Name: ____________________

1. (25 points) A small library has 100 books: 60 are history books, 50 are science, and 20 are others. Of the history and science books, 30 are on "history of science" and are counted in both categories. ($100 = 60 + 50 + 20 - 30$). A book is randomly selected (all books equally likely).
   (a) What is the probability it is a history book?
   (b) What is the probability it is "history of science" given it is a history book?
   (c) What is the probability it is "history of science" given it is history or science?
   (d) What is the probability it is history given it is science?
   (e) Are the events "it is a history book" and "it is a science book" independent? Why or why not?

a) $P(\text{history}) = \frac{60}{100} = 0.6$

b) $P(\text{history of science} \mid \text{history}) = \frac{30}{60} = 0.5$

c) $P(\text{hist of science} \mid \text{history or science}) = \frac{30}{60+50-30} = \frac{3}{8}$

d) $P(\text{history} \mid \text{science}) = \frac{P(\text{history} \cap \text{science})}{P(\text{science})} = \frac{30}{50} = 0.6$

e) $P(\text{history} \cap \text{science}) = \frac{30}{100} \stackrel{?}{=} P(\text{history})\,P(\text{science}) = \frac{60}{100} \times \frac{50}{100} = \frac{30}{100}$

Yes, they are independent

2. (25 points) A random variable $\boldsymbol{X}$ has probabilities $\Pr[\boldsymbol{X} = k] = 0.4, 0.2, 0.1, 0.1, 0.1, 0.1$ for $k = 0, 1, 2, 3, 4, 5$, respectively.

   (a) It is desired to code $\boldsymbol{X}$ with a binary code. How many bits are required for the naive code?

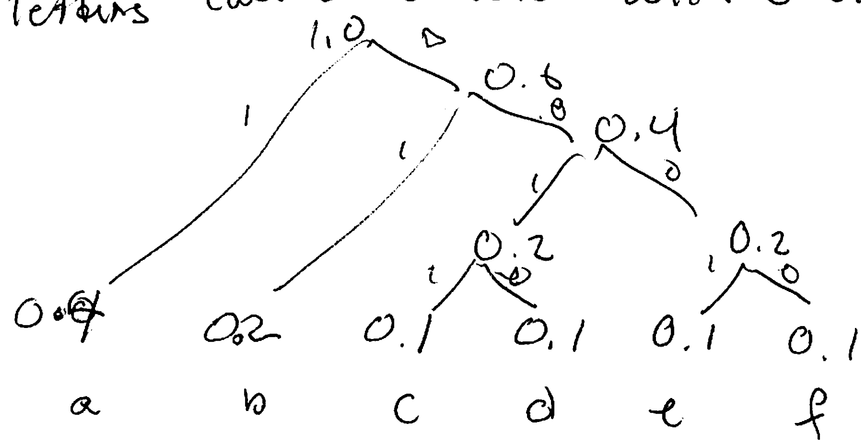
   (b) What is the Huffman code and how many bits does it require on average?

   (c) What is the entropy of $\boldsymbol{X}$? (Give a formula.)

   (d) What is the relationship between the average number of bits required and the entropy?

a). 6 letters can be encoded with 3 bits $(2^3 = 8 \geq 6)$

b)



		len
a	1	1
b	01	2
c	0011	4
d	0010	4
e	0001	4
f	0000	4

$EL(x) = 0.4 \times 1 + 0.2 \times 2 + (0.1 + 0.1 + 0.1 + 0.1) \times 4$

$= 2.4$ bits/letter

c) $H(x) = -0.4 \log_2 0.4 - 0.2 \log_2 0.2 - 0.1 \log_2 0.1 - 0.1 \log_2 0.1$
$- 0.1 \log_2 0.1 - 0.1 \log_2 0.1$ ($= 2.32$ bits/pixel)

d'). $EL(x) \geq H(x)$

3. (25 points) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ have joint PMF as in the table below:

y \ x	0	1	2
2	0.0	0.2	0.1
1	0.0	0.1	0.2
0	0.1	0.1	0.2

(a) What are the mean and variance of $\boldsymbol{X}$?

(b) What are the mean and variance of $\boldsymbol{Y}$?

(c) What are the correlation and covariance of $\boldsymbol{X}$ and $\boldsymbol{Y}$?

(d) What are $\Pr[\boldsymbol{X}=0|\boldsymbol{Y}=0]$ and $\Pr[\boldsymbol{Y}=0|\boldsymbol{X}=0]$?

(e) What is the PMF of $\boldsymbol{Z}=\boldsymbol{X}+\boldsymbol{Y}$?

a) $P_X = [0.1,\ 0.4,\ 0.5]$ $\quad EX = 0\times0.1 + 1\times0.4 + 2\times0.5 = 1.4$

$EX^2 = 0^2\times0.1 + 1^2\times0.4 + 2^2\times0.5 = 2.4$

$\mathrm{Var}X = 2.4 - (1.4)^2 = 0.44$

b) $P_Y = [0.4,\ 0.3,\ 0.3]$ $\quad EY = 0\times0.4 + 1\times0.3 + 2\times0.3 = 0.9$

$EY^2 = 0^2\times0.4 + 1^2\times0.3 + 2^2\times0.3 = 1.5$

$\mathrm{Var}Y = 1.5 - (0.9)^2 = 1.5 - 0.81 = 0.69$

c) $EXY = 1\times1\times0.1 + 1\times2\times0.2 + 1\times2\times0.2 + 2\times2\times0.1 =$

$= 0.1 + 0.4 + 0.4 + 0.4 = 1.3$

$\mathrm{Cov}(X,Y) = 1.3 - 1.4\times\overset{0.9}{\cancel{1.5}} = 1.3 - \overset{1.26}{\cancel{2.1}} = \cancel{-0.8}\ 0.04$

d) $P(X=0|Y=0) = \dfrac{0.1}{0.4} = \dfrac{1}{4}$ $\quad P(Y=0|X=0) = \dfrac{0.1}{0.1} = 1.0$

e) $P(Z=0) = 0.1$ $\quad P(Z=1) = 0.0 + 0.1 = 0.1$ $\quad P(Z=2) = 0.0 + 0.1 + 0.2 = 0.3$

$P(Z=3) = 0.2 + 0.2 = 0.4$ $\quad P(Z=4) = 0.1$

4. (25 points) Consider a simple betting game: a sack contains five marbles, two are red, two are blue, and one is green. The sack is shaken and two marbles are selected (without replacement). If both selected marbles are the same color (either red or blue), you win X dollars; if not, you lose 1 dollar. Assuming you are a profit seeking individual, how large must X be for you to play the game? Explain your reasoning.

$$P(\text{both same}) = 2 \times \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{2}{10}$$

$$E(\text{Wager}) = \frac{2}{10}X - \frac{8}{10} \times 1 = 0$$

$$\Rightarrow X = 4$$

Play if $X \geq 4$.