r}$$

Or $\Delta V_R = \varepsilon - Ir$

$$= \varepsilon - \left(\frac{\varepsilon}{R+r}\right) r = \varepsilon \cdot \frac{R}{R+r}$$

When $r \ll R$, $\Delta V_R \rightarrow \varepsilon$.

Example:



battery: $\mathcal{E} = 3\,V$

$r = 1\,\Omega$

$P = ?$ on light bulb?

$\Delta V = ?$

$$\Delta V = \mathcal{E} \cdot \frac{R}{R+r} = 3 \cdot \frac{6}{1+6}$$

$$= 2.57\,(V)$$

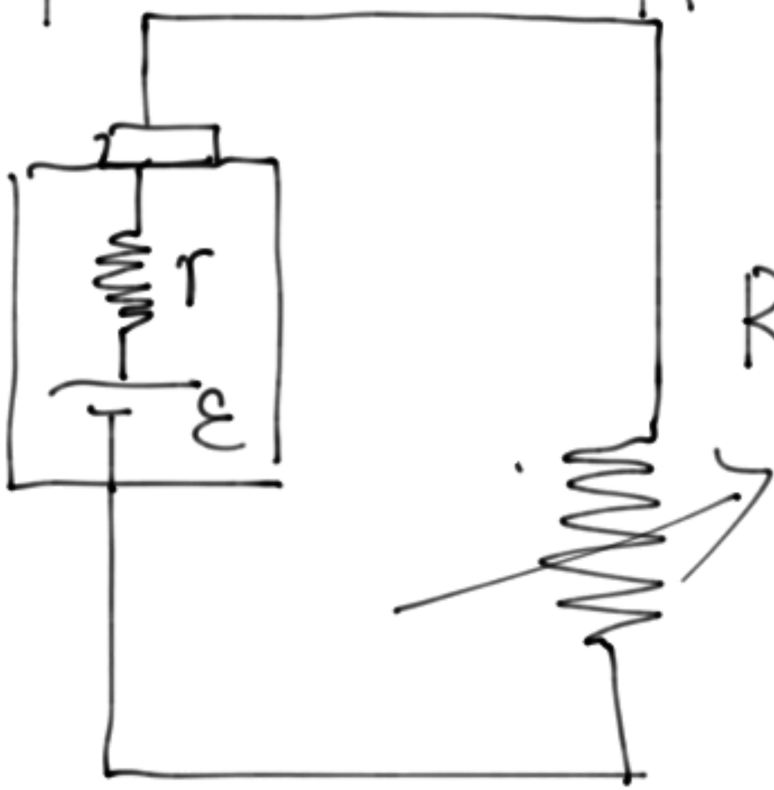
$$P = \frac{(\Delta V)^2}{R} = \frac{(2.57)^2}{6} = 1.10\,(W)$$

Alternative method:

$$I = \frac{\mathcal{E}}{R+r} = \frac{3}{1+6} = 0.428\,(A)$$

$$\Delta V = IR = 2.57\,(V) \qquad P = I^2 R = 1.10\,(W)$$

open circuit : $R \to \infty$



$R \to \infty$

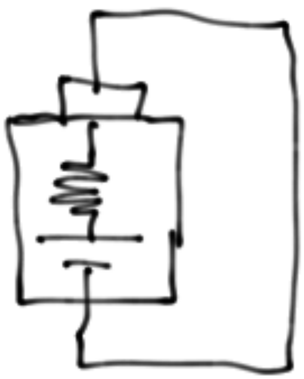
$$\Delta V_R = \varepsilon \cdot \frac{R}{R+r} = \varepsilon \cdot \frac{1}{\left(1+\frac{r}{R}\right)}$$

$R \to \infty$

$\Delta V_R \to \varepsilon$

For open circuit : $\Delta V_R = \varepsilon$

Short circuit :



$R \to 0$

$$\Delta V_R = \varepsilon \cdot \frac{1}{1 + r/R} \to 0$$

all voltage drops on internal resistor.

r is usually small.

For short circuit, current $I = \frac{\varepsilon}{r}$ is large.

power $P = \frac{\varepsilon^2}{r}$ is also large.

Consider a short circuited car battery:

$$\varepsilon = 12\,V, \quad r = 0.02\,\Omega$$

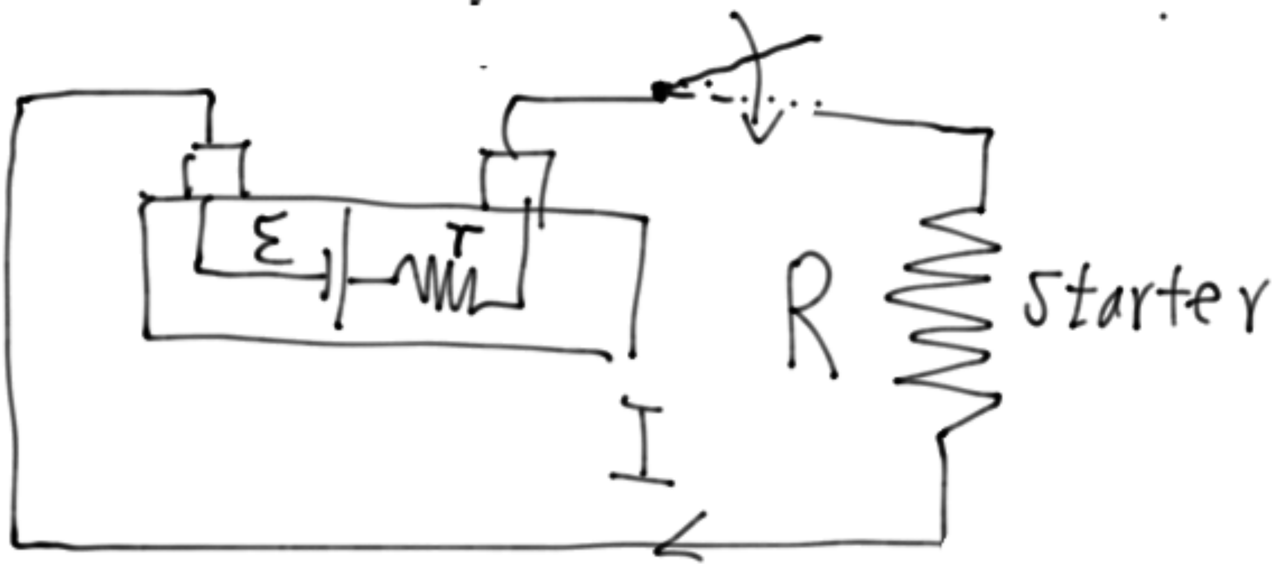
$$I = \frac{\varepsilon}{r} = \frac{12}{0.02} = 600\,(A)$$

$$P = \frac{\varepsilon^2}{r} = 7200\,(W)$$

chemical energy rapidly converts to heat.

