Sketching the root locus: More rules and an example

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Abstract

The root locus provides information on how the *closed-loop* poles move in the complex plane as a design parameter, e.g. a controller gain, varies. The rules described in this handout can be used to approximately draw the root locus without elaborate computations. More accurate graphs of the root locus can be constructed through the numerical computation of the roots of the characteristic equation for various values of the desired parameter or by using MATLAB's function `rlocus`. However, it is important to be able to check the validity of the numerical results, by knowing a priori (and with minimal computations) what to expect. The purpose of this handout is to fill some details on how to draw approximately the root locus.

1 Preliminaries

To apply the rules for approximately drawing the root locus of a feedback system, whose behavior depends on a (single) parameter of interest, we must first reduce the system to the block diagram shown in Figure 1.

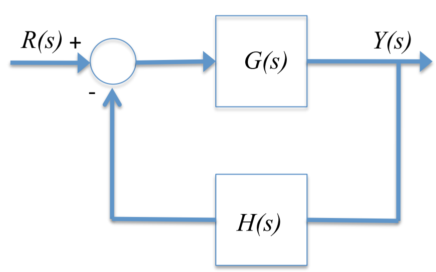

Figure 1: The general form of the block diagram for drawing the root locus.

In the block diagram of Figure 1 we can readily see that the *closed-loop* transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}. \tag{1}$$

Remember that the closed-loop transfer function (1) describes the dynamics involved in "transferring" the input $R(s)$ to the output $Y(s)$, and the roots of its denominator; that is, the solutions of

$$1 + G(s)H(s) = 0, \tag{2}$$

characterize the response of the system to the input $R(s)$. Equation (2) is the *characteristic equation* of the system and its roots are the *closed-loop poles*. The characteristic equation (2) can be written

$$G(s)H(s) = -1,$$

which is equivalent to the following conditions

$$|G(s)H(s)| = 1, \tag{3}$$

$$\angle G(s)H(s) = (2k+1)\pi, \quad k = 0, \pm 1, \pm 2, ... \tag{4}$$

known as the *norm and angle conditions*, respectively. Any point in the complex plane that satisfies the norm and angle conditions, also satisfies the characteristic equation (2), and hence it is a closed-loop pole of the transfer function (1).

From the above observations it is clear that the product $G(s)H(s)$ is important in determining the closed-loop poles since it participates in the norm (3) and angle (4) conditions. The product $G(s)H(s)$ has a special name; it is called the *loop transfer function*. It is easy to find the loop transfer function once the control system is reduced to the block diagram of Figure 1. Just follow the loop and multiply the transfer functions $G(s)$ and $H(s)$ of the forward and feedback parts of the loop, respectively. Generally, for a large family of systems, the loop transfer function can be written as the ratio of two polynomials $Z(s)$ and $P(s)$ multiplied with the parameter K of interest, i.e.

$$G(s)H(s) = K\frac{Z(s)}{P(s)}. \tag{5}$$

This is the form that we need to have in order to draw the root locus. Before we continue with an example of how one can sketch the paths of the roots of the characteristic equation (the closed-loop poles), a few remarks are in order.

Remark 1: One must be careful not to confuse the *closed-loop poles* with the *loop transfer function poles*. The closed-loop poles are the roots of the characteristic equation (2) and determine the response of the system, while the loop transfer function poles are the roots of the denominator $P(s)$ of the loop transfer function. To understand the relationship between the closed-loop poles and the loop transfer function poles, substitute (5) to (2) to get

$$1 + G(s)H(s) = 0 \Leftrightarrow 1 + K\frac{Z(s)}{P(s)} = 0 \Leftrightarrow P(s) + K\ Z(s) = 0,$$

which means that the roots of $1 + G(s)H(s) = 0$ are the same with the roots of $P(s) + K\ Z(s) = 0$. Hence, for $K = 0$ the closed-loop poles—that is, the roots of $1 + G(s)H(s) = 0$—coincide with the roots of $P(s) = 0$, which are the poles of the loop transfer function. In other words, the poles of the loop transfer function coincide with the closed-loop poles *only* when $K = 0$. Equivalently, the poles of the loop transfer function $G(s)H(s)$ represent just an instance of the motion of the closed-loop poles as K changes; namely, the initial locations from which the closed-loop poles start their journey in the complex plane as K changes.

