MEEG 332 Chapter 11 Coding Challenge 3 (2019)

Due May 10 by midnight

For full-credit, you must write a cover sheet (or sheets) that describes your process for each problem, and you must comment all variable assignment statements within your source code. Turn in your cover sheet(s), source code, any relevant command line output, and labeled plots.

In this coding assignment you will numerically determine the laminar boundary layer velocity profile over a flat plate and all properties that can be derived from that (i.e. boundary layer thickness, shear stress, and drag).

The equations defining the 'exact' solution to the boundary layer equations were derived in class as:

$$\frac{u}{U_\infty} = f'(\eta) \tag{1}$$

where η is a dimensionless "similarity parameter" defined as

$$\eta \equiv y \left(\frac{U_\infty}{\nu x}\right)^{1/2} \tag{2}$$

The function $f(\eta)$ is to be obtained by solving a simplified x-momentum equation:

$$f''' + \tfrac{1}{2} f f'' = 0 \tag{3}$$

Subject to three boundary conditions

$$f(0) = 0 \tag{4}$$
$$f'(0) = 0 \tag{5}$$
$$f'(\infty) \to 1 \tag{6}$$

1. Numerically solve Eq. 3 for $f(\eta)$ subject to boundary conditions Eq. 4-6 using a 'shooting method.'*

   Eq. 3 is a non-linear equation with no analytic solution. There are multiple ways to solve it numerically, but perhaps the best relies on standard ODE solvers such as the Runge-Kutta method† which are designed integrate initial value problems of the form $d\tilde{y}/dt = g(t, \tilde{y})$ forward in time from set of initial conditions ($\tilde{y}(0) = \tilde{y}_0$). The tilde indicates that the ODE solvers can integrate multiple unknown variables $\tilde{y} = (y_1(t), y_2(t), y_3(t), y_4(t), ..., y_N(t))$ at one time.

   (a) (10 points) Transform Eq. 3 into equations suitable for ODE integration by making the substitutions/definitions: $y_1 = f$, $y_2 = df/d\eta = dy_1/d\eta$, and $y_3 = d^2 f/d\eta^2 = dy_2/d\eta$. You should find three equations, which you can write in the form $d\tilde{y}/d\eta = g(\tilde{\eta})$.

   (b) (10 points) Write the three equations you find as a MATLAB function that returns the value of the derivatives $d\tilde{y}/d\eta$. Show that it runs by evaluating it at $y_1 = 0, y_2 = 0$, and $y_3 = 1$.

   (c) (30 points) Using either a built-in ODE integrator or writing your own (such as an Euler method), determine $\tilde{y}(\eta)$ from $0 < \eta < 10$, using the initial conditions $y_1(0) = f(0) = 0$, $y_2(0) = f'(0) = 0$, and $y_3(0) = 1$. Note that the last initial condition, $y_3(0) = f''(0) = 1$ is made-up (just a guess; not provided in the problems statement; the real 3rd condition was $f'(\infty) \to 1$), and the 'solution' you get for this guess won't give you the correct boundary value $f'(\infty) \to 1$. Don't worry about that for now, because we'll fix it in the next step. For now, just plot the solutions that are calculated ($y_1 = f(\eta), y_2 = f'(\eta), y_3 = f''(\eta)$) using these initial conditions.

*If you have forgotten what a shooting method is, take 5 minutes to remind yourself/go look it up on the internet.

†built-in to MATLAB as `ode45` and Python as `scipy.integrate.odeint` routines

(d) (10 points) In part c our guess for $y_3(0) = f''(0)$ did not give us the correct boundary value of $f'(\infty) \to 1$ that we were hoping for, but by viewing $y_3(0) = f''(0) = \alpha$ as an adjustable parameter, we can look for a value of α that would. To find it, 'shoot' for it. Make different guesses of α and see which guess 'hits the target' of $y_2 = f'(\infty) \to 1$. Note that you won't be able to integrate to $\eta \to \infty$, but $\eta = 10$ is close enough.

(e) (Bonus, 10 points) [if you do this] Write an *algorithm* to calculate the exact value of $y_3(0) = f''(0) = \alpha$, that yields the correct boundary condition, $y_2(\eta = 10) = 1$ (Hint: A cleverly devised `fzero` call can do this)

2. (10 points) Use your solution for α to calculate and plot the exact velocity profile $u/U_\infty = f'(\eta)$ from $0 < \eta < 5$. Compare that against the profile that we used for integral analysis in class

$$\begin{aligned} u/U_\infty &= \sin\left[\frac{\pi}{2}\left(\frac{\eta}{4.8}\right)\right] && 0 < \eta < 4.8 \\ &= 1 && 4.8 < \eta < \infty \end{aligned} \tag{7}$$

i.e. plot $f'(\eta)$ (your ODE solution) and Eq. 7 on the same plot.

3. (10 points) Using the velocity profile you calculated, determine the displacement thickness

$$\delta^*(x) = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy = \left(\frac{U_\infty}{\nu x}\right)^{-1/2} \underbrace{\int_0^\infty \left(1 - \frac{u}{U_\infty}\right) d\eta}_{A} \tag{8}$$

Report the value of A and compare it to the value given in your book.

4. (10 points) Using the velocity profile you calculated, determine the momentum thickness[‡] defined as

$$\theta(x) = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \left(\frac{U_\infty}{\nu x}\right)^{-1/2} \underbrace{\int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) d\eta}_{B} \tag{9}$$

Report the value of B and compare it to the value given in your book.

[‡]The drag on a plate of length x is $D(x) = \rho U_\infty^2 b\theta(x)$, so calculating $\theta(x)$ is equivalent to calculating the drag on the plate. Similarly, the shear stress at any point on the plate is $\tau_w(x) = \rho U_\infty^2 d\theta/dx$.