Example:

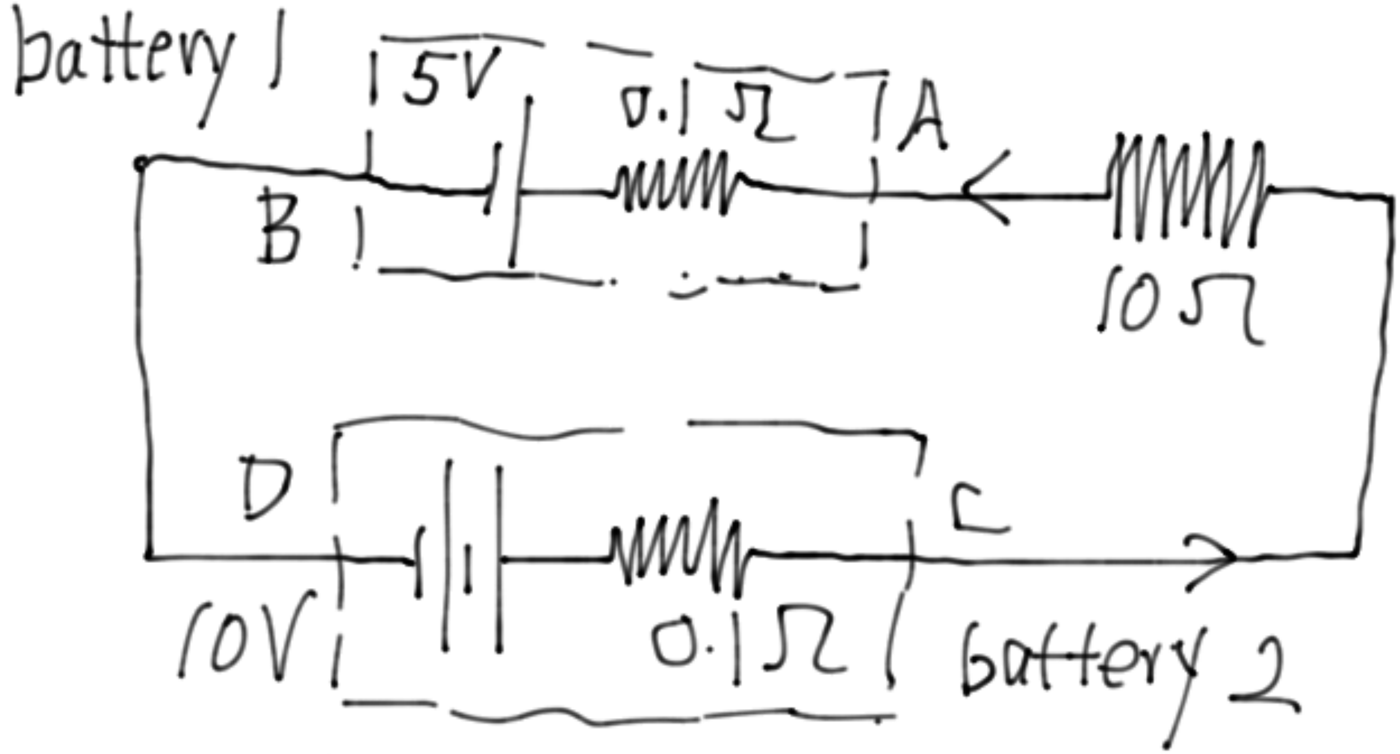
$\mathcal{E}=12$ V car battery. Starter (R) draws 95 A and terminal voltage drops to 8.4V. $R=?$ $r=?$



$$V_R = \mathcal{E} - Ir \quad , \quad r = \frac{\mathcal{E}-V_R}{I} = \frac{12-8.4}{95} = 0.038(\Omega)$$

$$R = \frac{V_R}{I} = \frac{8.4}{95} = 0.088(\Omega)$$

Is it possible to have terminal Voltage > $\mathcal{E}$?

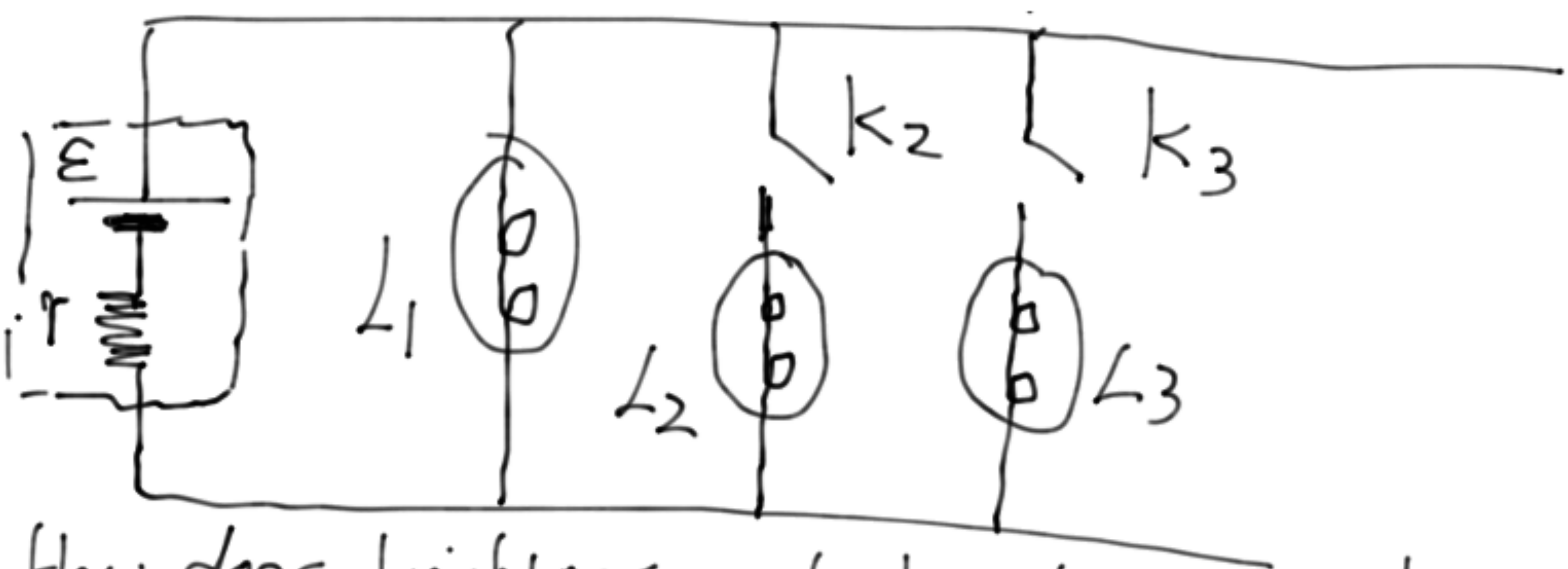


$$I = \frac{10-5}{10+0.1+0.1} = 0.5\text{ A}$$

terminal voltages $\Delta V_1 = V_A - V_B = 5 + 0.5 \times 0.1 = 5.05\text{V} > 5\text{V}$

$\Delta V_2 = V_C - V_D = 10 - 0.5 \times 0.1 = 9.95\text{V} < 10\text{V}$

Question L_1, L_2, L_3, identical light bulbs with resistance R



How does brightness of L_1 changes when k_2 (and then k_3) is closed?

