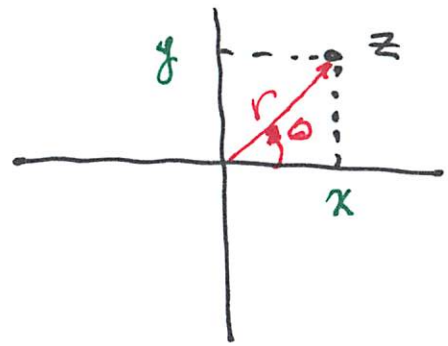
Complex Numbers and Euler's Relation

$j = \sqrt{-1}$ $\qquad z = x + jy$ rectangular

$= re^{j\theta}$ polar

real part $\quad x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$ magnitude

imaginary part $\quad y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x}$ phase



Euler's Relation $\quad e^{j\theta} = \cos\theta + j\sin\theta$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Numbers

complex number $z = \underbrace{x}_{Re\{z\}} + j\underbrace{y}_{Im\{z\}} = r(\cos\theta + j\sin\theta) = \underbrace{r}_{\text{magnitude}} e^{j\overbrace{\theta}^{\text{phase}}}$

$j = \sqrt{-1}$

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$

$$\theta = \tan^{-1}\frac{y}{x} \qquad y = r\sin\theta$$

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \text{Euler's Relation}$$

- conjugation = change the sign of the imaginary part

$$z = x + jy \qquad z = re^{j\theta}$$

$$z^* = x - jy \qquad z^* = re^{-j\theta}$$

- magnitude $|z|$

① $\sqrt{x^2 + y^2}$

or

② $\sqrt{zz^*}$

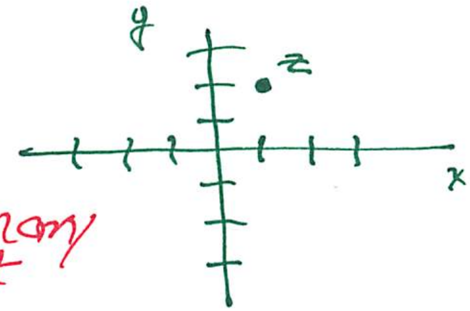
$$\left[(x+jy)(x-jy)\right]^{1/2} = \left[x^2 + \cancel{jxy} - \cancel{jxy} - \overbrace{(j)^2}^{-1} y^2\right]^{1/2}$$

$$= \sqrt{x^2 + y^2}$$

Complex Numbers
Example

a.) $z = 1 + 2j$

$\underbrace{1}_{\text{real part}} \quad \underbrace{2j}_{\text{imaginary part}}$



$z^* = 1 - 2j$ conjugate

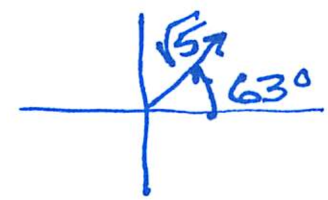
$|z| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$

$= \sqrt{z z^*} = \sqrt{(1+2j)(1-2j)} = \sqrt{1 + \cancel{2j} - \cancel{2j} - 4(j)^2}$ $\leftarrow -1$

$= \sqrt{5}$

$= r$ magnitude

$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} 2 \approx 63°$



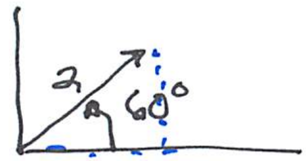
b.) $\frac{1+j}{1-j} = \frac{(1+j)(1+j)}{(1-j)(1+j)} = \frac{\cancel{1} + j + j \cancel{-1}}{1 \cancel{-j} \cancel{+j} + 1} = \frac{2j}{2} = j$ $\left(e^{j\frac{\pi}{2}}\right)$

"rationalize" the denominator

$e^{j\theta} = \cos\theta + j\sin\theta$

Complex Numbers - Examples

$$z = 2e^{j\frac{\pi}{3}}$$



a.) $x = r\cos\theta = 2\cos 60^\circ = 2(1/2) = 1$

$y = r\sin\theta = 2\sin 60^\circ = \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right) = \sqrt{3}$

$z = 1 + j\sqrt{3}$

b.) $z^* = 2e^{-j\pi/3}$

$= 1 - j\sqrt{3}$

c.) $|z| = |2e^{j\frac{\pi}{3}}| = |2|\underbrace{|e^{j\frac{\pi}{3}}|}_{|\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}| = \sqrt{\cos^2\frac{\pi}{3} + \sin^2\frac{\pi}{3}} = 1}$

$= \sqrt{zz^*} = \sqrt{2e^{j\frac{\pi}{3}} \cdot 2e^{-j\frac{\pi}{3}}}$

$= 2\sqrt{\underbrace{e^{j\frac{\pi}{3}}e^{-j\frac{\pi}{3}}}}$

add exponents $= 0 \Rightarrow e^{j0} = 1$

Geometric Series

↳ esp. sums of geometric series

$a, ar, ar^2, \ldots$

$$\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^{n})}{1-r}$$

# of terms

n terms

Special case: $n \to \infty$, $|r|<1$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Example: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

Geometric Series Example

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \underbrace{\cos\frac{\pi}{2}n}_{\frac{e^{j\frac{\pi}{2}n}+e^{-j\frac{\pi}{2}n}}{2}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n e^{j\frac{\pi}{2}n} + \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n e^{-j\frac{\pi}{2}n}$$

$$= \underbrace{\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\frac{\pi}{2}}\right)^n}_{\text{infinite geometric series}} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\frac{\pi}{2}}\right)^n$$

$r = \frac{1}{2}e^{j\frac{\pi}{2}} \Rightarrow |r| = \frac{1}{2} < 1$ converges $\frac{a}{1-r}$

$$= \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}j} + \frac{1}{2}\cdot\frac{1}{1+\frac{1}{2}j} = \frac{1}{2-j} + \frac{1}{2+j}$$

$$= \frac{2+\cancel{j}+2-\cancel{j}}{(2-j)(2+j)} = \frac{4}{4-2j+2j+1} = \frac{4}{5}$$

Why is Fourier analysis important?

i. An extremely broad class of signals can be represented as weighted sums or integrals of complex exponentials.

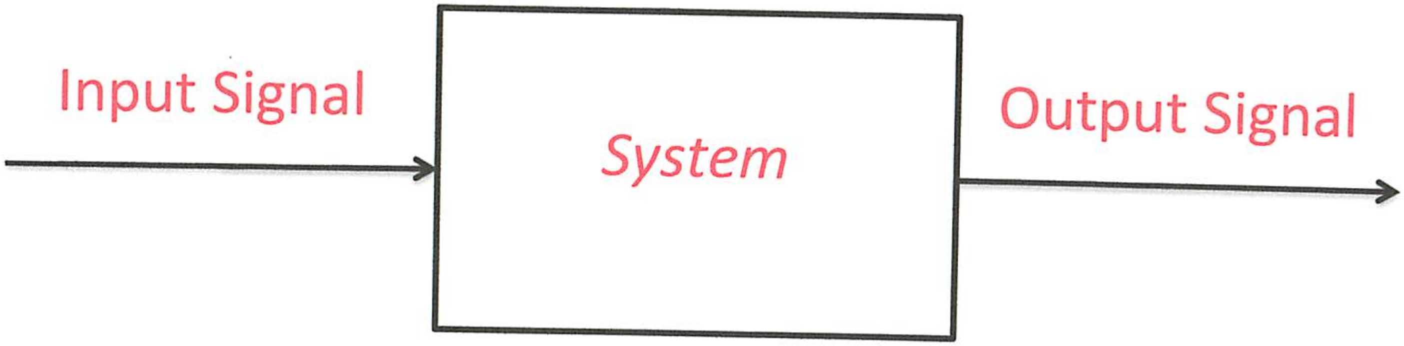
ii. The response of a linear, time-invariant system (LTI) to a complex exponential is the same exponential multiplied by a complex number characteristic of the system.

Why is Fourier analysis important?

i. Almost all signals can be represented as linear combinations of complex exponentials.

ii. Complex exponentials are eigenfunctions of LTI systems.

SIGNALS AND SYSTEMS
Chapter 1



- acts on input signal to produce an output signal

- time & frequency domain characterizations

goals:

1.) output for a given input

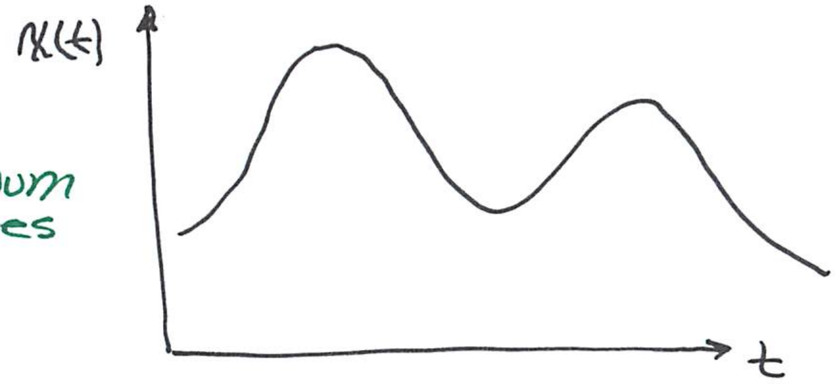
2.) design another system to fix the "distortion" the first system causes

Categorizing Signals

continuous vs. discrete-time

continuous-time signal $x(t)$

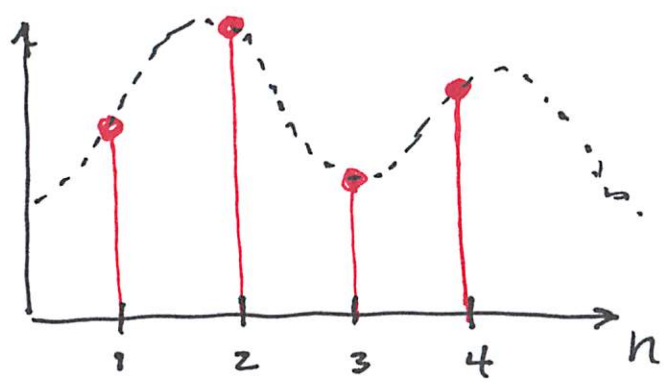
continuum of values



discrete-time signal $x[n]$

defined only for integer values for time

"sequence"



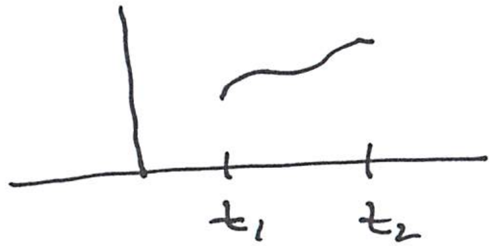
(Sampling Ch. 7)

Categorizing Signals

power v. energy

$t_1 \le t \le t_2$

- continuous-time signal $x(t)$, $t \in [t_1, t_2]$



E = energy ("joules")

$$\triangleq \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

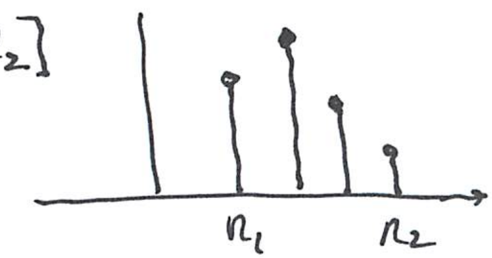
magnitude

$$P \triangleq \text{average power} = \frac{1}{t_2 - t_1} E$$

("watts = $\frac{\text{joules}}{\text{sec}}$")

observation interval

- discrete-time signal $x[n]$ $[n_1, n_2]$



$$E = \sum_{n=n_1}^{n_2} |x[n]|^2$$

$$P = \frac{1}{(\underbrace{\#\text{ of data points}}_{n_2 - n_1 + 1})} E$$

Categorizing Signals

power v. energy

Let time interval $\longrightarrow \infty$

continuous time

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

discrete time

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

If E_∞ is finite $\Longrightarrow$ energy signal

If P_∞ is finite $\Longrightarrow$ power signal

Power vs Energy Signals
Examples

- $x(t) = e^{-at} u(t)\,, \quad a > 0$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = -\frac{1}{2a} e^{-2at} \Big|_0^{\infty}$$

$$= 0 + \frac{1}{2a} = \frac{1}{2a} < \infty \Rightarrow \text{energy signal}$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \longrightarrow 0$$

- $x[n] = \left(\frac{1}{3}\right)^n u[n]$ ← unit step

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

infinite geometric series

$$= \frac{1}{1-\frac{1}{9}} = \frac{9}{8} < \infty \Rightarrow \text{energy signal}$$

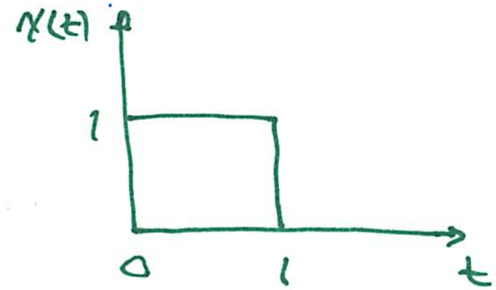
Categorizing Signals

power v. energy

- energy signals $\longrightarrow$ finite E_∞

  $\therefore \quad P_\infty = \lim_{T\to\infty} \frac{1}{2T} E_\infty = 0$ zero average power

  Example



$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^1 1\, dt$$

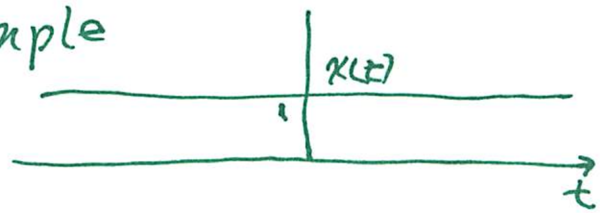
$$= 1$$

$$P_\infty = 0$$

- power signals $\longrightarrow$ finite P_∞ and non-zero $(P_\infty > 0)$

  $\therefore \quad E_\infty = \lim_{T\to\infty} 2T\,P \longrightarrow \infty$

  Example



$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \cdot 2T = 1$$

$$E_\infty \Rightarrow \infty$$

Power vs Energy Signals Example

- $x(t) = e^{-at}$, $a > 0$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2at} dt = -\frac{1}{2a} e^{-2at}\Big|_{-\infty}^{\infty}$$

$$= -\frac{1}{2a}\left(\cancel{\lim_{t\to\infty} e^{-2at}}^{\to 0} - \underbrace{\lim_{t\to-\infty} e^{-2at}}_{\to\infty} \right)$$

$\to \infty$ not an energy signal

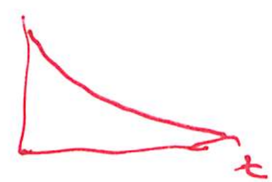
$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\left(-\frac{1}{2a}\right)\left(e^{-2aT} - e^{+2aT}\right)$$

$$= -\frac{1}{2a}\left[\underbrace{\lim_{T\to\infty} \frac{e^{-2aT}}{2T}}_{\to 0} - \underbrace{\lim_{T\to\infty} \frac{e^{2aT}}{2T}}_{\to\infty} \right]$$

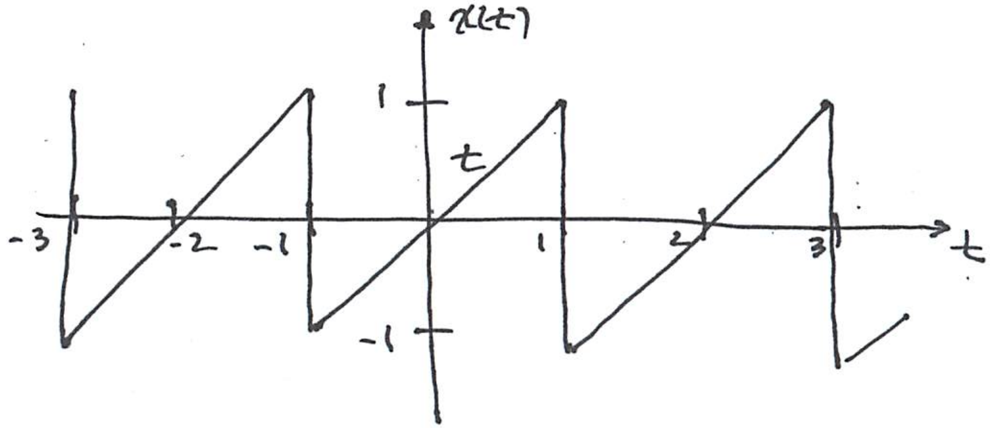
$\to \infty$ not a power signal

- $x(t) = e^{-at} \underbrace{u(t)}_{\text{unit step function}}$, $a > 0$

$$= \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$$



Example



energy or power signal?

