HW 2 Solution

2.1 $\Pr[HH \mid \text{at least one } H] = \dfrac{\Pr[HH]}{\Pr[HH] + \Pr[HT] + \Pr[TH]} = 1/3$

2.5 $\Pr[A] = \frac{1}{2}$ $\Pr[B] = \frac{2}{3}$ $\Pr[C] = \frac{2}{3}$ $\Pr[AB] = \frac{1}{3}$ $\Pr[BC] = \frac{1}{2}$ $\Pr[ABC] = \frac{1}{6}$

$$\Pr[ABC] = \Pr[A \mid BC]\, \Pr[B \mid C] \cdot \Pr[C] = \frac{\Pr[ABC]}{\Pr[BC]} \cdot \frac{\Pr[BC]}{\Pr[C]} \cdot \Pr[C]$$

$$= \frac{1/6}{1/2} \cdot \frac{1/2}{2/3} \cdot 2/3 = \frac{1}{6} = \Pr[ABC]$$

2.6

(a) $\Pr[H \mid S] = \dfrac{\Pr[HS]}{\Pr[S]} = \dfrac{20/250}{80/250} = \dfrac{20}{80} = \dfrac{1}{4}$

$\Pr[H \mid \bar{S}] = \dfrac{\Pr[H\bar{S}]}{\Pr[\bar{S}]} = \dfrac{80/250}{170/250} = \dfrac{8}{17}$

(b) $\Pr[S] = \dfrac{80}{250} = \dfrac{8}{25}$

$\Pr[H] = \dfrac{100}{250} = \dfrac{2}{5}$

$= \Pr[H \mid S]\, \Pr[S] + \Pr[H \mid \bar{S}]\, \Pr[\bar{S}]$

$= \dfrac{1}{4} \cdot \dfrac{8}{25} + \dfrac{8}{17} \cdot \dfrac{17}{25} = \dfrac{2}{5}$

(c) $\Pr[S \mid H] = \dfrac{\Pr[SH]}{\Pr[H]} = \dfrac{20/250}{100/250} = \dfrac{1}{5}$

(d) $\Pr[H \mid S] = \dfrac{\Pr[S \mid H]\, \Pr[H]}{\Pr[S]} = \dfrac{1/5 \cdot 2/5}{8/25} = \dfrac{1}{4}$

2.8 a. $Pr[AAB] = 0.9 \cdot 0.9 \cdot 0.8 = 0.65$

b. $Pr[AAB] + Pr[\bar{A}AB] + Pr[A\bar{A}B] + Pr[AA\bar{B}] = 0.954$

c $Pr[A \mid fail] = \dfrac{Pr[A \cdot fail]}{Pr[fail]} = \dfrac{2/3 \cdot 0.1}{2/3 \cdot 0.1 + 1/3 \cdot 0.2} = 0.5$

From B $1 - 0.5 = 0.5$

d $\dfrac{Pr[\bar{A}\bar{A}B]}{Pr[\bar{A}\bar{A}B] + Pr[\bar{A}A\bar{B}] + Pr[A\bar{A}\bar{B}]} = 0.182$

2.11 a. $Pr(A) = Pr(AB) + Pr(A\bar{B}) = 0.8$

$Pr(B) = Pr(\bar{A}B) + Pr(AB) = 0.6$

b. $Pr(A \mid B) = \dfrac{Pr(AB)}{Pr(B)} = 0.83$

$Pr(A \mid \bar{B}) = \dfrac{Pr(AB)}{Pr(\bar{B})} = 0.75$

$Pr(\bar{A} \mid B) = \dfrac{Pr(\bar{A}B)}{Pr(B)} = 0.1667$

$Pr(\bar{A} \mid \bar{B}) = \dfrac{Pr(\bar{A}\bar{B})}{Pr(\bar{B})} = 0.25$

c. $Pr(B \mid A) = \dfrac{Pr(A \mid B)\, Pr(B)}{Pr(A)} = 0.625$

2.15 Yes, should switch. For example, let "1" denote the prize. "$\downarrow$" denote first pick

if switch

$[\overset{\downarrow}{0}, 0, 1] \to [\overset{\downarrow}{0}, \ , 1]$

$[0, \overset{\downarrow}{0}, 1] \to [\ , \overset{\downarrow}{0}, 1]$

$[0, 0, \overset{\downarrow}{1}] \to [0, \ , \overset{\downarrow}{1}]$

$Pr(\text{get Prize if switch}) = \frac{2}{3}$

$Pr(\text{get Prize if not switch}) = \frac{1}{3}$

Solution to Problem 2.18

Denote the teams A, B, C, D and let "w. t." mean a team wins the tournament and "w1" and "w2" mean the team wins the first and second games, respectively. We note the probabilities of A and B winning the tournament do not depend on which team, C or D, wins the other game. However, the probability of C and D winning depends on whether A or B wins their game.

