Solution to HW56

5.1 (a) $p(x) = \sum_y p(x, y)$, so
(1 point)

$$p(x=0) = \sum_y p(x=0, y) = 0.1 + 0.1 + 0 = 0.2$$

Similarly, we have

X	0	1	2	3
p(x)	0.2	0.1	0.4	0.3

Y	0	1	2
P(Y)	0.5	0.3	0.2

(b) $P(X|Y) = \frac{P(X, Y)}{P(Y)}$, e.g. $P(X|Y=0) = \frac{P(X, Y=0)}{P(Y=0)}$

X	0	1	2	3
P(X|Y=0)	0	0.2	0.4	0.4
P(X|Y=1)	1/3	0	1/3	1/3
P(X|Y=2)	0.5	0	0.5	0

Similarly P(Y|X)

Y	0	1	2
P(Y|X=0)	0	0.5	0.5
P(Y|X=1)	1	0	0
P(Y|X=2)	0.5	0.25	0.25
P(Y|X=3)	2/3	1/2	0

Based on Baye's Theorem,

$$P(Y|X) = \frac{P(X|Y) \cdot P(Y)}{P(X)}$$

e.g. $P(Y=1|X=2) = \frac{P(X=2|Y=1) \cdot P(Y=1)}{P(X=2)}$

$$= \frac{1/3 \cdot 0.3}{0.4} = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4} = 0.25$$

We can easily derive other conditional probability $P(Y|X)$ based on the above example

5.2 (a) $E(X) = \sum_x x \cdot P(x) = 0\times0.2 + 1\times0.1 + 2\times0.4 + 3\times0.3$
(1 point)

$$= 0.1 + 0.8 + 0.9 = 1.8$$

$$E(Y) = \sum_y y \cdot P(y) = 0\times0.5 + 1\times0.3 + 2\times0.2$$
$$= 0.7$$

$$E(X^2) = \sum_x x^2 P(x) = 1^2\times0.1 + 2^2\times0.4 + 3^2\times0.3$$
$$= 0.1 + 1.6 + 2.7 = 4.4$$

$$Var(X) = E(X^2) - E(X)^2 = 4.4 - (1.8)^2 = 1.16$$

$$E(Y^2) = \sum_y y^2 \cdot P(y) = 1^2 \cdot 0.3 + 2^2 \cdot 0.2 = 1.1$$

$$Var(Y) = E(Y^2) - E(Y)^2 = 1.1 - (0.7)^2 = 0.61$$

$$E(XY) = \sum_{x,y} x \cdot y \cdot P(x,y) = 2 \cdot 0.1 + 3 \cdot 0.1 + 2 \cdot 2 \cdot 0.1$$
$$= 0.2 + 0.3 + 0.4 = 0.9$$

$$COV(X,Y) = E(XY) - E(X) \cdot E(Y) = 0.9 - 1.8 \cdot 0.7 = -0.36$$

(b) X and Y are not independent, since $COV(X,Y) \neq 0$

5-7 (a)

(2 point)

X	0	1	2	3	4
$P(X)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

Y	0	1	2
$P(Y)$	$\frac{5}{9}$	$\frac{3}{9}$	$\frac{1}{9}$

(b) $E(X) = \frac{2}{9} \times 1 + \frac{3}{9} \times 2 + \frac{2}{9} \times 3 + \frac{1}{9} \times 4$
$= \frac{1}{9}(2 + 6 + 6 + 4) = 2$

$E(Y) = \frac{3}{9} \times 1 + \frac{1}{9} \times 2 = \frac{5}{9}$

(c) $E(XY) = 1 \cdot \frac{1}{9} + 2 \cdot \frac{1}{9} + 3 \cdot \frac{1}{9} + 4 \cdot \frac{1}{9} = \frac{10}{9}$

$Cov(X,Y) = E(XY) - E(X) \cdot E(Y) = \frac{10}{9} - \frac{5}{9} \cdot 2 = 0$

However, X and Y are not independent, since
$P(X=0 \cap Y=1) = 0 \neq P(X=0) \cdot P(Y=1) = \frac{1}{9} \cdot \frac{3}{9}$