(periodic signal ⇒ power signals)

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \frac{1}{2}\int_{-1}^{1} t^2 dt$$

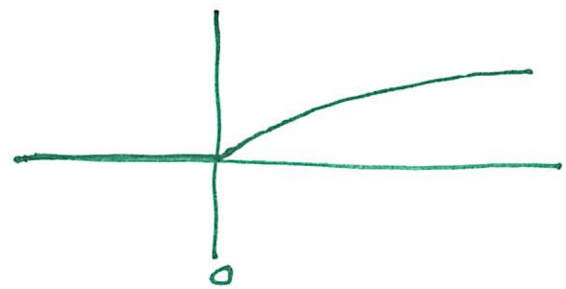
$$= \frac{t^3}{6}\Big|_{-1}^{1} = \frac{1}{3} < \infty$$ power signal

$E_\infty \to \infty$

Example

Neither power nor energy signal

- $x(t) = \sqrt{t} \quad t \geq 0$



$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt$$

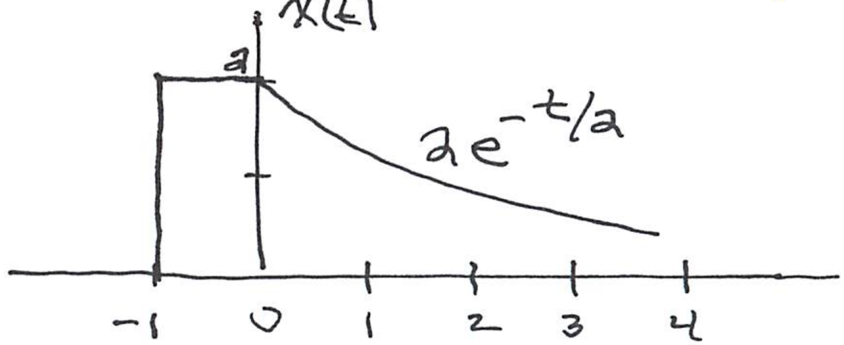
$$= \int_{0}^{\infty} t \, dt \longrightarrow \infty$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} t \, dt = \lim_{T \to \infty} \frac{1}{2\cancel{T}} \frac{T^{\cancel{2}}}{2}$$

$$= \lim_{T \to \infty} \frac{T}{4} \longrightarrow \infty$$

Example



energy or power signal ?

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-1}^{0} 4\,dt + \int_{0}^{\infty} 4e^{-t} dt$$

$$= 4t\Big|_{-1}^{0} + 4(-1)e^{-t}\Big|_{0}^{\infty}$$

$$= 4 + 4 = 8 \quad < \infty$$ energy signal

$$P_\infty = 0$$

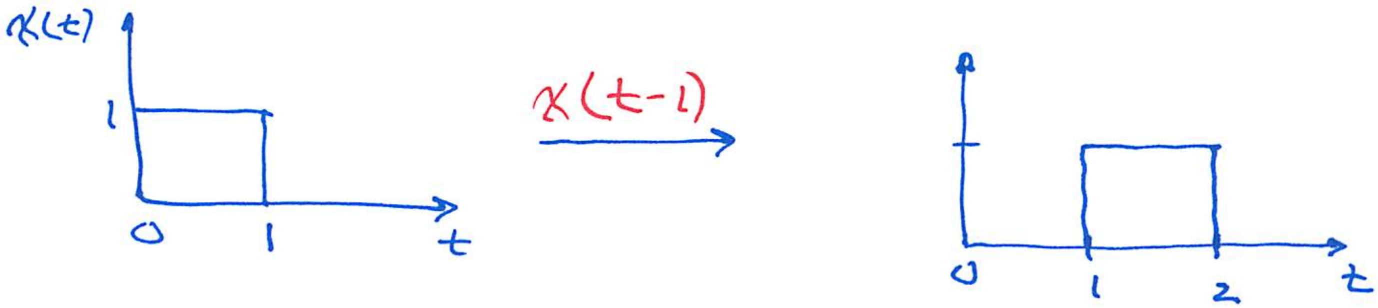
Section 1.2
Transformations of the Independent Variable

Transformations: Time Shift

$x(t)$ $\qquad\qquad$ $x(t-t_0)$

$x[n]$ $\qquad\qquad$ $x[n-n_0]$

- same shape
- t_0 positive, shifted to the right (delayed)
- t_0 negative, shifted to the left (advanced)



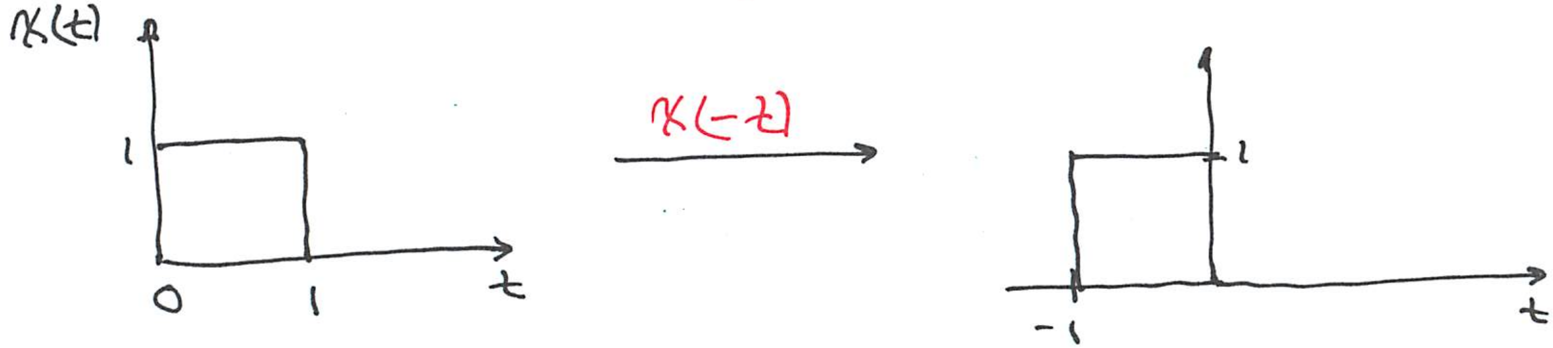
e.g. radar

$x(t-2t_0)$ $\qquad$ $t_0 = \frac{D}{c}$

Transformations: Time Reversal

$x(-t)$
$x[-n]$

flip around y-axis $(t=0)$

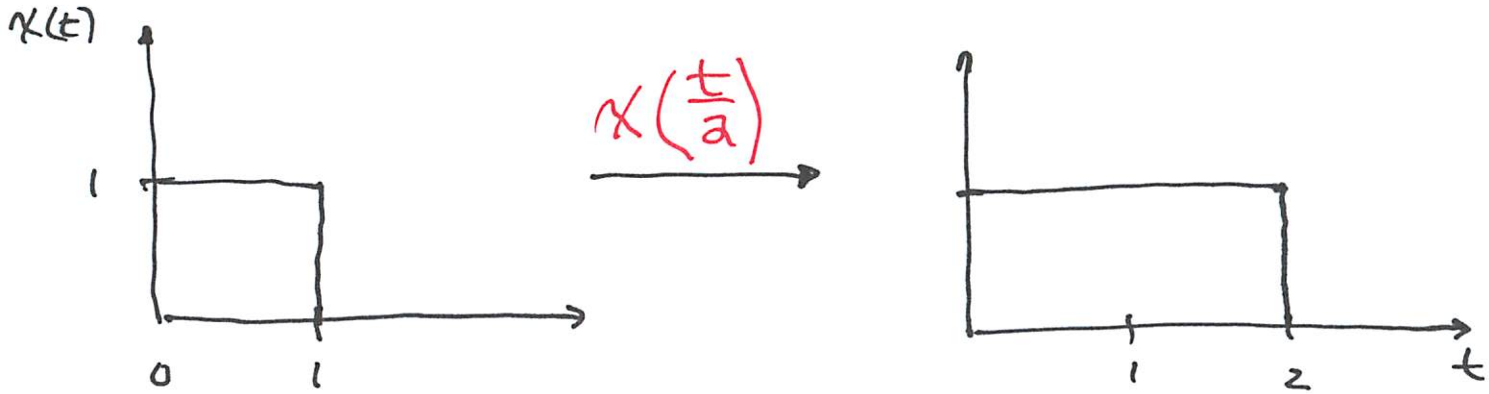


Transformations: Time Scaling

$$x(t) \longrightarrow x(\alpha t) \begin{cases} |\alpha| > 1 & \text{compressed} \\ |\alpha| < 1 & \text{stretched} \end{cases}$$

$$x[n] \longrightarrow x[kn]$$

careful!

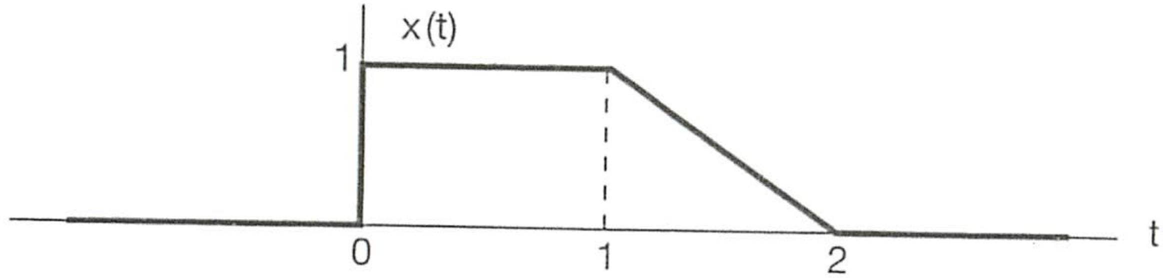


e.g. tape playback $x(t/2) \rightarrow$ half speed

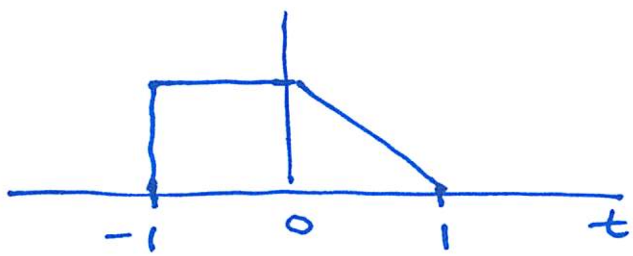
$x(2t) \rightarrow$ twice the speed

Combine $x(\overset{\text{scale}}{\alpha} t - \underset{\text{shift}}{\beta})$

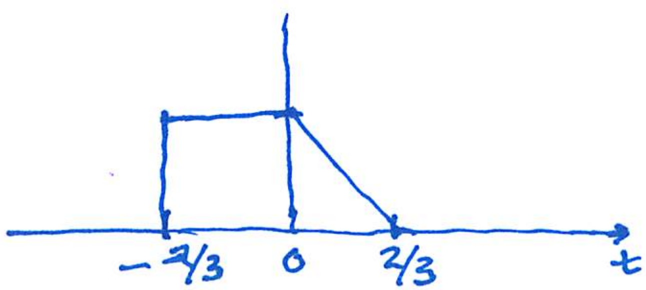
$x(\alpha t-\beta)$
"Shift and Scale" or "Scale and Shift"??



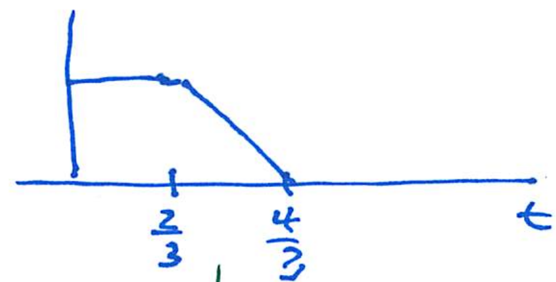
time shift first $x(t+1)$



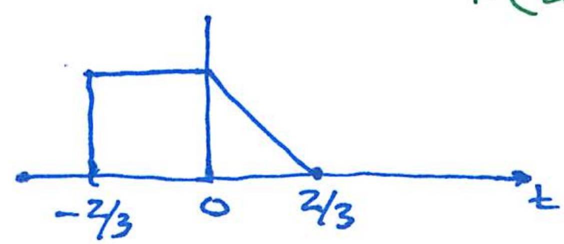
scale $t \to \frac{3}{2}t$



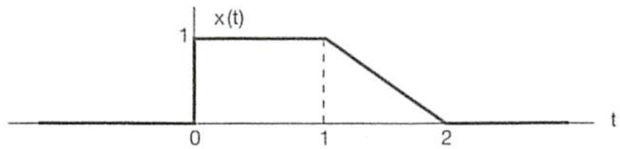
scale first $x(\frac{3}{2}t)$



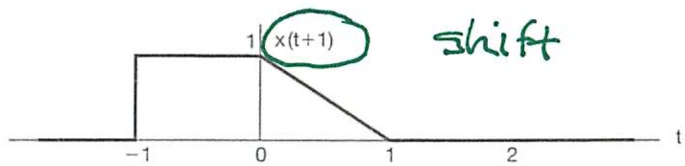
shift $x(\frac{3}{2}t+1) = x(\frac{3}{2}(t+\frac{2}{3}))$



Example 1.1 (p. 10)



(a)

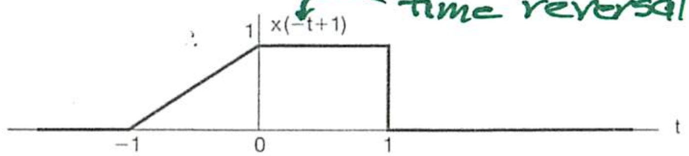


(b)

shift

$x(t-t_0) \Rightarrow t_0 = -1$

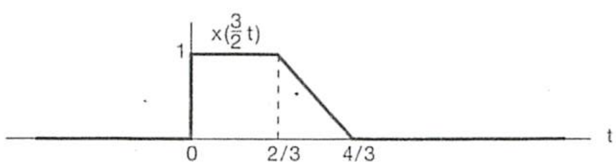
(shift to the left



(c)

time reversal

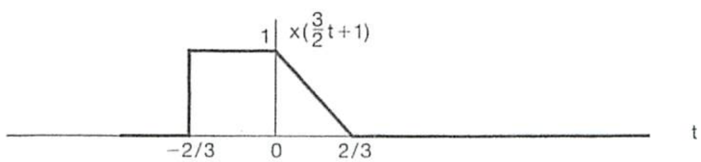
flip around 0



(d)

scaling $\alpha = \frac{3}{2} \Rightarrow$ compress (a)

*



(e)

scale and shift

$x\left(\frac{3}{2}\left(t+\frac{2}{3}\right)\right)$

Categorizing Signals

periodic v. aperiodic

- periodic $\quad x(t) = x(t+T)$ for <u>all</u> t $\quad$ continuous-time

  T ← period

  $x(t) = x(t+mT)$

  T_0 = fundamental period → smallest value of T for which $x(t) = x(t+T)$

  $f_0 = \frac{1}{T_0}$ = fund. freq.

- not periodic ⇒ aperiodic

- discrete time $\quad x[n] = x[n+N]$ $\quad N_0$ = fundamental period

  N ← period → integer!

