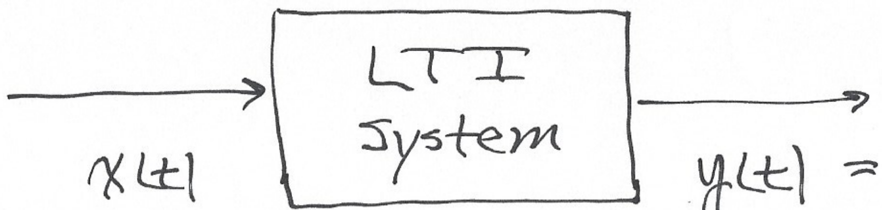
# Continuous-Time FT and LTI Systems

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X(j\omega)}\,\underline{e^{j\omega t}}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$




$x(t)$ $\qquad$ $h(t)$ imp resp $\qquad$ $y(t) = x(t) * h(t)$ $\quad$ conv. $\downarrow$

$X(j\omega)$ $\qquad$ $H(j\omega)$ freq response $\qquad$ $Y(j\omega) = X(j\omega)H(j\omega)$ $\quad$ mult.

# Section 6.1 Magnitude-Phase Representation of FT

continuous-time $\quad X(j\omega) = |X(j\omega)| e^{j \measuredangle X(j\omega)}$

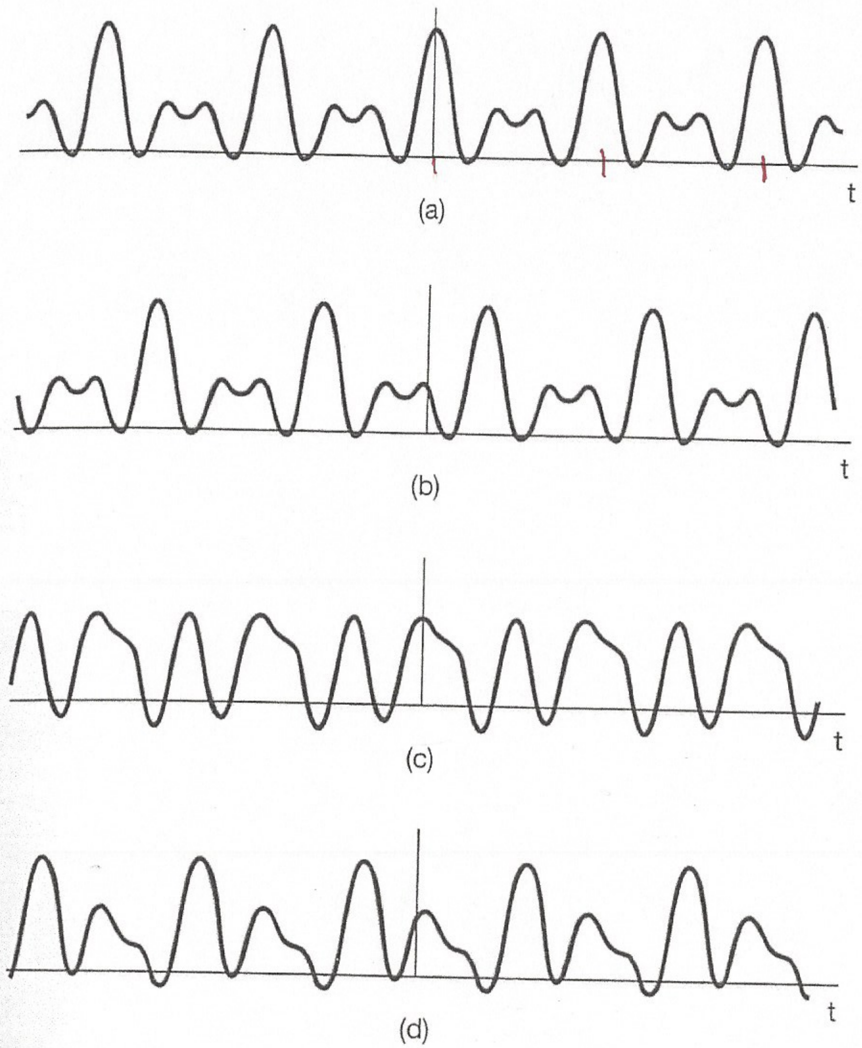
discrete-time $\quad X(e^{j\omega}) = |X(e^{j\omega})| e^{j \measuredangle X(e^{j\omega})}$

magnitude

phase

# Example: Importance of Phase (Fig. 6.1, p. 425)

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$



(a) (b) (c) (d)

