Math 342

Homework#4 solutions

Sec. 5.1 (Z):

21) Substituting $y = \sum_{n=0}^{\infty} c_n\, x^n$ into the equation, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=1}^{\infty} n\, c_n\, x^{n+1} + \sum_{n=0}^{\infty} c_n\, x^{n+1} &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n + \sum_{n=2}^{\infty} (n-1)\, c_{n-1}\, x^n + \sum_{n=1}^{\infty} c_{n-1}\, x^n &= 0 \\
2c_2 + (6c_3 + c_0)x + \sum_{n=2}^{\infty} \Big[(n+2)(n+1)\, c_{n+2} + n\, c_{n-1}\Big] x^n &= 0
\end{aligned}$$

which implies

$$c_2 = 0\,, \quad 6c_3 + c_0 = 0\,, \quad (n+2)(n+1)\, c_{n+2} + n\, c_{n-1} = 0 \quad \text{for} \quad n \geq 2$$

Therefore the recurrence relation is

$$c_2 = 0\,, \quad c_3 = -\frac{1}{6} c_0\,, \quad c_{n+2} = -\frac{n}{(n+2)(n+1)} c_{n-1} \quad \text{for} \quad n \geq 2$$

and the general solution can be expressed as

$$y(x) = c_0 \left(1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 + \cdots \right) + c_1 \left(x - \frac{1}{6}x^4 + \frac{5}{252}x^7 + \cdots \right)$$

24) Substituting $y = \sum_{n=0}^{\infty} c_n\, x^n$ into the equation, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1)\, c_n\, x^{n-2} + \sum_{n=1}^{\infty} n\, c_n\, x^n - \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
\sum_{n=1}^{\infty} (n+1)n\, c_{n+1}\, x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)\, c_{n+2}\, x^n + \sum_{n=1}^{\infty} n\, c_n\, x^n - \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
4c_2 - c_0 + \sum_{n=1}^{\infty} \Big[(n+1)n\, c_{n+1} + 2(n+2)(n+1)c_{n+2} + (n-1)c_n\Big] x^n &= 0
\end{aligned}$$

which implies

$$4c_2 - c_0 = 0\,, \quad (n+1)n\, c_{n+1} + 2(n+2)(n+1)c_{n+2} + (n-1)c_n \quad \text{for} \quad n \geq 1$$

Therefore the recurrence relation is

$$c_2 = \frac{1}{4} c_0\,, \quad c_{n+2} = -\frac{(n+1)n\, c_{n+1} + (n-1)c_n}{2(n+2)(n+1)} \quad \text{for} \quad n \geq 1$$

and the general solution can be expressed as

$$y(x) = c_0 \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \cdots\right) + c_1 x$$

Additional problems:

1)

(a) $y' = x^2 y$

Looking for power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\, x^n$ and inserting in the equation, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n\, c_n\, x^{n-1} &= \sum_{n=0}^{\infty} c_n\, x^{n+2} \\
\sum_{n=1}^{\infty} n\, c_n\, x^{n-1} &= \sum_{n=3}^{\infty} c_{n-3}\, x^{n-1} \\
c_1 + 2c_2\, x + \sum_{n=3}^{\infty} n\, c_n\, x^{n-1} &= \sum_{n=3}^{\infty} c_{n-3}\, x^{n-1} \\
c_1 = 0\,, \quad c_2 = 0\,, \quad n\, c_n &= c_{n-3}
\end{aligned}$$

Therefore the recurrence relation is

$$c_1 = 0\,, \quad c_2 = 0\,, \quad c_n = \frac{c_{n-3}}{n} \quad \text{for} \quad n \geq 3$$

and the radius of convergence is $R = \infty$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_n\, x^n}{c_{n-3}\, x^{n-3}} \right| = |x|^3 \lim_{n\to\infty} \frac{1}{n} < 1$$

The general solution is

$$y(x) = c_0 \left(1 + \frac{x^3}{3} + \frac{x^6}{2!3^2} + \frac{x^9}{3!3^3} + \cdots\right) = c_0\, e^{x^3/3}$$

(b) $2(x+1)y' = y$

Looking for power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\, x^n$ and inserting in the equation, we have

$$\begin{aligned}
2\sum_{n=1}^{\infty} n\, c_n\, x^n + 2\sum_{n=1}^{\infty} n\, c_n\, x^{n-1} &= \sum_{n=0}^{\infty} c_n\, x^n \\
2\sum_{n=0}^{\infty} n\, c_n\, x^n + 2\sum_{n=0}^{\infty} (n+1)\, c_{n+1}\, x^n &= \sum_{n=0}^{\infty} c_n\, x^n \\
2n\, c_n + 2(n+1)\, c_{n+1} &= c_n
\end{aligned}$$

Therefore the recurrence relation is

$$c_{n+1} = \frac{1}{2} c_n \frac{1-2n}{n+1} \quad \text{for} \quad n \geq 0$$

and the radius of convergence is $R = 1$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_{n+1}\, x^{n+1}}{c_n\, x^n} \right| = \frac{|x|}{2} \lim_{n\to\infty} \left| \frac{1-2n}{n+1} \right| = \frac{|x|}{2} \lim_{n\to\infty} \frac{2n}{n} = |x| < 1$$

Note: Using L'Hôpital's rule would give the same answer.

