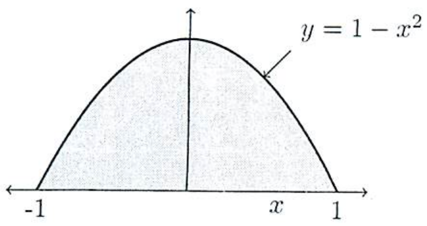
ELEG 310 Midterm ~~1~~ 2, ~~March 24~~, 2011

NAME:

1. Two random variables, X and Y, have density $f_{XY}(x, y) = ky$ in the shaded region below.
   a) What is k?
   b) What is $f_X(x)$?
   c) What is $f_{Y|X}(y|x)$?



a) $$1 = \int_{-1}^{1}\int_{0}^{1-x^2} ky\,dy\,dx = \int_{-1}^{1} \frac{k}{2}(1-x^2)^2\,dx$$

$$= \frac{k}{2}\int_{-1}^{1}(1-2x^2+x^4)\,dx = \frac{k}{2}\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_{-1}^{1}$$

$$= \frac{k}{2}\cdot 2\cdot\left(1-\frac{2}{3}+\frac{1}{5}\right) = k\frac{8}{15} \qquad \boxed{k = \frac{15}{8}}$$

b) $$f_X(x) = \int_{0}^{1-x^2} ky\,dy = \boxed{\frac{k}{2}(1-x^2)^2 \qquad -1<x<1}$$

c) $$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{ky}{\frac{k}{2}(1-x^2)^2} = \boxed{\frac{2y}{(1-x^2)^2} \qquad 0<y<1-x^2}$$

NAME:

2. Three random variables, X, Y, and Z have means 1, 2, and 3 and variances 4, 5, 6, respectively. In addition, the covariances are as follows: $\mathrm{Cov}[X, Y] = -1$, $\mathrm{Cov}[X, Z] = -2$, and $\mathrm{Cov}[Y, Z] = -3$. Let $S = X + Y + Z$. What are $E[S]$ and $\mathrm{Var}[S]$?

a) $E[S] = E[X+Y+Z] = E[X] + E[Y] + E[Z]$

$= 1+2+3 = 6$

$\mathrm{Var}[S] = \mathrm{Var}[X+Y+Z]$

$= \mathrm{Var}[X] + \mathrm{Var}[Y] + \mathrm{Var}[Z] + 2\,\mathrm{Cov}[X,Y]$

$+ 2\,\mathrm{Cov}[X,Z] + 2\,\mathrm{Cov}[Y,Z]$

$= 4+5+6 + 2(-1) + 2(-2) + 2(-3)$

$= 3$

NAME:

3. Let $\boldsymbol{X} \sim N(2,4)$, use the table on the right to find,

a) $\Pr[\boldsymbol{X} \leq 0]$.

b) $\Pr[-1 \leq \boldsymbol{X} \leq 1]$.

c) $\Pr[\boldsymbol{X} \geq 1]$.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	1.00	0.8413	2.00	0.9772	3.00	0.9987
0.05	0.5199	1.05	0.8531	2.05	0.9798	3.05	0.9989
0.10	0.5398	1.10	0.8643	2.10	0.9821	3.10	0.9990
0.15	0.5596	1.15	0.8749	2.15	0.9842	3.15	0.9992
0.20	0.5793	1.20	0.8849	2.20	0.9861	3.20	0.9993
0.25	0.5987	1.25	0.8944	2.25	0.9878	3.25	0.9994
0.30	0.6179	1.30	0.9032	2.30	0.9893	3.30	0.9995
0.35	0.6368	1.35	0.9115	2.35	0.9906	3.35	0.9996
0.40	0.6554	1.40	0.9192	2.40	0.9918	3.40	0.9997
0.45	0.6736	1.45	0.9265	2.45	0.9929	3.45	0.9997
0.50	0.6915	1.50	0.9332	2.50	0.9938	3.50	0.9998
0.55	0.7088	1.55	0.9394	2.55	0.9946	3.55	0.9998
0.60	0.7257	1.60	0.9452	2.60	0.9953	3.60	0.9998
0.65	0.7422	1.65	0.9505	2.65	0.9960	3.65	0.9999
0.70	0.7580	1.70	0.9554	2.70	0.9965	3.70	0.9999
0.75	0.7734	1.75	0.9599	2.75	0.9970	3.75	0.9999
0.80	0.7881	1.80	0.9641	2.80	0.9974	3.80	0.9999
0.85	0.8023	1.85	0.9678	2.85	0.9978	3.85	0.9999
0.90	0.8159	1.90	0.9713	2.90	0.9981	3.90	1.0000
0.95	0.8289	1.95	0.9744	2.95	0.9984	3.95	1.0000

