SOLUTION TO HOMEWORK #7

5.2

(a) Let $x[n] = \delta[n-1] + \delta[n+1]$. Then, using Eq. 5.9, the Fourier transform $X(e^{j\omega})$ of $x[n]$ is

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} (\delta[n-1] + \delta[n+1])e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \underbrace{\delta[n-1]}_{\text{only has value at } n=1} e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \underbrace{\delta[n+1]}_{\text{only has value at } n=-1} e^{-j\omega n} \\
&= e^{-j\omega} + e^{j\omega} \\
&= 2\cos(\omega)
\end{aligned}$$

(b) Let $x[n] = \delta[n+2] - \delta[n-2]$. Then, using Eq. 5.9, the Fourier transform $X(e^{j\omega})$ of $x[n]$ is

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} (\delta[n+2] - \delta[n-2])e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \delta[n+2]e^{-j\omega n} - \sum_{n=-\infty}^{\infty} \delta[n-2]e^{-j\omega n} \\
&= e^{2j\omega} - e^{-2j\omega} \\
&= 2j\sin(2\omega)
\end{aligned}$$

5.3

(a) The discrete-time periodic signal $x[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$ can be represented in terms of complex exponentials by Euler's relation as

$$x[n] = \frac{e^{j\frac{\pi}{3}n}e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{3}n}e^{-j\frac{\pi}{4}}}{2j} = \frac{e^{j\frac{\pi}{4}}}{2j}e^{j\frac{\pi}{3}n} - \frac{e^{-j\frac{\pi}{4}}}{2j}e^{-j\frac{\pi}{3}n}$$

From Eq. 5.18, we can immediately write

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \frac{\pi}{3} - 2\pi l)$$

So, in the range $-\pi \le \omega \le \pi$

$$X(e^{j\omega}) = \frac{\pi e^{j\frac{\pi}{4}}}{j}\delta(\omega - \frac{\pi}{3}) - \frac{\pi e^{-j\frac{\pi}{4}}}{j}\delta(\omega + \frac{\pi}{3})$$

But, $\frac{1}{j} = e^{-j\frac{\pi}{2}}$. Therefore

$$X(e^{j\omega}) = \pi e^{-j\frac{\pi}{4}}\delta(\omega - \frac{\pi}{3}) - \pi e^{-j\frac{3\pi}{4}}\delta(\omega + \frac{\pi}{3})$$

(b) Similarly, the discrete-time periodic signal $x[n] = 2+\cos(\frac{\pi}{6}n+\frac{\pi}{8})$ can be represented in terms of complex exponentials by Euler's relation as

$$x[n] = 2 + \frac{e^{j\frac{\pi}{6}n}e^{j\frac{\pi}{8}} + e^{-j\frac{\pi}{6}n}e^{-j\frac{\pi}{8}}}{2} = 2 + \frac{e^{j\frac{\pi}{8}}}{2}e^{j\frac{\pi}{6}n} + \frac{e^{-j\frac{\pi}{8}}}{2}e^{-j\frac{\pi}{6}n} \tag{1}$$

Therefore, again using Eq. 5.18 in the range $-\pi \leq \omega \leq \pi$

$$X(e^{j\omega}) = 4\pi\delta(\omega) + \pi e^{j\frac{\pi}{8}}\delta(\omega - \frac{\pi}{6}) + \pi e^{-j\frac{\pi}{8}}\delta(\omega + \frac{\pi}{6}) \tag{2}$$

5.21

(a) Let $x[n] = u[n-2] - u[n-6]$. This is a shifted rectangular pulse; it can be rewritten as the sum of impulses $x[n] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$. Then, using Eq. 5.9, the Fourier transform $X(e^{j\omega})$ of $x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \big(\delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]\big)e^{-j\omega n} \\ &= e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega} \end{aligned}$$

Alternatively, if we consider this as a finite geometric series,

$$\begin{aligned} X(e^{j\omega}) &= e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega} \\ &= \sum_{n=2}^{5} e^{-j\omega n} \\ &= e^{-2j\omega}\left(\frac{1-e^{-4j\omega}}{1-e^{-j\omega}}\right) \end{aligned}$$

To verify that

$$e^{-2j\omega}\left(\frac{1-e^{-4j\omega}}{1-e^{-j\omega}}\right) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}$$

we can use long division. Let $e^{-j\omega} = \beta^{-1}$, we have

$$\begin{aligned} e^{-2j\omega}\left(\frac{1-e^{-4j\omega}}{1-e^{-j\omega}}\right) &= \beta^{-2}\left(\frac{1-\beta^{-4}}{1-\beta^{-1}}\right) \\ &= \beta^{-2}\Big(1 + \beta^{-1} + \beta^{-2} + \beta^{-3}\Big) \\ &= \beta^{-2} + \beta^{-3} + \beta^{-4} + \beta^{-5} \end{aligned}$$

Given $e^{-j\omega} = \beta^{-1}$, we have $\beta^{-2}+\beta^{-3}+\beta^{-4}+\beta^{-5} = e^{-2j\omega}+e^{-3j\omega}+e^{-4j\omega}+e^{-5j\omega}$.

