ELEG 310 Chap 5 Multiple RVs 3/26/19

X, Y $\quad P(X=k \cap Y=l)$ $\quad k, l$ integer outcomes

$= p_{XY}(k,l)$ 2D PMF

$P(X=k \cap Y=l) = P(\{X=k\} \cap \{Y=l\}) = P(X=k, Y=l)$

intersection

CDF – $F_{XY}(u,v) = P(X \le u \cap Y \le v)$

Can get 1D PMF from 2D PMF –

$$P(X=k) = p_X(k) = \sum_l P(X=k \cap Y=l) = \sum_l p_{XY}(k,l)$$

$$p_Y(l) = \sum_k P(X=k \cap Y=l) = \sum_k p_{XY}(k,l)$$

Conditional Prob $\quad P(A|B) = \dfrac{P(AB)}{P(B)}$

$$P_{Y|X=k}(Y=\ell \mid X=k) = \frac{P(X=k, Y=\ell)}{P(X=k)} = \frac{P_{XY}(k,\ell)}{P_X(k)}$$

Expected Values – probabilistic averages

$g(X,Y)$ = some function

$$E[g(X,Y)] = \sum_k \sum_l g(k,\ell)\, P_{XY}(k,l)$$

$$\sum_\ell \sum_k P_{XY}(k,\ell) = 1 \quad \text{and} \quad P_{XY}(k,\ell) \geq 0$$

Ex: $E[c] = \sum_k \sum_\ell c\, p(k,l) = c$

Ex. $E[X] = \sum_k k\,p_X(k)$ $\qquad g(X,Y) = X$

$$= \sum_k \sum_l k\, p_{XY}(k,l) = \sum_k k \underbrace{\left(\sum_l p_{XY}(k,l) \right)}_{p_X(k)}$$

$$E[XY] = \sum_k \sum_l kl\, p_{XY}(k,l) = \text{correlation} = \mu_{XY}$$

$$E[\underbrace{(X-\mu_X)(Y-\mu_Y)}_{g(X,Y)}] = \text{covariance} = \sigma_{XY}$$

$\mu_X = E[X] \quad \mu_Y = E[Y]$ $\qquad \left[\sigma_X^2 = E[(X-\mu_X)^2] = \sigma_{XX} \right]$

Theorem: $\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y = \mu_{XY} - \mu_X\mu_Y$

Independence (A & B are ind $\Leftrightarrow$ $P(AB) = P(A)P(B)$)

X & Y are ind $\Leftrightarrow$ $P_{XY}(k,l) = P_X(k) P_Y(l)$ for all k, l

Ex. $g(X,Y) = g_1(X)\, g_2(Y)$ and X, Y ind,

$$E[g(X,Y)] = E[g_1(X) g_2(Y)] = \sum_k \sum_l g_1(k) g_2(l) \underbrace{P_{XY}(k,l)}_{P_X(k) P_Y(l)}$$

$$= \underbrace{\left(\sum_k g_1(k) P_X(k)\right)}\left(\sum_l g_2(l) P_Y(l)\right)$$

$$= E[g_1(X)]\; E[g_2(Y)]$$

if X & Y ind $\Rightarrow E[XY] = E[X]\,E[Y] \Rightarrow \sigma_{XY} = 0$

X & Y ind $\Rightarrow \sigma_{XY} = 0$

if $\sigma_{XY} = 0$, say X & Y are <u>uncorrelated</u>

ind $\Rightarrow$ uncorrelated

<u>But</u> unc $\not\Rightarrow$ ind

Def: $\rho = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$ ← dimensionless ⇒ number

(rho)

= correlation coefficient

$\sigma_X = \sqrt{\sigma_X^2}$ $\sigma_Y = \sqrt{\sigma_Y^2}$

σ_X = standard deviation of X

Theorem $-1 \le \rho \le 1$

Moments

Mean $\mu_X = E[X]$ $\quad\quad$ $\mu_Y = E[Y]$

Variance $\sigma_X^2 = Var[X] = E[(X-\mu_X)^2] = E[X^2] - \mu_X^2$

$\sigma_Y^2 = Var[Y]$ $\quad\quad$ std $\sigma_Y = \sqrt{\sigma_Y^2}$

correlation $\mu_{XY} = E[XY]$

covariance $\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \mu_{XY} - \mu_X\mu_Y$

correlation coeff $\rho = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

Sums of ind R.Vs

$$S = X_1 + X_2 + \cdots + X_n \qquad X_i = RV$$

$$X_i \,\&\, X_j \text{ ind } i \neq j$$

$$E[S] = E[X_1 + \cdots + X_n]$$
$$= E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$= \mu_1 + \mu_2 + \cdots + \mu_n$$

Always works. Does not need ind.

$$\mathrm{Var}[S] = E[(S - E[S])^2]$$
$$= E[((X_1 - \mu_1) + (X_2 - \mu_2) + \cdots + (X_n - \mu_n))^2]$$
$$= E[(X_1-\mu_1)^2 + (X_1-\mu_1)(X_2-\mu_2) + \cdots + (X_1-\mu_1)(X_n-\mu_n)$$
$$+ (X_2-\mu_2)^2 + (X_2-\mu_2)(X_1-\mu_1) + (X_2-\mu_2)(X_3-\mu_3) +$$
$$+ \cdots]$$

$$= E[(X_1-\mu_1)^2] + E[(X_2-\mu_2)^2] + \cdots + E[(X_n-\mu_n)^2]$$

$$+ 2\, E[(X_1-\mu_1)(X_2-\mu_2)] + 2\, E[(X_1-\mu_1)(X_3-\mu_3)]$$

$$+ \cdots$$

$$= \sigma_1^2 + \sigma_2^2 + \cdots \sigma_n^2 \leftarrow \text{sum of variances}$$

$$\underbrace{+\, 2\left(\sigma_{12} + \sigma_{13} + \cdots \sigma_{1n} + \sigma_{23} + \cdots + \sigma_{2n} + \cdots \sigma_{n-1,n}\right)}_{\text{twice sum of covariances}}$$

if $X_1, X_2, \ldots X_n$ are ind.

$$Var[S] = \sigma_S^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$$