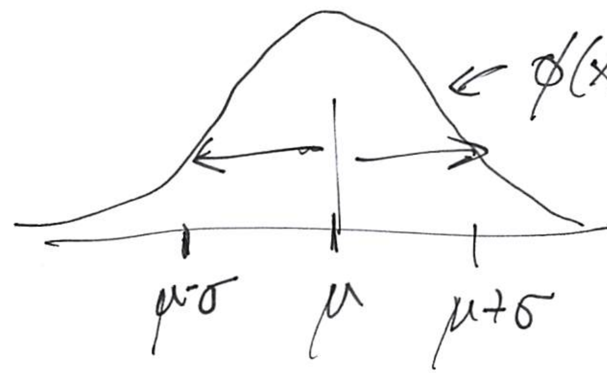
ELEG 310

4/24/2018

Gaussian Distribution (Normal)

$$X \sim N(\mu, \sigma^2)$$

mean var



$$\phi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$Z \sim N(0,1) \qquad X = \mu + \sigma Z \qquad Z = \frac{X-\mu}{\sigma}$$

$$F_Z(z) = P(Z \le z) = \Phi(z)$$

$$F_X(x) = P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$EX = \mu \quad Var\, X = \sigma^2 \quad \cancel{EX^3 = 0} \quad E$

$EZ^3 = 0$

$$E(X^3) = E(\mu + \sigma Z)^3 = E(\mu^3 + 3\mu^2\sigma Z + 3\mu\sigma^2 Z^2 + \sigma^3 Z^3)$$

$$= \mu^3 + 0 + 3\mu\sigma^2 + 0$$

$$M_Z(u) = E(e^{uZ}) = \int_{-\infty}^{\infty} e^{uz}\phi(z)\,dz = e^{u^2/2}$$

$$M_X(u) = E(e^{uX}) = E(e^{u\mu + u\sigma Z}) = E(e^{u\mu} e^{u\sigma Z})$$

$$= e^{u\mu} E(e^{u\sigma Z}) = e^{u\mu} e^{\sigma^2 u^2/2}$$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Central Limit Theorem

let $X_1\ X_2 \dots X_n$ be n IID RVs

with mean μ and Var $\sigma^2 < \infty$.

(Cauchy Distribution $f(x) = \frac{c}{1+x^2}$

$$EX = \int_{-\infty}^{\infty} c \frac{x}{1+x^2}\, dx$$

$$\operatorname{Var} X = \int_{-\infty}^{\infty} c \frac{x^2}{1+x^2}\, dx \to \infty$$
)

let $S_n = X_1 + X_2 + \dots + X_n$

$$E(S_n) = E(X_1 + X_2 + \dots + X_n)$$
$$= EX_1 + EX_2 + \dots + EX_n$$
$$= \mu + \mu + \dots + \mu = n\mu$$

$$\operatorname{Var}(S_n) = \operatorname{Var} X_1 + \dots + \operatorname{Var} X_n$$
$$+ 2\operatorname{Cov}(X_1, X_2) + \dots + 2\operatorname{Cov}(X_1, X_n)$$
$$+ 2\operatorname{Cov}(X_2, X_3) + \dots$$
$$= \sigma^2 + \sigma^2 + \dots + \sigma^2 + 0 + 0 \dots 0 = n\sigma^2$$

let $T_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$

The CLT says $T_n \sim N(0,1)$ for large n

$S_n \sim N(\mu n, n\sigma^2)$