(3/10)

$$P(W=w) = P(X-Y=w) = \sum_y P(X-Y=w \mid Y=y)\, P(Y=y)$$

$$= P(X=w \mid Y=0) \cdot P(Y=0) +$$

$$P(X-1=w \mid Y=1)\ P(Y=1) +$$

$$P(X-2=w \mid Y=2)\ P(Y=2)$$

$$W = \{-2\ \ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4\}$$

$$P(W=-2) = P(X=-2 \mid Y=0)\ P(Y=0) + P(X=-1 \mid Y=1)\ P(Y=1) + P(X=0 \mid Y=2) \cdot P(Y=2)$$

$$= 0$$

$$P(W=-1) = P(X=-1 \mid Y=0)\ P(Y=0) + P(X=0 \mid Y=1)\ P(Y=1) + P(X=1 \mid Y=2)\ P(Y=2)$$

$$= 0$$

$$P(W=0) = P(X=0 \mid Y=0)\ P(Y=0) + P(X=1 \mid Y=1)\ P(Y=1) + P(X=2 \mid Y=2)\ P(Y=2)$$

$$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$$

$$P(W=1) = P(X=1 \mid Y=0)\ P(Y=0) + P(X=2 \mid Y=1)\ P(Y=1) + P(X=3 \mid Y=2)\ P(Y=2)$$

$$= \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$P(W=2) = P(X=2 \mid Y=0)\ P(Y=0) + P(X=3 \mid Y=1)\ P(Y=1) + P(X=4 \mid Y=2)\ P(Y=2)$$

$$= \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$P(W=3) = P(X=3 \mid Y=0)\ P(Y=0) + P(X=4 \mid Y=1)\ P(Y=1) + P(X=5 \mid Y=2)\ P(Y=2)$$

$$= \frac{1}{9}$$

$$P(W=4) = P(X=4 \mid Y=0)\ P(Y=0) = \frac{1}{9}$$

$$E(W) = E(X) - E(Y) = 2 - \frac{5}{9} = \frac{13}{9}$$

$$E(W^2) = 0^2 \cdot \frac{3}{9} + 1^2 \times \frac{2}{9} + 2^2 \times \frac{2}{9} + 3^2 \times \frac{1}{9} + 4^2 \times \frac{1}{9} = \frac{35}{9}$$

$$Var(W) = E(W^2) - E(W)^2 = \frac{35}{9} - \left(\frac{13}{9}\right)^2 \approx 1.80$$

5.8 Consider the following PMF

(1 point)

y			
1	0.3	0	0.3
0	0	0.4	0
	-1	0	1

$E(X) = 0$ $E(Y) = 0.6$

$E(XY) = 0$; $Cov(X,Y) = E(XY) - E(X) \cdot E(Y) = 0$

But X and Y are dependent, since

$$P(X=0 \cap Y=1) = 0 \neq P(X=0) \cdot P(Y=1) = 0.24$$

5.10 Since X_i is IID uniform, the PMF of S is the convolution

(2 point) of various PMFs. Therefore we can generate a table to illustrate the convolution procedure.

For clarity, we multiply the PMFs of the X_i by 4.

We correct for this by dividing by $4^3 = 64$ at the end

$x_2 \backslash x_1$	1	1	1	1			
1	1	1	1	1			
1		1	1	1	1		
1			1	1	1	1	
1				1	1	1	1

$x_3 \backslash x_1+x_2$	1	2	3	4	3	2	1			
1	1	2	3	4	3	2	1			
1		1	2	3	4	3	2	1		
1			1	2	3	4	3	2	1	
1				1	2	3	4	3	2	1
$64 * P(S)$	1	3	6	10	12	12	10	6	3	1
S	3	4	5	6	7	8	9	10	11	12

Therefor $P(S=3) = 1/64$

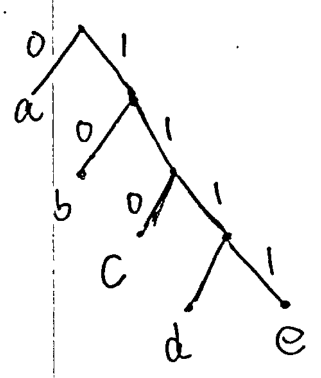
$P(S=4) = 4/64$

$P(S=5) = 6/64$

$\vdots$

5.21 (a) The coding tree is below

(1 point)



$E[L] = 0.3\times1 + 0.3\times2 + 0.2\times3 + 0.1\times4 + 0.1\times4 = 2.3$ bit per sample

(b) The worst tree is the same tree but with all labels reversed. namely, symbols a and b have codes of length 4 and symbol e has code of length one.