a) $P(X \leq 0) = P\left(\frac{X-2}{2} < \frac{0-2}{2}\right)$

$= P(Z < -1) = \Phi(-1)$

$= 1 - \Phi(1)$

$= 1 - 0.8413 = \boxed{0.1587}$

b) $P[-1 \leq X \leq 1] = P\left[\frac{-1-2}{2} \leq \frac{X-2}{2} \leq \frac{1-2}{2}\right]$

$= P[-1.5 \leq Z \leq -\frac{1}{2}] = \Phi(-\frac{1}{2}) - \Phi(-1.5)$

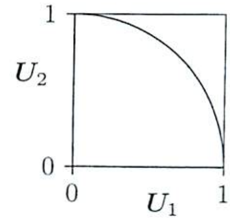
$= \Phi(1.5) - \Phi(0.5) = 0.9332 - 0.6915 = \boxed{0.2417}$

c) $P[X \geq 1] = P\left[\frac{X-2}{2} \geq \frac{1-2}{2}\right] = P[Z \geq -\frac{1}{2}]$

$= P[Z \leq \frac{1}{2}] = \boxed{0.6915}$

NAME:

Consider the following Monte Carlo experiment to compute the value of $\pi/4 \approx 0.79$. Generate a large number, n, of pairs of uniform $U(0,1)$ random variables, (U_1, U_2). For each pair, define $X_i = 1$ if the point falls inside the quarter circle, and $X_i = 0$ if not. Then let $S = (X_1 + X_2 + \cdots + X_n)/n$.



$$\begin{array}{r} 0.79 \\ \times 0.21 \\ \hline .0079 \\ .158 \\ \hline .1659 \end{array}$$

4.

a) What are the mean and variance of S?

b) What is the distribution of S? I.e., what is its PMF or its density?

c) ~~b~~) Estimate how large n must be so that $\Pr[-\epsilon < S - \pi/4 < \epsilon] \leq \delta$ for some $\epsilon > 0$ and $\delta > 0$. (You can't solve for n without knowing ϵ and δ, but solve the equation or equations as far as possible.)

a) $E S = E\left(\frac{X_1+X_2+\cdots+X_n}{n}\right) = \frac{1}{n}(EX_1 + EX_2 + \cdots EX_n) = \frac{np}{n} = p = \boxed{\frac{\pi}{4}}$

where $p = \frac{\pi}{4}$

$\text{Var } S = \frac{npq}{n^2} = \frac{n\frac{\pi}{4}(1-\frac{\pi}{4})}{n^2} = \frac{.166\,n}{n^2} = \boxed{\frac{0.166}{n}}$

b) $P(S = \frac{k}{n}) = \binom{n}{k} p^k q^{n-k}$ i.e. S = scaled binomial

nS = binomial

c) $P\left[-\varepsilon < S - \frac{\pi}{4} < \varepsilon\right] = P\left(\frac{-\varepsilon}{\sqrt{\frac{pq}{n}}} < \frac{S-\frac{\pi}{4}}{\sqrt{\frac{pq}{n}}} < \frac{\varepsilon}{\sqrt{\frac{pq}{n}}}\right)$

$= P\left(\frac{-\varepsilon\sqrt{n}}{\sqrt{pq}} \leq Z < \frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right) \approx 2\,\Phi\left(\frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right) - 1$ $\quad Z \sim N(0,1)$

$2\Phi\left(\frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right) - 1 \leq \delta \Rightarrow \Phi\left(\frac{\varepsilon\sqrt{n}}{\sqrt{pq}}\right) \leq \frac{\delta+1}{2} \Rightarrow \frac{\varepsilon\sqrt{n}}{\sqrt{pq}} \leq \Phi^{-1}\left(\frac{\delta+1}{2}\right)$

$\Rightarrow \sqrt{n} \leq \frac{\sqrt{pq}}{\varepsilon}\Phi^{-1}\left(\frac{\delta+1}{2}\right) \Rightarrow \boxed{n \leq \frac{pq}{\varepsilon^2}\left(\Phi^{-1}\left(\frac{\delta+1}{2}\right)\right)^2}$

FYI: if $\varepsilon = \delta = 0.05$, $n \approx 250$