Remark 2: One related question is where the closed-loop poles terminate their motion as K tends to infinity. This can be easily seen by noticing that

$$1 + G(s)H(s) = 0 \Leftrightarrow 1 + K\frac{Z(s)}{P(s)} = 0 \Leftrightarrow P(s) + K\ Z(s) = 0 \Leftrightarrow \frac{P(s)}{K} + Z(s) = 0,$$

which simply means that the roots of $1 + G(s)H(s) = 0$ are the same with the roots of $\frac{P(s)}{K} + Z(s) = 0$. Hence, as $K \to \infty$ the closed-loop poles approach the roots of $Z(s) = 0$; that is, the closed-loop poles approach the zeros of the loop transfer function.

2 An Example

We consider the example of Figure 2, in which the compensator[1]

$$G_\mathrm{c}(s) = K\frac{s+a}{s+b},$$

with $a = 6$ and $b = 4$, is used to control a plant with input/output dynamics described by the transfer function

$$G_\mathrm{p}(s) = \frac{1}{s(s^2+4s+8)}.$$

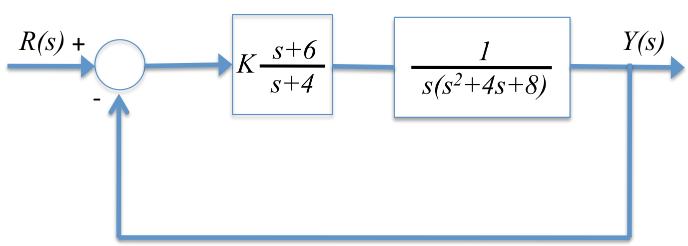

Figure 2: The control system used in the example for the sketching the root locus.

Preliminaries: The given system is in the form of the block diagram of Figure 2 with

$$G(s) = G_\mathrm{c}(s)G_\mathrm{p}(s) = K\frac{(s+6)}{s(s+4)(s^2+4s+8)} \quad \text{and} \quad H(s) = 1.$$

The loop transfer function is readily obtained in the form (5), i.e.

$$K\,\frac{Z(s)}{P(s)} = K\frac{(s+6)}{s(s+4)(s^2+4s+8)}. \tag{6}$$

[1]Note that when the zero of the compensator is to the left of its pole, as in the case of the example, we say that this is a *lag compensator*. Note that if the zero is much faster than the pole, this "slow" pole dominates the behavior of the compensator, which can be viewed as a PI controller.

In (6) there is one zero ($m = 1$) and four poles ($n = 4$). The (real) zero is located at position $s = -6$. Out of the four poles, one is located at the origin ($s = 0$), one at $s = -4$, and there is a complex conjugate pair located at $s = -2 \pm j2$. The position of the poles and zeros of (6) are shown in Figure 3. This is the starting point for sketching the root locus.

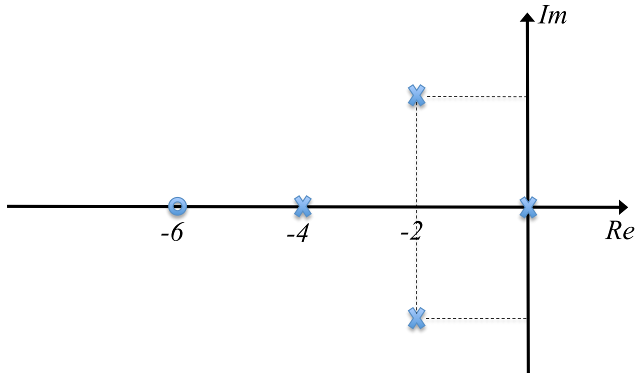

Figure 3: The poles and zeros of the loop transfer function; the first step for drawing the root locus.

To draw the root locus we follow the rules below:

Rule 1: *To each root of the characteristic equation* (2) *there corresponds exactly one branch of the root locus. Equivalently, the total number of branches in the root locus is equal to the number of poles of the loop transfer function.*

Application to our example: We have $n = 4$ poles and $m = 1$ zeros. Hence, we expect to have four branches in the root locus, as expected by the degree of the closed-loop characteristic equation (2).

