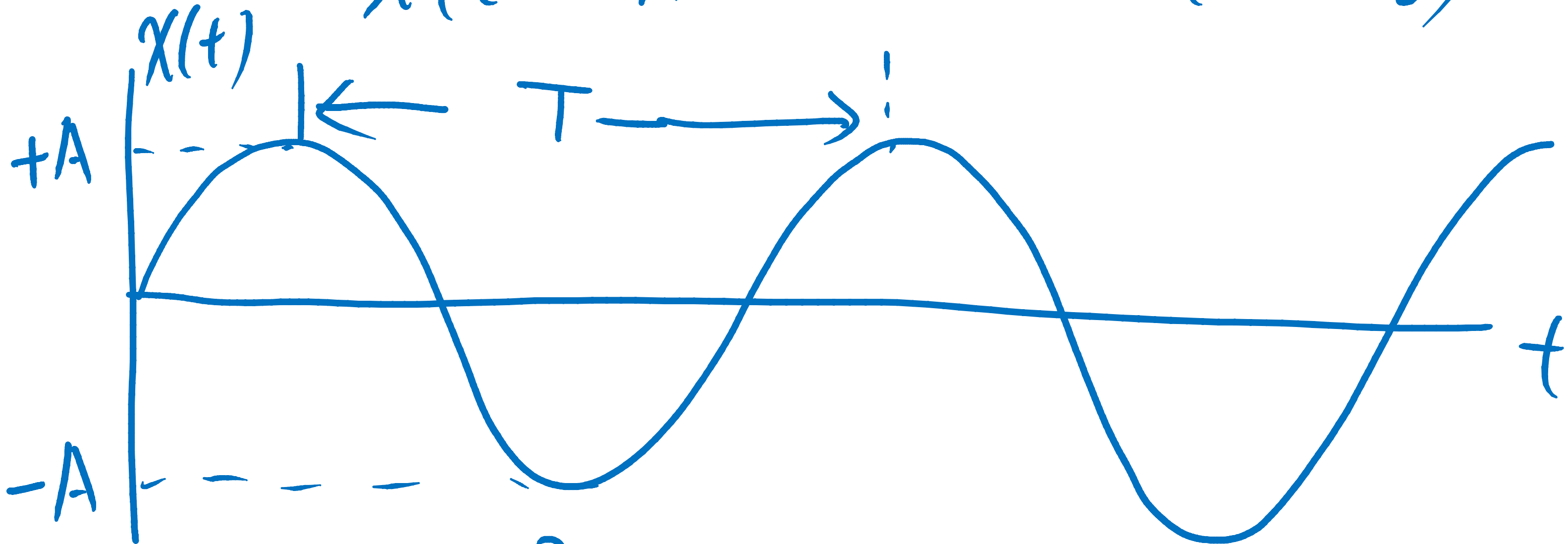
# Chapter 3. A.C. circuits

## * Sinusoidal waves

$$x(t) = A\sin\omega t = A\cos\left(\omega t - \frac{\pi}{2}\right)$$

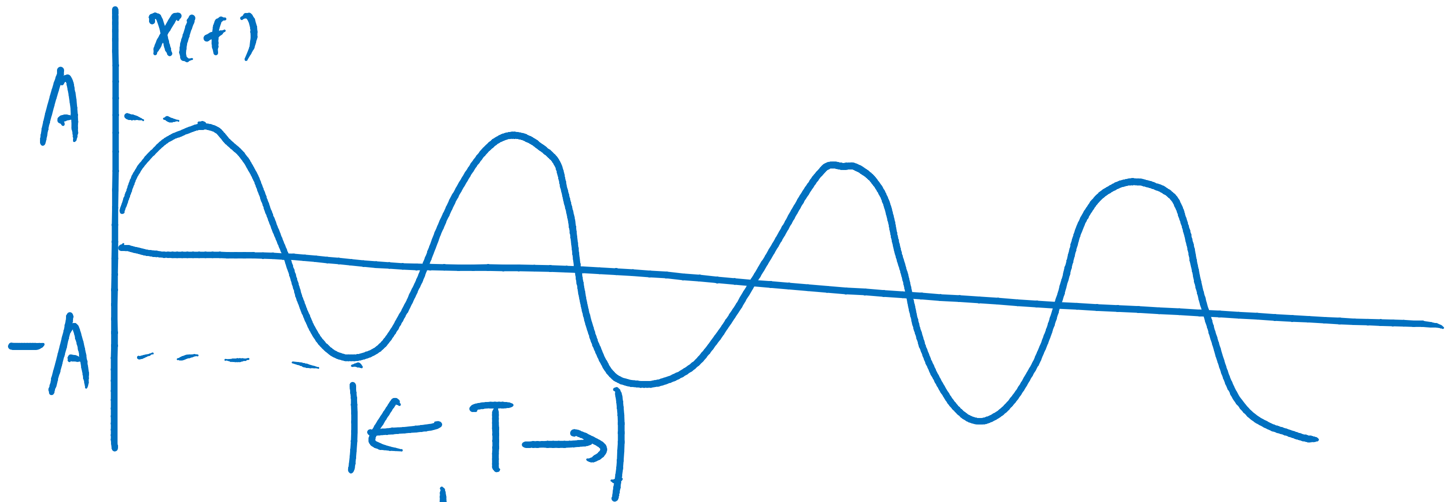



why sinusoids?

fundamental waves

Arbitrary periodic function can be constructed by using sinusoids with different frequencies)

(Fourier transform)




In general $x(t) = A\cos(\omega t + \varphi)$

$A$: amplitude

$T$: period. unit: s

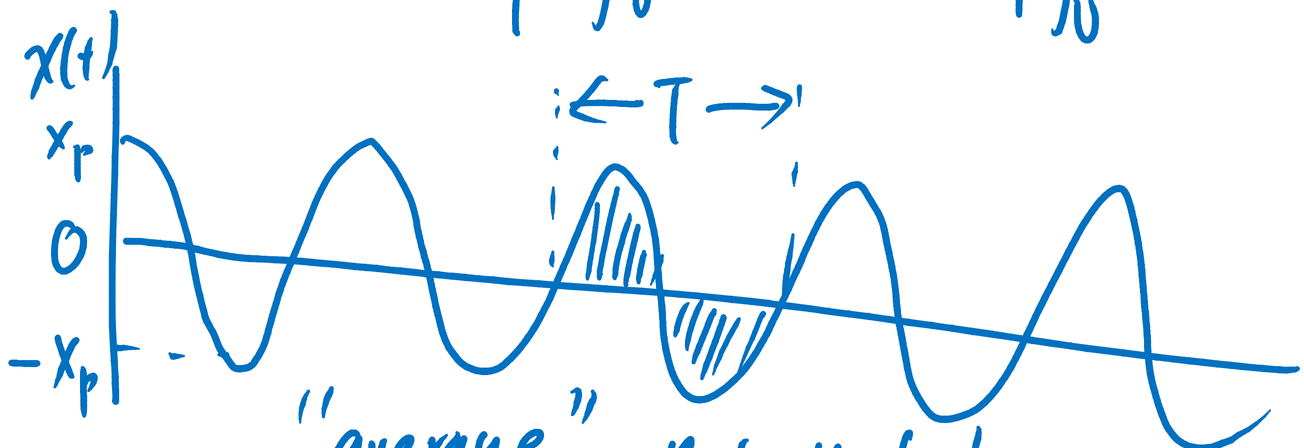
$f = \frac{1}{T}$ frequency. unit: $\frac{1}{s}$ or Hz

angular frequency $\omega = 2\pi f = \frac{2\pi}{T}$ unit: rad/s or 1/s

# Average and RMS values of a.c. signal.

— average

$$\langle x(t)\rangle = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_0^T A\cos(\omega t+\varphi)\,dt = 0$$




"average" not useful

— peak value $x_p$ is useful

— root-mean-square (rms) values

$$x_{rms} = \sqrt{\overline{x^2(t)}} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$

$$x(t) = A\sin\omega t$$

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T A^2\sin^2\omega t\,dt}$$

$$= \sqrt{\frac{A^2}{T}\int_0^T\left(\frac{1-\cos 2\omega t}{2}\right)dt}$$

$$= \sqrt{\frac{A^2}{\cancel{T}}\cdot\frac{\cancel{T}}{2}} = \frac{A}{\sqrt{2}}$$

If $A = V_p$, $V_{rms} = \frac{V_p}{\sqrt{2}}$, If $A = I_p$, $I_{rms} = \frac{I_p}{\sqrt{2}}$

$$p(t) = i^2(t)\,R$$

$$P_{ave} = \overline{p(t)} = \overline{i^2(t)}\,R = I_{rms}^2\,R$$

$$\boxed{P_{ave} = I_{rms}^2\,R}$$ (resistive)

general $$\boxed{P_{ave} = I_{rms}\,(I_{rms}\,R) = I_{rms}\cdot V_{rms}}$$

$$\boxed{P_{ave} = \frac{V_{rms}^2}{R}}$$ (resistive)

## * Complex Number Review

— a.c. Signal $x(t) = A\cos(\omega t + \varphi)$

$\omega$ does not change in circuits with R, C, L.

— $A$ and $\varphi$ characterize an a.c. Signal

— Complex number has two components, and can be used to represent a.c. signals.

$j^2 = -1$

Rectangular form: $a + jb$
(Cartesian)

real part ↵ ($a$) ↳ imaginary part ($b$)

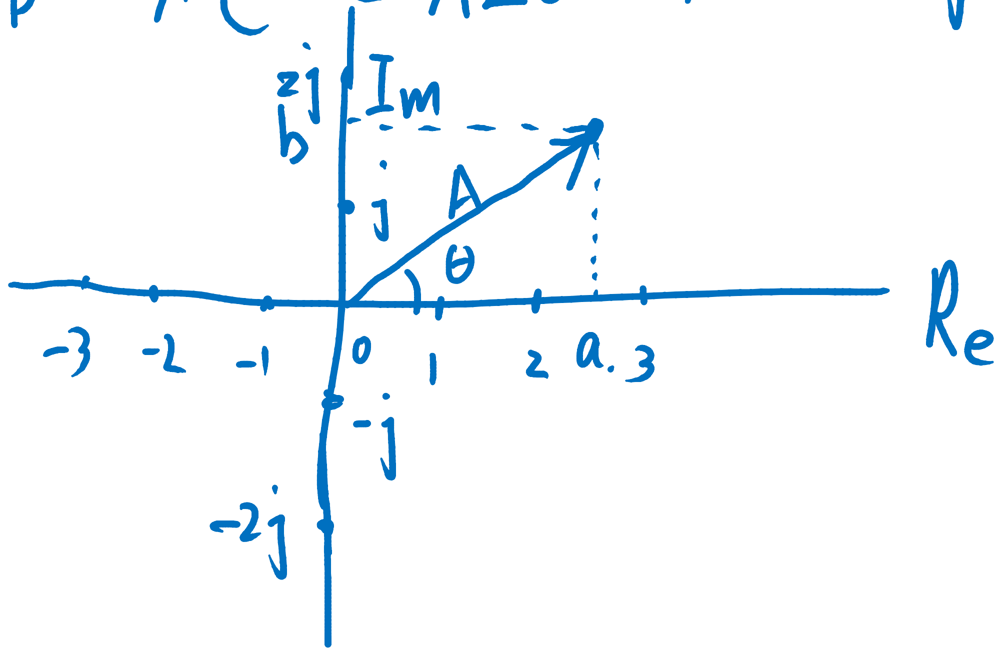
$$a+jb = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + j\frac{b}{\sqrt{a^2+b^2}}\right) \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$A = \sqrt{a^2+b^2}$
modulus

$$\frac{a}{\sqrt{a^2+b^2}} = \cos\theta \qquad \frac{b}{\sqrt{a^2+b^2}} = \sin\theta$$

$$a+jb = A(\cos\theta + j\sin\theta) = Ae^{j\theta} = A\angle\theta$$ (polar form)

$$a + jb = Ae^{j\theta} = A\angle\theta = A\cos\theta + jA\sin\theta$$



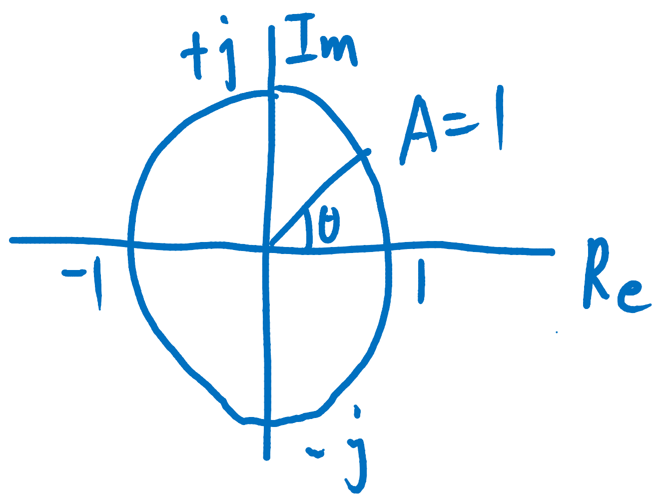

$A = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$a = A\cos\theta$

$b = A\sin\theta$

unit modulus $(A=1)$




$$1\angle\theta = 1e^{j\theta}$$
$$= \cos\theta + j\sin\theta$$

$a = \cos\theta$

$b = \sin\theta$

Operations of complex numbers

add or subtract: use rectangular form

$$C_1 = a_1 + jb_1 \qquad C_2 = a_2 + jb_2$$

$$C_1 + C_2 = (a_1 + a_2) + j(b_1 + b_2)$$

Multiply or divide: more convenient to use polar form.

$$A_1 = \sqrt{a_1^2 + b_1^2} \qquad \theta_1 = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

$$A_2 = \sqrt{a_2^2 + b_2^2} \qquad \theta_2 = \tan^{-1}\left(\frac{b_2}{a_2}\right)$$

$$C_1 = A_1 e^{j\theta_1} = A_1 \angle \theta_1$$

$$C_2 = A_2 e^{j\theta_2} = A_2 \angle \theta_2$$

$$C_1 \times C_2 = A_1 A_2 e^{j(\theta_1 + \theta_2)} = A_1 A_2 \angle (\theta_1 + \theta_2)$$

$$\frac{C_1}{C_2} = \frac{A_1}{A_2} e^{j(\theta_1 - \theta_2)} = \frac{A_1}{A_2} \angle (\theta_1 - \theta_2)$$

Multiply or Divide in rectangular form

$$C_1 = a_1 + jb_1 \qquad C_2 = a_2 + jb_2$$

$$C_1 \times C_2 = (a_1 + jb_1)(a_2 + jb_2)$$

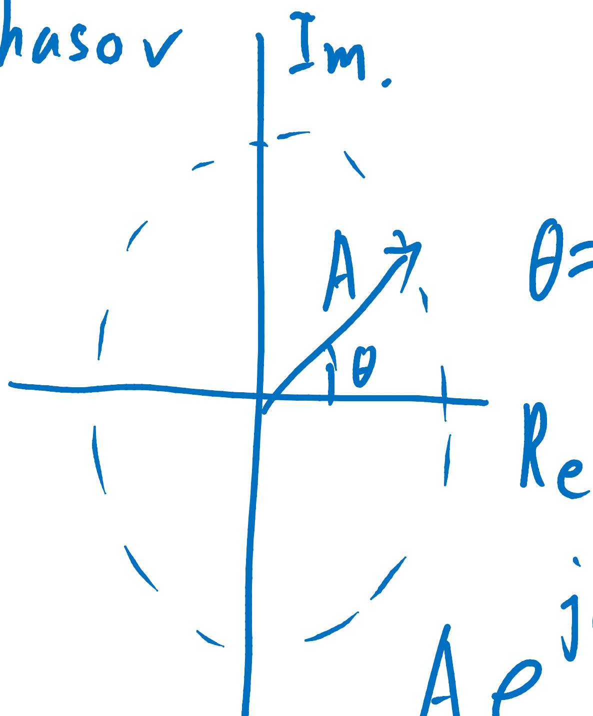
$$= a_1 a_2 + a_1 (jb_2) + (jb_1) a_2 + (jb_1)(jb_2)$$

$$= a_1 a_2 + j a_1 b_2 + j a_2 b_1 + (-b_1 b_2)$$

$$= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$$

$$\frac{C_1}{C_2} = \frac{(a_1 + jb_1)}{(a_2 + jb_2)} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} = \frac{(a_1 a_2 + b_1 b_2) + (a_2 b_1 - a_1 b_2) j}{a_2^2 + b_2^2}$$

* phasor




$\theta = \omega t + \varphi$ $\quad \omega = \frac{2\pi}{T}$

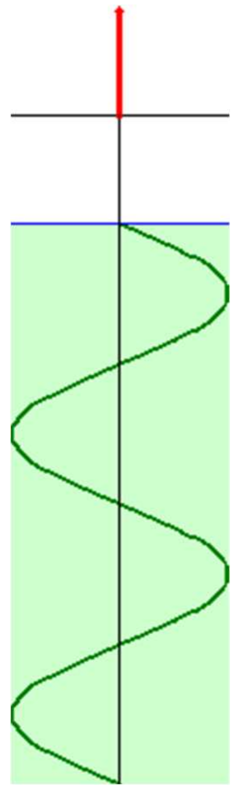
$$Ae^{j\theta} = A\angle\theta$$

$$= Ae^{j(\omega t+\varphi)}$$

$$= \underline{A\cos(\omega t+\varphi)} + jA\sin(\omega t+\varphi)$$

↳ real part is an a.c. signal

# Phasor Animation



sinPlotting1.swf

$$A\cos(\omega t+\varphi) = \mathrm{Re}\left[Ae^{j(\omega t+\varphi)}\right]$$

$$= \mathrm{Re}\left[Ae^{j\omega t} \cdot e^{j\varphi}\right]$$

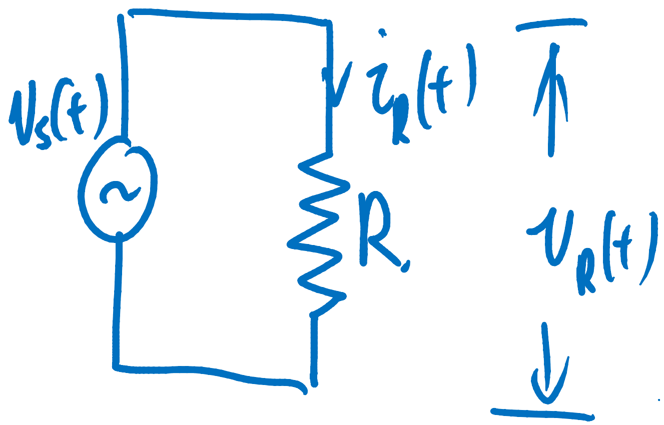
$e^{j\omega t}$ is same for all signals in the same circuit.

$$A\cos(\omega t+\varphi) \rightarrow A\underline{e^{j\omega t}}e^{j\varphi} \Rightarrow Ae^{j\varphi} = A\angle\varphi$$

$A\angle\varphi \rightarrow$ phasor representation (amplitude and phase) of an a.c. signal

- Phasor is a complex number in polar form used to assist the calculation of a.c. circuits.
- Note the specific frequency $\omega$ of the a.c. signal, since this is not explicit in a phasor.

* a.c. $i$–$v$ relationship for R, L, and C.




