UNIVERSITY *of* DELAWARE

Appendix E

Single Time Constant Circuits

Single-time-constant (STC) circuits

Circuits that are composed of or can be reduced to one reactive component (inductance or capacitance) and one resistance.

An STC circuit formed of an inductance L and a resistance R has a time constant $\tau = L/R$.

The time constant τ of an STC circuit composed of a capacitance C and a resistance R is given by $\tau = CR$.

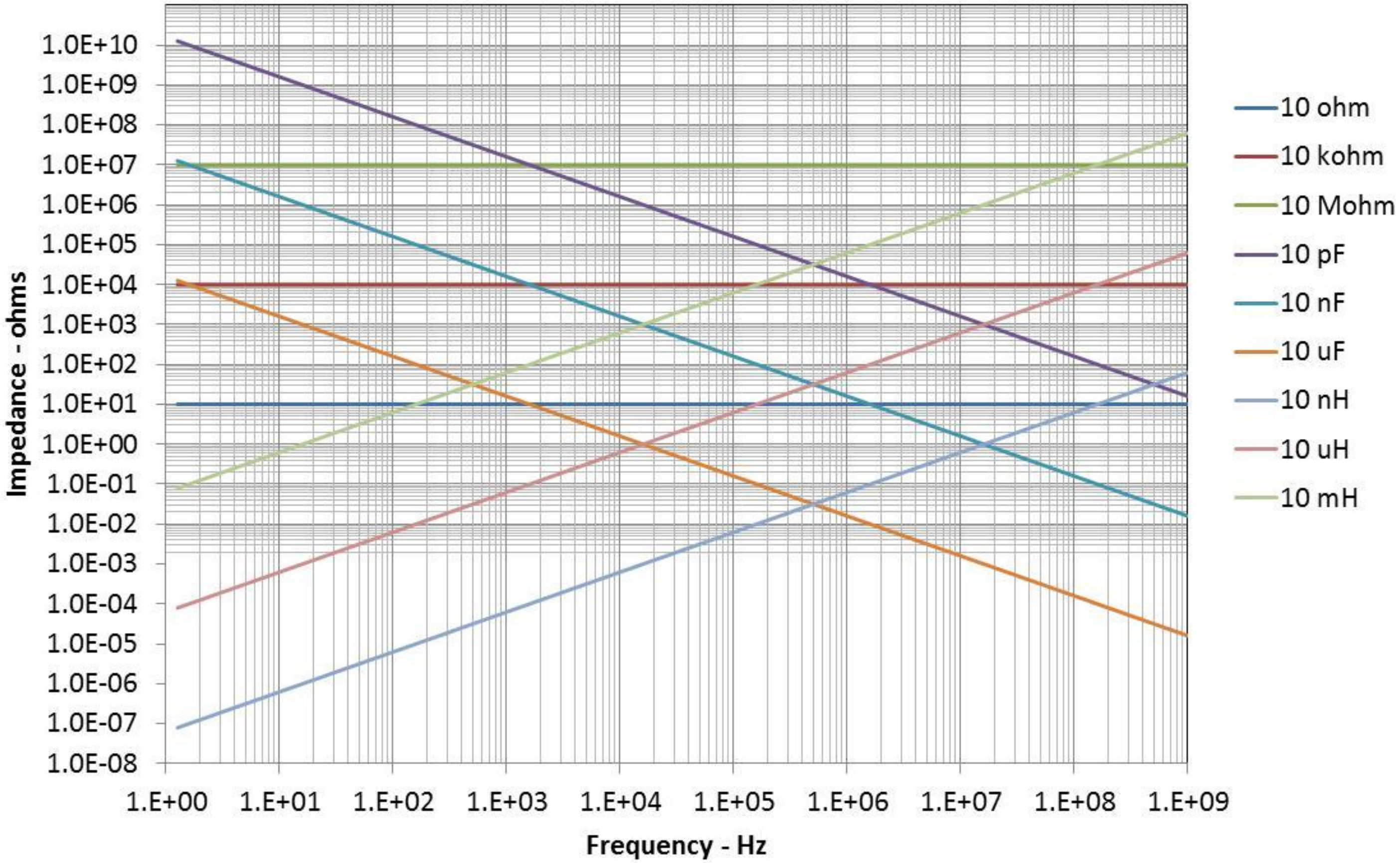
Although STC circuits are quite simple, they play an important role in the design and analysis of linear and digital circuits. For instance, the analysis of an amplifier circuit can usually be reduced to the analysis of one or more STC circuits.

Impedance vs. Frequency

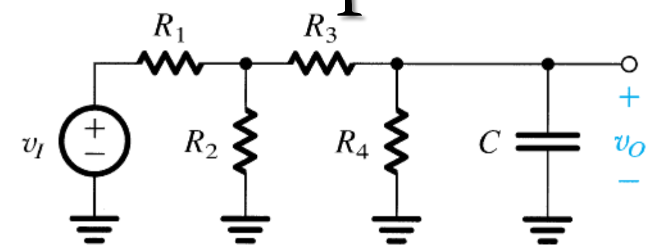
$$Z_R(\omega) = R$$

$$Z_C(\omega) = \frac{1}{j\omega C}$$

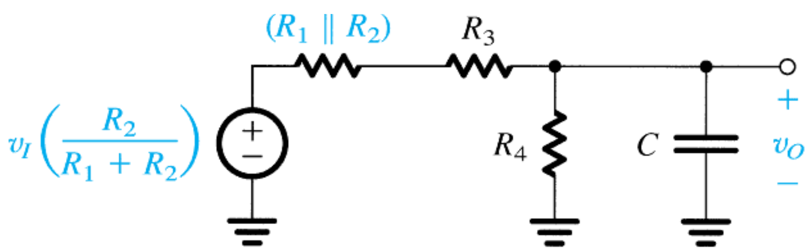
$$Z_L(\omega) = j\omega L$$



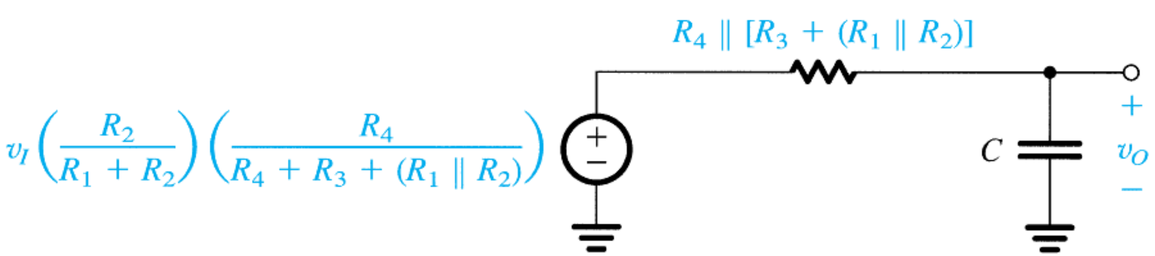
Example E.1



(a)



(b)



$$\tau = C\,\{R_4 \| [R_3 + (R_1 \| R_2)]\}$$

Evaluating the Time Constant, τ

1. Reduce Excitation
   – Short Voltage Sources
   – Open Current Sources
2. If only one reactive component, find the effective or equivalent resistance that it sees. The TC is then L/R_{eq} or CR_{eq}.
3. If only one resistance and multiple capacitors or inductors, find C_{eq} or L_{eq} seen by the resistor. The TC is then $C_{eq}R$ or L_{eq}/R.
4. If multiple resistors and reactive components are present more work is required to get the TC.

Example E.2

Example E.2

Find the time constant of the circuit in Fig. E.2.

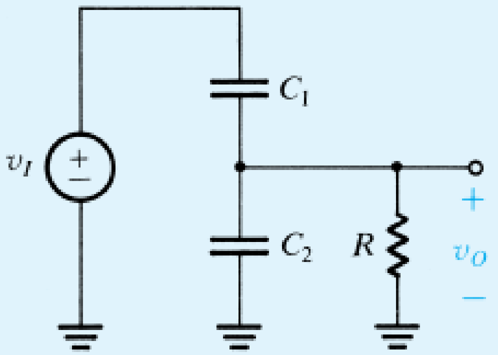
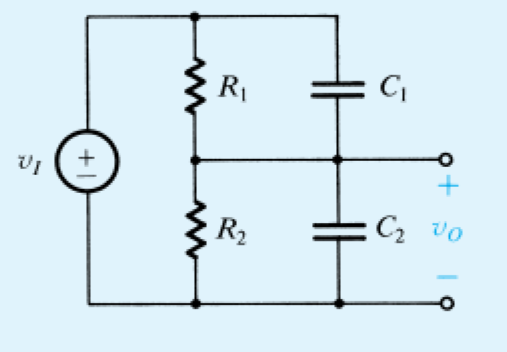


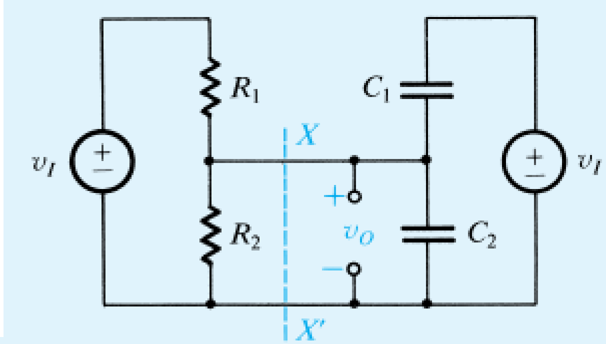
Figure E.2 Circuit for Example E.2.

$$\tau = R\,(C_1 + C_2)$$

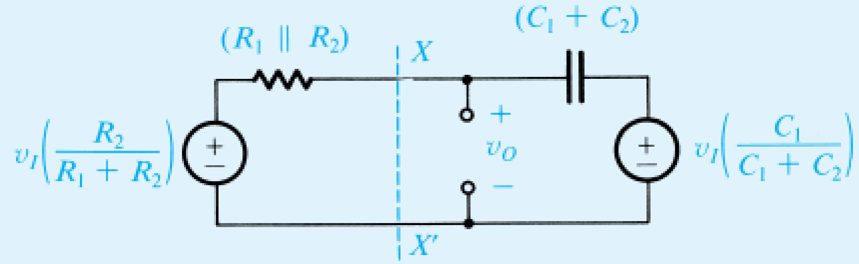
Example E.3



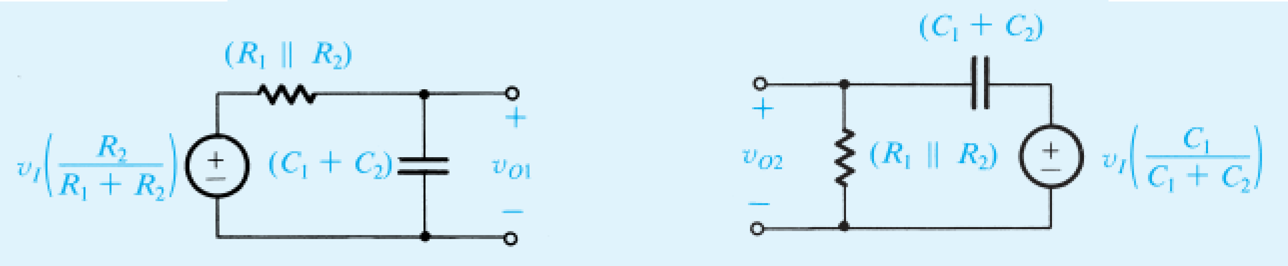
(a)



(b)



(c)



(d)

(e)

$$\tau = (C_1 + C_2)(R_1 \| R_2)$$

Classification of STC Circuits

STC circuits can be classified into two categories,

1. ***low-pass*** **(LP)**
2. ***high-pass*** **(HP)**

with each category displaying distinctly different signal responses.

The task of finding whether an STC circuit is of LP or HP type may be accomplished in a number of ways, the simplest of which uses the frequency domain response.

$$Z_c(\omega) = \frac{1}{j\omega C} \qquad Z_L(\omega) = j\omega L$$

Test frequency	Replace *C* with	Replace *L* with	Circuit is LP if	Circuit is HP if
$\omega = 0$	open-circuit	short-circuit	output is finite	output is zero
$\omega = \infty$	short-circuit	open-circuit	output is zero	output is finite

Transfer Functions and Bode Plots

A **transfer function** is a mathematical representation, in terms of spatial or temporal frequency, of the relation between the input and output of a linear time-invariant system.

http://en.wikipedia.org/wiki/Transfer_function

A **Bode plot** is a graph of the transfer function of a linear, time-invariant system versus frequency, plotted with a log-frequency axis, to show the system's frequency response. It is usually a combination of a Bode magnitude plot, expressing the magnitude of the frequency response gain, and a Bode phase plot, expressing the frequency response phase shift.

http://en.wikipedia.org/wiki/Bode_plot

Low-Pass STC Circuits

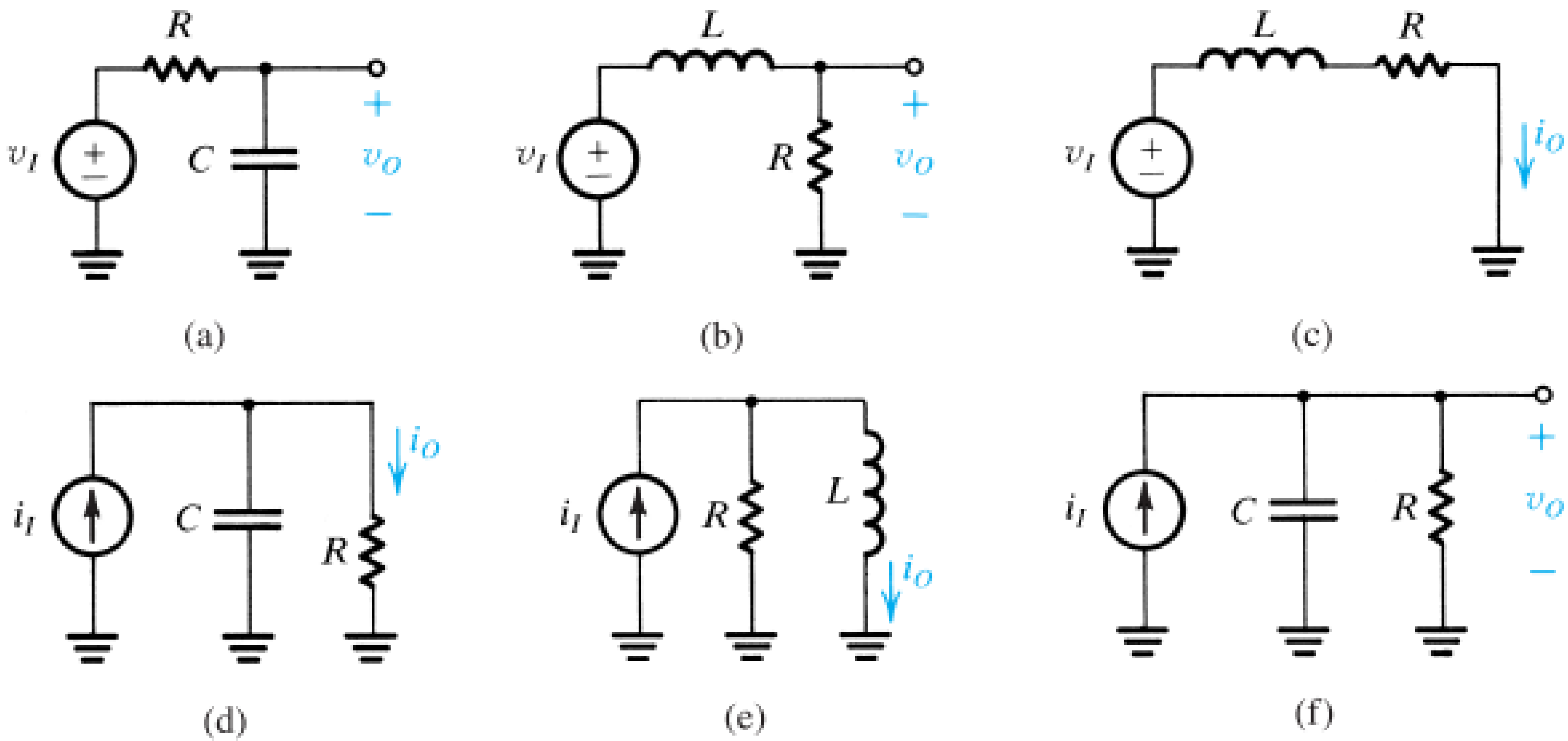


Figure E.4 STC circuits of the low-pass type.

