# Math 342

## Homework#2 solutions

**Sec. 6.5:**

2)

(a) By definition

$$\ker(T) = \{\text{all matrices}\, A \in M_{22}\, \text{such that}\, \operatorname{tr}(A) = 0\}$$

Only matrices (ii) and (iii) are in $\ker(T)$ because the sum of their diagonal entries equals zero.

(b) All three scalars (i), (ii) and (iii) are in $\operatorname{range}(T)$ because it is always possible to find matrices in $M_{22}$ whose trace equals the given scalars. For example

$$T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 + 0 = 0$$

$$T\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} = 2 + 0 = 2$$

$$T\begin{pmatrix} \sqrt{2}/2 & \pi \\ 12 & 0 \end{pmatrix} = \frac{\sqrt{2}}{2} + 0 = \frac{\sqrt{2}}{2}$$

(c) $\ker(T)$ is the set of all matrices in $M_{22}$ whose diagonal entries are negatives of each other. $\operatorname{range}(T)$ is all the real numbers.

4)

(a) By definition

$$\ker(T) = \{\text{all polynomials} \in \mathbb{P}_2\, \text{such that}\, xp'(x) = 0 \Longrightarrow p'(x) = 0\, \text{for all}\, x\}$$

or equivalently

$$\ker(T) = \{a + bx + cx^2 = 0 \in \mathbb{P}_2\, \text{such that}\, b + 2cx = 0 \Longrightarrow b = c = 0\, \text{for all}\, x\}$$

Therefore only polynomial (i) is in $\ker(T)$.

(b) We have

$$\operatorname{range}(T) = \{\text{all polynomials} \in \mathbb{P}_2\, \text{of the form}\, xp'(x)\}$$

or equivalently

$$\operatorname{range}(T) = \{a + bx + cx^2 = 0 \in \mathbb{P}_2\, \text{of the form}\, bx + 2cx^2\}$$

Only polynomials (ii) and (iii) are in $\operatorname{range}(T)$ because they have no constant term.

(c) $\ker(T)$ is the set of all constant polynomials. $\operatorname{range}(T)$ is the set of all polynomials with zero constant terms.

**Sec. 6.7:**

4) Let $y = x'$, then $x'' + x' - 12x = 0$ is equivalent to

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 12 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = 3 \Rightarrow \boldsymbol{p}_1 = (1, 3)^\top$
- $\lambda_2 = -4 \Rightarrow \boldsymbol{p}_2 = (1, -4)^\top$

Therefore the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \boldsymbol{p}_1 e^{\lambda_1 t} + c_2 \boldsymbol{p}_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-4t} \\ c_1 e^{3t} - 4 c_2 e^{-4t} \end{pmatrix}$$

Using the initial conditions $x(0) = 0$ and $y(0) = 1$, we find $c_1 = 1/7$ and $c_2 = -1/7$, therefore the solution is

$$x(t) = \frac{1}{7}(e^{3t} - e^{-4t})$$

6) Let $f = g'$, then $g'' - 2g = 0$ is equivalent to

$$\begin{pmatrix} g \\ f \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = \sqrt{2} \Rightarrow \boldsymbol{p}_1 = (1, \sqrt{2})^\top$
- $\lambda_2 = -\sqrt{2} \Rightarrow \boldsymbol{p}_2 = (1, -\sqrt{2})^\top$

Therefore the general solution is

$$\begin{pmatrix} g \\ f \end{pmatrix} = c_1 \boldsymbol{p}_1 e^{\lambda_1 t} + c_2 \boldsymbol{p}_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{t\sqrt{2}} + c_2 e^{-t\sqrt{2}} \\ \sqrt{2} c_1 e^{t\sqrt{2}} - \sqrt{2} c_2 e^{-t\sqrt{2}} \end{pmatrix}$$

Using the boundary conditions $g(0) = 1$ and $g(1) = 0$, we find $c_1 = -1/(e^{2\sqrt{2}} - 1)$ and $c_2 = e^{2\sqrt{2}}/(e^{2\sqrt{2}} - 1)$, therefore the solution is

$$g(t) = \frac{-e^{t\sqrt{2}} + e^{2\sqrt{2} - t\sqrt{2}}}{e^{2\sqrt{2}} - 1}$$

12) Let $f = h'$, then $h'' - 4h' + 5h = 0$ is equivalent to

$$\begin{pmatrix} h \\ f \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} h \\ f \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = 2 + i \Rightarrow \boldsymbol{p}_1 = (1, 2+i)^\top$

- $\lambda_2 = 2 - i \Rightarrow \boldsymbol{p}_2 = (1, 2-i)^\top$

Therefore the general solution is

$$\begin{pmatrix} h \\ f \end{pmatrix} = c_1 \mathrm{Re}(\boldsymbol{p}_1 e^{\lambda_1 t}) + c_2 \mathrm{Im}(\boldsymbol{p}_1 e^{\lambda_1 t}) = \begin{pmatrix} e^{2t}(c_1 \cos t + c_2 \sin t) \\ c_1 e^{2t}(2\cos t - \sin t) + c_2 e^{2t}(\cos t + 2\sin t) \end{pmatrix}$$

Using the initial conditions $h(0) = 0$ and $f(0) = -1$, we find $c_1 = 0$ and $c_2 = -1$, therefore the solution is

$$h(t) = -e^{2t} \sin t$$

**Additional problems:**

1) If $P$ diagonalizes $A$, then

$$P^{-1} A P = D$$

where $D$ is a diagonal matrix. It follows that

$$D^\top = (P^{-1} A P)^\top = P^\top A^\top (P^{-1})^\top = Q^{-1} A^\top Q$$

Therefore $Q = (P^{-1})^\top$ diagonalizes $A^\top$. Note that $D^\top = D$.