Load voltage $V_L = \varepsilon \cdot \dfrac{R_L}{R_L + r} = \varepsilon \cdot \dfrac{1}{1 + r/R_L}$

when k_2 closed, $R_L \searrow$ $V_L \searrow$, brightness $\searrow$

Similar with k_3.

Ammeter

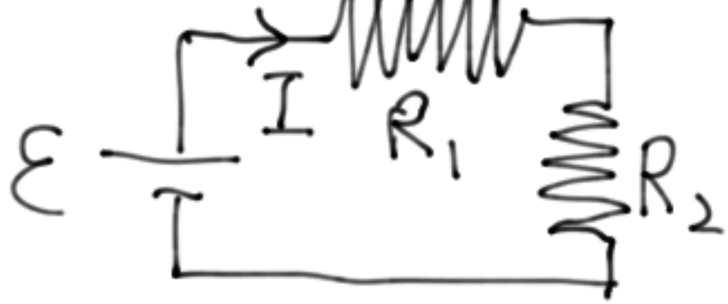
— Measure current

— put in series

— low internal resistance (compared to load)

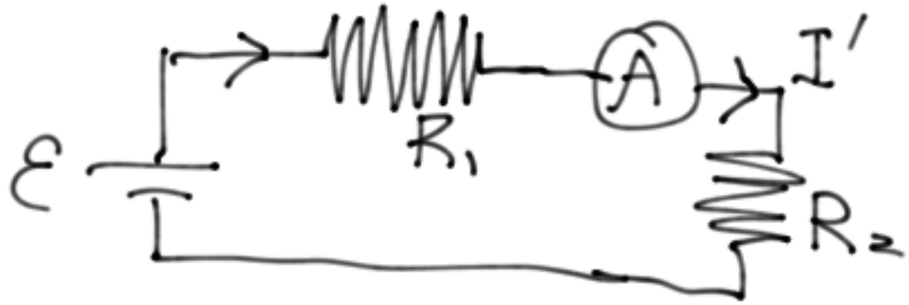
symbol for ammeter

A circuit to be measured



$$I = \frac{\varepsilon}{R_1 + R_2}$$

insert ammeter in series



$$I' = \frac{\varepsilon}{R_1 + R_2 + r_m}$$

when $r_m \ll R_1, R_2$, $I' \approx I$

Ideal ammeter

$r_m = 0$

Voltmeter

- Measure voltage
- put in parallel
- Large internal resistance (compared to load)

Ideal voltmeter

$r_m = \infty$

r_m: internal resistance.

symbol for voltmeter

A circuit to be measured

put (V) in parallel with R_2

V_2'

$$R_2' = \frac{R_2 r_m}{R_2 + r_m} = \frac{R_2}{(1+R_2/r_m)}$$

When $r_m \gg R_2$, $R_2' \approx R_2$, $V_2' \approx V_2$

Expand range of ammeter and voltmeter:

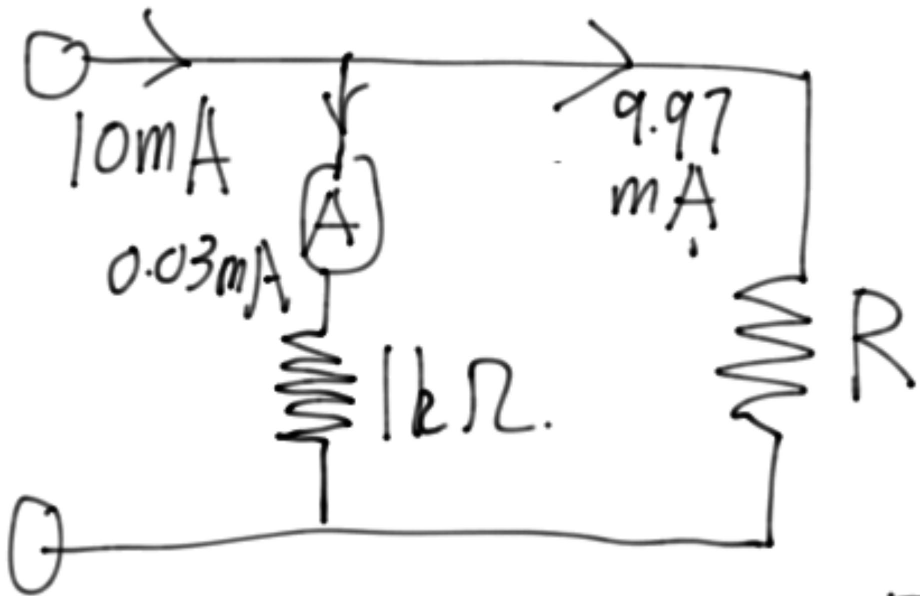
An ammeter has full range of $30\mu A$.
$r_m = 1k\Omega$. Can you expand range to $10mA$?

Question: Is $1k\Omega$ too large for ammeter?
Well, compare it to load resistance.
typical load for a circuit with $30\mu A$,
driven by $3V$ emf:

$$\frac{3V}{30\mu A} = \frac{3V}{3\times10^{-5}A} = 10^5\Omega = 100k\Omega. \gg 1k\Omega.$$

r_m is much smaller than load.

How to expand range to 10mA?
use a shunt resistor.



Method. #1

$$(1000\,\Omega)(0.03\,\text{mA}) = (10\,\text{mA})\frac{1000R}{1000+R}$$

$$3 = \frac{1000\,R}{1000+R}$$

$$R = 3.0\,\Omega$$

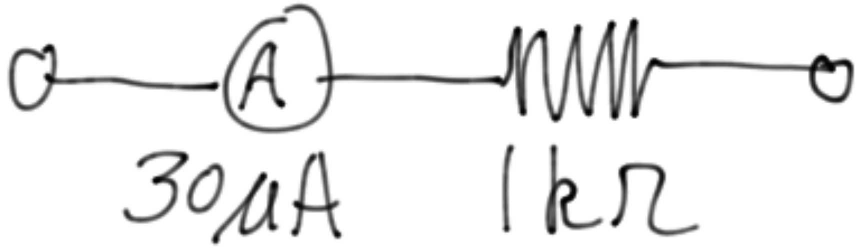
Method 2: $$10\,\text{mA}\cdot\frac{\frac{1}{1000}}{\frac{1}{1000}+\frac{1}{R}} = 0.03\,\text{mA}$$

Method 3: $$(9.97\,\text{mA})\,R = (0.03\,\text{mA})\,1000\,\Omega$$

Result: put a 3.0Ω resistor in parallel.