Rule 2: *Each branch in the root locus starts at a pole of* (6)*, where $K = 0$, and terminates at a zero of* (6)*, where $K \to \infty$. When the number of poles in* (6) *is larger than the number of zeros in* (6)*, i.e., $n > m$, there are $n - m$ poles that go to infinity by following asymptotes that are computed as in Rule 4.*

Application to our example: We have $n = 4$ poles and $m = 1$ zeros. Hence, one of the poles of the loop transfer function is attracted by the zero and we have $n - m = 3$ branches of the root locus that move to infinity as K increases to infinity.

Rule 3: *Segments of the real axis to the left of an odd number of poles or zeros are segments of the root locus. Remember that complex poles and complex zeros have no effect when applying this rule.*

Application to our example: We start from the rightmost end of the real axis and we scan the real axis to the left, noting that the points to the left of an odd number of (real) poles and (real) zeros (no distinction is made between poles and zeros here) must be included in the root locus. In doing so, complex poles or zeros are neglected; only real poles and zeros matter in the application of this rule. The result is shown in Figure 4.

Rule 4: *When the number of poles n is larger than the number of zeros m, as $K \to \infty$, $n - m$ branches of the root locus become straight lines that are called asymptotes. These asymptotes intersect with the real axis at angles*

$$\theta_k = \frac{(2k+1)180^0}{n-m} \ \textit{for} \ \ k = 0, 1, 2, ..., n-m-1. \tag{7}$$

The point at which the asymptotes intersect is given by

$$\sigma_\alpha = \frac{\sum p_i - \sum z_i}{n-m}, \tag{8}$$

where $\sum p_i$ s the sum of the real parts of the poles of (6) *including complex roots, and $\sum z_i$ is the sum of the real parts of the zeros of* (6) *including complex roots.*

Application to our example: As was mentioned above, we expect that $n - m = 3$ branches of the root locus tend to infinity. They do so by following three asymptotes that intersect the real axis at the angles:

$$\theta_k = \frac{(2k+1)180^0}{4-1} = (2k+1)60^0 \ \text{ for } \ k = 0, 1, 2; \tag{9}$$

equivalently, there are three asymptotes that intersect the real axis at angles $\theta_0 = 60^0$, $\theta_1 = 180^0$ and $\theta_2 = 300^0$. It is mentioned that the asymptote at $\theta_1 = 180^0$ corresponds to the real axis to the left of the zero at $s = -6$; see Figure 4. The asymptotes intersect the real axis at the point

$$\sigma_\alpha = \frac{(-4) + (-2 + j2) + (-2 - j2) - (-6)}{4-1} = -\frac{2}{3}. \tag{10}$$

Rule 5: *The branches of the root locus are symmetric with respect to the real axis*

By applying Rules 1 to 5, to our example, we arrive at the Figure 4. The closed-loop poles (roots of the characteristic equation (2)) start when $K = 0$ at the points denoted by "x," which correspond to the poles of (6) (roots of $P(s)$), and as K tends to infinity, they approach the zeros of (6) (roots of $Z(s)$) denoted by "o". Clearly, since we have four poles and one zero in (6), there are three poles that approach infinity along the three asymptotes computed by Rule 5; these are also shown in Figure 4.

The closed-loop poles that start at $s = 0$ and $s = -4$ for $K = 0$ move toward each other and they meet on the real axis. Since, each branch of the root locus much be unique, the two poles must leave the real axis and enter the complex plane. They do so at a point that is called a *breakaway point*. These two closed-loop poles will meet again at the branch of the root locus which is to the left of the zero located at $s = -6$ at a point that is called *break-in point*. After they meet again, one of the closed-loop poles moves to the right and approaches the zero at $s = -6$, while the other continues its journey to infinity along the asymptote $\theta_1 = 180^0$. Finally, the two complex poles located at $s = -2 \pm j2$ when $K = 0$ they move towards the asymptotes $\theta_0 = 60^0$ and $\theta_2 = 300^0$ and along them to infinity as K tends to infinity. Clearly, we need rules that will allow us to compute the locations of the breakaway and break-in points as well as the corresponding angles. The following two rules resolve the issue.