Source $v_s(t) = A\sin wt$

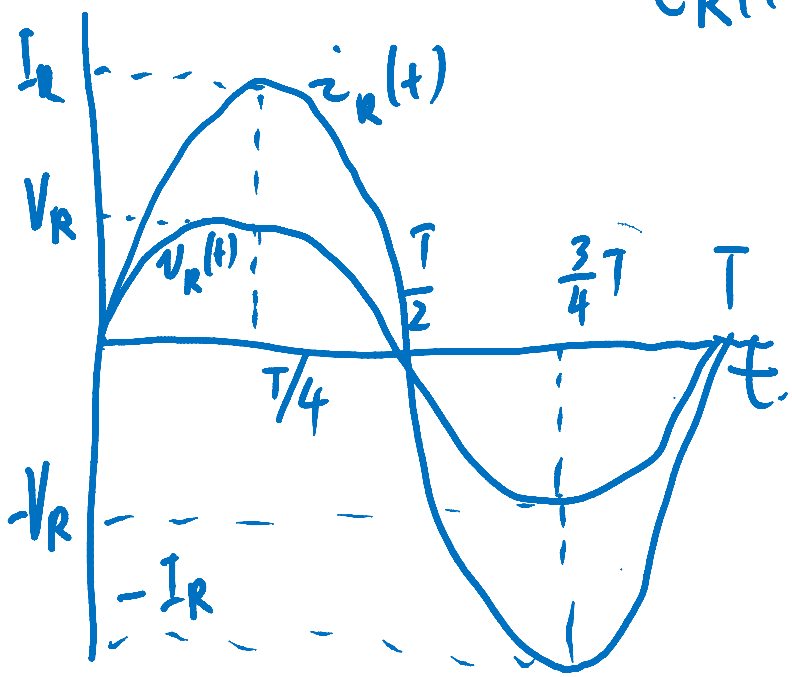
$v_R(t) = v_s(t) = A\sin wt$

$v_R(t) = A\cos(wt - \frac{\pi}{2})$

$\hookrightarrow 90^\circ$

$i_R(t) = \frac{v_R(t)}{R} = \frac{A}{R}\cos(wt - \frac{\pi}{2})$

$\varphi = -90^\circ = -\frac{\pi}{2}$ rad.




$v_R(t)$ and $i_R(t)$ are in phase.

Phasor representation

$$v_R(t) = A\cos(\omega t - 90^\circ)$$

$$V_R(j\omega) = A\angle -90^\circ = A\angle\left(-\frac{\pi}{2}\right)$$

$$i_R(t) = \frac{A}{R}\cos(\omega t - 90^\circ)$$

$$I_R(j\omega) = \frac{A}{R}\angle -90^\circ = \frac{A}{R}\angle\left(-\frac{\pi}{2}\right)$$

Impedance: phasor form of resistance.

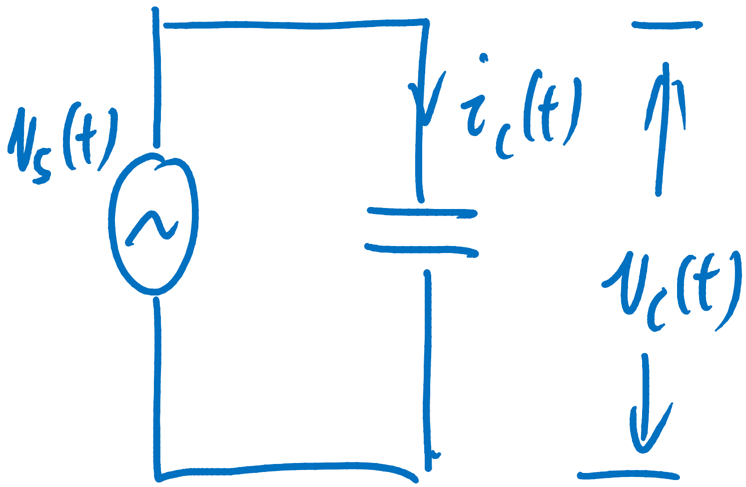
$$Z_R = \frac{V_R(j\omega)}{I_R(j\omega)} = \frac{A\angle -90^\circ}{\frac{A}{R}\angle -90^\circ} = R\angle 0^\circ = R$$

Generalized Ohm's law

$$V(j\omega) = Z\, I(j\omega)$$

$$Z = \frac{V(j\omega)}{I(j\omega)}$$

Capacitors.

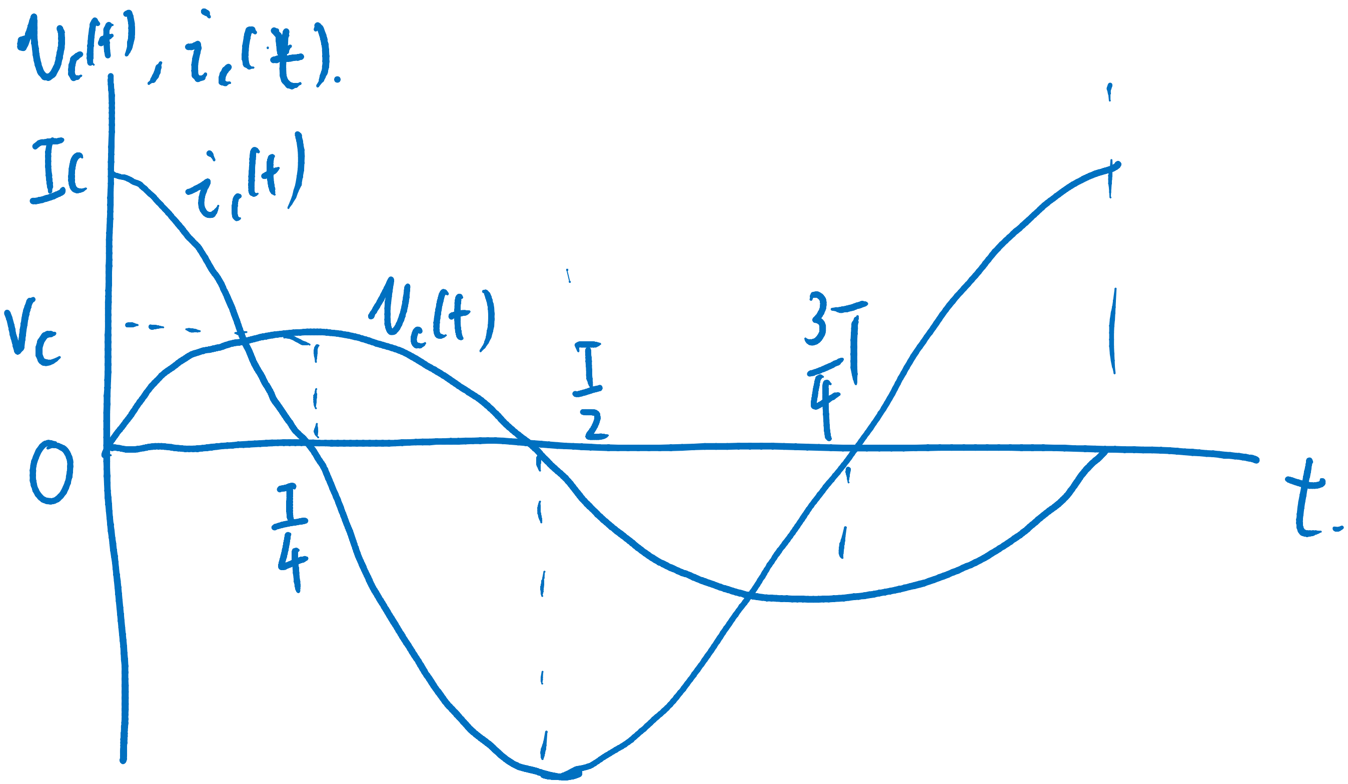



$$v_s(t) = A \sin \omega t$$

$$v_c(t) = v_s(t) = A \sin \omega t$$

$$i_c = C \frac{dv_c}{dt} = \omega C A \cos \omega t$$

$$v_c(t) = A \cos(\omega t - 90°)$$




current leads voltage by $\frac{\pi}{2}$ (90°)

use "ICE" to remember

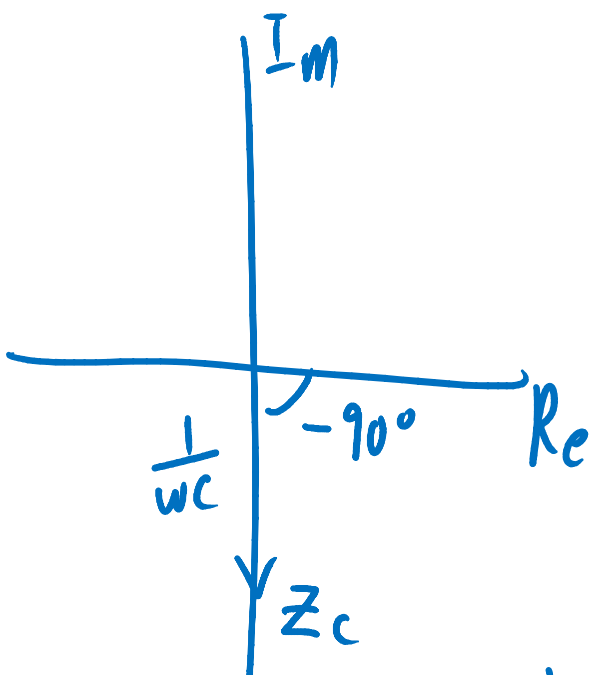
phasors

$$V_c(j\omega) = A\angle -90^\circ$$

$$I_c(j\omega) = \omega C A \angle 0^\circ$$

Impedance

$$Z_c = \frac{V_c(j\omega)}{I_c(j\omega)} = \frac{A\angle -90^\circ}{\omega C A\angle 0^\circ} = \frac{1}{\omega C}\angle -90^\circ$$




$$\frac{1}{j\omega C} = \frac{-jj}{j\omega c} = -\frac{j}{\omega C}$$

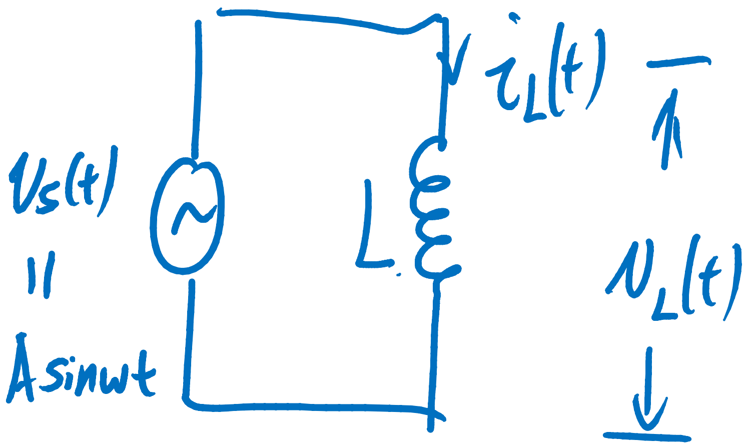
$$Z_c = \frac{1}{j\omega C} = \frac{1}{\omega C}\angle -90^\circ$$

$|Z_c| = \frac{1}{\omega C}$ $\quad \omega \uparrow \; |Z_c| \downarrow$

when $\omega = 0$ $\quad |Z_c| = \infty$

$\omega = \infty$ $\quad |Z_c| = 0$

# Inductive Load.




$$v_L(t) = v_s(t) = A\sin\omega t$$

$$\boxed{v_L(t) = A\cos(\omega t - 90^\circ)}$$

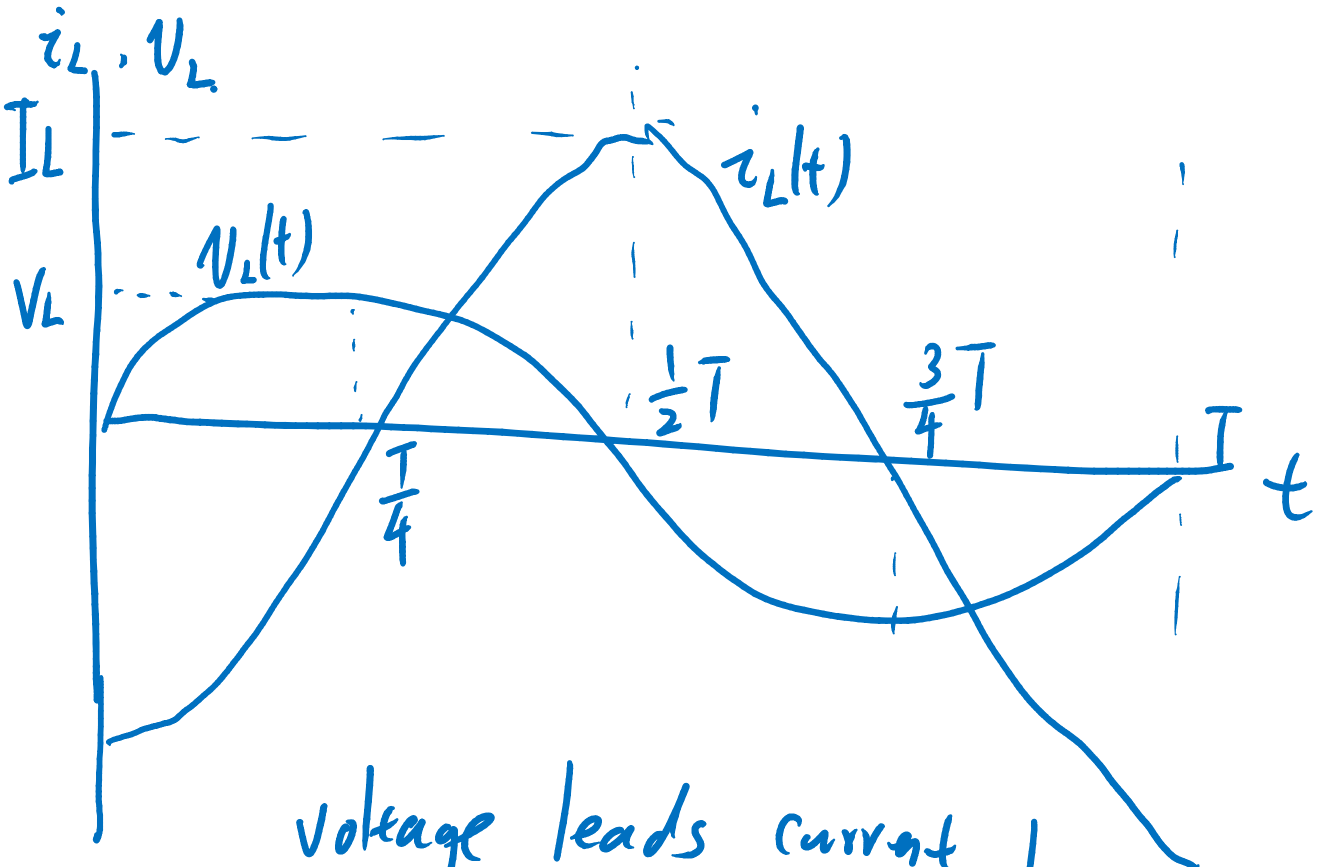
$$v_L = L\frac{di_L}{dt}$$

$$\frac{di_L}{dt} = \frac{v_L}{L} = \frac{A}{L}\sin\omega t$$

$$i_L = \int \frac{A}{L}\sin\omega t \, dt$$

$$= \frac{A}{L}\frac{(-\cos\omega t)}{\omega} = \frac{A}{L\omega}\cos(\omega t - 180^\circ)$$

$$\boxed{i_L(t) = \frac{A}{L\omega}\cos(\omega t - 180^\circ)}$$




voltage leads current by $\frac{\pi}{2}$ ($90°$)

remember it as "ELI"

E → voltage

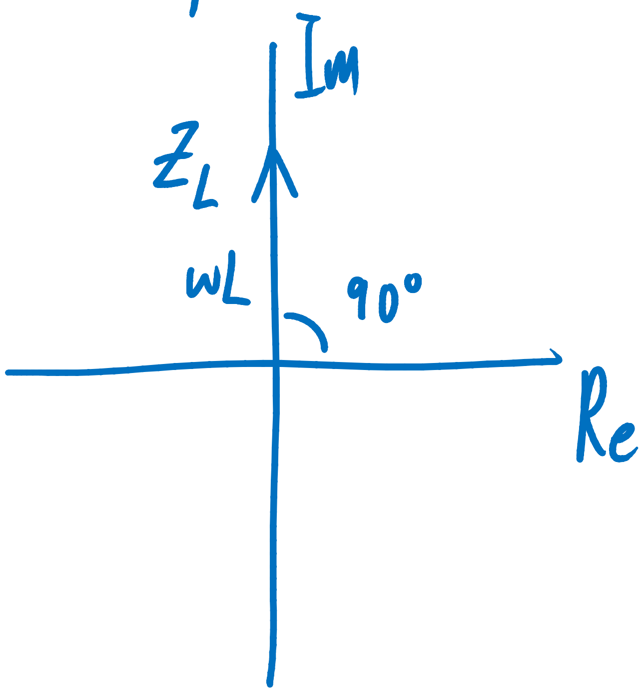
L → inductor

I → current

phasors $V_L(j\omega) = A\angle -90^\circ$

$$I_L(j\omega) = \frac{A}{\omega L}\angle -180^\circ$$

Impedance: $Z_L = \dfrac{V_L(j\omega)}{I_L(j\omega)} = \dfrac{A\angle -90^\circ}{\dfrac{A}{\omega L}\angle -180^\circ} = \omega L\angle 90^\circ$




$$Z_L = j\omega L$$

$|Z_L| = \omega L$ $\omega\uparrow$ $|Z_L|\uparrow$

when $\omega = 0$ $|Z_L| = 0$

$\omega = \infty$ $|Z_L| \to \infty$

* Impedance

$$Z_R = R \qquad Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} \qquad Z_L = j\omega L$$

$$V(j\omega) = I(j\omega)\, Z$$ generalized Ohm's law

- polar angle of $Z$ reflects phase difference between $V(j\omega)$ and $I(j\omega)$
- $Z_C$, $Z_L$ are related to $\omega$
- Everything we learnt from dc circuits can be used for phasors

* a.c. phasor current analysis procedure:

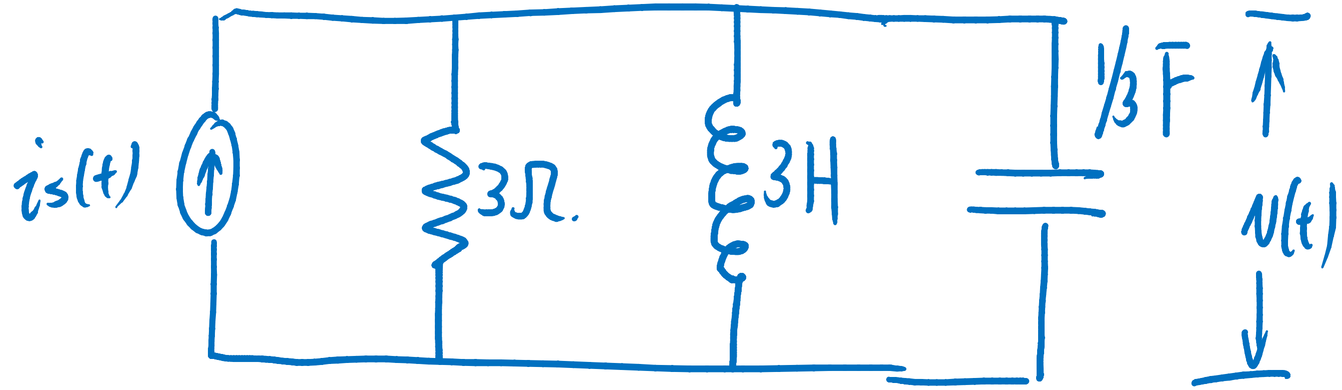
— Identify the sinusoidal and note the $\omega$

— Convert sources to phasor forms

— represent each circuit element by its impedance.