Exercise: How to expand range to 100mA?

— How to convert a ammeter (galvanometer) to voltmeter?

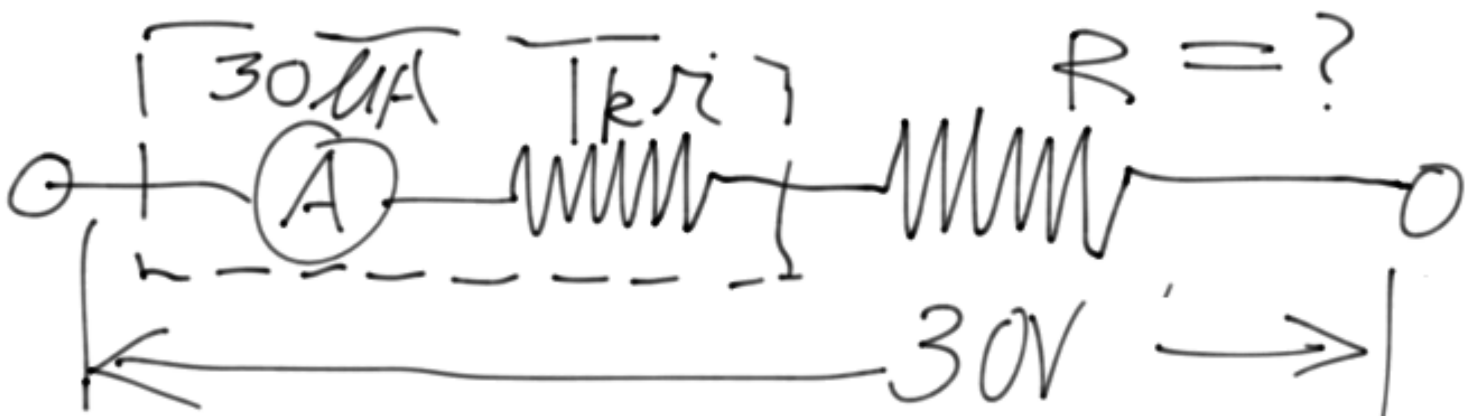


$0.03\,mA \times 1\,k\Omega = 0.03\,V$

original meter can only measure 0.03V.

How to expand range to measure 30V?

put a large resistor in series.



Method #1

$$\frac{30\,V}{R+1000} = 30\,\mu A = 3\times10^{-5}\,A$$

$$R = 10^{6}\,\Omega = 1\,M\Omega$$

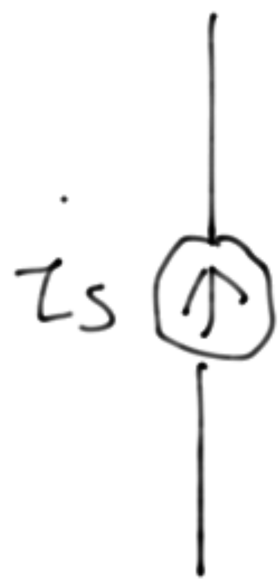
Method. #2

$$30V\cdot\frac{1000}{R+1000} = 0.03\,V$$

$$R = 1\,M\Omega.$$

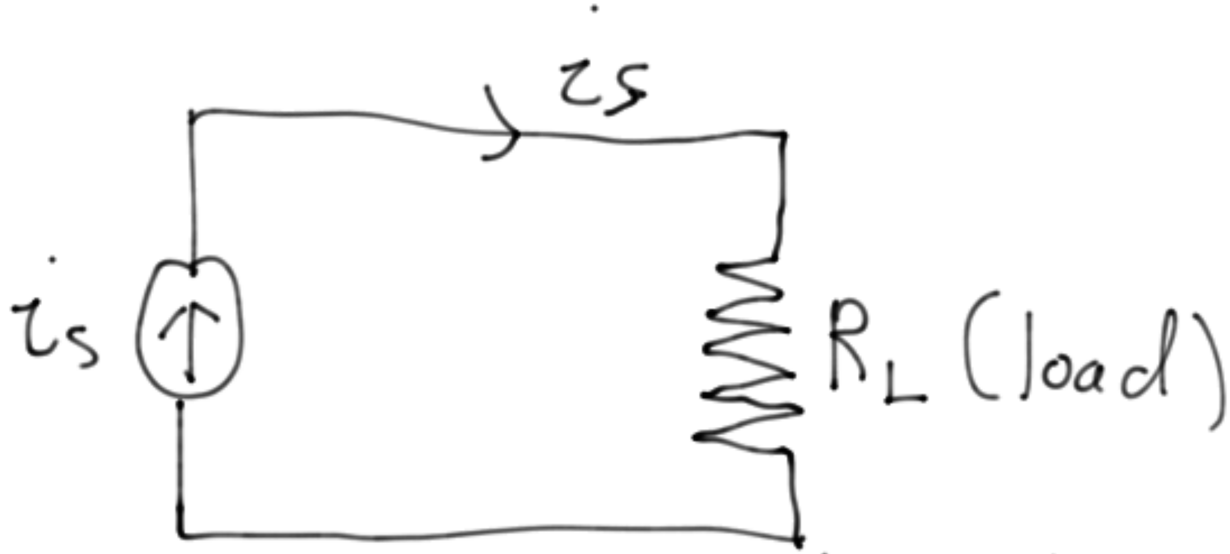
$$r_m' = r_m + R = 1000 + 10^{6} \simeq 10^{6}\,\Omega.$$

