SOLUTIONS TO EXAM #3 (5/14/19)

#1. a.) $y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = 5x[n] - \frac{5}{6}x[n-1]$

$$\Downarrow \mathcal{F}$$

$$Y(e^{j\omega}) + \frac{1}{6}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{6}e^{-2j\omega}Y(e^{j\omega})$$
$$= 5X(e^{j\omega}) - \frac{5}{6}e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 + \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}\right] = X(e^{j\omega})\left[5 - \frac{5}{6}e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{5 - \frac{5}{6}e^{-j\omega}}{1 + \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}}$$

b.) $h[n] = \mathcal{F}^{-1}\{H(e^{j\omega})\}$

$$H(e^{j\omega}) = \frac{5 - \frac{5}{6}e^{-j\omega}}{1 + \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}}$$
$$= \frac{5 - \frac{5}{6}e^{-j\omega}}{\left(1 + \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{-j\omega}\right)}$$
$$= \frac{A}{1 + \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - \frac{1}{3}e^{-j\omega}}$$

Let $v = e^{-j\omega}$,

$$H(e^{j\omega}) = \frac{A}{1 + \frac{1}{2}v} + \frac{B}{1 - \frac{1}{3}v}$$

$$A = H(e^{j\omega})\left(1 + \frac{1}{2}v\right)\Big|_{v=-2} = \frac{5 - \frac{5}{6}v}{1 - \frac{1}{3}v}\Bigg|_{v=-2}$$
$$= \frac{5 + \frac{5}{3}}{1 + \frac{2}{3}} = \frac{\frac{20}{\cancel{3}}}{\frac{5}{\cancel{3}}} = 4$$

$$B = H(e^{j\omega})\left(1 - \frac{1}{3}v\right)\Big|_{v=3} = \frac{5 - \frac{5}{6}v}{1 + \frac{1}{2}v}\Bigg|_{v=3}$$
$$= \frac{5 - \frac{5}{2}}{1 + \frac{3}{2}} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1$$

#1b. cont'd)

$$\therefore H(e^{j\omega}) = \frac{4}{1+\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$\Big\Downarrow \mathcal{F}^{-1}$$

$$h[n] = 4\left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

#2. (a) $X(e^{j\omega}) = \mathcal{F}\{x[n]\} \longrightarrow$ can use properties or defining sum

We will use properties here:

Let $y[n] = \left(\frac{1}{2}\right)^n u[n] \Rightarrow Y(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

Then, let $r[n] = \left(\frac{1}{2}\right)^2 \underbrace{y[n-2]}_{\text{delay of 2 units results in phase shift in freq.}}$

$$\therefore R(e^{j\omega}) = \left(\frac{1}{4}\right) e^{-2j\omega} Y(e^{j\omega})$$

Finally, $x[n] = \underbrace{e^{j\frac{\pi}{4}n}}_{\text{phase shift in time results shift in frequency}} r[n]$

$$\therefore X(e^{j\omega}) = R(e^{j(\omega-\frac{\pi}{4})})$$

$$= \left(\frac{1}{4}\right) e^{-j2(\omega-\frac{\pi}{4})} Y(e^{j(\omega-\frac{\pi}{4})})$$

$$= \frac{1}{4} \underbrace{e^{j\frac{\pi}{2}}}_{j} e^{-2j\omega} \cdot \frac{1}{1-\frac{1}{2}e^{j\frac{\pi}{4}}e^{-j\omega}}$$

$$= \frac{j}{4} e^{-2j\omega} \cdot \frac{1}{1-\frac{1}{2}e^{j\frac{\pi}{4}}e^{-j\omega}}$$

#2. cont'd) b.) $x[n] = \left(\frac{3}{4}\right)^{|n|} \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

$$Y(e^{j\omega}) = \underbrace{\frac{d}{d\omega}}_{\text{multiplication by } n}\Big\{ \underbrace{e^{-4j\omega}}_{\substack{\text{corresponds to shift in time}\\ \text{of 4 units}}} \Big[\underbrace{X(e^{j(\omega+\frac{\pi}{4})})}_{\substack{\text{shift in freq}\\ \Downarrow \\ e^{-j\frac{\pi}{4}n}x[n]}} - \underbrace{X(e^{j(\omega-\frac{\pi}{4})})}_{\substack{\text{shift in freq}\\ \Downarrow \\ e^{j\frac{\pi}{4}n}x[n]}} \Big]\Big\}$$