(a) $\phi_1 = \phi_2 = \phi_3 = 0$

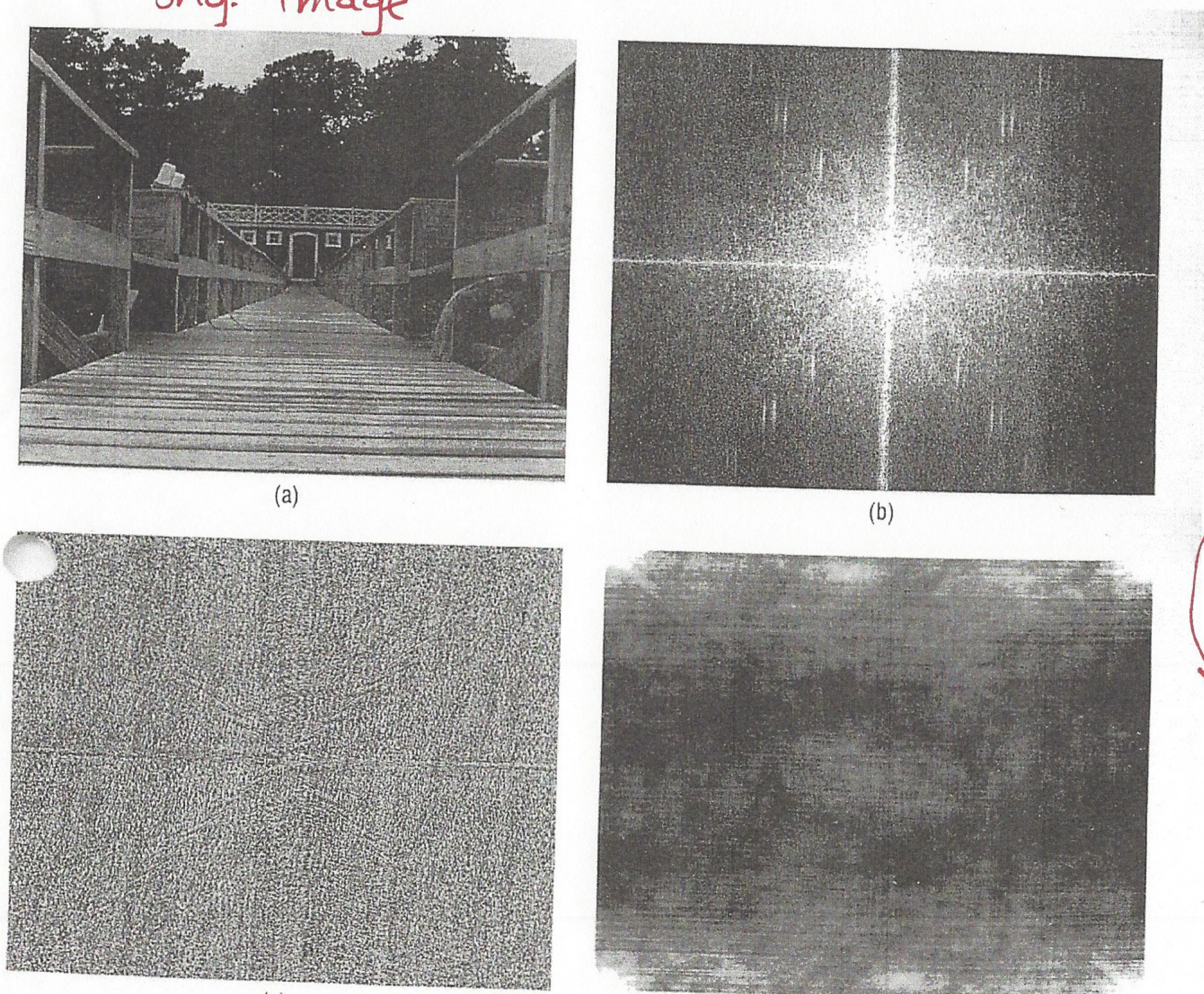
(b) $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad

(c) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad

(d) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad

# Example: Importance of Phase (Fig. 6.2, pp. 426-427)

orig. image



(a) (b)

(c) (d)

phase

2D FT magnitude

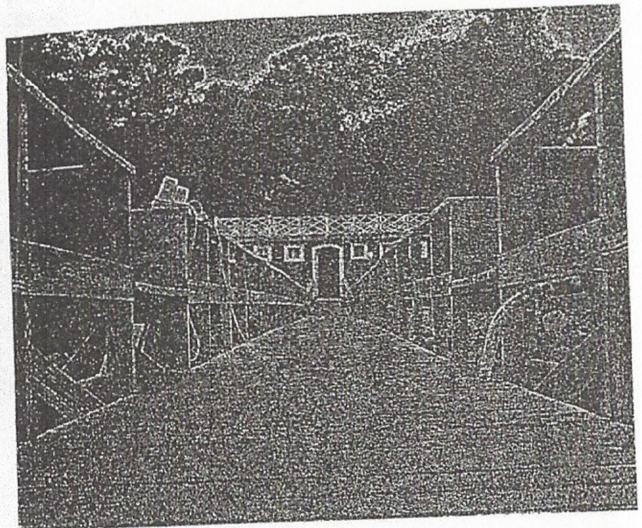
$|X(j\omega_1, j\omega_2)|$

$\angle X(\quad) = 0$

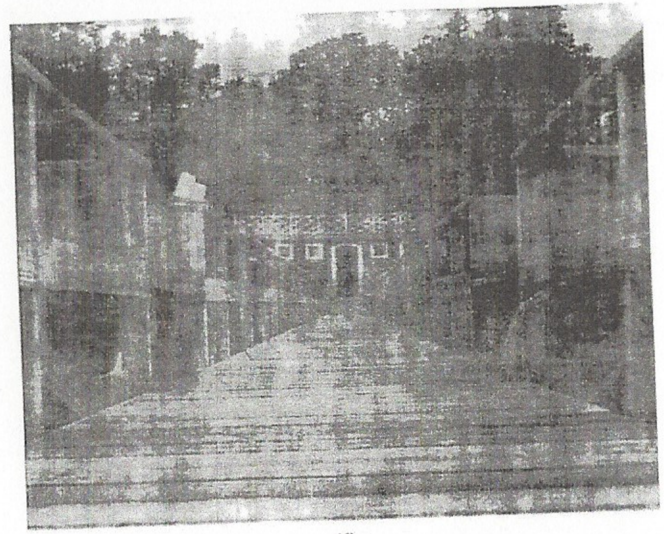
$\downarrow$

$x(t_1, t_2)$

# Example: Importance of Phase (Fig. 6.2, cont'd)



(e)



(f)

(g)

**Figure 6.2** (a) The image shown in Figure 1.4; (b) magnitude of the two-dimensional Fourier transform of (a); (c) phase of the Fourier transform of (a); (d) picture whose Fourier transform has magnitude as in (b) and phase equal to zero; (e) picture whose Fourier transform has magnitude equal to 1 and phase as in (c); (f) picture whose Fourier transform has phase as in (c) and magnitude equal to that of the transform of the picture shown in (g).

# Section 6.2 LTI Systems

## Ideal Response

continuous time $\quad Y(j\omega) = H(j\omega)\,X(j\omega) \qquad x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