— Solve the phasor circuit using previously learnt methods (Combination rules, mesh, Kirchhoff laws, Thevenin & Norton)

## Examples




a.c. Source current $i_s(t) = 10\cos 2t$ (A)

$$I_s(j\omega) = 10\angle 0^\circ = 10e^{j0^\circ} \qquad \omega = 2 \text{ rad/s}$$

$$Z_R = R = 3\Omega \qquad Z_L = j\omega L = j\cdot 2\cdot 3 = 6j\ (\Omega)$$

$$Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} = -\frac{j}{2\cdot\frac{1}{3}} = -1.5j\ (\Omega)$$

$Z_{eq} = ?$

$$\frac{1}{Z_{eq}} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L}$$

$$Z_{eq} = \frac{1}{\left(\frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_C}\right)}$$

$$= \frac{1}{\left(\frac{1}{3} + \frac{1 \cdot j}{(-1.5j)j} + \frac{1 \cdot j}{6j \cdot j}\right)} = \frac{1}{\frac{1}{3} + \frac{j}{1.5} - \frac{1}{6}j}$$

$$= \frac{1}{0.333 + 0.6667j - 0.1667j} = \frac{1}{0.333 + 0.5j}$$

$$= \frac{1\angle 0^\circ}{0.6007\angle 56.3^\circ}$$

$$A = \sqrt{(0.333)^2 + (0.5)^2} = 0.6007$$

$$\theta = \tan^{-1}\left(\frac{0.5}{0.333}\right) = 56.3^\circ$$

$$Z_{eq} = 1.667\angle(-56.3^\circ)$$

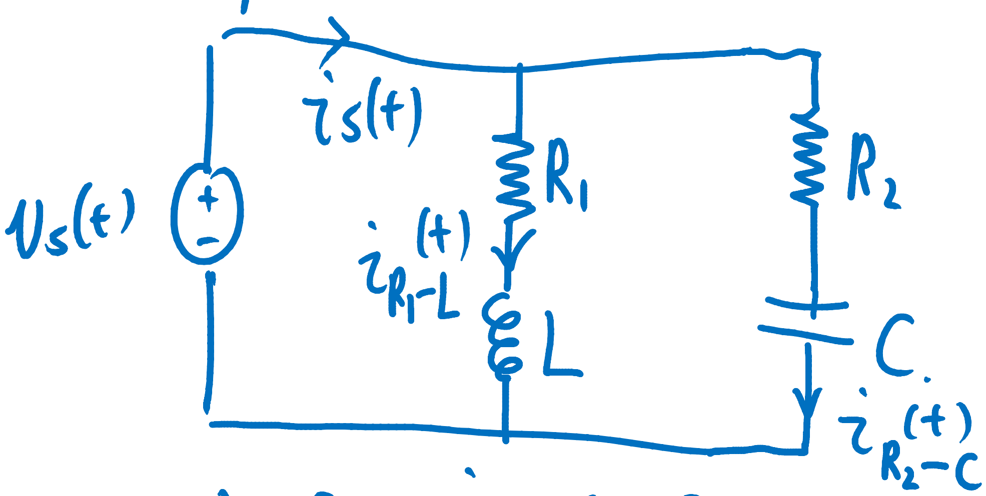
$$V(j\omega) = I(j\omega)\, Z_{eq}$$

$$= (10\angle 0^\circ)(1.667\angle -56.3^\circ)$$

$$V(j\omega) = 16.67\angle -56.3^\circ$$

$$v(t) = 16.67\cos(2t - 56.3^\circ) \quad (V)$$

**Example** Determine the current in each branch.




$v_s(t) = 636\cos\left(3000t + \frac{\pi}{12}\right) V$

$R_1 = 3.3k\Omega \quad R_2 = 22k\Omega \quad \frac{\pi}{12} = 15°$

$L = 1.9H \quad C = 6.8nF$

$= 6.8 \times 10^{-9} F$

$i_s(t) = ? \quad i_{R_1-L}(t) = ? \quad i_{R_2-C}(t) = ?$

$$V_s(j\omega) = 636 \angle 15°$$

$Z_{R_1} = 3300\Omega \quad Z_{R_2} = 22000\Omega$

$Z_L = j\omega L = j(3000 \times 1.9) = 5700j\ (\Omega)$

$$Z_C = -\frac{j}{\omega C} = -\frac{j}{(3000)\times(6.8\times10^{-9})} = -4.9\times10^4 j\ (\Omega)$$

$$Z_{eq} = (Z_C + Z_{R_2}) \,//\, (Z_L + Z_{R_1})$$

$$= \frac{(Z_C + Z_{R_2})(Z_L + Z_{R_1})}{Z_C + Z_{R_2} + Z_L + Z_{R_1}}$$

$$= \frac{(22000 - 4.9\times10^4 j)(5700j + 3300)}{22000 + 3000 - 4.9\times10^4 j + 5700j}$$

$$= \frac{(53712\angle -65.8^\circ)(6586\angle 59.9^\circ)}{(50149\angle -59.7^\circ)}$$

$$Z_{eq} = 7054\angle 53.8^\circ \ (\Omega)$$

$$I_s(j\omega) = \frac{V_s(j\omega)}{Z_{eq}} = \frac{636\angle 15^\circ}{7054\angle 53.8^\circ} = 0.090\angle -38.8^\circ$$

$$i_s(t) = 0.090 \cos(3000t - 38.8^\circ) \ (A)$$

$$i_{R_1-L}(t) = ?$$

$$I_{R_1-L}(j\omega) = \frac{V_s(j\omega)}{Z_L + Z_{R_1}} = \frac{636\angle 15^\circ}{6586\angle 59.9^\circ} = 0.096\angle -44.9^\circ$$

$$i_{R_1-L}(t) = 0.096 \cos(3000t - 44.9^\circ)$$

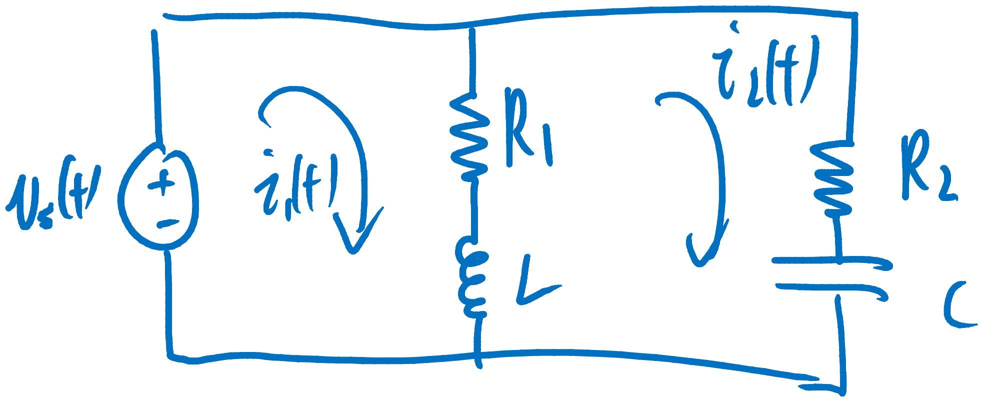
$$i_{R_2-C} = ?$$

$$I_{R_2-C}(j\omega) = \frac{V_s(j\omega)}{Z_C + Z_{R_2}} = \frac{636\angle 15^\circ}{53712\angle -65.8^\circ} = 0.012\angle 80.8^\circ$$

$$i_{R_2-C}(t) = 0.012 \cos(3000t + 80.8^\circ)$$

Also try current divider rule as exercise.

Same circuit. use mesh analysis.




$V_s(j\omega) = 636 \angle 15^\circ$

$I_1(j\omega) = ?$ $\quad I_2(j\omega) = ?$

$$(Z_{R_1} + Z_L) I_1(j\omega) - (Z_{R_1} + Z_L) I_2(j\omega) = V_s(j\omega) \quad (1)$$

$$(Z_{R_1} + Z_L + Z_{R_2} + Z_C) I_2(j\omega) - (Z_{R_1} + Z_L) I_1(j\omega) = 0 \quad (2)$$

$(1)+(2)$
$$(Z_{R_2} + Z_C) I_2(j\omega) = V_s(j\omega)$$

$$I_2(j\omega) = \frac{V_s(j\omega)}{Z_{R_2} + Z_C} = \frac{636 \angle 15^\circ}{53712 \angle -65.8^\circ} = 0.012 \angle 80.8^\circ$$

$$i_2(t) = 0.012 \cos(3000t + 80.8^\circ) = i_{R_2-C}(t)$$

$$I_1(j\omega) = \frac{V_s(j\omega)\,(Z_{R_1} + Z_L + Z_{R_2} + Z_C)}{(Z_{R_2} + Z_C)\,(Z_{R_1} + Z_L)}$$

$$= \frac{(636\angle 15^\circ)\,(50149\angle -59.7^\circ)}{(53712\angle -65.8^\circ)\,(6586\angle 59.9^\circ)}$$

$$= 0.090\angle -38.8^\circ$$

$$i_1(t) = 0.090\cos(3000t - 38.8^\circ) = i_s(t)$$

$$I_{R_1-L}(j\omega) = I_1(j\omega) - I_2(j\omega) = 0.090\angle -38.8^\circ - 0.012\angle 80.8^\circ$$

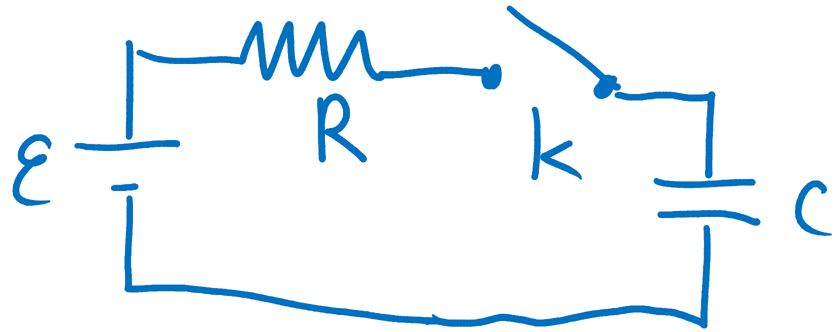
$$= (0.070 - 0.056j) - (0.0019 + 0.00118j)$$

$$= 0.068 - 0.068j = 0.096\angle -45^\circ$$

$$i_{R_1-L} = 0.096\cos(3000t - 45^\circ)$$

# * Transient Circuits

— transient: transition region between two steady states.




initially no charge on C

$$V_C = \frac{Q_C}{C} = 0$$

$t < 0$ Switch is open

at $t = 0$ switch is closed,

what happens at $t = 0^+$ ?

what happens at $t = \infty$ ? (steady state)

what happens in between $t = 0^+$ and $t = \infty$ ?

$$i_c = \frac{dq_c}{dt} = C\frac{dv_c}{dt}$$

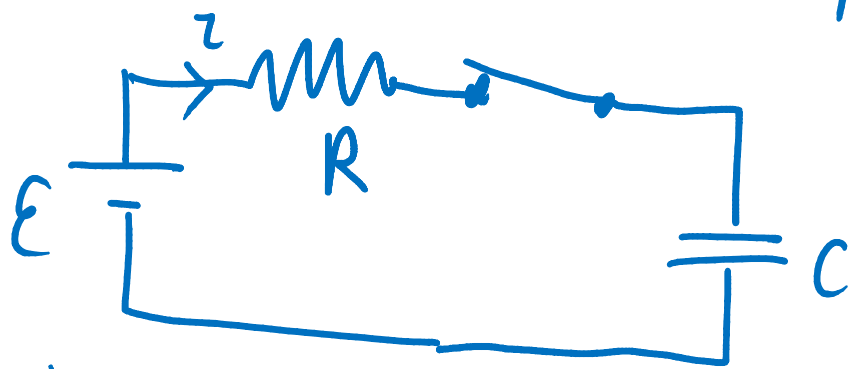
if $v_c$ suddenly changes

$$i_c = C\frac{dv_c}{dt} = \infty$$

(unphysical)

No sudden jump of voltage on capacitor

At $t = 0^+$ what happens




KVL

$$\mathcal{E} - iR - V_c = 0$$

~~At~~ At $t < 0$ $V_c = 0$

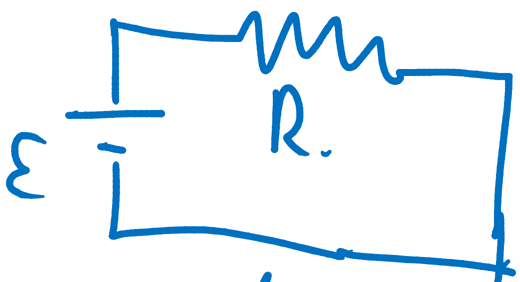
At $t = 0^+$ $V_c = 0$ (no sudden jump)

$$\mathcal{E} - iR = 0$$

$$i(t = 0^+) = \frac{\mathcal{E}}{R}$$

$i\big|_{t=0^+} = \frac{\varepsilon}{R}$ at $t=0^+$

At $\boxed{t=0^+}$ the capacitor is equivalent to a piece of wire.



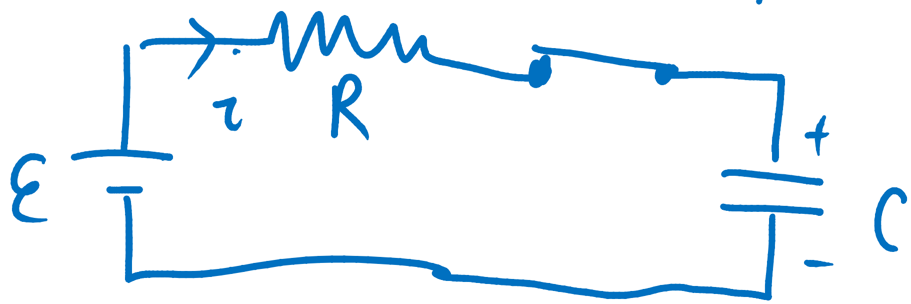

$i = \frac{\varepsilon}{R}$

From impedance perspective:

$Z_c = \frac{1}{j\omega C}$ $\quad |Z_c| = \frac{1}{\omega C}$

sudden change $\Leftrightarrow$ $\omega = \infty$ $\quad |Z_c| = 0$ only true for $t=0^+$

At $t=\infty$ what happens?




At $t=\infty$ C is fully charged.

$q_c = \text{const}$

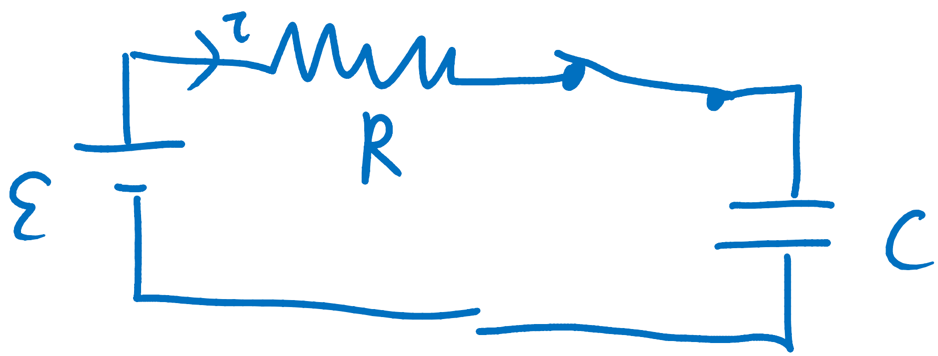
$i = \frac{dq_c}{dt} = 0 \quad (t=\infty)$

KVL $\quad \varepsilon - \underset{\overset{\parallel}{0}}{i}R - V_c = 0$

$V_c = \varepsilon$ at $t=\infty$

$Q_c = C \cdot V_c = C\varepsilon$

between $t=0^+$ and $t=\infty$




KVL $\quad \varepsilon - iR - V_c = 0$

$$i = \frac{\varepsilon - V_c}{R} \quad (1)$$

$$i = C\frac{dV_c}{dt} \quad (2)$$

$$\frac{\varepsilon - V_c}{R} = C\frac{dV_c}{dt} \qquad V_c(t) = ?$$

$$\frac{dV_c}{\varepsilon - V_c} = \frac{dt}{RC} \qquad \frac{d(\varepsilon - V_c)}{\varepsilon - V_c} = -\frac{dt}{RC}$$

$$\ln(\varepsilon - v_c) = -\frac{t}{RC} + A$$

$$\varepsilon - v_c = e^{A} \cdot e^{-t/RC} \qquad \text{Let } e^{A} = B$$

$$\varepsilon - v_c = B \cdot e^{-t/RC}$$

At $t = 0^{+}$ $\quad v_c = 0$ , $\quad B = \varepsilon$

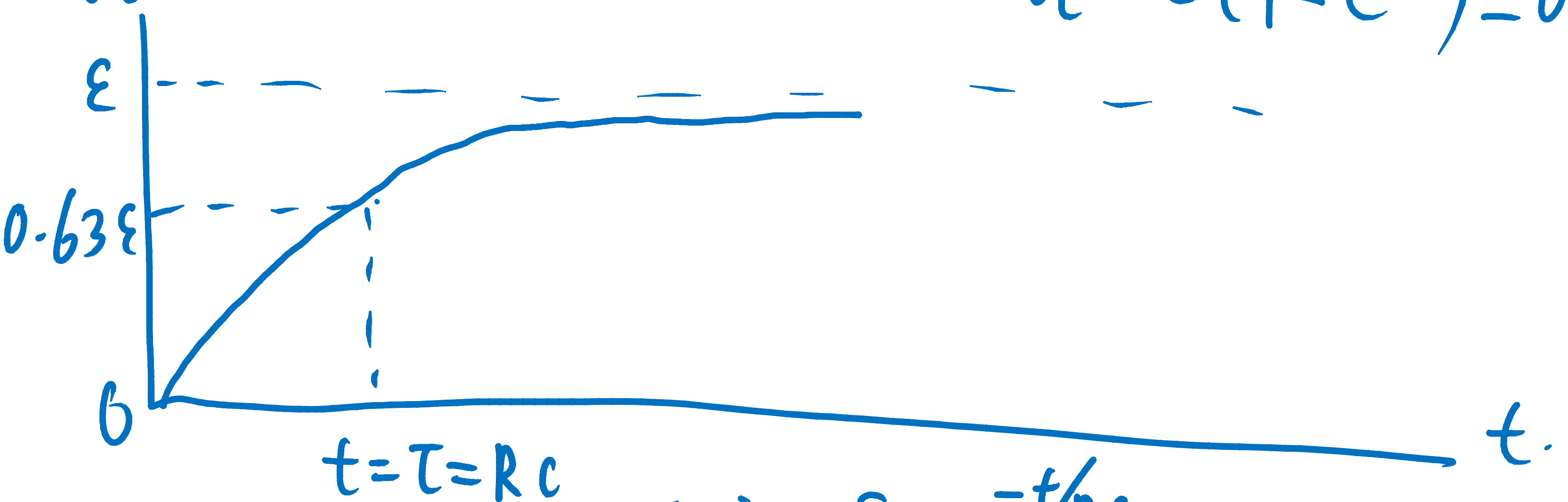
$$\varepsilon - v_c = \varepsilon\, e^{-t/RC}$$