2)

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

- eigenvalue $\lambda = 1 \Rightarrow$ eigenvector $(3, -2)$
- eigenvalue $\lambda = 2 \Rightarrow$ eigenvector $(2, -1)$

$\Rightarrow$ diagonalizing matrix

$$P = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

$$\Rightarrow A^6 = P\, D^6\, P^{-1} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 253 & 378 \\ -126 & -188 \end{pmatrix}$$

3)

(a)

$$A = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

- eigenvalue $\lambda = 1 \Rightarrow$ eigenvector $(1, 0)$
- eigenvalue $\lambda = -1 \Rightarrow$ eigenvector $(1, 1)$

$\Rightarrow$ diagonalizing matrix

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow e^{tA} = P\,e^{tD}\,P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & -e^t + e^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

$$\Rightarrow Y = e^{tA} Y_0 = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

- eigenvalue $\lambda = 1 \Rightarrow$ eigenvector $(1, 0, 0)$
- eigenvalue $\lambda = -1 \Rightarrow$ eigenvector $(0, -1, 1)$
- eigenvalue $\lambda = 0 \Rightarrow$ eigenvector $(-1, 1, 0)$

$\Rightarrow$ diagonalizing matrix

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow e^{tA} = P\,e^{tD}\,P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^t - 1 & e^t - 1 \\ 0 & 1 & -e^{-t} + 1 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

$$\Rightarrow Y = e^{tA} Y_0 = \begin{pmatrix} 3e^t - 2 \\ 2 - e^{-t} \\ e^{-t} \end{pmatrix}$$

4)

(a)

$$\begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= -y_1 - 3y_2 \end{aligned}$$

The corresponding coefficient matrix is

$$A = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = -2 \Rightarrow \boldsymbol{p}_1 = (1, -1)^\top$

- $\lambda_2 = 1 \Rightarrow \boldsymbol{p}_2 = (4, -1)^\top$

Therefore the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \boldsymbol{p}_1 e^{\lambda_1 t} + c_2 \boldsymbol{p}_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{-2t} + 4c_2 e^t \\ -c_1 e^{-2t} - c_2 e^t \end{pmatrix}$$

(b)

$$\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= y_1 + y_2 \end{aligned}$$

The corresponding coefficient matrix is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = 1 - i \Rightarrow \boldsymbol{p}_1 = (1, i)^\top$

- $\lambda_2 = 1 + i \Rightarrow \boldsymbol{p}_2 = (1, -i)^\top$

Therefore the general real-valued solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \mathrm{Re}(\boldsymbol{p}_1 e^{\lambda_1 t}) + c_2 \mathrm{Im}(\boldsymbol{p}_1 e^{\lambda_1 t}) = \begin{pmatrix} e^t(c_1 \cos t - c_2 \sin t) \\ e^t(c_1 \sin t + c_2 \cos t) \end{pmatrix}$$

5)

(a)

$$\begin{aligned} y_1' &= -y_1 + 2y_2 \\ y_2' &= 2y_1 - y_2 \end{aligned}$$

The corresponding coefficient matrix is

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = -3 \Rightarrow \boldsymbol{p}_1 = (1, -1)^\top$

- $\lambda_2 = 1 \Rightarrow \boldsymbol{p}_2 = (1, 1)^\top$

Therefore the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \boldsymbol{p}_1 e^{\lambda_1 t} + c_2 \boldsymbol{p}_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{-3t} + c_2 e^t \\ -c_1 e^{-3t} + c_2 e^t \end{pmatrix}$$

Using the initial conditions $y_1(0) = 3$ and $y_2(0) = 1$, we find $c_1 = 1$ and $c_2 = 2$, therefore the solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{-3t} + 2e^t \\ -e^{-3t} + 2e^t \end{pmatrix}$$

(b)

$$\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= 2y_1 + y_2 \end{aligned}$$

The corresponding coefficient matrix is

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are:

- $\lambda_1 = 1 - 2i \Rightarrow \boldsymbol{p}_1 = (i, -1)^\top$

- $\lambda_2 = 1 + 2i \Rightarrow \boldsymbol{p}_2 = (-i, -1)^\top$

Therefore the general real-valued solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \mathrm{Re}(\boldsymbol{p}_1 e^{\lambda_1 t}) + c_2 \mathrm{Im}(\boldsymbol{p}_1 e^{\lambda_1 t}) = \begin{pmatrix} e^t(c_1 \sin 2t + c_2 \cos 2t) \\ e^t(-c_1 \cos 2t + c_2 \sin 2t) \end{pmatrix}$$

Using the initial conditions $y_1(0) = 1$ and $y_2(0) = -2$, we find $c_1 = 2$ and $c_2 = 1$, therefore the solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^t(2 \sin 2t + \cos 2t) \\ e^t(-2 \cos 2t + \sin 2t) \end{pmatrix}$$