ELEG 310 Midterm 1, March 22, 2012

NAME:

1. From a standard 52 card, well shuffled, deck of cards, draw three cards (without replacement). What are the following? (For each, give an expression. You do not have to evaluate it.)

a) Pr[three Aces]

b) Pr[two Aces and one King]

c) Pr[two Aces and another card that isn't an Ace]

d) Pr[three hearts]

e) Pr[three different suits]

a) $P(3 \text{ Aces}) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{4}{\frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1}} = \frac{4 \cdot 6}{52 \times 51 \times 50}$

b) $P(2 \text{ Aces} \,\&\, 1 \text{ King}) = \frac{\binom{4}{2}\binom{4}{1}}{\binom{52}{3}} = \frac{6 \times 4}{\frac{52 \times 51 \times 50}{3 \times 2 \times 1}} = \frac{6 \times 4}{52 \times 17 \times 25}$

c) $P(2 \text{ Aces} \,\&\, 1$ ~~non-King~~ non-ace$) = \frac{\binom{4}{2}\cancel{\binom{4}{0}}\cancel{\binom{44}{1}}\binom{48}{1}}{\binom{52}{3}} = \frac{6 \times \cancel{44}\,48}{\frac{52 \times 51 \times 50}{3 \times 2 \times 1}} = \frac{36 \times \cancel{44}\,48}{52 \times 51 \times 50}$

d) $P(3 \text{ hearts}) = \frac{\binom{13}{3}}{\binom{52}{3}} = \frac{\frac{13 \times 12 \times 11}{\cancel{3 \times 2 \times 1}}}{\frac{52 \times 51 \times 50}{\cancel{3 \times 2 \times 1}}} = \frac{13 \times 12 \times 11}{52 \times 51 \times 50}$

e) $P(3 \text{ diffirent suits}) = 4 \times \frac{\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{3}} = \frac{\cancel{4} \times \cancel{13} \times 13 \times 13 \times 6}{\cancel{52} \times 51 \times 50}$

NAME:

2. $\boldsymbol{X}$ and $\boldsymbol{Y}$ have joint PMF below.

y \ x	0	1	2	3
1	0.3	0.1	0.0	0.2
0	0.1	0.1	0.1	0.1

What are the following?

a) $E[\boldsymbol{X}]$

b) $E[\boldsymbol{Y}]$

c) $\mathrm{Cov}[\boldsymbol{X},\boldsymbol{Y}]$

d) $\Pr[\boldsymbol{Y}=1|\boldsymbol{X}=3]$

e) Are $\boldsymbol{X}$ and $\boldsymbol{Y}$ independent? Why or why not?

a) $P_x = [0.4, 0.2, 0.1, 0.3]$ $EX = 0\times 0.4 + 1\times 0.2 + 2\times 0.1 + 3\times 0.3 = 1.3$

b) $P_y = [0.4, 0.6]$ $EY = 0\times 0.4 + 1\times 0.6 = 0.6$

c) $\cancel{\mathrm{Cov}[X,Y]}$ $EXY = 1\times 1\times 0.1 + 2\times 1\times 0.0 + 3\times 1\times 0.2 = 0.7$

$\mathrm{Cov}[X,Y] = 0.7 - 1.3\times 0.6 = -0.08$

d) $P(Y=1|X=3) = \dfrac{0.2}{0.3} = \dfrac{2}{3}$

e) NO. if ind, then $\mathrm{cov}(X,Y] = 0$

Since $\mathrm{cov}(X,Y) \neq 0$ then X & Y are dependent

NAME:

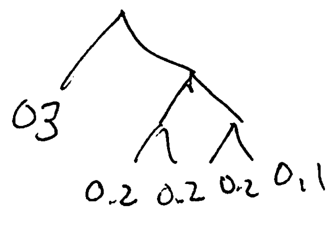
3. A five letter alphabet has probabilities $[0.3, 0.2, 0.2, 0.2, 0.1]$.

a) Find a coding tree that has average code length less than 3 bits per symbol (but not the optimal tree).

b) Find the optimal coding tree. What is it's average code length?

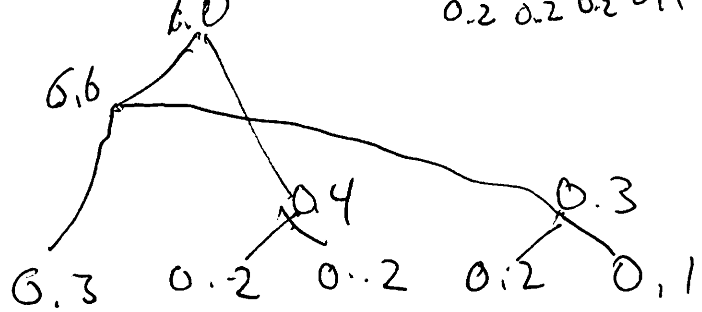
c) What is the entropy of this source? How is the entropy related to the average code length?

a) Here's one.



$EL = 1\times 0.3 + 3\times 0.7 = 2.4$ bits/~~pixel~~ letter

b)



$EL = 2\times 0.3 + 2\times(0.2+0.2) + 3\times(0.2+0.1) = 0.6+0.8+0.9 = 2.3$ bits/~~pix~~ letter

c) $H(X) = -0.3\times\log_2 0.3 - 0.2\log_2 0.2 - 0.2\log_2 0.2 - 0.2\log_2 0.2 - 0.1\log_2 0.1$

$EL(X) \geq H(X)$ for any decodable code

NAME:

4. There are two urns. The first urn has 1 red marble and 2 blue marbles. The second urn has 1 red marble and 4 blue marbles. The first urn is selected ($U = 1$) with probability 1/3; the second urn is selected ($U = 2$) with probability 2/3. After selecting an urn, the person reaches in and blindly selects a marble. What are the following?

a) $\Pr[\text{blue marble}]$

b) $\Pr[U = 1|\text{blue marble}]$

c) $\Pr[\text{blue marble}|U = 1]$

a) $P(\text{Blue}) = P(\text{Blue}|U1)\,P(U1) + P(\text{Blue}|U2)\,P(U2)$

$$= \frac{2}{3} \times \frac{1}{3} + \frac{4}{5} \times \frac{2}{3} = \frac{2}{9} + \frac{8}{15} = \frac{10+24}{45} = \frac{34}{45}$$

b) $P(U1 | \text{Blue}) = \dfrac{P(\text{Blue} \cap U1)}{P(\text{Blue})} = \dfrac{P(\text{Blue}|U1)\,P(U1)}{P(\text{Blue})}$

$$= \frac{\cancel{\frac{2}{3}}}{\cancel{\frac{34}{45}}} = \cancel{\frac{2}{3} \cdot \frac{45}{34}} = \cancel{\frac{15}{17}}$$

$$= \frac{\frac{2}{3} \cdot \frac{1}{3}}{\frac{34}{45}} = \frac{2}{9} \cdot \frac{45}{34} = \frac{10}{34}$$

c) $P(\text{blue}|U1) = \frac{2}{3}$