# SOLUTION TO HOMEWORK #8

## 9.5

**(a)**

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3} = \frac{2(s+2)}{(s+1)(s+3)}$$

A zero is the value of $s$ for which $X(s) = 0$. From the equation above, $X(s)$ has a zero at $s = -2$. Also, $X(s) \to 0$ as $s \to \infty$. So, there is a zero at $s = \infty$. Therefore, $X(s)$ has two zeros: one zero in the finite s-plane (s=-2) and one zero at infinity.

**(b)**

$$X(s) = \frac{s+1}{s^2-1} = \frac{1}{s-1}$$

From the equation above, $X(s)$ has no zeros in the finite s-plane. However, as $s \to \infty$, $X(s) \to 0$. Therefore, $X(s)$ has one zero.

**(c)**

$$X(s) = \frac{(s-1)(s^2+s+1)}{s^2+s+1} = s-1$$

From the equation above, $X(s)$ has one zero at $s = 1$. Also, $X(s)$ has no zeros at $\infty$.

## 9.6

**(a)** No. From Property 3 in Section 9.2 we know that for a finite-length signal, the ROC is the entire s-plane. Therefore, there can be no poles in the finite s-plane for a finite-length signal. So, we cannot have a pole at $s = 2$. This means that the signal cannot be of finite duration.

**(b)** Yes. Since the signal is absolutely integrable, the Fourier transform exists and, so, the ROC must include the $j\omega$-axis. Furthermore, $X(s)$ has a pole at $s = 2$. Therefore, one valid ROC for the signal would be $\mathfrak{Re}\{s\} < 2$. From Property 5 in Section 9.2, this would correspond to a left-sided signal.

**(c)** No. Since the signal is absolutely integrable, the Fourier transform exists and, so, the ROC must include the $j\omega$-axis. Furthermore, $X(s)$ has a pole at $s = 2$. Therefore, we can never have a ROC of the form $\mathfrak{Re}\{s\} > \alpha$, where $\alpha$ is a positive number. From Property 4 in Section 9.2, $x(t)$ cannot be a right-sided signal.

**(d)** Yes. Since the signal is absolutely integrable, the Fourier transform exists and, so, the ROC must include the $j\omega$-axis. Furthermore, $X(s)$ has a pole at $s = 2$. Therefore, one valid ROC for the signal would be $\alpha < \mathfrak{Re}\{s\} < 2$, where $\alpha < 0$. From Property 6 in Section 9.2, this would correspond to a two-sided signal.

## 9.21

**(a)** $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} (e^{-2t} + e^{-3t})e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(2+s)t}dt + \int_{0}^{\infty} e^{-(3+s)t}dt \\ &= -\frac{e^{-(s+2)t}}{s+2}\bigg|_0^\infty - \frac{e^{-(s+3)t}}{s+3}\bigg|_0^\infty \end{aligned}$$

For convergence of the first term, we require that $\mathfrak{Re}\{-(s+2)\} < 0$, or $\mathfrak{Re}\{s\} > -2$. For convergence of the second term, we require that $\mathfrak{Re}\{-(s+3)\} < 0$, or $\mathfrak{Re}\{s\} > -3$. The ROC is the intersection of these two regions, $\mathfrak{Re}\{s\} > -2$. Then,

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{s^2+5s+6}, \ \mathfrak{Re}\{s\} > -2$$

There are two poles ($s = -2$, $s = -3$) and two zeros ($s = -5/2$, $s = \infty$).

**(b)** $x(t) = e^{-4t}u(t) + e^{-5t}\sin(5t)u(t)$

$$\begin{aligned} X(s) &= \int_{0}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} (e^{-4t}u(t) + e^{-5t}\sin(5t)u(t))e^{-st}dt \\ &= \int_{0}^{\infty} e^{-4t}e^{-st}dt + \int_{0}^{\infty} e^{-5t}\sin(5t)e^{-st}dt \end{aligned}$$

Using Euler's relation,

$$\begin{aligned} X(s) &= \int_{0}^{\infty} e^{-4t}e^{-st}dt + \int_{0}^{\infty} e^{-5t}\frac{e^{5jt}-e^{-5jt}}{2j}e^{-st}dt \\ &= -\frac{e^{(-s-4)t}}{s+4}\bigg|_0^\infty - \frac{e^{(-s-5+5j)t}}{2j(s+5-5j)}\bigg|_0^\infty + \frac{e^{(-s-5-5j)t}}{2j(s+5+5j)}\bigg|_0^\infty \end{aligned}$$