Aperiodic vs Periodic
Example: Problem #1.9d

$x(t) = x(t+T)$
$x[n] = x[n+N]$

$$x[n] = 3e^{j3\pi(n+1/2)/5}$$

- periodic?
- period?

$$= \underbrace{3}_{\text{magnitude}}\,\underbrace{e^{j\frac{3\pi n}{5}}}_{*}\,e^{j\left(\frac{3\pi}{10}\right)}$$

phase

CT $e^{j\frac{3\pi t}{5}}$, $\omega_0 = \frac{3\pi}{5}$ $\quad T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{3\pi/5} = \frac{10}{3}$

DT $\rightarrow$ periodic if the value of $\omega_0/2\pi$ is <u>rational</u> (the ratio of integers)

- $\frac{\omega_0}{2\pi} = \frac{3}{10} \longrightarrow$ yes, it is periodic

- period $N = 10$

$$\frac{\omega_0}{2\pi} = \frac{m}{N} \Rightarrow N = \frac{2\pi m}{\omega_0} = \frac{10}{3}\cdot m$$

Categorizing Signals

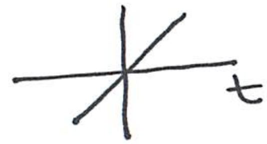
even and odd

even $\quad x(-t) = x(t)$ $\quad$ e.g cosine,

$x[-n] = x[n]$

odd $\quad x(-t) = -x(t)$ $\quad$ e.g. sine,

$x[-n] = -x(n]$



$$x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\}$$

$$= \underbrace{x_e(t)}_{x_e(-t) = x_e(t)} + \underbrace{x_o(t)}_{x_o(-t) = -x_o(t)}$$

$$x(t) = x_e(t) + x_o(t)$$

$$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$$

$$\therefore \quad x(t) + x(-t) = 2x_e(t) \Rightarrow x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_o(t) = \frac{1}{2}(x(t) - x(-t))$$

Section 1.3
Exponential and Sinusoidal Signals

Exponential Signals

continuous-time

$$x(t) = C e^{at}$$

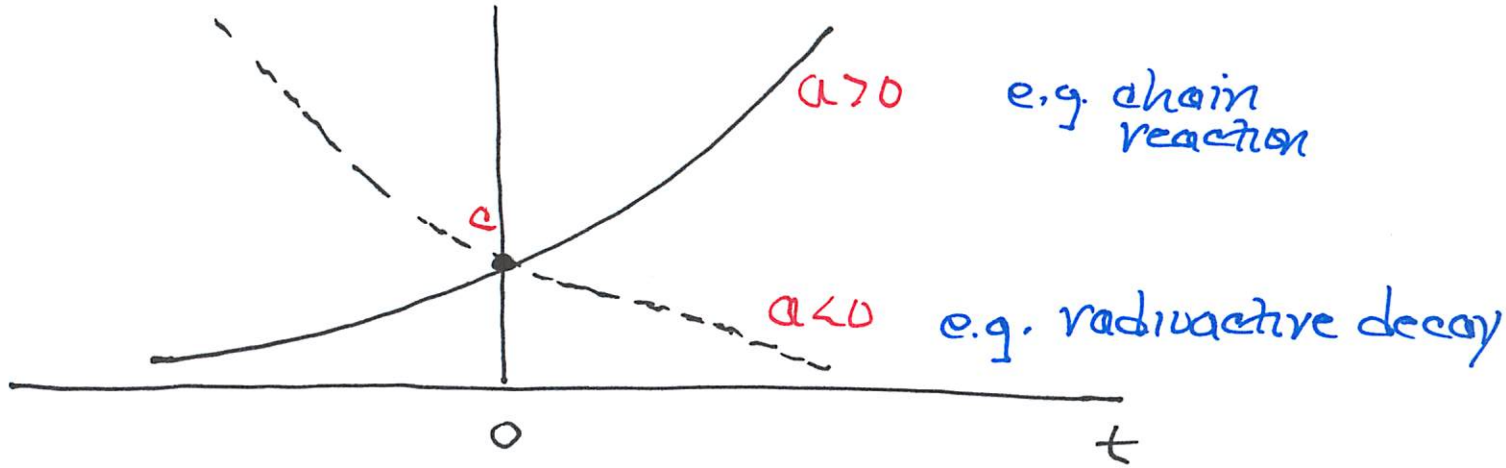
continuous-time complex exponential

Exponential Signals

continuous-time (c and a real)

$$x(t) = c e^{at}$$

$x(t)$ is real



$e^{j2\pi} = 1$
$e^{j\pi} = -1$
$\Big\} \int e^{j\theta} = \cos\theta + j\sin\theta$

ω = radian frequency
$f = \frac{\omega}{2\pi}$

Exponential Signals

continuous-time (c real, a purely imaginary)

$$x(t) = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t \quad (\text{let } c=1)$$

↳ periodic signal

$\therefore \quad x(t) = x(t+T)$

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

$$\cancel{e^{j\omega_0 t}} = \cancel{e^{j\omega_0 t}} e^{j\omega_0 T}$$

$$\therefore \quad \boxed{e^{j\omega_0 T} = 1}$$

↳ $\sin\omega_0 T = 0$, $\cos\omega_0 T = 1$ $\Rightarrow |\omega_0| T = 2\pi$

$$\therefore \quad T_0 = \frac{2\pi}{|\omega_0|} \quad \text{fundamental period}$$

Exponential Signals

sets of harmonically related complex exponentials

$$e^{j\omega T_0} = 1$$

$$\Downarrow$$

$$\omega T_0 = 2\pi k \qquad k = 0, \pm 1, \pm 2, \dots$$

$$\omega = \underbrace{\frac{2\pi}{T_0}}_{\omega_0} k = k\omega_0$$

$$\boxed{\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \dots}$$

Exponential Signals

continuous-time (c and a arbitrary)

$$x(t) = c\, e^{at}$$

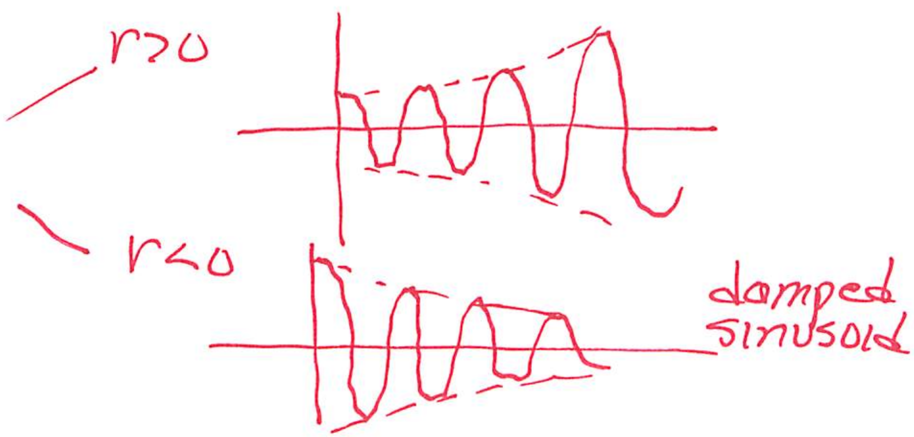
$$c = |c|e^{j\theta} \qquad a = r + j\omega_0$$

$$x(t) = \underbrace{|c|\, e^{rt}}_{\text{envelope (real)}}\ \underbrace{e^{j(\omega_0 t + \theta)}}_{\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)}$$

envelope (real)
- decaying or growing

$\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)$
(periodic)

$$\mathrm{Re}\{x(t)\} = |c|\, e^{rt}\cos(\omega_0 t + \theta)$$



Exponential Signals

discrete-time

$$x[n] = C e^{\beta n} = C \alpha^n$$

integer

$C \quad e^{\beta}$

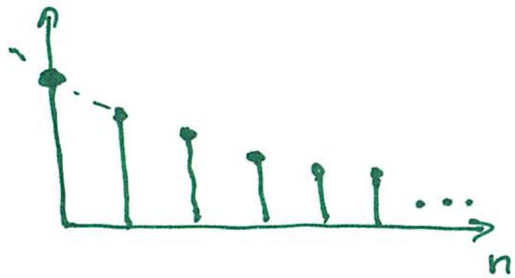
discrete-time complex exponential

Exponential Signals

discrete-time (c and α real)

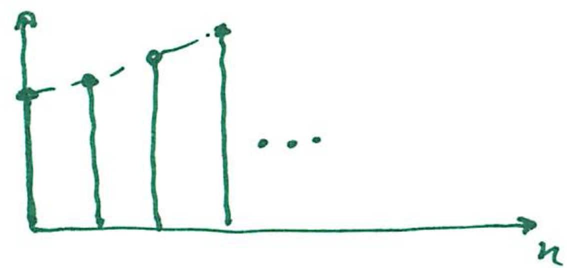
$x[n] = c\,\alpha^n$ → real
→ geometric series

$0 < \alpha < 1$



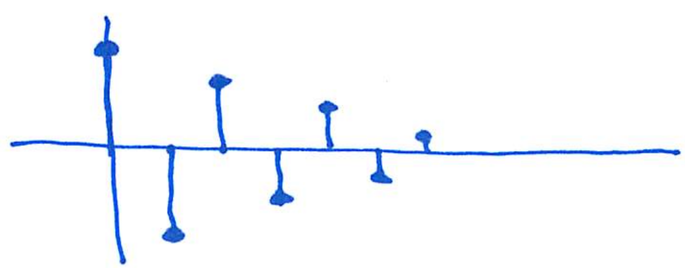
e.g. $\left(\frac{1}{2}\right)^n$

$\alpha > 1$



e.g. $(2)^n$

α negative (consider $-1 < \alpha < 0$)



alternates sign $(-1)^n$

Exponential Signals

discrete-time (β pure imaginary)

$$x[n] = e^{j\omega_0 n}$$

integer

* similar to continuous-time but many important differences (integer!)

$\downarrow$

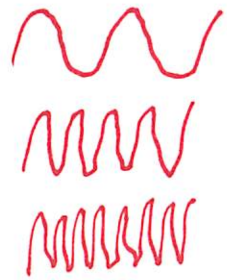
compare $e^{j\omega t}$ and $e^{j\omega n}$

Exponential Signals

$e^{j\omega t}$ v $e^{j\omega n}$

rate of oscillation

$e^{j\omega_0 t}$ $\longrightarrow$ rate of oscillation increases as ω_0 increases



$\omega_0 = 2\pi(1\ Hz)$

$\omega_0 = 2\pi(3\ Hz)$

$\omega_0 = 2\pi(6\ Hz)$

$e^{j\omega_0 n}$?

- suppose increase ω_0 by 2π

$$e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}\,\cancel{e^{j2\pi n}}^{\,1} = e^{j\omega_0 n}$$

$\Longrightarrow$ increasing ω_0 by 2π results in the same signal!

$\therefore$ only need to consider a frequency interval of length 2π

e.g. $0 \leq \omega_0 \leq 2\pi$

$-\pi \leq \omega_0 \leq \pi$

Exponential Signals

rate of oscillation

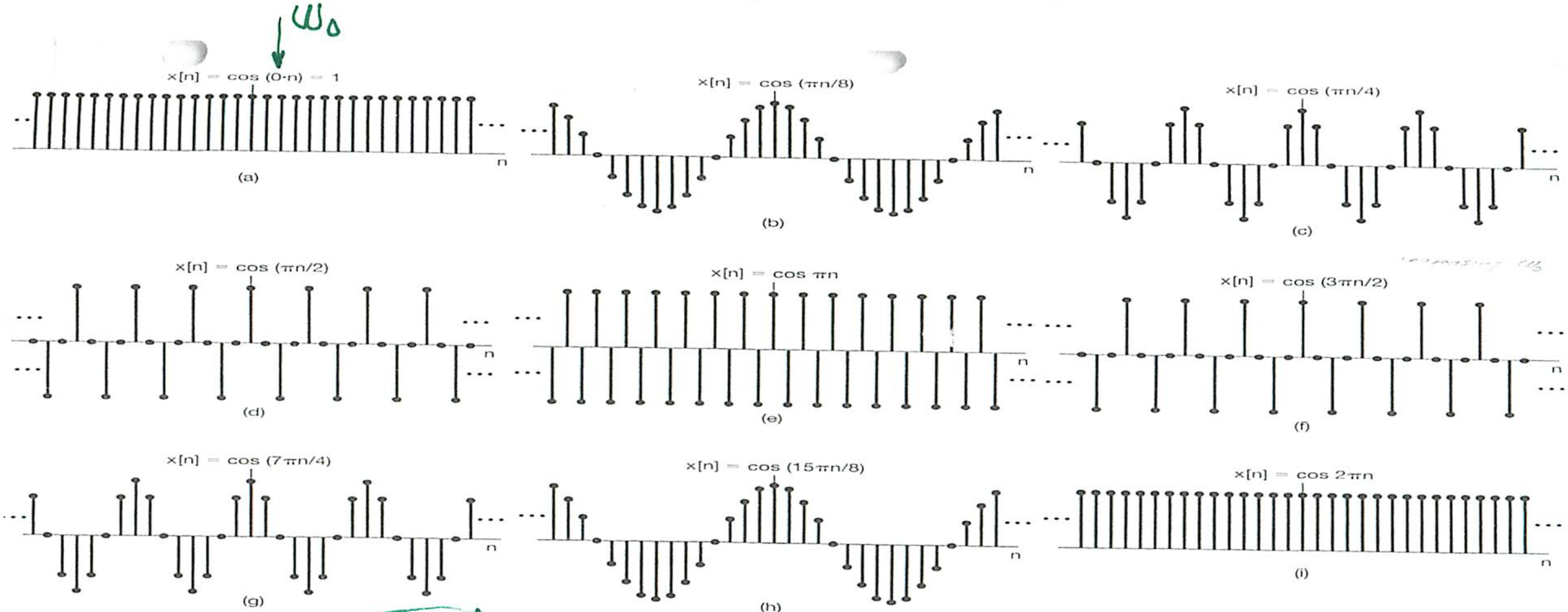


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

→ Chapter 7 – aliasing

Exponential Signals

$e^{j\omega t}$ v $e^{j\omega n}$

periodicity

- continuous time $e^{j\omega_0 t}$ is periodic for any value of ω_0
  ($T_0 = 2\pi/\omega_0$)
- discrete time $e^{j\omega_0 n}$?

integer!

To be periodic, $x[n] = x[n+N]$ for all n

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

$$\cancel{e^{j\omega_0 n}}\, e^{j\omega_0 N} = \cancel{e^{j\omega_0 n}}$$

$$\boxed{e^{j\omega_0 N} = 1}$$

integer!!

$$\therefore \quad \omega_0 N = 2\pi m$$

$$\frac{\omega_0}{2\pi} = \frac{m}{N} \Rightarrow \text{rational number} \quad (\text{ratio of integers})$$

Exponential Signals

sets of harmonically related complex exponentials

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, \quad k = 0, \pm 1, \pm 2, \cdots$$

but not all distinct

$$\phi_{k+N}[n] = e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\left(\frac{2\pi}{N}\right)n} \cancel{e^{j2\pi n}}^{\,1}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$= \phi_k[n]$$ $\Rightarrow$ only N distinct complex exponentials

$$\phi_0[n] = 1$$

$$\phi_1[n] = e^{j\frac{2\pi}{N}n}$$

$$\phi_2[n]$$

$$\vdots$$

$$\phi_{N-1}[n] = e^{j\frac{2\pi(N-1)}{N}n} = \cancel{e^{j2\pi n}}^{\,1} e^{-j\frac{2\pi}{N}n} = e^{-j\frac{2\pi}{N}n}$$

$$= \phi_{-1}[n]$$

Discrete-Time Exponentials Examples

$\cos[\omega_0 n]$ periodic?