$$v_c = \varepsilon\left(1 - e^{-t/RC}\right)$$

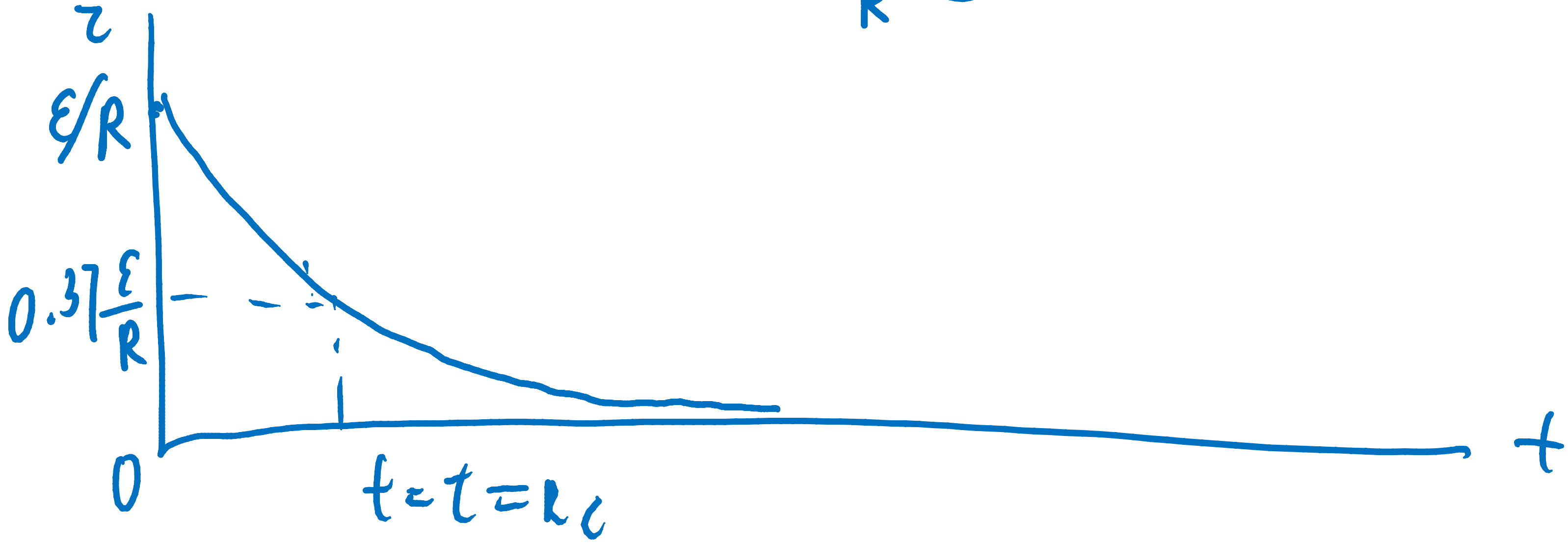
$$i_c = C\frac{dv_c}{dt} = \frac{\varepsilon}{R}\, e^{-t/RC}$$

$$v_c = \varepsilon\left(1 - e^{-t/RC}\right) \qquad t = \tau = RC$$

$$v_c = \varepsilon\left(1 - e^{-1}\right) = 0.63\,\varepsilon$$




$$i = \frac{\varepsilon}{R} e^{-t/RC}$$

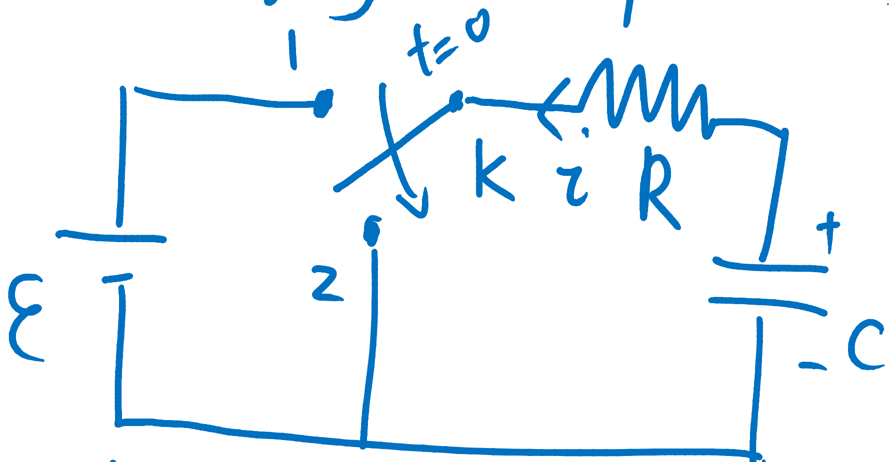


— Time constant $\tau = RC$
time to charge a capacitor to 63% of its full charge capacity

— The larger the $\tau = RC$, the longer it takes to charge the capacitor.

( large R means smaller charging current
larger C ~~means~~ more charge )

Discharging a capacitor.




Switch to 1.

charging $V_C = \varepsilon$

($t<0$, $v_C = \varepsilon$)

At $t=0$, switch to 2 : discharging

at $t=0^+$ (right after switching to 2)

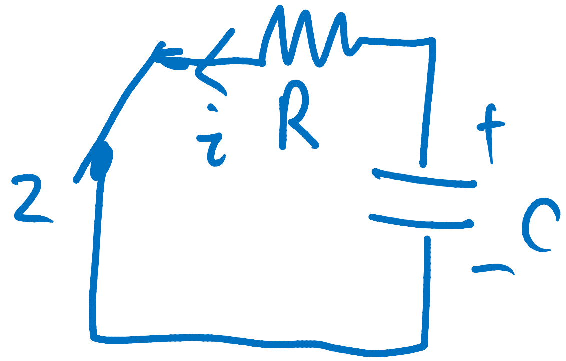
$v_c = \varepsilon$ (no sudden jump of V on C)

KVL $v_c - iR = 0$

$$i\big|_{t=0^+} = \frac{v_c}{R} = \frac{\varepsilon}{R}$$

At $t=\infty$ $i\big|_{t=\infty} = 0$ $v_c\big|_{t=\infty} = 0$ completely discharged

Between $t = 0^+$ and $t = \infty$




$v_c(t)$

$$v_c - iR = 0$$

$$i = \frac{v_c}{R} \quad (1)$$

$$i = -\frac{dq_c}{dt} = -C\frac{dv_c}{dt} \quad (2)$$

$$-C\frac{dv_c}{dt} = \frac{v_c}{R}$$

$$\frac{dv_c}{v_c} = -\frac{dt}{RC}$$

$$\ln v_c = -\frac{t}{RC} + A$$

$$v_c = e^{-t/RC} \cdot e^{A}$$

Let $e^{A} = B$

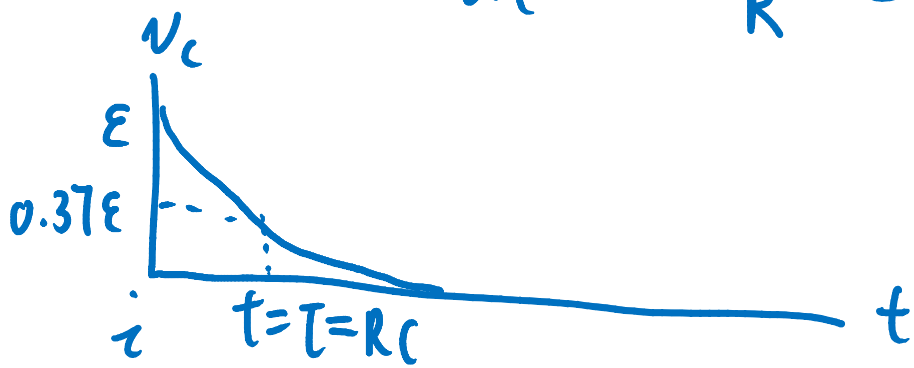
$$v_c = B e^{-t/RC}$$

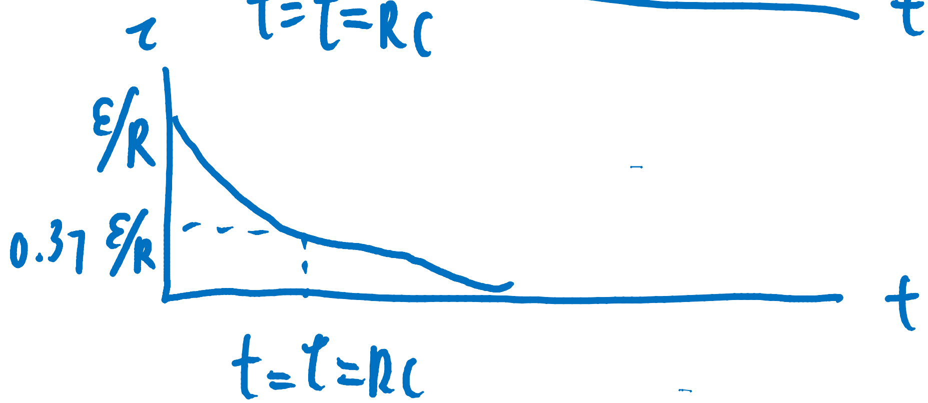
At $t = 0^+$ $v_c = \mathcal{E}$

$B = \mathcal{E}$

$$v_c(t) = \mathcal{E} \cdot e^{-t/RC}$$

$$i = -C\frac{dv_c}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC}$$

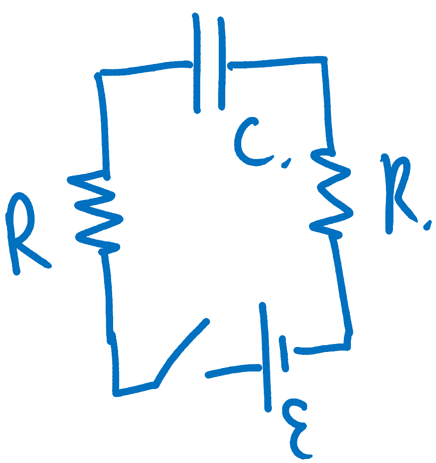


General form of transient variables in circuits

$$x(t) = x(\infty) + \left(x(0^+) - x(\infty)\right)e^{-t/\tau}$$

Example




C initially uncharged.

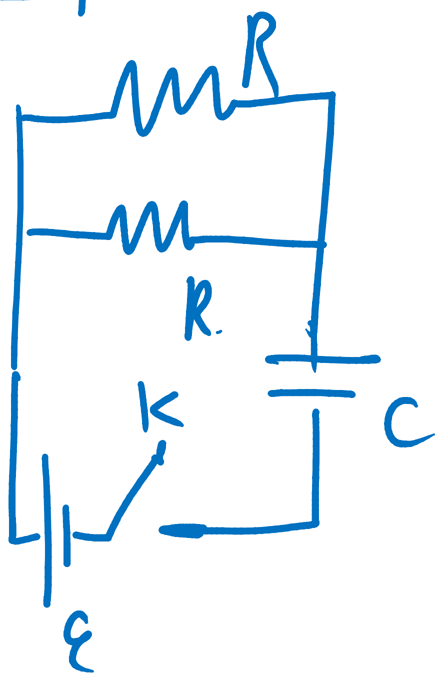
$$i\big|_{t=0^+} = \frac{\varepsilon}{2R}. \qquad v_c\big|_{t=0^+} = 0$$

$$i\big|_{t=\infty} = 0 \qquad v_c\big|_{t=\infty} = \varepsilon$$

$$\tau = 2RC$$

$$i = \frac{\varepsilon}{2R} e^{-t/2RC} \qquad v_c = \varepsilon\left(1 - e^{-t/2RC}\right)$$

## Example

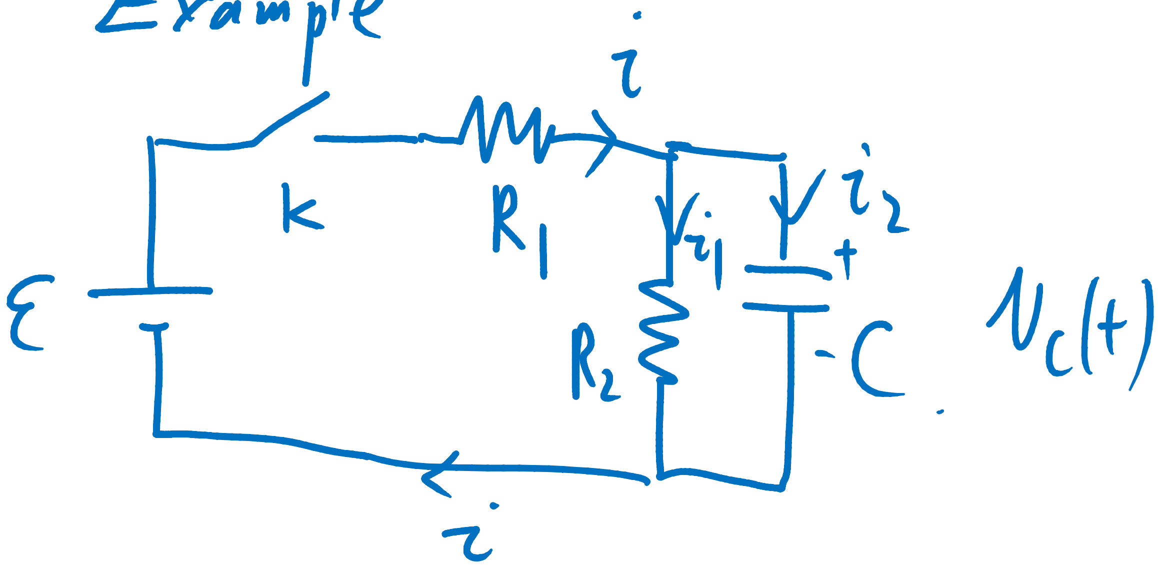



$$i\Big|_{t=0^+} = \frac{\varepsilon}{\left(\frac{R}{2}\right)}$$

time constant

$$\tau = \frac{R}{2} C$$

## Example




$$
\begin{cases}
i = i_1 + i_2 & (1) \\
i_2 = C\dfrac{dv_c}{dt} & (2) \\
\varepsilon - iR_1 - i_1 R_2 = 0 & (3) \\
i_1 R_2 = v_c & (4)
\end{cases}
$$

(1) in to (3)

$$\varepsilon - (i_1 + i_2)R_1 - i_1 R_2 = 0 \quad (5)$$

(2) (4) into (5)

$$\mathcal{E} - \frac{v_c}{R_2} R_1 - C \frac{dv_c}{dt} R_1 - v_c = 0$$

$$-CR_1 \frac{dv_c}{dt} - \frac{R_1+R_2}{R_2} v_c + \mathcal{E} = 0$$

$$CR_1 \frac{dv_c}{dt} = -\frac{R_1+R_2}{R_2} v_c + \mathcal{E}$$

$$\frac{dv_c}{-\frac{R_1+R_2}{R_2} v_c + \mathcal{E}} = \frac{dt}{CR_1}$$

$$\frac{d\left(-\frac{R_1+R_2}{R_2} v_c + \mathcal{E}\right)}{-\frac{R_1+R_2}{R_2} v_c + \mathcal{E}} = -\frac{dt}{C\left(\frac{R_1 R_2}{R_1+R_2}\right)}$$

$$\ln\left[-\frac{R_1+R_2}{R_2} v_c + \mathcal{E}\right) = -\frac{t}{C\left(\frac{R_1 R_2}{R_1+R_2}\right)} + A$$

Let $B = e^{A}$

$$\mathcal{E} - \frac{R_1+R_2}{R_2} v_c = B e^{-\frac{t}{C\left(\frac{R_1R_2}{R_1+R_2}\right)}}$$

$t = 0^+ \qquad v_c = 0$

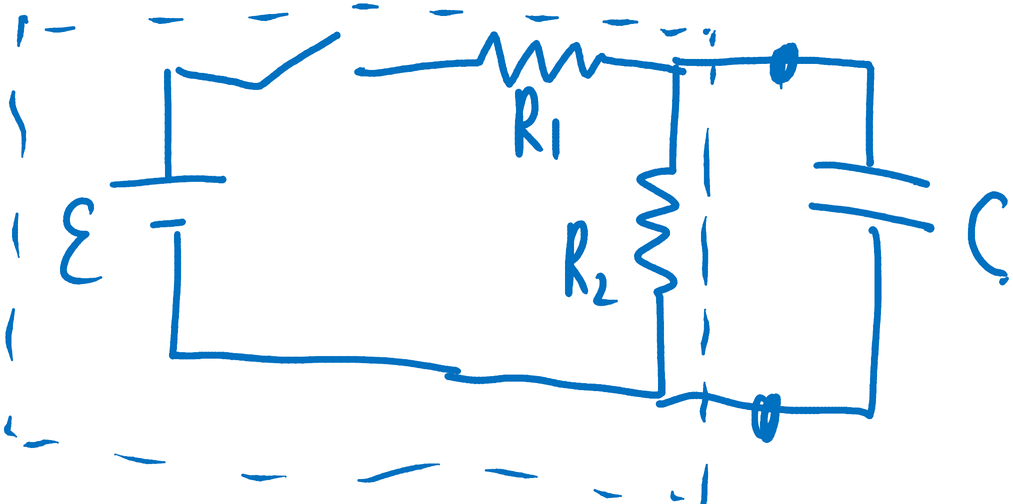
$$B = \mathcal{E}$$

$$v_c(t) = \frac{R_2}{R_1+R_2}\mathcal{E}\left(1 - e^{-\frac{t}{C\left(\frac{R_1R_2}{R_1+R_2}\right)}}\right)$$

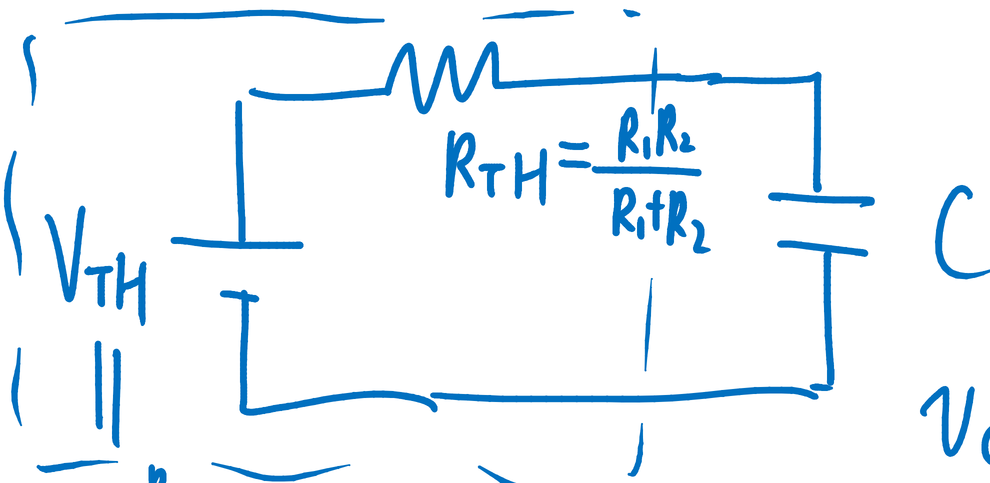
$t = 0^+ \qquad v_c = 0$

$t = \infty \qquad v_c = \frac{R_2}{R_1+R_2}\mathcal{E}$

$$\tau = \left(\frac{R_1R_2}{R_1+R_2}\right)C$$




Thevenin equivalent.



