ELEG 310 Midterm 1 March 19, 2013

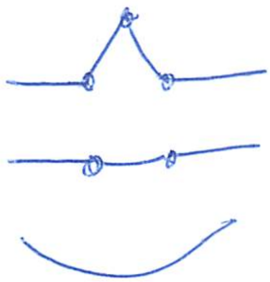
Name: ______________________________

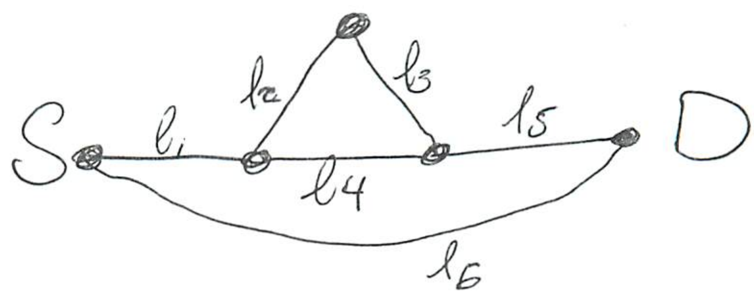
1. (25 points) For the network below, assume each link works with probability p independently of the other links.

   (a) What is the probability S can send a message to D?

   (b) Use the Law of Total Probability by conditioning on the status of link l_4 to calculate the probability S can send a message to D?

a) let $A =$, $B =$, $C =$





$$P(S\to D) = P[A\cup B\cup C] = P[A]+P[B]+P[C) - P[AB]-P[AC]-P[BC]+P[ABC]$$

$$= p^4+p^3+p-p^5-p^5-p^4+p^6$$

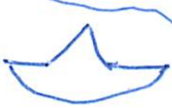
$$\boxed{= p+p^3-2p^5+p^6}$$

Check if $p=0$,

$P(S\to D) = 0+0-2\cdot 0+0 = 0$

if $p=1$, $P(S\to D) = 1+1-2+1 = 1$

b) $$P(S\to D) = \underbrace{P(S\to D\,|\,l_4=1)}_{(p^2+p-p^3)}\underbrace{P(l_4=1)}_{(p)} + \underbrace{P(S\to D\,|\,l_4=0)}_{(p^4+p-p^5)}\underbrace{P(l_4=0)}_{(1-p)}$$



$$= (p^2+p-p^3)p + (p^4+p-p^5)(1-p)$$

$$= p^3+\cancel{p^2}-\cancel{p^4}+\cancel{p^4}+p-p^5-p^5-\cancel{p^2}+p^6$$

$$\boxed{= p+p^3-2p^5+p^6}$$

2. (25 points) Consider the PMF for $\boldsymbol{X}$ below.
   (a) What are $E[\boldsymbol{X}]$ and $\mathrm{Var}[\boldsymbol{X}]$?
   (b) What are $\Pr[\boldsymbol{X} = k | \boldsymbol{X} \geq 2]$ for $k = 0, 1, 2, 3$.

k	0	1	2	3
$\Pr[\boldsymbol{X} = k]$	0.1	0.2	0.3	0.4

a) $E[X] = 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.4$

$= 0 + 0.2 + 0.6 + 1.2 = \boxed{2.0}$

$E[X^2] = 0^2 \times 0.1 + 1^2 \times 0.2 + 2^2 \times 0.3 + 3^2 \times 0.4$

$= 0 + 0.2 + 1.2 + 3.6 = 5.0$

$Var[X] = E[X^2] - (E[X])^2 = 5.0 - (2.0)^2 = \boxed{1.0}$

b) $P[X=0 | X \geq 2] = \dfrac{P[X=0 \cap X \geq 2]}{P[X \geq 2]} = \boxed{0}$

since $\{X=0\} \cap \{X \geq 2\} = \emptyset$

$P[X=1 | X \geq 2] = \boxed{0}$ since $\{X=1\} \cap \{X \geq 2\} = \emptyset$

$P[X=2 | X \geq 2] = \dfrac{P[X=2]}{P[X=2] + P[X=3]} = \dfrac{0.3}{0.3+0.4} = \boxed{\dfrac{3}{7}}$

$P[X=3 | X \geq 2] = \dfrac{P[X=3]}{P[X=2] + P[X=3]} = \dfrac{0.4}{0.3+0.4} = \boxed{\dfrac{4}{7}}$

3. (25 points) Let X_1, X_2, ..., X_n be a sequence of independent and identically distributed (IID) Bernoulli random variables (i.e., $\Pr[X = 1] = p$ and $\Pr[X = 0] = 1-p$). Let $S = X_1 + X_2 + \cdots + X_n$.

(a) What are the mean and variance of S?

(b) What are the mean and variance of $T = S/n$?

$$E[X] = 1\cdot p + 0\cdot(1-p) = p$$

$$E[X^2] = 1^2\cdot p + 0^2(1-p) = p$$

$$Var[X] = E[X^2] - E[X]^2 = p - p^2 = p(1-p)$$

a) $E[S] = E[X_1 + X_2 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = \boxed{np}$

$Var[S] = Var[X_1 + X_2 + \cdots + X_n] = \underbrace{Var[X_1] + Var[X_2] + \cdots + Var[X_n]}_{\text{by independence}}$

$= p(1-p) + p(1-p) + \cdots + p(1-p)$

$= \boxed{np(1-p)}$

b) $E[T] = E\left[\frac{S}{n}\right] = \frac{1}{n}E[S] = \frac{np}{n} = \boxed{p}$

$Var[T] = Var\left[\frac{S}{n}\right] = \frac{1}{n^2}Var[S] = \frac{np(1-p)}{n^2} = \boxed{\frac{p(1-p)}{n}}$

4. (25 points) Consider a simplified version of Blackjack with a deck with 5 cards: 10, J, Q, K, A. Assuming the deck is thoroughly shuffled, you are dealt two cards. If you get Blackjack (an A and any other card) you win X dollars; if you do not get Blackjack (i.e., no A) you lose one dollar. Assuming you are a profit seeking individual, how large must X be for you to play the game? Why?

$$P[\text{BlackJack}] = \frac{\binom{4}{1}\binom{1}{1}}{\binom{5}{2}} = \frac{4 \cdot 1}{10} = \frac{4}{10}$$

$$E[\text{Win}] = X \cdot P[\text{BlackJack}] - 1 \cdot P[\overline{\text{BlackJack}}]$$

$$= \frac{4}{10}X - \frac{6}{10}$$

$$E[\text{Win}] \geq 0 \Rightarrow \frac{4}{10}X - \frac{6}{10} \geq 0 \Rightarrow X \geq \frac{6}{4} = \boxed{\$1.50}$$

If X is ~~$~~ \$1.50 ~~or~~ ~~to~~ the wager is even money.

If $X > \$1.50$ the player is favored and should play.