Frequency Response of STC Low-Pass Circuits

The transfer function $T(s)$ of an STC low-pass circuit can always be written in the form

$$T(s) = \frac{K}{1 + \left(s/\omega_0\right)}$$

which, for physical frequencies, where $s = j\omega$, becomes

$$T(j\omega) = \frac{K}{1 + j\left(\omega/\omega_0\right)}$$

where K is the magnitude of the transfer function at $\omega = 0$ (dc) and ω_0 is defined by

$$\omega_0 = 1/\tau$$

with τ being the time constant. Thus the magnitude response is given by

$$\left|T(j\omega)\right| = \frac{K}{\sqrt{1 + \left(\omega/\omega_0\right)^2}}$$

and the phase response is given by

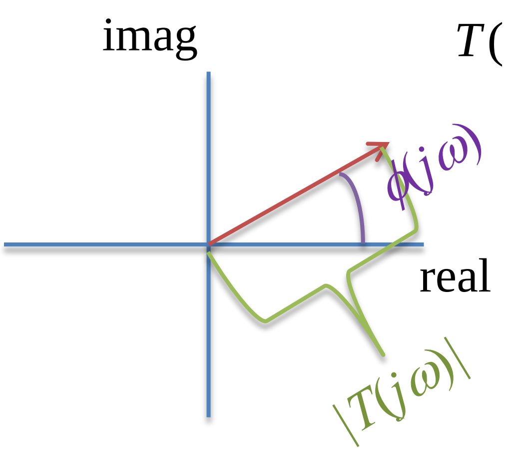
$$\varphi(\omega) = -\tan^{-1}\left(\omega/\omega_0\right)$$

Derivation of Mag and Phase for LP

$$T(j\omega) = \frac{K}{1 + j(\omega/\omega_0)}$$

The transfer function $T(j\omega)$ of an STC circuit is complex (i.e. a real and imaginary component)

$$T(j\omega) = \frac{K}{1 + j(\omega/\omega_0)} \frac{(1 - j(\omega/\omega_0))}{(1 - j(\omega/\omega_0))} = \overbrace{\left[\frac{K}{1 + (\omega/\omega_0)^2}\right]}^{\text{Re}} + j\overbrace{\left[\frac{-K(\omega/\omega_0)}{1 + (\omega/\omega_0)^2}\right]}^{\text{Im}}$$

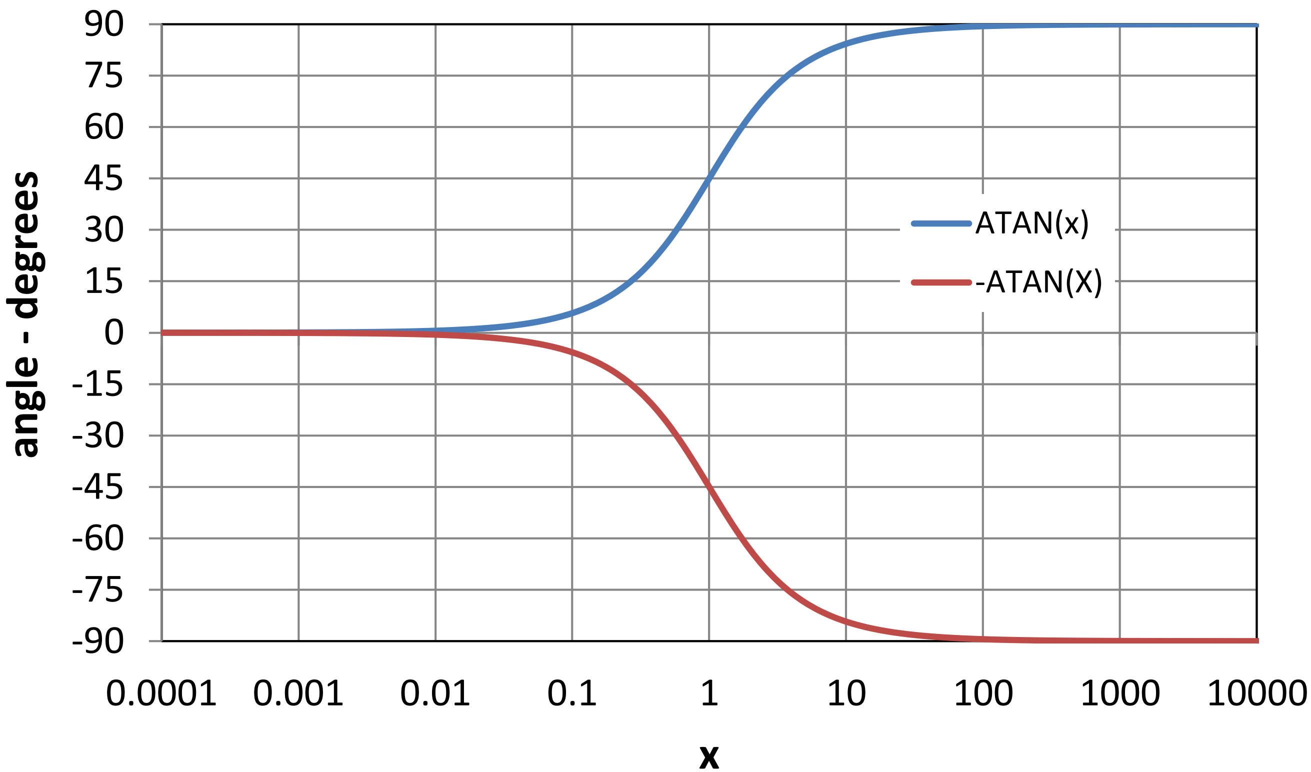


$$|T(j\omega)| = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{\left[\frac{K}{1 + (\omega/\omega_0)^2}\right]^2 + \left[\frac{-K(\omega/\omega_0)}{1 + (\omega/\omega_0)^2}\right]^2}$$

$$= \sqrt{\frac{K^2\left(1 + (\omega/\omega_0)^2\right)}{\left(1 + (\omega/\omega_0)^2\right)^2}} = \frac{K}{\sqrt{1 + (\omega/\omega_0)^2}}$$

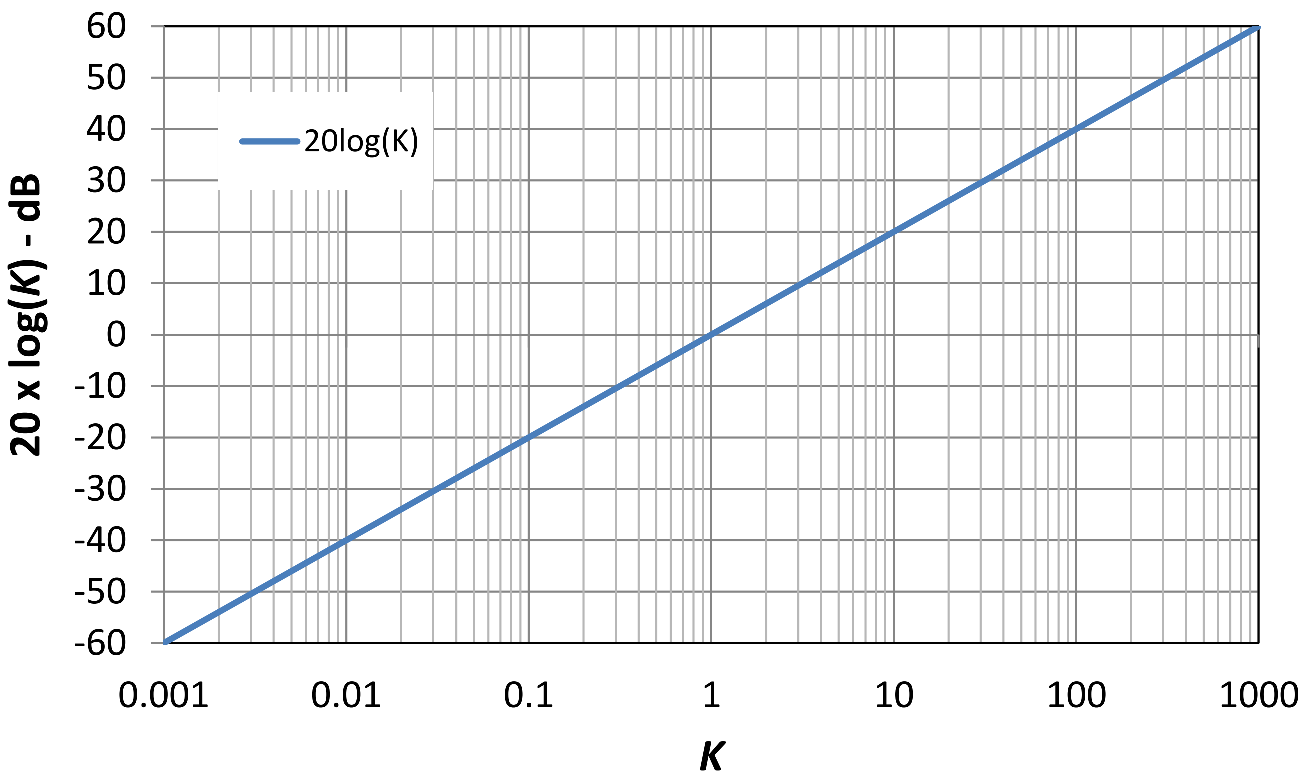
$$\tan(\phi(\omega)) = \frac{\text{Im}}{\text{Re}} = \left[\frac{-K(\omega/\omega_0)}{1 + (\omega/\omega_0)^2}\right]\left[\frac{1 + (\omega/\omega_0)^2}{K}\right] = -\frac{\omega}{\omega_0} \qquad \Rightarrow \phi(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

$\tan^{-1}(x)$, or ATAN(x)



x	$\tan^{-1}(x)$
0.001	0.06°
0.01	0.57°
0.1	5.71°
1	45°
10	84.29°
100	89.43°
100	89.94°

20 x log(K)



K	20 x log(K)
0.001	-60
0.01	-40
0.1	-20
0.5	-6.02
0.707	-3.01
1	0.0
1.414	3.01
2	6.02
10	20
100	40
100	60

Frequency Response of STC Low-Pass Circuits

$$\left|T(j\omega)\right| = \frac{K}{\sqrt{1+\left(\omega/\omega_0\right)^2}}$$

$$\varphi(\omega) = -\tan^{-1}\left(\omega/\omega_0\right)$$

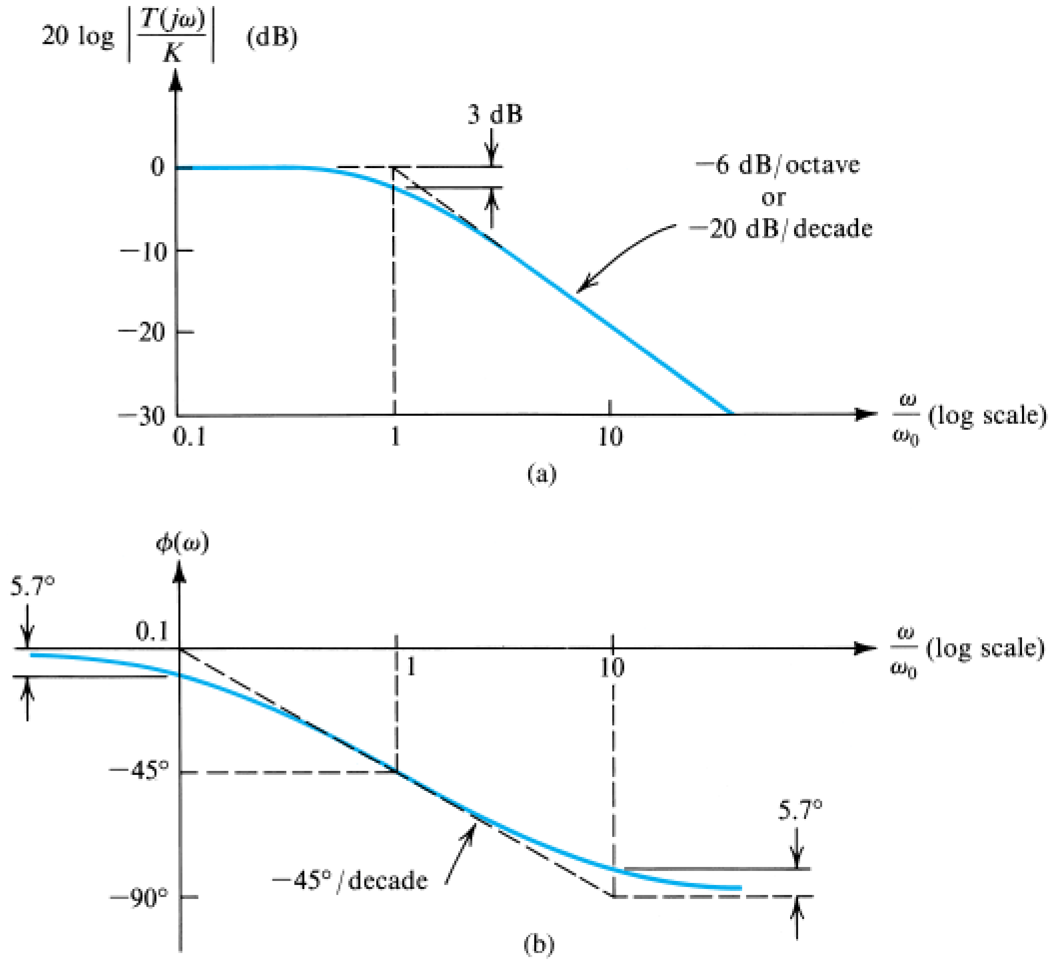
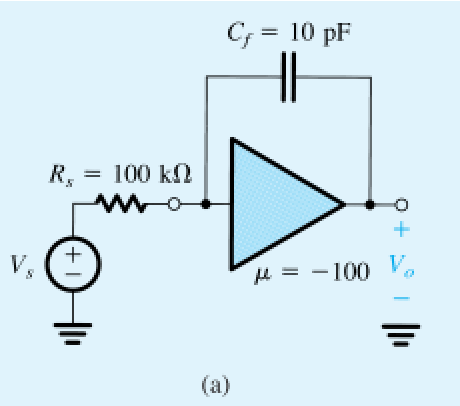


Figure E.6 (a) Magnitude and (b) phase response of STC circuits of the low-pass type.

Example E.4

Consider the circuit shown in Fig. E.7(a), where an ideal voltage amplifier of gain μ = –100 has a small (10-pF) capacitance connected in its feedback path. The amplifier is fed by a voltage source having a source resistance of 100 kΩ. Show that the frequency response V_o/V_s of this amplifier is equivalent to that of an STC circuit, and sketch the magnitude response.



$$V_o = \mu V_{in} \qquad \chi_C = \frac{1}{sC_f} \qquad i_{in} = \frac{V_s - V_{in}}{R_s}$$

$$V_o = V_{in} - i_{in}\chi_C \qquad V_o = \frac{V_o}{\mu} - \left(\frac{V_s - {V_o}/{\mu}}{R_s}\right)\frac{1}{sC_f}$$

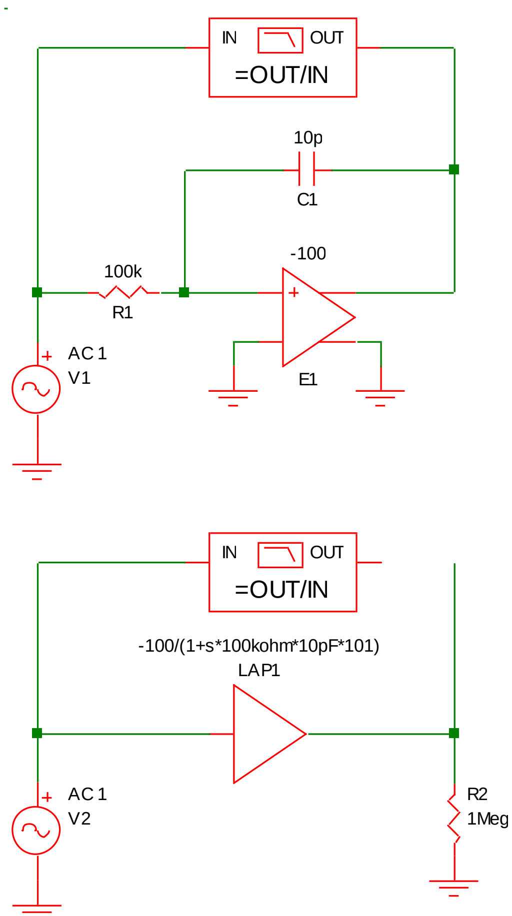
$$T(s) = \frac{V_o}{V_s} = \frac{\mu}{1 + sR_sC_f(-\mu + 1)}$$