(b) Let $x[n] = (\frac{1}{2})^{-n}u[-n-1]$. Then, using Eq. 5.9, the Fourier transform $X(e^{j\omega})$ of $x[n]$ is

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{-n} u[-n-1]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}e^{j\omega}\right)^{-n}
\end{aligned}$$

Letting $m = -n$, the summation becomes a geometric series with $|\alpha| = |\frac{1}{2}e^{j\omega}| < 1$. Therefore we get

$$\begin{aligned}
\sum_{n=-\infty}^{-1} \left(\frac{1}{2}e^{j\omega}\right)^{-n} &= \sum_{m=1}^{\infty} \left(\frac{1}{2}e^{j\omega}\right)^{m} \\
&= \frac{\frac{1}{2}e^{j\omega}}{1 - \frac{1}{2}e^{j\omega}}
\end{aligned}$$

Another way to solve this problem is by using the properties. Let $x_1[n] = \left(\frac{1}{2}\right)^n u[n-1]$ and $x_2[n] = \left(\frac{1}{2}\right)^n u[n]$. Then, we have $x[n] = x_1[-n]$ and $x_1[n] = \frac{1}{2}x_2[n-1]$. We know that

$$x_2[n] \xrightarrow{DTFT} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

By using the time-shifting property,

$$x_1[n] \xrightarrow{DTFT} \frac{1}{2}e^{-j\omega}\frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

By using the time-reversal property,

$$x[n] \xrightarrow{DTFT} X_1(e^{-j\omega}) = \frac{1}{2}e^{j\omega}\frac{1}{1 - \frac{1}{2}e^{j\omega}}$$

(d) Let $x[n] = 2^n \sin(\frac{\pi}{4}n)u[-n]$. Then, using Eq. 5.9, the Fourier transform $X(e^{j\omega})$ of $x[n]$ is

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} 2^n \sin(\frac{\pi}{4}n)u[-n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{0} 2^n \sin(\frac{\pi}{4}n)e^{-j\omega n}
\end{aligned}$$

Using Euler's formula $\sin(\frac{\pi}{4}n) = \frac{1}{2j}(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n})$, we have

$$
\begin{aligned}
\sum_{n=-\infty}^{0} 2^n \sin(\frac{\pi}{4}n)e^{-j\omega n} &= \sum_{n=-\infty}^{0} 2^n \frac{1}{2j}(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n})e^{-j\omega n} \\
&= \frac{1}{2j}\sum_{n=-\infty}^{0} 2^n e^{j\frac{\pi}{4}n} e^{-j\omega n} - \frac{1}{2j}\sum_{n=-\infty}^{0} 2^n e^{-j\frac{\pi}{4}n} e^{-j\omega n} \\
&= \frac{1}{2j}\sum_{n=-\infty}^{0} \left(\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}\right)^{-n} - \frac{1}{2j}\sum_{n=-\infty}^{0} \left(\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}\right)^{-n}
\end{aligned}
$$

Letting $m = -n$, we get

$$
\begin{aligned}
&\frac{1}{2j}\sum_{n=-\infty}^{0} \left(\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}\right)^{-n} - \frac{1}{2j}\sum_{n=-\infty}^{0} \left(\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}\right)^{-n} \\
=&\frac{1}{2j}\sum_{m=0}^{\infty} \left(\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}\right)^{m} - \frac{1}{2j}\sum_{m=0}^{\infty} \left(\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}\right)^{m} \\
=&\frac{1}{2j}\left(\frac{1}{1-\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{1-\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}}\right)
\end{aligned}
$$

