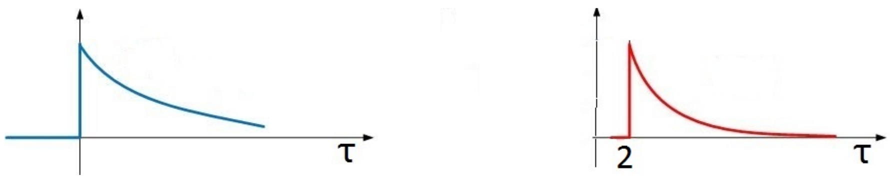
SOLUTION TO HOMEWORK #6

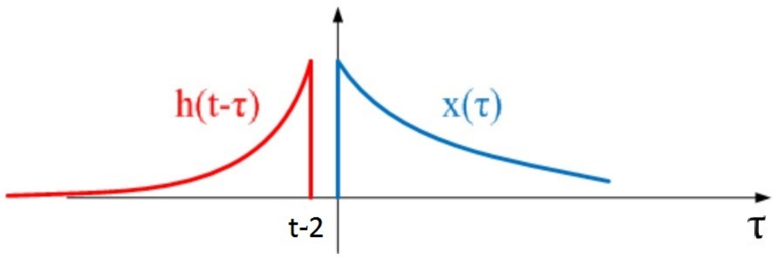
#1

(a) Compute the convolution integral using the graphical approach used in class. Here we have



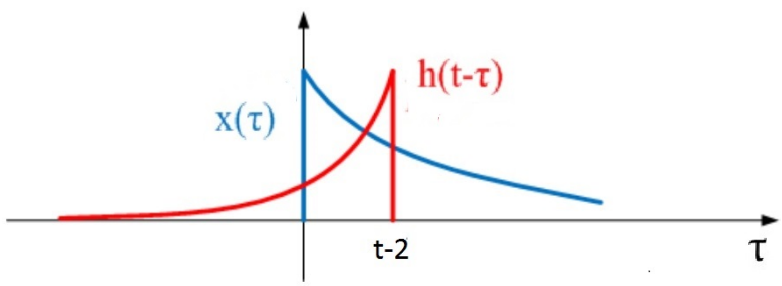
$x(\tau) = e^{-\alpha\tau}u(\tau)$ $\qquad$ $h(\tau) = e^{-\beta(\tau-2)}u(\tau-2)$

- If $t < 2$,



and there is no overlap. So the product $x(\tau)h(t-\tau) = 0$ and $y(t) = 0$.

- If $t \geq 2$,



when $\alpha \neq \beta$

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t-2} e^{-\alpha\tau}e^{-\beta(t-\tau-2)}d\tau \\
&= \int_{0}^{t-2} e^{(-\alpha+\beta)\tau}e^{-\beta(t-2)}d\tau = e^{-\beta(t-2)}\int_{0}^{t-2} e^{(-\alpha+\beta)\tau}d\tau \\
&= e^{-\beta(t-2)}\left[-\frac{1}{\alpha-\beta}e^{-(\alpha-\beta)\tau}\right]\Big|_{0}^{t-2} \\
&= e^{-\beta(t-2)}\left(\frac{1}{\beta-\alpha}\right)\left[e^{-(\alpha-\beta)(t-2)}-1\right] \\
&= \frac{e^{-\alpha(t-2)}-e^{-\beta(t-2)}}{\beta-\alpha}
\end{aligned}$$

Therefore, $y(t) = \left[\frac{e^{-\alpha(t-2)}-e^{-\beta(t-2)}}{\beta-\alpha}\right]u(t-2)$

(b)

$$x(t) = e^{-\alpha t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{j\omega+\alpha}$$

$$h(t) = e^{-\beta(t-2)}u(t-2) \longleftrightarrow H(j\omega) = \frac{e^{-j2\omega}}{j\omega+\beta}$$

Then, the frequency characteristic of the output is

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{e^{-j2\omega}}{(j\omega+\beta)(j\omega+\alpha)}$$

(c) Let $Z(j\omega) = \frac{1}{(j\omega+\beta)(j\omega+\alpha)} = (\frac{A}{j\omega+\alpha}+\frac{B}{j\omega+\beta})$

$$A = Z(j\omega)(j\omega+\alpha)\Big|_{j\omega=-\alpha} = \frac{1}{j\omega+\beta}\Big|_{j\omega=-\alpha} = \frac{1}{\beta-\alpha}$$

$$B = Z(j\omega)(j\omega+\beta)\Big|_{j\omega=-\beta} = \frac{1}{j\omega+\alpha}\Big|_{j\omega=-\beta} = \frac{1}{\alpha-\beta}$$

So, $z(t) = Ae^{-\alpha t}u(t) + Be^{-\beta t}u(t) = \frac{1}{\beta-\alpha}(e^{-\alpha t}-e^{-\beta t})u(t)$.

