HW9 Solution

Problem 9.2

If $\boldsymbol{X} \sim N(-1,5)$, what are the following:

a) $\Pr[\boldsymbol{X} < 0]$

b) $\Pr[\boldsymbol{X} > 0]$

c) $\Pr[-2 < \boldsymbol{X} < 0]$

d) $\Pr[-2 < \boldsymbol{X} < 2]$

Solution to Problem 9.2

a) $\Pr[\boldsymbol{X} < 0] = \Pr[(\boldsymbol{X}+1)/\sqrt{5} < (0+1)/\sqrt{5}] = \Phi(1/\sqrt{5}) = \Phi(0.447) \approx 0.67$

b) $\Pr[\boldsymbol{X} > 0] = 1.0 - \Pr[\boldsymbol{X} < 0] \approx 0.33$

c) $\Pr[-2 < \boldsymbol{X} < 0] = \Pr[(-2+1)/\sqrt{5} < (\boldsymbol{X}+1)/\sqrt{5} < (0+1)/\sqrt{5}] = \Phi(1/\sqrt{5}) - \Phi(-1/\sqrt{5}) = 2\Phi(1/\sqrt{5}) - 1 = 0.34$

d) $\Pr[-2 < \boldsymbol{X} < 2] = \Pr[(-2+1)/\sqrt{5} < (\boldsymbol{X}+1)/\sqrt{5} < (2+1)/\sqrt{5}] = \Phi(3/\sqrt{5}) - \Phi(-1/\sqrt{5}) = \Phi(1.342) + \Phi(0.447) - 1 \approx 0.91 + 0.67 - 1 = 0.58$

Problem 9.4

If $\boldsymbol{X} \sim N(3,16)$, and $\boldsymbol{Y} = 3\boldsymbol{X} + 4$, what are the mean and variance of $\boldsymbol{Y}$? What is the density of $\boldsymbol{Y}$?

Solution to Problem 9.4

$$E[\boldsymbol{Y}] = E[3\boldsymbol{X}+4] = 3 \times 3 + 4 = 13$$
$$\mathrm{Var}[\boldsymbol{Y}] = 3^2\mathrm{Var}[\boldsymbol{X}] = 9 \times 16 = 144$$
$$f_{\boldsymbol{Y}}(y) = \frac{1}{12\sqrt{2\pi}} \exp\left((y-13)^2/24\right)$$

Problem 9.7

For $\boldsymbol{X} = \sigma\boldsymbol{Z} + \mu$ with $\boldsymbol{Z} \sim N(0,1)$, use (9.13) and (9.14) to calculate $E[\boldsymbol{X}]$, $E[\boldsymbol{X}^2]$, $E[\boldsymbol{X}^3]$, and $E[\boldsymbol{X}^4]$?

Solution to Problem 9.7

$$E[\boldsymbol{X}] = E[\sigma\boldsymbol{Z}+\mu] = \sigma E[\boldsymbol{Z}] + \mu = \mu$$
$$E[\boldsymbol{X}^2] = E[(\sigma\boldsymbol{Z}+\mu)^2] = \sigma^2 E[\boldsymbol{Z}^2] + 2\sigma\mu E[\boldsymbol{Z}] + \mu^2 = \sigma^2 + 0 + \mu^2$$
$$E[\boldsymbol{X}^3] = \sigma^3 E[\boldsymbol{Z}^3] + 3\sigma^2\mu E[\boldsymbol{Z}^2] + 3\sigma\mu^2 E[\boldsymbol{Z}] + \mu^3 = 0 + 3\sigma^2\mu + 0 + \mu^3$$
$$E[\boldsymbol{X}^4] = \sigma^4 E[\boldsymbol{Z}^4] + 4\sigma^3\mu E[\boldsymbol{Z}^3] + 6\sigma^2\mu^2 E[\boldsymbol{Z}^2] + 4\sigma\mu^3 E[\boldsymbol{Z}] + \mu^4 = 3\sigma^4 + 0 + 6\sigma^2\mu^2 + 0 + \mu^4$$

