Exam#1 (Math 342)

Oct 10 2018

STUDENT NAME: ______________________________

Instructions:

The duration of the test is 50 minutes. The test consists of 5 questions and the points are specified next to each question. Total grade = 50. Detail your calculations. No aid sheet, class notes or calculator allowed.

1) [5] Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\boldsymbol{y}_1 = \boldsymbol{x}_2 - \boldsymbol{x}_1 \ , \quad \boldsymbol{y}_2 = \boldsymbol{x}_3 - \boldsymbol{x}_2 \ , \quad \boldsymbol{y}_3 = \boldsymbol{x}_3 - \boldsymbol{x}_1$$

Are $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ linearly independent? Justify your answer.

Solution:

If $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ are linearly dependent, then they must satisfy

$$c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_2\boldsymbol{y}_3 = \mathbf{0}$$

with c_1, c_2 and c_3 not all zero. However, we can rewrite this equation as

$$\begin{aligned} c_1(\boldsymbol{x}_2 - \boldsymbol{x}_1) + c_2(\boldsymbol{x}_3 - \boldsymbol{x}_2) + c_3(\boldsymbol{x}_3 - \boldsymbol{x}_1) &= \mathbf{0} \\ -(c_1 + c_3)\boldsymbol{x}_1 + (c_1 - c_2)\boldsymbol{x}_2 + (c_2 + c_3)\boldsymbol{x}_3 &= \mathbf{0} \end{aligned}$$

Because $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent, this implies that

$$c_1 + c_3 = 0 \ , \quad c_1 - c_2 = 0 \ , \quad c_2 + c_3 = 0$$

leading to infinitely many solutions parameterized by $c_1 = c_2 = t$ and $c_3 = -t$ where t is a free parameter, not necessarily zero. Therefore, $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ are linearly dependent.

2) [10] For the 2×2 matrix

$$A = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$$

(a) Compute e^A

(b) Compute A^{100}

via the method of diagonalization.

Hint: $e^A = Pe^D P^{-1}$ and $A^k = PD^k P^{-1}$.

Solution:

- eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -6 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - \lambda = \lambda(\lambda - 1) = 0$$

$\Rightarrow \lambda = 0, 1$

- eigenvectors:

- $\lambda = 0$

$$\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x + 3y = 0$$

  Therefore a possible eigenvector is $(3, -1)^\top$

- $\lambda = 1$

$$\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x + 2y = 0$$

  Therefore a possible eigenvector is $(2, -1)^\top$

- diagonalizing matrix

$$P = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$$

(a)

$$e^A = P\,e^D\,P^{-1} = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 3-2e & 6-6e \\ -1+e & -2+3e \end{pmatrix}$$

(b)

$$A^{100} = P\,D^{100}\,P^{-1} = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} = A$$

3) [12] Determine whether the following vectors are linearly dependent. Justify your answer.

(a)

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{in} \quad \mathbb{R}^3$$

(b) $\left\{2,\, x^2,\, x,\, 2x+3\right\}$ in $\mathbb{P}_3$

(c) $\left\{1,\, e^x + e^{-2x},\, e^x - e^{-2x}\right\}$ in $C^2[0,1]$

Solution:

(a)

$$\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 8 + 40 - 48 = 0$$

$\Rightarrow$ the vectors are linearly dependent

(b) If we use the Wronskian

$$\begin{vmatrix} 2 & x^2 & x & 2x+3 \\ 0 & 2x & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{identically so we cannot infer}$$

If we solve $2c_1 + c_2x^2 + c_3x + c_4(2x+3) = 0 \Rightarrow (2c_1 + 3c_4) + (c_3 + 2c_4)x + c_2x^2 = 0$

Therefore $2c_1 + 3c_4 = 0,\ c_3 + 2c_4 = 0,\ c_2 = 0$

$\Rightarrow c_1, c_3, c_4$ (not necessarily zero) are related so the vectors are linearly dependent

Remark: Or we can simply note that the three vectors 2, x and $2x+3$ are linearly related

(c)

$$\begin{vmatrix} 1 & e^x + e^{-2x} & e^x - e^{-2x} \\ 0 & e^x - 2e^{-2x} & e^x + 2e^{-2x} \\ 0 & e^x + 4e^{-2x} & e^x - 4e^{-2x} \end{vmatrix} = -12e^{-x} \neq 0 \text{ for all } x \in [0,1]$$

$\Rightarrow$ the functions are linearly independent on $C^2[0,1]$

4) [12] Determine whether the following are linear transformations. Justify your answer.

(a)

$$\begin{array}{rccc} T: & \mathbb{P}_2 & \longrightarrow & \mathbb{P}_2 \\ & p(x) & \longmapsto & x\,p'(x) \end{array}$$

(b)

$$\begin{array}{rccc} T: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & 2\,x + x^2 \end{array}$$

(c)

$$\begin{array}{rccc} T: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x_1, x_2) & \longmapsto & (1 + x_1, x_2) \end{array}$$

Solution:

(a) T is a linear transformation because

- addition:

$$T(p_1(x) + p_2(x)) = x(p_1'(x) + p_2'(x)) = xp_1'(x) + xp_2'(x) = T(p_1(x)) + T(p_2(x))$$

- scalar multiplication:

$$T(\alpha p(x)) = x(\alpha p'(x)) = \alpha x p'(x) = \alpha T(p(x))$$

(b) T is not a linear transformation because

- addition:

$$\begin{aligned} T(x_1 + x_2) &= 2(x_1 + x_2) + (x_1 + x_2)^2 = 2x_1 + x_1^2 + 2x_2 + x_2^2 + 2x_1x_2 \\ &= T(x_1) + T(x_2) + 2x_1x_2 \neq T(x_1) + T(x_2) \end{aligned}$$

- scalar multiplication:

$$T(\alpha x) = 2\alpha x + (\alpha x)^2 = 2\alpha x + \alpha^2 x^2 \neq \alpha T(x) = 2\alpha x + \alpha x^2$$

(c) T is not a linear transformation because

- addition:

$$\begin{aligned} T((x_1, x_2) + (y_1, y_2)) &= T((x_1 + y_1, x_2 + y_2)) = (1 + x_1 + y_1, x_2 + y_2) = (1 + x_1, x_2) + (y_1, y_2) \\ &\neq (1 + x_1, x_2) + (1 + y_1, y_2) = T((x_1, x_2)) + T((y_1, y_2)) \end{aligned}$$

- scalar multiplication:

$$T((\alpha x_1, \alpha x_2) = (1 + \alpha x_1, \alpha x_2) \neq \alpha(1 + x_1, x_2) = \alpha T((x_1, x_2))$$

5) [11] Via diagonalization, find the general real-valued solution to the following system of differential equations

$$\begin{cases} y_1' &= \quad y_1 + y_2 \\ y_2' &= -y_1 + y_2 \end{cases}$$

Give an expression as explicit as possible.

Solution:

- eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm i$$

- eigenvectors:

- $\lambda = 1 + i \Rightarrow -ix + y = 0,\ -x - iy = 0 \Rightarrow$ eigenvector $(1, i)^\top$
- $\lambda = 1 - i \Rightarrow ix + y = 0,\ -x + iy = 0 \Rightarrow$ eigenvector $(1, -i)^\top$

Therefore the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1-i)t}$$

The real-valued solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \operatorname{Re}\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t} \right\} + c_2 \operatorname{Im}\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t} \right\}$$

More explicitly,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^t$$