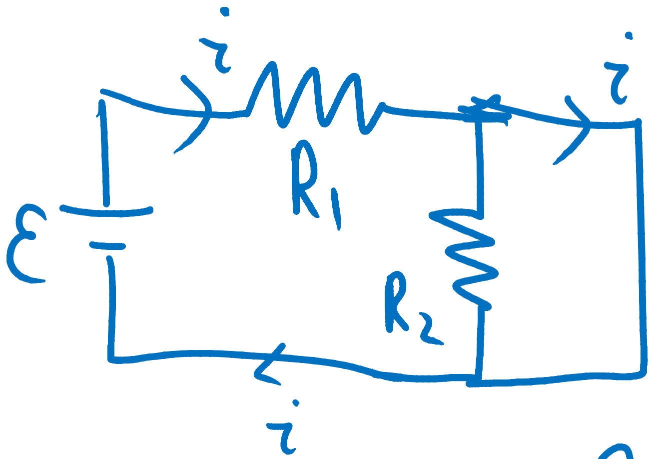

$$\parallel$$

$$\varepsilon \cdot \frac{R_2}{R_1+R_2}$$

$$V_C = V_T\left(1 - e^{-\frac{t}{R_T C}}\right)$$

$$V_C = \varepsilon \frac{R_2}{R_1+R_2}\left(1 - e^{-\frac{t}{C\left(\frac{R_1 R_2}{R_1+R_2}\right)}}\right)$$

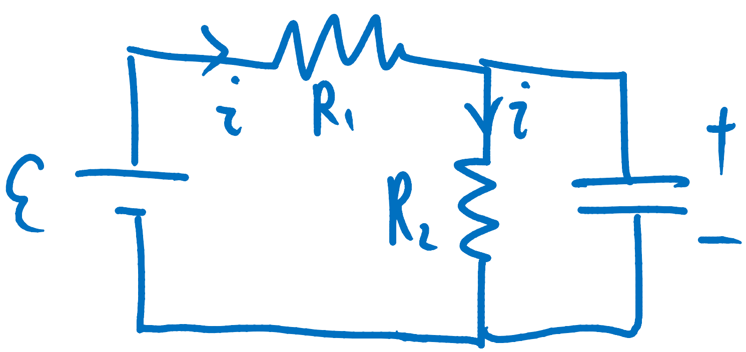
At $t=0^+$ and $t=\infty$ What is the current through the battery $\mathcal{E}$




$t=0^+$

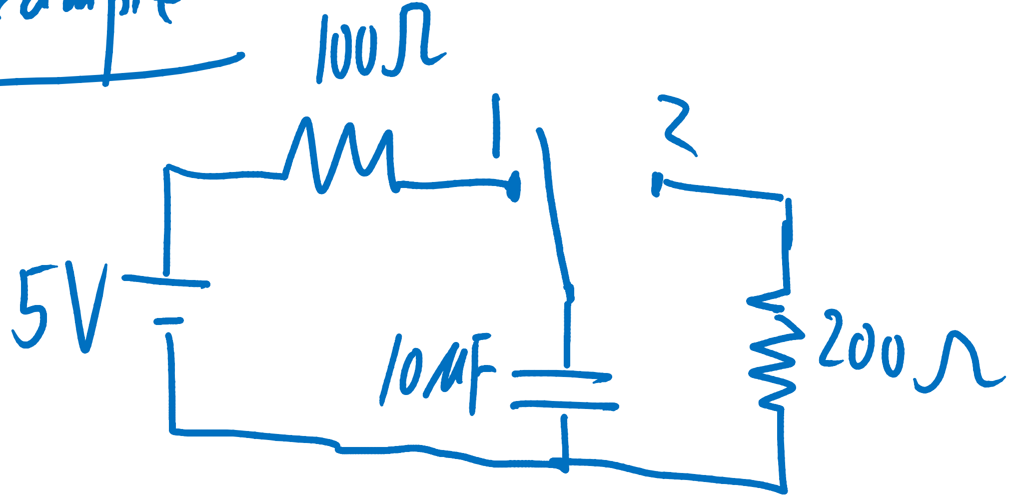
$$i\Big|_{t=0^+} = \frac{\mathcal{E}}{R_1}$$

At $t=\infty$




$$i\Big|_{t=\infty} = \frac{\mathcal{E}}{R_1+R_2}$$

Example




switch to 1 charging: $\tau_1 = R_1 C$

$$= (100\,\Omega) \times (10 \times 10^{-6} F)$$

$$= 1 \times 10^{-3} (s) = 1 ms$$

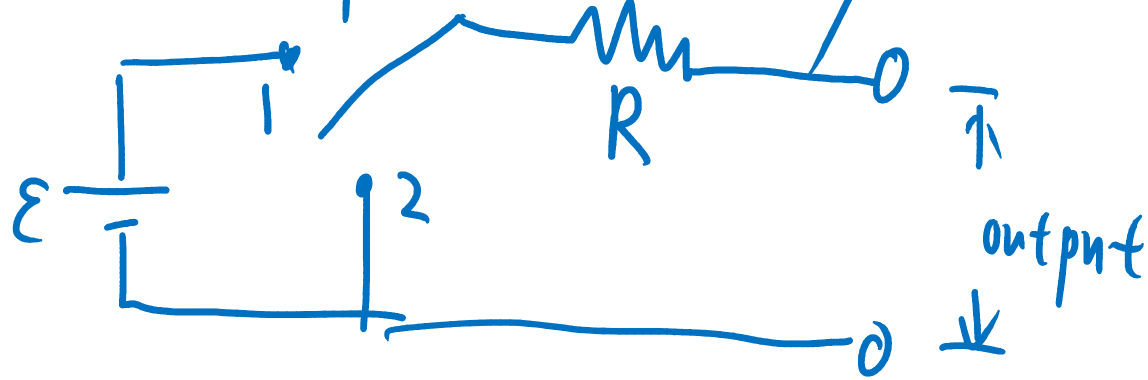
switch to 2 discharging: $\tau_2 = R_2 C$

$$= 200 \times 10 \times 10^{-6}$$

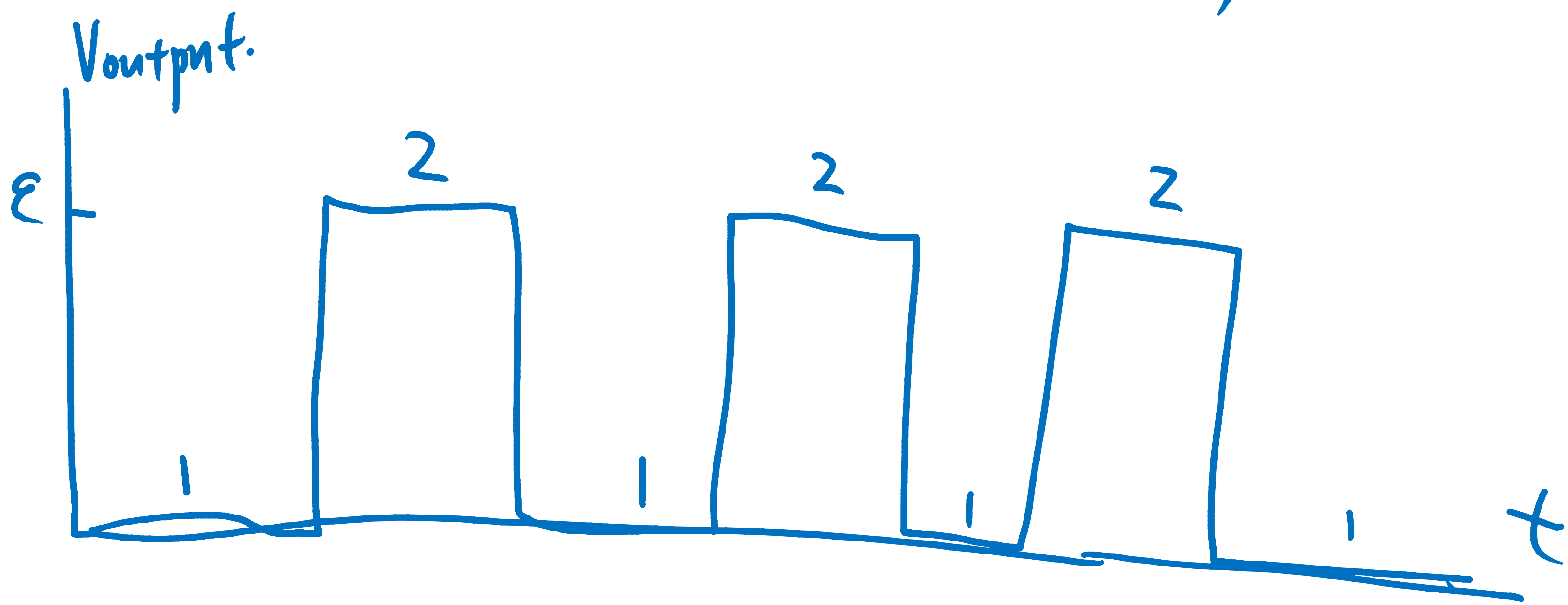
$$= 2 ms$$

RC circuits provide a delay

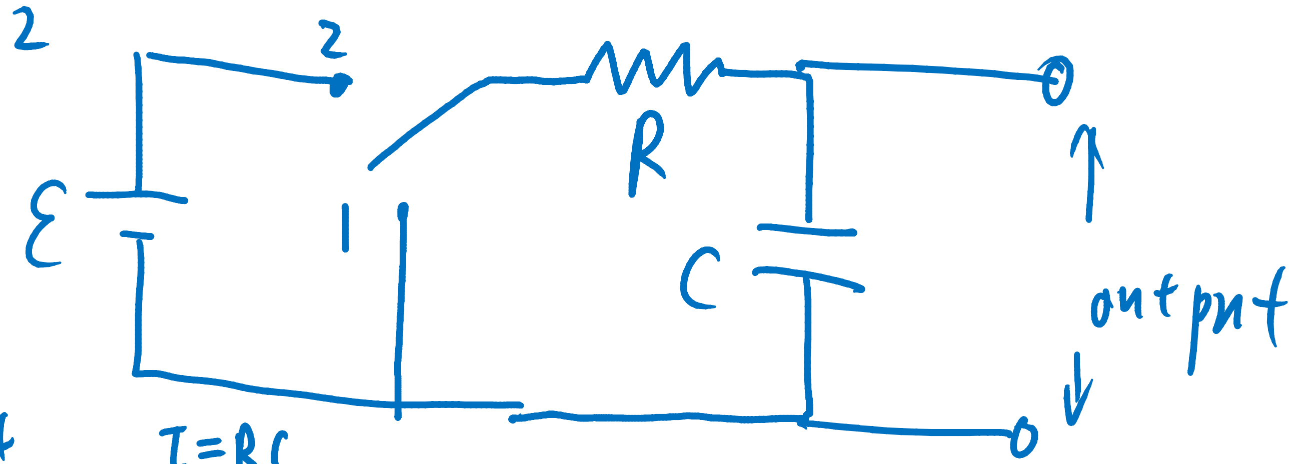
Circuit #1

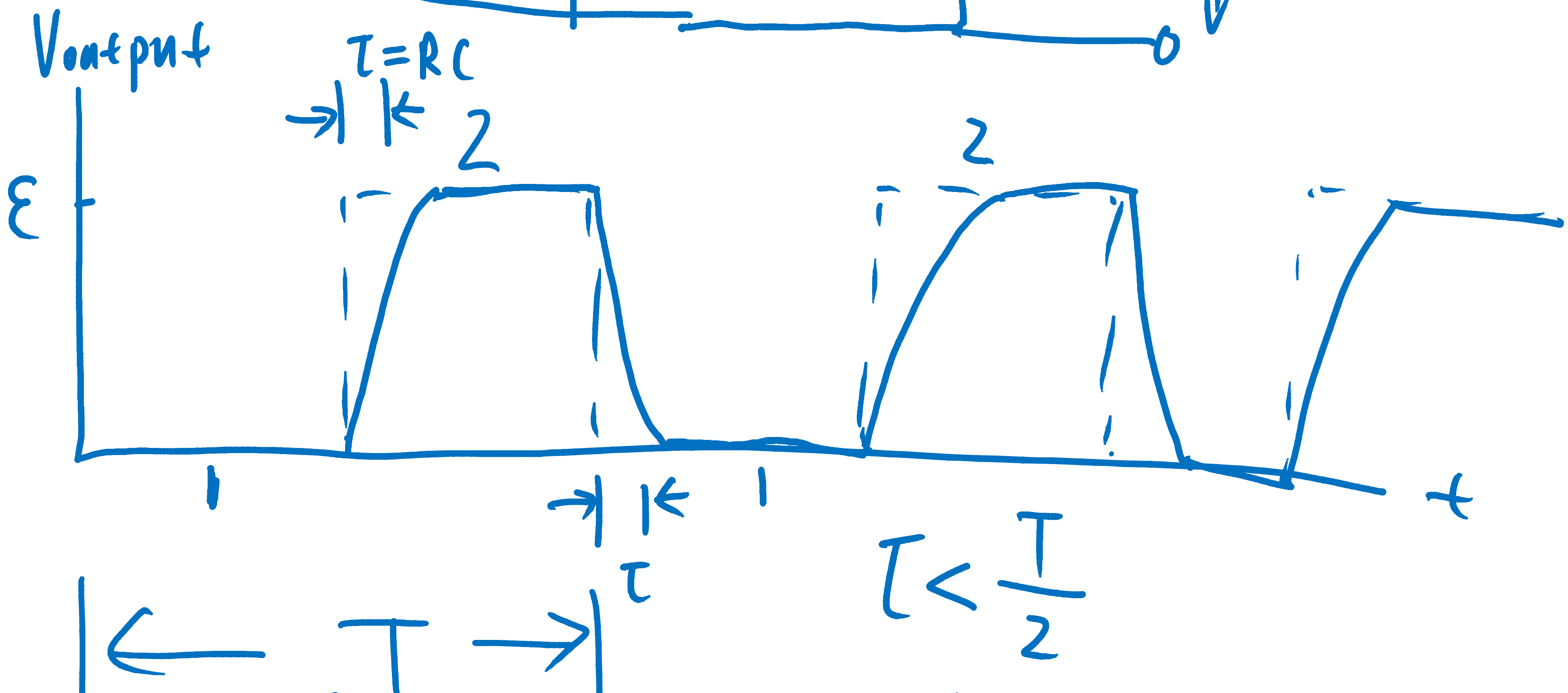



toggles between 1 and 2 periodically



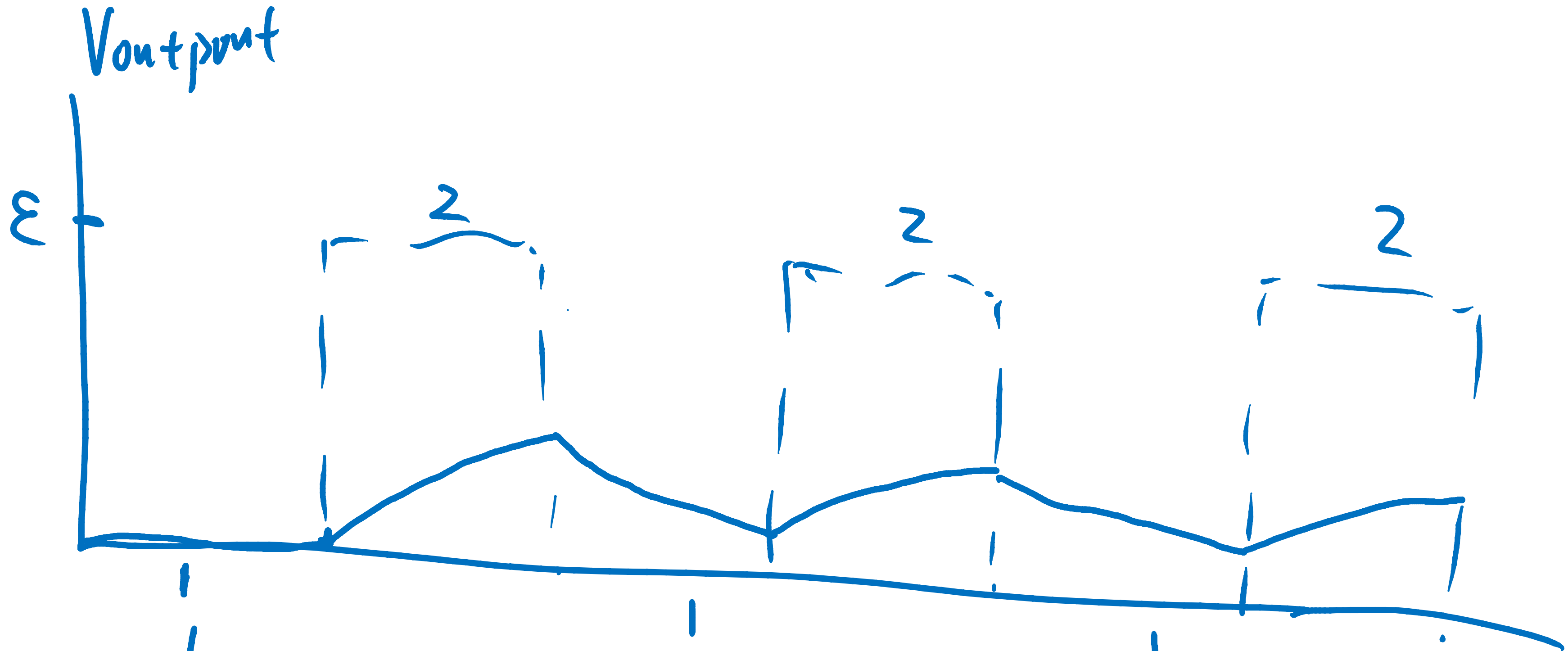
Circuit # 2



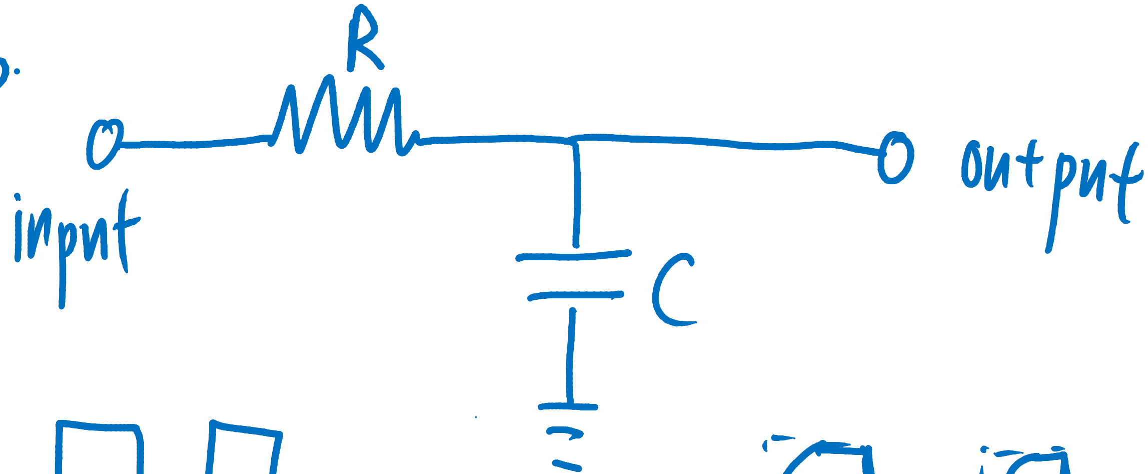

Response is delayed in RC circuit. Because there is no abrupt jump of voltage on capacitor.

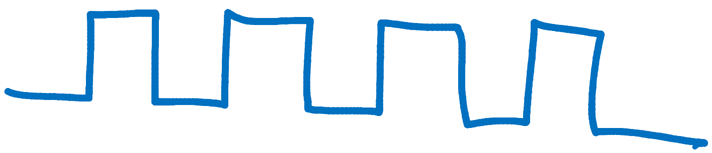
Imagine RC value increases so that $\tau = RC > \frac{T}{2}$

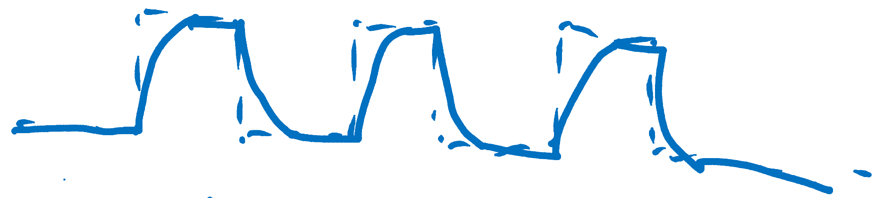



The square wave is "smoothed out"

In Lab.