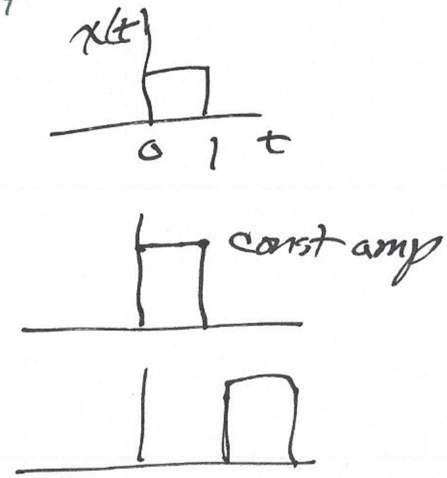
gain

magnitude $\quad |Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|$

phase $\quad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$

phase shift

ideal response
- constant amplitude
- linear phase $\longrightarrow$ delay



# Section 6.2 LTI Systems

$X(j\omega) = \int \delta(t) e^{-j\omega t} dt = 1$

## Linear Phase

$H(j\omega) = e^{-j\omega t_0}$

$|H(j\omega)| = 1$ all pass

impulse response $h(t) = \delta(t - t_0)$

$\measuredangle H(j\omega) = -\omega t_0$

linear function of $\omega$

$y(t) = x(t - t_0)$

# Section 6.2 LTI Systems

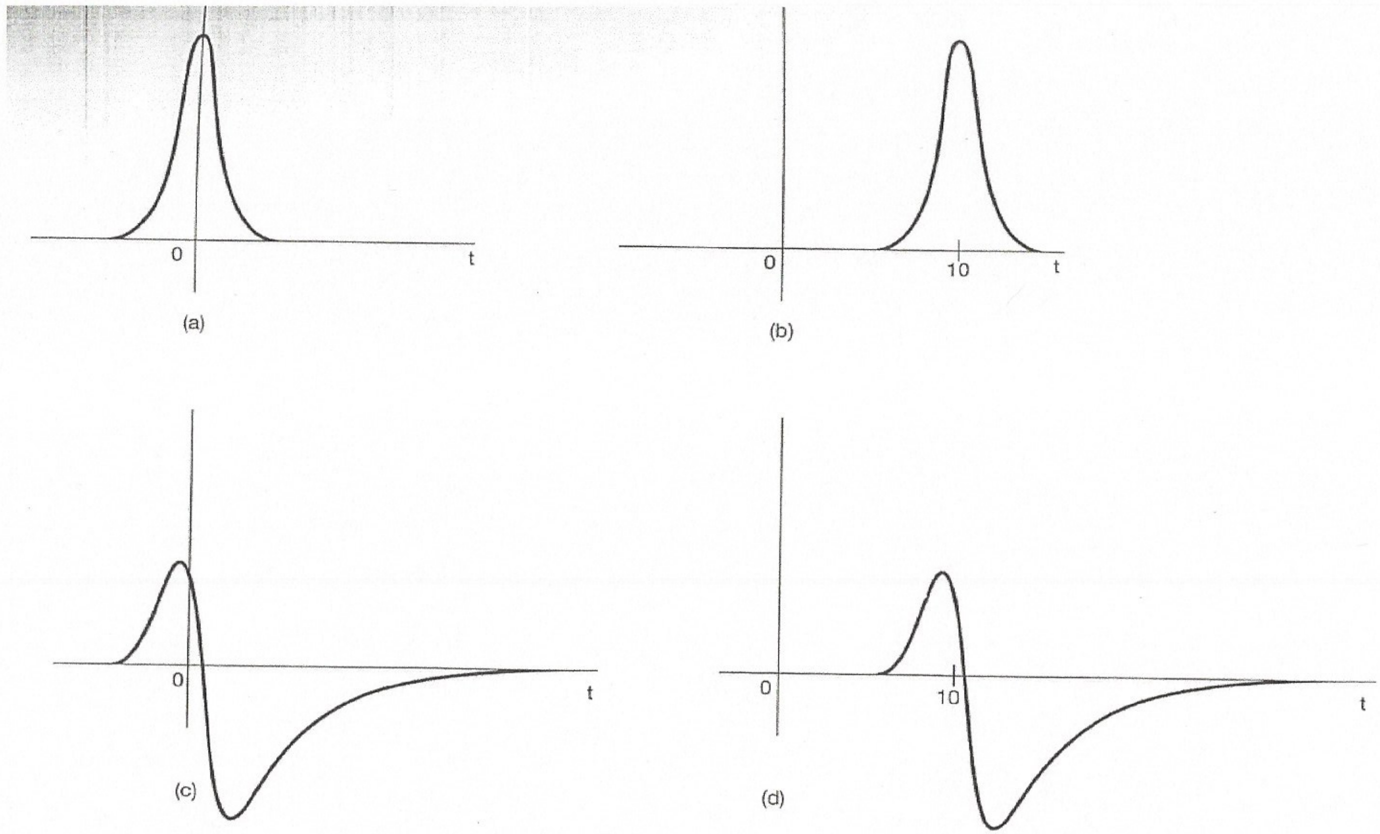
## Group Delay

linear phase $\quad \measuredangle H(j\omega) = -\omega t_0 \implies$ delay $= t_0$ (slope of linear phase)

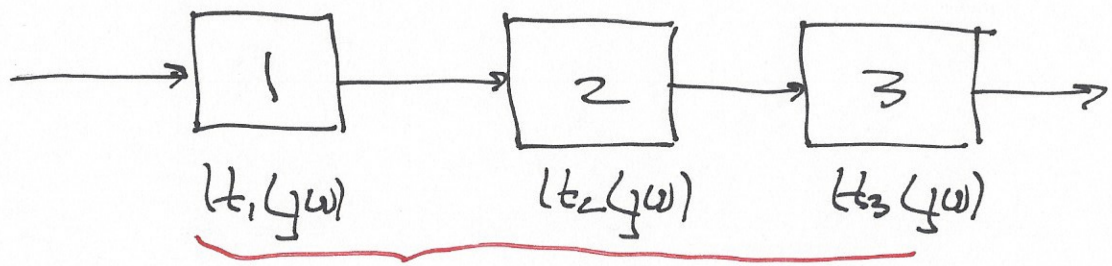
nonlinear phase $\implies$ define delay as derivative of phase $\quad \tau(\omega) =$ group delay

