HW7 Solution

6.29 a. $1 - np \le (1-p)^n$

$1 \le (1-p)^n + np$

Let $h(x) = (1-x)^n + nx$

$\frac{d\,h(x)}{dx} = -n(1-x)^{n-1} + n = 0$

$x = 0$

$h(x)_{min} = h(0) = 1$

Thus $1 \le h(x)$

$1 \le (1-p)^n + np$

$1 - np \le (1-p^n)$

b. $(1-p)^n \le e^{-np}$

$\log(1+x) \le x$

$n\log(1+x) \le nx$

Let $x = -p$ $\quad n\log(1-p) \le -np$

take exp bothside $\quad (1-p)^n \le e^{-np}$

Problem 6.31

Here's a problem first solved by Isaac Newton (he did it without calculators or computers). Which is more likely, (a) getting at least one six in a throw of six dice, (b) getting at least two sixes in a throw of twelve dice, or (c) getting at least three sixes in a throw of eighteen dice?

Solution to Problem 6.31

It is easier to calculate 1 minus the complementary probability.

a) $\Pr[\text{at least one 6 in 6 dice}] = 1 - (5/6)^6 = 0.665$

b) $\Pr[\text{at least two 6's in 12 dice}] = 1 - (5/6)^{12} - \binom{12}{1}(1/6)(5/6)^{11} = 0.619$

c) $\Pr[\text{at least three 6's in 18 dice}] = 1 - (5/6)^{18} - \binom{18}{1}(1/6)(5/6)^{17} - \binom{18}{2}(1/6)^2(5/6)^{16} =$ 0.597

Thus, one six in a throw of six dice is most likely.

Problem 7.2

Let $\boldsymbol{X}$ have density $f(x) = cx^2$ for $0 \le x \le 2$ and $f(x) = 0$ for other values of x.

a) What is c?

b) What is $F(x)$?

c) What are $E[\boldsymbol{X}]$ and $\mathrm{Var}[\boldsymbol{X}]$?

Solution to Problem 7.2

a) $1 = \int_0^2 f(x)\,dx = \int_0^2 cx^2\,dx$ implies $c = 3/8$.

b)

$$F(x) = \Pr[\boldsymbol{X} \le x] = \int_{-\infty}^{x} f(v)\,dv = \begin{cases} 0 & x < 0 \\ x^3/8 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

c) $E[\boldsymbol{X}] = \int_0^2 x f(x)\,dx = \int_0^2 (3/8)x^3\,dx = 3/2$

[redacted]

d) $E[\boldsymbol{X}^2] = \int_0^2 x^2 (3/8) x^2\,dx = 12/5$, $\mathrm{Var}[\boldsymbol{X}] = E[\boldsymbol{X}^2] - E[\boldsymbol{X}]^2 = 3/20$.

Problem 7.3

Let $\boldsymbol{X}$ have density $f(x) = cx(1-x)$ for $0 \le x \le 1$ and $f(x) = 0$ for other values of x.

a) What is c?

b) What is $F(x)$?

c) What are $E[\boldsymbol{X}]$ and $\mathrm{Var}[\boldsymbol{X}]$?

Solution to Problem 7.3

a) $1 = \int_0^1 f(x)\,dx = \int_0^1 cx(1-x)\,dx$ implies $c = 6$.

b)

$$F(x) = \Pr[\boldsymbol{X} \le x] = \int_{-\infty}^{x} f(v)\,dv = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

c) $E[\boldsymbol{X}] = \int_0^1 x f(x)\,dx = \int_0^1 6x^2(1-x)\,dx = 1/2$

d) $E[\boldsymbol{X}^2] = \int_0^1 6x^3(1-x)\,dx = 3/10$, $\mathrm{Var}[\boldsymbol{X}] = E[\boldsymbol{X}^2] - E[\boldsymbol{X}]^2 = 1/20$.

Solution to Problem 7.4

[redacted]

$$\begin{aligned}
F_{\boldsymbol{Y}}(y) &= \Pr[\boldsymbol{Y} \le y] = \Pr[\boldsymbol{X}^2 \le y] \\
&= \Pr[-\sqrt{y} \le \boldsymbol{X} \le \sqrt{y}] \\
&= \sqrt{y} \text{ for } 0 \le y \le 1 \\
f_{\boldsymbol{Y}}(y) &= \frac{d}{dy} F_{\boldsymbol{Y}}(y) = \frac{1}{2\sqrt{y}} \text{ for } 0 \le y \le 1 \\
E[\boldsymbol{Y}] &= \int_0^1 y \frac{1}{2\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 \sqrt{y}\,dy = \frac{1}{2}\cdot\frac{2}{3}\,y^{3/2}\Big|_{y=0}^{1} = \frac{1}{3} \\
&= E[\boldsymbol{X}^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_{x=0}^{1} = \frac{1}{3} \\
E[\boldsymbol{Y}^2] &= \int_0^1 y^2 \frac{1}{2\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 y^{3/2}\,dy = \frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5} \\
&= E[\boldsymbol{X}^4] = \int_0^1 x^4\,dx = \frac{x^5}{5}\Big|_{x=0}^{1} = \frac{1}{5} \\
\mathrm{Var}[\boldsymbol{Y}] &= \frac{1}{5} - \frac{1}{3^2} = \frac{4}{45}
\end{aligned}$$

Problem 7.6

Let $\boldsymbol{X}$ be exponential with parameter λ.

a) What are $F_{\boldsymbol{X}}(x|\boldsymbol{X} < x_0)$ and $f_{\boldsymbol{X}}(x|\boldsymbol{X} < x_0)$?

b) What is the conditional mean $E\left[\boldsymbol{X}|\boldsymbol{X} < x_0\right]$?