* Current sources



ideal current source

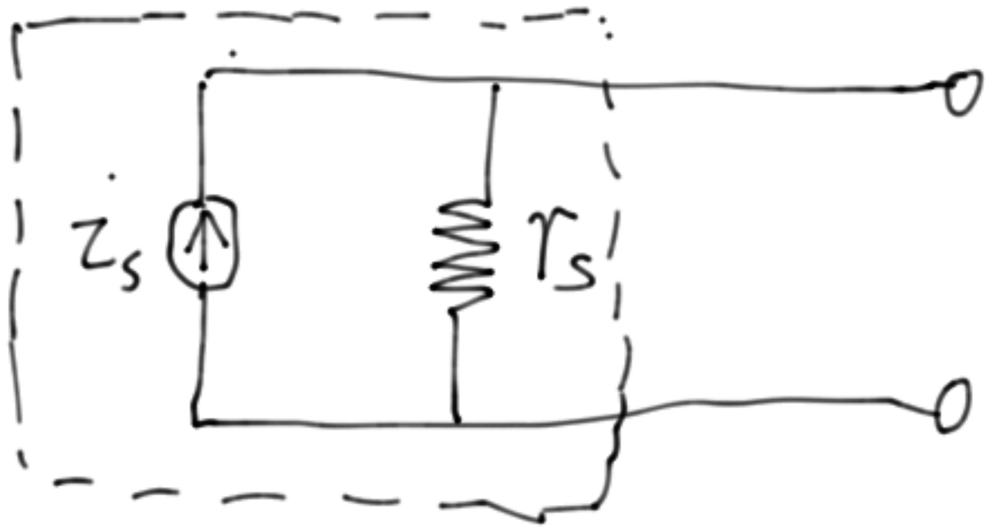
internal resistance $r_s = \infty$



The output current supplied by ideal current source does not depend on load.

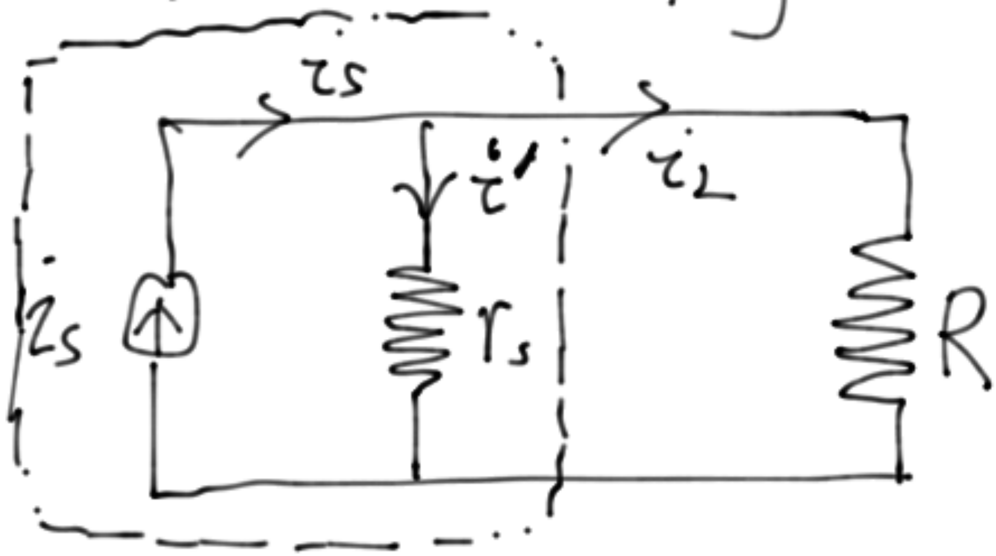
Practical current source:

ideal current source in parallel with finite r_s.



$r_s < \infty$

when connecting to a load resistor,



$r_s // R$

output current:

$$i_L = i_s \frac{1/R}{1/r_s + 1/R}$$

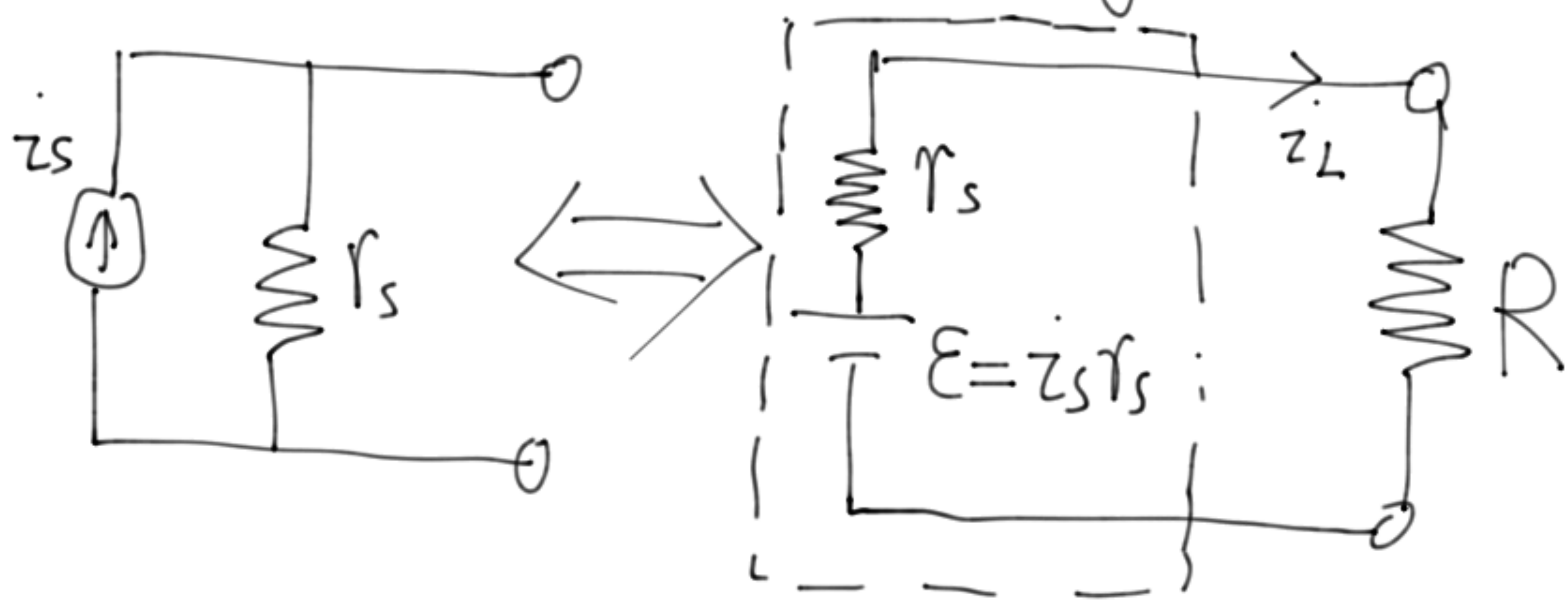
$$i_L = i_S \cdot \frac{r_s}{R + r_s} = i_S \frac{1}{\left(\frac{R}{r_s} + 1\right)}$$

i_L depends on R_L

When $R \ll r_s$, $i_L \approx i_S$

So a good current source typically has a large internal resistance (compared to load resistance).

Current source is equivalent to a voltage source in series with a large resistor.



