ELEG 305
SOLUTIONS TO EXAM #2 (4/17/18)

#1. a.) The Fourier series coefficients are periodic. Therefore, they must represent a discrete-time signal.

b.) $|a_k| = |a_{-k}|$ (even), $\measuredangle a_k = -\measuredangle a_{-k}$ (odd) $\Big\rangle$ $a_k = a_{-k}^*$ conjugate symmetry

$\therefore$ $x[n]$ is real

c.) Use Parseval's Relation:

$$\text{average power} = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

$$= |1|^2 + \left|-\frac{1}{2j}\right|^2 + \left|\frac{1}{2j}\right|^2 + |1|^2$$

$$= 1 + \frac{1}{4} + \frac{1}{4} + 1$$

$$= 2\tfrac{1}{2}$$

d.) $\omega_0 = 2\pi/N = 2\pi/9$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = 1\cdot e^{-j3\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n} + \frac{1}{2j}e^{j\omega_0 n} + 1\cdot e^{j3\omega_0 n}$$

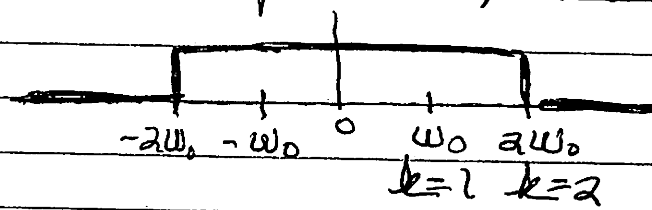
<use [-4, 4]

$$= 2\cos 3\omega_0 n + \sin \omega_0 n$$

$$= 2\cos\frac{2\pi}{3}n + \sin\frac{2\pi}{9}n$$

#1 cont'd) e.) $x[n] \rightarrow \boxed{\text{filter}} \rightarrow y[n]$

ideal lowpass filter, cutoff $2\omega_0$



Cutoff at $2\omega_0$ filters out the $\cos \frac{2\pi}{3} n$ term, leaving

$$y[n] = \sin \frac{2\pi}{9} n$$

#2 a.) Use Parseval's Relation:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\therefore \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 2\pi \sum_{n=-2}^{2} |n^2|^2$$

$$= 2\pi \sum_{n=-2}^{2} n^4$$

$$= 2\pi \left[16 + 1 + 0 + 1 + 16 \right]$$

$$= 68\pi$$

b.) $y(t) = x(t) \cos \omega_0 t$

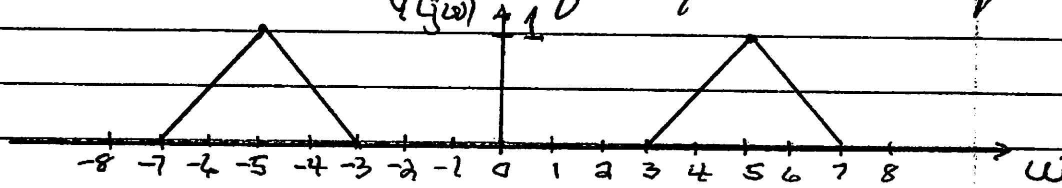
$$= x(t) \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right)$$

$$= \frac{x(t)}{2} e^{j\omega_0 t} + \frac{x(t)}{2} e^{-j\omega_0 t}$$

#2b contd) Use the frequency-shift property (4.3.6)

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

$$\therefore Y(j\omega) = \underbrace{\frac{1}{2} X(j(\omega-\omega_0))}_{\text{shift up in frequency}} + \underbrace{\frac{1}{2} X(j(\omega+\omega_0))}_{\text{shift down in frequency}}$$



c) Let $x_1(t) = e^{-3t} u(t)$

and $x_2(t) = e^{-t} u(t-2)$

Then,

$$x(t) = \frac{d}{dt}\left[x_1(t) * x_2(t)\right]$$

$$\therefore X(j\omega) = j\omega \cdot X_1(j\omega) \cdot X_2(j\omega)$$

$$X_1(j\omega) = \mathcal{F}\left[e^{-3t} u(t)\right] = \frac{1}{j\omega+3}$$

$$X_2(j\omega) = \mathcal{F}\left[e^{-t} u(t-2)\right]$$

$$= \int_2^{\infty} e^{-t} e^{-j\omega t}\, dt$$

$$= \frac{1}{j\omega+1} e^{-(1+j\omega)2}$$

$$\therefore X(j\omega) = \frac{j\omega\, e^{-(1+j\omega)2}}{(j\omega+3)(j\omega+1)}$$