$$= -\frac{d}{d\omega}\{\measuredangle H(j\omega)\}$$

# Section 6.2 LTI Systems
# Nonlinear Phase (Fig. 6.3, p. 429)



# Example 6.1 (pp. 432-434)




$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)\, H_3(j\omega)$$

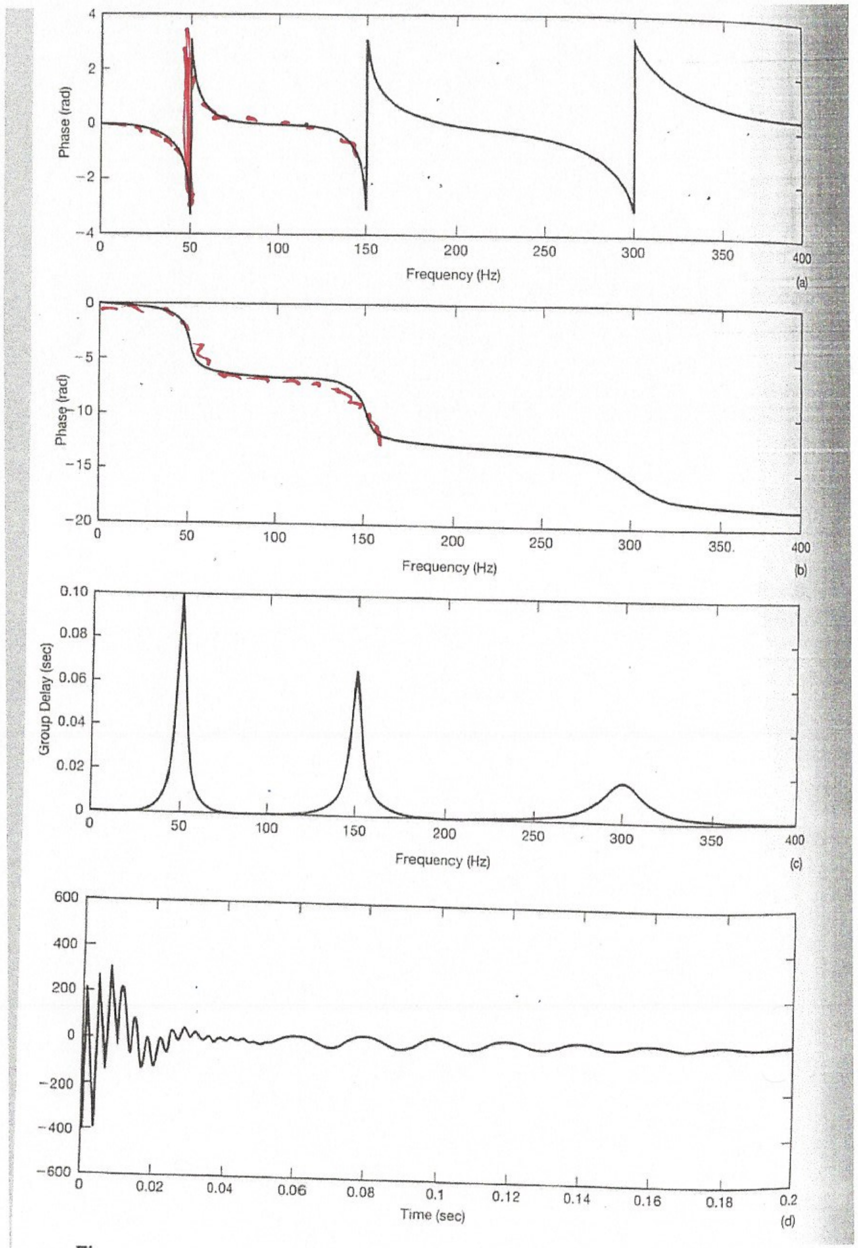
$$H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\,\xi_i\,(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\,\xi_i\,(\omega/\omega_i)}$$

| | $\lvert H_i(j\omega)\rvert$ | $\omega_i$ (rad/sec) | $\xi_i$ | $f_i = \omega_i/2\pi$ |
|---|---|---|---|---|
| 1 | 1 | 315 | 0.066 | ~50 Hz |
| 2 | 1 | 943 | 0.033 | ~150 Hz |
| 3 | 1 | 1888 | 0.058 | ~300 Hz |

# Example 6.1 (pp. 432-434) (Fig. 6.5, p. 434)

$\pi$

$-\pi$




$$\measuredangle H_i(j\omega) = \tan^{-1}\left[\frac{-2\xi_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right] - \tan^{-1}\left[\frac{2\xi_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right]$$

$$\measuredangle H(j\omega) = \measuredangle H_1(j\omega) + \measuredangle H_2(j\omega) + \measuredangle H_3(j\omega)$$