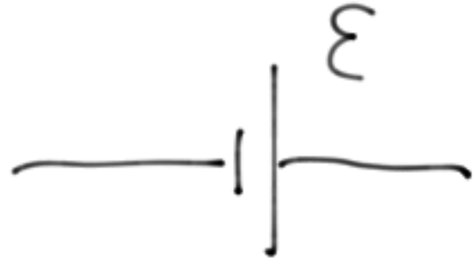
$$i_L = \frac{\varepsilon}{r_s + R} = \frac{i_s r_s}{r_s + R} = i_s \cdot \frac{1}{1 + R/r_s}$$

when $r_s \gg R$, $i_L \approx i_s$

Same as i_L for current source.

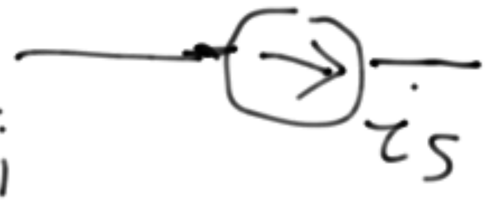
Recap on sources

ideal voltage source



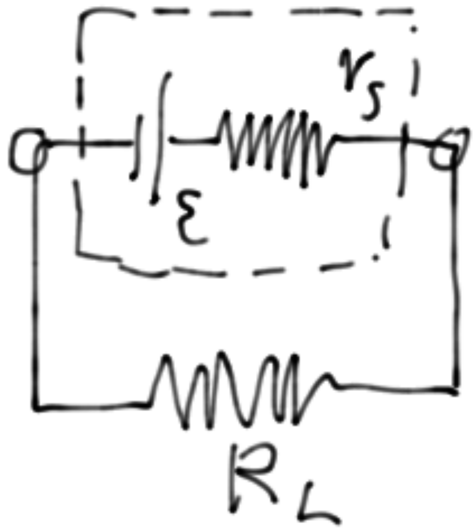
$r_s = 0$

ideal Current Source



$r_s = \infty$

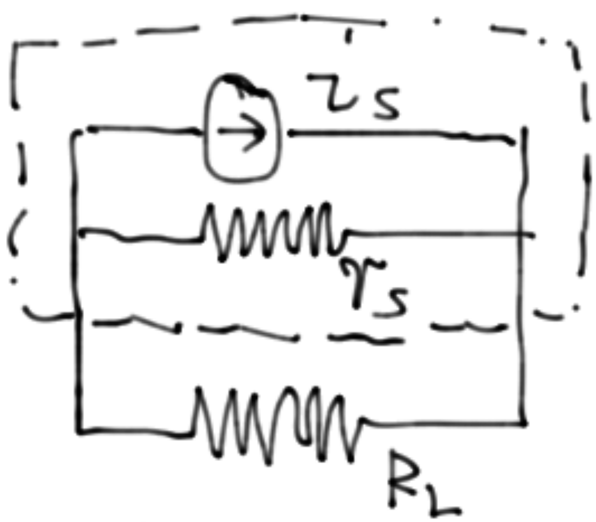
practical voltage source



$r_s > 0$

$$V_L = \varepsilon \cdot \frac{R_L}{r_s + R_L}$$

practical Current Source:



$r_s < \infty$

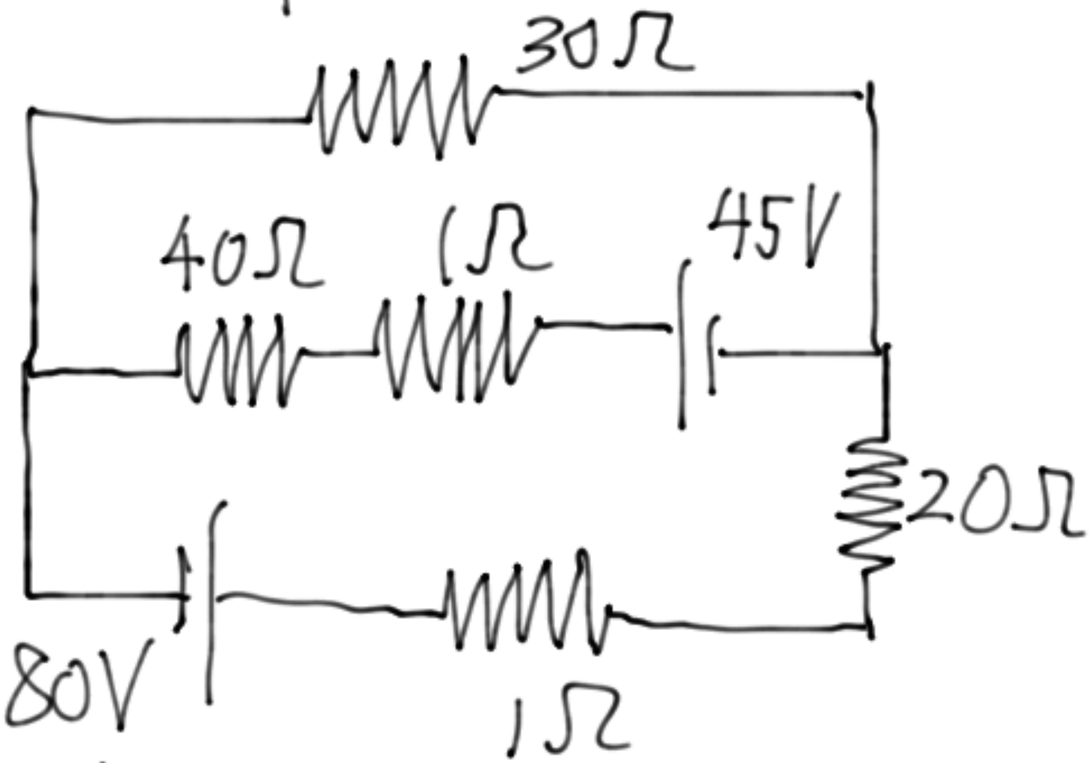
$$i_L = i_s \cdot \frac{r_s}{R_L + r_s}$$

They are equivalent when $\varepsilon = i_s \cdot r_s$

* Kirchhoff's laws

Some circuits are too complicated for combination rules.

Two emfs in two branches



Instead, we use Kirchhoff's laws.