The general solution is

$$y(x) = c_0 \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots \right) = c_0 \sqrt{1+x}$$

2)

(a) $y'' + y = x$

Looking for power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\, x^n$ and inserting in the equation, we have

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^n &= x \\ 2c_2 + c_0 + (6c_3 + c_1 - 1)x + \sum_{n=0}^{\infty} \Big[(n+2)(n+1) c_{n+2} + c_n \Big] x^n &= 0 \end{aligned}$$

which implies

$$2c_2 + c_0 = 0 \;, \quad 6c_3 + c_1 - 1 = 0 \;, \quad (n+2)(n+1)c_{n+2} + c_n = 0$$

Therefore the general recurrence formula is

$$c_{n+2} = -\frac{c_n}{(n+1)(n+2)} \quad \text{with} \quad c_2 = -\frac{c_0}{2} \;, \quad c_3 = -\frac{c_1 - 1}{6}$$

and the general solution is

$$\begin{aligned} y(x) &= x + c_0 \Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Big) + (c_1 - 1) \Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \Big) \\ &= x + c_0 \cos x + (c_1 - 1) \sin x \end{aligned}$$

The radius of convergence is $R = \infty$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_{n+2}\, x^{n+2}}{c_n\, x^n} \right| = x^2 \lim_{n\to\infty} \frac{1}{(n+2)(n+1)} < 1$$

(b) $y'' + xy' + y = 0$

Looking for power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\, x^n$ and inserting in the equation, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + \sum_{n=1}^{\infty} n\, c_n\, x^n + \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n + \sum_{n=0}^{\infty} n\, c_n\, x^n + \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
(n+2)(n+1)\, c_{n+2} + n\, c_n + c_n &= 0
\end{aligned}$$

Therefore the recurrence relation is $c_{n+2} = -c_n/(n+2)$ for $n \geq 0$ and the radius of convergence is $R = \infty$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_{n+2}\, x^{n+2}}{c_n\, x^n} \right| = x^2 \lim_{n\to\infty} \frac{1}{n+2} < 1$$

The general solution is

$$\begin{aligned}
y(x) &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! 2^n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} \\
&= c_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots \right) + c_1 \left(x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \cdots \right)
\end{aligned}$$

which can be viewed as the linear combination of two linearly independent power series (even and odd powers)

(c) $(x^2 - 1)y'' + 8xy' + 12y = 0$

Looking for power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\, x^n$ and inserting in the equation, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^n - \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} + 8 \sum_{n=1}^{\infty} n\, c_n\, x^n + 12 \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
\sum_{n=0}^{\infty} n(n-1)\, c_n\, x^n - \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n + 8 \sum_{n=0}^{\infty} n\, c_n\, x^n + 12 \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
n(n-1)\, c_n - (n+2)(n+1)\, c_{n+2} + 8n\, c_n + 12 c_n &= 0 \\
(n^2 + 7n + 12)\, c_n - (n+2)(n+1)\, c_{n+2} &= 0 \\
(n+3)(n+4)\, c_n - (n+2)(n+1)\, c_{n+2} &= 0
\end{aligned}$$

Therefore the recurrence relation is

$$c_{n+2} = c_n \frac{(n+3)(n+4)}{(n+1)(n+2)} \quad \text{for} \quad n \geq 0$$

and the radius of convergence is $R = 1$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_{n+2}\, x^{n+2}}{c_n\, x^n} \right| = x^2 \lim_{n\to\infty} \frac{(n+3)(n+4)}{(n+1)(n+2)} = x^2 \lim_{n\to\infty} \frac{n^2}{n^2} = x^2 < 1$$

Note: Again, using L'Hôpital's rule would give the same answer.

The general solution is

$$
\begin{aligned}
y(x) &= c_0 \sum_{n=0}^{\infty} (n+1)(2n+1)x^{2n} + \frac{c_1}{3} \sum_{n=0}^{\infty} (n+1)(2n+3)x^{2n+1} \\
&= c_0 \left(1 + 6x^2 + 15x^4 + 28x^6 + \cdots\right) + \frac{c_1}{3} \left(3x + 10x^3 + 21x^5 + 36x^7 + \cdots\right)
\end{aligned}
$$

3) $(x^2 - x + 1)y'' - y' - y = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1$

Looking for power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\, x^n$ and inserting in the equation, we have

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^n - \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-1} + \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} - \sum_{n=1}^{\infty} n\, c_n\, x^{n-1} - \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
\sum_{n=0}^{\infty} n(n-1)\, c_n\, x^n - \sum_{n=0}^{\infty} (n+1)n\, c_{n+1}\, x^n + \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n - \sum_{n=0}^{\infty} (n+1)\, c_{n+1}\, x^n & \\
- \sum_{n=0}^{\infty} c_n\, x^n &= 0 \\
n(n-1)\, c_n - (n+1)n\, c_{n+1} + (n+2)(n+1)\, c_{n+2} - (n+1)\, c_{n+1} - c_n &= 0 \\
(n^2 - n - 1)\, c_n - (n+1)^2\, c_{n+1} + (n+2)(n+1)\, c_{n+2} &= 0
\end{aligned}
$$

Therefore the recurrence relation is

$$
c_{n+2} = \frac{(n+1)^2\, c_{n+1} + (1 + n - n^2)\, c_n}{(n+2)(n+1)} \quad \text{for} \quad n \geq 0
$$

The initial conditions correspond to $y(0) = 0 \Rightarrow c_0 = 0$ and $y'(0) = 1 \Rightarrow c_1 = 1$

Therefore the general solution is

$$
y(x) = \sum_{n=0}^{\infty} c_n\, x^n = x + \frac{x^2}{2} + \frac{x^3}{2} + \cdots
$$