Math 342

Homework#1 solutions

Sec. 6.2:

19)

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} c_1 + c_3 + c_4 &= 0 \\ -c_2 + c_3 + c_4 &= 0 \\ c_2 + c_3 + c_4 &= 0 \\ c_1 + c_3 - c_4 &= 0 \end{cases}$$

$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0 \Rightarrow$ linearly independent

Since there are four of them and $\dim M_{22} = 4$, it follows that they span M_{22} and thus form a basis.

20)

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = -2c_4$, $c_2 = -c_4$, $c_3 = c_4 \Rightarrow$ linearly dependent and thus do not form a basis.

22) Since $\dim \mathbb{P}_2 = 3$, if these vectors are linearly independent then they span and thus form a basis. To see if they are linearly independent, we want to solve

$$c_1 x + c_2(1+x) + c_3(x - x^2) = c_2 + (c_1 + c_2 + c_3)x - c_3 x^2 = 0 \quad \text{for all } x$$

which implies $c_2 = c_3 = 0$ and therefore $c_1 = 0$ as well. The three polynomials are linearly independent and form a basis for $\mathbb{P}_2$.

25) Since $\dim \mathbb{P}_2 = 3$ but there are four vectors, they must be linearly dependent, so cannot form a basis.

Sec. 6.4:

2) T is not a linear transformation because

$$\begin{aligned} T\left\{ \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} + \begin{pmatrix} w_2 & x_2 \\ y_2 & z_2 \end{pmatrix} \right\} &= T\left\{ \begin{pmatrix} w_1 + w_2 & x_1 + x_2 \\ y_1 + y_2 & z_1 + z_2 \end{pmatrix} \right\} \\ &= \left\{ \begin{matrix} 1 & w_1 + w_2 - z_1 - z_2 \\ x_1 + x_2 - y_1 - y_2 & 1 \end{matrix} \right\} \end{aligned}$$

while

$$\begin{aligned} T\left\{ \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} \right\} + T\left\{ \begin{pmatrix} w_2 & x_2 \\ y_2 & z_2 \end{pmatrix} \right\} &= \begin{pmatrix} 1 & w_1 - z_1 \\ x_1 - y_1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & w_2 - z_2 \\ x_2 - y_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & w_1 + w_2 - z_1 - z_2 \\ x_1 + x_2 - y_1 - y_2 & 2 \end{pmatrix} \end{aligned}$$

26) We have

$$(S\circ T)(3+2x-x^2) = S\big(T(3+2x-x^2)\big) = S(2-2x) = 2-4x^2$$

$$(S\circ T)(a+bx+cx^2) = S\big(T(a+bx+cx^2)\big) = S(b+2cx) = b+(b+2c)x+4cx^2$$

Since the domain of S is $\mathbb{P}_1$ which equals the codomain of T, we can compute $T\circ S$ as

$$(T\circ S)(a+bx) = T\big(S(a+bx)\big) = T(a+(a+b)x+2bx^2) = (a+b)+4bx$$

Additional problems:

1)

(a)

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ linearly independent

(b)

$$\begin{cases} c_1 + c_3 + c_4 &= 0 \\ c_2 + 2c_4 &= 0 \\ c_2 + c_3 + 3c_4 &= 0 \end{cases}$$

$\Rightarrow c_1 = 0,\ c_2 = -2c_4,\ c_3 = -c_4 \Rightarrow$ linearly dependent

(c)

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix} = 8 - 12 + 4 = 0$$

$\Rightarrow$ linearly dependent

(d)

$$\begin{vmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{vmatrix} = 0$$

$\Rightarrow$ linearly dependent

(e)

$$\begin{cases} c_1 &= 0 \\ c_1 + 2c_2 &= 0 \\ 3c_1 + c_2 &= 0 \end{cases}$$

$\Rightarrow c_1 = 0,\ c_2 = 0 \Rightarrow$ linearly independent.
Another possible answer: by inspection, the two vectors are not multiple of each other.

2)

(a)

$$c_1\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = 0,\ c_2 = 0 \Rightarrow$ linearly independent.
Another possible answer: by inspection, the two matrices are not multiple of each other.

(b)

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0 \Rightarrow$ linearly independent

(c)

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = -2c_3,\ c_2 = -3c_3 \Rightarrow c_1 = -2\alpha,\ c_2 = -3\alpha,\ c_3 = \alpha$ (α scalar) $\Rightarrow$ linearly dependent

3) If $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ are linearly dependent, then they must satisfy

$$c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_2\boldsymbol{y}_3 = \mathbf{0}$$

with c_1, c_2 and c_3 not all zero. However, we can rewrite this equation as

$$\begin{aligned} c_1(\boldsymbol{x}_1 + \boldsymbol{x}_2) + c_2(\boldsymbol{x}_2 + \boldsymbol{x}_3) + c_3(\boldsymbol{x}_3 + \boldsymbol{x}_1) &= \mathbf{0} \\ (c_1 + c_3)\boldsymbol{x}_1 + (c_1 + c_2)\boldsymbol{x}_2 + (c_2 + c_3)\boldsymbol{x}_3 &= \mathbf{0} \end{aligned}$$

Because $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent, this implies that

$$c_1 + c_3 = 0\ , \quad c_1 + c_2 = 0\ , \quad c_2 + c_3 = 0$$

leading to $c_1 = c_2 = c_3 = 0$ (all zero). Therefore, $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ are linearly independent

4)

(a)

$$\text{Wronskian} \quad W = \begin{vmatrix} \cos(\pi x) & \sin(\pi x) \\ -\pi\sin(\pi x) & \pi\cos(\pi x) \end{vmatrix} = \pi \neq 0 \quad \text{for all } x \in [0,1]$$

$\Rightarrow$ linearly independent

(b)

$$W = \begin{vmatrix} x^{3/2} & x^{5/2} \\ \frac{3}{2}x^{1/2} & \frac{5}{2}x^{3/2} \end{vmatrix} = x^3$$

For example, $W = 1 \neq 0$ for $x = 1 \Rightarrow$ linearly independent.
Another possible answer: the two functions $x^{3/2}$ and $x^{5/2}$ are not multiple of each other on $C[0,1]$.