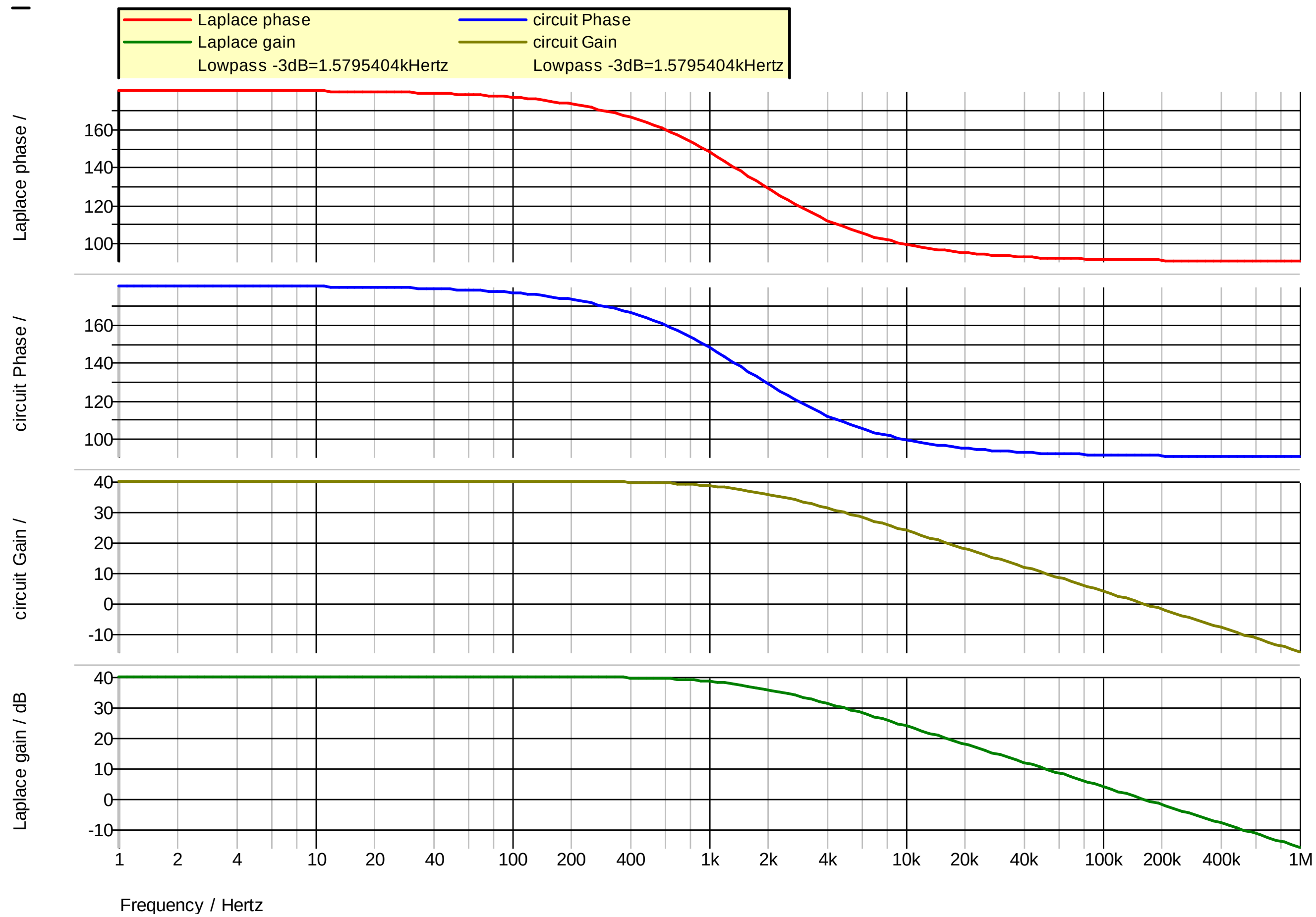
dc gain (K): $= \mu = -100 \quad = 40$ dB

time constant (τ): $= R_sC_f(-\mu + 1) \quad = 101$ us

corner frequency: $\omega_0 = {1}/{\tau} = 9.901 \times 10^3\ {\text{rad}}/{s} \qquad f_0 = 1.576$ kHz

Example E.4


IN
OUT
=OUT/IN
10p
C1
-100
100k
R1
AC 1
V1
E1
IN
OUT
=OUT/IN
-100/(1+s*100kohm*10pF*101)
LAP1
AC 1
V2
R2
1Meg


Laplace phase
Laplace gain
Lowpass -3dB=1.5795404kHertz
circuit Phase
circuit Gain
Lowpass -3dB=1.5795404kHertz
Laplace phase /
circuit Phase /
circuit Gain /
Laplace gain / dB
160
140
120
100
40
30
20
10
0
-10
1
2
4
10
20
40
100
200
400
1k
2k
4k
10k
20k
40k
100k
200k
400k
1M
Frequency / Hertz

The Miller or Inverting Integrator (ELEG309)

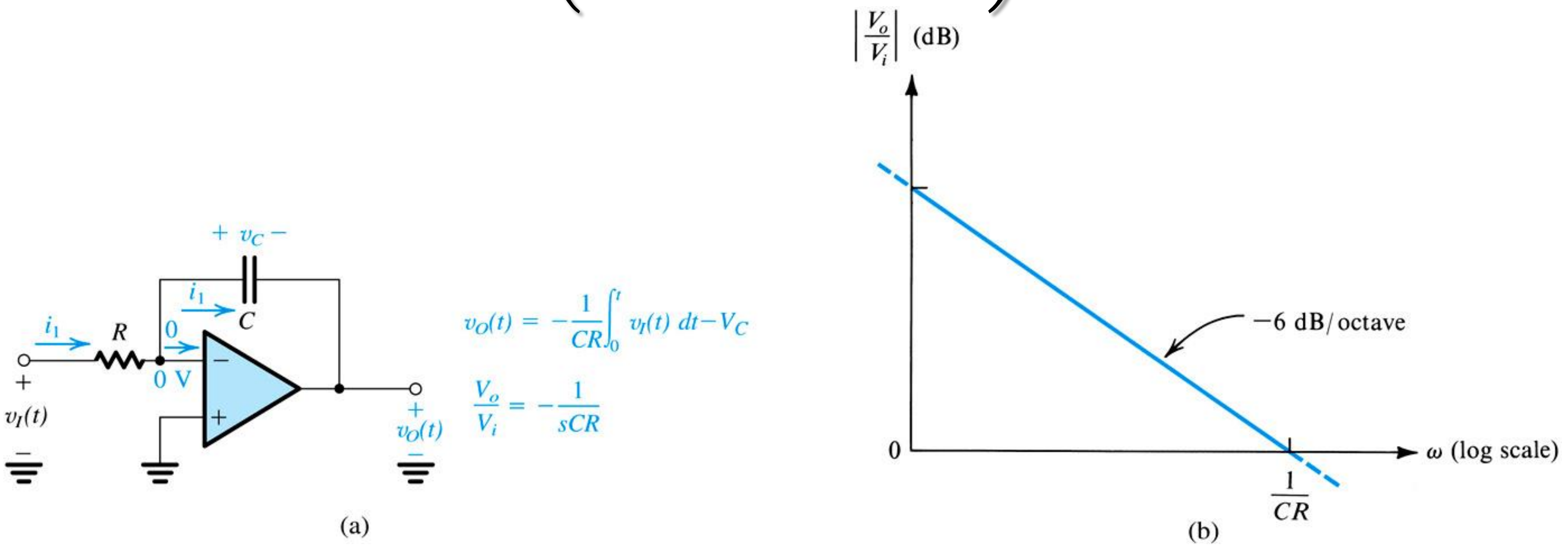


Figure 2.24 (a) The Miller or inverting integrator. **(b)** Frequency response of the integrator.

$$\frac{V_o(s)}{V_i(s)} = -\frac{1}{sCR} \qquad \left|\frac{V_o}{V_i}\right| = \frac{1}{\omega CR}$$

Exercise E.3

Find the dc transmission, the corner frequency f_0, and the transmission at f = 2 MHz for the low-pass STC circuit shown in Fig. EE.3.

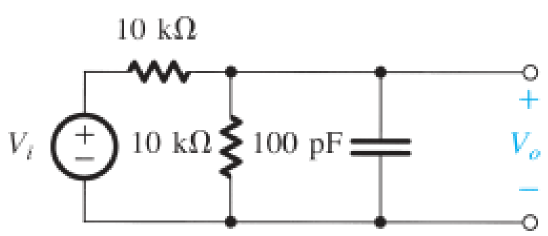


Figure EE.3

dc gain (K): $=$ 0.5 = -6 dB

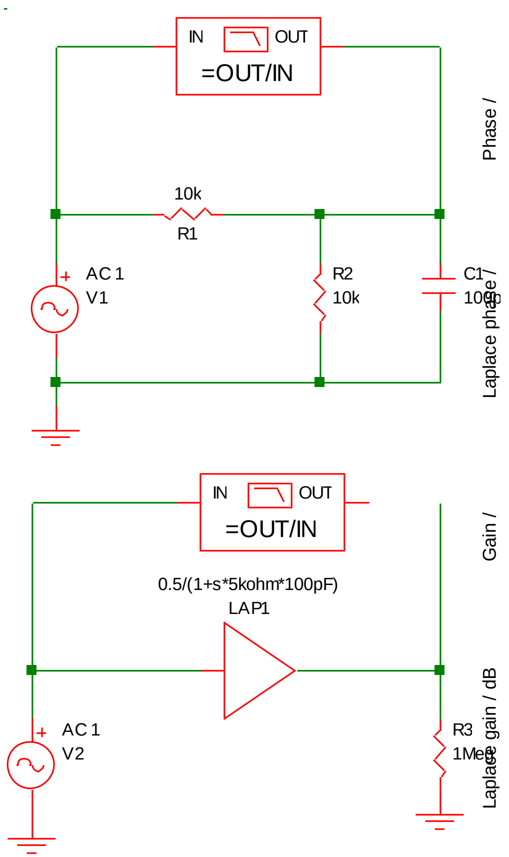
$$T(s) = \frac{V_o}{V_i} = \frac{0.5}{1 + sC(R_1 \parallel R_2)} \qquad |T(j\omega)| = \frac{K}{\sqrt{1 + \left({}^{\omega}/_{\omega_0}\right)^2}}$$

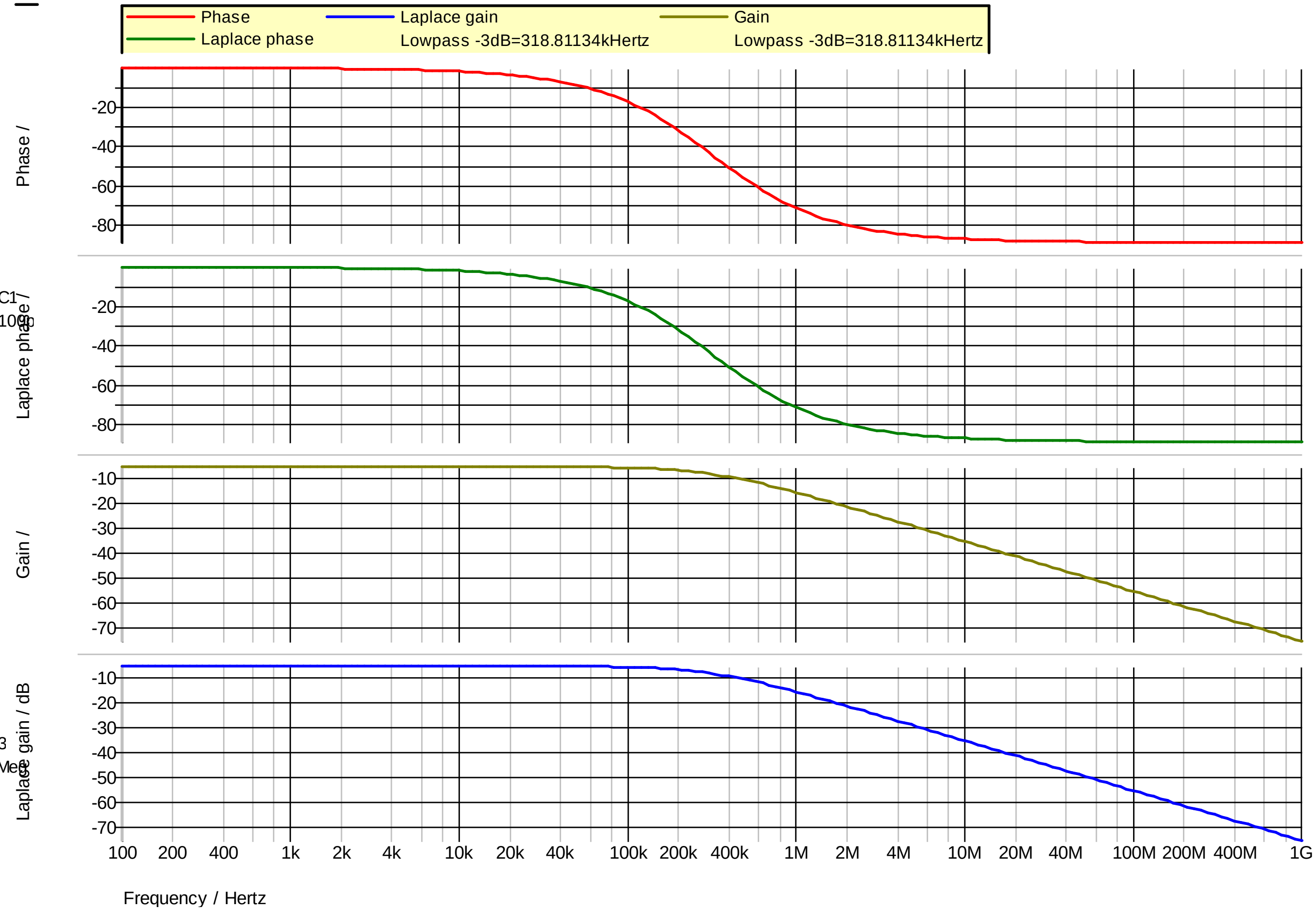
time constant (τ): $= C(R_1 \parallel R_2)$ =100 pF x 5 kΩ = 500 ns

corner frequency: $\omega_0 = {}^{1}/_{\tau} = 2 \times 10^6\ {}^{\text{rad}}/_{s}$ $\quad f_0$ = 318.3 kHz

$$|T(2\text{MHz})| = \frac{0.5}{\sqrt{1 + \left({}^{2\text{MHz}}/_{318.3k\text{Hz}}\right)^2}} = 0.079 = -22.1\text{ dB}$$

Exercise E.3


IN
OUT
=OUT/IN
10k
R1
AC 1
V1
R2
10k
C1
100p
IN
OUT
=OUT/IN
0.5/(1+s*5kohm*100pF)
LAP1
AC 1
V2
R3
1Meg


Phase
Laplace phase
Laplace gain
Lowpass -3dB=318.81134kHertz
Gain
Lowpass -3dB=318.81134kHertz
Phase /
Laplace phase /
Gain /
Laplace gain / dB
-20
-40
-60
-80
-10
-30
-50
-70
100
200
400
1k
2k
4k
10k
20k
40k
100k
200k
400k
1M
2M
4M
10M
20M
40M
100M
200M
400M
1G
Frequency / Hertz

High-Pass STC Circuits

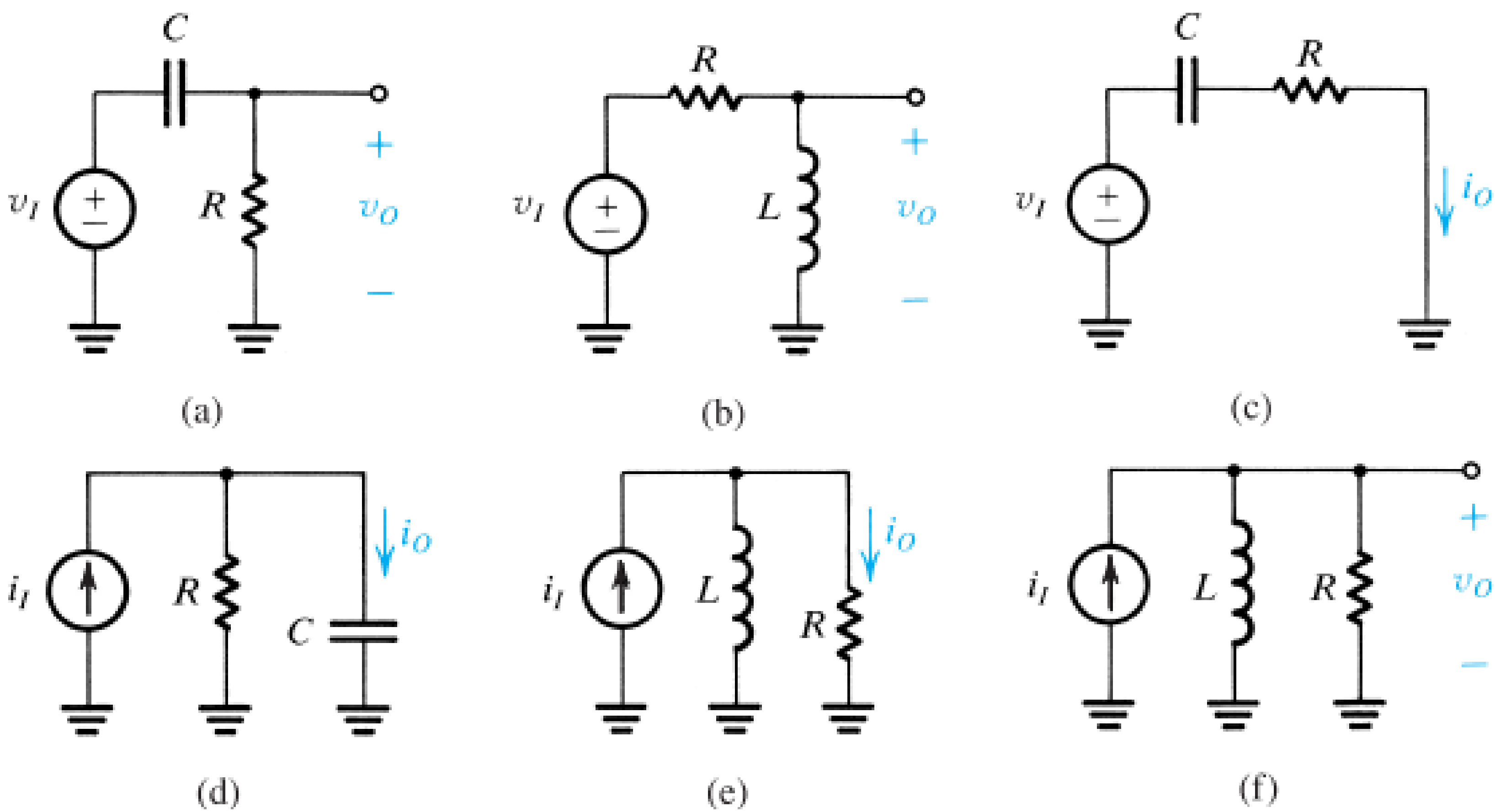


Figure E.5 STC circuits of the high-pass type.

