Math 342

Homework#5 solutions

Sec. 4.2 (Z):

6)

$$F(s) = \frac{(s+2)^2}{s^3} = \frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3} \Rightarrow f(t) = 1 + 4t + 2t^2$$

34) Taking the Laplace transform of the initial value problem, we get

$$\begin{aligned} 2sY(s) - 2y(0) + Y(s) &= 0 \\ Y(s) &= -\frac{6}{2s+1} = -\frac{3}{s+1/2} \end{aligned}$$

so the solution is $y(t) = -3e^{-t/2}$

Additional problems:

1)

(a)

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^1 e^{-st}\,dt = -\left.\frac{e^{-st}}{s}\right|_0^1 = \frac{1-e^{-s}}{s} \qquad s > 0$$

(b)

$$f(t) = \sin 3t \cos 3t = \frac{1}{2}\sin 6t \Rightarrow F(s) = \frac{3}{s^2+36} \qquad s > 0$$

(c)

$$f(t) = (1+t)^3 = 1 + 3t + 3t^2 + t^3 \Rightarrow F(s) = \frac{1}{s} + \frac{3}{s^2} + \frac{6}{s^3} + \frac{6}{s^4} \qquad s > 0$$

(d)

$$\int_0^\infty te^t e^{-st}\,dt = \int_0^\infty te^{(1-s)t}\,dt = \left.\frac{te^{(1-s)t}}{1-s}\right|_0^\infty - \frac{1}{1-s}\int_0^\infty e^{(1-s)t}\,dt = -\left.\frac{e^{(1-s)t}}{(1-s)^2}\right|_0^\infty = \frac{1}{(1-s)^2} \qquad s > 1$$

(e)

$$f(t) = \sinh^2 3t = \frac{1}{4}(e^{6t} - 2 + e^{-6t}) = \frac{1}{4}\left(\frac{1}{s-6} - \frac{2}{s} + \frac{1}{s+6}\right) = \frac{1}{2}\left(\frac{s}{s^2-36} - \frac{1}{s}\right)$$

2)

(a)

$$F(s) = \frac{1}{s+5} \Rightarrow f(t) = e^{-5t}$$

(b)

$$F(s)=\frac{10s-3}{25-s^2}=\frac{3}{s^2-25}-\frac{10s}{s^2-25}\Rightarrow f(t)=\frac{3}{5}\sinh 5t-10\cosh 5t$$

3)

(a) $y''+8y'+15y=0\,,\quad y(0)=2\,,\quad y'(0)=-3$

$$\begin{aligned}(s^2+8s+15)Y(s)-(s+8)y(0)-y'(0)&=0\\ (s^2+8s+15)Y(s)-2(s+8)+3&=0\\ Y(s)&=\frac{2s+13}{s^2+8s+15}=\frac{2s+13}{(s+3)(s+5)}\\ \text{(by partial fractions)}\quad Y(s)&=\frac{A}{s+3}+\frac{B}{s+5}\end{aligned}$$

By identification

$$\frac{2s+13}{(s+3)(s+5)}=\frac{A(s+5)+B(s+3)}{(s+3)(s+5)}=\frac{(A+B)s+5A+3B}{(s+3)(s+5)}$$

$\Rightarrow A+B=2, 5A+3B=13\Rightarrow A=7/2, B=-3/2$

$$Y(s)=\frac{7/2}{s+3}+\frac{-3/2}{s+5}\Rightarrow y(t)=\frac{7}{2}e^{-3t}-\frac{3}{2}e^{-5t}$$

(b) $y''+y=\cos 3t\,,\quad y(0)=1\,,\quad y'(0)=0$

$$\begin{aligned}(s^2+1)Y(s)-sy(0)-y'(0)&=\mathcal{L}\{\cos 3t\}\\ (s^2+1)Y(s)-s&=\frac{s}{s^2+9}\\ Y(s)&=\frac{s}{(s^2+9)(s^2+1)}+\frac{s}{s^2+1}\\ \text{(by partial fractions)}\quad Y(s)&=\frac{As+B}{s^2+9}+\frac{Cs+D}{s^2+1}+\frac{s}{s^2+1}\end{aligned}$$