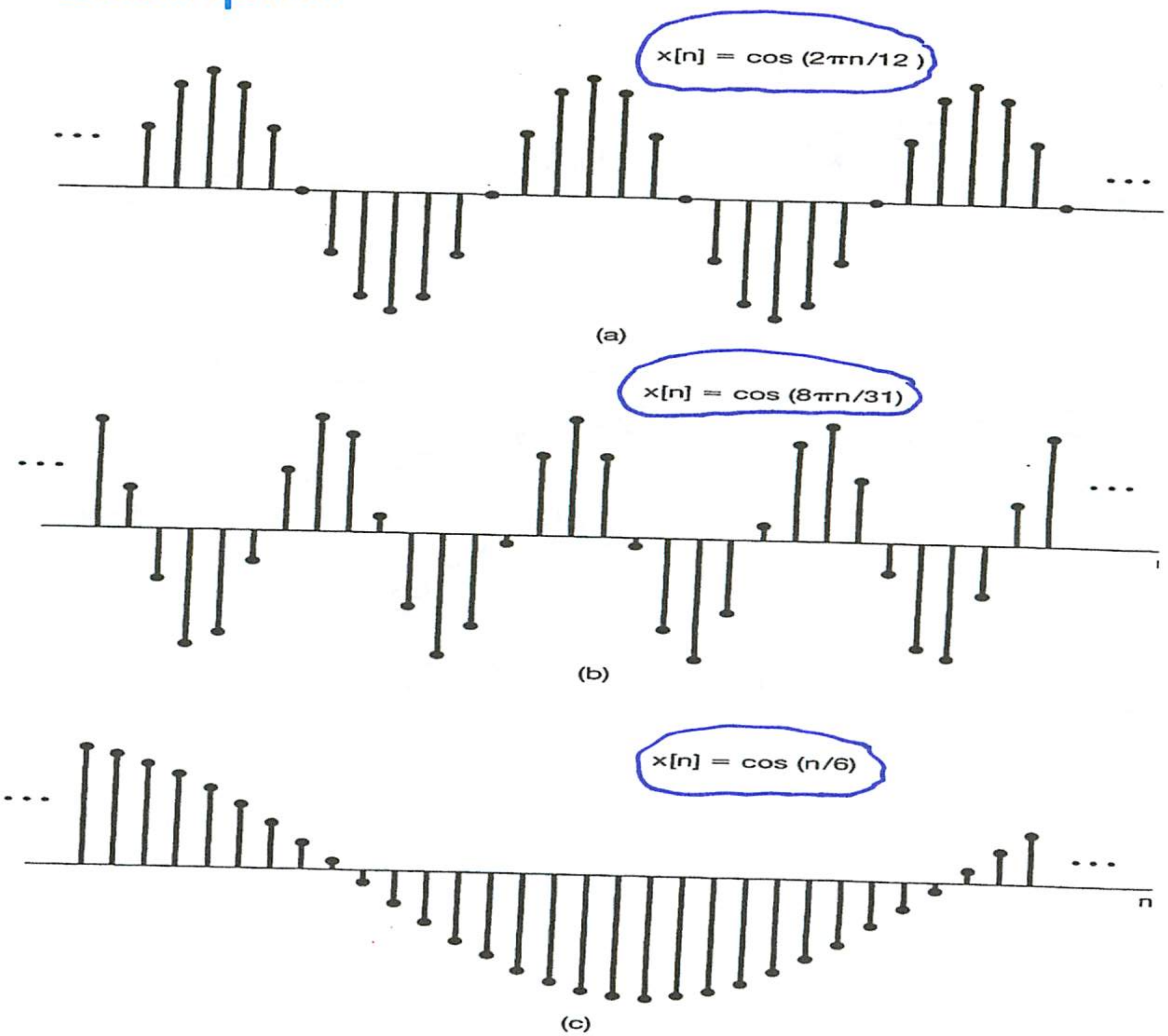


Figure 1.25 Discrete-time sinusoidal signals.

looks periodic

$\omega_0 = \frac{2\pi}{12} \Rightarrow \frac{\omega_0}{2\pi} = \frac{1}{12}$ ✓

$N = \frac{2\pi}{\omega_0} = 12$

periodic

$\omega_0 = \frac{8\pi}{31} \Rightarrow \frac{\omega_0}{2\pi} = \frac{4}{31}$ ✓

periodic

– what is period?

CT $T = 31/4$

DT $N = 31 \Rightarrow$ must be integer

CT $\cos(t/6)$

$T_0 = \frac{2\pi}{\omega_0} = 12\pi \Rightarrow$

not an integer, irrational, but still periodic

DT not periodic

$\frac{\omega_0}{2\pi} = \frac{1}{12\pi} \rightarrow$ not rational

Examples
Problems #1.25 and 1.26

$f =$ fund. freq in Hz $= \frac{1}{T}$

$\omega = 2\pi f =$ radian freq. $= \frac{2\pi}{T}$
(rad/sec)

1. $3\cos\left(4t + \frac{\pi}{3}\right)$

periodic

period $= \frac{2\pi}{4} = \frac{\pi}{2}$

2. $\left[\cos\left(2t - \frac{\pi}{3}\right)\right]^2 = \frac{1}{2} + \frac{1}{2}\cos\left(4t + \frac{2\pi}{3}\right)$

periodic

period $= \frac{\pi}{2}$

3. $\sin\left(\frac{6\pi}{7}n + 1\right)$ periodic?

- if continuous-time, $\sin\left(\frac{6\pi}{7}t + 1\right) \longrightarrow$ periodic,
period $= \frac{2\pi}{6\pi/7} = \frac{7}{3}$

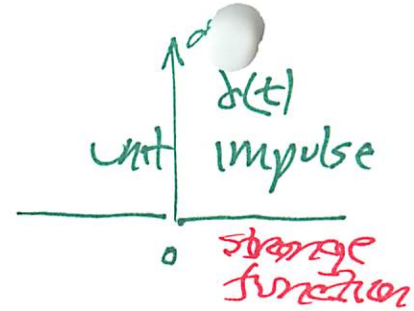
$\hookrightarrow$ but discrete time, period must be an integer

$\rightarrow$ periodic, period $= 7$

Section 1.4
Unit Impulse and Unit Step Functions

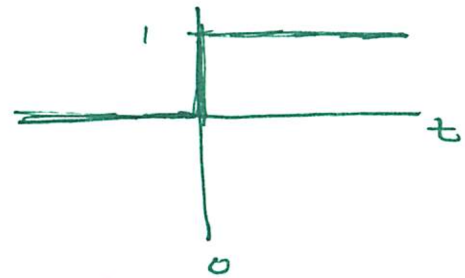
Unit Step Functions

(Section 1.4)



Continuous time

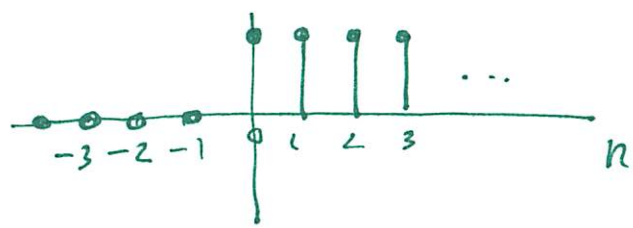
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



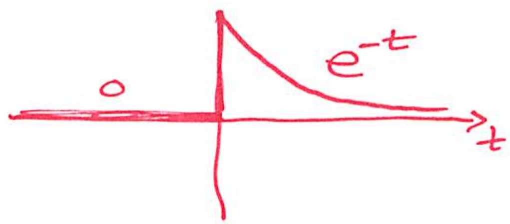
discontinuous at $t = 0$

discrete time

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



Example: $x(t) = e^{-t} u(t)$

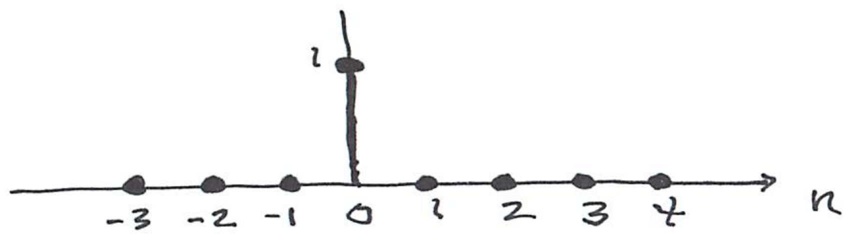


Unit Impulse and Unit Step

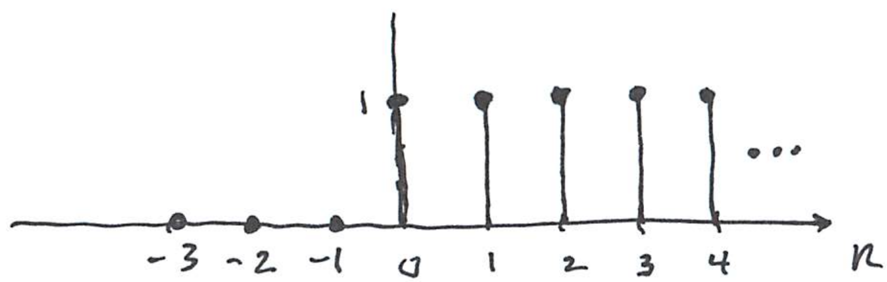
discrete-time

unit impulse

$$\delta[n] = \begin{cases} 0 & , \ n \neq 0 \\ 1 & , \ n = 0 \end{cases}$$



unit step



$$u[n] = \begin{cases} 0 & , \ n < 0 \\ 1 & , \ n \geq 0 \end{cases}$$

Unit Impulse and Unit Step

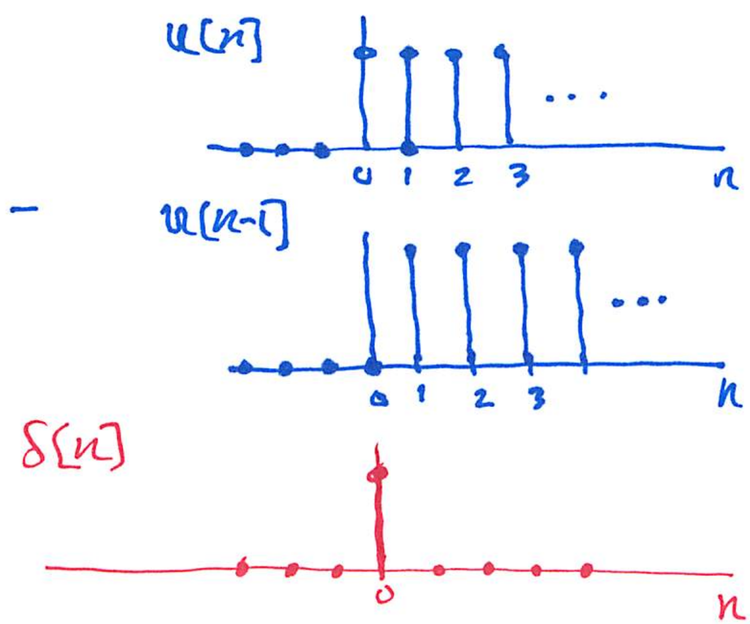
discrete-time $\longrightarrow$ useful relationships

- $u[n] = \sum_{m=-\infty}^{n} \delta[m]$
  "sum"

$$\begin{array}{ll} n<0 & u[n]=0 \\ n=0 & u[n]=1 \\ n>0 & u[n]=1 \end{array}$$

$$\overset{k=n-m}{=} \sum_{k=\infty}^{0} \delta[n-k] = \sum_{k=0}^{\infty} \underbrace{\delta[n-k]}_{=\begin{cases} 1 & n=k \\ 0 & n\neq k \end{cases}}$$

- $\delta[n] = u[n] - u[n-1]$
  "difference"

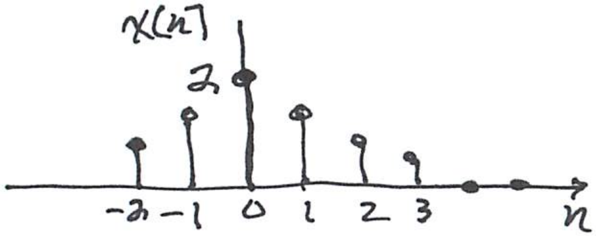


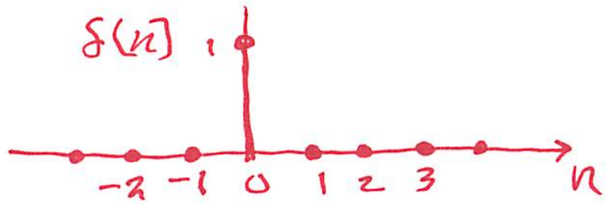
Unit Impulse

discrete-time

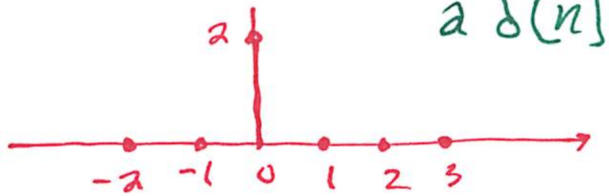
What is $x[n]\,\delta[n]$?

e.g





$\parallel$



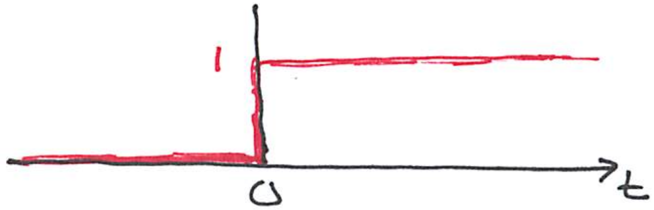
$2\,\delta[n] = x[0]\,\delta[n]$ $\Rightarrow$ sample

In general, $x[n] \cdot \underbrace{\delta[n-n_0]}_{\text{at } n_0} = x[n_0]\,\delta[n-n_0]$

Unit Impulse and Unit Step

continuous-time

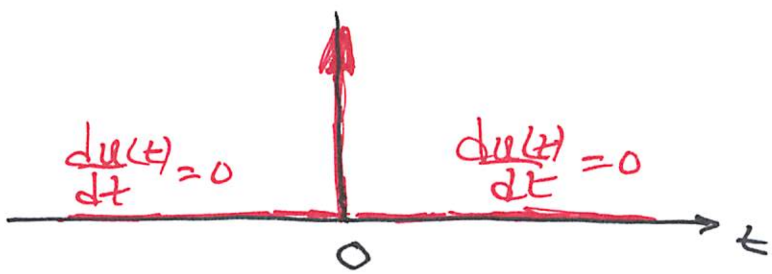
unit step



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

discontinuous at $t=0$

→ unit impulse ? $\dashrightarrow \sim \dfrac{du(t)}{dt}$

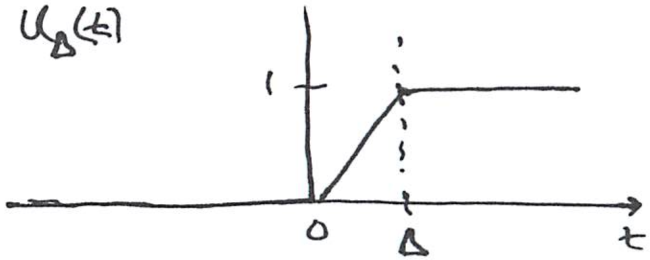


generalized function

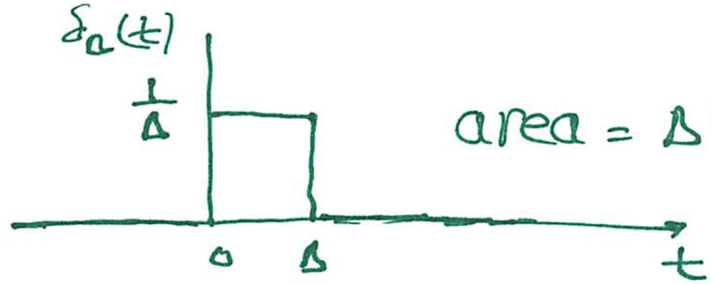
singularity function

Unit Impulse and Unit Step

continuous-time



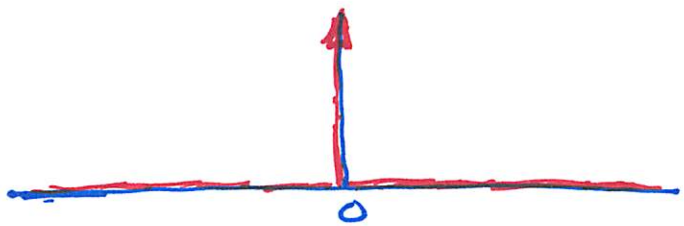
$$\xrightarrow[\quad]{\frac{du_\Delta(t)}{dt}}$$



$\text{area} = \Delta \cdot \frac{1}{\Delta} = 1$

Let $\Delta \to 0$ $\Rightarrow$ peak $\to \infty$ but area remains $= 1$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



area = 1

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

	discrete	continuous	
sum	$u[n] = \sum_{m=-\infty}^{n} \delta[m]$	$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$	integral
difference	$\delta[n] = u[n] - u[n-1]$	$\delta(t) = \frac{du(t)}{dt}$	derivative

Unit Impulse and Unit Step

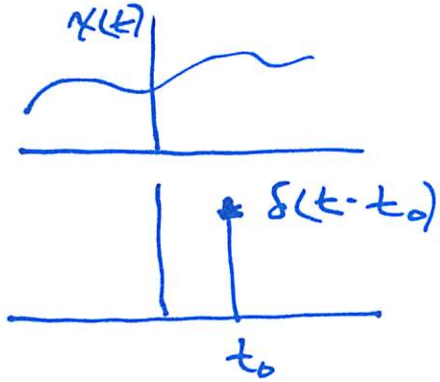
continuous-time → useful relationships

- $$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \overset{\sigma = t-\tau}{=\!=} \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$
  $$= \int_{0}^{\infty} \delta(t-\sigma)\,d\sigma$$