9.12 $X \sim N(0,1)$

$P[X \geq 1] = 1 - \Phi(1) = 0.1586$

$P[X \geq 2] = 1 - \Phi(2) = 0.0227$

$P[X \geq 3] = 1 - \Phi(3) = 0.0013$

$P[X \geq 4] = 1 - \Phi(4) = 0.00003$

$X \sim \text{Laplacian}(u=0, \lambda=1)$

$P[X \geq k] = \int_k^\infty \frac{1}{2} e^{-x} dx = \frac{1}{2} e^{-k}$

$P[X \geq 1] = \frac{1}{2} e^{-1} = 0.1839$

$P[X \geq 2] = \frac{1}{2} e^{-2} = 0.0676$

$P[X \geq 3] = \frac{1}{2} e^{-3} = 0.0248$

$P[X \geq 4] = \frac{1}{2} e^{-4} = 0.009157$

9.14 $X \sim N(u, \sigma^2)$, $Y = e^X$

$$F_Y(Y) = P(Y \le y] = P[e^X \le y] = P[X \le \log y] = F_X(\log y)$$

$$= P[Z \le \frac{\log y - u}{\sigma}]$$

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\left(\frac{\log y - u}{\sigma}\right)^2/2}$$

9.17 a. $z > z_0 \Rightarrow \frac{z}{z_0} > 1 \Rightarrow \Phi(z) \le \frac{z}{z_0}\Phi(z)$

$$\Rightarrow \int_{z_0}^{\infty} \Phi(z)dz \le \int_{z_0}^{\infty} \frac{z}{z_0}\Phi(z)dz$$

b. $$\int_{z_0}^{\infty} \frac{z}{z_0}\Phi(z)dz = \frac{1}{z_0}\int_{z_0}^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{-1}{z_0\sqrt{2\pi}} e^{-z^2/2}\Big|_{z_0}^{\infty} = \frac{-1}{z_0\sqrt{2\pi}}\left(-e^{-z_0^2/2}\right)$$

$$P[Z > z_0] = 1 - \Phi(z_0) \le \frac{1}{z_0\sqrt{2\pi}} e^{-z_0^2/2}$$

c.

z_0	$1 - \Phi(z_0)$	$\frac{1}{z_0\sqrt{2\pi}} e^{-z_0^2/2}$
1	1.586×10^{-1}	2.419×10^{-1}
2	2.275×10^{-2}	2.699×10^{-2}
3	1.349×10^{-3}	1.477×10^{-3}
4	3.167×10^{-5}	3.345×10^{-5}
5	2.866×10^{-7}	2.973×10^{-7}

Problem 9.18

What are the mean and variance of $\boldsymbol{W} = a\boldsymbol{X} + b$ if $\boldsymbol{X} \sim N(\mu, \sigma^2)$? What is the density and distribution function of $\boldsymbol{W}$? What is the MGF of $\boldsymbol{W}$?

Solution to Problem 9.18

$$E[\boldsymbol{W}] = aE[\boldsymbol{X}] + b = a\mu + b$$
$$\mathrm{Var}[\boldsymbol{W}] = a^2\mathrm{Var}[\boldsymbol{W}] = a^2\sigma^2$$
$$\boldsymbol{W} \sim N(a\mu + b, a^2\sigma^2)$$
$$M_{\boldsymbol{W}}(u) = E[e^{u\boldsymbol{W}}] = \exp\left((a\mu + b)u + a^2\sigma^2u^2/2\right)$$

9.25 a. binomial : $\sum_{k=3}^{6}\binom{10}{k}0.5^{k}\,0.5^{10-k} = 0.7735$

9.16 : $\Phi\left(\frac{6-10\times0.5}{\sqrt{10\times0.5^2}}\right) - \Phi\left(\frac{3-10\times0.5}{\sqrt{10\times0.5^2}}\right) = 0.6366$

9.17 : $\Phi\left(\frac{6-10\times0.5+0.5}{\sqrt{10\times0.5^2}}\right) - \Phi\left(\frac{3-10\times0.5-0.5}{\sqrt{10\times0.5^2}}\right) = 0.7741$

b. binomial : $\sum_{k=3}^{6}\binom{10}{k}0.2^{k}\,0.8^{10-k} = 0.3213$

9.16 $\Phi\left(\frac{6-10\times0.2}{\sqrt{10\times0.2\times0.8}}\right) - \Phi\left(\frac{3-10\times0.2}{\sqrt{10\times0.2\times0.8}}\right) = 0.2111$