$$
\begin{aligned}
\Pr[\text{A w. t.}] &= \Pr[\text{A w2}|\text{A w1}]\Pr[\text{A w1}] = 0.7 \times 0.7 = 0.49 \\
\Pr[\text{B w. t.}] &= \Pr[\text{B w2}|\text{B w1}]\Pr[\text{B w1}] = 0.5 \times 0.3 = 0.15 \\
\Pr[\text{C w. t.}] &= \Pr[\text{C w2}|\text{C w1} \cap \text{A w1}]\Pr[\text{C w1} \cap \text{A w1}] \\
&\quad + \Pr[\text{C w2}|\text{C w1} \cap \text{B w1}]\Pr[\text{C w1} \cap \text{B w1}] \\
&= 0.3 \times 0.7 \times 0.5 + 0.5 \times 0.5 \times 0.3 = 0.18 \\
\Pr[\text{D w. t.}] &= \Pr[\text{D w2}|\text{D w1} \cap \text{A w1}]\Pr[\text{D w1} \cap \text{A w1}] \\
&\quad + \Pr[\text{D w2}|\text{D w1} \cap \text{B w1}]\Pr[\text{D w1} \cap \text{B w1}] \\
&= 0.3 \times 0.7 \times 0.5 + 0.5 \times 0.5 \times 0.3 = 0.18
\end{aligned}
$$

Check: $0.49 + 0.15 + 0.18 + 0.18 = 1.0$.

Solution to Problem 2.21

Let $T = 1$ denote a positive test and $U = 1$ denote that the person is positive for whatever is being tested. Then, for a true positive rate of 0.1,

$$
\begin{aligned}
\Pr[T=1] &= \Pr[T=1|U=1]\Pr[U=1] + \Pr[T=1|U=0]\Pr[U=0] \\
&= 0.9 \times 0.1 + 0.05 \times 0.9 = 0.135
\end{aligned}
$$

$$
\Pr[U=1|T=1] = \frac{\Pr[T=1|U=1]\Pr[U=1]}{\Pr[T=1]} = \frac{0.9 \times 0.1}{0.135} = \frac{0.09}{0.135} = 0.666
$$

For a true positive rate of 0.01,

$$
\begin{aligned}
\Pr[T=1] &= \Pr[T=1|U=1]\Pr[U=1] + \Pr[T=1|U=0]\Pr[U=0] \\
&= 0.9 \times 0.01 + 0.05 \times 0.99 = 0.0585
\end{aligned}
$$

$$
\Pr[U=1|T=1] = \frac{\Pr[T=1|U=1]\Pr[U=1]}{\Pr[T=1]} = \frac{0.9 \times 0.01}{0.0585} = \frac{0.009}{0.0585} = 0.153
$$

Solution to Problem 2.24

a) We can assume the one of the two error rates is $1 - 0.973 = 0.027$ and other is less that value or less. To be conservative, we assume both error rates equal 0.027. Approximately, 2.7% of all people will be misidentified.

b) With $n = 70{,}000$ spectators, one of whom is the "bad guy", approximately $0.027 \times 69{,}999 = 1890$ "non-bad guys" will be falsely identified. With high likelihood, the "bad guy" will also be identified. Therefore, of the 1890+1 = 1891 people identified as possible "bad guys", 1890 will be innocent.

If the "bad guy" is not present, 1890 will be falsely identified as possible "bad guys".

These numbers are all approximate due to the randomness of the test. As we see in Chapter 6, the typical number of matches is in the range from about 1890-86 = 1804 to 1890+86 = 1976. In other words, the expected number of false matches is about 1900, plus or minus about 100.

Facial recognition at the rate of 97.3% correct is almost useless for identifying a "bad guy" in a crowd of 70000 "good guys". Even a 99.9% correct test would still result in about 70 false matches.