Unwrap phase

# Section 6.2 LTI Systems
# Log-Magnitude and Phase Plots

$$|Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|$$

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

$$dB = 20\log_{10}|\cdot|$$

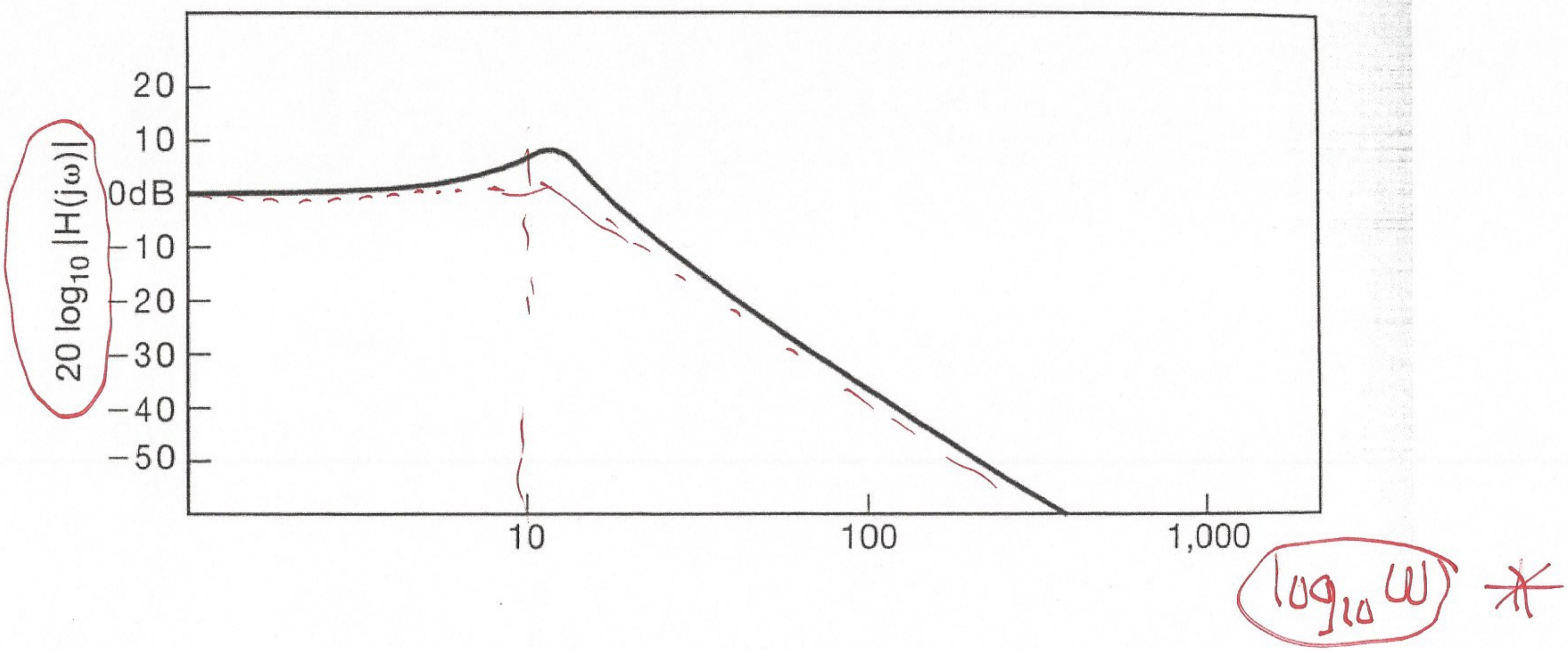
0 dB gain 1

6 dB gain 2

$$10\log_{10}|\ |^2$$

power

# Section 6.2 LTI Systems
## Bode Plots



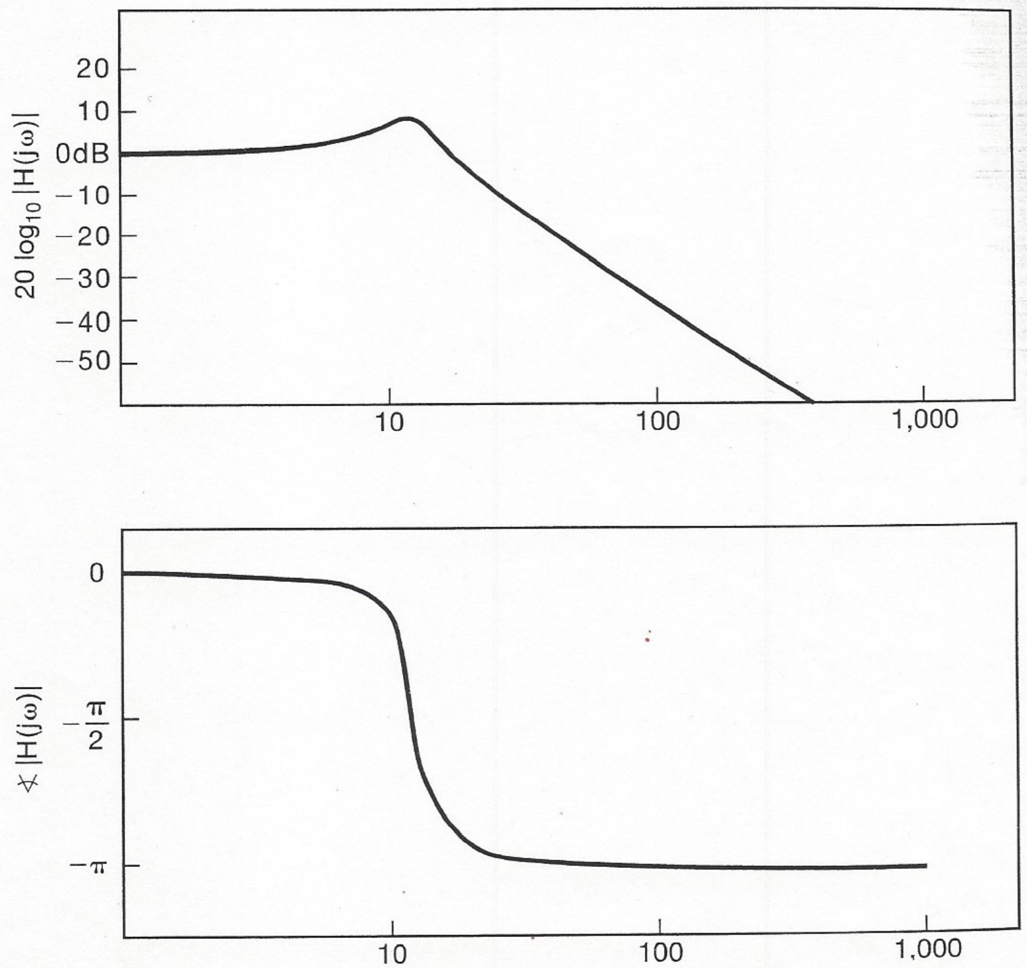


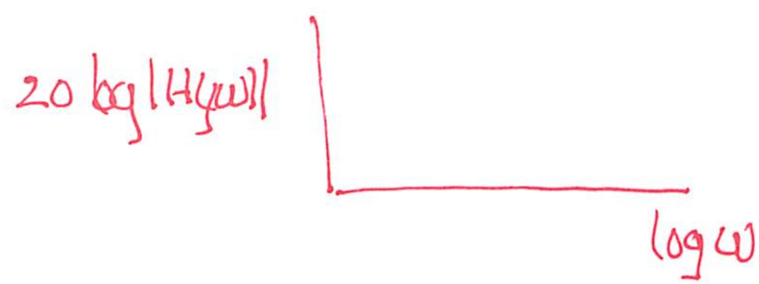

**Figure 6.8** A typical Bode plot. (Note that $\omega$ is plotted using a logarithmic scale.)

of a particular response curve doesn't change if the frequency is scaled. (See Probl 6.30.) Furthermore for continuous-time LTI systems described by differential equatio an approximate sketch of the log magnitude vs. log frequency can often be easily obtair through the use of asymptotes. In Section 6.5, we will illustrate this by developing si ple piecewise-linear approximate Bode plots for first- and second-order continuous-ti systems.