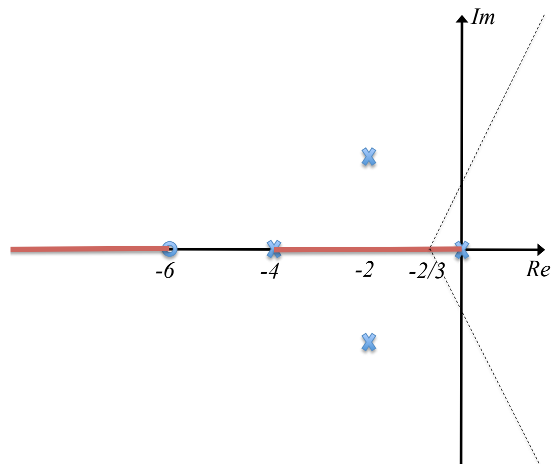

Figure 4: This plot shows the segments of the real axis that are part of the root locus according to Rule 3. It also shows the asymptotes, as these are computed by the equations of Rule 4.

Rule 6: *A breakaway point is obtained when K is at a local maximum and a break-in point is obtained when K is at a local minimum. To compute the points at which K has a critical value, we use the characteristic equation to obtain*

$$P(s) + KZ(s) = 0 \Leftrightarrow K = -\frac{P(s)}{Z(s)}. \tag{11}$$

Breakaway and break-in points are then computed by the condition

$$\frac{dK}{ds} = -\frac{P'(s)Z(s) - P(s)Z'(s)}{Z^2(s)} = 0. \tag{12}$$

Note here that if $s = s_1$ is a root of (12)*, it does not automatically mean that s_1 is a breakaway or break-in point. A simple rule to determine whether or not s_1 is to be accepted as a breakaway or break-in point is the following. Once a root s_1 of* (12) *is computed, we substitute s_1 to the expression* (11) *for K. If the resulting K is positive, then s_1 is a breakaway or break-in point; if not, s_1 is discarded and we proceed with the next root of* (12).

Application to our example: The characteristic equation in our system is

$$1 + G(s)H(s) = 0 \Leftrightarrow 1 + \frac{K(s+6)}{s(s+4)(s^2+4s+8)} = 0,$$

form which we obtain

$$K = -\frac{s(s+4)(s^2+4s+8)}{s+6} \Leftrightarrow \frac{dK}{ds} = -\frac{3s^4 + 40s^3 + 168s^2 + 288s + 192}{(s+6)^2}$$

Differentiating this expression and setting the derivative equal to zero results in a fourth-degree polynomial equal to zero, which has four roots. We must compute the roots of this polynomial. Then, we substitute each root to (11) and we keep only those roots that result in positive gains. In this example, the breakaway point is located (approximately) to -3, while the break-in point is located (approximately) to -7.3. Note that the equation $dK/ds = 0$ has two more roots, which though are complex and have no meaning in locating the breakaway and break-in points.

Note: It is evident that the application of Rule 6 typically involves finding the roots of a high-degree polynomial, a task that is generally cumbersome and contradicts the general philosophy of the root locus, which aims at sketching the path of the closed-loop poles without having to perform elaborate computations.

Rule 7: *Two branches of the root locus (not asymptotes) depart from or arrive at the real axis at angles* $\pm 90^0$. *When three or more branches of the root locus leave the real axis, then the angles of departure and arrival are computed through the use of the angle condition; see Rule 8 below for how this can be done.*

Application to our example: The result of applying the Rules 1-7 is shown in Figure 5.