But $Y(j\omega) = Z(j\omega)e^{-j2\omega}$. So, using the time-shift property,

$$y(t) = z(t-2) = \frac{1}{\beta-\alpha}\left[e^{-\alpha(t-2)}-e^{-\beta(t-2)}\right]u(t-2)$$

#2

(a)

$$X(j\omega) = \underbrace{\frac{2\sin(\omega-2)}{\omega-2}}_{X_1(j\omega)} * \underbrace{\frac{e^{-2j\omega}\sin 2\omega}{\omega}}_{X_2(j\omega)}$$

Since convolution-in-frequency is multiplication-in-time (see Property 4.5), $x(t) = 2\pi x_1(t)x_2(t)$.

Let $Y_1(j\omega) = \frac{2\sin\omega}{\omega}$, then, $X_1(j\omega) = Y_1(j(\omega-2))$, and, using Property 4.3.6, $x_1(t) = y_1(t)e^{j2t}$. Inverting $Y_1(j\omega)$, we get

$$y_1(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| < 1 \end{cases}$$

Similarly, let $Y_2(j\omega) = \frac{2\sin 2\omega}{\omega}$. Then, $X_2(j\omega) = \frac{1}{2}e^{-2j\omega}Y_2(j\omega)$, and using Property 4.3.2, $x_2(t) = \frac{1}{2}y_2(t-2)$. Inverting $Y_2(j\omega)$, we get

$$y_2(t) = \begin{cases} 1, & |t| < 2 \\ 0, & |t| < 2 \end{cases}$$

Therefore, $x(t) = 2\pi x_1(t)x_2(t) = [2\pi y_1(t)e^{j2t}][\frac{1}{2}y_2(t-2)]$, and

$$x(t) = \begin{cases} \pi e^{j2t}, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

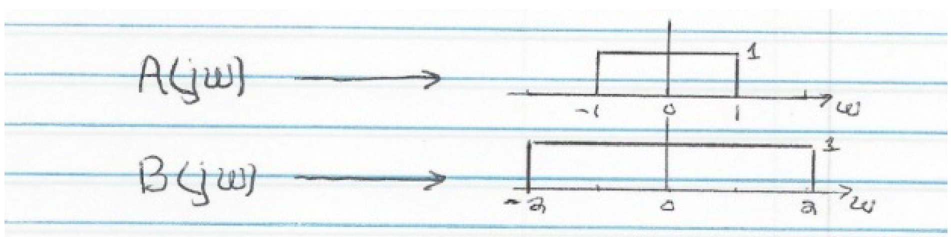
(b)

$$y(t) = x(t-2)$$

$$\therefore\ Y(j\omega) = X(j\omega)e^{-2j\omega}$$

$$x(t) = \underbrace{\frac{\sin\ t}{\pi t}}_{a(t)} * \frac{d}{dt}[\underbrace{\frac{\sin\ 2t}{\pi t}}_{b(t)}]$$

$$\therefore\ X(j\omega) = A(j\omega)\cdot j\omega B(j\omega)$$



$$\therefore\ Y(j\omega) = \begin{cases} j\omega e^{-2j\omega}, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) (This was a clicker question in class.) Use Parseval's theorem

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

We have the Fourier transform pair

$$X(j\omega) = \frac{2}{j\omega + 2} \rightarrow x(t) = 2e^{-t}u(t)$$

Therefore,

$$\int_{-\infty}^{\infty} |\frac{2}{j\omega + 2}|^2 d\omega = 2\pi \cdot 4 \int_{0}^{\infty} e^{-4t} dt = 8\pi(\frac{1}{4}) = 2\pi$$

#3

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is a periodic function (draw graph if not sure) with period T. To determine the Fourier series coefficients a_k, we use Eq. (3.39) in the textbook and select the interval of integration to be $[-T/2, T/2]$, avoiding the placement of impulses at the integration limits. Within this interval, $x(t) = \delta(t)$, and thus

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T} \text{ for all } k$$

Substituting this value for a_k in Eq. (4.42) yields

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$$

#4

$$H(j\omega) = \frac{2}{-\omega^2 + 3j\omega + 2} = \frac{2}{(j\omega)^2 + 3j\omega + 2}$$