In discrete time, the magnitudes of Fourier transforms and frequency responses often displayed in dB for the same reasons that they are in continuous time. Howev in discrete time a logarithmic frequency scale is not typically used, since the range frequencies to be considered is always limited and the advantage found for differen equations (i.e., linear asymptotes) does not apply to difference equations. Typical grap cal representations of the magnitude and phase of a discrete-time frequency response shown in Figure 6.9. Here, we have plotted $\sphericalangle H(e^{j\omega})$ in radians and $|H(e^{j\omega})|$ in decib [i.e., 20 $\log_{10}|H(e^{j\omega})|$] as functions of $\omega$. Note that for $h[n]$ real, we actually need p

# Section 6.5
# First-Order and Second-Order CT Systems

+ Bode Plots



# First-Order Systems

## Frequency Response

$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

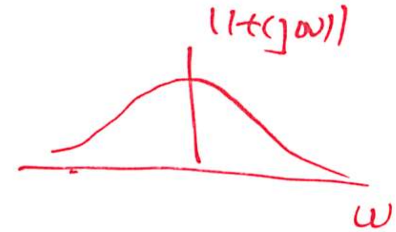
$$\Big\downarrow \mathcal{F}$$

$$j\omega\tau\, Y(j\omega) + Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega\tau + 1}$$

$$Y = HX$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

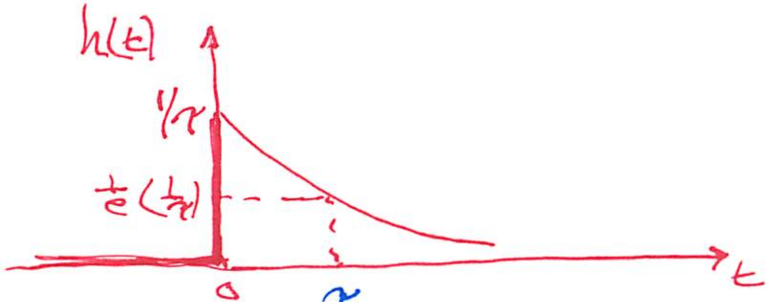



$$\measuredangle H(j\omega) = -\tan^{-1} \omega\tau$$

# First-Order Systems
## Impulse and Step Responses

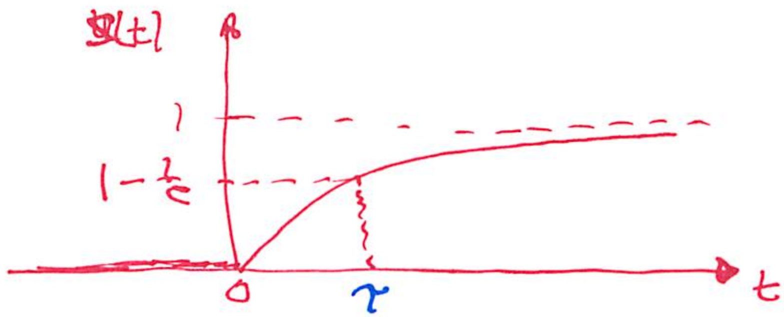
$$H(j\omega) = \frac{1}{j\omega\tau + 1} \longrightarrow h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

impulse response




$\tau$ decreases $h(t) \to \delta(t)$

$\tau$ time constant $\longrightarrow$ controls system response time




step response

$$s(t) = \int_{-\infty}^{t} h(x)\,dx$$

$$= \left[1 - e^{-t/\tau}\right] u(t)$$

$\tau$ decrease $s(t) \to u(t)$

# First-Order Systems

## Bode Plot → log of magnitude v. log. of frequency

$$20 \log_{10} |H(j\omega)| = -10 \log_{10}[(\omega\tau)^2 + 1]$$

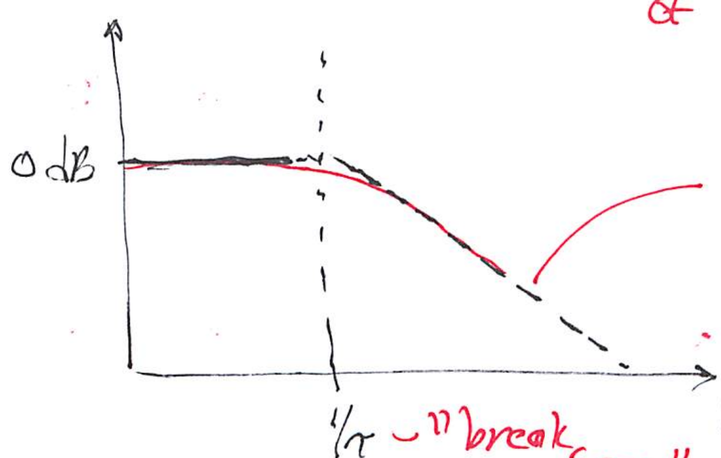
① lowpass response

② approximate using asymptotes

low freq asymptote $\omega \ll \frac{1}{\tau}$ $\quad 20 \log_{10}|H(j\omega)| \approx 0 \text{ dB}$

high freq asymptote $\omega \gg \frac{1}{\tau}$ $\quad 20 \log_{10}|H(j\omega)| = -20 \log_{10}(\omega\tau)$

linear function of $\log \omega$ $\quad = -20 \log_{10}\tau - 20 \log_{10}\omega$




for every decade in $\omega$ (i.e. a factor of 10), the magnitude changes by −20 dB

③ only an approximation ⇒ at $\omega = \frac{1}{\tau}$, $|H(j\omega)| = \frac{1}{\sqrt{2}}$ → ≈ −3 dB

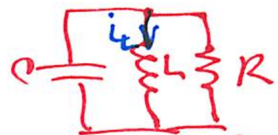
# Second-Order Systems

## Frequency Response

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

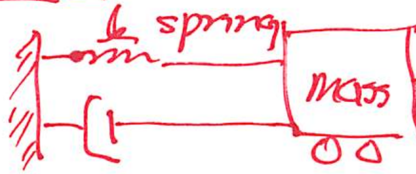
$\frac{1}{2RC}$, $\omega_n = \frac{1}{\sqrt{LC}}$

e.g RLC circuit




$$\frac{d^2 i_L}{dt^2} + 2\alpha\frac{di_L}{dt} + \omega_n^2 i_L(t) = \omega_n^2 x(t)$$

e.g mechanical circuit




$x(t)$ = applied force

$$m\frac{d^2y}{dt^2} = x(t) - ky(t) - b\frac{dy(t)}{dt}$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

$$c_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$

$$c_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

# Second-Order Systems

## Impulse Response (case a)

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

$$c_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$
$$c_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

$\hookrightarrow \xi \neq 1 \Rightarrow c_1 \neq c_2$

$$H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2}$$

where $$M = \frac{\omega_n}{2\sqrt{\xi^2 - 1}}.$$

could be complex

$$h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right]u(t)$$

# Second-Order Systems

## Impulse Response (case b)

$\zeta = 1 \Rightarrow c_1 = c_2 = -\omega_n$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

$$h(t) = \omega_n^2 \, t \, e^{-\omega_n t} u(t)$$

# Second-Order Systems

## Comments on Frequency Response

$$H(j\omega) = \frac{1}{(j\,\omega/\omega_n)^2 + 2\xi(j\,\omega/\omega_n) + 1}$$

$\dfrac{\omega}{\omega_n} \longrightarrow$ changing $\omega_n$ is like time/frequency scaling