For convergence of the first term, we require that $\mathfrak{Re}\{-s-4\} < 0$, or $\mathfrak{Re}\{s\} > -4$. For convergence of the second term, we require that $\mathfrak{Re}\{(-s-5+5j)\} < 0$, or $\mathfrak{Re}\{s\} > -5$. For convergence of the third term, we require that $\mathfrak{Re}\{(-s-5-5j)\} < 0$ or $\mathfrak{Re}\{s\} > -5$. The ROC is the intersection of these regions, which is $\mathfrak{Re}\{s\} > -4$. Then,

$$\begin{aligned} X(s) &= \frac{1}{s+4} + \frac{1}{2j(s+5-5j)} - \frac{1}{2j(s+5+5j)} \\ &= \frac{1}{s+4} + \frac{5}{(s+5-5j)(s+5+5j)} \\ &= \frac{s^2+15s+70}{(s+4)(s+5-5j)(s+5+5j)}, \ \mathfrak{Re}\{s\} > -4 \end{aligned}$$

There are three poles ($s = -4$, $s = -5 + 5j$, $s = -5 - 5j$) and two zeros ($s = -7.5 + j3.7081$, $s = -7.5 - j3.7081$).

**(c)** $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{0} (e^{2t} + e^{3t})e^{-st}dt \\ &= \int_{-\infty}^{0} e^{(2-s)t}dt + \int_{-\infty}^{0} e^{(3-s)t}dt \\ &= -\frac{e^{(2-s)t}}{s-2}\bigg|_{-\infty}^{0} - \frac{e^{(3-s)t}}{s-3}\bigg|_{-\infty}^{0} \end{aligned}$$

For convergence of the first term, we require that $\mathfrak{Re}\{2 - s\} > 0$, or $\mathfrak{Re}\{s\} < 2$. For convergence of the second term, we require that $\mathfrak{Re}\{3-s\} > 0$, or $\mathfrak{Re}\{s\} < 3$. The ROC is the intersection of these two regions, $\mathfrak{Re}\{s\} < 2$. Then,

$$X(s) = -\frac{1}{s-2} - \frac{1}{s-3} = -\frac{2s-5}{s^2-5s+6},\ \mathfrak{Re}\{s\} < 2$$

There are two poles ($s = 2$, $s = 3$) and two zeros ($s = 5/2$, $s = \infty$).

**(i)** $x(t) = \delta(t) + u(t)$

$$\begin{aligned} X(s) &= \int_{0}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} (\delta(t) + u(t))e^{-st}dt \\ &= \int_{0}^{\infty} \delta(t)e^{-st}dt + \int_{0}^{\infty} u(t)e^{-st}dt \\ &= 1 - \frac{e^{(-s)t}}{s}\bigg|_{0}^{\infty} \end{aligned}$$

For convergence of the second term, we require that $\mathfrak{Re}\{-s\} < 0$. The ROC is $\mathfrak{Re}\{s\} > 0$. Then,

$$X(s) = 1 + \frac{1}{s} = \frac{s+1}{s},\ \mathfrak{Re}\{s\} > 0$$

There is a pole ($s = 0$) and a zero ($s = -1$).

## 9.22

**(a)** We will use partial fraction expansion on $X(s)$ to find the inverse. The polynomial $s^2+9$ can be factored (using the quadratic formula) as $(s-3j)(s+3j)$.

$$
\begin{aligned}
&X(s) = \frac{1}{s^2+9} = \frac{1}{(s-3j)(s+3j)}, \quad \mathfrak{Re}\{s\} > 0 \\
&\text{Assume } \; X(s) = \frac{1}{(s-3j)(s+3j)} = \frac{A}{s-3j} + \frac{B}{s+3j} \\
&\text{So, cross-multiplying} \Rightarrow A(s+3j) + B(s-3j) = 1 \\
&\Rightarrow A3j + B(-3j) = 1; \quad A + B = 0 \\
&\Rightarrow \quad A = \frac{1}{6j}; \quad B = -\frac{1}{6j} \\
&\Rightarrow \; X(s) = \frac{1}{6j}\frac{1}{s-3j} - \frac{1}{6j}\frac{1}{s+3j}
\end{aligned}
$$

We know that $e^{-\alpha t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+\alpha}$, $\mathfrak{Re}\{s\} > -\alpha$ and $-e^{-\alpha t}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s+\alpha}$, $\mathfrak{Re}\{s\} < -\alpha$. In this problem, $\mathfrak{Re}\{s\} > 0$, therefore,

$$
x(t) = \frac{1}{6j}\left(e^{3jt}u(t) - e^{-3jt}u(t)\right) = \frac{1}{3}\left[\frac{e^{3jt} - e^{-3jt}}{2j}\right]u(t) = \frac{1}{3}\sin(3t)u(t)
$$

