Solution to homework 1

1.2. Probabilities are real numbers between 0 and 1

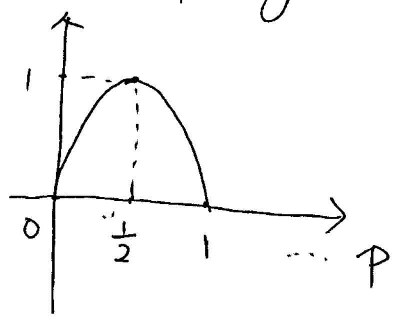
(a) $\sqrt{p}$ True

(b) $1/p$ False, e.g., $1/p = 2 > 1$ if $p = \frac{1}{2}$

(c) $(p-1)^3$ False, e.g., $(p-1)^3 = -\frac{1}{8} < 0$ if $p = \frac{1}{2}$

(d) $\sin(\pi p)$ True.

(e) $4p(1-p)$ True, the curve corresponding to $4p(1-p)$ is



That means $0 \le 4p(1-p) \le 1$

1.4 Eeperiments: flipping a coin twice.

Outcomes: (H, H), (H T), (T. H), and (T. T). H denotes Head, T denotes Tail

Event: $A = \{(H, H)\}$ $P(A) = 0.5 \cdot 0.5 = 0.25$

$B = \{(H, T), (T, H)\}$ $P(B) = 0.5 \cdot 0.5 + 0.5 \cdot 0.5 = 0.5$

$C = \{(T, T)\}$ $P(C) = 0.5 \cdot 0.5 = 0.25$

1.5. a. There are infinite outcomes

b. $Pr[N=k] = \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^k$

c. $\sum_{k=1}^{\infty} Pr[N=k] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$

Based on the summation of geometric series

we can derive that $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k = \lim_{K\to\infty} \sum_{k=1}^{K} \left(\frac{1}{2}\right)^k$

$$= \lim_{K\to\infty} \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^K}{1-\frac{1}{2}}$$

$$= \lim_{K\to\infty} \left[1-\left(\frac{1}{2}\right)^K\right]$$

$$= 1 - \lim_{K\to\infty}\left(\frac{1}{2}\right)^K$$

$$= 1 - 0 = 1$$

(d) Define an event A as $N \geq l$:

$A = \{N=l,\ N=l+1,\ \cdots\}$

$Pr[A] = \sum_{k=l}^{\infty}\left(\frac{1}{2}\right)^k = \lim_{K\to\infty}\sum_{k=l}^{K}\left(\frac{1}{2}\right)^k = \lim_{K\to\infty}\frac{\left(\frac{1}{2}\right)^l - \left(\frac{1}{2}\right)^l\left(\frac{1}{2}\right)^{(K-l)}}{1-\frac{1}{2}}$

Since l is a constant, we can derive that

$Pr[A] = \frac{\left(\frac{1}{2}\right)^l}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{2}}\lim_{K\to\infty}\left(\frac{1}{2}\right)^K$

$= \left(\frac{1}{2}\right)^{l-1} - 0 = \left(\frac{1}{2}\right)^{l-1}$

(e). Define an event B as $N \le l$

$$B = \{N=1,\ N=2 \ \dots\ N=l\}$$

$$Pr\{B\} = \sum_{k=1}^{l} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^l}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^l$$

(f). No. $Pr\{A\} + Pr\{B\} = \left(\frac{1}{2}\right)^{l-1} + 1 - \left(\frac{1}{2}\right)^l$

$$= 2 \cdot \left(\frac{1}{2}\right)^l + 1 - \left(\frac{1}{2}\right)^l$$

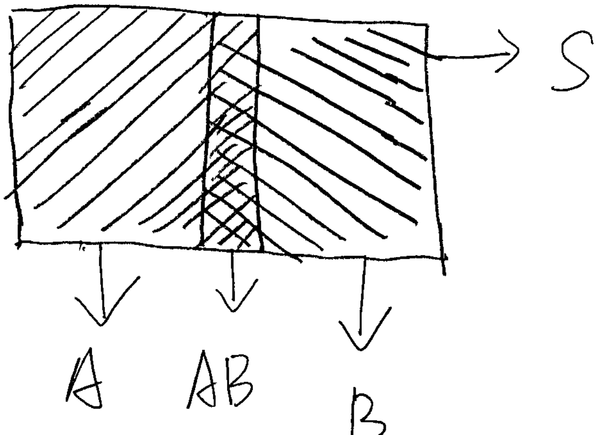
$$= 1 + \left(\frac{1}{2}\right)^l$$

Since $\left(\frac{1}{2}\right)^l \neq 0$ for all $l \ge 1$ Therefore

$Pr\{A\} + Pr\{B\} \neq 1$

No. They should not be 1. Since they are not complement events. with each other.

We can find more initutive explanation for this problem



$A = \{N \ge l\}$ $B = \{N \le l\}$

$AB = \{N = l\}$

1.8. (a) $Pr\{A\} = Pr[\{1, 2, 3\}]$

$= Pr[1] + Pr[2] + Pr[3]$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$Pr\{B\} = Pr[2] + Pr\{4\} + Pr\{6\} = \frac{1}{2}$

$Pr\{C\} = Pr\{2\} + Pr\{3\} + Pr\{4\} + Pr\{5\} = \frac{2}{3}$

(b). $AB = A \cap B = \{2\}$

$Pr\{AB\} = Pr\{2\} = \frac{1}{6}$

$AC = A \cap C = \{2, 3\}$

$Pr\{AC\} = Pr\{2\} + Pr\{3\} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$BC = B \cap C = \{2, 4\}$