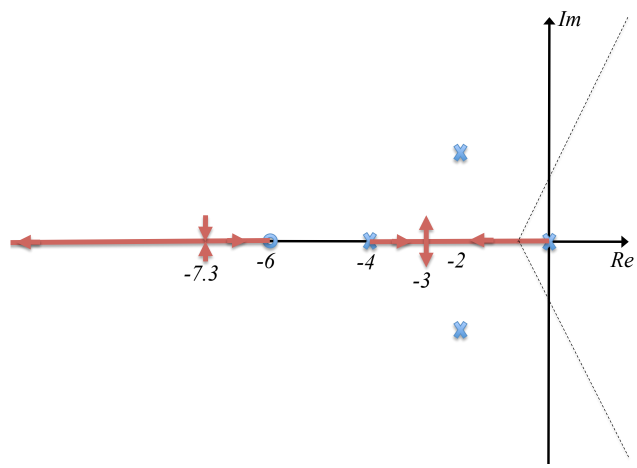

Figure 5: This plot shows the breakaway and break-in points computed by Rule 6. Note that the branches of the root locus that leave and approach the real axis do so at angles $\pm 90^0$ as Rule 7 specifies.

To complete the information needed to sketch the root locus, we could like to know the angles at which the branches depart from the complex poles $-2 \pm j2$. This is done by using the angle condition as detailed in Rule 8 below. Note that this rule can be applied to compute the angle at which a branch of the root locus departs from or arrives to a point on the real axis as well; this is useful when Rule 7 cannot be used.

Rule 8: *To compute the angle of departure from a complex pole or arrival to a complex zero we use the angle condition as follows. We select a test point very close to the complex pole or zero at which the angle of the root locus is of interest. This is done in a way so that the angles of all the other poles and zeros to the test point are the same with their angles to the pole or zero in consideration. Then we apply the angle condition.*

Application to our example: Consider the complex pole at $s = -2 + j2$. Select a test point very close to that pole; the test point must be so close to $s = -2 + j2$ so that angles of the poles and zeros (of the loop transfer function) to this test point are equal to the angles of the poles and zeros to $s = -2 + j2$; see Figure 6. Applying the angle condition gives

$$-\varphi_d - \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4 = 180^0, \tag{13}$$

where ϕ_d is the angle of the departure that we want to compute. The angles $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ can be either measured or computed. For illustration let's compute one of these angles, say φ_3. The test point, let's call it s_{test} almost coincides with the pole at $-2 + j2$ and so $s_{\text{test}} \approx -2 + j2$. Then, as shown in Figure 6 $\varphi_3 = \angle(s_{\text{test}} + 4)$ is the phase of the complex number that starts at -4 and points at s_{test} and we have

$$\tan\varphi_3 = \frac{\text{Im}(s_{\text{test}} + 4)}{\text{Re}(s_{\text{test}} + 4)} = \frac{\text{Im}(-2 + j2 + 4)}{\text{Re}(-2 + j2 + 4)} = \frac{\text{Im}(2 + j2)}{\text{Re}(2 + j2)} = \frac{2}{2} = 1 \Leftrightarrow \varphi_3 = 45^0$$

The values of the rest of the angles are given in the caption of Figure 6 (you should try to compute them all). Substitution of these values for $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ in (13) results in $\phi_d = -63.4^0$ or 296.6^0. By Rule 5, the angle of departure from the complex conjugate pole $s = -2 - j2$ does not need to be computed since the branches of the root locus are symmetric with respect to the real axis.

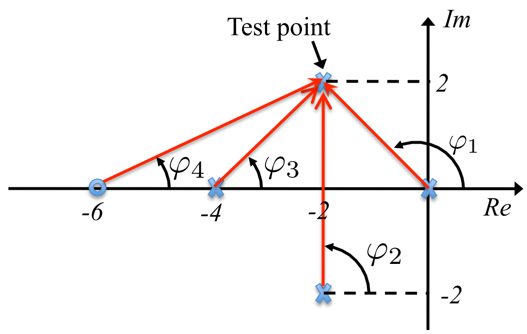

Figure 6: The geometry for computing the angle of departure from the complex pole $-2 + j2$. The test point is selected to be so that it is almost indistinguishable from $-2 + j2$. The angles involved can be computed or measured to be $\varphi_1 = 135^0$, $\varphi_2 = 90^0$, $\varphi_3 = 45^0$ and $\varphi_4 = 26.6^0$.

The complete picture of the root locus is shown in Figure 7 while MATLAB's result is shown in Figure 8.