Another way to solve this problem is by using the properties. Let $x_1[n] = 2^{-n}\sin(-\frac{\pi}{4}n)u[n]$ and $x_2[n] = (\frac{1}{2})^n u[n]$. Then, we have $x[n] = x_1[-n]$ and $x_1[n] = \frac{1}{2j}(e^{-j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}n})x_2[n]$. We know that

$$
x_2[n] \xrightarrow{DTFT} \frac{1}{1-\frac{1}{2}e^{-j\omega}}
$$

By using the frequency-shifting property,

$$
\begin{aligned}
X_1(e^{j\omega}) &= \frac{1}{2j}[X_2(e^{j(\omega+\frac{\pi}{4})}) - X_2(e^{j(\omega-\frac{\pi}{4})})] \\
&= \frac{1}{2j}\left(\frac{1}{1-\frac{1}{2}e^{-j(\omega+\frac{\pi}{4})}} - \frac{1}{1-\frac{1}{2}e^{-j(\omega-\frac{\pi}{4})}}\right)
\end{aligned}
$$

By using the time-reversal property,

$$
\begin{aligned}
X(e^{j\omega}) &= X_1(e^{-j\omega}) \\
&= \frac{1}{2j}\left(\frac{1}{1-\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{1-\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}}\right)
\end{aligned}
$$

(g) The discrete-time signal $x[n] = \sin(\frac{\pi}{2}n) + \cos(n)$ can be represented in terms of complex exponentials by Euler's relation as

$$
x[n] = \frac{1}{2j}(e^{j\pi/2n} - e^{-j\pi/2n}) + \frac{1}{2}(e^{jn} + e^{-jn})
$$

Therefore, as we did in the problem 5.3, using Eq. 5.18 in the range $-\pi \leq \omega \leq \pi$, we have

$$\begin{aligned}
X(e^{j\omega}) &= \frac{\pi}{j}(\delta[\omega - \pi/2] - \delta[\omega + \pi/2]) + \pi(\delta[\omega - 1] - \delta[\omega + 1]) \\
&= \pi e^{-j\frac{\pi}{2}}(\delta[\omega - \pi/2] - \delta[\omega + \pi/2]) + \pi(\delta[\omega - 1] - \delta[\omega + 1])
\end{aligned}$$

5.22

(a) Let $X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} \leq |\omega| \leq \pi, 0 \leq |\omega| \leq \frac{\pi}{4} \end{cases}$. Then, using Eq. 5.8, the inverse Fourier transform $x[n]$ of $X(e^{j\omega})$ is

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\bigg|_{-3\pi/4}^{-\pi/4} + \frac{1}{jn}e^{j\omega n}\bigg|_{\pi/4}^{3\pi/4}\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{jn}\left(e^{-j\pi n/4} - e^{-j3\pi n/4}\right) + \frac{1}{jn}\left(e^{3j\pi n/4} - e^{j\pi n/4}\right)\right] \\
&= \frac{1}{\pi n}\left[\frac{\left(e^{j3\pi n/4} - e^{-j3\pi n/4}\right)}{2j} + \frac{\left(e^{-j\pi n/4} - e^{j\pi n/4}\right)}{2j}\right] \\
&= \frac{1}{\pi n}\left(\sin\frac{3\pi n}{4} - \sin\frac{\pi n}{4}\right)
\end{aligned}$$

(c) Let $X(e^{j\omega}) = e^{-j\omega/2}$ for $-\pi \leq \omega \leq \pi$. Then, using Eq. 5.8, the inverse Fourier transform $x[n]$ of $X(e^{j\omega})$ is

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{j\omega}{2}}e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\frac{1}{2})}d\omega \\
&= \frac{1}{2\pi}\frac{1}{j(n-1/2)}e^{j\omega(n-\frac{1}{2})}\bigg|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\frac{1}{j(n-1/2)}\left[e^{j\pi(n-1/2)} - e^{-j\pi(n-1/2)}\right] \\
&= \frac{\sin\pi(n-1/2)}{\pi(n-1/2)}
\end{aligned}$$

(e) This is the Fourier transform of a periodic signal with fundamental frequency $\frac{\pi}{2}$. Therefore, its fundamental period is four. Also, the Fourier series coefficients of this signal are $a_k = (-1)^k$. Therefore, the signal is given by

$$x[n] = \sum_{k=0}^{3}(-1)^k e^{jk(\pi/2)n} = 1 - e^{j\pi n/2} + e^{j\pi n} - e^{j3\pi n/2}$$

(g) Given the Fourier transform

$$\begin{aligned} X(e^{j\omega}) &= \frac{1 - \frac{1}{3}e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega}} \\ &= \frac{1 - \frac{1}{3}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega})}, \end{aligned}$$

let

$$\frac{1 - \frac{1}{3}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega})} = \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 + \frac{1}{4}e^{-j\omega}}$$