**(c)** We will use (from Table 9.2) the following Laplace transform pair on $X(s)$ to find the inverse.

$$
e^{t}\cos(3t)u(t) \xrightarrow{\mathcal{L}} \frac{s-1}{(s-1)^2+9}, \; \mathfrak{Re}\{s\} > 1
$$

Using the time-scaling property, applied to $x(t) = e^{t}\cos(3t)u(t)$, i.e., $x(-t) \xrightarrow{\mathcal{L}} X(-s)$, we obtain

$$
e^{-t}\cos(-3t)u(-t) \xrightarrow{\mathcal{L}} \frac{-s-1}{(-s-1)^2+9}, \; \mathfrak{Re}\{-s\} > 1
$$

or equivalently

$$
e^{-t}\cos(-3t)u(-t) \xrightarrow{\mathcal{L}} -\frac{s+1}{(s+1)^2+9}, \; \mathfrak{Re}\{s\} < -1
$$

Therefore,

$$
x(t) = -e^{-t}\cos(3t)u(-t)
$$

**(d)** We will use partial fraction expansion on $X(s)$ to find the inverse. The polynomial $s^2+7s+12$ can be factored as $(s+3)(s+4)$.

$$X(s) = \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+3)(s+4)}, \quad -4 < \mathfrak{Re}\{s\} < -3$$
Assume $X(s) = \frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$

So, cross-multiplying $\Rightarrow A(s+4) + B(s+3) = s+2$

$\Rightarrow A4 + B3 = 2; \quad A + B = 1$

$\Rightarrow A = -1; \quad B = 2$

$\Rightarrow X(s) = \frac{2}{s+4} - \frac{1}{s+3}$

We then take the inverse transform of each term,

$$\frac{2}{s+4}, \ \mathfrak{Re}\{s\} > -4 \xrightarrow{\mathcal{L}} 2e^{-4t}u(t),$$
$$\frac{1}{s+3}, \ \mathfrak{Re}\{s\} < -3 \xrightarrow{\mathcal{L}} -e^{-3t}u(-t),$$

Therefore,

$$X(s) = \frac{2}{s+4} - \frac{1}{s+3}, \quad -4 < \mathfrak{Re}\{s\} < -3 \quad \xrightarrow{\mathcal{L}} 2e^{-4t}u(t) + e^{-3t}u(-t)$$

**(e)** We will use partial fraction expansion on X(s) to find the inverse. The polynomial $s^2 + 5s + 6$ can be factored as $(s+2)(s+3)$.

$$X(s) = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)}, \ -3 < \mathfrak{Re}\{s\} < -2$$
Assume $X(s) = \frac{s+1}{(s+2)(s+3)} = \frac{A}{s+3} + \frac{B}{s+2}$

So, cross-multiplying $\Rightarrow A(s+2) + B(s+3) = s+1$

$\Rightarrow A + B = 1; \quad 3A + 2B = 1$

$\Rightarrow A = 2; \quad B = -1$

$\Rightarrow X(s) = \frac{2}{s+3} - \frac{1}{s+2}$

We then take the inverse transform of each term,

$$\frac{2}{s+3} \xrightarrow{\mathcal{L}} 2e^{-3t}u(t), \qquad \mathfrak{Re}\{s\} > -3$$
$$\frac{1}{s+2} \xrightarrow{\mathcal{L}} -e^{-2t}u(-t), \qquad \mathfrak{Re}\{s\} < -2$$

Therefore,

$$x(t) = 2e^{-3t}u(t) + e^{-2t}u(-t)$$

## 9.26

The function $y(t)$ is the convolution of $x_1(t-2)$ and $x_2(-t+3)$, thus $Y(s)$ is the multiplication of the Laplace transform of $x_1(t-2)$ and the Laplace transform of $x_2(-t+3)$. Furthermore, the Laplace transform of $x_1(t-2)$ and $x_2(-t+3)$ can be related to the Laplace transform of $x_1(t)$ and $x_2(t)$, respectively, by applying the time-shifting (Section 9.5.2) and time-scaling (Section 9.5.4) properties.
Using Table 9.2 or Eq. (9.3), we have

$$x_1(t) = e^{-2t}u(t) \xrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s+2}, \qquad \mathfrak{Re}\{s\} > -2$$

and

$$x_2(t) = e^{-3t}u(t) \xrightarrow{\mathcal{L}} X_2(s) = \frac{1}{s+3}, \qquad \mathfrak{Re}\{s\} > -3$$