## Second-Order Systems

## Comments on Impulse Response

$\xi$ = damping ratio

$\omega_n$ = undamped natural frequency

$$c_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

— for $0 < \xi < 1$, $c_1$ and $c_2$ are complex

$$c_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$c_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$$

$$\therefore\ e^{c_1 t} = \underbrace{e^{-\xi\omega_n t}}_{\text{exponential decay}} \underbrace{e^{j\omega_n\sqrt{1-\xi^2}\,t}}_{\text{oscillation}}, \quad M = \frac{\omega_n}{2j\sqrt{1-\xi^2}}$$

$$\therefore\ h(t) = \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\,t\right) u(t)$$

damped oscillation $\Rightarrow$ "under-damped"

# Second-Order Systems

## Comments on Impulse Response

- for $\xi > 1$, $c_1$ and $c_2$ are real and negative

$$h(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{-(\xi-\sqrt{\xi^2-1})\omega_n t} - e^{-(\xi+\sqrt{\xi^2-1})\omega_n t}\right]u(t)$$

decaying exponentials $\Rightarrow$ "overdamped"

- for $\xi = 1$, $c_1 = c_2 = -\omega_n$

$$h(t) = \omega_n^2 t\, e^{-\omega_n t} u(t)$$ $\Rightarrow$ "critically damped"

# Second-Order Systems

## Comments on Step Response

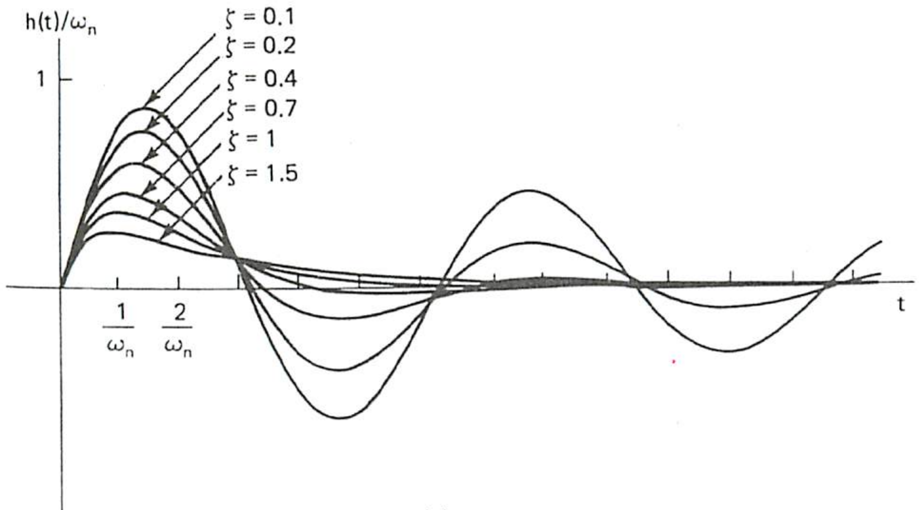
$\xi \neq 1 \qquad s(t) = \left\{ 1 + M\left[ \frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2} \right] \right\} u(t)$

underdamped → exhibits overshoot and ringing

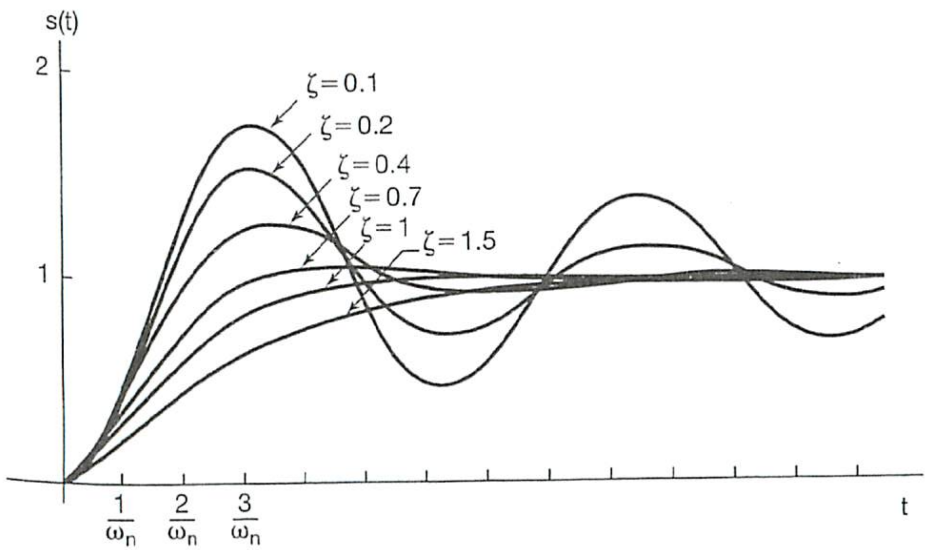
$\xi = 1 \qquad s(t) = \left[ 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t} \right] u(t)$

no overshoot

# Second-Order Systems




(a)




(b)

# Second-Order Systems
## Comments

- $\omega_n$ – controls the time scale of the responses
  – for $\xi < 1$, $\omega_n$ larger $\rightarrow$ more compressed impulse response

frequency of oscillation $= \omega_n \sqrt{1-\xi^2}$

only have oscillation if underdamped

$\neq \omega_n$ (except if $\xi = 0$)

no oscillation if $\xi \geq 1$

- $\measuredangle H(j\omega) = -\tan^{-1}\left(\dfrac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$