Then, multiplying by $(1 - \frac{1}{2}e^{-j\omega})(1 + \frac{1}{4}e^{-j\omega})$, we have

$$(A + B) + \left(\frac{1}{4}A - \frac{1}{2}B\right) e^{-j\omega} = 1 - \frac{1}{3}e^{-j\omega}$$

which gives

$$\begin{aligned} A + B &= 1 \\ \frac{1}{4}A - \frac{1}{2}B &= -\frac{1}{3} \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= \frac{2}{9} \\ B &= \frac{7}{9} \end{aligned}$$

We know $a^n u[n] \xrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$, so the time-domain signal is

$$x[n] = \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{9}\left(-\frac{1}{4}\right)^n u[n]$$

5.23

(b) Let $x_1[n] = x[n+2]$, then $x_1[n]$ is real and even, which indicates that $X_1(e^{j\omega})$ is also even and real. This implies that the phase of $X_1(e^{j\omega})$, $\sphericalangle X_1(e^{j\omega}) = 0$. Also, we know that $X_1(e^{j\omega}) = e^{2j\omega}X(e^{j\omega})$; therefore $\sphericalangle X(e^{j\omega}) = -2\omega$.

(c) $x[n] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$. This implies that $\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = 2\pi x[n]$. So, by setting $n = 0$, we obtain

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = 4\pi$$

(d) Using the figure and the definition of the Fourier transform, we have

$$
\begin{aligned}
X(e^{j\pi}) &= X(e^{j\omega})\Big|_{\omega=\pi} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\Bigg|_{\omega=\pi} \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} \\
&= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\
&= x[-3]\cdot(-1)^{-3} + x[-1]\cdot(-1)^{-1} + x[0]\cdot(-1)^{0} + x[1]\cdot(-1)^{1} \\
&\quad + x[3]\cdot(-1)^{3} + x[4]\cdot(-1)^{4} + x[5]\cdot(-1)^{5} + x[7]\cdot(-1)^{7} \\
&= -1\cdot(-1) + 1\cdot(-1) + 2\cdot 1 + 1\cdot(-1) + 1\cdot(-1) + 2\cdot(1) + 1\cdot(-1) - 1\cdot(-1) \\
&= 2
\end{aligned}
$$

(f) (i) From Parseval's theorem we have

$$\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 28\pi$$

(ii) Using the differentiation-in-frequency property of the DTFT, we obtain $nx[n] \xrightarrow{DTFT} j\frac{dX(e^{j\omega})}{d\omega}$. Therefore, we have

$$\int_{-\infty}^{\infty} |\frac{dX(e^{j\omega})}{d\omega}|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |nx[n]|^2 = 316\pi$$

5.33

(a) Taking the Fourier transform of the difference equation $y[n] + \frac{1}{2}y[n-1] = x[n]$, we obtain

$$Y(e^{j\omega}) + \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

(b) (ii) Let $x[n] = (-\frac{1}{2})^n u[n]$. Then, using Eq. 5.9, the Fourier transform $X(e^{j\omega})$ of

$x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}(-\frac{1}{2})^n u[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(-\frac{1}{2})^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(-\frac{1}{2}e^{-j\omega})^n \\ &= \frac{1}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Then, the Fourier transform of the output $y[n]$ is

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)^2} \end{aligned}$$

Given the Fourier transform pair (in Table 5.2)

$$(n+1)a^n u[n] \xrightarrow{DTFT} \frac{1}{\left(1-ae^{-j\omega}\right)^2}$$

we get

$$y[n] = (n+1)(-\frac{1}{2})^n u[n]$$

5.34

(a) Consider the two LTI systems, with frequency responses $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$. The frequency response of the cascade of these two systems is given by

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \left[\frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}}\right] = \frac{2-e^{-j\omega}}{1+\frac{1}{8}e^{-j3\omega}}$$

But, we have

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2-e^{-j\omega}}{1+\frac{1}{8}e^{-j2\omega}} \Rightarrow Y(e^{j\omega})(1+\frac{1}{8}e^{-j3\omega}) = X(e^{j\omega})(2-e^{-j\omega})$$

Therefore, by taking the inverse Fourier transform, we obtain he difference equation

$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1]$$

Additional Problems on Sampling

Problem #1

We know

$$\frac{\sin(Wt)}{\pi t} \longleftrightarrow X(j\omega) = \begin{cases} 1, |\omega| < W \\ 0, |\omega| > W \end{cases}$$

Therefore,

$$\left(\frac{\sin(Wt)}{\pi t}\right)^4 \longleftrightarrow X(j\omega) * X(j\omega) * X(j\omega) * X(j\omega)$$

Since the bandwidth of $\frac{\sin(Wt)}{\pi t}$ is $W = 100\pi$ Hz, the bandwidth of $\left(\frac{\sin(Wt)}{\pi t}\right)^4$ is $\omega_M = 400\pi$ Hz. Therefore, the Nyquist rate $\omega_s = 2\omega_M = 800\pi$Hz.