Using the time-shifting property (Section 9.5.2)

$$x_1(t-2) \xrightarrow{\mathcal{L}} e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}, \qquad \mathfrak{Re}\{s\} > -2$$

Using the time-scaling (Section 9.5.4) and time-shifting properties

$$x_2(-t+3) \xrightarrow{\mathcal{L}} e^{-3s}X_2(-s) = \frac{e^{-3s}}{-s+3}, \qquad \mathfrak{Re}\{s\} < 3$$

Therefore,

$$y(t) \xrightarrow{\mathcal{L}} \frac{e^{-2s}}{s+2}\frac{e^{-3s}}{3-s}, \qquad -2 < \mathfrak{Re}\{s\} < 3$$

## 9.28

**(a)** There are three poles: $-2, -1, +1$. So, the possible ROCs are

(i) $\mathfrak{Re}\{s\} > 1$ (right of rightmost pole)

(ii) $-1 < \mathfrak{Re}\{s\} < 1$ (strip between poles)

(iii) $-2 < \mathfrak{Re}\{s\} < -1$ (strip between poles)

(iv) $\mathfrak{Re}\{s\} < -2$ (left of leftmost pole)

**(b)** By using the properties of the ROC, the corresponding systems are

(i) Unstable (does not contain $j\omega$-axis) and causal (right half plane).

(ii) Stable (contains $j\omega$-axis) and not causal (two-sided).

(iii) Unstable (does not contain $j\omega$-axis) and not causal (two-sided).

(iv) Unstable (does not contain $j\omega$-axis) and not causal (left half plane).

## 9.31

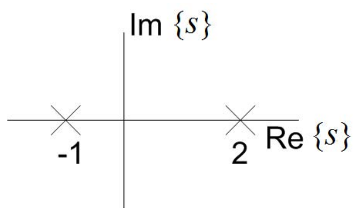
**(a)** The input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Taking the Laplace transform of both sides of the differential equation, we obtain

$$\begin{aligned}
&s^2Y(s) - sY(s) - 2Y(s) = X(s)\\
&(s^2 - s - 2)Y(s) = X(s)\\
&H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}
\end{aligned}$$

Therefore, $H(s)$ has two poles $(2, -1)$ and two zeros $(\infty, \infty)$. The pole-zero plot for $H(s)$ is




**(b)** Using partial fraction expansion, we obtain

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

(i) If the system is stable, the ROC includes the $j\omega$-axis. This means that the ROC should be $-1 < \mathfrak{Re}\{s\} < 2$ (the intersection of $\mathfrak{Re}\{s\} < 2$, corresponding to a left-sided signal, and $\mathfrak{Re}\{s\} > -1$, corresponding to a right-sided signal) and this corresponds to a two-sided signal. Therefore,

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$$

(ii) If the system is causal, the ROC should be a right-half plane, which means that the ROC should be $\mathfrak{Re}\{s\} > 2$ and this corresponds to a right-sided signal. Therefore, both terms in $X(s)$ correspond to right-sided signals and

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)$$

(iii) If the system is neither stable nor causal, then the ROC should be $\mathfrak{Re}\{s\} < -1$, and that corresponds to a left-sided signal. Therefore, both terms in $X(s)$ correspond to left-sided signals and

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)$$

## 9.44

**(a)** The signal $x(t) = \sum_{n=0}^{\infty} e^{-nT}\delta(t-nT)$ has Laplace transform

$$X(s) = \sum_{n=0}^{\infty} e^{-nT} \int_{0}^{\infty} \delta(t-nT)e^{-st}dt$$
$$= \sum_{n=0}^{\infty} e^{-nT}e^{-snT} = \sum_{n=0}^{\infty} (e^{-T(s+1)})^n$$

Using the geometric series formula, we obtain

$$X(s) = \frac{1}{1-e^{-T(s+1)}}$$

The poles occur when

$$e^{-T(s+1)} = 1 \text{ or } e^{-T(s_k+1)} = e^{j2\pi k}, k = 0, \pm 1, \pm 2, \ldots$$

Then, taking the logarithm on both sides of $e^{-T(s_k+1)} = e^{j2\pi k}$, we get

$$s_k = -1 + \frac{j2\pi k}{T}\ , k = 0, \pm 1, \pm 2, \ldots$$

Therefore, the poles all lie on a vertical line (parallel to the $\mathfrak{Im}\{s\}$ axis) passing through $s = -1$. Since the signal is right-sided, the ROC is $\mathfrak{Re}\{s\} > -1$.

**(b)**