ELEG 310

20 Feb 2018

Combinatorics – mathematics of counting

Basic Principle: if Exp1 can be done in n_1 ways and Exp2 in n_2 ways the ~~join~~ compound Exp (Exp1, Exp2) can be done in $n_1 \times n_2$ ways

Case 1: n items select k ordered and with replacement

$n! = n$ factorial $= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$0! = 1$

ways to select k items from n, w/o repl, ordered

$$= n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = (n)_k$$

= # permutations of k items from n

$$= {}_nP_k$$

Ordered $6\diamondsuit, 5\clubsuit \neq 5\clubsuit, 6\diamondsuit$

Unordered $6\diamondsuit\ 5\clubsuit = 5\clubsuit\ 6\diamondsuit$

With replacement

Without replacement

Ex. select 2 cards from deck of 52

with replacement, ~~un~~ordered $= 52 \times 52$ n^k

w/o repl, ~~un~~ordered $= 52 \times 51$

Select k items from n unordered w/o repl

Eg. 2 from 52 # ways $= \frac{52 \times 51}{2}$

3 from 52 # ways $= \frac{52 \times 51 \times 50}{3!}$ $3! = 6$

general answer

k from n # ways $= \frac{n(n-1) \cdots (n-k+1)}{k!}$

ways $= \binom{n}{k}$ = "n choose k"

= binomial coefficient

$= \frac{n!}{(n-k)!\,k!} = {}_nC_k$

Select k from n unordered with repl

$$= \binom{n+k-1}{k}$$

Binomial coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k \text{ terms}}}{\underbrace{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1}_{k \text{ terms}}}$$

$$\binom{n}{k} \geq 0$$

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

$$\binom{5}{1} = \binom{5}{4} \qquad \binom{52}{2} = \binom{52}{50}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

n slots

place k 1's

$n-k$ 0's

$$\overbrace{\underbrace{\underline{1}\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{0}}_{k \text{ slots out of } n-1}}^{n-1} \Big|\ \overbrace{\underline{0}}^{1}$$

$$\binom{n-1}{k}$$

$n=8$

$k=3$

$$\underbrace{\underline{0}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{0}}_{\binom{n-1}{k-1}} \Big|\ \underline{1}$$

$$\Rightarrow \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's Triangle

$n=1$ 1 1

$n=2 \longrightarrow$ 1 2 1

$n=3 \longrightarrow$ 1 3 3 1

$n=4 \rightarrow$ 1 4 6 4 1

$n=5 \rightarrow$ 1 5 10 10 5 1

$n=6 \rightarrow$ 1 6 15 20 15 6 1

$k=0$ $k=1$ $k=2$ $k=3$ $k=4$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$(a+b)^1 = a+b = \sum_{k=0}^{1} \binom{1}{k} a^k b^{n-k}$$

$$= (a+b)^{n-1}(a+b)$$ ← assume true for $n-1$

$$= (a+b) \sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-1-k}$$

$$\vdots$$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

$$\binom{n}{k} \leq 2^n$$

Binomial Probabilities

$$1^n = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$q = 1-p$

Prob getting k heads in n flips of coin with prob p of a heads.

Multinomial Coefficients

$$\binom{n}{k_1\, k_2 \cdots k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

$$k_1 + k_2 + \cdots + k_m = n$$

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{k's} \binom{n}{k_1\, k_2 \cdots k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$$