$E[L] = 0.3\times4 + 0.3\times4 + 0.2\times3 + 0.1\times2 + 0.1\times1 = 3.3$ bits per symbol

5.26 (2 point)

Assume X takes on values $\{x_n\}_{n=1}^{N}$

Y takes on values $\{y_m\}_{m=1}^{M}$

Therefore Z takes on values $\{x_1 \cdots x_N, y_1 \cdots y_M\}$

Z	x_1	$\cdots$	x_N	y_1	$\cdots$	y_M
$p(z)$	$p\cdot p(x_1)$	$\cdots$	$p\cdot p(x_N)$	$(1-p)\cdot p(y_1)$	$\cdots$	$(1-p)\cdot p(y_M)$

$$E(Z) = \sum_{i=1}^{M+N} z_i\, p(z_i) = p\cdot\sum_{i=1}^{N} x_i\cdot p(x_i) + (1-p)\cdot\sum_{i=N+1}^{N+M} p(y_{i-N})\cdot y_{i-N}$$

$$= p\cdot \mu_x + (1-p)\cdot \mu_y$$

$\cdot\ E(Z^2) = \sum_{i=1}^{M+N} z_i^2\, p(z_i) = p \cdot \sum_{i=1}^{N} x_i^2 \cdot p(x_i) + (1-p) \sum_{i=N+1}^{N+M} p(y_{i-N}) \cdot y_{i-N}^2$

$= p \cdot E(X^2) + (1-p) \cdot E(Y^2)$

since $Var(x) = \sigma_x^2$ and $E(X^2) = Var(x) + E(x)^2$

We can derive that $E(X^2) = \sigma_x^2 + \mu_x^2$

Similiarly $E(Y^2) = \sigma_y^2 + \mu_y^2$

Therefore $E(Z^2) = p(\sigma_x^2 + \mu_x^2) + (1-p) \cdot (\sigma_y^2 + \mu_y^2)$

$\therefore Var(Z) = E(Z^2) - E(Z)^2$

$= p(\sigma_x^2 + \mu_x^2) + (1-p)(\sigma_y^2 + \mu_y^2) - [p \cdot \mu_x + (1-p) \cdot \mu_y]^2$

$= p \cdot \sigma_x^2 + p\mu_x^2 + (1-p)\sigma_y^2 + (1-p)\mu_y^2 - (p^2\mu_x^2 + 2p(1-p)\mu_x\mu_y + (1-p)^2\mu_y^2)$

$= p\sigma_x^2 + (1-p)\sigma_y^2 + (p - p^2)\mu_x^2 + [(1-p) - (1-p)^2]\mu_y^2 - 2p(1-p)\mu_x\mu_y$

6.1 With $n=4$ and $p=0.33$.

(1 point) $$\Pr[N=k] = \binom{4}{k}\cdot(0.33)^k(1-0.33)^{4-k}$$

k	0	1	2	3	4
Pr	0.202	0.397	0.293	0.096	0.012

6.7 Pr[at least 20 questions correct]

(1 point)
$$= \sum_{k=20}^{40} \binom{40}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{40-k} = 2.17\times 10^{-5}$$

Pr[at least 32 questions correct]

$$= \sum_{k=32}^{40} \binom{40}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{40-k} = 5.90\times 10^{-16}$$

6.11 (a). Let N be the number of points

(1 point) $$E[N] = 0\times(1-p) + 1\times p\cdot(1-p) + 2\times p^2 = p+p^2$$

(b) $$E[N] = 0\times(1-p)^2 + 1\times 2p(1-p) + 2\times p^2 = 2p$$

(c) The difference is $2p-p-p^2 = p(1-p)$

Taking derivative and let it equal to 0, we have $1-2p=0$

The answer is 0.5

(9/10)

6.24. Since all flips are independent,

(1 point) Pr{at least k of the n students are still flipping after t flips}

$$= \sum_{m=k}^{n} \binom{n}{m} \cdot P_t^{m} \cdot (1-P_t)^{n-m}$$

Where P_t is the probability of one student is still flipping after t flips, We can derive that

$$P_t = p^t$$

Therefore $Pr = \sum_{m=k}^{n} \binom{n}{m} p^{tm} \cdot (1-p^t)^{n-m}$

6.32 (a)

(1 point)

X	-1	1	2	3
P	$\frac{125}{6^3}$	$\frac{75}{6^3}$	$\frac{15}{6^3}$	$\frac{1}{6^3}$

$$P(X=-1) = \binom{3}{0}\cdot\left(\frac{1}{6}\right)^0\cdot\left(\frac{5}{6}\right)^3 = \frac{125}{6^3}$$

$$P(X=1) = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{6^3}$$

$$P(X=2) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \frac{15}{6^3}$$

$$P(X=3) = \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{6^3}$$

$$E(X) = \frac{1}{6^3}(-125 + 75 + 2\times 15 + 3)$$
$$= -\frac{17}{6^3}$$

(b)

X	-1	1	2	10
P	$\frac{125}{6^3}$	$\frac{75}{6^3}$	$\frac{15}{6^3}$	$\frac{1}{6^3}$

$$E(X) = \frac{1}{6^3}(-125+75+30+10)$$
$$= -\frac{10}{6^3}$$

(10/10)