Frequency Response of STC High-Pass Circuits

The transfer function $T(s)$ of an STC high-pass circuit can always be written in the form

$$T(s) = \frac{Ks}{s+\omega_0}$$

which, for physical frequencies, where $s = j\omega$, becomes

$$T(j\omega) = \frac{K}{1 - j(\omega_0/\omega)}$$

where K is the magnitude of the transfer function as s or ω approaches infinity and ω_0 is defined by

$$\omega_0 = 1/\tau$$

with τ being the time constant. Thus the magnitude response is given by

$$|T(j\omega)| = \frac{K}{\sqrt{1+(\omega_0/\omega)^2}}$$

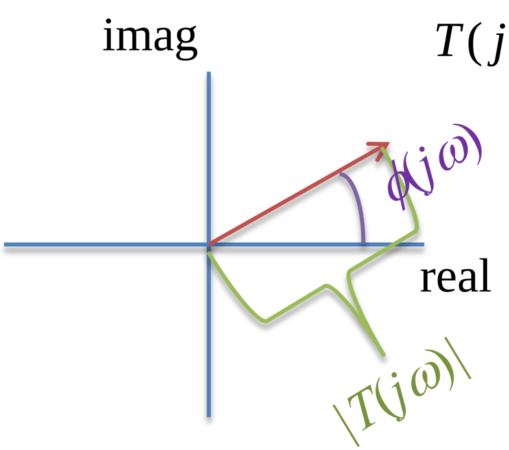
and the phase response is given by

$$\varphi(\omega) = \tan^{-1}(\omega_0/\omega)$$

Derivation of Mag and Phase for HP

$$T(j\omega) = \frac{K}{1 - j\left(\omega_0/\omega\right)}$$

The transfer function $T(j\omega)$ of an STC circuit is complex (i.e. a real and imaginary component)



$$T(j\omega) = \frac{K}{1 - j\left(\omega_0/\omega\right)}\left[\frac{\left(1 + j\left(\omega_0/\omega\right)\right)}{\left(1 + j\left(\omega_0/\omega\right)\right)}\right] = \overbrace{\left[\frac{K}{1+\left(\omega_0/\omega\right)^2}\right]}^{\text{Re}} + j\overbrace{\left[\frac{K\left(\omega_0/\omega\right)}{1+\left(\omega_0/\omega\right)^2}\right]}^{\text{Im}}$$

$$|T(j\omega)| = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{\left[\frac{K}{1+\left(\omega_0/\omega\right)^2}\right]^2 + \left[\frac{K\left(\omega_0/\omega\right)}{1+\left(\omega_0/\omega\right)^2}\right]^2}$$

$$= \sqrt{\frac{K^2\left(1+\left(\omega_0/\omega\right)^2\right)}{\left(1+\left(\omega_0/\omega\right)^2\right)^2}} = \frac{K}{\sqrt{1+\left(\omega_0/\omega\right)^2}}$$

$$\tan\left(\phi(\omega)\right) = \frac{\text{Im}}{\text{Re}} = \left[\frac{K\left(\omega_0/\omega\right)}{1+\left(\omega_0/\omega\right)^2}\right]\left[\frac{1+\left(\omega_0/\omega\right)^2}{K}\right] = \frac{\omega_0}{\omega} \qquad \Rightarrow \phi(\omega) = \tan^{-1}\left(\frac{\omega_0}{\omega}\right)$$

Frequency Response of STC High-Pass Circuits

$$\left|T(j\omega)\right| = \frac{K}{\sqrt{1+(\omega_0/\omega)^2}}$$

$$\varphi(\omega) = \tan^{-1}(\omega_0/\omega)$$

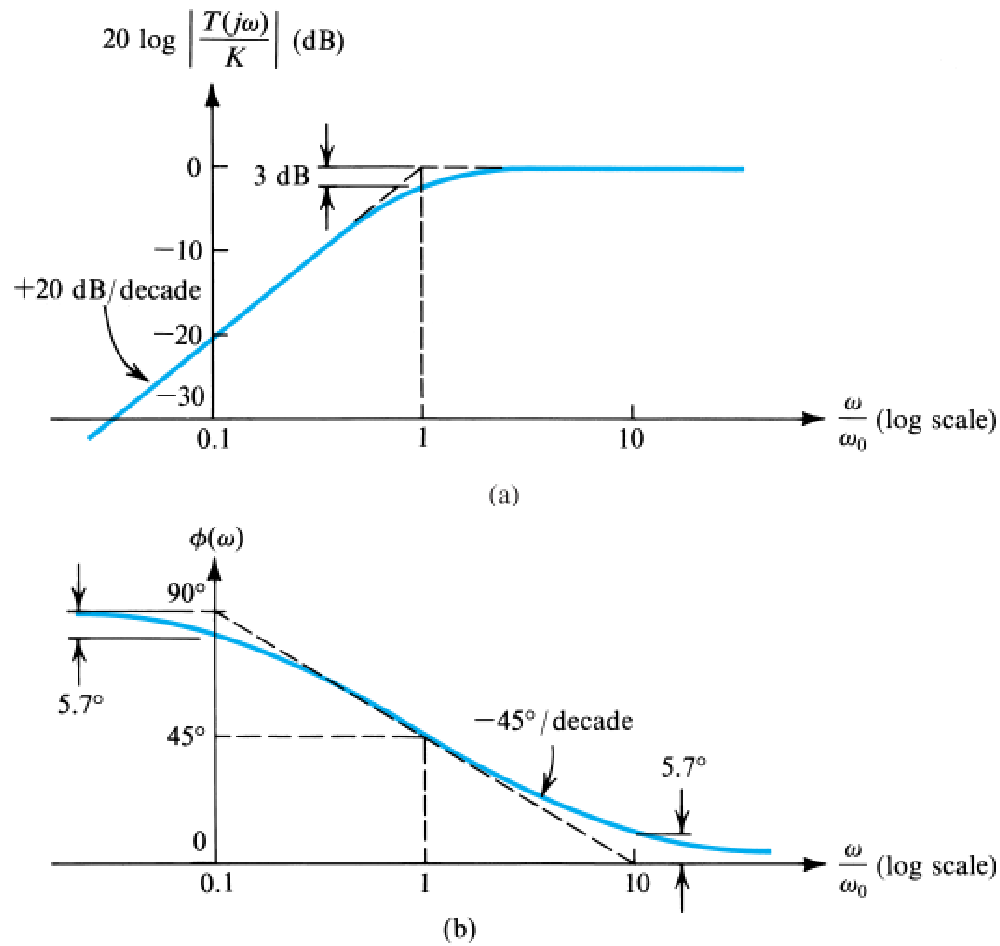


Figure E.8 (**a**) Magnitude and (**b**) phase response of STC circuits of the high-pass type.

Exercise E.6

Find the high-frequency gain, the 3-dB frequency f_0, and the gain at f = 1 Hz of the capacitively coupled amplifier shown in Fig. EE.6. Assume the voltage amplifier to be ideal.

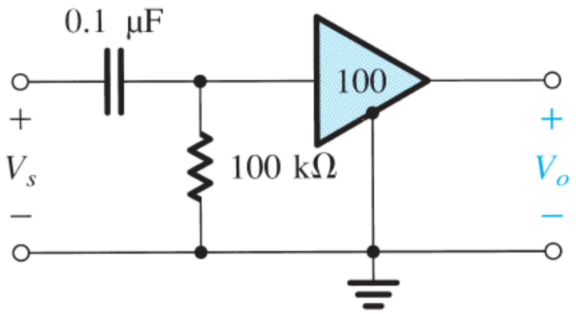


Figure EE.6

infinity gain (K): = 100 = 40 dB

$$T(s) = \frac{V_o}{V_i} = \frac{100s}{s + 1/(100k\Omega \times 0.1uF)}$$

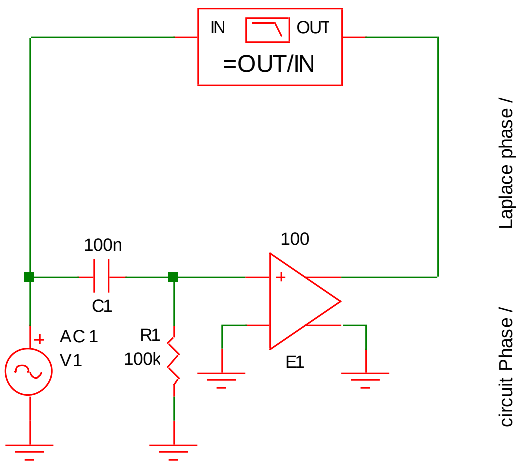
$$|T(j\omega)| = \frac{K}{\sqrt{1 + \left(\omega_0/_{\omega}\right)^2}}$$

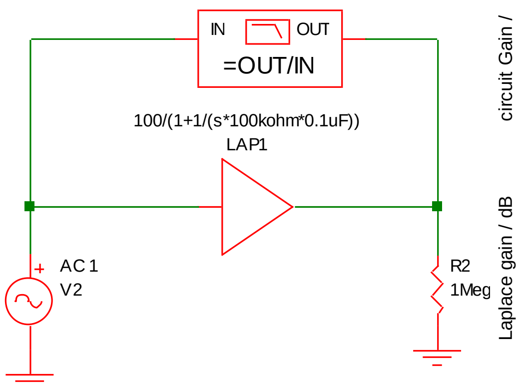
time constant (τ): $= RC$ = 0.1 uF x 100 kΩ = 10 ms

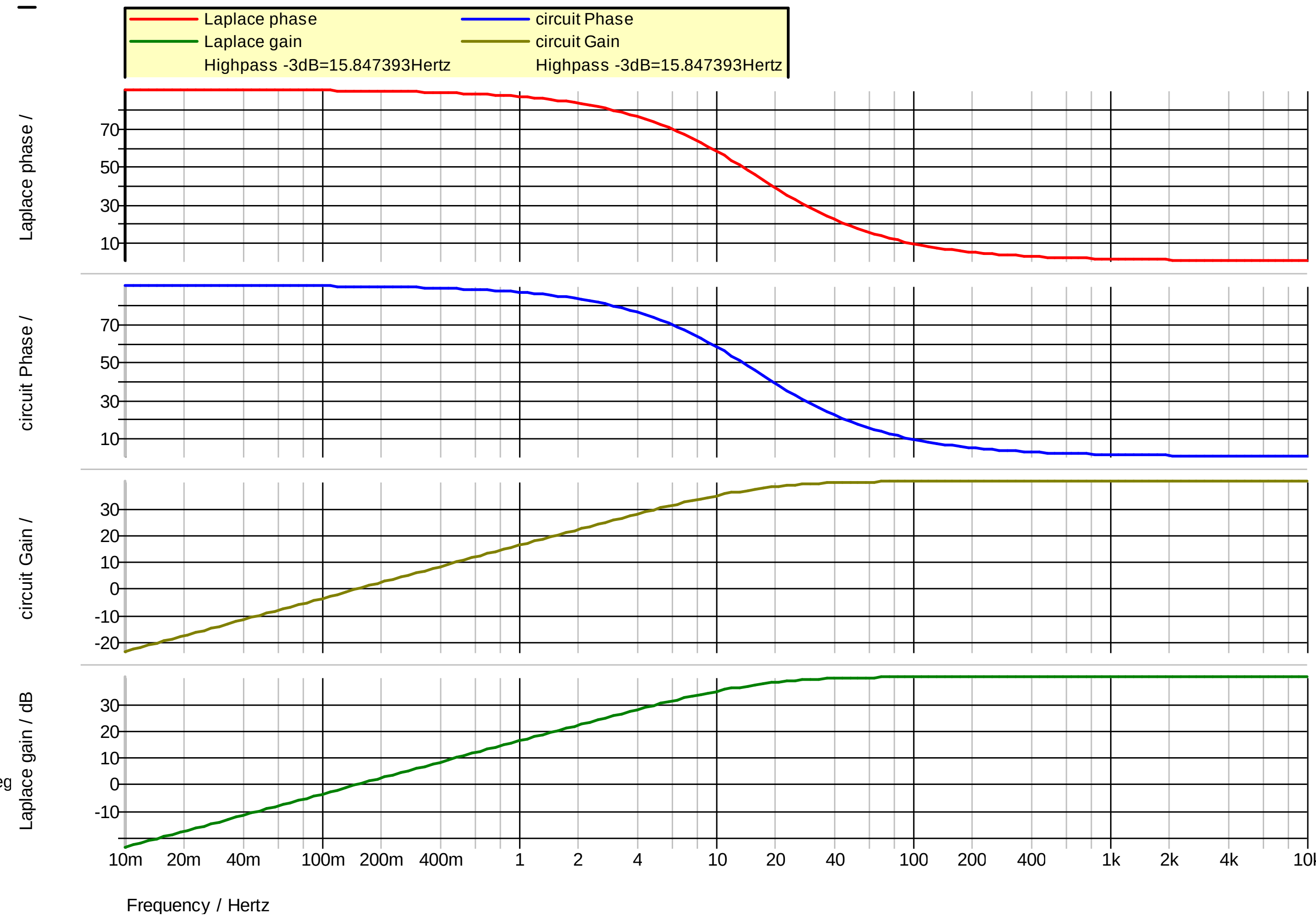
corner frequency: $\omega_0 = {}^{1}/_{\tau} = 100\, {}^{\text{rad}}/_{s}$ $\quad f_0$ = 15.915 Hz

$$|T(1\text{Hz})| = \frac{100}{\sqrt{1 + \left({}^{15.915\text{Hz}}/_{1\text{Hz}}\right)^2}} = 6.277 = 15.96\text{ dB}$$

Exercise E.6


IN
OUT
=OUT/IN
100n
C1
100
+
AC 1
V1
R1
100k
E1


IN
OUT
=OUT/IN
100/(1+1/(s*100kohm*0.1uF))
LAP1
+
AC 1
V2
R2
1Meg


Laplace phase
circuit Phase
Laplace gain
circuit Gain
Highpass -3dB=15.847393Hertz
Highpass -3dB=15.847393Hertz
Laplace phase /
circuit Phase /
circuit Gain /
Laplace gain / dB
70
50
30
10
70
50
30
10
30
20
10
0
-10
-20
30
20
10
0
-10
10m
20m
40m
100m
200m
400m
1
2
4
10
20
40
100
200
400
1k
2k
4k
10k
Frequency / Hertz

Step Response of STC Circuits

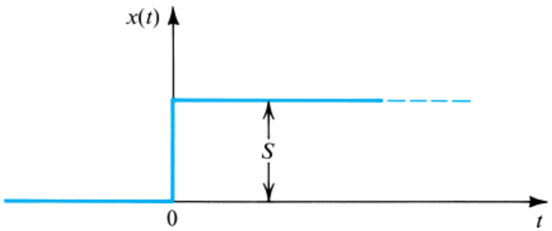


Figure E.9 A step-function signal of height S.

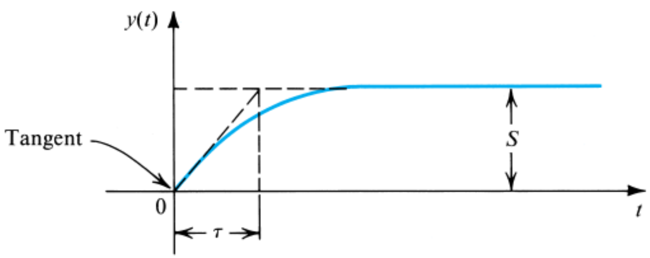


Figure E.10 The output $y(t)$ of a low-pass STC circuit excited by a step of height S.

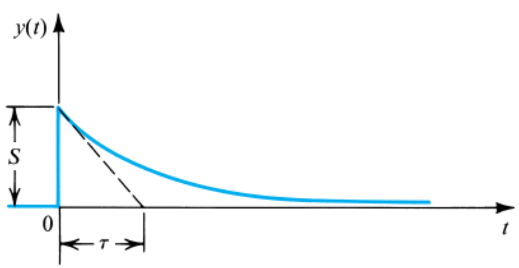


Figure E.11 The output $y(t)$ of a high-pass STC circuit excited by a step of height S.

In response to an input step signal of height S, a low-pass STC circuit (with a dc gain $K = 1$) produces the waveform shown in Fig. E.10.

$$y(t) = Y_\infty - (Y_\infty - Y_{0+})e^{-t/\tau}$$

$$y(t) = S\left(1 - e^{-t/\tau}\right)$$

The response of an STC high-pass circuit (with a high-frequency gain $K = 1$) to an input step of height S is shown in Fig. E.11.

$$y(t) = Se^{-t/\tau}$$

Pulse Response of low-pass STC Circuits

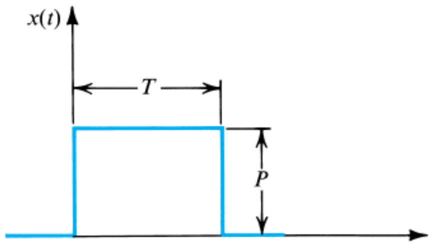
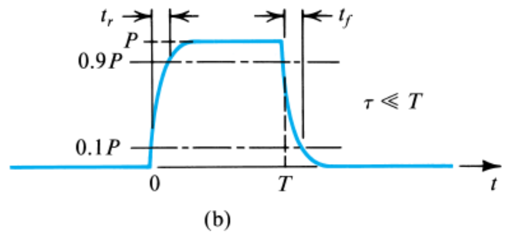


Figure E.12 A pulse signal with height P and width T.