(c)

$$W = \begin{vmatrix} 1 & e^x + e^{-x} & e^x - e^{-x} \\ 0 & e^x - e^{-x} & e^x + e^{-x} \\ 0 & e^x + e^{-x} & e^x - e^{-x} \end{vmatrix} = -4 \neq 0 \quad \text{for all } x \in [0,1]$$

$\Rightarrow$ linearly independent

(d)

$$W = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = -6e^{2x} \neq 0 \quad \text{for all } x \in [0,1]$$

$\Rightarrow$ linearly independent

5) For any finite set of vectors containing the zero vector, say $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \mathbf{0}\}$, we can always write

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots c_n\boldsymbol{v}_n + c_0\mathbf{0} = \mathbf{0}$$

with $c_1 = c_2 = \cdots = c_n = 0$ but $c_0 \neq 0$ (not all zero). Therefore the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \mathbf{0}\}$ is linearly dependent.

6)

- addition $\Rightarrow L(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{a} = (\boldsymbol{x} + \boldsymbol{a}) + \boldsymbol{y} \neq L(\boldsymbol{x}) + L(\boldsymbol{y})$

- scalar multiplication $\Rightarrow L(\alpha\boldsymbol{x}) = \alpha\boldsymbol{x} + \boldsymbol{a} \neq \alpha(\boldsymbol{x} + a) = \alpha L(\boldsymbol{x})$

$\Rightarrow L$ is not a linear transformation

7)

(a)

- addition $\Rightarrow L((x_1, x_2)^\top + (y_1, y_2)^\top) = (x_1 + y_1, x_2 + y_2, 1)^\top = (x_1, x_2, 0)^\top + (y_1, y_2, 1)^\top$
$\neq L((x_1, x_2)^\top) + L((y_1, y_2)^\top)$

- scalar multiplication $\Rightarrow L((\alpha x_1, \alpha x_2)^\top) = (\alpha x_1, \alpha x_2, 1)^\top \neq (\alpha x_1, \alpha x_2, \alpha)^\top = \alpha L((x_1, x_2)^\top)$

$\Rightarrow L$ is not a linear transformation

(b)

- addition $\Rightarrow L((x_1, x_2)^\top + (y_1, y_2)^\top) = (x_1 + y_1, x_2 + y_2, x_1 + y_1 + 2(x_2 + y_2))^\top$
$= (x_1, x_2, x_1 + 2x_2)^\top + (y_1, y_2, y_1 + 2y_2)^\top = L((x_1, x_2)^\top) + L((y_1, y_2)^t op)$

- scalar multiplication $\Rightarrow L((\alpha x_1, \alpha x_2)^\top) = (\alpha x_1, \alpha x_2, \alpha x_1 + 2\alpha x_2)^\top = \alpha L((x_1, x_2)^\top)$

$\Rightarrow L$ is a linear transformation

(c)

- addition $\Rightarrow L((x_1, x_2)^\top + (y_1, y_2)^\top) = (x_1 + y_1, 0, 0)^\top$
$= (x_1, 0, 0)^\top + (y_1, 0, 0)^\top = L((x_1, x_2)^\top) + L((y_1, y_2)^\top)$

- scalar multiplication $\Rightarrow L((\alpha x_1, \alpha x_2)^\top) = (\alpha x_1, 0, 0)^\top = \alpha L((x_1, x_2)^\top)$

$\Rightarrow L$ is a linear transformation

(d)

- addition $\Rightarrow L((x_1, x_2)^\top + (y_1, y_2)^\top) = (x_1 + y_1, x_2 + y_2, (x_1 + y_1)^2 + (x_2 + y_2)^2)^\top$
$\neq (x_1, x_2, x_1^2 + x_2^2)^\top + (y_1, y_2, y_1^2 + y_2^2)^\top = L((x_1, x_2)^\top) + L((y_1, y_2)^\top)$

- scalar multiplication $\Rightarrow L((\alpha x_1, \alpha x_2)^\top) = (\alpha x_1, \alpha x_2, \alpha^2 x_1^2 + \alpha^2 x_2^2)^\top = \alpha L((x_1, x_2)^\top)$

$\Rightarrow L$ is not a linear transformation

8)

(a)

- addition $L(p_1(x) + p_2(x)) = x^2 + p_1(x) + p_2(x) = (x^2 + p_1(x)) + p_2(x)$
$\neq x^2 + p_1(x) + x^2 + p_2(x) = L(p_1(x)) + L(p_2(x))$

- scalar multiplication $L(\alpha p_1(x)) = x^2 + \alpha p_1(x) \neq \alpha(x^2 + p_1(x)) = \alpha L(p_1(x))$

$\Rightarrow L$ is not a linear transformation

(b)

- addition $L(p_1(x) + p_2(x)) = p_1(x) + p_2(x) + x(p_1(x) + p_2(x)) + x^2(p_1'(x) + p_2'(x))$
$= (p_1(x) + xp_1(x) + x^2 p_1'(x)) + (p_2(x) + xp_2(x) + x^2 p_2'(x)) = L(p_1(x)) + L(p_2(x))$

- scalar multiplication $L(\alpha p_1(x)) = \alpha p_1(x) + x\alpha p_1(x) + x^2 \alpha p_1'(x) = \alpha L(p_1(x))$

$\Rightarrow L$ is a linear transformation