Short questions: Circle your answer so there is no ambiguity.

1. (5 points) If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent with $\boldsymbol{X} \sim N(1,2)$ and $\boldsymbol{Y} \sim N(-1,6)$, what is the variance of $\boldsymbol{Z} = 2\boldsymbol{X} + 3\boldsymbol{Y}$?

$$Var(Z) = 2^2 Var(X) + 3^2 Var(Y) = 4 \times 2 + 9 \times 6 = \boxed{62}$$

2. (5 points) If $f(x) = cx^2$ for $0 < x < 4$ and $f(x) = 0$ elsewhere, what is c?

$$1 = \int_0^4 cx^2\,dx = \frac{cx^3}{3}\Big|_0^4 = \frac{64}{3}c \Rightarrow \boxed{c = \frac{3}{64}}$$

3. (5 points) If $F(x) = 0$ for $-\infty < x < \infty$, $F(x) = x^2$ for $0 < x < 1$ and $F(x) = 1$ for $x > 1$, what is $E(\boldsymbol{X})$?

$$f(x) = \frac{dF}{dx} = 2x \qquad 0 < x < 1$$

$$EX = \int_0^1 x \cdot 2x\,dx = \frac{2x^3}{3}\Big|_0^1 = \boxed{\frac{2}{3}}$$

4. (5 points) If $F(x) = 1 - e^{-2x}$ for $x > 0$, what is $f(x)$?

$$f(x) = \frac{dF}{dx} = \boxed{2e^{-2x} \quad x > 0}$$

5. (5 points) If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent discrete random variables with $\Pr[\boldsymbol{X} = k] = 0.25$ for $k = 0, 1, 2, 3$ and $\Pr[\boldsymbol{Y} = l] = 0.5$ for $l = 0, 1$, what is the PMF of $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$?

	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\frac{1}{2}$		$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

6. (5 points) If $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent Bernoulli random variables with parameters p_1 and p_2, respectively, what is the PMF of $\boldsymbol{Z} = \boldsymbol{X}_1 + \boldsymbol{X}_2$?

	q_1	p_1	
q_2	$q_2 q_1$	$q_2 p_1$	
p_2		$p_2 q_1$	$p_2 p_1$
	$q_2 q_1$	$q_2 p_1 + p_2 q_1$	$p_2 p_1$

i.e. $P(Z=1) = q_2 p_1 + p_2 q_1$

7. (5 points) If $\boldsymbol{X} \sim U(0, 1)$, what is the density of $\boldsymbol{Y} = -\lambda \log(\boldsymbol{X})$?

$$F_Y(y) = P(Y \le y) = P(-\lambda \log X \le y) = P\left(\log X \ge \frac{-y}{\lambda}\right)$$
$$= P(X \ge e^{-y/\lambda}) = 1 - e^{-y/\lambda}$$

$$\boxed{f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda} \qquad y > 0}$$

8. (5 points) If $\boldsymbol{X}_i$ for $i = 1, 2, \ldots, n$ are IID with mean μ and variance σ^2, what are the mean and variance of $\boldsymbol{S} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_n$?

$$\boxed{\begin{aligned} ES &= n\mu \\ \operatorname{Var} S &= n\sigma^2 \end{aligned}}$$

9. (5 points) If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are IID with density $f(u) = 2u$ for $0 < u < 1$ (and $f(u) = 0$ elsewhere), what is the density ~~of $\boldsymbol{X} = \boldsymbol{Z} + \boldsymbol{Y}$~~? $Z = X + Y$?

$$f_Z = f_X * f_Y$$

$$f_Z(z) = \int_{-\infty}^{\infty} 2u\,(2z - 2u)\,du = \begin{cases} \int_0^z 2u(2z-2u)\,du = \frac{2}{3}z^3 & 0<z<1 \\ \int_{z-1}^{2} 2u(2z-2u)\,du = & 1<z<2 \end{cases}$$

10. (5 points) Let $\boldsymbol{X}_i$ for $i = 1, 2, \ldots, 12$ be 12 IID $U(0,1)$ random variables, and let $\boldsymbol{S} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \cdots + \boldsymbol{X}_{12}$. What is a reasonable approximation to the CDF of $\boldsymbol{S}$?

$$ES = 6 \quad \operatorname{Var} S = 12 \operatorname{Var}(X) = 12 \cdot \frac{1}{12} = 1$$

$$F_S(s) = P(S \le s) = P(S-6 \le s-6) = \boxed{\Phi(s-6)}$$

11. (5 points) If $\boldsymbol{X} \sim N(1,4)$, what is $\Pr[\boldsymbol{X} < 0]$?

$$P(X<0) = P\left(\frac{X-1}{2} \le \frac{-1}{2}\right) = \Phi\left(-\frac{1}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right)$$

$$= 1 - 0.6915 = \boxed{0.3085}$$

12. (5 points) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ have joint PMF below. What is $E(\boldsymbol{XY})$?