# Second-Order Systems

## Approximate Bode Plot

$$|H(j\omega)| = \frac{1}{\sqrt{\left(1-(\omega/\omega_n)^2\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

$$20\log|H(j\omega)| = -10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

- asymptotes

low freq $\omega \ll \omega_n$ $\quad 20\log_{10}|H(j\omega)| \approx 0$ dB

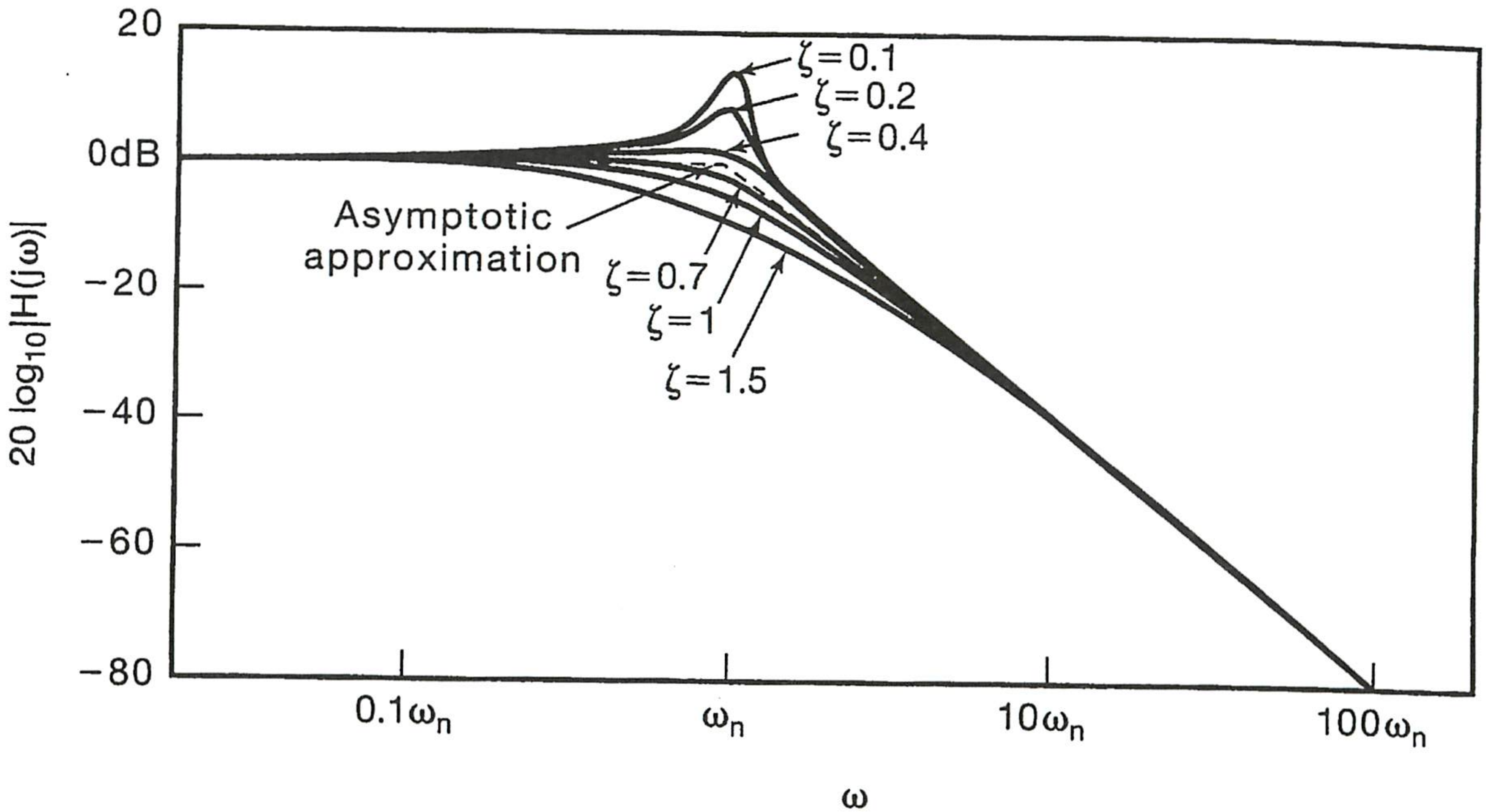
high freq $\omega \gg \omega_n$ $\quad 20\log_{10}|H(j\omega)| \approx -10\log_{10}\left[(\omega/\omega_n)^4\right]$

$$= -40\log_{10}\omega + 40\log_{10}\omega_n$$

slope = −40 dB/decade

# Second-Order Systems

## Real Bode Plot (Fig. 6.23, p. 455)



# Example 6.5 (pp. 459-460)

$$H(j\omega) = \frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)} = \underbrace{\left(\frac{1}{10}\right)}_{\text{constant}} \underbrace{\left(\frac{1}{1+j\omega/10}\right)}_{\text{first order}} \underbrace{\left(\frac{1}{1+j\omega/100}\right)}_{\text{first order}} \underbrace{(1+j\omega)}_{\text{reciprocal of first order}}$$

$$|H(j\omega)| = \frac{1}{10} \cdot \sqrt{\frac{1}{1+\left(\frac{\omega}{10}\right)^2}} \cdot \sqrt{\frac{1}{1+\left(\frac{\omega}{100}\right)^2}} \cdot \sqrt{1+\omega^2}$$

$$20\log_{10}|H(j\omega)| = \underbrace{-20\text{ dB}}_{\text{offset}} - 10\log_{10}\left[1+\left(\frac{\omega}{10}\right)^2\right] - 10\log_{10}\left[1+\left(\frac{\omega}{100}\right)^2\right] + 10\log_{10}\left[1+\omega^2\right]$$

positive slope

3 break frequencies $\omega_n = 1, 10, 100$

NOTE: using what we have described so far, we can construct Bode plots for arbitrary rational frequency responses.

Example: (#6.5, p. 459–460)

$$H(j\omega) = \frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)}$$

$$= \underbrace{\left(\frac{1}{10}\right)}_{\text{constant}} \underbrace{\left(\frac{1}{1+j\omega/10}\right)}_{\text{first-order}} \underbrace{\left(\frac{1}{1+j\omega/100}\right)}_{\text{first-order}} \underbrace{(1+j\omega)}_{\substack{\text{reciprocal of} \\ \text{first order}}}$$

$$|H(j\omega)| = \frac{1}{10} \cdot \sqrt{\frac{1}{1+(\omega/10)^2}} \cdot \sqrt{\frac{1}{1+(\omega/100)^2}} \cdot \sqrt{1+\omega^2}$$

$$20\log|H(j\omega)| = \underbrace{-20\,dB}_{\text{offset}} - 10\log\left[1+\left(\frac{\omega}{10}\right)^2\right] - 10\log\left[1+\left(\frac{\omega}{100}\right)^2\right] + 10\log\left[1+\omega^2\right]$$

positive slope

break frequencies: $\omega_n = 1, 10, 100$