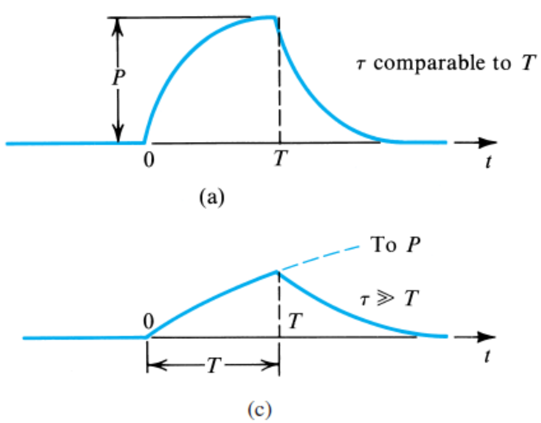


Figure E.13 Pulse responses of three STC low-pass circuits.

The distortion of a pulse signal by a parasitic (i.e., unwanted) low-pass circuit is measured by its *rise time* and *fall time*. The rise time is conventionally defined as the time taken by the amplitude to increase from 10% to 90% of the final value. Similarly, the fall time is the time during which the pulse amplitude falls from 90% to 10% of the maximum value. By use of the exponential equations of the rising and falling edges of the output waveform, it can be easily shown that

$$t_r = t_f = 2.2\tau \approx 0.35/f_0$$

Pulse Response of high-pass STC Circuits

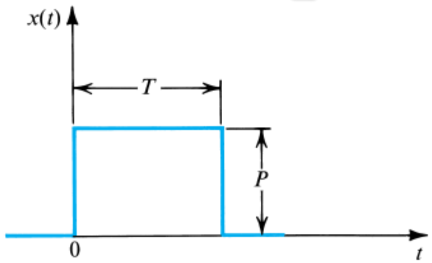
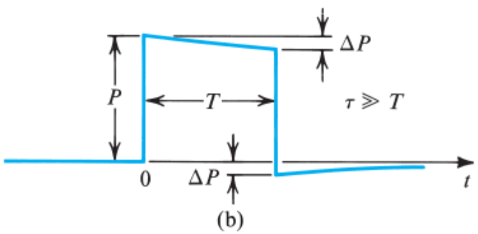


Figure E.12 A pulse signal with height P and width T.



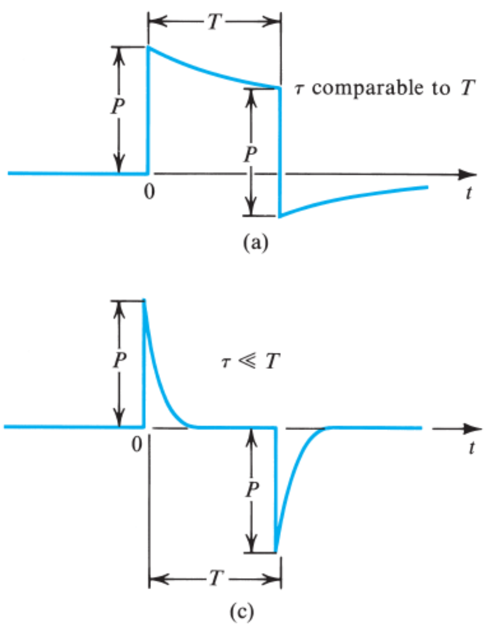


Figure E.14 Pulse responses of three STC high-pass circuits.

The distortion effect of the high-pass circuit on the input pulse is usually specified in terms of the per-unit or percentage loss in pulse height. This quantity is taken as an indication of the “sag” in the output pulse,

$$\text{Percentage sag} \equiv \Delta P / P \times 100$$

$$\text{Percentage sag} \equiv T/\tau \times 100$$

Homework #5

- Read Appendix E, Single-Time-Constant Circuits
- Read Appendix F, s-Domain Analysis

1. Problem E2
2. Problem E3
3. Problem E4
4. Problem E6

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Appendix F

S-Domain Analysis: Poles, Zeros, and Bode Plots

S-Domain Analysis

In analyzing the frequency response of an amplifier, most of the work involves finding the amplifier voltage gain as a function of the complex frequency s. In this **s-domain analysis**, a capacitance C is replaced by an admittance sC, or equivalently an impedance $1/sC$, and an inductance L is replaced by an impedance sL. Then, using usual circuit-analysis techniques, one derives the voltage transfer function $T(s) \equiv V_o(s)/V_i(s)$.

Once the transfer function $T(s)$ is obtained, it can be evaluated for **physical frequencies** by replacing s by $j\omega$. The resulting transfer function $T(j\omega)$ is in general a complex quantity whose magnitude gives the magnitude response (or transmission) and whose angle gives the phase response of the amplifier.

Laplace Transform/s-domain

Definition of the Laplace Transform $\mathcal{L} = \int_{-\infty}^{\infty} e^{-st} f(t)dt$

- For a capacitor

$$Q_c(t) = CV_c(t)$$

$$\frac{dQ_c(t)}{dt} \equiv i_c(t) = C\frac{dV_c(t)}{dt}$$

$$\mathcal{L}\{i_c(t)\} \equiv I_c(s) = CsV_c(s)$$

$$Z_c(s) = \frac{V_c(s)}{I_c(s)} = \frac{1}{sC}$$

- For an inductor

$$v(t) = L\frac{di(t)}{dt}$$

$$\mathcal{L}\{v(t)\} \equiv V(s) = sLI(s)$$

$$Z_L(s) = \frac{V(s)}{V(s)} = sL$$

Transfer Function

$$T(s) \equiv \frac{V_o(s)}{V_I(s)}$$

In general, *T*(*s*) can be expressed in the form

$$T(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

where the coefficients *a* and *b* are real numbers, and the order *m* of the numerator is smaller than or equal to the order *n* of the denominator; the latter is called the **order of the network**. Furthermore, for a **stable circuit**—that is, one that does not generate signals on its own—the denominator coefficients should be such that *the roots of the denominator polynomial all have negative real parts*. The problem of amplifier stability is studied in Chapter 11.

Poles and Zeros

$$T(s) \equiv \frac{V_o(s)}{V_I(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

$T(s)$ can also be expressed in the form

$$T(s) = a_m \frac{(s - Z_1)(s - Z_2) \cdots (s - Z_m)}{(s - P_1)(s - P_2) \cdots (s - P_n)}$$

where a_m is a multiplicative constant, $Z_1, Z_2, \ldots, Z_m$ are the roots of the numerator polynomial, and $P_1, P_2, \ldots, P_n$ are the roots of the denominator polynomial. $Z_1, Z_2, \ldots, Z_m$ are called the **transfer-function zeros** or **transmission zeros**, and $P_1, P_2, \ldots, P_n$ are the **transfer-function poles** or the **natural modes** of the network. A transfer function is completely specified in terms of its poles and zeros together with the value of the multiplicative constant.

First-Order Transfer Functions

$$T(s) \equiv \frac{V_o(s)}{V_I(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

General form for a first-order transfer function

$$T(s) = \frac{a_1 s + a_0}{s + \omega_0}$$

ω_0 is the pole frequency = $1/\tau$

Case 1: a_1=0

$$T(s) = \frac{a_0}{s + \omega_0}$$

low-pass

Case 2: a_0=0

$$T(s) = \frac{a_1 s}{s + \omega_0}$$

high-pass

Bode Plots 101

Bode Plots

A simple technique exists for obtaining an approximate plot of the magnitude and phase of a transfer function given its poles and zeros. The technique is particularly useful in the case of real poles and zeros. The method was developed by Hendrik Wade Bode, and the resulting diagrams are called **Bode plots**.

Hendrik Wade Bode, (24 December 1905 – 21 June 1982) was an American engineer, researcher, inventor, author and scientist, of Dutch ancestry. As a pioneer of modern control theory and electronic telecommunications he revolutionized both the content and methodology of his chosen fields of research.

http://en.wikipedia.org/wiki/Hendrik_Wade_Bode

Bode Plots

$$T(s) = a_m \frac{(s - Z_1)(s - Z_2)\cdots(s - Z_m)}{(s - P_1)(s - P_2)\cdots(s - P_n)}$$

A transfer function consists of a product of factors of the form $s + a$, where such a factor appears on top if it corresponds to a zero and on the bottom if it corresponds to a pole. It follows that the magnitude response in decibels of the network can be obtained by summing together terms of the form $20\log_{10}\sqrt{a^2+\omega^2}$, and the phase response can be obtained by summing terms of the form $\tan^{-1}(\omega/a)$. In both cases the terms corresponding to poles are summed with negative signs. For convenience we can extract the constant a and write the typical magnitude term in the form $20\log_{10}\sqrt{1+(\omega/a)^2}$.

Complex Frequency and the s-Plane

The Laplace Transform of a time-domain function gives us a function of the complex frequency variable ‘s’

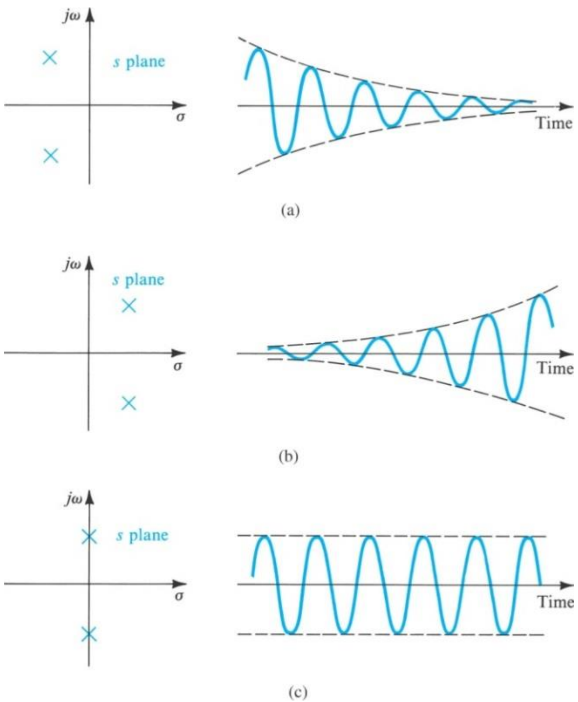


Figure 10.35 Relationship between pole location and transient response.

$$H(s)=\int_0^\infty h(t)e^{-st}dt \qquad \text{where } h(t)=\frac{V_o(t)}{V_i(t)}$$

$$s=\sigma+j\omega$$

For an amplifier or any other system to be stable, its poles should lie in the left half of the *s* plane. Poles in the right half of the *s* plane give rise to growing oscillations. A pair of complex-conjugate poles on the $j\omega$ axis gives rise to sustained sinusoidal oscillations. For an amplifier with a pole pair at $s=\sigma_0 \pm j\omega_n$

$$v(t)=e^{\sigma_0 t}\left[e^{+j\omega_n t}+e^{-j\omega_n t}\right]=2e^{\sigma_0 t}\cos(\omega_n t)$$

Envelope function, σ_0 must be less than 0

Bode Magnitude Plot

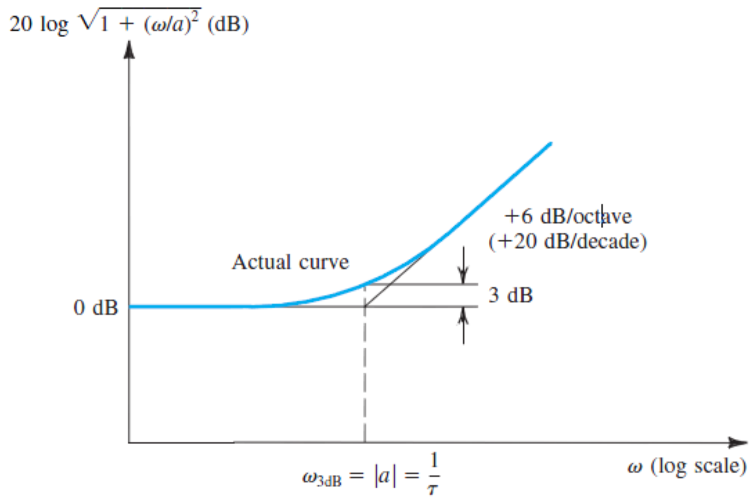


Figure F.1 Bode plot for the typical magnitude term. The curve shown applies for the case of a zero. For a pole, the high-frequency asymptote should be drawn with a –6-dB/octave slope.

On a plot of decibels versus log frequency this term gives rise to the curve and straight-line asymptotes shown in Fig. F.1. Here the low-frequency asymptote is a horizontal straight line at 0-dB level and the high-frequency asymptote is a straight line with a slope of 6 dB/octave or, equivalently, 20 dB/decade. The two asymptotes meet at the frequency $\omega = |a|$, which is called the **corner frequency**.

Bode Phase Plot

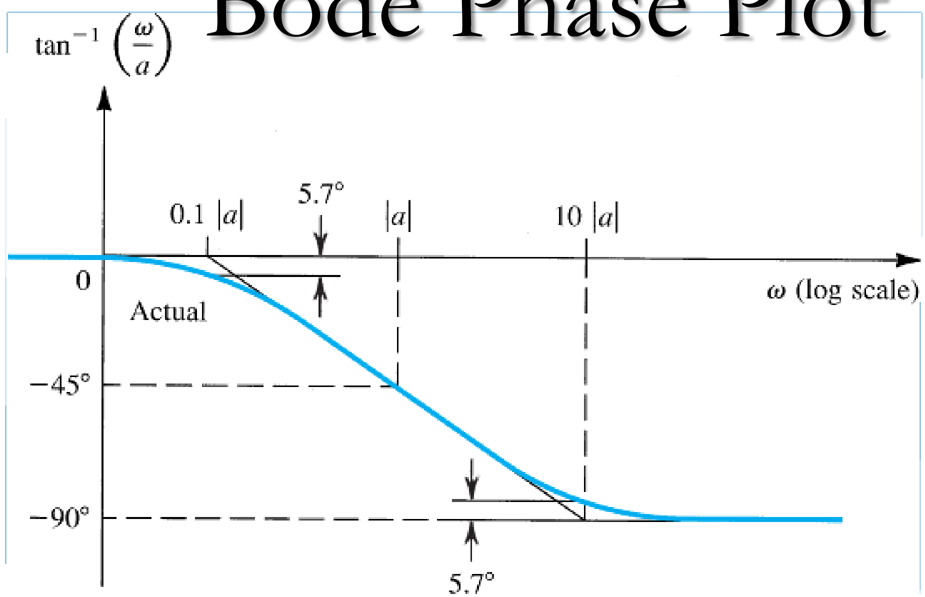


Figure F.3 Bode plot of the typical phase term $\tan^{-1}(\omega/a)$ when a is negative.

We next consider the Bode phase plot. Figure F.3 shows a plot of the typical phase term $\tan^{-1}(\omega/a)$, assuming that a is negative. Also shown is an asymptotic straight-line approximation of the arctan function. The asymptotic plot consists of three straight lines. The first is horizontal at $\varphi = 0$ and extends up to $\omega = 0.1\,|a|$. The second line has a slope of –45°/decade and extends from $\omega = 0.1\,|a|$ To $\omega = 10\,|a|$. The third line has a zero slope and a level of $\varphi = -90°$. The complete phase response can be obtained by summing the asymptotic Bode plots of the phase of all poles and zeros.

Poles and Zeros

A transfer function can be factored into the following form.

$$T(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ \ldots\ (s+p_n)}$$

The roots of the numerator polynomial are called <u>zeros.</u>

The roots of the denominator polynomial are called <u>poles</u>.

Determine the Transfer Function and Factor to put in the Standard Form

$$T(s) \equiv \frac{V_o(s)}{V_i(s)} = \frac{A(s+z_1)(s+z_2)\cdots(s+z_N)}{(s+p_1)(s+p_2)\cdots(s+p_M)}$$

$$= \frac{Az_1z_2\cdots z_N}{p_1p_2\cdots p_M}\frac{\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\cdots\left(1+\frac{s}{z_N}\right)}{\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\cdots\left(1+\frac{s}{p_M}\right)} = \frac{K\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\cdots\left(1+\frac{s}{z_N}\right)}{\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\cdots\left(1+\frac{s}{p_M}\right)}$$

For a Bode magnitude plot we plot 20log|T(s)| versus log(ω or f). Note that

$$\log(a \times b) = \log(a) + \log(b)$$

so we can add the 20log terms.

