ELEG 310

1 March 2018

Midterm March 22

Study Abroad – Jan 2019 London
interest meeting 3/16 4pm ISuite
4/6

Discrete Random Variables, X, N X_1 X_2 X_3

$$P(N=k), \quad P(\underbrace{N=k}_{A} \mid \underbrace{N \le \ell}_{B}) = \frac{P(N=k \text{ and } N \le \ell)}{P(N \le \ell)}$$

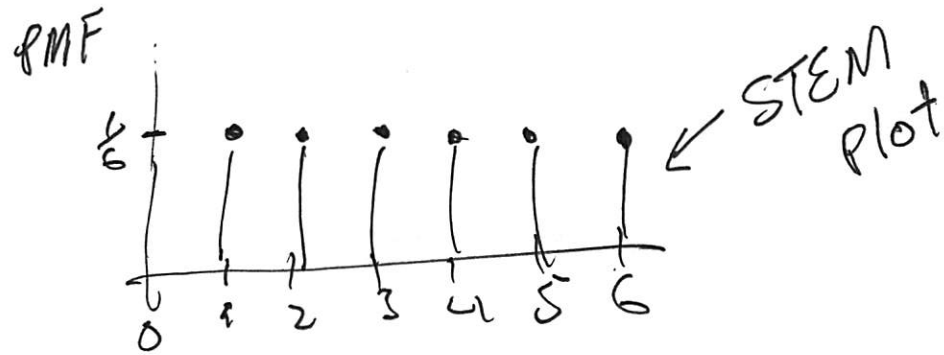
$$P(A|B) = \frac{P(AB)}{P(B)}$$

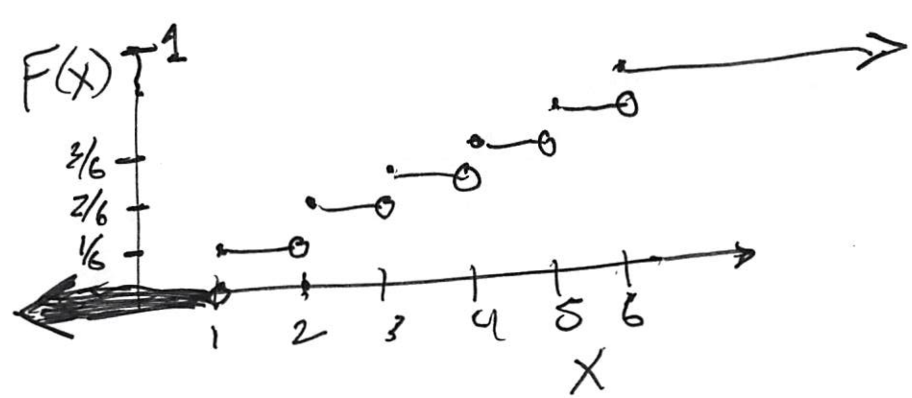
Probability mass function (PMF) $p_k = P(N=k)$

PMF $P(N=k)$ for all k

Cumulative Distribution Function (CDF)

$$F_N(x) = P(N \le x) \qquad x \text{ is cts}$$





$$P(k < N \le l) = \sum_{i=k+1}^{l} p_i = \sum_{i=0}^{l} p_i - \sum_{i=0}^{k} p_i = F_N(l) - F_N(k)$$

$$P(N=k \mid N \le \ell) = \frac{P(N=k \text{ and } N \le \ell)}{P(N \le \ell)}$$

$$P(N=k \text{ and } N \le \ell) = \begin{cases} P(N=k) & \text{if } k \le \ell \\ 0 & \text{o.w.} \end{cases}$$

$$P(N \le \ell) = \sum_{i \le \ell} P(N=i) = \sum_{i \le \ell} p_i$$

$$P(N=k \mid N \le \ell) = \begin{cases} \dfrac{P(N=k)}{\sum_{i \le \ell} P(N=i)} & k \le \ell \\ 0 & \text{o.w} \end{cases}$$