Kirchhoff's current law (KCL)
(junction rule)

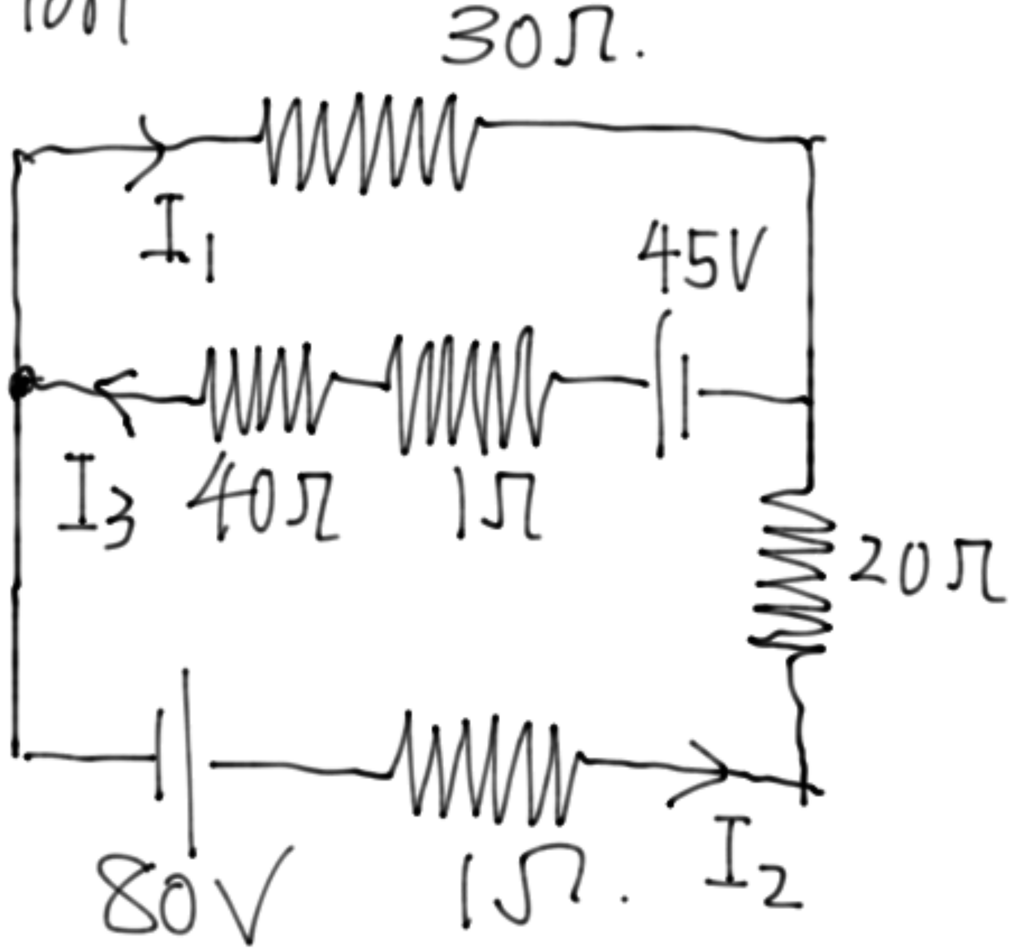
At any junction point, the sum of all currents entering the junction must equal the sum of all currents leaving the junction.

Kirchhoff's voltage law (KVL)
(loop rule)

The sum of the changes in potential around any closed loop of a circuit must be zero

KCL is based on charge conservation.
KVL is based on energy conservation.

Approach and solution



1. Label the currents and directions in each branch
   Direction: arbitrary.
2. Identify the unknowns: I_1, I_2, I_3

So need three equations.

3. KCL

$$I_3 = I_1 + I_2 \quad (1)$$

4. KVL

upper loop cw $\sum \Delta V = 0$

$$-30I_1 + 45 - 1 \cdot I_3 - 40I_3 = 0$$

$$-30I_1 + 45 - 41I_3 = 0 \quad (2)$$

lower loop cw

$$+40I_3 + I_3 - 45 + 20I_2 + I_2 - 80 = 0$$

$$21I_2 + 41I_3 - 125 = 0 \quad (3)$$

substitute (1) into (2) and (3)

$$71 I_1 + 41 I_2 = 45$$

$$41 I_1 + 62 I_2 = 125 \Rightarrow I_2 = 2.02 - 0.66 I_1$$

$$I_1 = -0.86 \ (A)$$

$\hookrightarrow$ negative sign, I_1 is opposite to assumed direction.

$$I_2 = 2.59 \ (A) \quad I_3 = 1.73 \ (A)$$

Note:

you can choose the outer loop instead of the lower loop for Eq. (3)

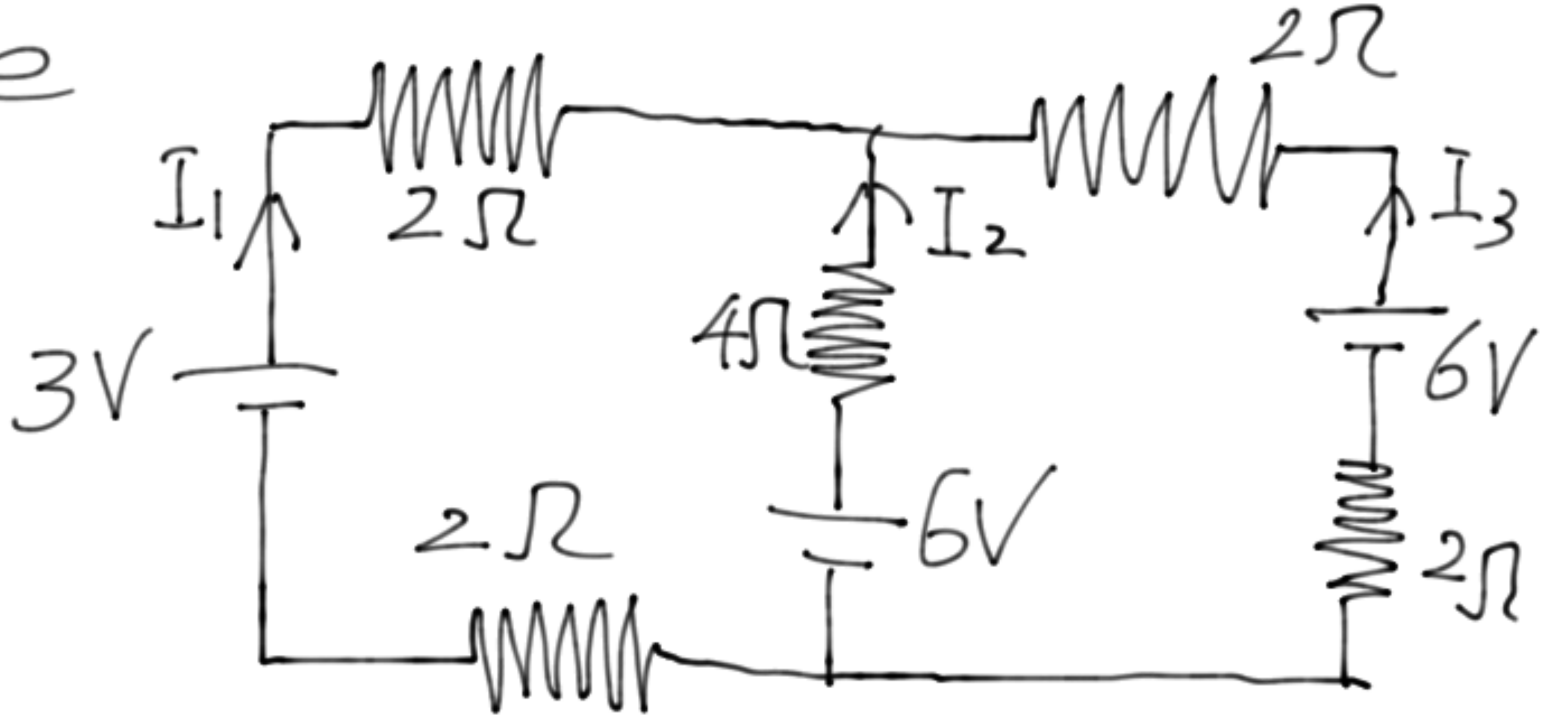
outer loop. (CW)