4 possible components

1. Constant terms, K
2. Poles and Zeros at the origin (s = 0)
3. Poles and Zeros not at the origin
4. Complex Poles and Zeros

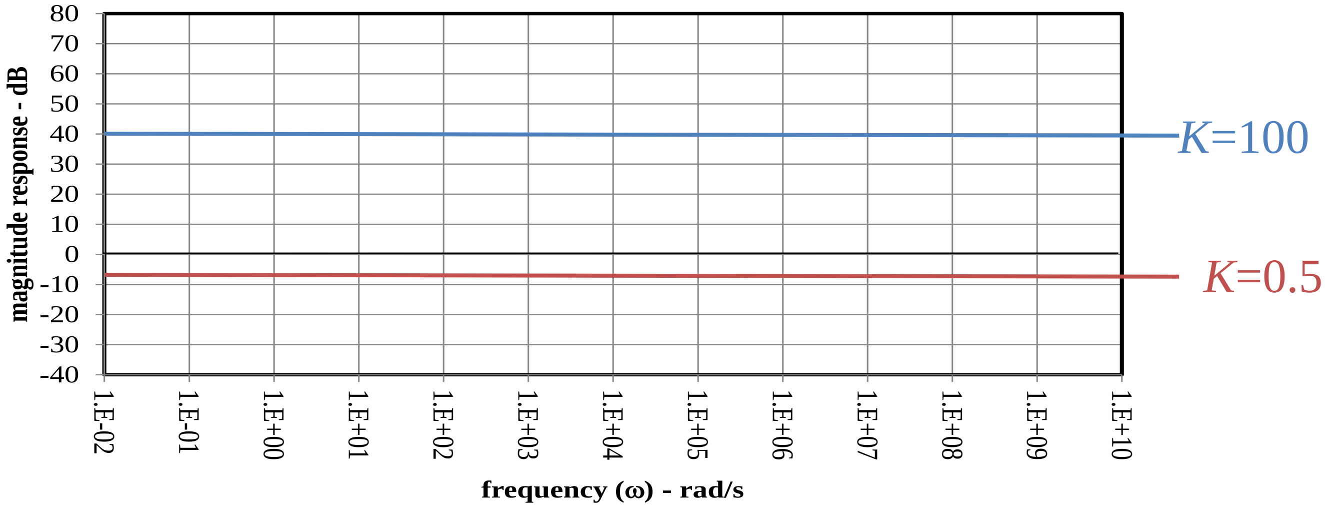
Complex Frequency and the s-Plane

The Laplace Transform of a time-domain function gives us a function of the complex frequency variable ‘s’

$$H(s) = \int_0^{\infty} h(t)e^{-st}dt \qquad \text{where } h(t) = \frac{V_o(t)}{V_i(t)}$$

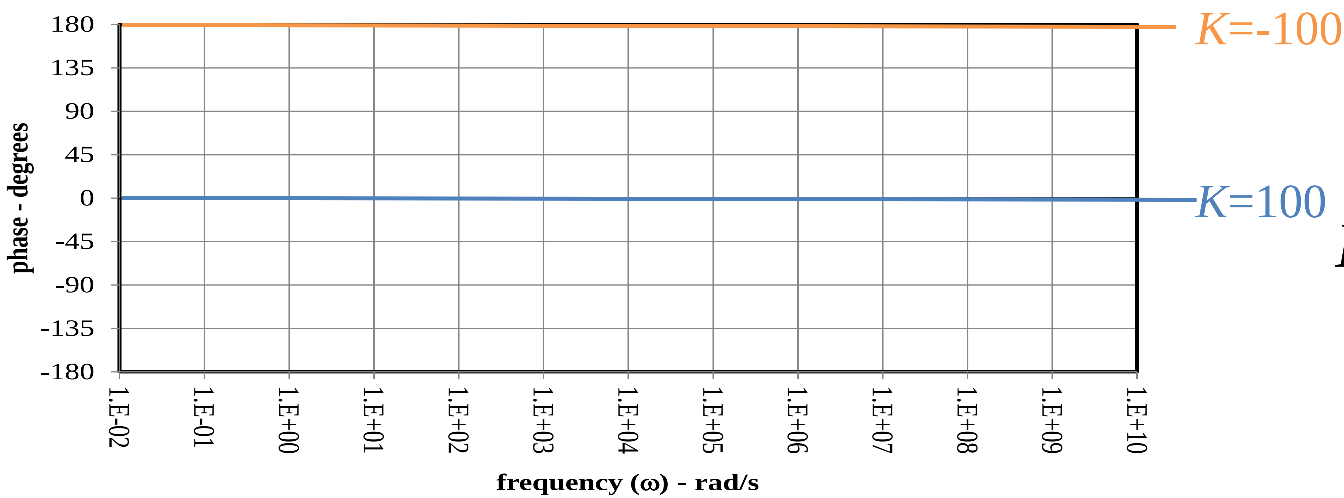
$$s = \sigma + j\omega$$

Constant Terms, K



Magnitude

$20\log(K)$



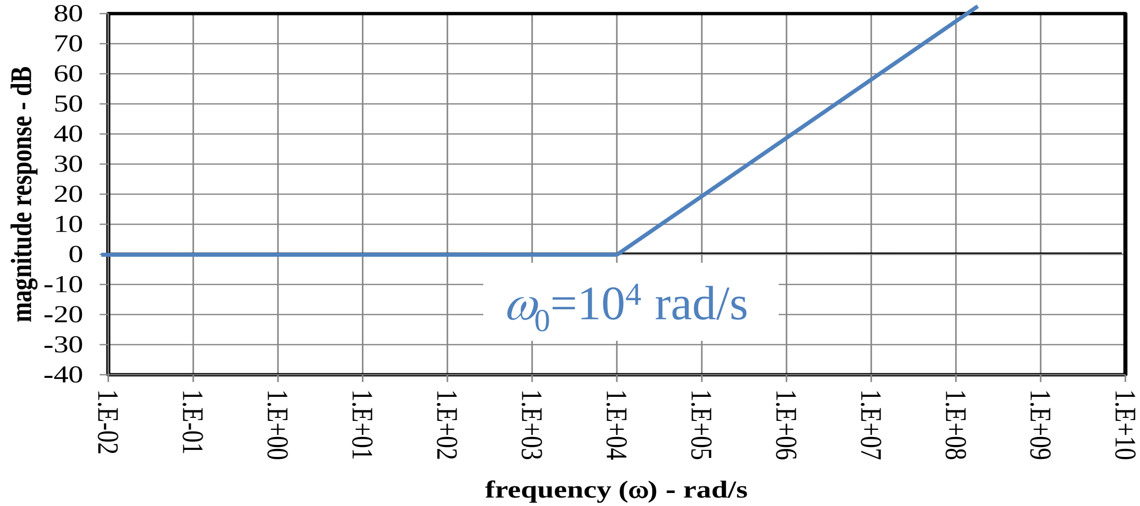
Phase

$K > 0;\ \phi = 0°$

$K < 0;\ \phi = \pm 180°$

Zero not at the Origin

$$\left(1+\frac{s}{z_1}\right)$$

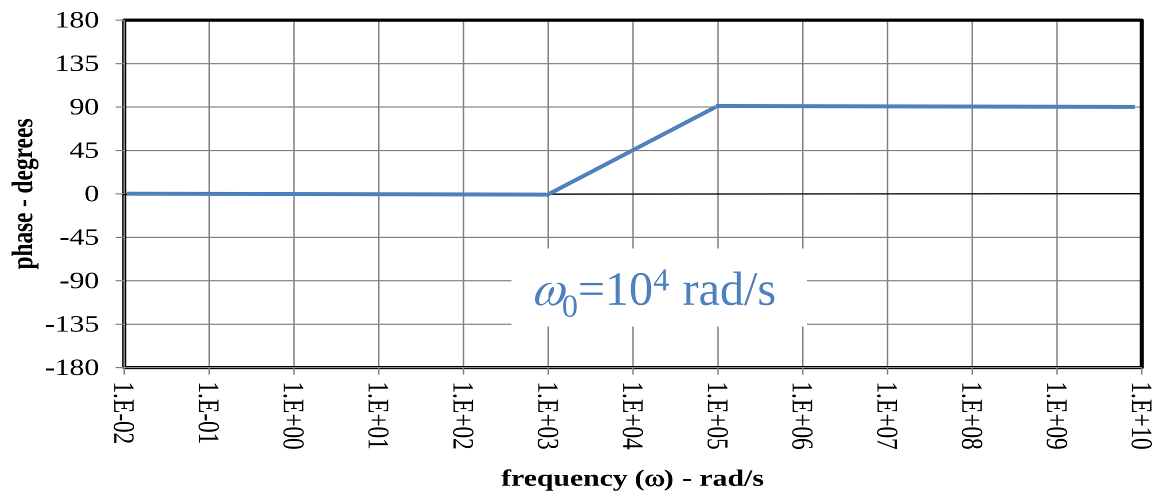


Magnitude

$\omega \ll \omega_0$; $|T(j\omega)|$ = 0 dB

$\omega = \omega_0$; $|T(j\omega)|$ = 3 dB

$\omega \gg \omega_0$; $|T(j\omega)|$ = +20 dB/dec



Phase

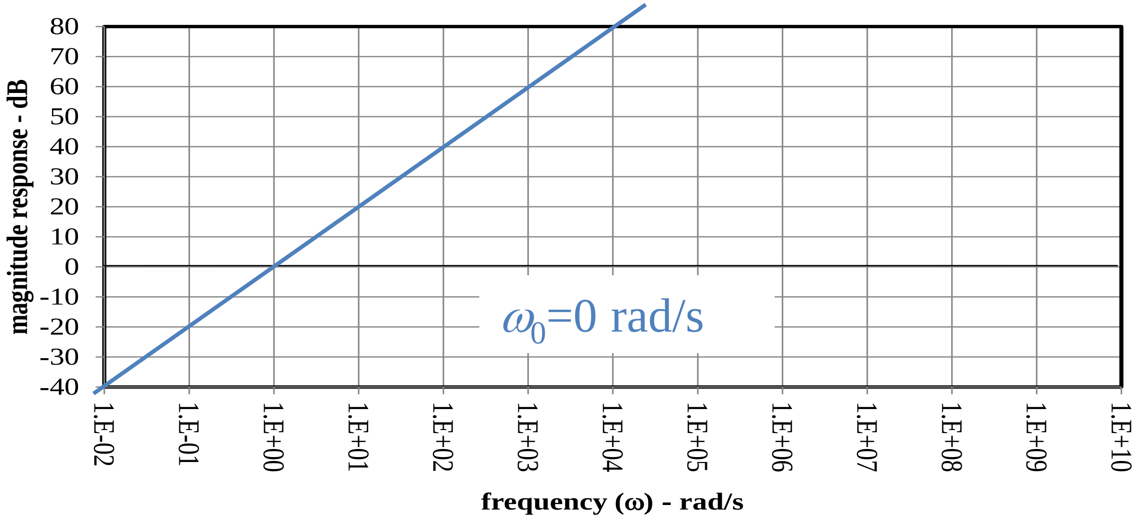
$\omega \ll \omega_0$; $\phi(\omega) = 0°$