- sampling

  $$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

- $$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = ?$$



$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt$$
$$= \int_{-\infty}^{\infty} \underbrace{x(t_0)}_{\text{constant}}\,\delta(t-t_0)\,dt$$
$$= x(t_0) \underbrace{\int_{-\infty}^{\infty} \delta(t-t_0)\,dt}_{1}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)}$$ sifting property

Examples
Unit Impulse and Unit Step

$$x(t_0) = \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt$$

- $$\int_{-2}^{2} (3t^2+1)\left[\delta(t) + 2\delta(t+1)\right] dt$$

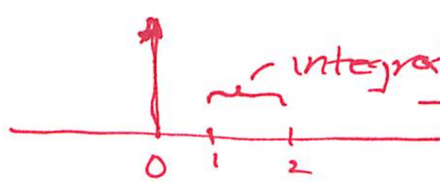
$$= \int_{-2}^{2} (3t^2+1)\,\underbrace{\delta(t)}_{\text{at } t=0}\,dt + 2\int_{-2}^{2} (3t^2+1)\,\underbrace{\delta(t+1)}_{\text{at } t=-1}\,dt$$

$$= 1 + 2\left(3(-1)^2+1\right) = 9$$

- $$\int_{1}^{2} (3t^2+1)\left[\delta(t) + 2\delta(t+1)\right] dt = ?$$

$$\int_{1}^{2} \delta(t)\,dt = ?$$

$$= 0$$



integration interval – doesn't include impulse function

Examples
Unit Impulse and Unit Step

$$\int_0^2 e^{-t}\left[\underbrace{\delta(t+1)}_{\text{at } t=-1} + \underbrace{\delta(t-1)}_{\text{at } t=1}\right] dt$$

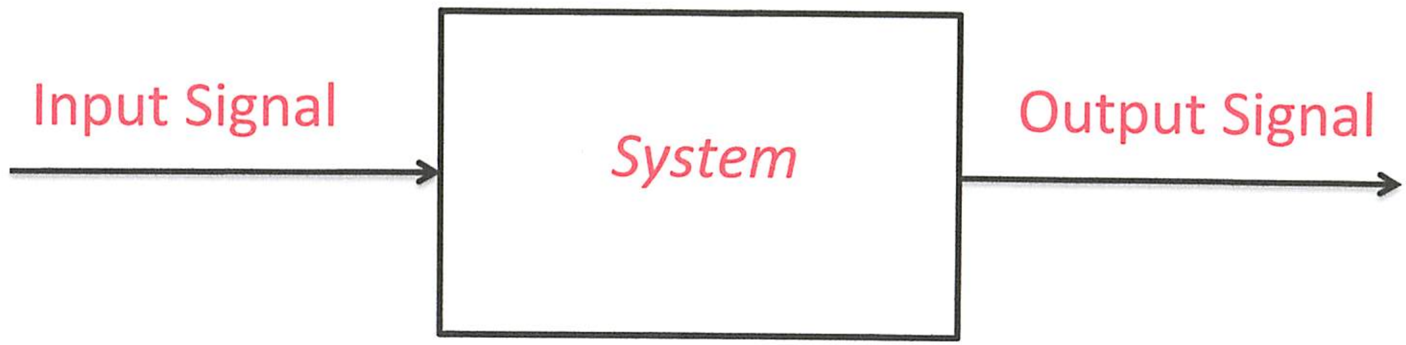
$\Downarrow$

integrates to 0

$$= \int_0^2 e^{-t}\,\delta(t-1)\,dt = e^{-1} = \frac{1}{e}$$

Section 1.6
System Properties

System Properties



- Memoryless
- Invertible
- Causal
- Stable
- Time-invariant
- Linear

System Properties
Memory

Output for each value of the independent variable at a given time is dependent on the input signal at only that same time.

e.g. $y[n] = x^2[n] + x[n] \rightarrow$ memoryless

$y[n] = x[n-1] \rightarrow$ memory

delay

- output at time n is dependent on input at a different time $(n-1)$

Systems with memory $\rightarrow$ past (and future) inputs are needed to compute the current output

delay $y[n] = x[n-1]$

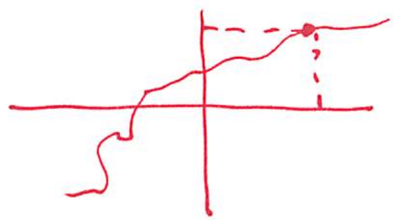
accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$

Voltage across a capacitor $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau \rightarrow$ stores energy

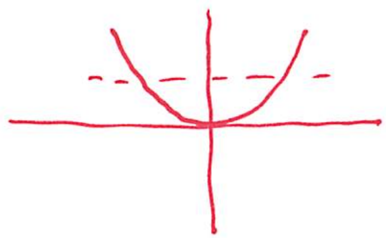
System Properties
Invertibility

Same as usual inverse function definition in calculus

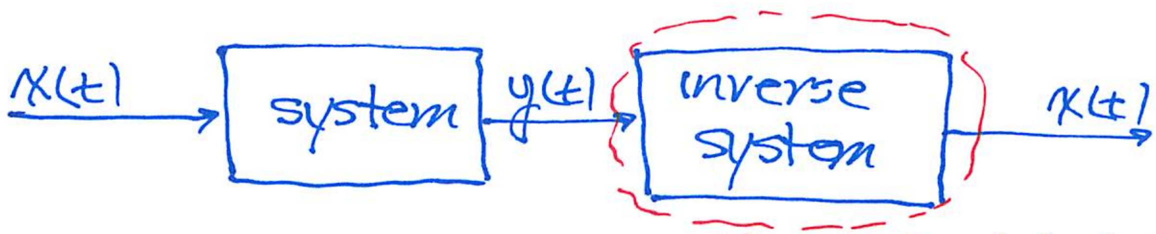
Invertible ⇒ System is invertible if distinct inputs lead to distinct outputs



invertible ("one-to-one")



not invertible
(but still a function)



(goal of a lot of design applications)

e.g. $y(t) = e^{x(t)}$

$\ln y(t) = x(t)$

System Properties

Causality

A system is causal if the output at any time depends on values of the input at only present and past times.

$\Downarrow$

"no output before input applied"

e.g. $y[n] = x[n] - x[n+1]$

not causal

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

causal

System Properties
Causality (cont'd)

- memoryless $\Rightarrow$ causal
- need to look at all time samples to prove causality

? $y[n] = x[-n]$ causal or not causal ?

$n > 0$ $y[1] = x[-1]$

$y[2] = x[-2]$

$\vdots$

* $n < 0$ $y[-1] = x[1] \Rightarrow$ future

NOT CAUSAL

- causality is only a function of the input & output — not of any other time function

e.g. $y(t) = x(t)\,(t+1)$ causal

$y(t) = x(t+1)\,g(t)$ not causal

System Properties

Stability

Stable → If input is bounded, output must also be bounded.

How do we check stability?

- pick a specific bounded input that gives an unbounded output → unstable

- if none can be found, then must use a <u>nonspecific</u> input

Stability
Example (#1.13 pp. 49-50)

- $y(t) = t\,x(t)$

  $\underbrace{x(t) = K}_{\text{constant}}$, $K < \infty$ bounded

  $y(t) = Kt$

  $\to \infty$ unstable

- $y(t) = e^{x(t)}$

  $|x(t)| < B < \infty$ bounded input

  $-B < x(t) < B$

  $\underbrace{e^{-B}}_{<\infty} < y(t) < \underbrace{e^{B}}_{<\infty}$

  $\hookrightarrow$ bounded output $\Rightarrow$ stable

System Properties

Time-Invariance

A system is time invariant if the behavior and characteristics of the system are fixed over time.

system behaves the same way everytime we run the experiment

$$x[n] \longrightarrow y[n]$$

$$x[n-n_0] \longrightarrow y[n-n_0]$$

time shift in input results in identical time shift in output signal

Time-Invariance
Examples (#1.14-#1.16, pp. 51-52)

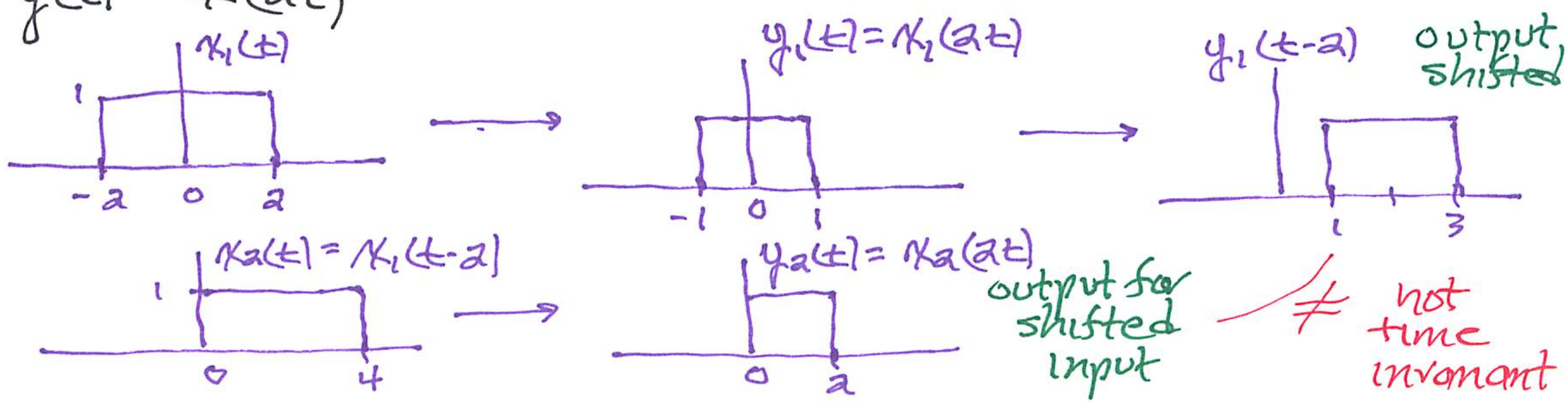
i) $y(t) = \sin[x(t)]$

$x_1(t)$ | $x_2(t) = x_1(t-t_0)$
$y_1(t) = \sin[x_1(t)]$ | $y_2(t) = \sin[x_2(t)]$
$y_1(t-t_0) = \sin[x_1(t-t_0)]$ $\longleftrightarrow$ $y_2(t) = \sin[x_1(t-t_0)]$

time invariant

ii) $y[n] = n\,x[n]$
(time varying gain $\rightarrow$ not time invariant

iii) $y(t) = x(2t)$



Time-Invariance Example

$y(t) = x(2-t)$

① $x_1(t) \longrightarrow y_1(t)$

$x_1(t-t_0) \Rightarrow x_2(t)$

$x_2(t) \longrightarrow y_2(t)$

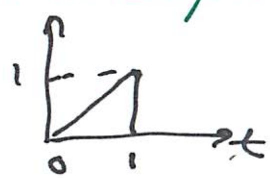
output for shifted input

② $y_3(t) = y_1(t-t_0)$

shifted output

If time-invariant, $y_3(t) = y_2(t) \Rightarrow$ shift in input results in same shift in output

$\rightarrow$ Graphical: consider specific $x(t) = \begin{cases} t, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$



System Properties

Linearity

"A linear system is a system that possesses the important property of superposition: If an input consists of a weighted sum of several signals then the output is the superposition – that is, the weighted sum – of the responses of the system to each of those signals." (p. 53 Section 1.6.6)

$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$

Linear if

i) $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$ superposition

ii) $a x_1(t) \longrightarrow a y_1(t)$ homogeneity

In general,

$$\sum_i a_i x_i(t) \longrightarrow \sum_i a_i y_i(t)$$

NOTE: $x(t) = 0 \longrightarrow y(t) = 0$

$y = ax + b \Rightarrow$ NOT LINEAR (AFFINE)

Linearity
Example (Problem #1.19, p. 59)

a.) $y(t) = t^2 x(t-1)$

$x_1(t),\ x_2(t)$

$y_1(t) = t^2 x_1(t-1)$

$y_2(t) = t^2 x_2(t-1)$

$x_3(t) = a x_1(t) + b x_2(t)$

$$y_3(t) = t^2 x_3(t-1) = a \underbrace{t^2 x_1(t-1)}_{y_1(t)} + b \underbrace{t^2 x_2(t-1)}_{y_2(t)}$$

$\therefore\ y_3(t) = a y_1(t) + b y_2(t) \Rightarrow$ linear

b.) $y[n] = x^2[n-2]$

$y_1[n] = x_1^2[n-2]$

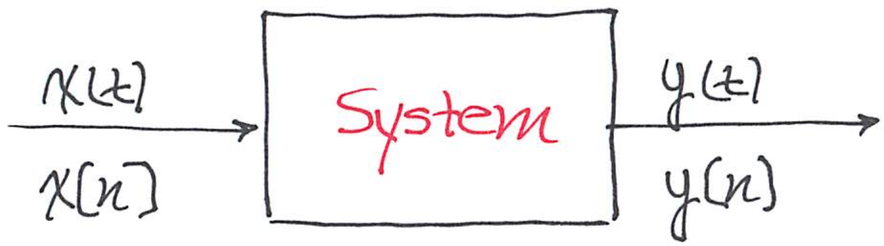
$y_2[n] = x_2^2[n-2]$

$x_3[n] = x_1[n] + x_2[n]$

$$y_3[n] = \underbrace{x_1^2[n-2] + x_2^2[n-2]}_{y_1(n) + y_2(n)} + 2 x_1[n-2]\, x_2[n-2]$$

Not linear

Chapter 2
Linear, Time-Invariant Systems



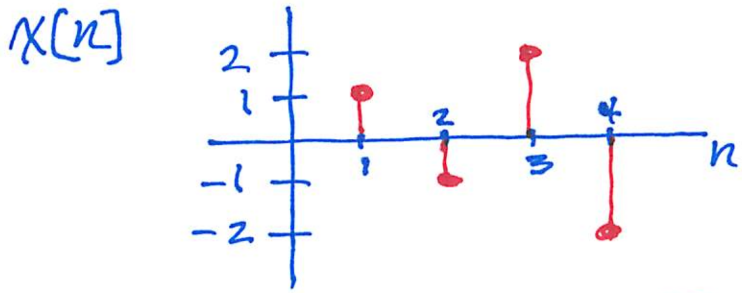
① impulse response & convolution

② differential equations for CT
difference equations for DT

Section 2.1
Discrete-Time LTI Systems
Convolution Sum

Discrete-Time Signal

A discrete-time signal can be represented as a linear combination of shifted unit impulses



$$x[n] = 1\,\delta[n-1] - 1\,\delta[n-2] + \underset{\text{weighting}}{2}\,\delta[\underset{\text{shift}}{n-3}] - 2\,\delta[n-4] \quad \text{sum}$$

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

e.g $x[n] = u[n] \Rightarrow u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

Unit Impulse Response

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$x[n] = \sum x[k]\,\delta[n-k]$$

The response to an input $x[n]$ for an LTI system is then

linearity

i) superposition of the scaled responses of the system to each of the shifted impulses

time invariance

ii) responses are simply time-shifted versions of one another

$\delta[n]$ input $\longrightarrow$ impulse response (output) $h[n]$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$

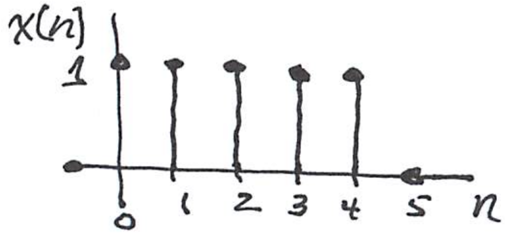
convolution sum

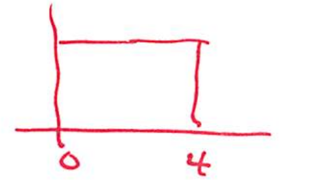
linear TI

$$= x[n] * h[n]$$

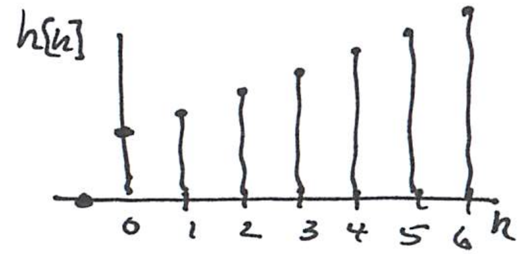
Convolution
Example (#2.4, pp. 85-88)