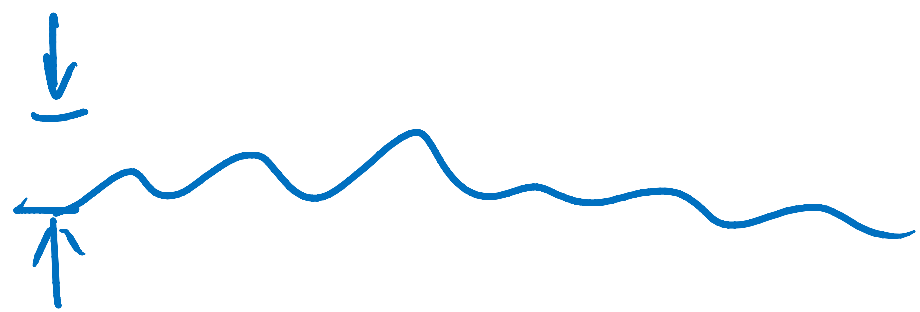





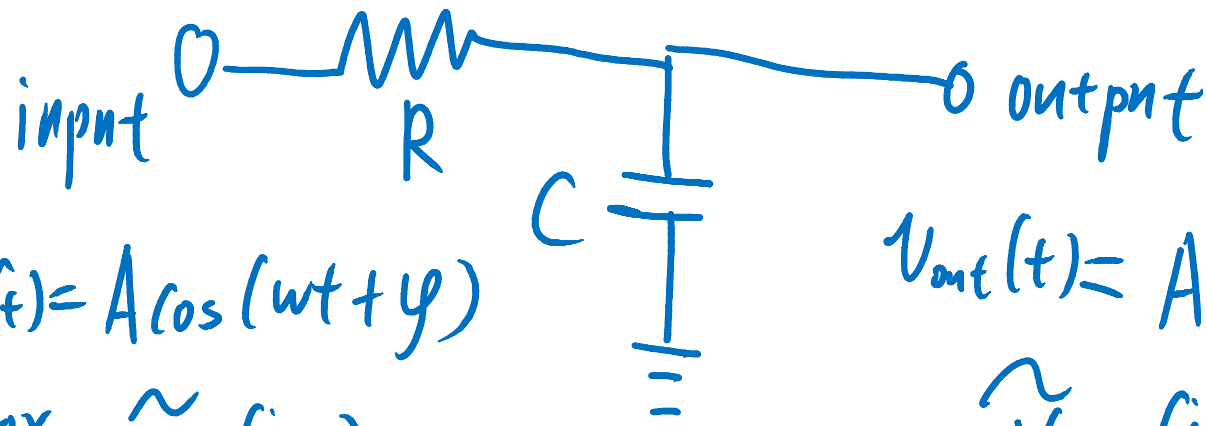
RC circuit can be used to smooth out "voltage spikes"

input :

output:



RC circuits can be used as low pass filter




$v_{in}(t) = A\cos(wt+\varphi)$ $\qquad$ $v_{out}(t) = A'\cos(wt+\varphi')$

phasor $\tilde{V}_{in}(jw)$ $\qquad$ $\tilde{V}_{out}(jw)$

$$\tilde{V}_{out}(jw) = \tilde{V}_{in}(jw)\cdot\frac{Z_C}{Z_C+Z_R} = \tilde{V}_{in}(jw)\frac{1}{1+Z_R/Z_C}$$

$Z_R = R$ $\qquad$ $Z_C = \frac{1}{jwC}$

$w\downarrow$ $\quad |Z_C|\uparrow$ $\quad |\tilde{V}_{out}|\uparrow$ $\qquad$ $w\to 0$ $\quad |\tilde{V}_{out}| = |\tilde{V}_{in}|$

$w\uparrow$ $\quad |Z_C|\downarrow$ $\quad |\tilde{V}_{out}|\downarrow$ $\qquad$ $w\to\infty$ $\quad |\tilde{V}_{out}| = 0$

RL circuits

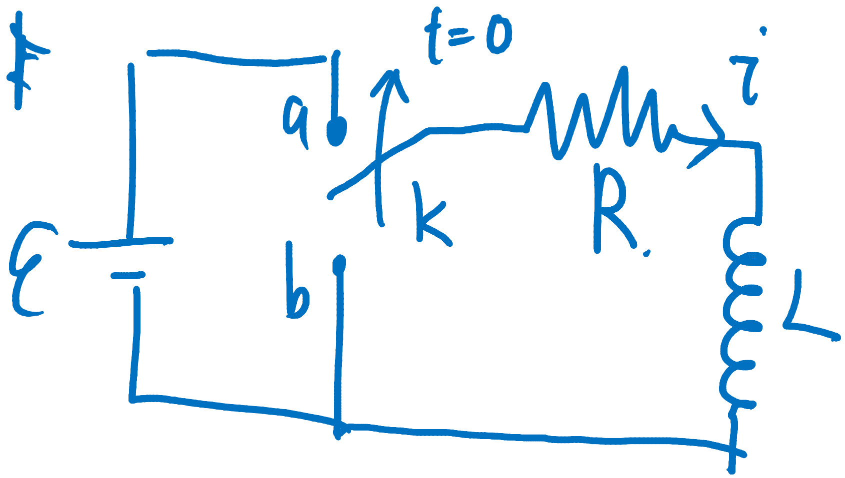
$$v_L = L\frac{di}{dt}$$

if $i$ changes abruptly $\frac{di}{dt} \to \infty$

then $v_L = \infty$ (physically impossible)

KVL can not be satisfied.

So there should be no abrupt change of current in $L$.




At t=0 switch to a.

charge the inductor

$t<0 \quad i=0$

$v_L = 0$

at $t=0^+ \quad i=? \quad v_L=?$

$$i\big|_{t=0^+} = i\big|_{t<0} = 0$$

KVL $\quad \varepsilon - \underbrace{iR}_{=0} - v_L = 0$

$$v_L\big|_{t=0^+} = \varepsilon$$

The initially uncharged inductor behaves like an open circuit at $t=0^+$

At $t=\infty$ $\quad i=?\quad v_L=?$

eventually circuit stablizes $\quad i=\text{const}$

$$v_L = L\frac{di}{dt} = 0$$

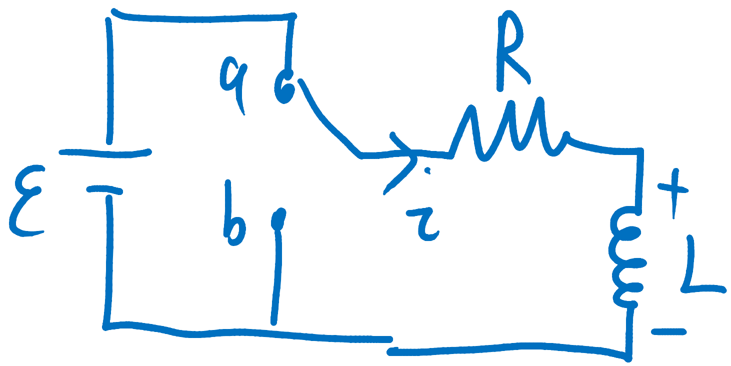
KVL $\quad \varepsilon - iR - \underset{\overset{\parallel}{0}}{v_L} = 0$

$$\varepsilon = iR$$

$$i\Big|_{t=\infty} = \frac{\varepsilon}{R} \qquad v_L\Big|_{t=\infty} = 0$$

At $t=\infty$, inductor behaves like a short circuit

What happens between $t=0^+$ and $t=\infty$




KVL $\varepsilon - iR - v_L = 0$

$$v_L = \varepsilon - iR$$

$$v_L = L\frac{di}{dt}$$

$$L\frac{di}{dt} = \varepsilon - iR$$

$$\frac{di}{\varepsilon - iR} = \frac{dt}{L}$$

$$\frac{d(\varepsilon - iR)}{\varepsilon - iR} = -\frac{dt}{\left(\frac{L}{R}\right)}$$

$$\ln(\varepsilon - iR) = -\frac{t}{\left(\frac{L}{R}\right)} + A$$

$$\varepsilon - iR = e^{-t/(\frac{L}{R})} \cdot e^{A}$$

Let $e^{A} = B$

$$\varepsilon - iR = B \cdot e^{-t/(\frac{L}{R})}$$

$i = 0$ at $t = 0^{+}$

$$B = \varepsilon$$

$$\varepsilon - iR = \varepsilon \cdot e^{-t/(\frac{L}{R})}$$

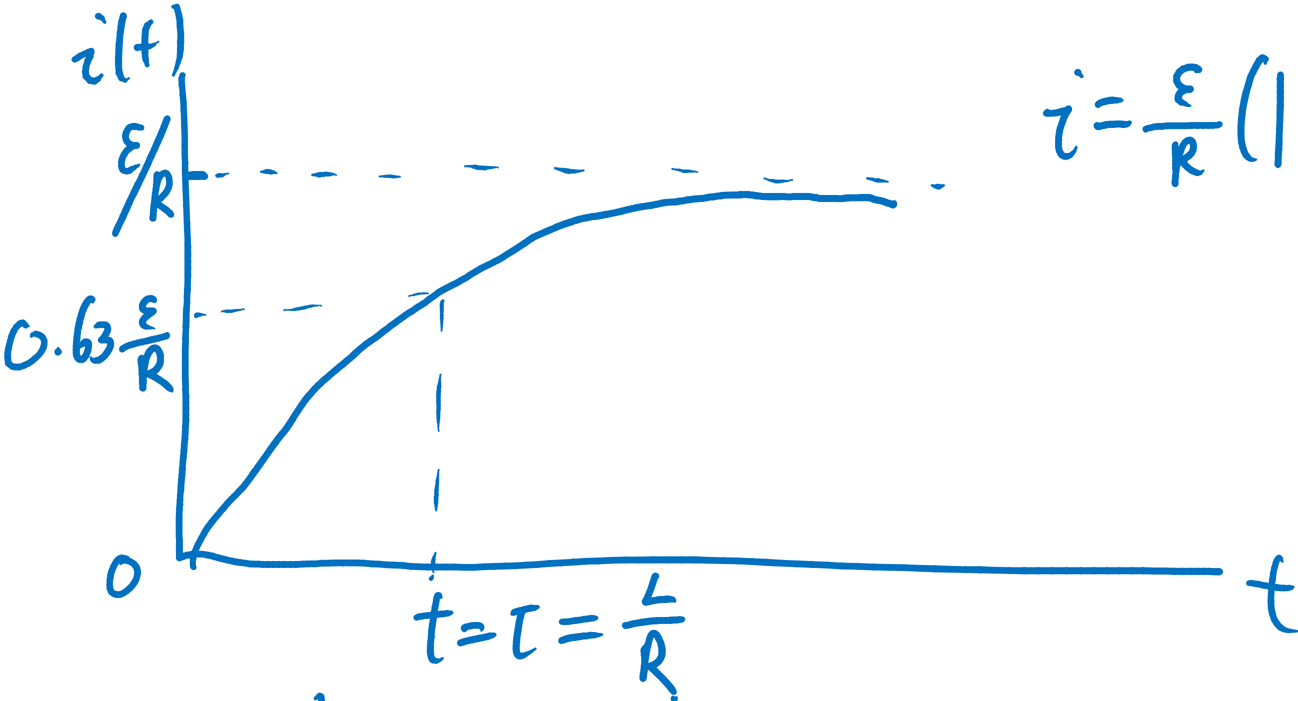
$$\tau = \frac{L}{R}$$

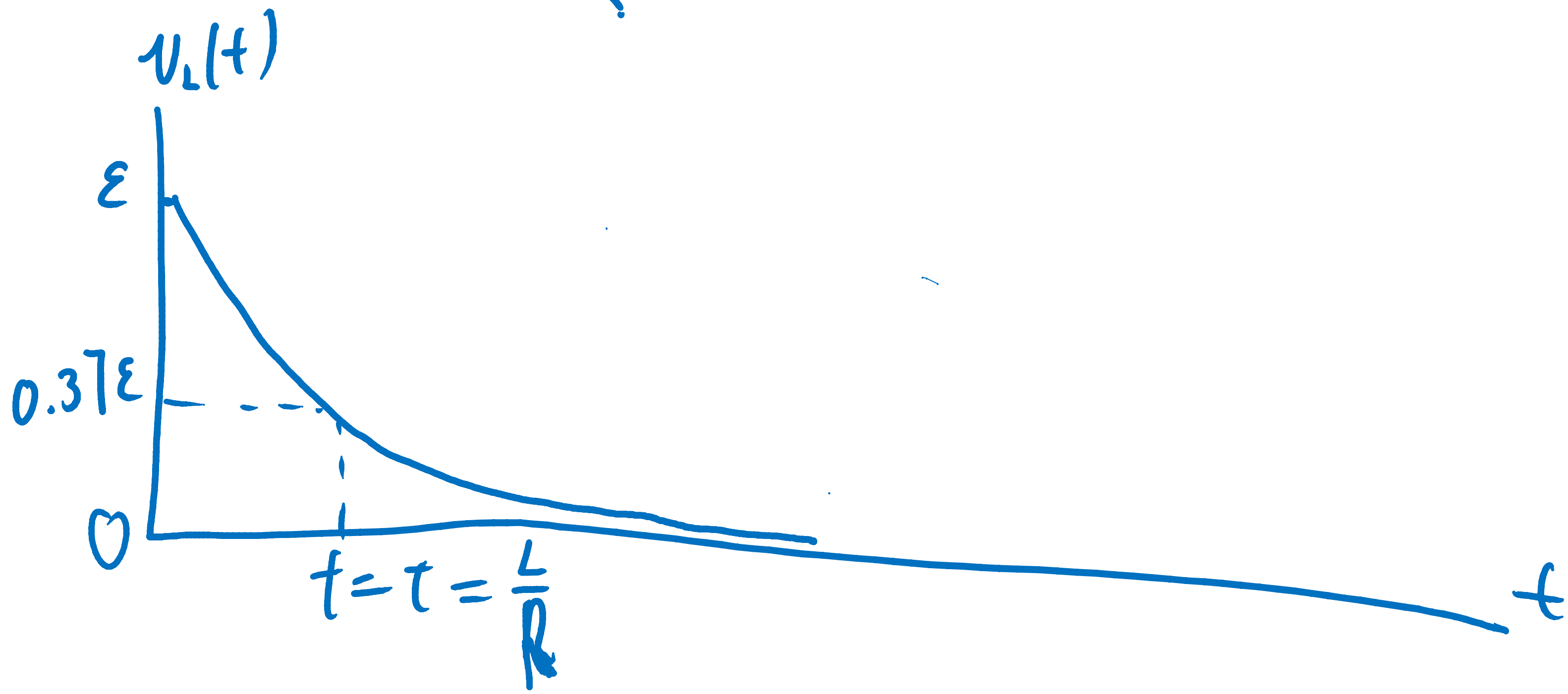
$$i(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/(\frac{L}{R})}\right)$$

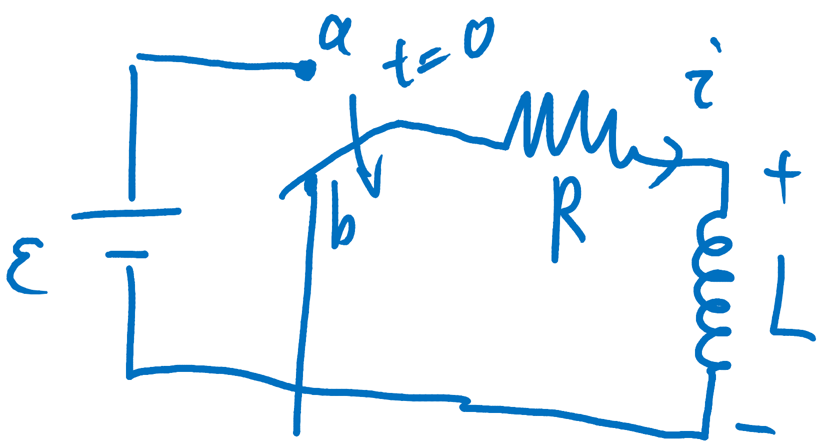
$$v_L = \varepsilon - iR = \varepsilon \cdot e^{-t/(\frac{L}{R})}$$

Satisfy general form $x(t) = x(\infty) + (x(0^{+}) - x(\infty))\, e^{-t/\tau}$

$$i = \frac{\varepsilon}{R}\left(1 - e^{-t/(\frac{L}{R})}\right)$$






At $t=0$ switch to b

discharge inductor

$t<0$ $\quad i = \frac{\varepsilon}{R}$, $\quad v_L = 0$

At $\quad t=0^+ \quad i = ? \quad v_L = ?$

no sudden jump of $i$ on inductor

$$i\Big|_{t=0^+} = i\Big|_{t<0} = \frac{\varepsilon}{R}$$

KVL $\quad -iR - v_L = 0$

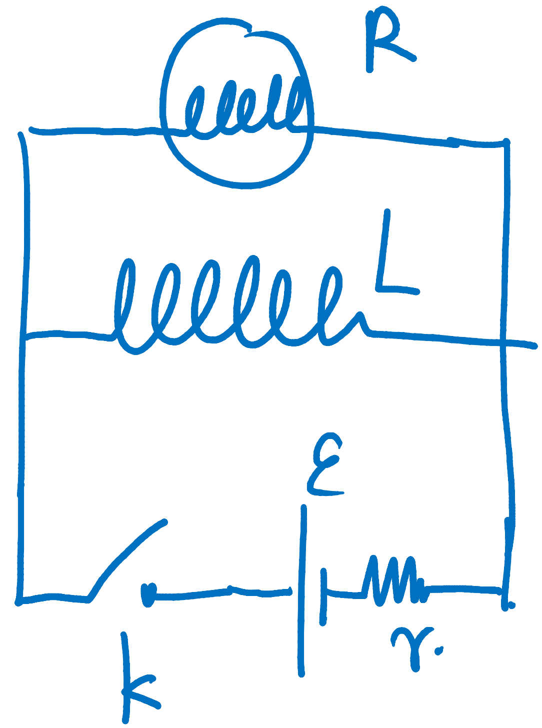
$$v_L\Big|_{t=0^+} = -\varepsilon$$

At $t=\infty \quad i=0 \quad v_L=0$

$$i(t) = \frac{\varepsilon}{R}\, e^{-t/\left(\frac{L}{R}\right)}$$

$$v_L(t) = -\varepsilon\, e^{-t/\left(\frac{L}{R}\right)}$$

Example



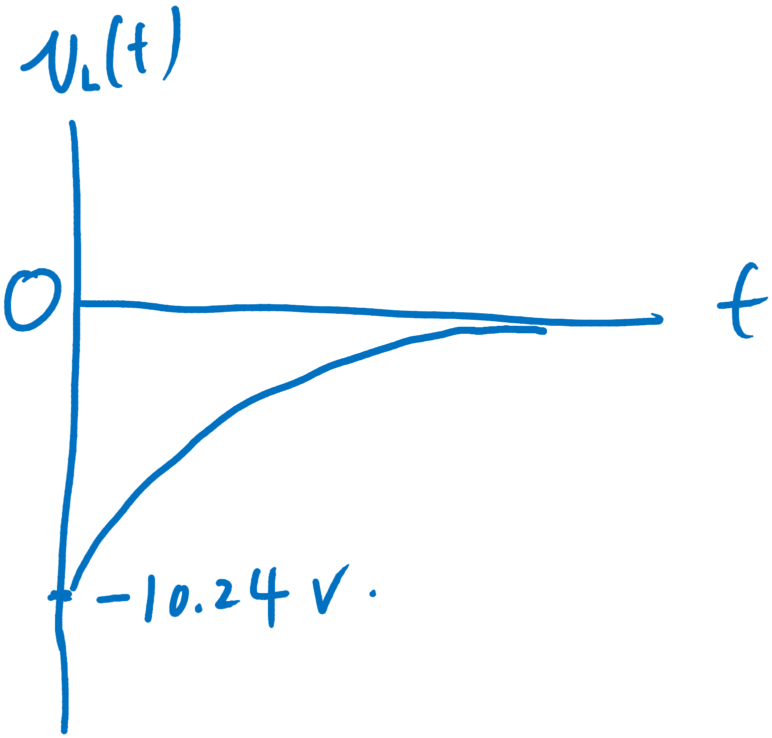

when the switch is closed, light bulb flares up brightly, then dims and goes out.