Ex: roll 6-sided die

$$P(N=2 \mid N \le 4) = \frac{P(N=2)}{P(N \le 4)} = \frac{1/6}{4/6} = \frac{1}{4}$$

Expected Value – statistical Average

Ex. $N \sim$ roll of die m-sided die

$$EN = \sum_{k=1}^{m} k\, P(N=k) = \sum_{k=1}^{m} k\, p_k = \underline{\text{mean}}$$

Ex. m-sided die

Ex. $\frac{1+2+3+4+5+6}{6} = 3.5$

$$EN = \frac{m+1}{2} = \mu$$

Variance $\quad E\left((N-EN)^2\right) = \sum_{k=1}^{m} (k-\mu)^2 p_k = \sigma^2$

$EN = \text{mean} = \mu = \sum_{k=1}^{\infty} k\, p_k$ ⟵ Statistical average

$\text{Var}\, N = \text{variance} = \sigma^2 = \sum_{k=1}^{\infty} (k-\mu)^2 p_k$ ⟵ squared deviation from average

Standard Deviation $= \sigma = \sqrt{\sigma^2}$

Theorem: $\sigma^2 = E N^2 - \mu^2$

$E(N)$ EN

$$\sigma^2 = E((N-\mu)^2) = E(N^2 - 2\mu N + \mu^2)$$

$$= E(N^2) - E(2\mu N) + \underbrace{E(\mu^2)}_{\mu^2}$$

$$= E(N^2) - 2\mu^2 + \mu^2$$

$E(N^2) - \mu^2$

in general $E g(N) = \sum_{k=1}^{\infty} g(k)\, p_k$

Expected value algebra

$$E(aX+b) = a\,EX + b$$

$$E(g_1(X) + g_2(X)) = Eg_1(X) + Eg_2(X)$$

$$E\left(g_1(X)\,g_2(X)\right) \neq E(g_1(X))\,E(g_2(X))$$

$$\mu = EX$$

$$\sigma^2 = E(X-\mu)^2 = EX^2 - \mu^2$$

Moment Generating Function (MGF)

$$M(u) = E\left(e^{uX}\right) = \sum_{k=1}^{\infty} e^{uk} p_k$$

moments EX^l $l=1 \Rightarrow$ mean

$$M(u) = E(e^{uX})$$

$$\frac{dM}{du} = \frac{d}{du} E(e^{uX}) = E\left(\frac{d}{du} e^{uX}\right) = E(Xe^{uX})$$

$$\left.\frac{dM}{du}\right|_{u=0} = \left.E(Xe^{u\cancel{X}})\right|_{u=0} = EX$$

$$\left.\frac{d^2M}{du^2}\right|_{u=0} = X^2$$

$$\frac{d^k}{du^k} M(u) = EX^k$$

Ex. $X \sim \text{Geometric}(p)$ $\quad P(X=k) = q^{k-1}p \quad k=1,2,\dots$

$$EX = \sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} k q^{k-1} p \qquad q = 1-p$$

$$M(u) = \sum_{k=1}^{\infty} e^{uk} p_k = \sum_{k=1}^{\infty} e^{uk} q^{k-1} p \qquad \ell = k-1$$

$$k = \ell + 1$$

$$= \sum_{\ell=0}^{\infty} e^{u\ell} e^{u} q^{\ell} p = pe^{u} \sum_{\ell=0}^{\infty} (qe^{u})^{\ell}$$

$$= \frac{pe^{u}}{1-qe^{u}} = pe^{u}(1-qe^{u})^{-1}$$

$$\left.\frac{dM}{du}\right|_{u=0} = p\,\cancel{e^{u}}^{\,1}(1-q\,\cancel{e^{u}}^{\,1})^{-1} + p\,\cancel{e^{u}}^{\,1}(-1)(1-q\,\cancel{e^{u}}^{\,1})^{-2}\,q\,\cancel{e^{u}}^{\,1}(-1) = \frac{1}{p}$$