- Let $R(e^{j\omega}) = X(e^{j(\omega+\frac{\pi}{4})}) - X(e^{j(\omega-\frac{\pi}{4})})$

$$\therefore\ r[n] = e^{-j\frac{\pi}{4}n}x[n] - e^{j\frac{\pi}{4}n}x[n]$$

$$= \left(e^{-j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}n}\right)x[n]$$

- Let $S(e^{j\omega}) = e^{-4j\omega}R(e^{j\omega})$

$$\therefore\ s[n] = r[n-4]$$

$$= \left(e^{-j\frac{\pi}{4}(n-4)} - e^{j\frac{\pi}{4}(n-4)}\right)x[n-4]$$

$$= \left(e^{-j\frac{\pi}{4}n}\underbrace{e^{j\pi}}_{-1} - e^{j\frac{\pi}{4}n}\underbrace{e^{-j\pi}}_{-1}\right)x[n-4]$$

$$= \left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)x[n-4]$$

- Finally, let

$$Y(e^{j\omega}) = \frac{d}{d\omega}S(e^{j\omega})$$

#2b cont'd)

$$\therefore y[n] = jn\,s[n]$$

$$= jn\underbrace{\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)}_{2j\sin\frac{\pi}{4}n} x[n-4]$$

$$y[n] = -2n\left(\sin\tfrac{\pi}{4}n\right)\left(\tfrac{3}{4}\right)^{|n-4|}$$

c) (i) $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}$$

$$\therefore X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}\Big|_{\omega=0} = \sum_{n=-\infty}^{\infty} x[n]$$

$$= 0$$

(ii) $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega$$

$$\therefore \int_{-\pi}^{\pi} X(e^{j\omega})\,e^{4j\omega}\,d\omega = 2\pi\,x[n]\Big|_{n=4}$$

$$= 2\pi(1)$$

$$= 2\pi$$

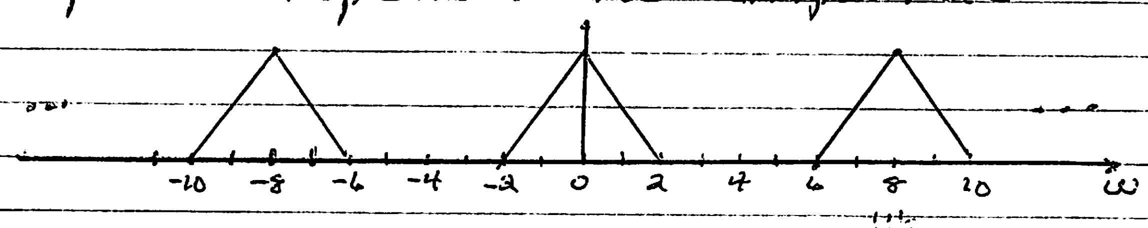
#3. a) Nyquist rate $= 2\omega_M$, where ω_M is the highest frequency in the signal

$\omega_M = 2$ rad/sec

$\therefore$ Nyquist rate $= 2\omega_M = 4$ rad/sec

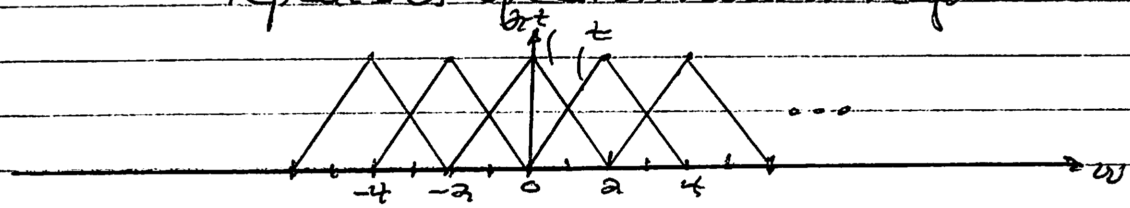
b.) Sample at twice the Nyquist rate $\Rightarrow \omega_s = 8$ rad/sec

Spectrum repeats at the sample rate

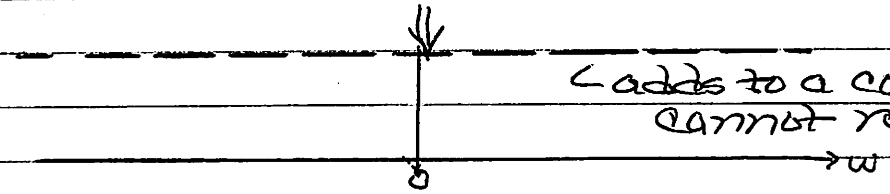


#3. cont'd) c.) Sample at half the Nyquist rate $\Rightarrow \omega_s = 2$ rad/sec