It should be emphasized that the part of the root locus which is in the neighborhood of the imaginary axis is important, since it is close to the stability boundary. In the example above we see that as the gain K increases, two of the closed-loop poles move in the right-hand side of the complex plane. This implies that the closed-loop system is *conditionally stable*. It is important to know the critical value of the gain K where this happens and make sure that in tuning our controller we do not exceed this value. One way to compute the critical value of the gain K beyond which instability occurs is to substitute $s = j\omega$ in the characteristic equation (this is because at this point the closed-loop poles *intersect* the imaginary axis) so that

$$1 + G(s)H(s) = 0 \Leftrightarrow (j\omega)^4 + 8(j\omega)^3 + 24(j\omega)^2 + (32 + K)(j\omega) + 6K = 0, \tag{14}$$

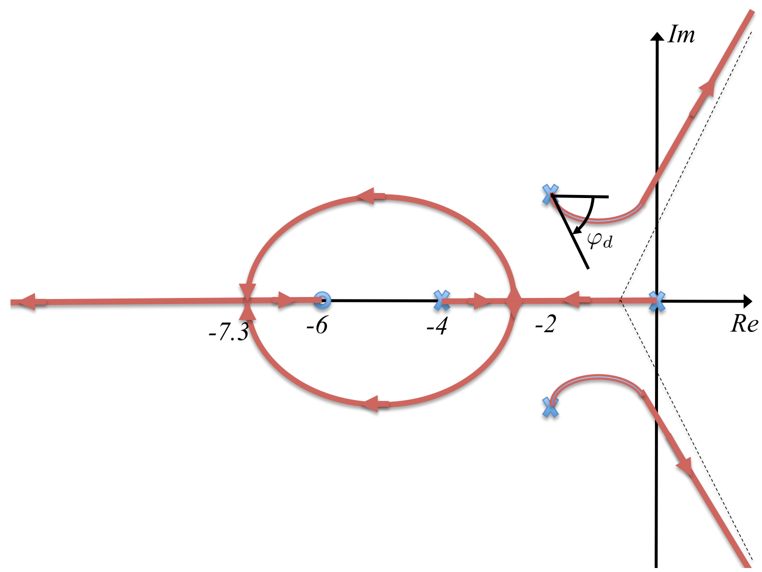

Figure 7: The complete sketch of the root locus. Note that φ_d is the angle of departure computed by (13).

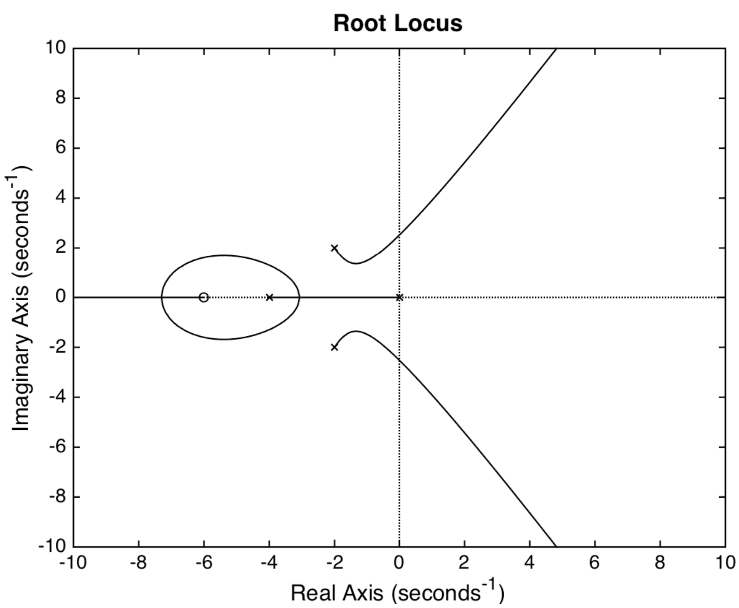

Figure 8: The exact root locus as it is computed by MATLAB.

and solve for ω and K by setting the real and imaginary parts of the equation above to zero. The other way to find the region where K must lie in order not to cause any instability is to use Routh's criterion.