(a) Using partial fraction expansion,

$$H(j\omega) = \frac{2}{(j\omega)^2 + 3j\omega + 2} = \frac{2}{(j\omega + 1)(j\omega + 2)} = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2}$$

$$A = H(j\omega)(j\omega + 1)\Big|_{j\omega=-1} = \frac{2}{j\omega + 2}\Big|_{j\omega=-1} = 2$$

$$B = H(j\omega)(j\omega+2)\Big|_{j\omega=-2} = \frac{2}{j\omega+1}\Big|_{j\omega=-2} = -2$$

$$\therefore\ H(j\omega) = \frac{2}{j\omega+1} - \frac{2}{j\omega+2}$$

Taking the inverse Fourier transform of $H(j\omega)$ gives

$$h(t) = 2e^{-t}u(t) - 2e^{-2t}u(t) = 2(e^{-t} - e^{-2t})u(t)$$

(b)

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{(j\omega)^2 + 3j\omega + 2}$$

$$(j\omega)^2 Y(j\omega) + 3j\omega Y(j\omega) + 2Y(j\omega) = 2X(j\omega)$$

Taking the inverse Fourier transform of this equation gives

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

#5 Consider an LTI system described by the following differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt}$$

(a) Taking the Fourier transform of the differential equation, we get

$$(j\omega)^2 Y(j\omega) + 4(j\omega)Y(j\omega) + 3Y(j\omega) = X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 4j\omega + 3\right] = X(j\omega)$$

Thus,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega}{(j\omega)^2 + 4j\omega + 3}$$

(b) $h(t) =$ Impulse response $= \mathcal{F}^{-1}\{H(j\omega)\}$. Using partial fraction expansion,

$$H(j\omega) = \frac{j\omega}{(j\omega+1)(j\omega+3)} = \frac{A}{j\omega+3} + \frac{B}{j\omega+1}$$

We can compute the coefficients A and B using the method in Problem # 4a. An alternative is to simply cross multiply. Using this approach, we get

$$j\omega = A(j\omega+1) + B(\omega+3)$$

$$j\omega = (A+B)j\omega + (A+3B)$$

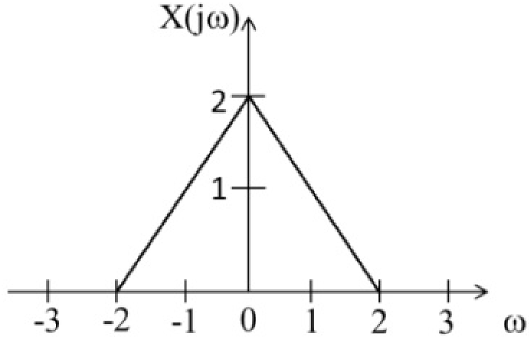
$$A + 3B = 0,\ A + B = 1 \Rightarrow A = \frac{3}{2},\ B = -\frac{1}{2}$$

$$H(j\omega) = \frac{3/2}{j\omega+3} - \frac{1/2}{j\omega+1}.$$

Taking the inverse transform of $H(j\omega)$ gives

$$h(t) = \frac{3}{2}e^{-3t}u(t) - \frac{1}{2}e^{-t}u(t)$$

#6



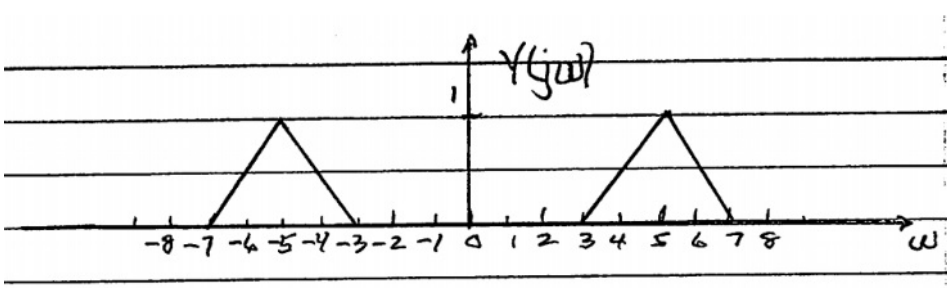
The message $x(t)$ is to be transmitted using AM modulation. Its frequency characteristic is shown above. From the diagram in the homework statement, the output $y(t)$ is

$$\begin{aligned} y(t) &= x(t)\cos\omega_o t,\ \omega_0{=}5 \text{ radians/sec} \\ &= x(t)\left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right) \\ &= \frac{x(t)}{2}e^{j\omega_0 t}+\frac{x(t)}{2}e^{-j\omega_0 t} \end{aligned}$$

