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Basics of Information Theory

(Search and Data Mining)

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Information Theory

- Information theory deals with uncertainty and the transfer or storage of quantified information in the form of bits.
- Useful concepts for text analysis
  - Entropy
  - KL divergence
  - Mutual information

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Entropy

- Measuring uncertainty of a random variable

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Entropy: A Motivating Example

- We use random variable X to represent the outcomes of a coin flipping.

$$X = \begin{cases} 1 & if\ heads \\ 0 & if\ tails \end{cases}$$

- P(X=1)=0.5 is more random than P(X=1)=0.9.
- How does one quantitatively measure the randomness of a random variable like X?

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Entropy: Definition

- Entropy H(X) measures the uncertainty/randomness of random variable X.

$$H(X) = H(p) = \sum_{x \in \Omega} -p(x)\log p(x) \quad \Omega = all\ \ possible\ values$$
$$Define\ \ 0\log 0 = 0,\ \ \log = \log_2$$

- Interpretation of Entropy:
  - Minimum average number of bits to compress values of X
  - The more random X is, the harder to compress

$$"Information\ of\ x" = "\#bits\ to\ code\ x" = -\log p(x) \quad H(X) = E_p[-\log p(x)]$$

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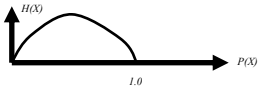
An example of Entropy (X: Head)

- For a fair coin, we have p(X=1) = P(X=0) =0.5.
  So, we get $H(X) = -0.5 * log0.5 - 0.5 * log0.5 = 1$
- For a complete biased coin with p(X=1) =1, P(X=0) =0.
  So, we get $H(X) = -0 * log0 - 1 * log1 = 0$

A fair coin has the maximum information, and is hardest to compress
A biased coin has some information, and can be compressed to <1 bit on average
A completely biased coin has no information, and needs only 0 bit



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Entropy of words

- Let W be the random variable that denotes whether a word occurs in a document.
  - W=1 if the word occurs
  - W=0 otherwise
- How is the $value\ of\ H(W_{the})$ compared with the value of $H(W_{computer})$?

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Mutual Information:

- measuring the correlation of two random variables

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Conditional Entropy

- Conditional entropy is used to quantify uncertainties of conditional probabilities.
- H(X|Y): the expected uncertainty of X given that we observe Y.
  - H(X|Y)=0, if X is completely determined by Y.
  - H(X|Y)=H(X), if X and Y are independent.
- Mathematically, we have

$$H(Y \mid X) = \sum_{x \in \Omega_X} p(x) H(Y \mid X = x)$$
$$= -\sum_{x \in \Omega_X} p(x) \sum_{y \in \Omega_Y} p(y \mid x) \log p(y \mid x)$$
$$= -\sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} p(x, y) \log p(y \mid x) \ = -E(\log p(Y \mid X))$$

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Mutual Information

- It is defined as the reduction of entropy X due to knowledge Y, i.e, I(X;Y).

$$I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = H(X) - H(X \mid Y) = H(Y) - H(Y \mid X)$$

- It can measure the correlation of two random variables.
- Properties:
  - I(X;Y)≥0;
  - I(X;Y)=I(Y;X);
  - I(X;Y)=0 iff X & Y are independent

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Kullback-Leibler (KL) divergence

- comparing two distributions

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Cross Entropy H(p,q)

Assume X has the distribution p, but we encode X with a code optimized for a wrong distribution q, what it the expected number of bits?

$$H(p,q) = E_p[-\log q(x)] = -\sum_{x \in \Omega} p(x) \log q(x)$$

Intuitively, H(p,q) ≥ H(p).

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Kullback-Leibler (KL) Divergence

Assume X has the distribution p, but we encode it with a code optimized for a wrong distribution q. How many bits would we waste?

$$D(p\,\|\,q) = H(p,q) - H(p) = \sum_{x\in\Omega} p(x)\log\frac{p(x)}{q(x)}$$

Properties:

- *D(p||q)≥0*
- *D(p||q)≠D(q||p)*
- *D(p||q)=0 iff p=q*

⇒ **KL-divergence is often used to measure the distance between two distributions**

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