ELEG 310

4/10/2018

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Continuous RVs

density $f_X(u)$ (1) $f_X(u) \geq 0$

(2) $\int_{-\infty}^{\infty} f_X(u)\,du = 1$

CDF $F_X(u) = P(X \leq u) = \int_{-\infty}^{u} f(x)\,dx$

$F(-\infty) = 0$ $F(v) \geq F(u)$ for $v \geq u$

$F(\infty) = 1$

$f(x) = \frac{d}{dx} F(x)$

$$E(g(X)) = \int_{-\infty}^{\infty} g(v)\, f(v)\, dv$$

$$\mu = E(X) = \int_{-\infty}^{\infty} v\, f(v)\, dv$$

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (v-\mu)^2 f(v)\, dv = E(X^2) - \mu^2$$

MGF $\quad M(u) = E(e^{uX}) = \int_{-\infty}^{\infty} e^{ux} f(x)\, dx$

$$E(X) = \left.\frac{dM}{du}\right|_{u=0} \qquad E(X^2) = \left.\frac{d^2M}{du^2}\right|_{u=0} \qquad E(X^k) = \left.\frac{d^k}{du^k} M(u)\right|_{u=0}$$

# Change of Variable

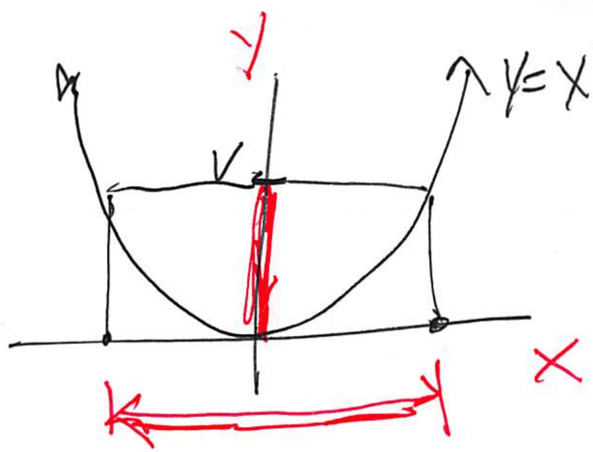
have $X$ let $Y = X^2$ Compute $f_Y(y)$

$$F_X(x) = \int_{-\infty}^{x} f_X(w)\,dw = P(X \le w)$$

$$\boxed{F_Y(v) = P(Y \le v)} = P(X^2 \le v) = P(-\sqrt{v} \le X \le \sqrt{v})$$

$Y \le v$

$$\boxed{= F_X(\sqrt{v}) - F_X(-\sqrt{v})}$$




$$f_Y(v) = \frac{d}{dv} F_Y(v) = \frac{d}{dv}\left(F_X(\sqrt{v}) - F_X(-\sqrt{v})\right)$$

$$= f_X(\sqrt{v})\,\frac{1}{2}v^{-\frac{1}{2}} - f_X(-\sqrt{v})\left(-\frac{v^{-\frac{1}{2}}}{2}\right)$$

$$= \frac{f_X(\sqrt{v})\,v^{-\frac{1}{2}} + f_X(-\sqrt{v})\,v^{-\frac{1}{2}}}{2}$$

$$\frac{d}{dv} F(g(v)) = \frac{dF}{dv}(g(v)) \frac{dg}{dv}$$

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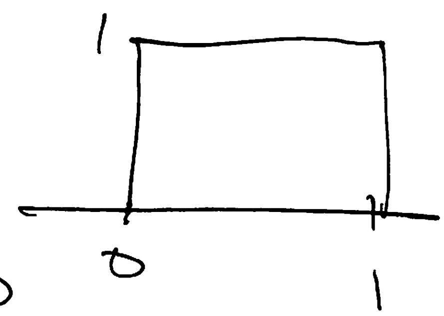
Ex. $X \sim U(0,1)$




$$F_X(u) = \begin{cases} 0 & u < 0 \\ u & 0 < u < 1 \\ 1 & u > 1 \end{cases}$$

$Y = X^2$ $\quad P(X^2 \le v) = P(-\sqrt{v} \le X \le \sqrt{v})$

$$= F_X(\sqrt{v}) - F_X(-\sqrt{v})$$

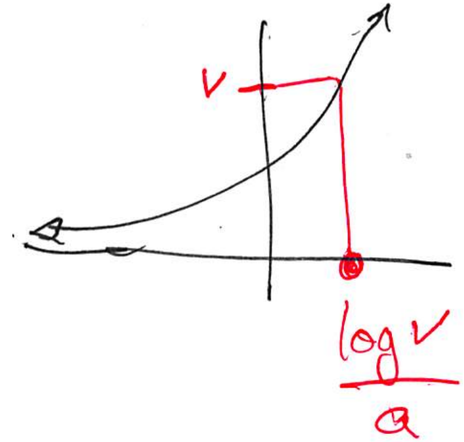
$$= \sqrt{v} - 0$$

$0 < v < 1$

$$F_Y(v) = \begin{cases} 0 & v < 0 \\ \sqrt{v} & 0 < v < 1 \\ 1 & v \ge 1 \end{cases}$$

$$f_Y(v) = \begin{cases} 0 & v < 0 \\ \frac{1}{2} v^{-\frac{1}{2}} & 0 < v < 1 \\ 0 & v > 1 \end{cases}$$

$X \sim U(0,1)$ $\qquad Y = e^{aX}$




$F_Y(v) = P(Y \le v) = P(e^{aX} \le v)$

$\log(e^{ax}) = ax \le \log(v)$

$x =$

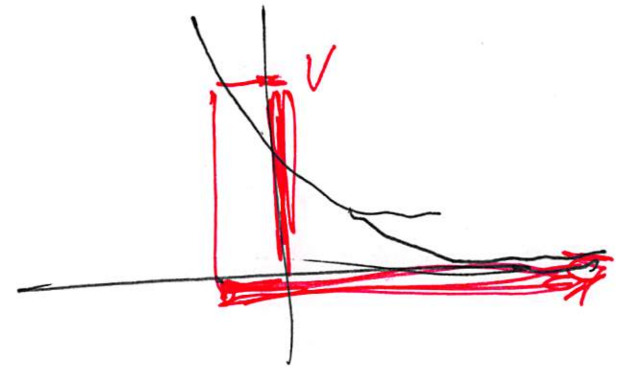
$= P(aX \le \log v)$

$= P\left(X \le \frac{\log v}{a}\right) = F_X\left(\frac{\log v}{a}\right) = \begin{cases} 0 & \frac{\log v}{a} < 0 \\ \frac{\log v}{a} & 1 \le v \le e^a \\ 1 & \frac{\log v}{a} > 1 \end{cases}$

$1 \le v \le e^a$

$f_Y(v) = \frac{1}{av} \qquad 1 \le v \le e^a$

$X \sim U(0,1)$ $\quad Y = e^{aX}$ $\quad a < 0$




$P(Y \le v)$

$$\boxed{\{Y \le v\} = \left\{X \ge \frac{\log v}{a}\right\}}$$

$$F_Y(v) = P\left(X \ge \frac{\log v}{a}\right) = 1 - F_X\left(\frac{\log v}{a}\right)$$

$$f_Y(v) = \frac{d}{dv}\left(1 - F_X\left(\frac{\log v}{a}\right)\right) = -f_X\left(\frac{\log v}{a}\right)\frac{1}{av}$$

$$= \frac{1}{|a|v} f_X\left(\frac{\log v}{a}\right)$$