Spectrum repeats at the sample rate $\rightarrow$ repeats of spectrum will overlap



$\Downarrow$



← adds to a constant → cannot recover original signal

#4. a)

$$H(s) = \frac{5s+16}{s^2+6s+8} = \frac{Y(s)}{X(s)}$$

$$Y(s)(s^2+6s+8) = X(s)(5s+16)$$

$$s^2Y(s) + 6sY(s) + 8Y(s) = 5sX(s) + 16X(s)$$

$$\Downarrow \mathcal{L}^{-1}$$

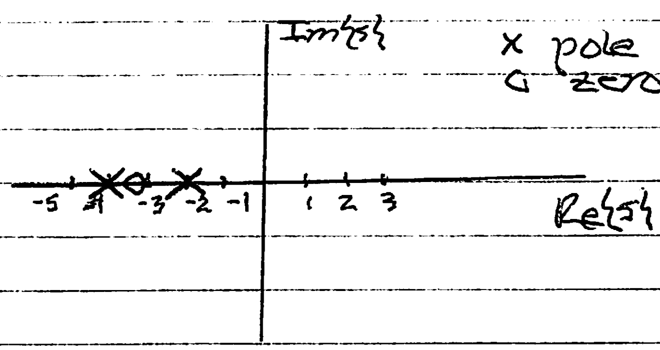
$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(s) = 5\frac{dx(t)}{dt} + 16x(t)$$

b.)

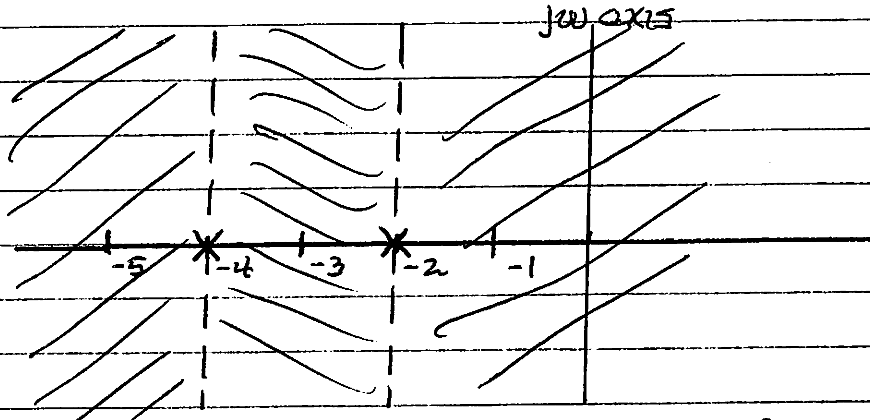
$$H(s) = \frac{5s+16}{(s+4)(s+2)}$$

poles: $H(s) \rightarrow \infty$ $\quad s = -2,\ s = -4$

zeros: $H(s) \rightarrow 0$ $\quad s = -16/5,\ s = \infty$



4.6 cont'd) c.)



left of leftmost pole $Re\{s\} < -4$
LS signal

right of rightmost pole
$Re\{s\} > -2$
RS signal

causal and stable

between the poles
$-4 < Re\{s\} < -2$
2S signal

d.) $h(t) = \mathcal{L}^{-1}\{H(s)\}$ ROC: $Re\{s\} > -2$
causal

$$H(s) = \frac{5s+16}{s^2+6s+8} = \frac{5s+16}{(s+4)(s+2)}$$

$$= \frac{A}{s+4} + \frac{B}{s+2}$$

$$A = H(s)(s+4)\Big|_{s=-4} = \frac{5s+16}{s+2}\Big|_{s=-4} = \frac{-4}{-2} = 2$$

$$B = H(s)(s+2)\Big|_{s=-2} = \frac{5s+16}{s+4}\Big|_{s=-2} = \frac{6}{2} = 3$$

$$H(s) = \frac{2}{s+4} + \frac{3}{s+2}, \quad Re\{s\} > -2$$

$$\Downarrow$$

$$h(t) = 2e^{-4t}u(t) + 3e^{-2t}u(t)$$

#5. a.)