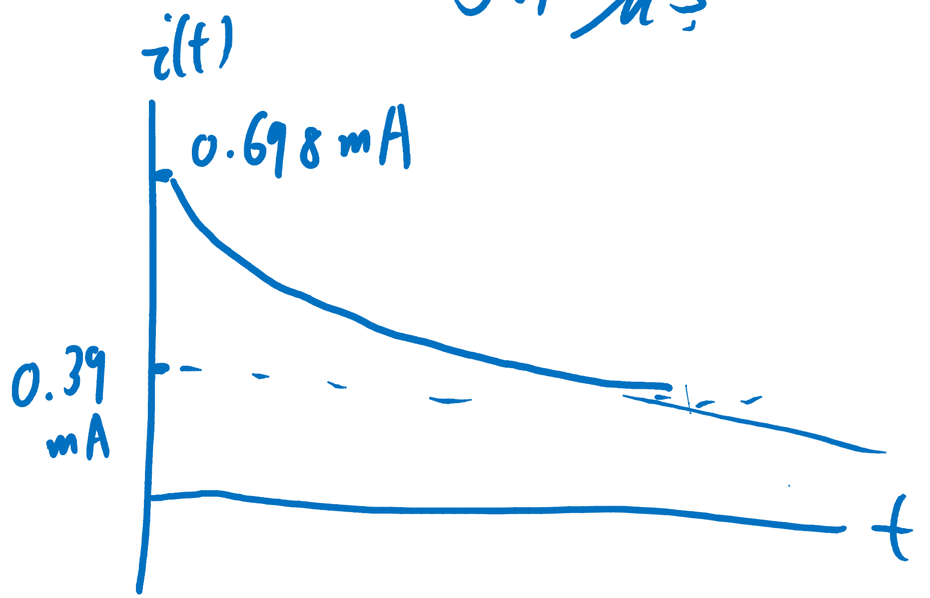
at $t=\infty$ $v_L=0$

$$i=\frac{13}{(4300+29000)}=0.39\ (mA)$$

$$\tau=\frac{L}{R}=\frac{170\times10^{-3}}{(4300+29000)}=5.1\times10^{-6}\ (s)$$

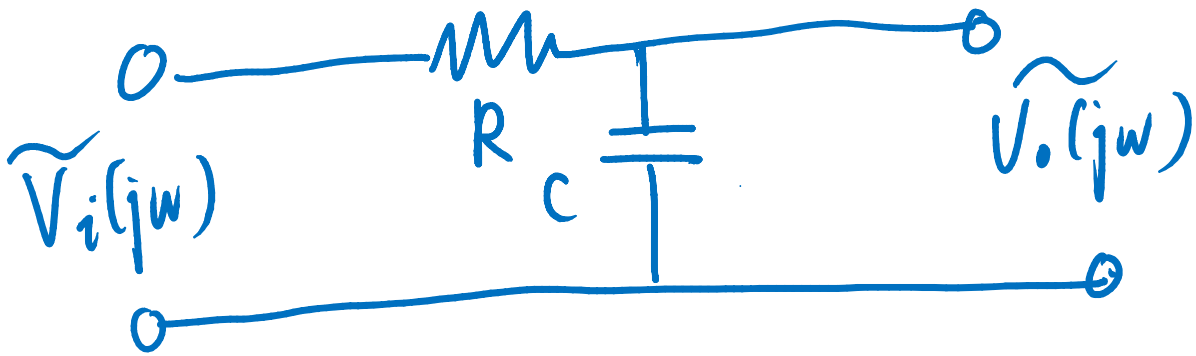
$$=5.1\ \mu s$$

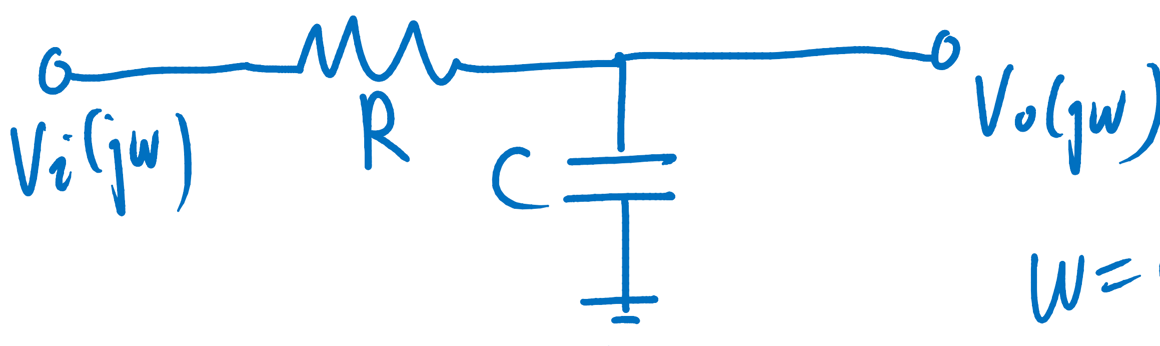


* Filter

low pass filter (LPF): a circuit that preserves lower frequencies while attenuating higher frequencies.




$\omega = 0 \quad |V_o| = |V_i|$

$\omega = \infty \quad |V_o| = 0$

$$\tilde{V}_o(j\omega) = \tilde{V}_i(j\omega) \cdot \frac{Z_c}{Z_c + Z_R}$$

$$H_v(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{Z_c}{Z_c + Z_R} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{-\frac{j}{\omega C}}{R - \frac{j}{\omega C}}$$

$$= \frac{\left(-\frac{1}{\omega C} j\right)\left(R + \frac{1}{\omega C} j\right)}{\left(R - \frac{j}{\omega C}\right)\left(R + \frac{1}{\omega C} j\right)} = \frac{\frac{1}{\omega^2 C^2} - \frac{R}{\omega C} j}{R^2 + \frac{1}{\omega^2 C^2}}$$

$$= \frac{1 - R\omega C j}{R^2\omega^2 C^2 + 1} = \frac{\sqrt{1 + (R\omega C)^2}\, e^{j[\tan^{-1}(-R\omega C)]}}{1 + (R\omega C)^2}$$

$$H_v(j\omega) = \frac{1}{\sqrt{1 + (\omega R C)^2}} e^{-j\tan^{-1}(R\omega C)}$$

$$= \frac{1}{\sqrt{1 + (\omega R C)^2}} \angle\left[-\tan^{-1}(R\omega C)\right] \qquad |H_v(j\omega)| \leq 1$$

$|H_v(j\omega)| = 1$ means 100% transmission

$|H_v(j\omega)|$ is a measure of signal transmission

$$\theta = -\tan^{-1}(R\omega C)$$

phase shift between $V_o(j\omega)$ and $V_i(j\omega)$

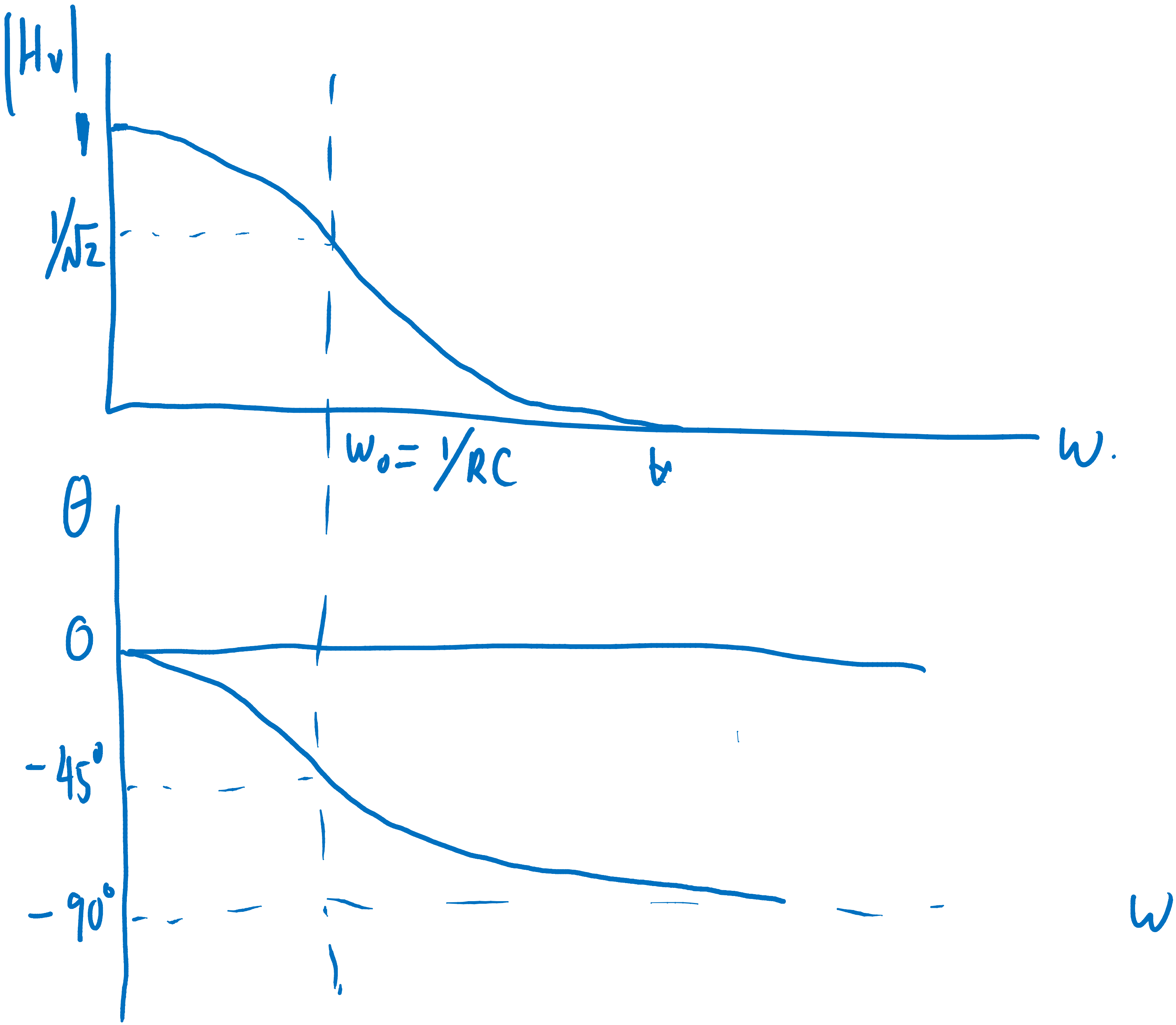
Define $\omega_o = \frac{1}{RC}$

$$H_v(j\omega) = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_o}\right)^2}}\, e^{-j\tan^{-1}\left(\frac{\omega}{\omega_o}\right)}$$

$\omega = 0$ $\quad |H_v| = 1$ $\quad \theta = 0^\circ$

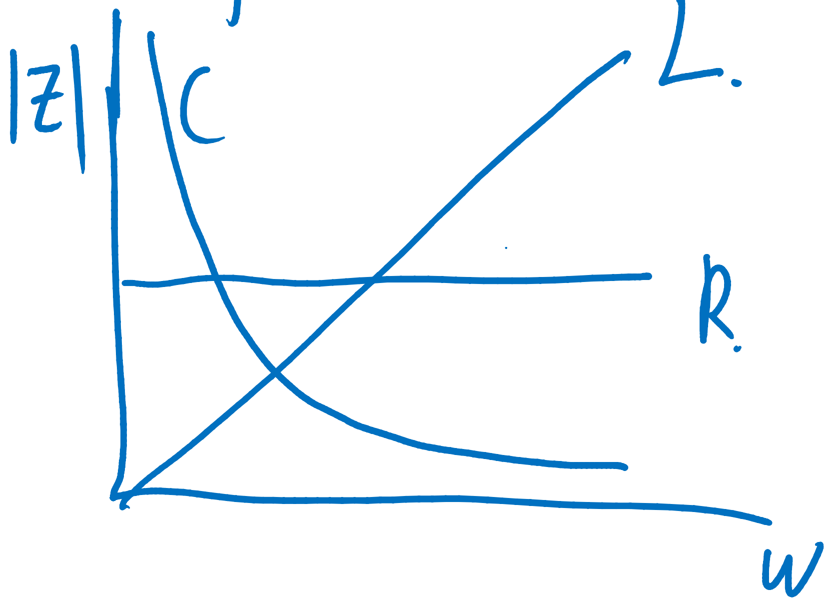
$\omega = \infty$ $\quad |H_v| = 0$ $\quad \theta = -90^\circ$

$\omega = \omega_o$ $\quad |H_v| = 1/\sqrt{2}$ $\quad \theta = -45^\circ$



|Hv|
1
1/√2
w₀= 1/RC
w.
θ
0
-45°
-90°
w

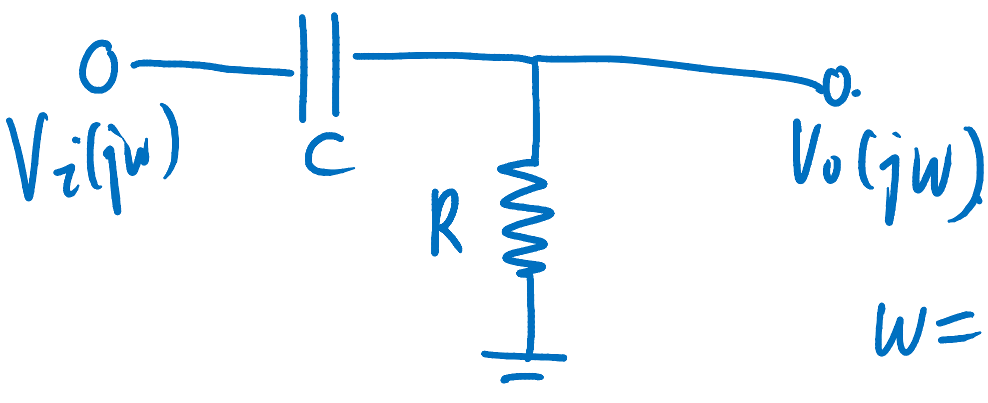
Other filters




$$Z_C = \frac{1}{jwC}$$

$$Z_L = jwL$$

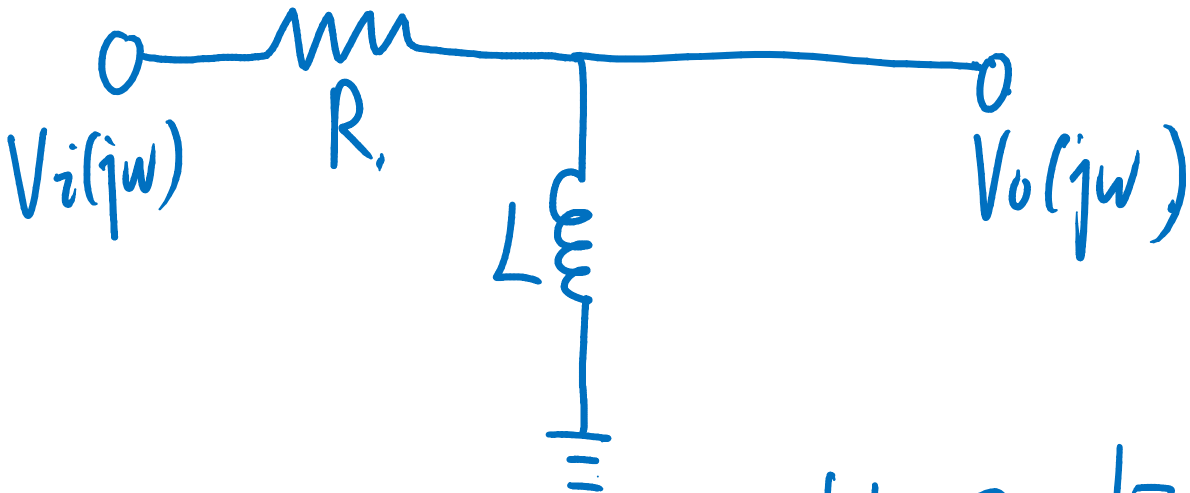
$$Z_R = R$$




$w = 0 \quad |Z_C| = \infty \quad |V_o| = 0$

$w = \infty \quad |Z_C| = 0 \quad |V_o| = |V_i|$

high pass filter (HPF)



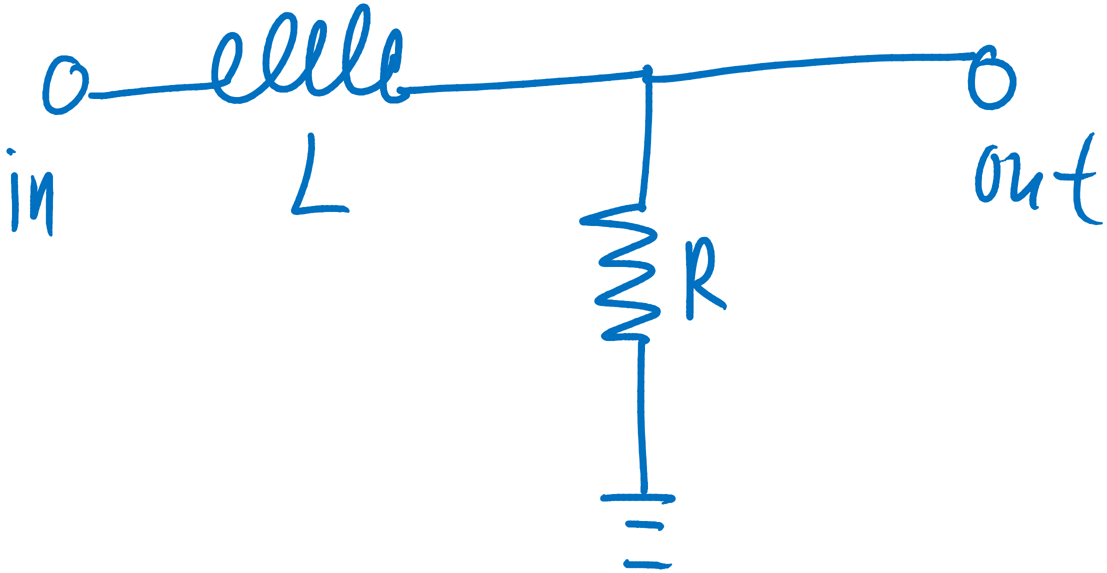

$w = 0 \quad |Z_L| = 0 \quad |V_o| = 0$

$w = \infty \quad |Z_L| = \infty \quad |V_o| = |V_i|$

high pass filter (HPF)

$$H_v(jw) = \frac{V_o}{V_i} = \frac{Z_L}{Z_L + Z_R}.$$

Example

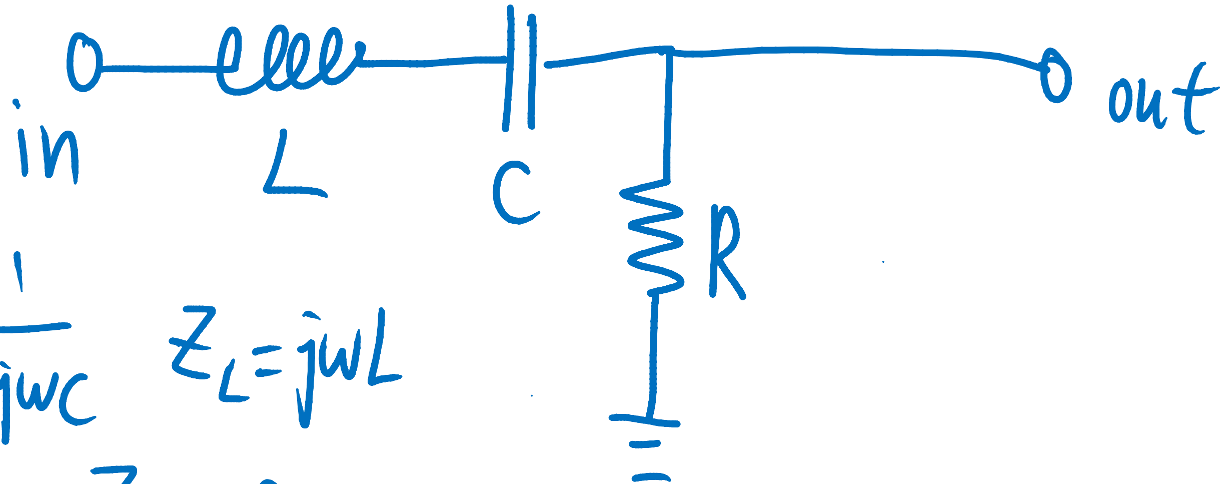



$\omega = 0 \qquad |Z_L| = 0 \quad |V_o| = |V_i|$

$\omega = \infty \qquad |Z_L| = \infty \quad |V_o| = 0$

low pass filter (LPF)

