ELEG 310 Chap 3 Combinatorics 3/7/19

selection of k items from n distinguishable items

ordered $abc \neq cab$ unordered $abc = cab$

w/ repl • can get aaa w/o repl ~~aaa~~

• ordered, w/ repl $n^k = n \cdot n \cdot n \cdots n = \cancel{(n)_k}$

• ordered w/o repl $n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

$(n)_k = \frac{n!}{(n-k)!}$

if $k=n$

$(n)_k = n!$

$n!$ = n factorial

$= n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$

permutations

$0! = 1$

• unordered, w/o repl combinations $\frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

"n choose k" = binomial coefficient

- unordered, w/ repl $\binom{n+k-1}{k-1}$

	ordered	unordered
w/o	$(n)_k$	$\binom{n}{k}$
w/	n^k	$\binom{n+k-1}{k-1}$

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

Ex. 0 0 1 1 0 1 0 ← n slots $n=7$ $k=3$

$$\binom{7}{3} = \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}\cdot 3\cdot 2\cdot 1} = 35 \qquad = \frac{7!}{4!\,3!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}\,3!}$$

$$= \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}$$

Comment on computation $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ Do not compute $n!$, $(n-k)!$, $k!$

E.g. $\binom{100}{2} = \frac{100!}{98!\,2!} \approx \frac{9.33\times 10^{157}}{9.42\times 10^{153}\times 2} \approx \frac{9904.45}{2}$

$$= \frac{100\cdot 99}{2\cdot 1} = 50\times 99 = 4950$$

$$\binom{100}{5} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$x = 100 \quad x / 1 \quad x = 99 \quad x / 2 \quad x = 98 \quad x / 3 \ldots$

Binomial Theorem & Pascals Triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\underbrace{\overbrace{1\ 1\ \_\ 1\ \_\ \_}^{k\ 1's}}_{n \text{ slots}}$$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$\binom{2}{0}\binom{2}{1}\binom{2}{2}$

$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$$

$$= (a+b)^{n-1}(a+b) = \left(\sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-1-k}\right) a + \left(\sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-k-1}\right) b$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 3\cdot 2\cdot 1}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$0^n = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots \pm \binom{n}{n}$$

Multinomial Coefficient & Theorem

Binomial divide n items into 2 piles, k in first, $n-k$ in second.

Ex. k_1 in 1st pile, k_2 in 2nd pile, k_3 in 3rd pile $\Big\}$ $k_1 + k_2 + k_3 = n$

$$\binom{n}{k_1, k_2, k_3} = \binom{n}{k_1} \binom{k_2 + k_3}{k_2} = \frac{n!}{k_1!\, k_2!\, k_3!}$$

Ex. $\binom{6}{3\ 2\ 1} = \dfrac{6 \cdot 5 \cdot \cancel{4}^{2} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot \cancel{2} \cdot 1 \cdot 1} = 60$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k,\ n-k}$$

$$(a+b+c)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1\ k_2\ k_3} a^{k_1} b^{k_2} c^{k_3}$$

$$= \sum_{k_0=0}^{n} \sum_{k_1=0}^{n-k_0} \sum_{k_2=0}^{n-k_0-k_1} \binom{n}{k_0\ k_1\ k_2} a^{k_0} b^{k_1} c^{k_2}$$

Birthday Problem –

$$P(\text{no common birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \left(\frac{365-k+1}{365}\right)$$

in general $\dfrac{n}{n} \cdot \dfrac{n-1}{n} \cdot \dfrac{n-2}{n} \cdots \dfrac{n-k+1}{n} = \dfrac{(n)_k}{n!}$