$\omega = \omega_0$; $\phi(\omega) = 45°$

$\omega \gg \omega_0$; $\phi(\omega) = 90°$

Zero at the Origin, $s = 0$

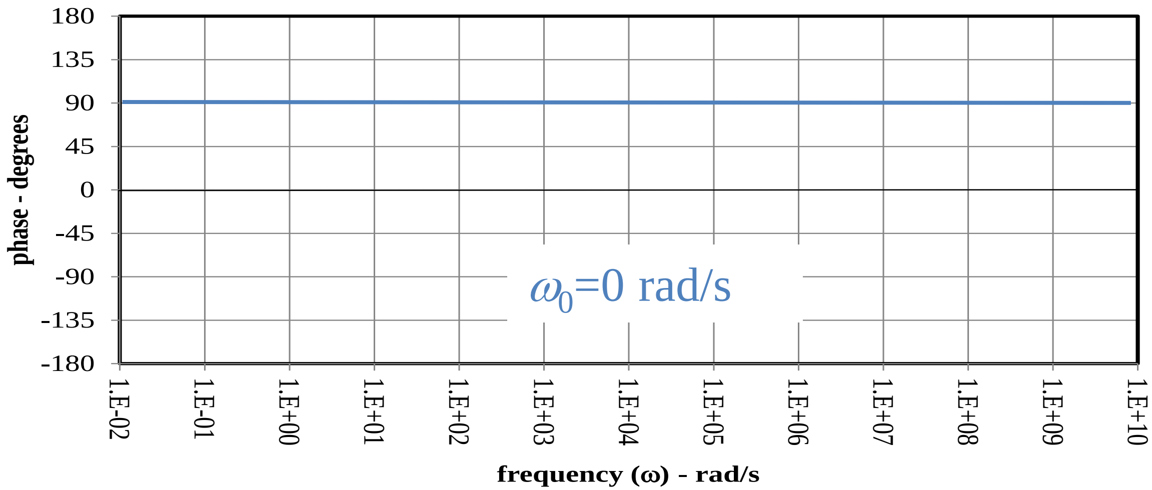
s



Magnitude

$\omega = 1$; $|T(j\omega)| = 0$ dB

Slope of $|T(j\omega)| = +20$ dB/dec

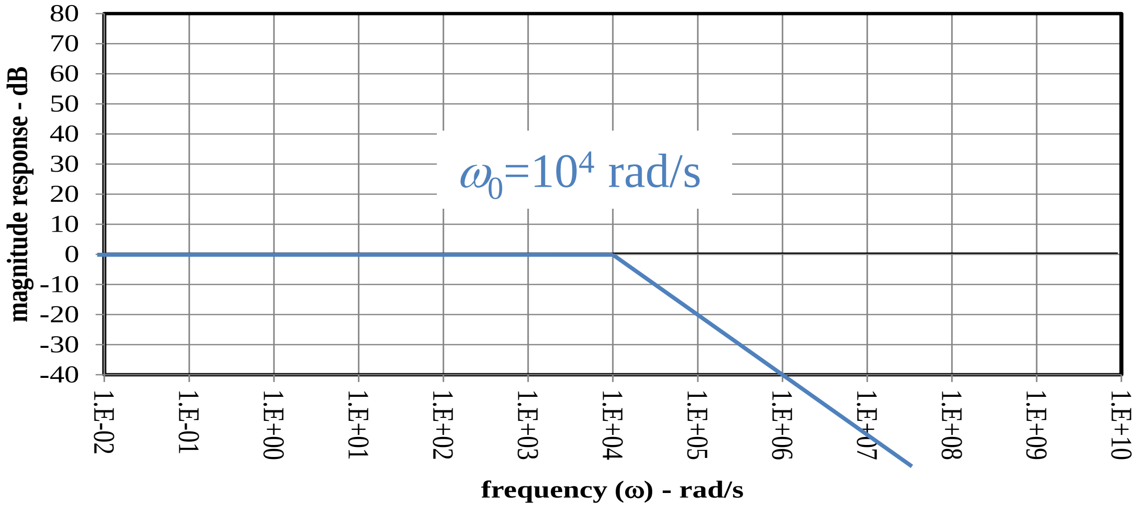


Phase

$\phi(\omega) = 90°$

Pole not at the Origin

$$\frac{1}{\left(1+\dfrac{s}{p_1}\right)}$$

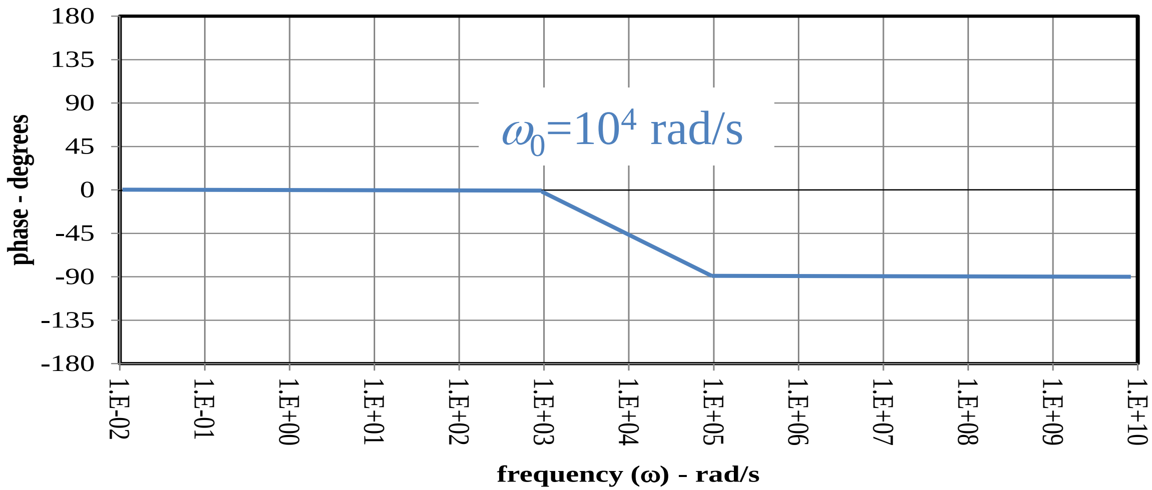


Magnitude

$\omega \ll \omega_0$; |T(jω)|= 0 dB

$\omega = \omega_0$; |T(jω)|= -3 dB

$\omega \gg \omega_0$; |T(jω)|= -20 dB/dec



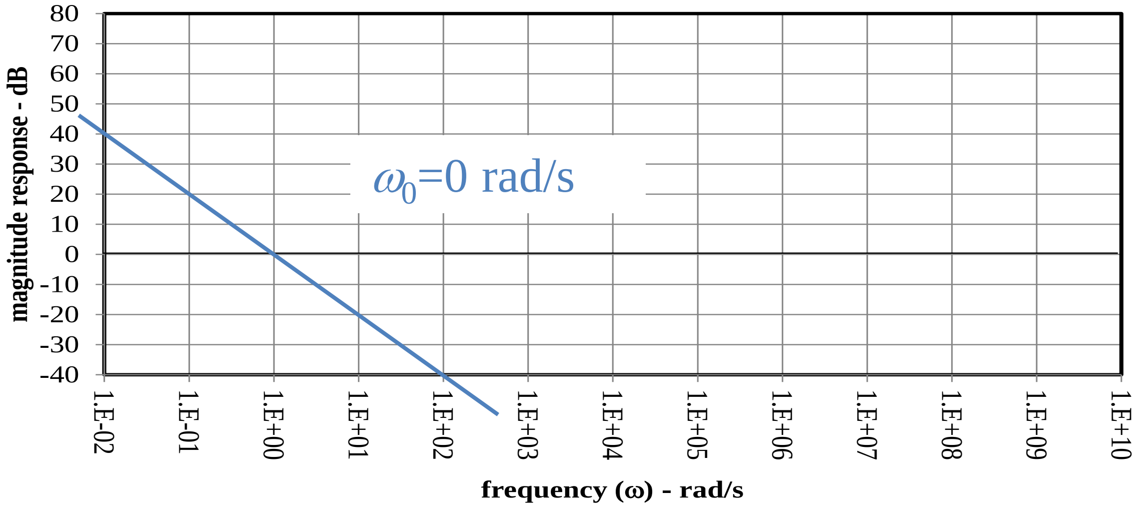
Phase

$\omega \ll \omega_0$; $\phi(\omega)$ = 0°

$\omega = \omega_0$; $\phi(\omega)$ = -45°

$\omega \gg \omega_0$; $\phi(\omega)$ = -90°

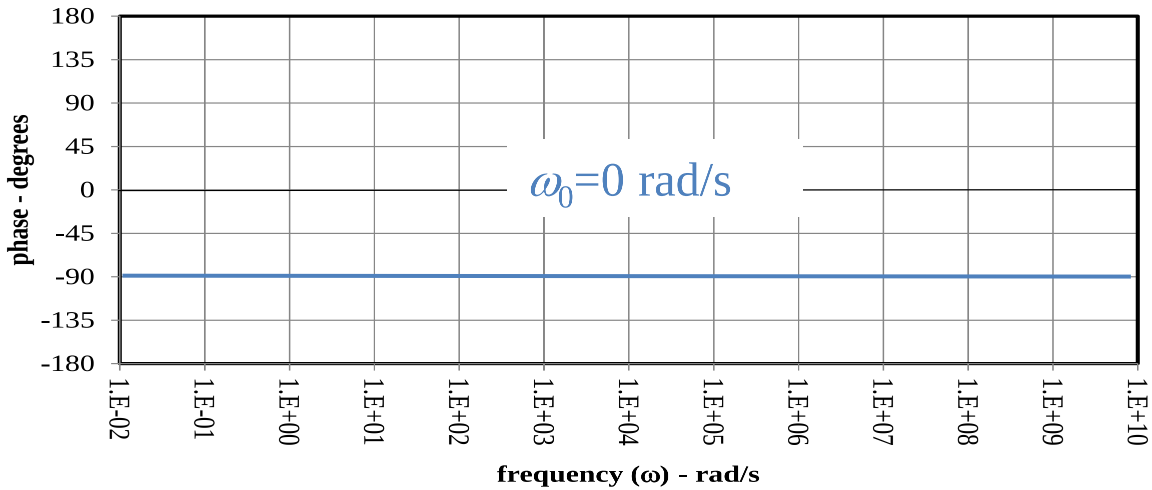
Pole at the Origin, $s = 0$ $\frac{1}{s}$



Magnitude

$\omega = 1$; $|T(j\omega)| = 0$ dB

Slope of $|T(j\omega)| = -20$ dB/dec



Phase

$\phi(\omega) = -90°$

Example: STC Low-Pass Circuits

$$T(s)=\frac{K}{1+(s/\omega_0)}$$

$$T(j\omega)=\frac{K}{1+j(\omega/\omega_0)}$$

Single pole at ω_0

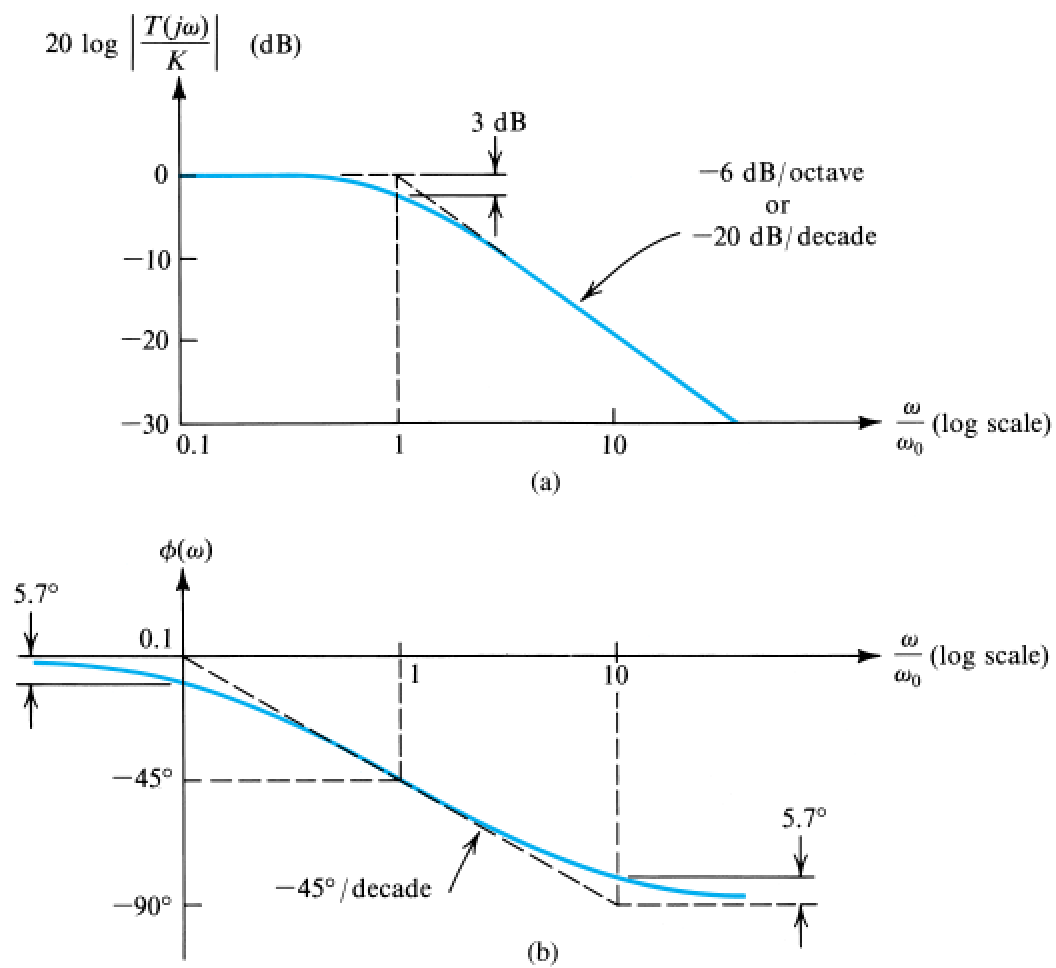


Figure E.6 (a) Magnitude and (b) phase response of STC circuits of the low-pass type.

Example: STC High-Pass Circuits

$$T(s)=\frac{Ks}{s+\omega_0}$$

$$T(j\omega)=\frac{K}{1-j\left(\omega_0/\omega\right)}$$

Single zero at $\omega = 0$ rad/s and a single pole at ω_0

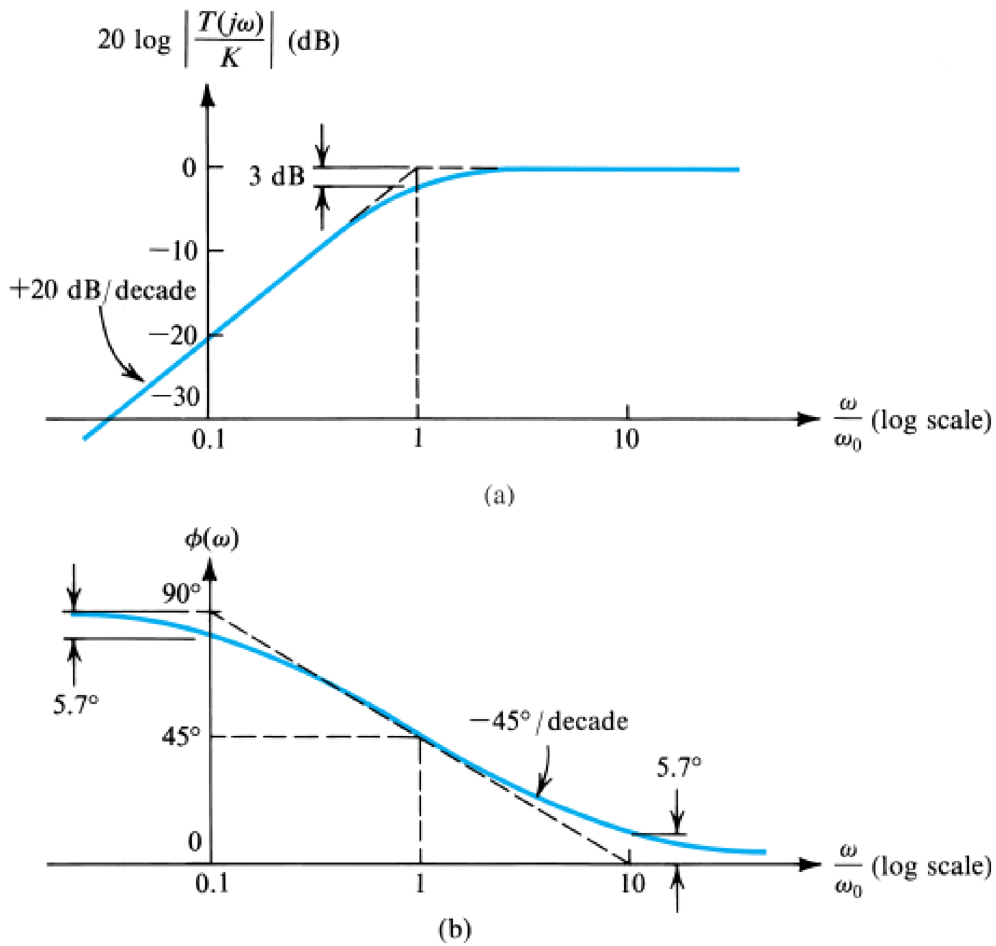


Figure E.8 (**a**) Magnitude and (**b**) phase response of STC circuits of the high-pass type.

Example F.1

An amplifier has the voltage transfer function

$$T(s) = \frac{10s}{(1 + s/10^2)(1 + s/10^5)}$$

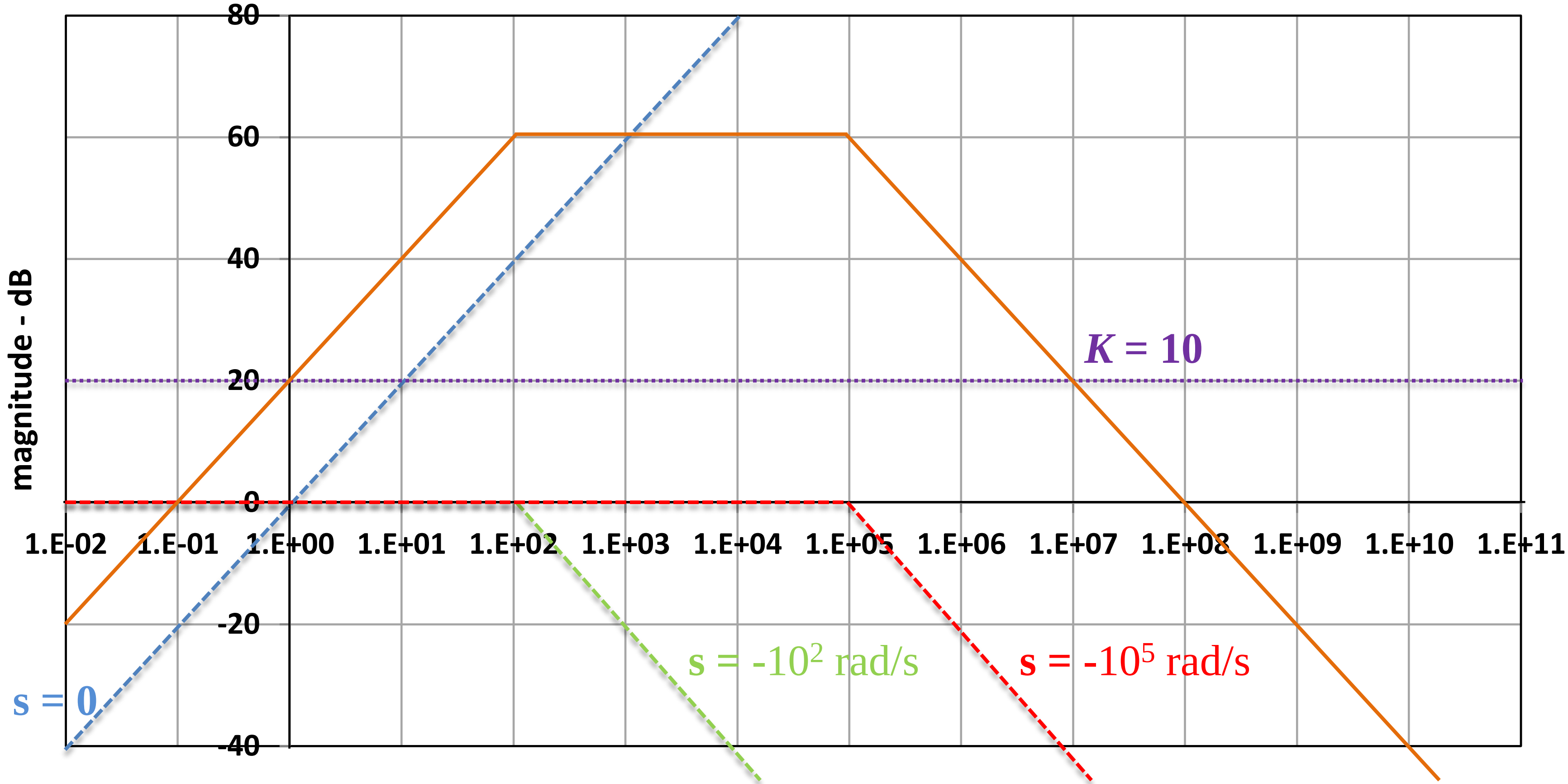
Find the poles and zeros and sketch the magnitude of the gain versus frequency. Find approximate values for the gain at ω = 10, 10^3, and 10^6 rad/s.

zeros: $s = 0$, $s = \infty$

poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s

Example F.1

zeros: $s = 0$, $s = \infty$ poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s



Example F.1

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Find the poles and zeros and sketch the magnitude of the gain versus frequency. Find approximate values for the gain at ω = 10, 10^3, and 10^6 rad/s.

zeros: $s = 0$, $s = \infty$

poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s

ω	Approximate gain	Actual gain
10	40 dB	39.96 dB
10^3	60 dB	59.96 dB
10^6	40 dB	39.96 dB

Example F.2

Find the Bode plot for the phase of the transfer function of the amplifier considered in Example F.1.