$$-30I_1 + 20I_2 + I_2 - 80 = 0$$

$$-30I_1 + 21I_2 - 80 = 0 \quad (3')$$

Solve (1), (2), (3') for I_1, I_2, I_3.

Example



$$I_1 + I_2 + I_3 = 0 \quad (1)$$

left loop cw $\quad 3 - 2I_1 + 4I_2 - 6 - 2I_1 = 0$

$$4I_1 - 4I_2 + 3 = 0 \quad (2)$$

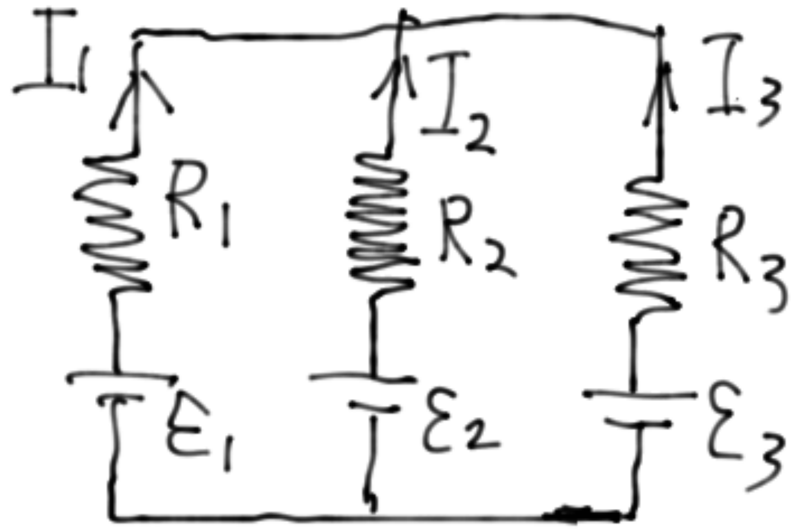
right loop cw $\quad 6 - 4I_2 + 2I_3 - 6 + 2I_3 = 0$

$$I_2 = I_3 \quad (3)$$

$I_1 = -0.5(A) \quad I_2 = I_3 = 0.25\,(A)$

* Mesh analysis

In Kirchhoff's laws, we use branch currents as variables.

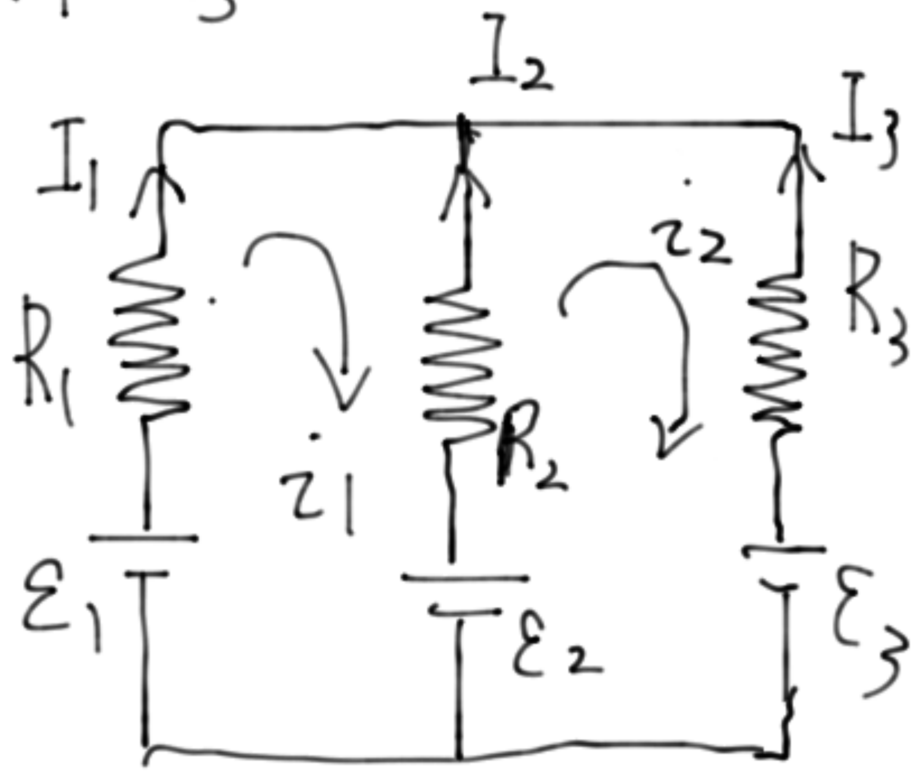


3 variables.

3 equations.

$$I_1 + I_2 + I_3 = 0$$

Define mesh currents i_1 and i_2



$$I_1 = i_1$$

$$I_3 = -i_2$$

$$I_2 = i_2 - i_1$$

$$I_1 + I_2 + I_3 = i_1 + (-i_2) + (i_2 - i_1) = 0$$

By defining mesh currents i_1 and i_2, KCL is automatically satisfied.

Loop equations (KVL)

left mesh: $\varepsilon_1 - I_1 R_1 + I_2 R_2 - \varepsilon_2 = 0$

$$\varepsilon_1 - i_1 R_1 + (i_2 - i_1) R_2 - \varepsilon_2 = 0$$

$$i_1 (R_1 + R_2) - i_2 R_2 = \varepsilon_1 - \varepsilon_2 \quad (1)$$

Right mesh: $-R_2 i_1 + (R_2 + R_3) i_2 = \varepsilon_2 - \varepsilon_3 \quad (2)$

similarly

2 equations, 2 variables