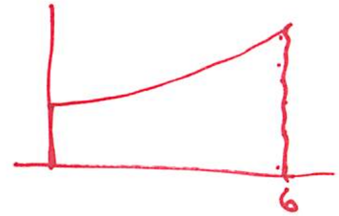
$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

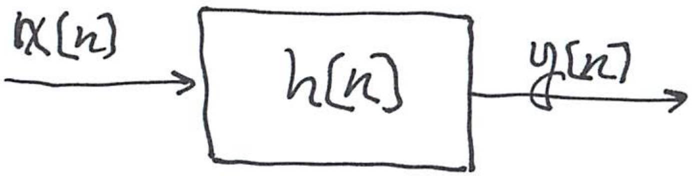




$$h[n] = \begin{cases} a^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}$$

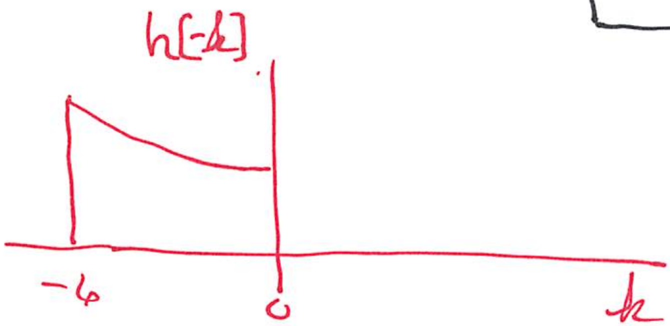


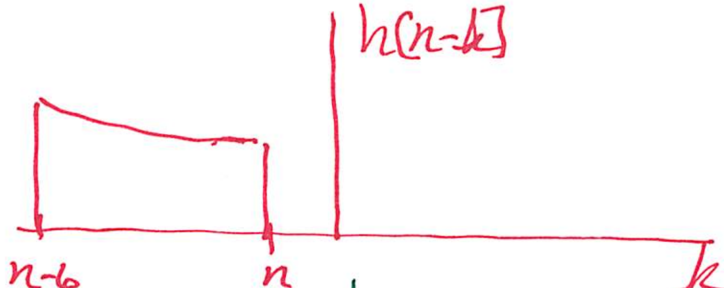


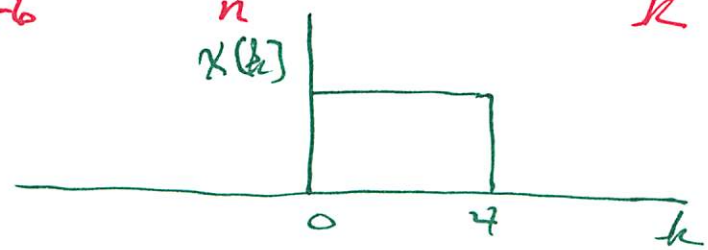


$$y[n] = \sum x[k]\, h[n-k]$$

shift flip







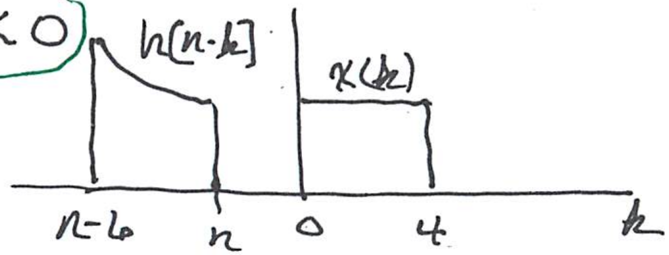
n slowly increase & perform sum over k

Convolution Example (#2.4, pp. 85-88)

$$\sum x[k]h[n-k]$$

$$h[n] = \alpha^n, \quad 0 \le n \le 6$$

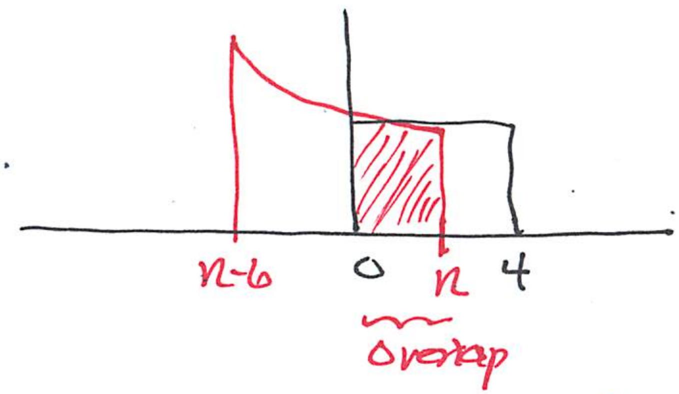
① $n < 0$



$x[k]h[n-k] = 0$ no overlap

$\therefore\ y[n] = 0, \quad n < 0$

② $n \ge 0$



$$\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=0}^{n} x(k)\,h[n-k]$$

limits → overlap

breakpoints → "picture"

$$= \sum_{k=0}^{n} \alpha^{n-k} = \alpha^n \sum_{k=0}^{n} \alpha^{-k}$$

geometric series

$n+1$ terms

easier

$r = n-k$

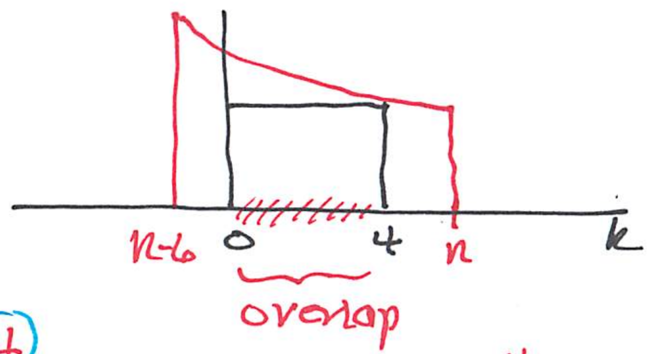
$$= \sum_{r=n}^{0} \alpha^r = \sum_{r=0}^{n} \alpha^r = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Convolution
Example (#2.4, pp. 85-88)

→ this applies until "picture" changes → until $n>4$

② $0 \le n \le 4$

③ $n > 4$

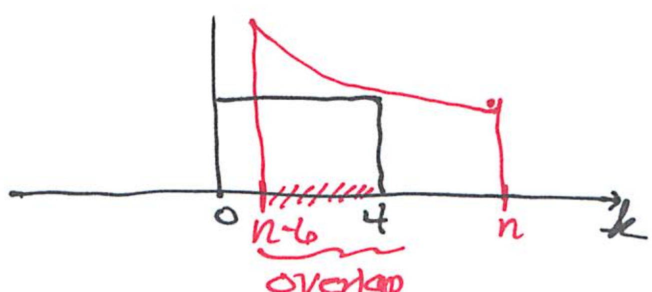


$$\sum_{k=0}^{4} x[k]\,h[n-k] = \sum_{k=0}^{4} \alpha^{n-k} = \alpha^{n-4}\,\frac{1-\alpha^5}{1-\alpha}$$

→ same picture until $n-6=0$

$\therefore$ $4 < n \le 6$

④ $n > 6$



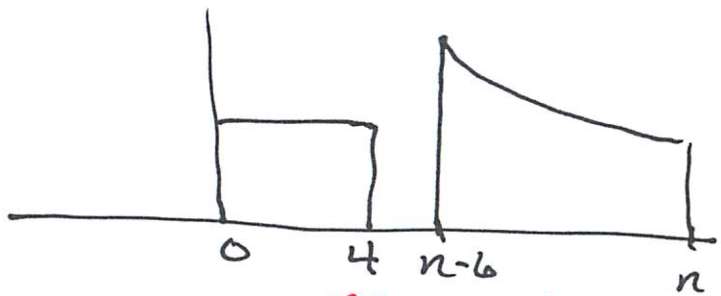
$$\sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}$$

Convolution
Example (#2.4, pp. 85-88)

↳ same picture until $n-6=4 \Rightarrow n=10$

$6 < n \leq 10$

5 $n > 10$



no overlap

$y[n]=0$

n

n]

n

(2.14)

lly, changing

(2.15)

ate eq. (2.15).
n in eq. (2.15)

(2.16)

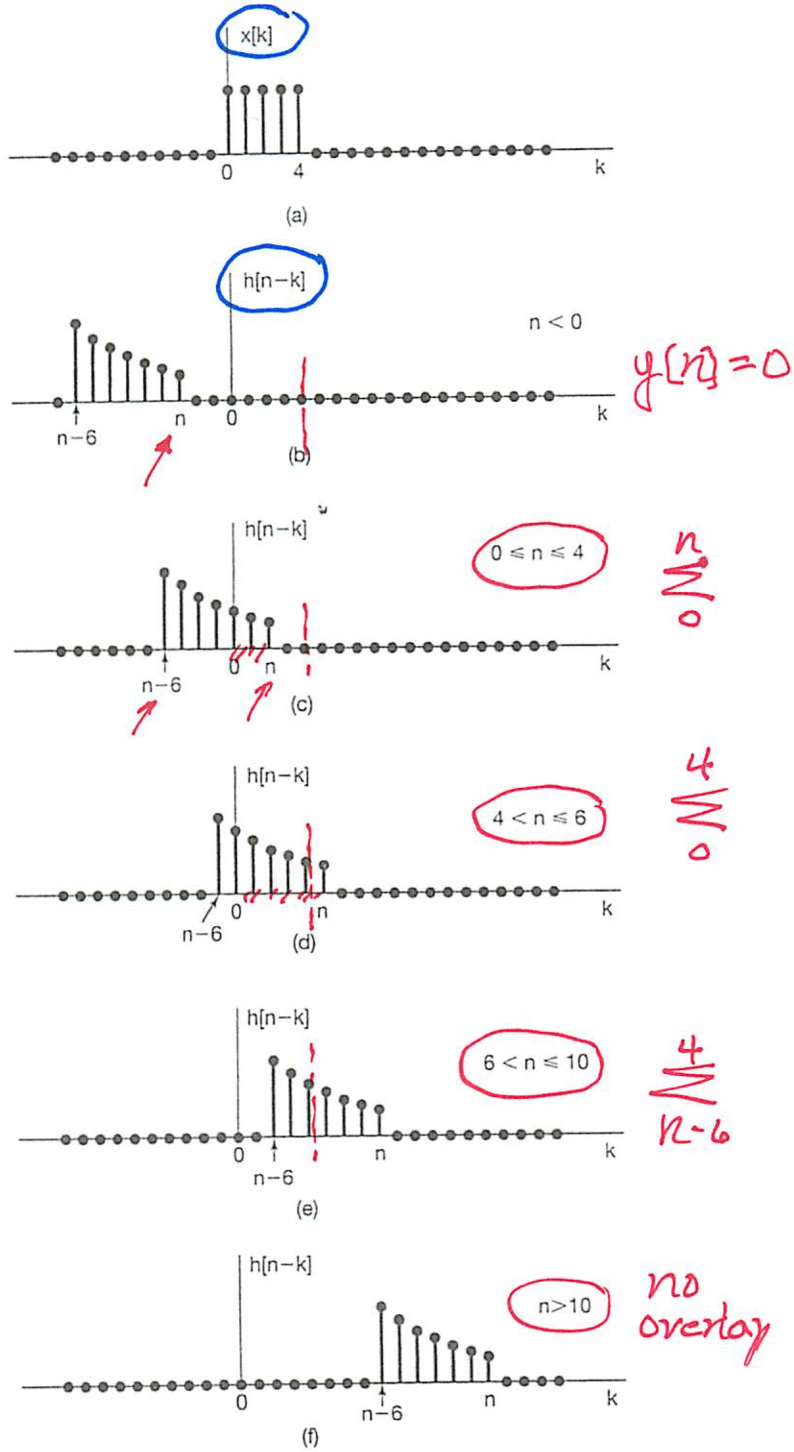


Figure 2.9 Graphical interpretation of the convolution performed in Example 2.4.

so that

$$y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}.$$

We can again use eq. (2.13) to evaluate this summation. Letting $r = k-n+6$, we obtain

$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^6 \sum_{r=0}^{10-n} (\alpha^{-1})^r = \alpha^6 \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^7}{1-\alpha}.$$

Interval 5. For $n-6 > 4$, or equivalently, $n > 10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and hence,

$$y[n] = 0.$$

Summarizing, then, we obtain

$$\boxed{y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \dfrac{\alpha^{n-4}-\alpha^7}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases}},$$

which is pictured in Figure 2.10.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

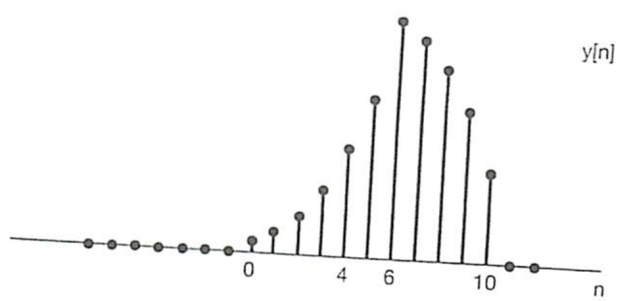


Figure 2.10 Result of performing the convolution in Example 2.4.

Example 2.5

Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows:

$$x[n] = 2^n u[-n], \tag{2.17}$$

$$h[n] = u[n]. \tag{2.18}$$

Impulse Response

"Response of the system to an input which is an impulse function"

Examples:

$$y(t) = x(t-1) + x(t-2)$$

impulse response $h(t) = ?$

$$h(t) = \delta(t-1) + \delta(t-2)$$

$$y[n] = x[n-a] + \alpha\, x[n+b]$$

impulse response $h[n] = ?$

$$h[n] = \delta[n-a] + \alpha\, \delta[n+b]$$

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = ?$$

$$x[n] = \sum x[k]\,\delta[n-k]$$

$$y[n] = x[n-2]$$

$$?\quad h[n] = y[n]\Big|_{x[n]=\delta[n]}$$

unit impulse ~~response~~

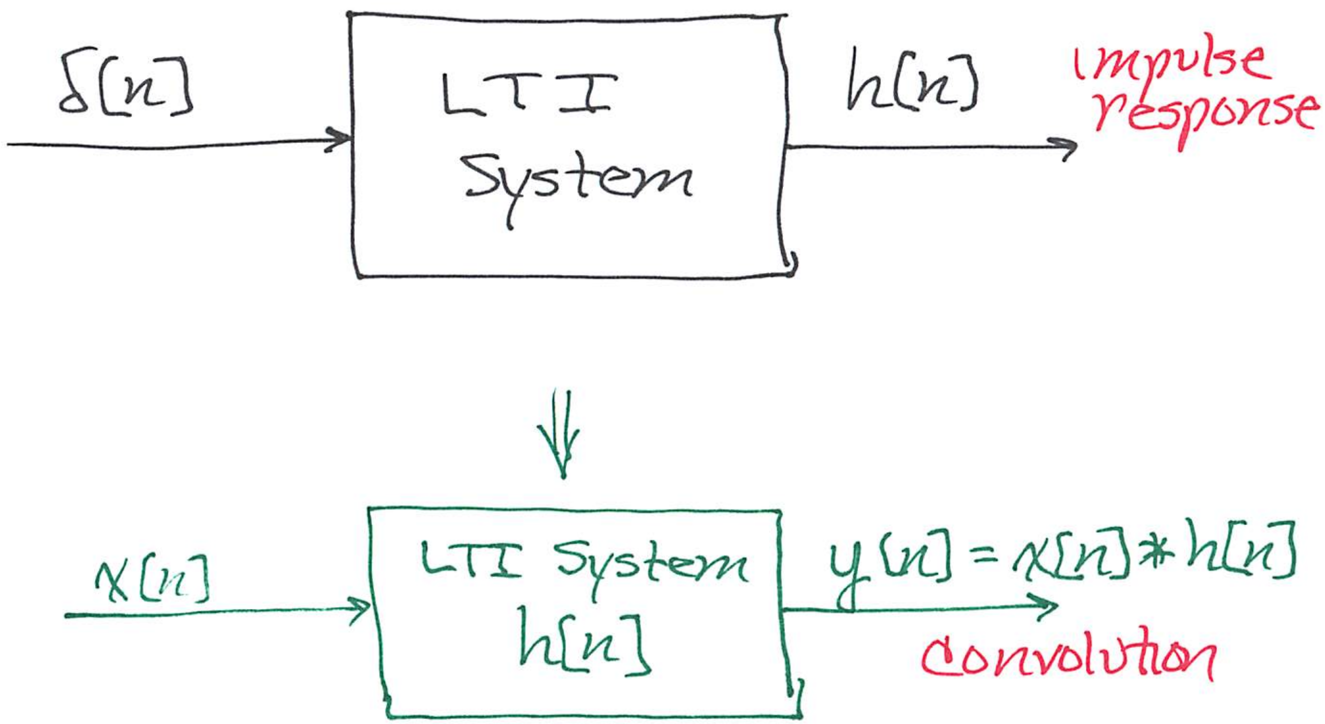
$$\underset{\text{input}}{x[n] = \delta[n]} \longrightarrow \boxed{\text{System}} \longrightarrow \underset{\text{output}}{y[n] = h[n]} \rightarrow \text{impulse response}$$