$$x(t) = \underbrace{e^{t}\underbrace{\frac{d}{dt}\Big[\underbrace{e^{-2t}u(-t)}_{y_1(t)}\Big]}_{z_1(t)}}_{x_1(t)} * \underbrace{\frac{d^2}{dt^2}\Big[\underbrace{-e^{-t}u(-t)}_{y_2(t)}\Big]}_{x_2(t)}$$

$$X(s) = X_1(s)\,X_2(s)$$

$$X_1(s) = Z_1(s-1)$$

$$Z_1(s) = s\,Y_1(s)$$

$$Y_1(s) = -\frac{1}{s+2},\ Re\{s\} < -2$$

$$Y_2(s) = \frac{1}{s+1},\ Re\{s\} < -1$$

$$X_2(s) = s^2\,Y_2(s)$$

$$= \frac{s^2}{s+1},\ Re\{s\} < -1$$

$$X_1(s) = -\frac{s-1}{s-1+2} = -\frac{(s-1)}{s+1}$$

$Re\{s\} < -1$ (shifted)

$$\therefore\ X(s) = X_1(s)\,X_2(s) = \frac{(1-s)s^2}{(s+1)^2},\ Re\{s\} < -1$$

b.) There are three poles: $-1,\ 1+j,\ 1-j$

(i) Causal? Yes, this system can be causal. If the ROC is to the right of the rightmost pole, i.e., $Re\{s\} > 1$, the system is causal.

(ii) Stable? Yes, this system can be stable. If the ROC contains the $j\omega$-axis (i.e., $-1 < Re\{s\} < 1$), the system is stable.

(iii) Both causal and stable? No. For a system to be causal and stable, all the poles must be in the LHP.

5 cont'd) c.) $H(s) = \frac{2s+3}{s(s+5)}$

(i) $h(t)$ as $t \to 0^+$

Use Initial Value Theorem

$$h(0^+) = \lim_{s \to \infty} s\,X(s)$$

$$= \lim_{s \to \infty} \cancel{s}\,\frac{2s+3}{\cancel{s}(s+5)}$$

$$= \lim_{s \to \infty} \frac{2s+3}{s+5} = 2$$

(ii) $h(t)$ as $t \to \infty$

Use Final Value Theorem

$$\lim_{t \to \infty} h(t) = \lim_{s \to 0} s\,X(s)$$

$$= \lim_{s \to 0} \cancel{s}\,\frac{2s+3}{\cancel{s}(s+5)}$$

$$= \lim_{s \to 0} \frac{2s+3}{s+5} = \frac{3}{5}$$

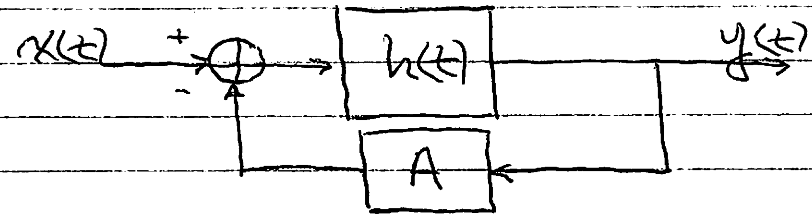
extra credit #1

$h(t) = e^{t}u(t)$ is not stable $\left(\int_{-\infty}^{\infty} |h(t)|\,dt \to \infty\right)$

$\Downarrow \mathcal{L}$

$H(s) = \frac{1}{s-1}$, $Re\{s\} > 1$ $\Rightarrow$ not stable, ROC does not contain $j\omega$-axis

So, we must move the pole to the LHP using feedback



#1 (cont'd)

From the feedback structure,

$$Y(s) = H(s)\left(X(s) - AY(s)\right)$$

$$Y(s)\left(1 + AH(s)\right) = H(s)X(s)$$

$$\therefore H_{feed}(s) = \frac{H(s)}{1 + AH(s)}$$

$$= \frac{1/(s-1)}{1 + A/(s-1)}$$

$$= \frac{1}{s + (A-1)}$$

To make this system stable, the pole must be moved to the left half plane

$$\Downarrow$$

$$A - 1 > 0$$

$$\therefore A > 1$$

extra credit #2

When an impulse is input to a second-order system, based on the value of the damping ratio, three reactions are possible. As described in class, the three possible responses are called *underdamped*, *critically damped* and *overdamped*. In an underdamped system, because the damping ratio is low, the response rises and oscillates around the steady state. The amplitude of the oscillation decreases over time to zero. In a critically damped system, the damping ratio is just right and the response reaches the steady state as fast as possible without oscillating. In an overdamped system, because the damping ratio is high, it takes a longer time for the impulse response to achieve the final steady state.