$Pr\{BC\} = Pr\{2\} + Pr\{4\} = \frac{1}{3}$

$ABC = A \cap B \cap C = \{2\}$

$Pr\{ABC\} = Pr\{2\} = \frac{1}{6}$

(c) $A \cup B = \{1. 2. 3. 4. 6\}$

$Pr\{A \cup B\} = Pr\{1\} + Pr\{2\} + Pr\{3\} + Pr\{4\} + Pr\{6\} = \frac{5}{6}$

$A \cup C = \{1. 2. 3. 4. 5\}$

$Pr\{A \cup C\} = \frac{5}{6}$

$B \cup C = \{ 2. 3. 4. 5. 6\}$

$Pr\{B \cup C\} = \frac{5}{6}$

$A \cup B \cup C = \{1. 2. 3. 4. 5. 6\}$

$Pr\{A \cup B \cup C\} = 1$

(d) DeMorgan's laws: $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{AB} = \bar{A} \cup \bar{B}$, $\bar{A} = \{4. 5. 6\}$ $\bar{B} = \{1. 3. 5\}$

$\bar{A} \cup \bar{B} = \{1. 3. 4. 5. 6\}$

$P(\overline{AB}) = P(\bar{A} \cup \bar{B}) = \frac{5}{6}$

$\overline{A \cup B} = \bar{A} \cap \bar{B} = \{5\}$

$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) = \frac{1}{6}$

Directly, $AB = A \cap B = \{2\}$

$\overline{AB} = \{1, 3, 4, 5, 6\}$

$P(\overline{AB}) = \frac{5}{6}$

$A \cup B = \{1, 2, 3, 4, 6\}$

$\overline{A \cup B} = \{5\}$

$P(\overline{A \cup B}) = \frac{1}{6}$

1.12. (a) $Pr[A] = Pr[\{1, 2\}]$

$= Pr[1] + Pr[2]$

$= 0.4 + 0.3 = 0.7$

$Pr[B] = Pr[\{2, 3\}]$

$= Pr[2] + Pr[3]$

$= 0.3 + 0.2 = 0.5$

$Pr[C] = Pr[\{1, 4\}]$

$= Pr[1] + Pr[4]$

$= 0.4 + 0.1 = 0.5$

1.12 (b) $AB = A \cap B = \{2\}$

$Pr\{AB\} = Pr\{2\} = 0.3$

$A \cup B = \{1, 2, 3\}$

$Pr\{A \cup B\} = Pr\{\{1, 2, 3\}\}$

$= Pr\{1\} + Pr\{2\} + Pr\{3\}$

$= 0.4 + 0.3 + 0.2$

$= 0.9$

(c) $ABC = A \cap B \cap C$

$= \phi$

$Pr\{ABC\} = 0$

$A \cup B \cup C = \{1, 2, 3, 4\}$

$Pr\{A \cup B \cup C\} = 1$

1.18 Let A be the event the upper path consisting of the serial connection of L_1 and L_2 works

Let B be the event the bottom path consisting of the single link L_3 works.

$$Pr\{A\} = Pr\{L_1 = 1 \cap L_2 = 1\}$$

$$= Pr\{L_1 = 1\} \cdot Pr\{L_2 = 1\}$$

$$= p_1 \cdot p_2$$

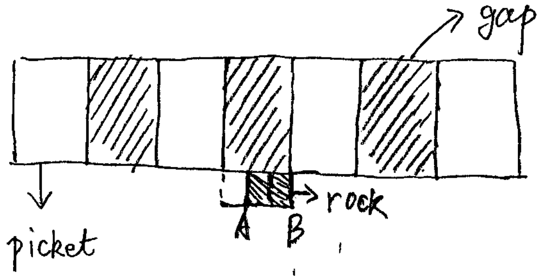
$$Pr\{B\} = Pr\{L_3 = 1\} = p_3$$

$$Pr\{S \to D\} = Pr\{A \cup B\} = Pr\{A\} + Pr\{B\} - Pr\{A \cap B\}$$

$$= p_1 \cdot p_2 + p_3 - p_1 \cdot p_2 \cdot p_3$$

$$= p_3 + p_1 \cdot p_2 - p_1 \cdot p_2 \cdot p_3$$

1.19



The above figure shows the general geometrical relationship between picket, gap and rock. We can find that the rock can pass through a gap only its right side is inside of the space between A and B, which occupies $\frac{2}{3}$ the space of the gap.

Assuming there are N pickets, which generate $N-1$ gaps. Based on the above discussion, the whole space allowing a rock pass through is $(N-1)\cdot\frac{2}{3}\cdot 3$ inches.

Moreover, the space of the fence is $(N+N-1)\cdot 3$ inches,

Therefore $Pr[\text{a rock can pass through the fence}]$

$$= \frac{2(N-1)}{3(2N-1)} = \frac{2(N-1)}{6(N-1)+3} = \frac{1}{3+\frac{3}{2(N-1)}} < \frac{1}{3}$$

1.24: Experiment: Bob paints Alice's house a coat

Outcome a: the surface is not covered by Bob

Therefore. We have $P(a) = 2\%$ $P(\bar{a}) = 98\%$

Question: how much surface is not covered by Bob after two coats

This event has four outcomes $\{\bar{a}_1\bar{a}_2, \bar{a}_1 a_2, a_1\bar{a}_2, a_1 a_2\}$

a_1 means the surface is not covered by Bob's first paint

a_2 means the surface is not covered by Bob's Second paint

The question is equivalent to calculate $P[\{a_1 a_2\}]$

If we assume a_1 and a_2 are independent,

we can derive that $P[\{a_1 a_2\}] = 2\%\ 2\% = 0.04\%$

However, a_2 could be depend on a_1. in practice.

For example, Bob only paint the surface that is missed by the first time, the missed surface can be very small.

So. the probability of Bob misses the second paint can be much smaller than 2%.

Therefore. We can derive much smaller result than 0.04%