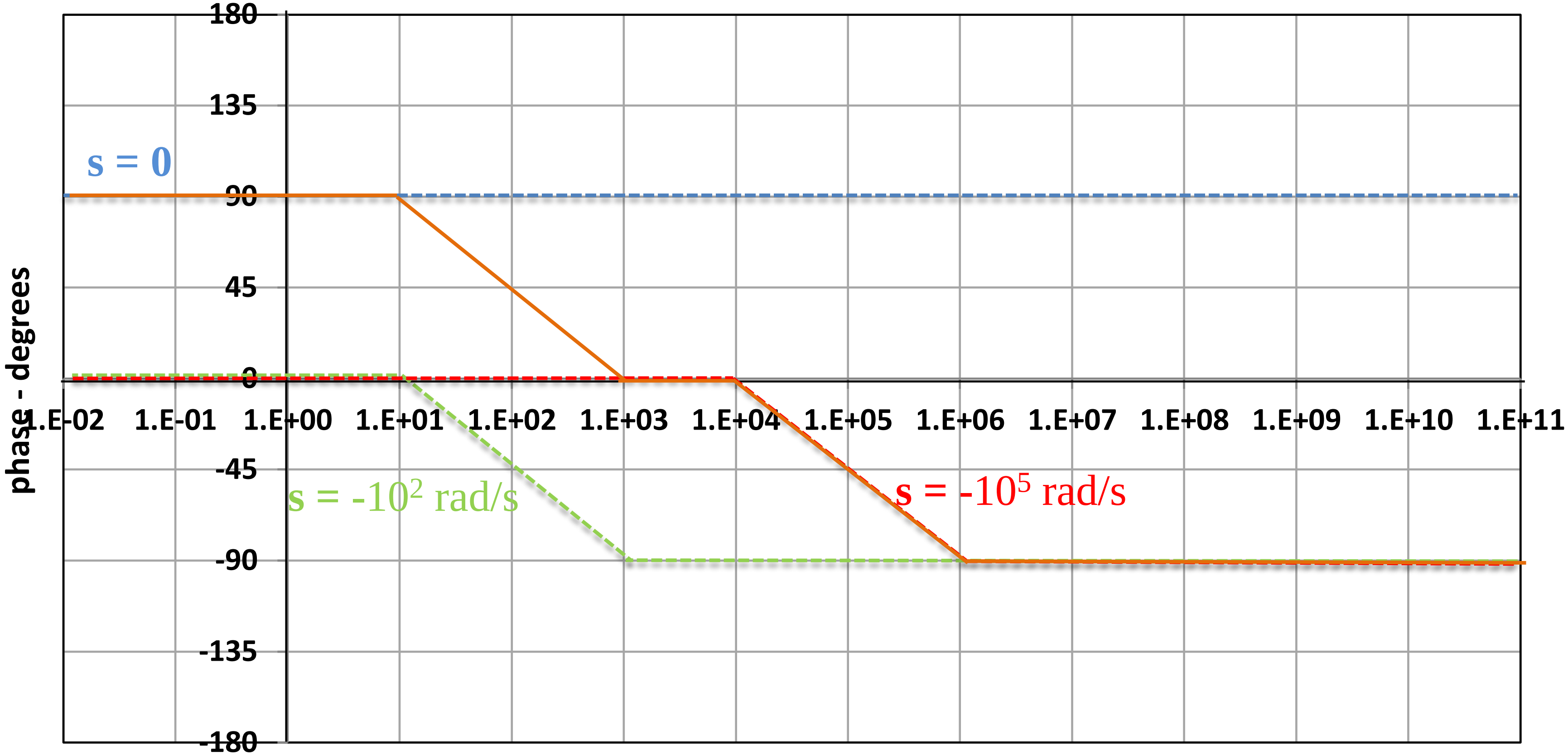
$$T(s) = \frac{10s}{(1 + s/10^2)(1 + s/10^5)}$$

zeros: $s = 0$, $s = \infty$

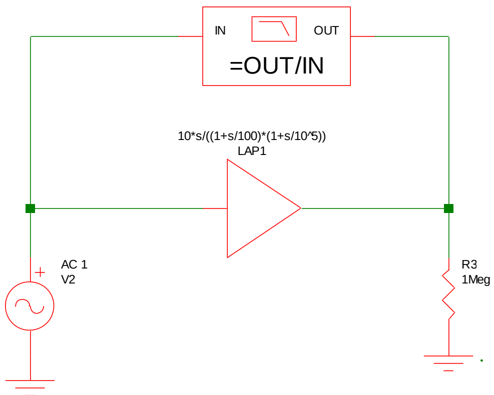
poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s

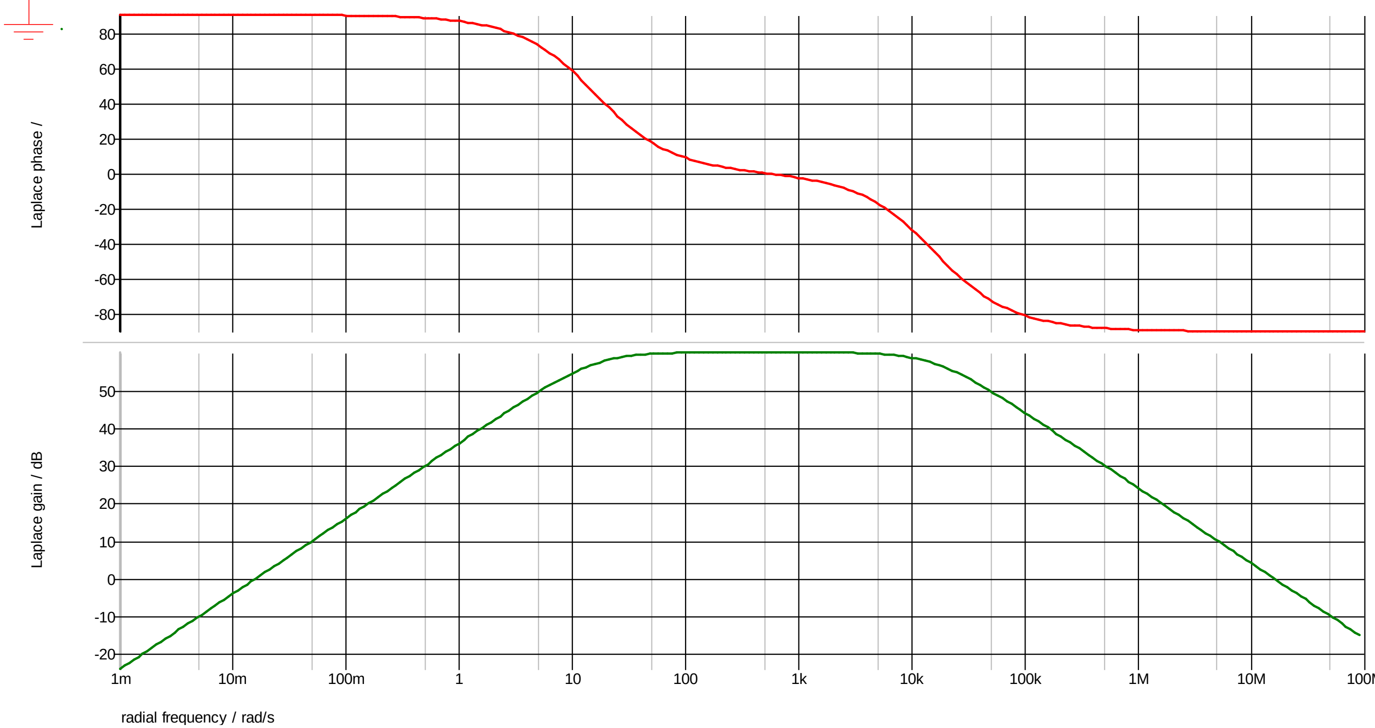
Example F.2

zeros: $s = 0$, $s = \infty$ poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s



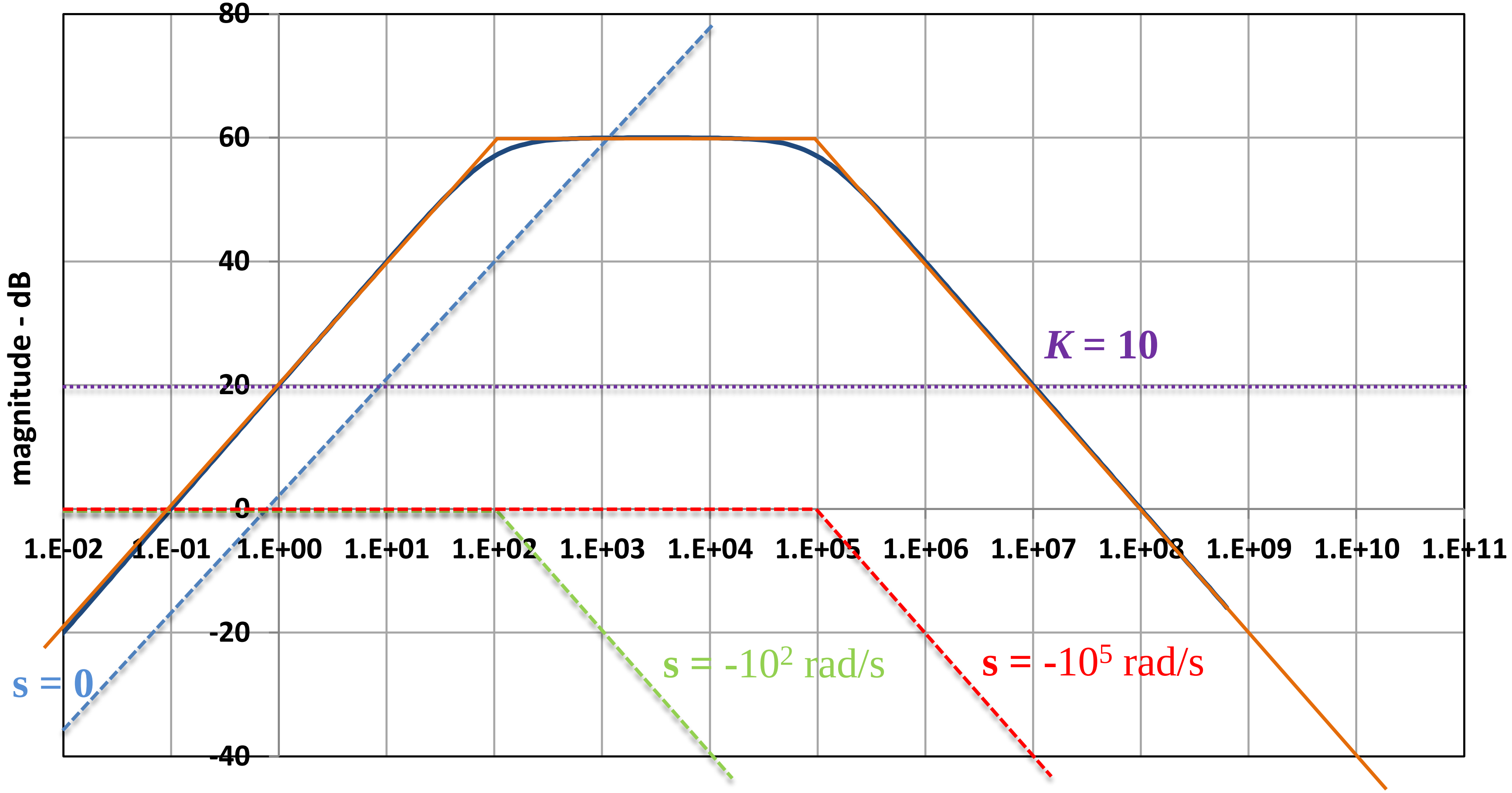
Example F.1 & F.2





Example F.1

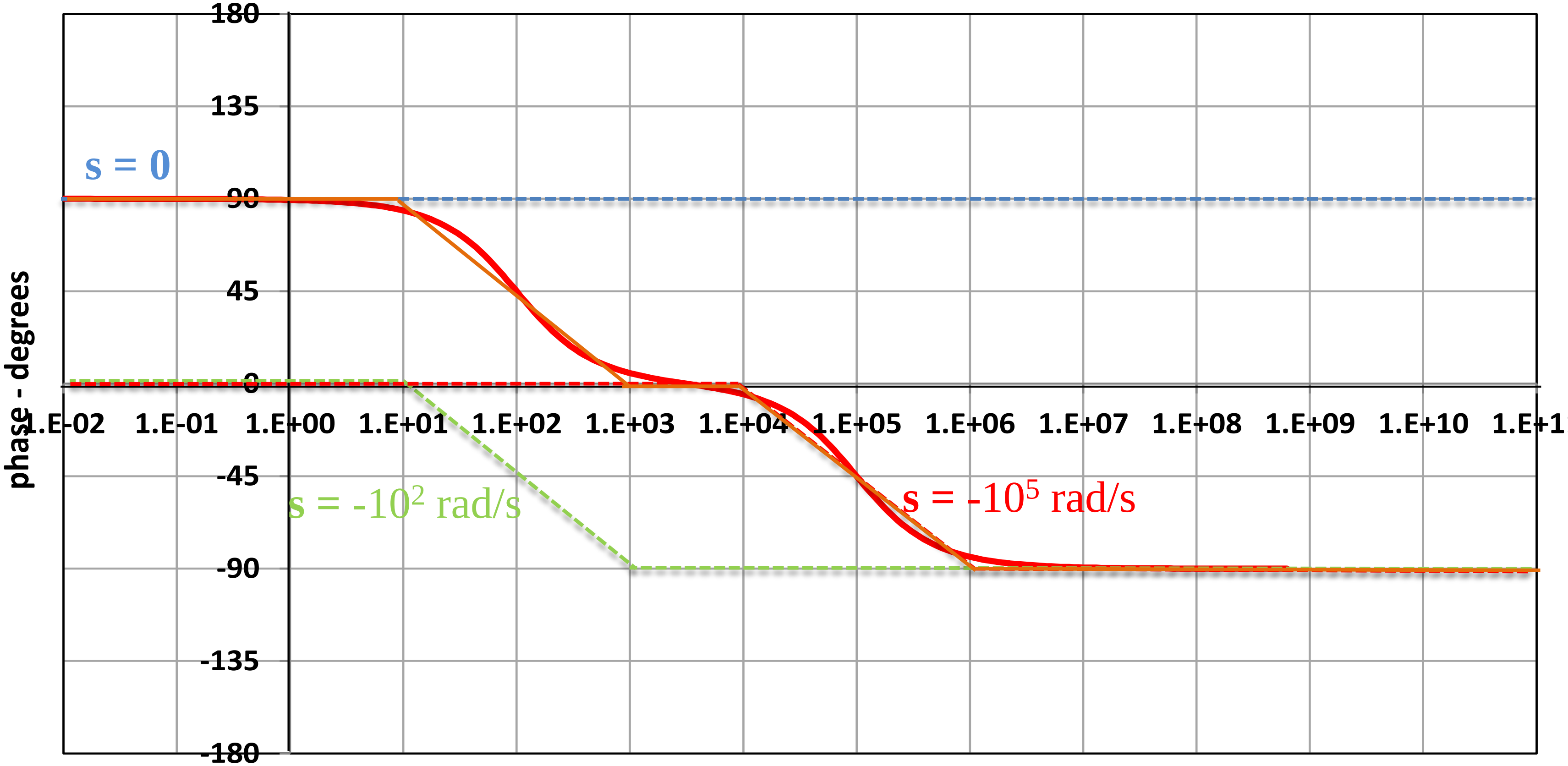
zeros: $s = 0$, $s = \infty$ poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s



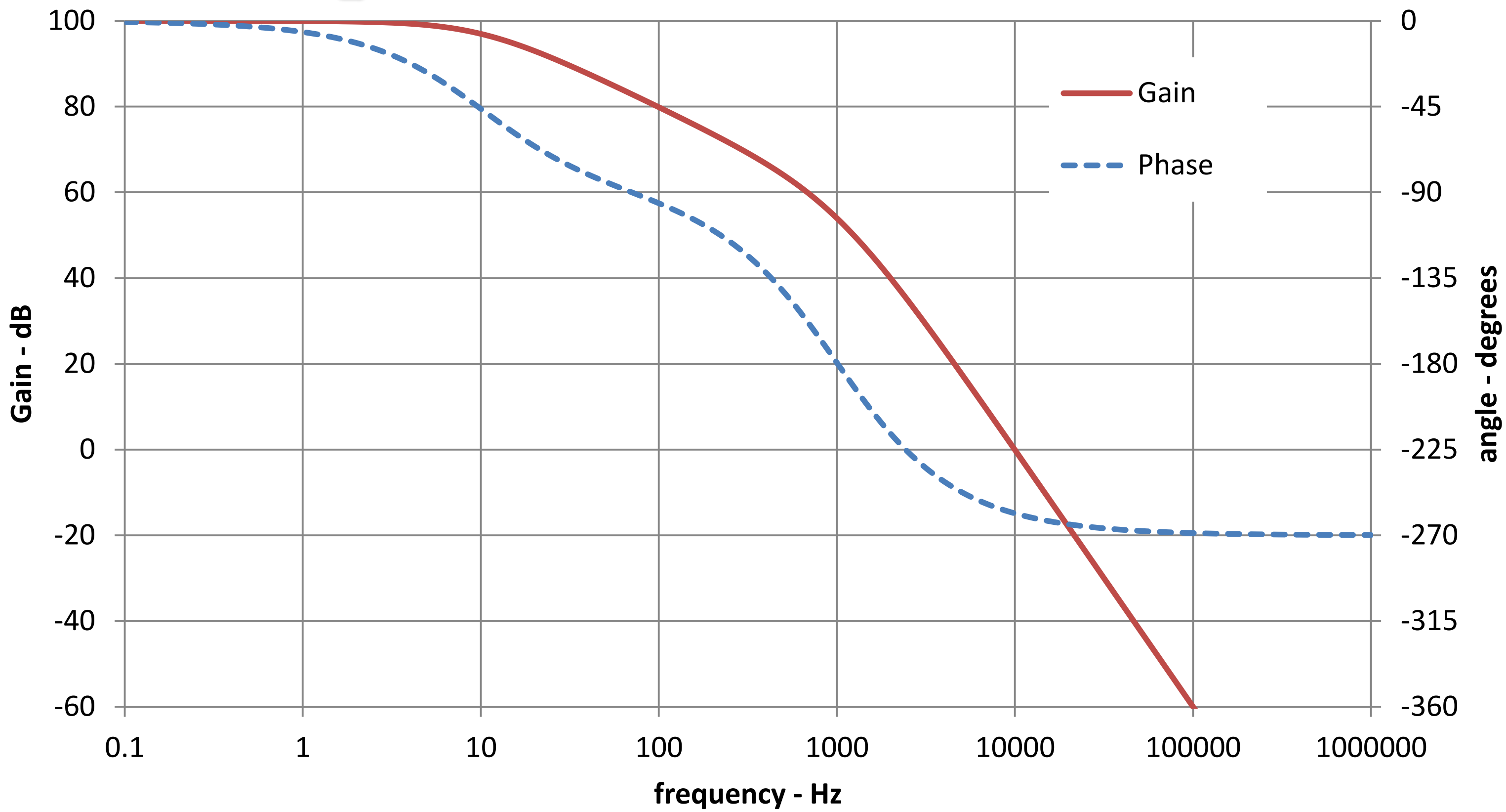
Example F.2

zeros: $s = 0$, $s = \infty$

poles: $s = -10^2$ rad/s, $s = -10^5$ rad/s



Example from 2011 Final Exam



The Bode plot for an amplifier is shown above. List the poles and zeros for the amplifier and the transfer function of the amplifier. (10 points)

Homework #6

- Read Appendix F, s-Domain Analysis: Poles, Zeros, and Bode Plots

1. Problem F1
2. Problem F2
3. Problem F10
4. Problem F11