Solution to Problem 7.6

$$
\begin{aligned}
F_{\boldsymbol{X}}(x|\boldsymbol{X} < x_0) &= \frac{\Pr\left[\boldsymbol{X} \le x \cap \boldsymbol{X} \le x_0\right]}{\Pr\left[\boldsymbol{X} \le x_0\right]} \\
&= \begin{cases} \dfrac{\Pr\left[\boldsymbol{X} \le x\right]}{\Pr\left[\boldsymbol{X} \le x_0\right]} & 0 \le x \le x_0 \\ \dfrac{\Pr\left[\boldsymbol{X} \le x_0\right]}{\Pr\left[\boldsymbol{X} \le x_0\right]} & x_0 < x \end{cases} \\
&= \begin{cases} \dfrac{1-e^{-\lambda x}}{1-e^{-\lambda x_0}} & 0 \le x \le x_0 \\ 1 & x_0 < x \end{cases} \\
f_{\boldsymbol{X}}(x|\boldsymbol{X} < x_0) &= \frac{d}{dx} F_{\boldsymbol{X}}(x|\boldsymbol{X} < x_0) = \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x_0}} \quad 0 \le x \le x_0
\end{aligned}
$$

$$
\begin{aligned}
E\left[\boldsymbol{X}|\boldsymbol{X} < x_0\right] &= \int_0^{x_0} x \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x_0}}\, dx \\
&= \frac{1}{\lambda(1-e^{-\lambda x_0})} \int_0^{\lambda x_0} y e^{-y}\, dy && (y = \lambda x) \\
&= \frac{1}{\lambda(1-e^{-\lambda x_0})} \left(-y e^{-y}\Big|_{y=0}^{\lambda x_0} - \int_0^{\lambda x_0} -e^{-y}\, dy \right) && (\text{by parts, } u = y,\ dv = e^{-y}\, dy) \\
&= \frac{1-(\lambda x_0 + 1)e^{-\lambda x_0}}{\lambda(1-e^{-\lambda x_0})} \\
&= \frac{1}{\lambda} - x_0 \left(\frac{e^{-\lambda x_0}}{1-e^{-\lambda x_0}} \right) \\
&= \text{(original mean) - (correction for eliminating tail)}
\end{aligned}
$$

Problem 7.7

If $\boldsymbol{X}$ is exponential with parameter λ, what are the density and distribution of $\boldsymbol{Y} = \boldsymbol{X}^3$?

Solution to Problem 7.7

$$
\begin{aligned}
F_{\boldsymbol{Y}}(y) &= \Pr\left[\boldsymbol{Y} \le y\right] = \Pr\left[\boldsymbol{X}^3 \le y\right] = \Pr\left[\boldsymbol{X} \le y^{1/3}\right] = 1 - e^{-\lambda y^{1/3}} \\
f_{\boldsymbol{Y}}(y) &= \frac{d}{dy} F_{\boldsymbol{Y}}(y) = \frac{d}{dy}\left(1 - e^{-\lambda y^{1/3}}\right) = \frac{\lambda}{3 y^{\frac{2}{3}} e^{\lambda \sqrt[3]{y}}}
\end{aligned}
$$

Problem 7.10

If $\boldsymbol{X} \sim U(0,1)$, what are the density and distribution of $\boldsymbol{Y} = -\log \boldsymbol{X}$?

Solution to Problem 7.10

$$
\begin{aligned}
F_{\boldsymbol{Y}}(y) &= \Pr\left[-\log(\boldsymbol{X}) \le y\right] = \Pr\left[\boldsymbol{X} \ge e^{-y}\right] = 1 - e^{-y} \\
f_{\boldsymbol{Y}}(y) &= \frac{d}{dy} F_{\boldsymbol{Y}}(y) = e^{-y} \text{ for } y > 0
\end{aligned}
$$

$\boldsymbol{Y}$ has the exponential distribution.

Problem 7.14

Using the life table in Example 7.2, estimate the following:

a) The probability a male lives to age 50?

b) The probability a female lives to age 50?

c) The conditional probability a male lives to 50 given they were alive at age 40?

d) The conditional probability a female lives to 50 given they were alive at age 40?

Solution to Problem 7.14

a) $\Pr[\text{Male lives to 50}] = 92588/100000 = 0.926$

b) $\Pr[\text{Female lives to 50}] = 95633/100000 = 0.956$

c) $\Pr[\text{Male lives to 50} | \text{alive at 40}] = 92588/95770 = 0.967$

d) $\Pr[\text{Female lives to 50} | \text{alive at 40}] = 95633/97679 = 0.979$

Problem 7.19

To reflect population statistics, the life table numbers must be updated with the birthrates. In the USA, 105 males are born for every 100 females. Using the life table as indicative of population dynamics, what is the probability a randomly selected 50 year old person is male?

Solution to Problem 7.19

From Table 7.6, the expected number of 50 year old males per 100000 births is 92588 and the number of females is 95633. Adjusting the males to the birthrate, $92588 \times 1.05 = 97217$. The estimated probability a randomly selected 50 year old is male is $97217/(97217+95633) = 0.504$.