#3. a.) $x(t) = e^{-3t}u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = e^{-3(t-2)}u(t-2)$

i.) Approach #1 : It is clear that $y(t) = x(t-2)$
Therefore, $h(t) = \delta(t-2)$

ii.) Approach #2 :

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$Y(j\omega) = \frac{1}{j\omega+3}e^{-2j\omega}$$

$$X(j\omega) = \frac{1}{j\omega+3}$$

$$\therefore H(j\omega) = e^{-2j\omega} \longleftrightarrow h(t) = \delta(t-2)$$

b.) $$H(e^{j\omega}) = \frac{2 + \frac{1}{4}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{4}e^{-j\omega})}$$

$$= \frac{2+\frac{1}{4}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega})(1-\tfrac{1}{4}e^{-j\omega}-\tfrac{1}{8}e^{-2j\omega}) = X(e^{j\omega})(2+\tfrac{1}{4}e^{-j\omega})$$

$$Y(e^{j\omega}) - \frac{1}{4}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{8}e^{-2j\omega}Y(e^{j\omega})$$

$$= 2X(e^{j\omega}) + \frac{1}{4}e^{-j\omega}X(e^{j\omega})$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 2x[n] + \frac{1}{4}x[n-1]$$

#3 cont'd) a.) $y[n-2]+6y[n] = 8x[n-1]+18x[n]$

$$\downarrow \mathcal{F}$$

$$Y(e^{j\omega})e^{-2j\omega}+6Y(e^{j\omega}) = 8X(e^{j\omega})e^{-j\omega}+18X(e^{j\omega})$$

$$Y(e^{j\omega})(e^{-2j\omega}+6) = X(e^{j\omega})(8e^{-j\omega}+18)$$

$\therefore$ frequency response $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

$$= \frac{8e^{-j\omega}+18}{e^{-2j\omega}+6}$$

#4. LTI system

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t) = 3\frac{dx(t)}{dt}+8x(t)$$

$$\downarrow \mathcal{F}$$

a.)

$$(j\omega)^2Y(j\omega)+5j\omega Y(j\omega)+6Y(j\omega) = 3j\omega X(j\omega)+8X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2+5j\omega+6\right] = X(j\omega)\left[3j\omega+8\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3j\omega+8}{(j\omega)^2+5j\omega+6}$$

b.) $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$

$$H(j\omega) = \frac{3j\omega+8}{(j\omega)^2+5j\omega+6} = \frac{3j\omega+8}{(j\omega+3)(j\omega+2)}$$

$$= \frac{A}{j\omega+3}+\frac{B}{j\omega+2}$$

#4b. cont'd)

$$A = H(j\omega)(j\omega+3)\Big|_{j\omega=-3} = \frac{3j\omega+8}{j\omega+2}\Big|_{j\omega=-3} = \frac{-1}{-1} = 1$$

$$B = H(j\omega)(j\omega+2)\Big|_{j\omega=-2} = \frac{3j\omega+8}{j\omega+3}\Big|_{j\omega=-2} = \frac{2}{1} = 2$$

$$\therefore H(j\omega) = \frac{1}{j\omega+3} + \frac{2}{j\omega+2}$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$h(t) = e^{-3t}u(t) + 2e^{-2t}u(t)$$

Extra Credit:

(i) $h(t) = 0$ for $t<0 \Rightarrow$ system is causal

(ii) $\int_{-\infty}^{\infty} |h(t)|^2 dt < \infty \Rightarrow$ system is stable

#5. Call the impulse response of the synapse block

$h_s[n] \Rightarrow \delta[n-1]$ because just a delay of one unit.

a.) cascade $h[n] = h_1[n] * h_s[n] * h_2[n]$

$$\therefore H(e^{j\omega}) = H_1(e^{j\omega}) \cdot H_s(e^{j\omega}) \cdot H_2(e^{j\omega})$$

- $H_1(e^{j\omega}) = \mathcal{F}\{(\frac{1}{2})^n u[n]\} = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$
- $H_2(e^{j\omega}) = \mathcal{F}\{(\frac{1}{4})^n u[n]\} = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$
- $H_s(e^{j\omega}) = \mathcal{F}\{\delta[n-1]\} = e^{-j\omega}$

#5a. cont'd)

$$\therefore H(e^{j\omega}) = \frac{e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$ just a delay in time

b.) • Approach #1: First find $h'[n] = h_1[n] * h_2[n]$

$H'(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$

Then $h[n] = h'[n-1] \longrightarrow$ just delay by one unit

$$H'(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

$$A = H'(e^{j\omega})(1-\tfrac{1}{2}e^{-j\omega})\Big|_{e^{-j\omega}=2}$$

$$= \frac{1}{1-\frac{1}{4}e^{-j\omega}}\Big|_{e^{-j\omega}=2} = \frac{1}{1-\frac{1}{2}} = 2$$

$$B = H'(e^{j\omega})(1-\tfrac{1}{4}e^{-j\omega})\Big|_{e^{-j\omega}=4}$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}}\Big|_{e^{-j\omega}=4} = \frac{1}{1-2} = -1$$

$$\therefore H'(e^{j\omega}) = \frac{2}{1-\frac{1}{2}e^{-j\omega}} - \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$h'[n] = 2\left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{4}\right)^n u[n]$$

$$\therefore h[n] = h'[n-1]$$

$$= 2\left(\tfrac{1}{2}\right)^{n-1} u[n-1] - \left(\tfrac{1}{4}\right)^{n-1} u[n-1] \quad *$$

#4b cont'd

- Approach #2:

$$H(e^{j\omega}) = \frac{e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

$$A = \left.\frac{e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}\right|_{e^{-j\omega}=2} = \frac{2}{\frac{1}{2}} = 4$$

$$B = \left.\frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right|_{e^{-j\omega}=4} = \frac{4}{-1} = -4$$

$$H(e^{j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{4}{1-\frac{1}{4}e^{-j\omega}}$$

$$\big\downarrow \mathcal{F}^{-1}$$

$$\therefore h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{4}\right)^n u[n] \quad \star$$

NOTE: $*$ and $\star$ are the same function. Simply compute the values of each solution for different values of n.

n	$*$	$\star$
0	0	0
1	1	1
2	3/4	3/4
⋮	⋮	

Extra Credit: The complete system in this problem is

$$\xrightarrow[x[n]]{\text{Neuron \#1}} \boxed{h[n]} \xrightarrow[y(n)]{\text{Neuron \#2}}$$

$H(e^{j\omega})$ → we need to compute inverse of this function

$y(n)$ → to recover $x(n)$, we need to pass this through system that undoes $h[n]$

#5 Ec cont'd

$$y[n] \longrightarrow \boxed{h^{inv}[n]} \longrightarrow x[n]$$

so, if we want to recover $x[n]$, simply multiply $Y(e^{j\omega})$ by

$$H^{inv}(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

$$H^{inv}(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}{e^{-j\omega}}$$

$$= e^{j\omega}\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)$$

$$= e^{j\omega} - \frac{3}{4} + \frac{1}{8}e^{-j\omega}$$

$$\therefore h^{inv}[n] = \underbrace{\delta[n+1]}_{\text{not causal}} - \frac{3}{4}\delta[n] + \frac{1}{8}\delta[n-1]$$