- time-invariance
  $$x[n] = \delta[n-k]$$
  $$\Downarrow$$
  $$y[n] = h[n-k]$$

- linearity $\rightarrow$ sum of inputs gives sum of individual outputs

$$\sum x[k]\,\delta[n-k] \Rightarrow \sum x[k]\,h[n-k]$$

Unit Impulse Response



Section 2.2
Continuous-Time LTI Systems
Convolution Integral

Unit Impulse Response

input $\delta(t)$ output $h(t)$

impulse response

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \xrightarrow[\text{time invariance}]{\text{linearity,}} y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

convolution integral

$\Downarrow$

$$y(t) = x(t) * h(t)$$

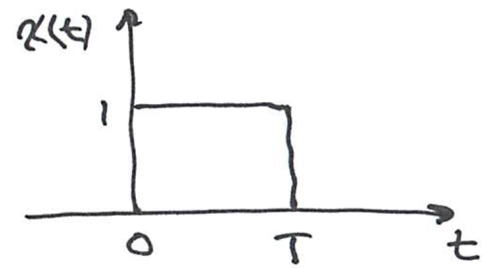
? $y(t) = x(t-1)$ What is the impulse response?

impulse response = system response to an input $x(t) = \delta(t)$

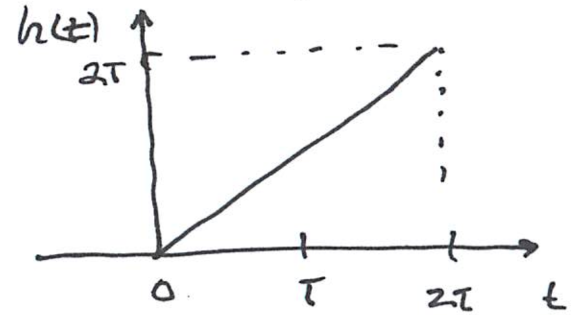
$\therefore h(t) = \delta(t-1)$

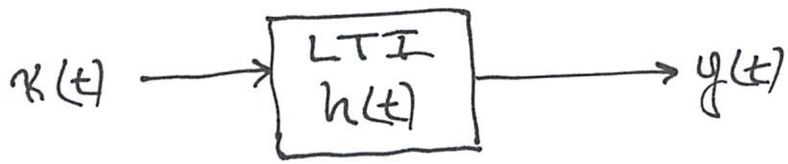
Convolution
Example (#2.7, pp. 99-101)

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$



$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$





$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

shift

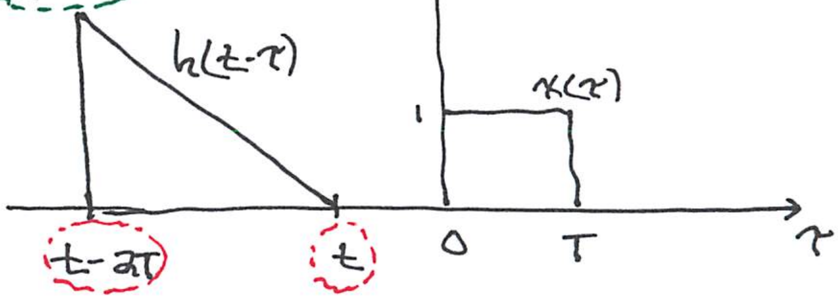
flip

1. time reverse $h(\tau)$
2. shift or slide $h(\tau)$ through signal
3. multiply & integrate

- limits
- breakpoints

Convolution
Example (#2.7, pp. 99-101)

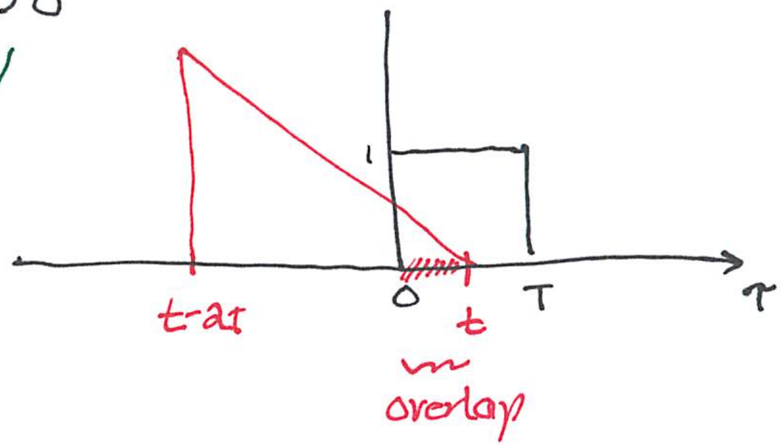
① $t < 0$



no overlap

$y(t) = 0$

② $t > 0$



$$y(t) = \int_{0}^{t} 1 \cdot (t-\tau)\, d\tau$$

$$= \left(t\tau - \frac{\tau^2}{2}\right)\Big|_0^t$$

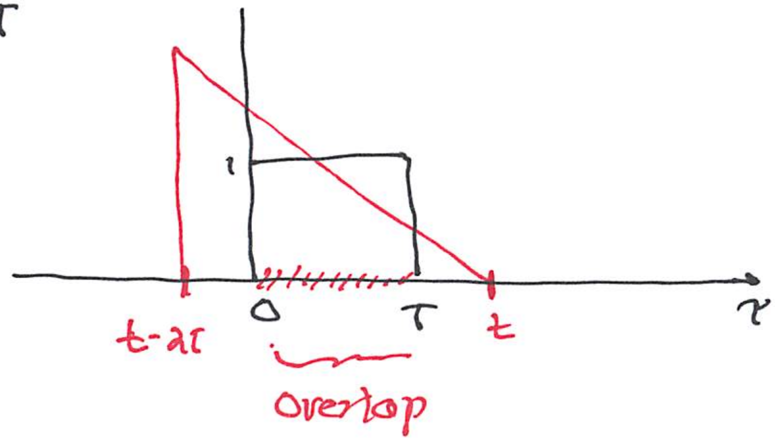
$$= \frac{t^2}{2}$$

this "picture" (limits on integral) remains until $t = T$

$0 < t < T$

Convolution
Example (#2.7, pp. 99-101)

③ $t > T$



this picture (limits) apply until $t - 2T > 0$

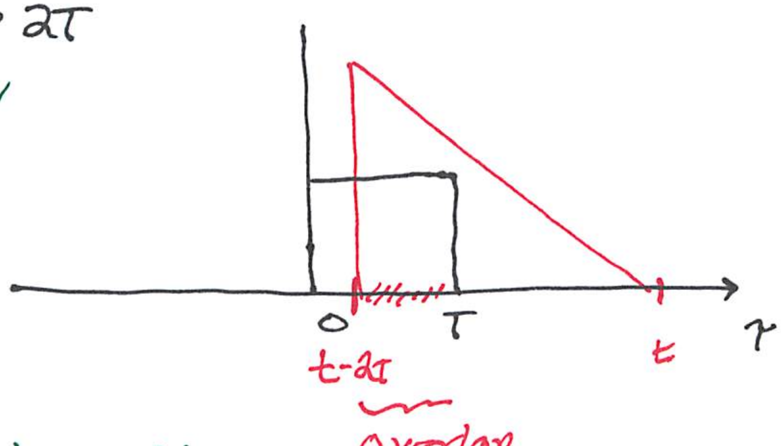
$\therefore \quad T < t < 2T$

$$y(t) = \int_{0}^{T} (t-\tau)\,d\tau$$

$$= \left(t\tau - \frac{\tau^2}{2}\right)\Big|_0^T$$

$$= tT - \frac{T^2}{2}$$

④ $t > 2T$



applies until $t - 2T > T$

$\therefore \quad 2T < t < 3T$

$$y(t) = \int_{t-2T}^{T} (t-\tau)\,d\tau$$

$$= -\frac{t^2}{2} + Tt + \frac{3T^2}{2}$$

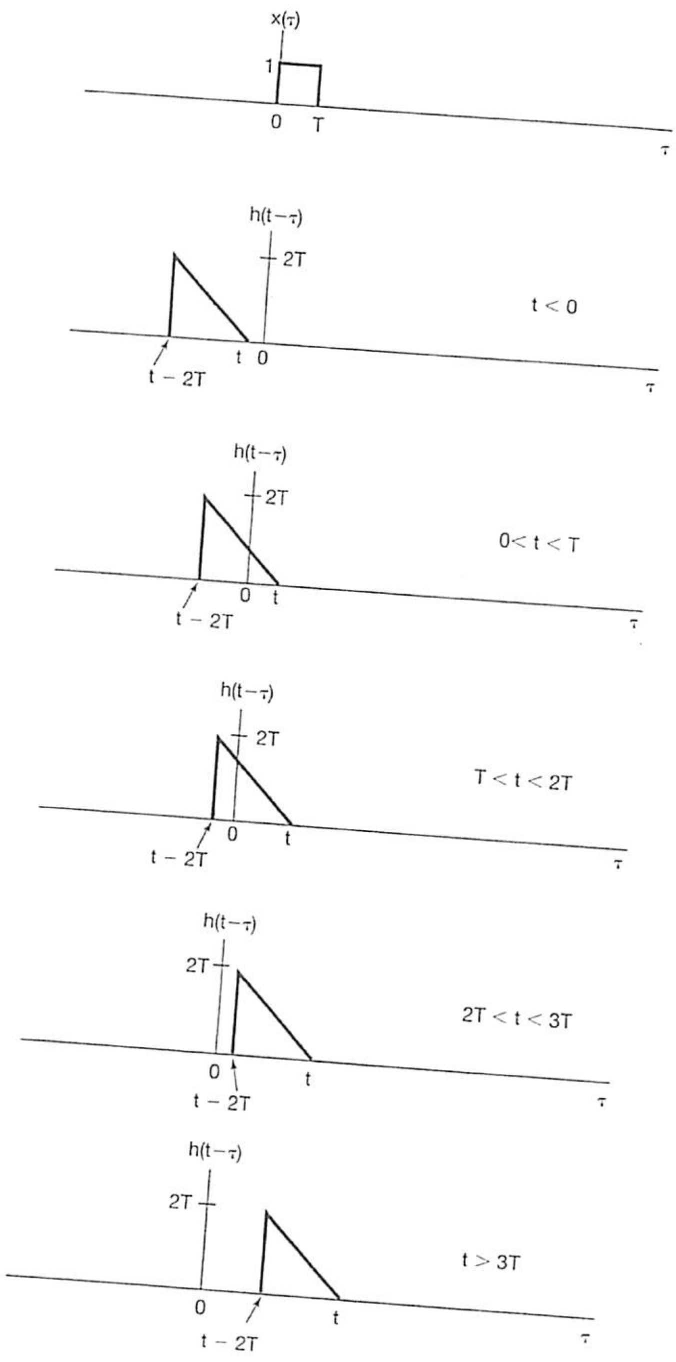


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

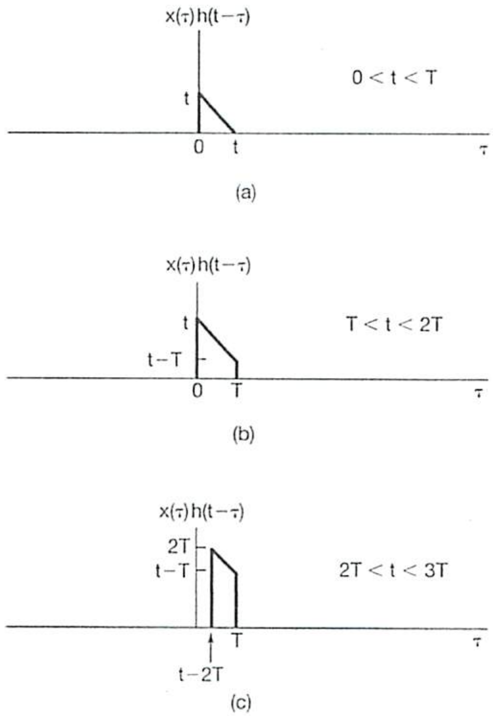


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

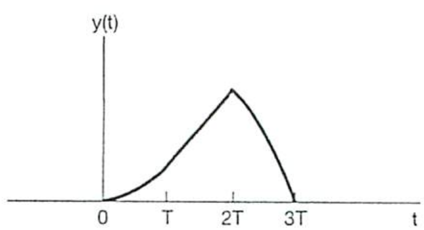


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

NOTE: result is equal to length of input plus length of system impulse response

Example 2.8

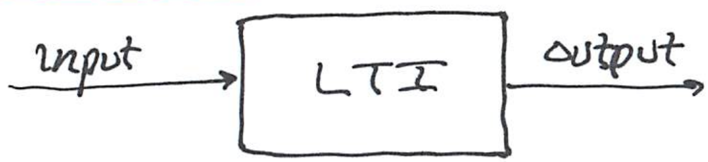
Let $y(t)$ denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t), \tag{2.35}$$

$$h(t) = u(t-3). \tag{2.36}$$

The signals $x(\tau)$ and $h(t-\tau)$ are plotted as functions of τ in Figure 2.22(a). We first observe that these two signals have regions of nonzero overlap, regardless of the value