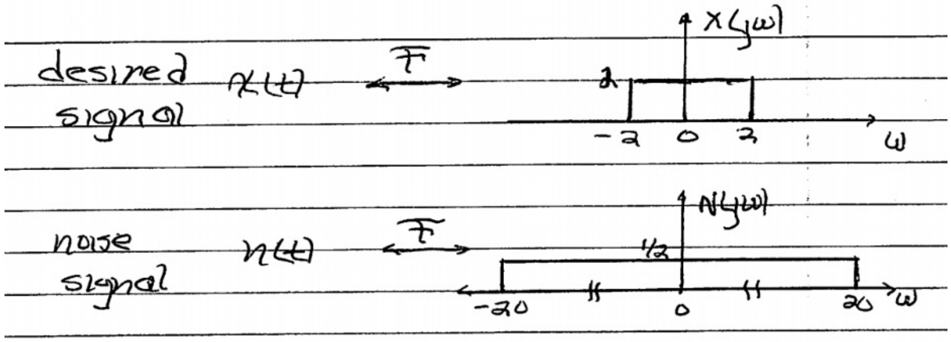
Use the frequency-shift property (4.3.6)

$$e^{j\omega_0 t}x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

$$\therefore\ Y(j\omega) = \underbrace{\frac{1}{2}X(j(\omega-\omega_0))}_{\text{shift up in frequency}} + \underbrace{\frac{1}{2}X(j(\omega+\omega_0))}_{\text{shift down in frequency}}$$



#7



The received signal is $y(t) = x(t) + n(t)$.

(a) Noise energy $= \int_{-\infty}^{\infty} |n(t)|^2 dt \overset{\text{Parseval}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} |N(j\omega)|^2 d\omega = \frac{1}{2\pi}(\frac{1}{4}40) = \frac{5}{\pi}$

(b) Desired signal energy $= \int_{-\infty}^{\infty} |x(t)|^2 dt \overset{\text{Parseval}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi}(4 \cdot 4) = \frac{8}{\pi}$

(c) The received signal $y(t) = x(t) + n(t)$, where the first term extends from $\omega = -2$ to 2, and the second term from $\omega = -20$ to 20.

So, if we lowpass filter $y(t)$ with a cutoff frequency at $\omega_c = 2$ rad/sec, the signal will pass unaffected, while the noise energy will be reduced to

$$\frac{1}{2\pi} \int_{-2}^{2} (1/2)^2 d\omega = \frac{1}{2\pi} 1 = \frac{1}{2\pi}$$

$\therefore$ The noise energy has been reduced by a factor of 10 (10 dB).

Conceptual

(a) A lowpass filter will cutoff the higher frequencies and not permit the image to contain regions which vary rapidly $\rightarrow$ so the edges will be blurred.

(b) A highpass filter will cutoff the low frequencies, and accentuate the high frequencies (that is, where the image changes rapidly) $\rightarrow$ so the edges will be enhanced.

Math Review

(a) The given integral can be written as

$$\int_{\alpha}^{\infty} e^{-t} dt = -e^{-t}\Big|_{\alpha}^{\infty} = -\left(e^{-\infty} - e^{-\alpha}\right) = e^{-\alpha}$$

For the integral to exist, α can be any value except $\alpha = -\infty$.

(b) $\int_0^\infty e^{-(\alpha+j\omega)t}dt = \frac{1}{-(\alpha+j\omega)}e^{-(\alpha+j\omega)t}\Big|_0^\infty = \frac{1}{\alpha+j\omega}\left[1-\lim_{t\to\infty} e^{-j\omega t}e^{-\alpha t}\right]$

For the integral to exist, the second term must remain finite as $t \to \infty$. Therefore, α must be non-negative, $\alpha \geq 0$.

(c) The given integral can be written as

$$\begin{aligned}\int_{-\infty}^{0} e^{-(\alpha+1)t}dt &= \frac{1}{-(\alpha+1)}e^{-(\alpha+1)t}\Big|_{-\infty}^{0} \\ &= -\frac{1}{\alpha+1}\left[1-\lim_{t\to-\infty} e^{-(\alpha+1)t}\right]\end{aligned}$$

For the integral to exist, the second term must remain finite as $t \to -\infty$. Therefore, $(\alpha+1)$ must be negative. Therefore, $\alpha+1<0$, and $\alpha<-1$.

At $\alpha=-1$, we have $\int_{-\infty}^{0} 1dt$ which does not converge.