By identification

$$\frac{s}{(s^2+9)(s^2+1)}=\frac{(As+B)(s^2+1)+(Cs+D)(s^2+9)}{(s^2+9)(s^2+1)}=\frac{(A+C)s^3+(B+D)s^2+(A+9C)s+B+9D}{(s^2+9)(s^2+1)}$$

$\Rightarrow A+C=0, B+D=0, A+9C=1, B+9D=0\Rightarrow A=-1/8, B=0, C=1/8, D=0$

$$Y(s)=\frac{-s/8}{s^2+9}+\frac{9s/8}{s^2+1}\Rightarrow y(t)=-\frac{1}{8}\cos 3t+\frac{9}{8}\cos t$$

(c) $y'' + 4y' + 3y = 1\,, \quad y(0) = 0\,, \quad y'(0) = 0$

$$\begin{aligned}
(s^2+4s+3)Y(s) - (s+4)y(0) - y'(0) &= \mathcal{L}\{1\} \\
(s^2+4s+3)Y(s) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+1)(s+3)} \\
\text{(by partial fractions)} \quad Y(s) &= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}
\end{aligned}$$

By identification

$$\frac{1}{s(s+1)(s+3)} = \frac{A(s+1)(s+3) + Bs(s+3) + Cs(s+1)}{s(s+1)(s+3)} = \frac{(A+B+C)s^2 + (4A+3B+C)s + 3A}{s(s+1)(s+3)}$$

$\Rightarrow A + B + C = 0, 4A + 3B + C = 0, 3A = 1 \Rightarrow A = 1/3, B = -1/2, C = 1/6$

$$Y(s) = \frac{1/3}{s} + \frac{-1/2}{s+1} + \frac{1/6}{s+3} \Rightarrow y(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

4)

(a) $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\cos kt\}$

$$\begin{aligned}
\mathcal{L}\{(t\cos kt)''\} &= -2k\mathcal{L}\{\sin kt\} - k^2\mathcal{L}\{t\cos kt\} \\
&= s^2\mathcal{L}\{t\cos kt\} - sf(0) - f'(0) \\
&= s^2\mathcal{L}\{t\cos kt\} - 1
\end{aligned}$$

$$\begin{aligned}
-2k\mathcal{L}\{\sin kt\} - k^2\mathcal{L}\{t\cos kt\} &= s^2\mathcal{L}\{t\cos kt\} - 1 \\
\mathcal{L}\{t\cos kt\} &= \frac{s^2-k^2}{(s^2+k^2)^2}
\end{aligned}$$

(b) $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\sinh kt\}$

$$\begin{aligned}
\mathcal{L}\{(t\sinh kt)''\} &= 2k\mathcal{L}\{\cosh kt\} + k^2\mathcal{L}\{t\sinh kt\} \\
&= s^2\mathcal{L}\{t\sinh kt\} - sf(0) - f'(0) \\
&= s^2\mathcal{L}\{t\sinh kt\}
\end{aligned}$$

$$\begin{aligned}
2k\mathcal{L}\{\cosh kt\} + k^2\mathcal{L}\{t\sinh kt\} &= s^2\mathcal{L}\{t\sinh kt\} \\
\mathcal{L}\{t\sinh kt\} &= \frac{2ks}{(s^2-k^2)^2}
\end{aligned}$$

5)

$$\cos(2t - \pi/4) = \cos(2t)\ \cos(\pi/4) + \sin(2t)\ \sin(\pi/4) = (\cos 2t + \sin 2t)/\sqrt{2}$$

so

$$\mathcal{L}\{\cos 2(t - \pi/8)\} = \frac{1}{\sqrt{2}} \frac{s+2}{s^2+4}$$

$$\mathcal{L}\{e^{-t/2}\ \cos 2(t - \pi/8)\} = \frac{1}{\sqrt{2}} \frac{(s+1/2)+2}{(s+1/2)^2+4} = \frac{1}{\sqrt{2}} \frac{2s+5}{4s^2+4s+17}$$

6)

$$F(s) = \frac{3s+5}{s^2-6s+25} = 3\frac{s-3}{(s-3)^2+16} + \frac{7}{2}\frac{4}{(s-3)^2+16}$$

so

$$f(t) = e^{3t}\left[3\cos 4t + \frac{7}{2}\sin 4t\right]$$