Problem #2

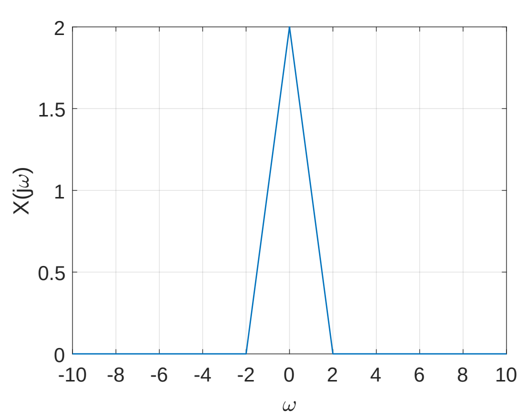
Consider the continuous-time signal $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$ with $W = 100\pi$ Hz. Let $X_1(j\omega)$ and $X_2(j\omega)$ be the Fourier transforms of the first and second terms of $x(t)$, respectively. Then $x(t)$ has Fourier transform

$$\begin{aligned} X(j\omega) &= X_1(j\omega)X_2(j\omega) \\ &= \frac{1}{6 + j\omega} \cdot \begin{pmatrix} 1, & |\omega| \leq W \\ 0, & \text{elsewhere} \end{pmatrix} \end{aligned}$$

Therefore, $X(jw)$ is bandlimited to W, and the Nyquist rate $\omega_s = 2W = 200\pi$ rads/sec. So, the minimum sampling frequency is $f_s = \frac{\omega_s}{2\pi} = 100$ Hz.

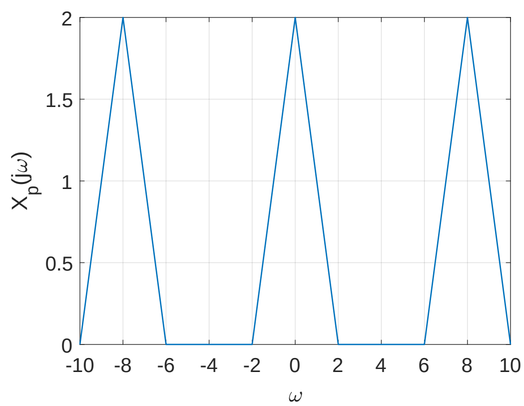
Problem #3

Consider the continuous-time signal $x(t)$ with the frequency characteristic shown below

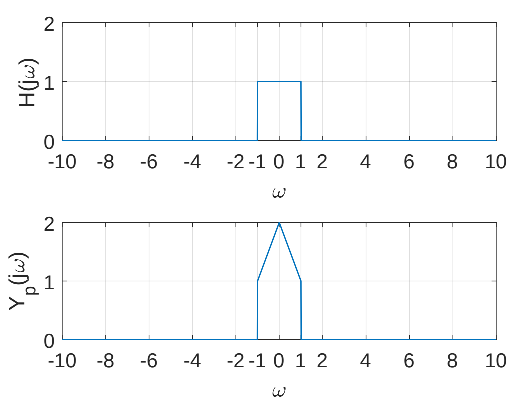


(a) The Nyquist rate is twice the bandwidth of $x(t)$ (i.e., the maximum frequency in $X(j\omega)$). So, by looking at the figure above, the Nyquist rate $\omega_s = 2\omega_{\rm M} = 4$ rads/sec.

(b) After $x(t)$ is sampled at twice the Nyquist rate using impulse-train sampling, the spectrum of the sampled signal $x_p(t)$ is periodic with period $W = 8$ rads/sec. The drawing is presented below.



(c) When $x_p(t)$ is passed through an ideal lowpass filter, with frequency response $H(j\omega)$, the spectrum of the output signal $y_p(t)$ can be written as $Y_p(j\omega) = H(j\omega)X_p(j\omega)$. The frequency characteristic of the ideal lowpass filter and the output signal is shown below.



As it can be seen from the figure above, ω_M for $Y(j\omega)$ is 1 rad/sec. Therefore, $\omega_s = 2\omega_M = 2$ rads/sec.