9.17 $\Phi\left(\frac{6-10\times0.2+0.5}{\sqrt{10\times0.2\times0.8}}\right) - \Phi\left(\frac{3-10\times0.2-0.5}{\sqrt{10\times0.2\times0.8}}\right) = 0.344$

c d , Same as above

9.37 $P[\text{correct} \mid A\text{ sent}]$

$= P[R_x \geq 2\Delta \cap R_y \geq 2\Delta \mid A\text{sent}]$

$= P[(3\Delta+N_x \geq 2\Delta)\cap(3\Delta+N_y \geq 2\Delta) \mid A\text{sent}]$

$= P[(N_x \geq -\Delta)\cap(N_y \geq -\Delta)]$

$= P[\frac{N_x}{\sigma} \geq -\frac{\Delta}{\sigma}]\cdot P[\frac{N_y}{\sigma} \geq -\frac{\Delta}{\sigma}]$

$= (1-\Phi(-\frac{\Delta}{\sigma}))^2 = \Phi(\frac{\Delta}{\sigma})^2$

$P[\text{correct} \mid B\text{ sent}]$

$= P[R_x \geq 2\Delta \cap 0 \leq R_y \leq 2\Delta \mid B\text{sent}]$

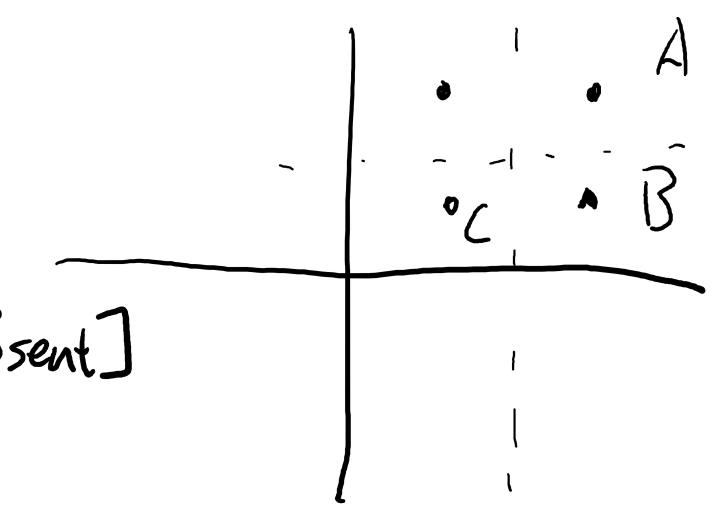
$= P[(3\Delta+N_x \geq 2\Delta)\cap(0 \leq \Delta+N_y \leq 2\Delta) \mid B\text{sent}]$

$= P[(N_x \geq -\Delta)\cap(-\Delta \leq N_y \leq \Delta)]$

$= P[\frac{N_x}{\sigma} \geq -\frac{\Delta}{\sigma}]\cdot P[-\frac{\Delta}{\sigma} \leq \frac{N_y}{\sigma} \leq \frac{\Delta}{\sigma}]$

$= (1-\Phi(-\frac{\Delta}{\sigma}))\cdot(\Phi(\frac{\Delta}{\sigma})-\Phi(-\frac{\Delta}{\sigma}))$

$= 2\Phi(\frac{\Delta}{\sigma})^2 - \Phi(\frac{\Delta}{\sigma})$



$P[\text{correct} \mid C\text{ sent}]$

$= P[(0 \leq R_x \leq 2\Delta)\cap(0 \leq R_y \leq 2\Delta) \mid C\text{sent}]$

$= P[(0 \leq \Delta+N_x \leq 2\Delta)\cap(0 \leq \Delta+N_y \leq 2\Delta) \mid C\text{sent}]$

$= P[-\frac{\Delta}{\sigma} \leq N_x \leq \frac{\Delta}{\sigma}]\cdot P[-\frac{\Delta}{\sigma} \leq N_y \leq \frac{\Delta}{\sigma}]$

$= (\Phi(\frac{\Delta}{\sigma}) - \Phi(-\frac{\Delta}{\sigma}))^2$

$P[\text{correct}] = 4\,P[\text{correct} \mid A\text{sent}]\,P[A\text{sent}] + 8P[\text{correct} \mid B\text{sent}]\,P[B\text{sent}]$
$+ 4\,P[\text{correct} \mid C\text{sent}]\,P[C\text{sent}]$

$= \frac{9}{4}\Phi(\frac{\Delta}{\sigma})^2 - \frac{3}{2}\Phi(\frac{\Delta}{\sigma}) + \frac{1}{4}$

$P[\text{error}] = 1 - P[\text{correct}]$