# RLC Band-pass filter.

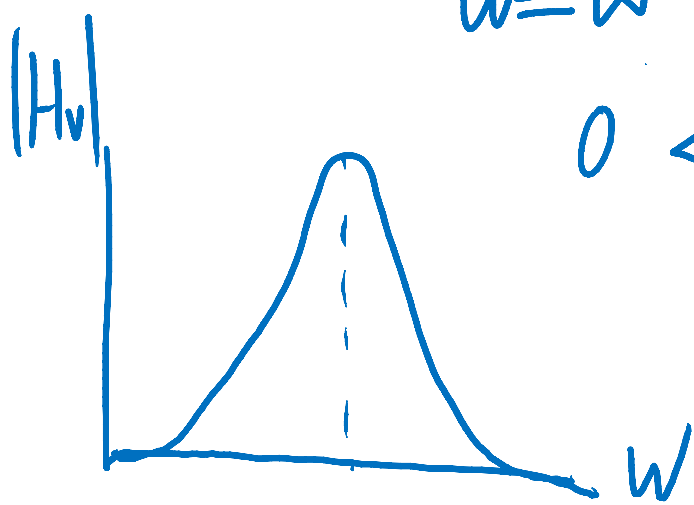



$$Z_C = \frac{1}{jwC} = -\frac{j}{wC} \qquad Z_L = jwL \qquad Z_R = R$$

$w = 0 \quad |Z_L| = 0 \quad |Z_C| = \infty \quad |V_o| = 0$

$w = \infty \quad |Z_L| = \infty \quad |Z_C| = 0 \quad |V_o| = 0$

$0 < w < \infty \quad |V_o| > 0$



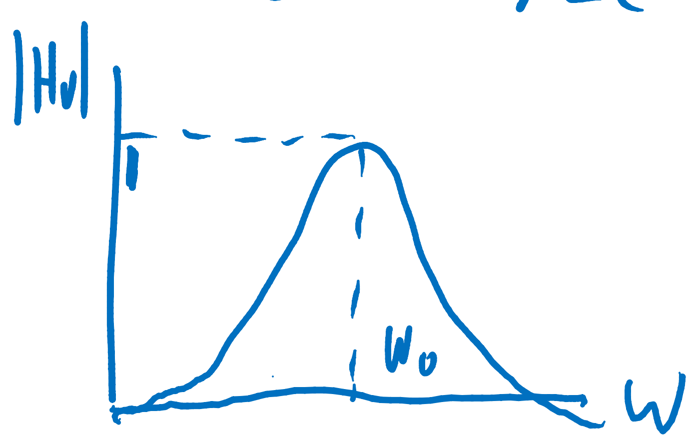
$$V_o = V_i \frac{Z_R}{Z_C + Z_L + Z_R}$$

$$H_v = \frac{V_o}{V_i} = \frac{Z_R}{Z_C + Z_L + Z_R} = \frac{R}{\frac{1}{j\omega C} + j\omega L + R}$$

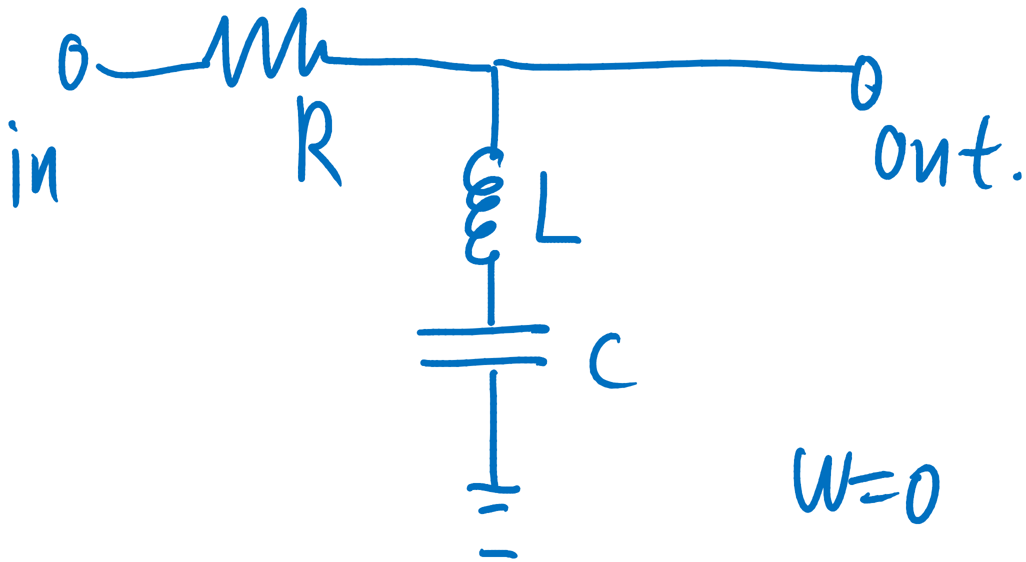
$$= \frac{R}{R + \underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{=0} j}$$

$$\omega L = \frac{1}{\omega C} \qquad \omega = \sqrt{\frac{1}{LC}} = \omega_0$$

$$H_v \Big|_{\omega = \omega_0} = 1$$



Band stop filter



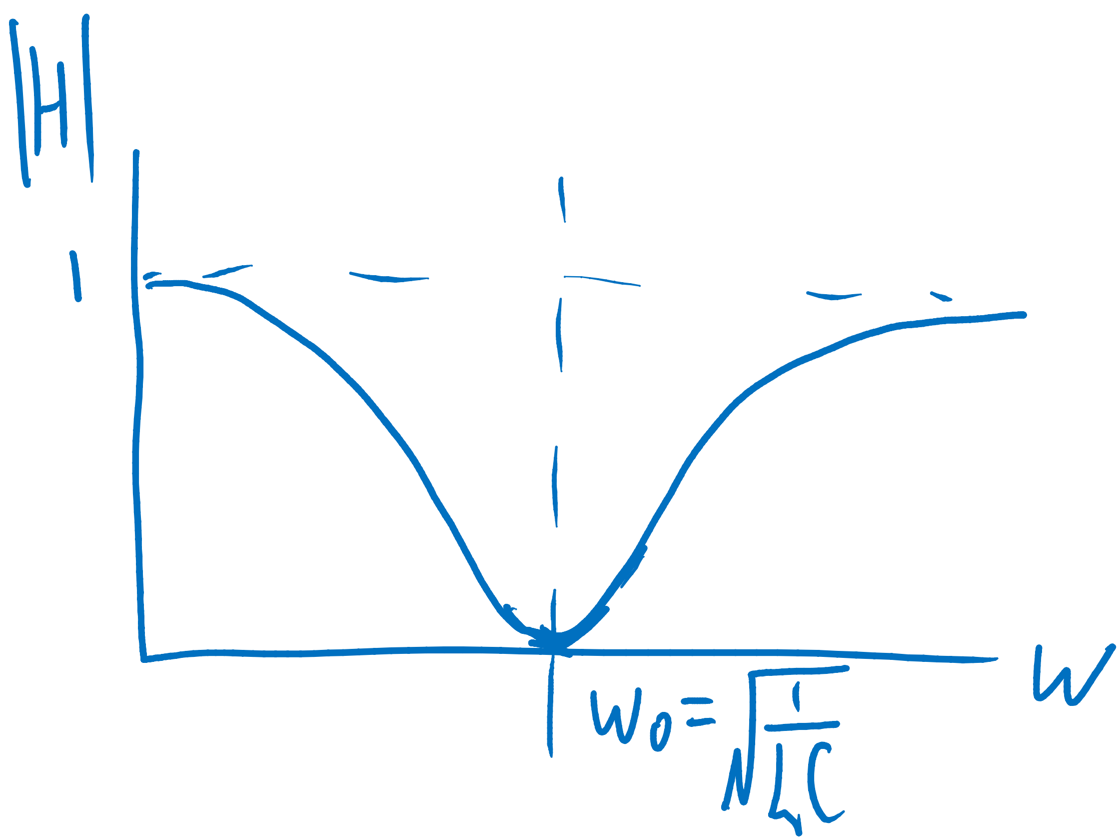

$w = 0 \quad |Z_c| = \infty \quad |Z_L| = 0 \quad |V_0| = |V_i|$

$w = \infty \quad |Z_c| = 0 \quad |Z_L| = \infty \quad |V_0| = |V_i|$

$0 < w < \infty \quad |V_0| < |V_i|$

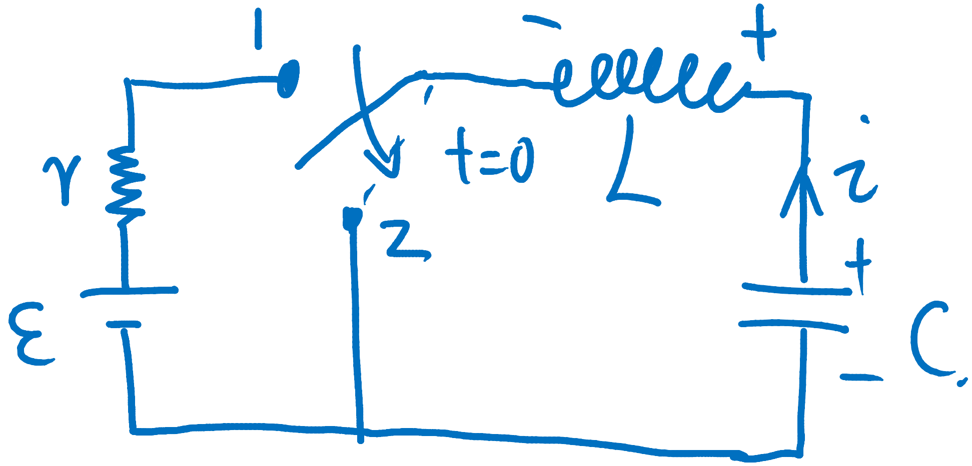
$$H_V = \frac{V_o}{V_i} = \frac{Z_L + Z_C}{Z_L + Z_C + R} = \frac{j\left(wL - \frac{1}{wC}\right)}{R + j\left(wL - \frac{1}{wC}\right)}$$

$$wL = \frac{1}{wC} \qquad w = \sqrt{\frac{1}{LC}} = w_o \qquad H_V\Big|_{w=w_o} = 0$$




Band stop filter

LC resonance circuit




$i=0 \quad t<0$

$t=0$ Switch from 1 to 2

$t>0 \quad i \neq 0$

$t>0$ KVL $V_C - V_L = 0$

$$\frac{Q}{C} - L\frac{di}{dt} = 0$$

$$i = -\frac{dQ}{dt}$$

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{1}{LC} \Leftrightarrow \frac{k}{m}$$

$$Q = Q_0 \cos(\omega t + \varphi)$$

$Q_0$, $\varphi$ determined by initial conditions.

$$-\omega^2 Q_0 \cos(\omega t + \varphi) + \frac{Q_0}{LC} \cos(\omega t + \varphi) = 0$$

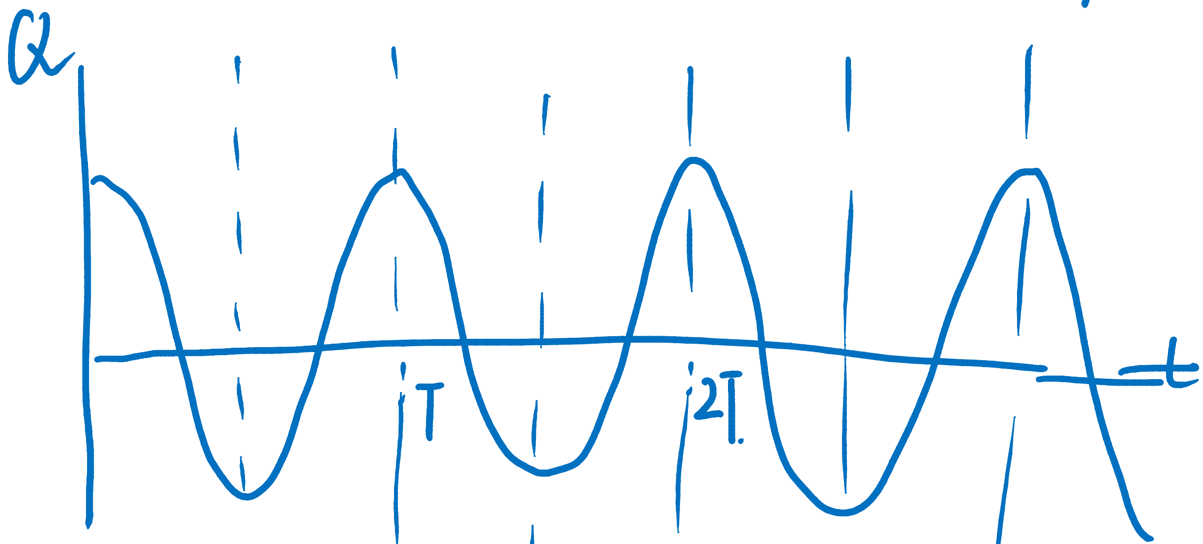
$$\left(-\omega^2 + \frac{1}{LC}\right) \cos(\omega t + \varphi) = 0$$

true for anytime $t$

$$-\omega^2 + \frac{1}{LC} = 0$$

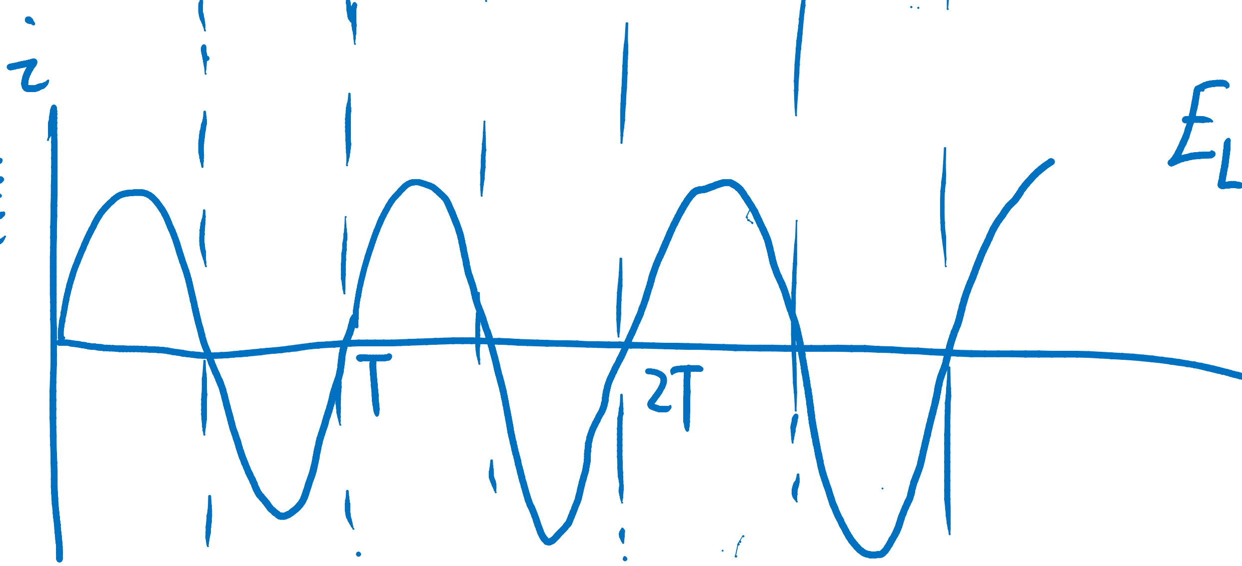
$$\omega = \sqrt{\frac{1}{LC}}$$ resonance frequency

Q and $i$ oscillates sinusoidally.




$$E_C = \frac{Q^2}{2C}$$

$$i = -\frac{dQ}{dt}$$




$$E_L = \frac{1}{2} L i^2$$

when $Q$ is max, $i = 0$

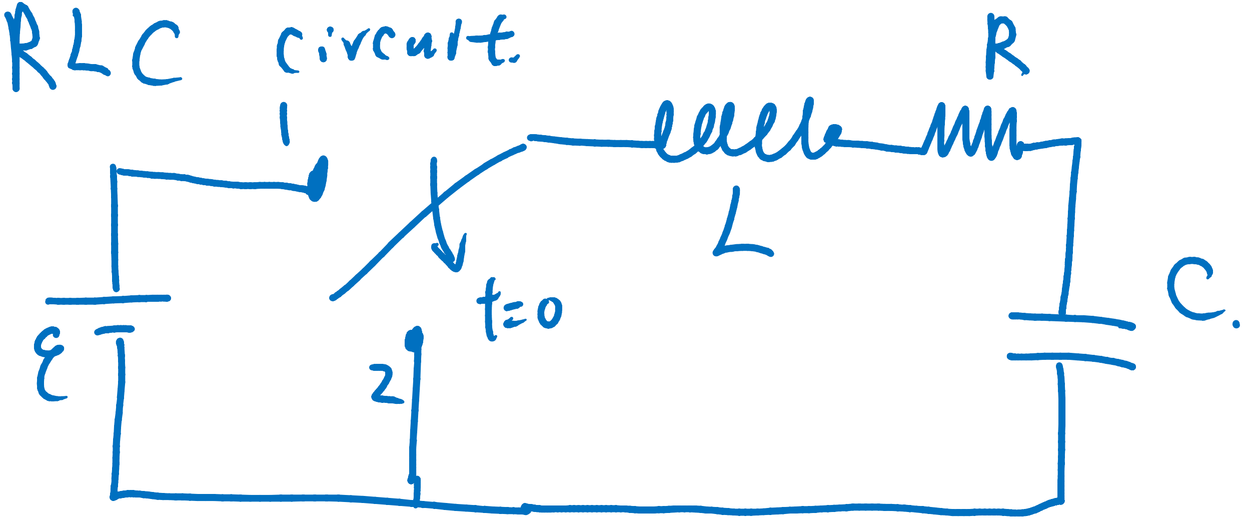
when $Q = 0$, $i$ is max

$$E_C = \frac{Q^2}{2C} \qquad E_L = \frac{1}{2} L i^2$$

when $E_C$ is max, $E_L = 0$

when $E_C = 0$, $E_L$ is max,

RLC circuit




R : energy dissipation.

damped oscillation.