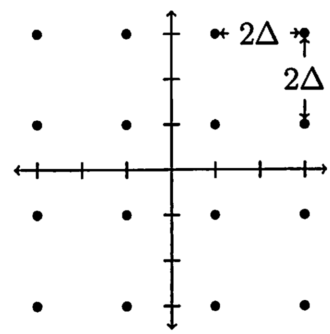
y \ x	0	1	2	3
1	0.1	0.2	0.1	0.2
0	0.0	0.1	0.1	0.2

$$E(XY) = 1\times1\times0.2 + 1\times2\times0.1 \times 1\times3\times0.2 = \boxed{1.0}$$

14. (20 points) For 16 QAM, assume the points are equally likely to be transmitted and the noises N_X and N_Y are IID $N(0, \sigma^2)$.

    (a) What is the average transmitter power?

    (b) What is the probability of error?



a) Ave Power $= E(D^2)$

$$= \frac{4}{16}(\Delta^2+\Delta^2) + \frac{8}{16}(\Delta^2 + 3^2\Delta^2) + \frac{4}{16}(3^2\Delta^2 + 3^2\Delta^2)$$

$$= \frac{8+80+72}{160}\Delta^2 = \boxed{10\Delta^2}$$

b) $P(\text{error}) \boxed{= 1 - P(\text{correct})}$

$$P(\text{correct}) = \frac{4}{16} P(-\Delta < N_x < \Delta \cap -\Delta < N_y < \Delta)$$

$$+ \frac{8}{16} P(N_x > -\Delta \cap -\Delta < N_y < \Delta)$$

$$+ \frac{4}{16} P(N_x > -\Delta \cap N_y > -\Delta)$$

$$\boxed{= \frac{4}{16}\left(2\Phi\left(\frac{\Delta}{\sigma}\right)-1\right)^2 + \frac{8}{16}\Phi\left(\frac{\Delta}{\sigma}\right)\left(2\Phi\left(\frac{\Delta}{\sigma}\right)-1\right) + \frac{4}{16}\Phi\left(\frac{\Delta}{\sigma}\right)^2}$$

$P(N > -\Delta) = P(N < \Delta) = \Phi\left(\frac{\Delta}{\sigma}\right)$ $\quad P(-\Delta < N < \Delta) = \Phi\left(\frac{\Delta}{\sigma}\right) - \Phi\left(\frac{-\Delta}{\sigma}\right) = 2\Phi\left(\frac{\Delta}{\sigma}\right) - 1$

13. (20 points) 3. Let $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, …, $\boldsymbol{X}_5$ be IID exponential random variables with density $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.

(a) What is $\Pr\big[\boldsymbol{X} \geq 1\big]$?

(b) What is $\Pr\big[\text{at least 3 of the 5 } \boldsymbol{X}\text{'s are} \geq 1\big]$?

(c) What is $\Pr\big[\max(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_5) \geq 1\big]$? (where $\max(2,5) = 5$ is the maximum function)

a) $P(X \geq 1) = \int_1^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda}\Big|_1^\infty = \boxed{e^{-\lambda}}$

b) $P(\text{at least 3 of 5 } X\text{s} \geq 1) = \sum_{k=3}^{5} P(\text{exactly } k \text{ of 5 are} \geq 1)$

$$\boxed{= \sum_{k=3}^{5} \binom{5}{3} p^k q^{5-k} \qquad p = e^{-\lambda} \quad q = 1-p}$$

c) $P(\max(X_1, \ldots, X_5) \geq 1) = 1 - P(\text{all } X_i \leq 1) = \boxed{1-(1-e^{-\lambda})^5}$

$= \sum_{k=1}^{5} P(\text{exactly } k \geq 1) = \boxed{\sum_{k=1}^{5} \binom{5}{k} p^k q^{5-k}}$ ← OR, both are correct

Table 1: Values of the Standard Normal Distribution Function

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	1.00	0.8413	2.00	0.9772	3.00	0.9987
0.05	0.5199	1.05	0.8531	2.05	0.9798	3.05	0.9989
0.10	0.5398	1.10	0.8643	2.10	0.9821	3.10	0.9990
0.15	0.5596	1.15	0.8749	2.15	0.9842	3.15	0.9992
0.20	0.5793	1.20	0.8849	2.20	0.9861	3.20	0.9993
0.25	0.5987	1.25	0.8944	2.25	0.9878	3.25	0.9994
0.30	0.6179	1.30	0.9032	2.30	0.9893	3.30	0.9995
0.35	0.6368	1.35	0.9115	2.35	0.9906	3.35	0.9996
0.40	0.6554	1.40	0.9192	2.40	0.9918	3.40	0.9997
0.45	0.6736	1.45	0.9265	2.45	0.9929	3.45	0.9997
0.50	0.6915	1.50	0.9332	2.50	0.9938	3.50	0.9998
0.55	0.7088	1.55	0.9394	2.55	0.9946	3.55	0.9998
0.60	0.7257	1.60	0.9452	2.60	0.9953	3.60	0.9998
0.65	0.7422	1.65	0.9505	2.65	0.9960	3.65	0.9999
0.70	0.7580	1.70	0.9554	2.70	0.9965	3.70	0.9999
0.75	0.7734	1.75	0.9599	2.75	0.9970	3.75	0.9999
0.80	0.7881	1.80	0.9641	2.80	0.9974	3.80	0.9999
0.85	0.8023	1.85	0.9678	2.85	0.9978	3.85	0.9999
0.90	0.8159	1.90	0.9713	2.90	0.9981	3.90	1.0000
0.95	0.8289	1.95	0.9744	2.95	0.9984	3.95	1.0000