Section 2.3
Properties of LTI Systems



what are implications
for impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

impulse response $\Longrightarrow$ $h[n] = y[n]\Big|_{x[n]=\delta[n]}$

e.g. $y[n] = \frac{1}{3}\left[x[n] + x[n-2] + x[n-4]\right]$

$h[n] = \frac{1}{3}\left[\delta[n] + \delta[n-2] + \delta[n-4]\right]$

Properties of LTI Systems

Commutative

$a \,\square\, b = b \,\square\, a$ order

- $$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

  $$\overset{r=n-k}{=} \sum_{r=+\infty}^{-\infty} x[n-r]\,h[r]$$

  $$= \sum_{r=-\infty}^{\infty} h[r]\,x[n-r]$$

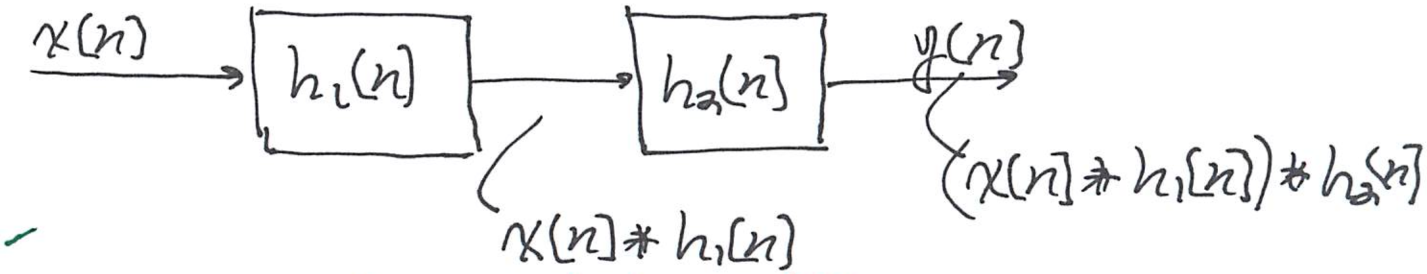
  $$= h[n] * x[n]$$

- $$x(t) * h(t) = h(t) * x(t)$$

Properties of LTI Systems

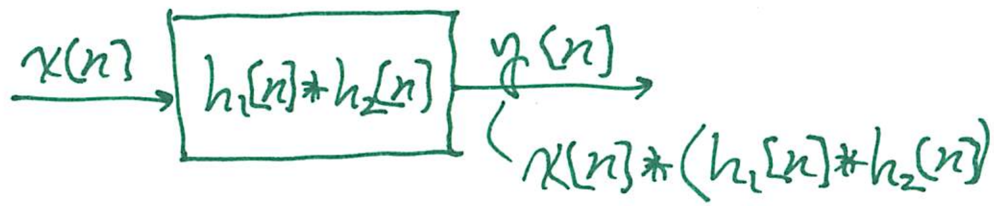
Associative

$$a \,\square\, (b \,\square\, c) = (a \,\square\, b) \,\square\, c$$



Series interconnection of LTI systems

equivalent to product of impulse responses



NOTE: generally not true for nonlinear systems

e.g. $y(n) = 4x^2[n] \neq (4x[n])^2$

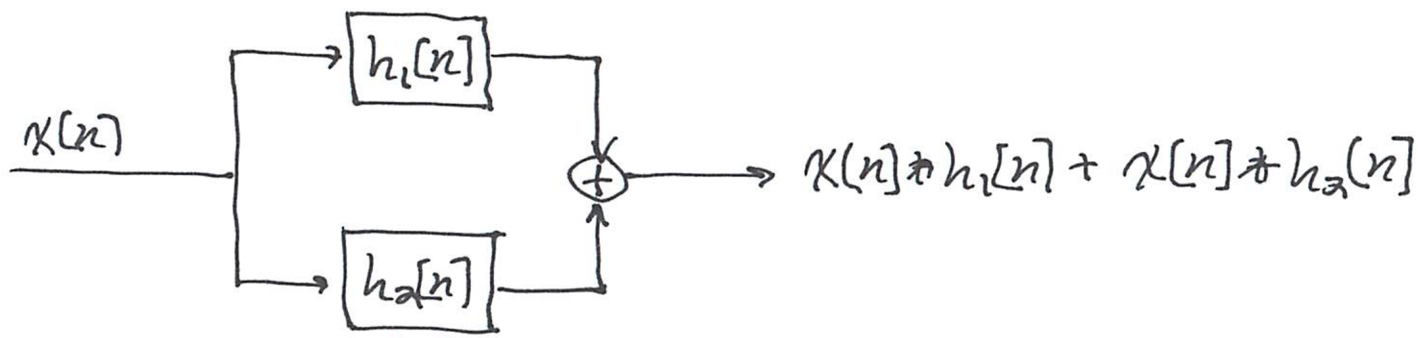
Properties of LTI Systems

Distributive

$$a \square (b+c) = a \square b + a \square c$$

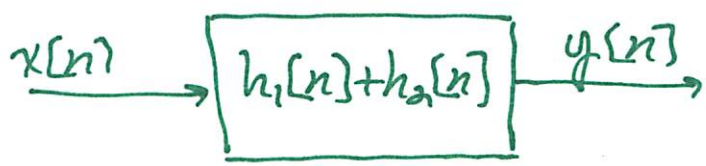
$$x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$$

MOTIVATION: arrive at a solution by the easiest route



parallel interconnection of LTI systems

equivalent to sum of impulse responses



Properties of LTI Systems

Memory

Memoryless $\longrightarrow$ output at any time depends only on the value of the input at the same time

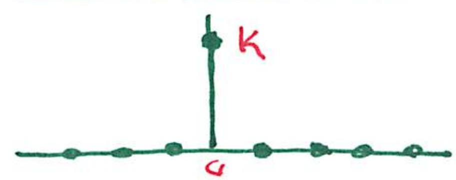
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$\overset{\text{comm.}}{=} \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] = \cdots \cancel{x[n+1]\, h[-1]}^{\,0} + \boxed{x[n]\, h[0]} + \cancel{x[n-1]\, h[1]}^{\,0} + \cdots$$

only memoryless if

$h[k] = K \quad k=0$

$\boxed{h[k] = 0 \quad k \neq 0}$



$\therefore$ memoryless $\Rightarrow h[n] = K\delta[n]$ $\Rightarrow$ if $h[n] \neq 0$ for some value of $n \neq 0 \Rightarrow$ memory

$y[n] = K x[n]$

Properties of LTI Systems

Invertibility

system is invertible if

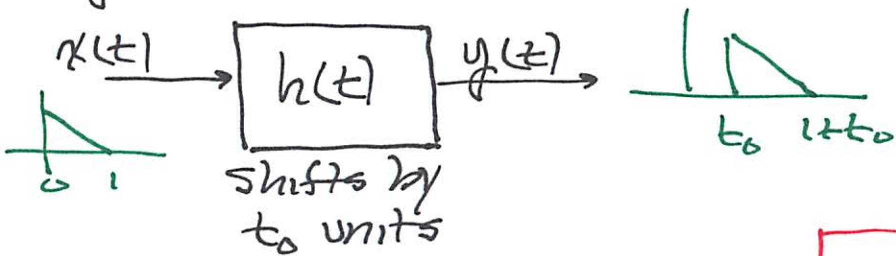
series interconnection

$\therefore$ invertible if

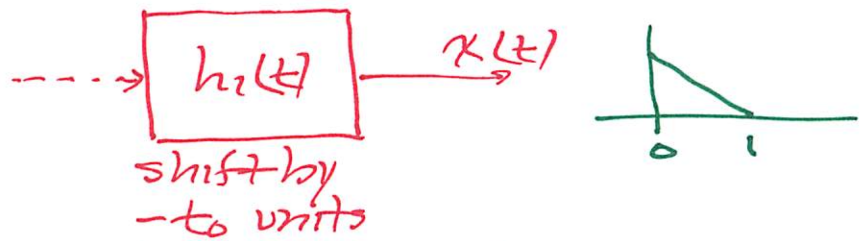
$$h(t) * h_1(t) = \delta(t)$$

Invertibility
Example (#2.11 and 2.12, pp. 110-111)

#2.11 $y(t) = x(t - t_0)$



$h(t) = \delta(t - t_0)$



$h_1(t) = \delta(t + t_0)$

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

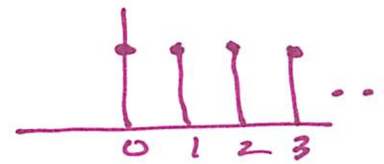
Invertibility
Example (#2.11 and 2.12, pp. 110-111)

#2.12 summer/accumulator $y(n) = \sum_{k=-\infty}^{n} x[k]$

impulse response = ?

Let $x(n) = \delta(n)$,

$$\therefore h(n) = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$



How do we get $x[n]$ back again? $x \to \boxed{h(n)} \xrightarrow{y} \boxed{h_1(n)} \xrightarrow{x}$

$\Downarrow$ first difference

$$\text{output} = y(n) - y(n-1) = \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x[k] = x[n]$$

$$\therefore h_1(n) = \delta(n) - \delta(n-1)$$

$$* \quad h(n) * h_1[n] = u[n] * (\delta[n] - \delta(n-1]) = u[n]\delta[n] - u[n] * \delta(n-1] = u[n] - u[n-1] = \delta(n)$$

Properties of LTI Systems

Causality

causal $\longrightarrow$ output depends only on the present and past values of the input

discrete time LTI $\Longrightarrow$ $y[n]$ must not depend on $x[k]$ for $k > n$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[\underbrace{n-k}]$$

$k > n$, argument is negative $\longrightarrow$ not permitted if causal

$\therefore$ causal $\longrightarrow$ $h[n] = 0$ for $n < 0$

NOTE:- causal = no output before input applied
- impulse response $\longrightarrow$ input applied at $n = 0$

Properties of LTI Systems
Stability

stable $\rightarrow$ a bounded input produces a bounded output

if $|x[n]| \le B < \infty$ for all n $\rightarrow$ $|y[n]|$ bounded

output $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|$$

$$\le \sum_{k=-\infty}^{\infty} |h[k]|\,\underbrace{|x[n-k]|}_{\le B}$$

$$\le B \sum_{k=-\infty}^{\infty} |h[k]|$$

must be finite for system to be stable

$1(-1)+2(-2)+3(4)=7$

$1\cdot 1+2\cdot 2+3\cdot 4=17$

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ absolutely summable

$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$ absolutely integrable

Stability Examples

i.) time shift $h(t) = \delta(t-t_0)$

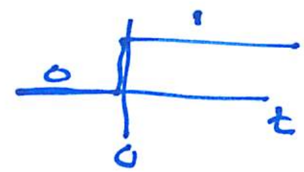
$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$ ⇒ stable

ii.) accumulation $h[n] = u[n]$

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} 1 \to \infty$$ ⇒ not stable

iii.) Problem #2.14a $h(t) = e^{-(1-2j)t}\,\underset{\text{starts at 0}}{u(t)}$

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} |e^{-t}||e^{2jt}|\,u(t)\,dt$$



$$= \int_{0}^{\infty} e^{-t}\,dt = 1$$ ⇒ stable

Section 2.4

Causal LTI Systems

Differential and Difference Equations

Linear, Constant-Coefficient Differential Equations

assume causal

no output before input applied

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Nth order

- $y(t) = y_p(t) + y_h(t)$

"particular" "homogeneous" → when input = 0

forced response natural response

- auxiliary conditions (e.g., initial conditions)

Linear, Constant-Coefficient Differential Equations Example (#2.14, pp. 118-119)

First order

$$\frac{dy(t)}{dt} + a y(t) = x(t) \quad , \quad x(t) = Ke^{3t}u(t)$$

homogeneous $\quad x(t) = 0$

$$\frac{dy(t)}{dt} + a y(t) = 0$$

→ many ways to solve this

separation of variables

$$y_h' + a y_h = 0$$

$$y' = -ay$$

$$\frac{y'}{y} = -a$$

$$\int \frac{y'}{y}\, dt = -a \int dt \Rightarrow \ln y_h(t) = -at + c$$

$$\therefore \quad y_h(t) = e^{-at+c} = Ae^{-at}$$

Linear, Constant-Coefficient Differential Equations Example (#2.14, pp. 118-119)

particular solution ("forced response")

$$x(t) = Ke^{3t}u(t)$$

Guess $\quad y_p(t) = Ye^{3t}$ for $t>0$

$$y_p' + a y_p = x(t)$$

$$3Y\cancel{e^{3t}} + aY\cancel{e^{3t}} = K\cancel{e^{3t}} \qquad t>0$$

$$5Y = K$$

$$Y = K/5$$

$\therefore \quad y(t) = A e^{-at} + \frac{K}{5} e^{3t}, \quad t>0$

determined by auxiliary conditions

e.g. for a system to be causal, $y(t) = 0$ for $t=0$

$$A + \frac{K}{5} = 0 \Rightarrow A = -\frac{K}{5}$$

$$\therefore \quad y(t) = \frac{K}{5}\left[e^{3t} - e^{-at}\right]u(t)$$

Linear, Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Nth order

- homogeneous solution $\quad x[n]=0$

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

$$y[n] = \frac{1}{a_0}\left[-\sum_{k=1}^{N} a_k y[n-k]\right]$$

recursive equation (output depends on past output)

- Can solve the general difference equation in a similar way

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

Linear, Constant-Coefficient Difference Equations

- Special case $N=0$

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} \underbrace{\left(\frac{b_k}{a_0}\right)}_{h[k]} \underbrace{x(n-k]}_{\text{input}}$$

non-recursive

"finite impulse response (FIR) system"

- $N \geq 1 \Rightarrow$ impulse response will have infinite duration

"infinite impulse response (IIR) system"

Linear, Constant-Coefficient Difference Equations
Example (#2.15, p. 123)

$$y[n] - \frac{1}{2} y[n-1] = x[n], \qquad x[n] = K\delta[n]$$

$$y[n] = x[n] + \frac{1}{2} y[n-1] \qquad \text{recursive}$$

causal $\Rightarrow$ $y[-1] = 0$

n	$y[n]$	
0	$\underbrace{x[0]}_{K\delta[n]} + \frac{1}{2} \cancelto{0}{y[-1]}$	$= K$
1	$x[1] + \frac{1}{2} y[0]$	$= 0 + \frac{K}{2} = \frac{K}{2}$
2	$x[2] + \frac{1}{2} y[1]$	$= 0 + \frac{K}{4} = \frac{K}{4}$
$\vdots$		

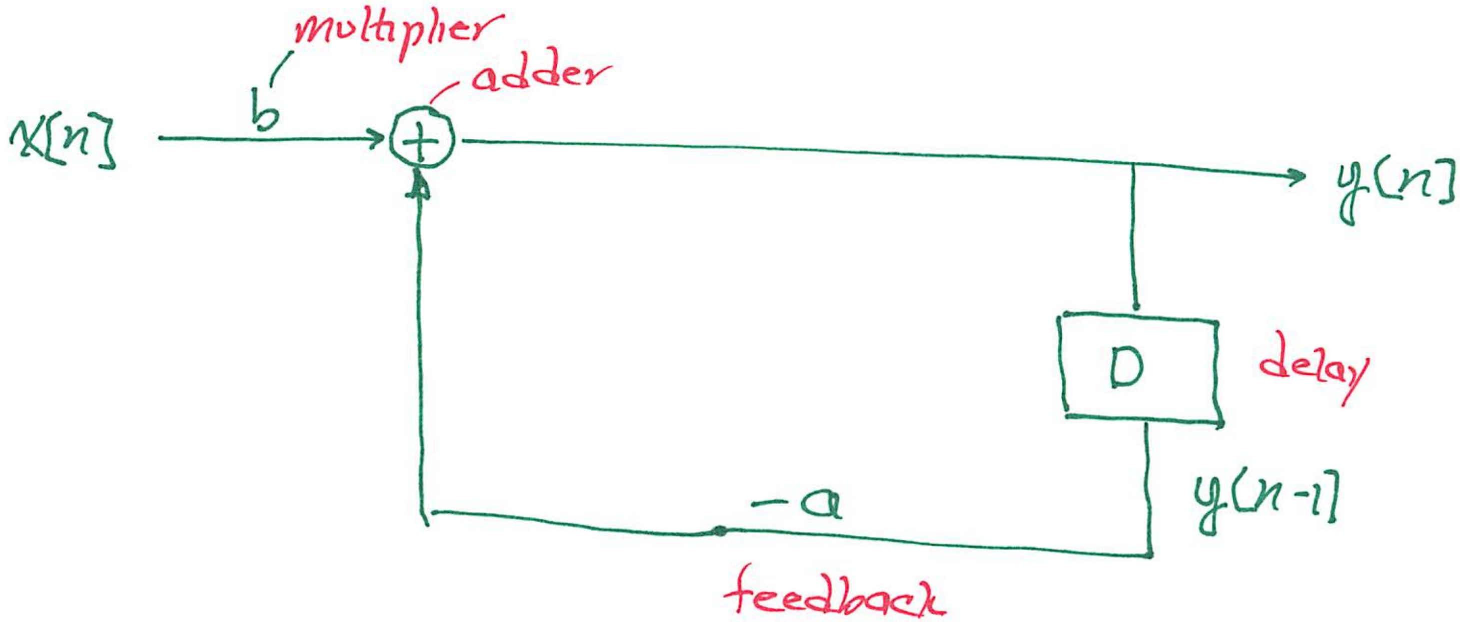
$$y[n] = \left(\frac{1}{2}\right)^n K \, u[n]$$

Block Diagrams for First-Order Systems

Discrete-Time System

$$y[n] + a y[n-1] = b x[n]$$

$$y[n] = b x[n] - a y[n-1]$$



Block Diagrams for First-Order Systems

Continuous-Time System

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$$

$$y(t) = \frac{1}{a}\left[b\,x(t) - \underbrace{\frac{dy(t)}{dt}}_{\text{problems.}}\right]$$

$$y(t) = \int_{-\infty}^{t} \left(b\,x(\tau) - a\,y(\tau)\right) d\tau$$

- adder
- multiplier
- integrator