ELEG 305
SOLUTIONS TO EXAM #3 (5/11/17)

#1. a.)

$$\frac{d^2 y(t)}{dt^2} - 9\,y(t) = x(t) + \frac{dx(t)}{dt}$$

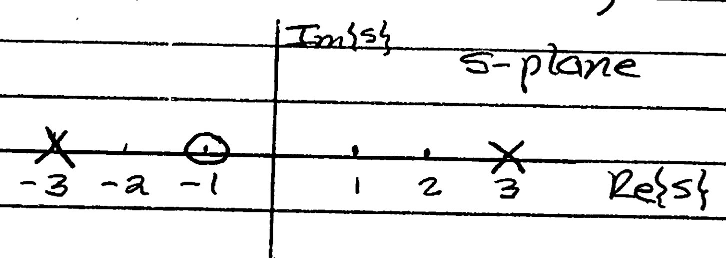
$$\Downarrow \mathcal{L}$$

$$s^2 Y(s) - 9\,Y(s) = X(s) + s\,X(s)$$
$$(s^2 - 9)\,Y(s) = X(s)\,(s+1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2-9} = \frac{s+1}{(s+3)(s-3)}$$

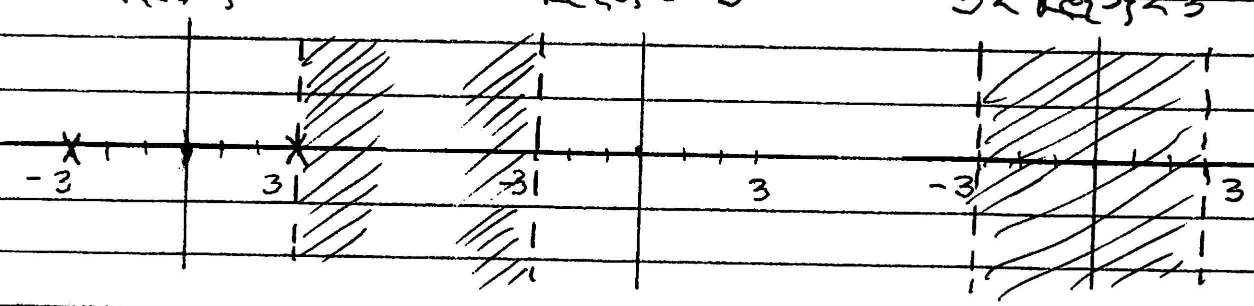
b.) poles : $H(s) \to \infty$ $\quad s = -3,\ s = 3$

zeros : $H(s) \to 0$ $\quad s = -1,\ s = \infty$



c.) H(s) is rational

$Re\{s\} > 3$ $\qquad Re\{s\} < -3$ $\qquad -3 < Re\{s\} < 3$



right of rightmost pole $\Downarrow$ right-sided signal

left of leftmost pole $\Downarrow$ left-sided signal

strip bounded by poles $\Downarrow$ two-sided signal

#4. cont'd) d.) $h(t) = ?$ (for stable system)

$$H(s) = \frac{s+1}{(s+3)(s-3)} \stackrel{PFE}{=} \frac{A}{s+3} + \frac{B}{s-3}$$

$$A = H(s)(s+3)\Big|_{s=-3} = \frac{s+1}{s-3}\Big|_{s=-3} = \frac{-2}{-6} = \frac{1}{3}$$

$$B = H(s)(s-3)\Big|_{s=3} = \frac{s+1}{s+3}\Big|_{s=3} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore H(s) = \frac{1/3}{s+3} + \frac{2/3}{s-3}$$

For a stable system, the ROC must contain the $j\omega$-axis (Fourier transform must exist). This is only true for $-3 < Re\{s\} < 3$, and corresponds to a two-sided signal, with

$$H(s) = \underbrace{\frac{1/3}{s+3}}_{\substack{\text{corresponds}\\ \text{to RS signal}\\ \text{with } Re\{s\} > -3}} + \underbrace{\frac{2/3}{s-3}}_{\substack{\text{corresponds}\\ \text{to LS signal}\\ \text{with } Re\{s\} < 3}}$$

$$\therefore h(t) = \frac{1}{3}e^{-3t}u(t) - \frac{2}{3}e^{3t}u(-t)$$

e.) A causal system is one for which $h(t) = 0$ for $t < 0$, i.e., a right-sided signal. So, $Re\{s\} > 3$.

#2. Discrete-time LTI system with impulse response

$$h[n] = \underbrace{\left(\tfrac{1}{4}\right)^n u[n]}_{\substack{\text{right sided}\\ \text{sequence}}} + \underbrace{\tfrac{4}{3}\left(\tfrac{1}{3}\right)^n u[-n-1]}_{\substack{\text{left sided}\\ \text{sequence}}}$$

a.) $H(z) = \mathcal{Z}\{h[n]\}$

$$= \mathcal{Z}\left\{\left(\tfrac{1}{4}\right)^n u[n]\right\} + \mathcal{Z}\left\{\tfrac{4}{3}\left(\tfrac{1}{3}\right)^n u[-n-1]\right\}$$

$$= \sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n z^{-n} + \tfrac{4}{3}\sum_{n=-\infty}^{-1}\left(\tfrac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\tfrac{1}{4}z^{-1}\right)^n + \tfrac{4}{3}\sum_{m=1}^{\infty}(3z)^m \qquad m=-n$$

$$= \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{4/3\,(3z)}{1-3z}$$

$$\left|\tfrac{1}{4}z^{-1}\right| < 1 \qquad |3z| < 1$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$|z| > \tfrac{1}{4} \;\cap\; |z| < \tfrac{1}{3}$$

$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{4}{3}\,\frac{3z}{1-3z}$$

divide numerator and denominator by $3z$

$$= \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{4}{3}\,\frac{1}{\frac{1}{3}z^{-1}-1}$$

$$= \frac{1}{1-\frac{1}{4}z^{-1}} - \frac{4}{3}\,\frac{1}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{1-\cancel{\frac{1}{3}z^{-1}}-\frac{4}{3}+\cancel{\frac{1}{3}z^{-1}}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

#2a. cont'd)

$$H(z) = \frac{-1/3}{1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2}}$$

ROC: $\frac{1}{4} < |z| < \frac{1}{3}$

b.)

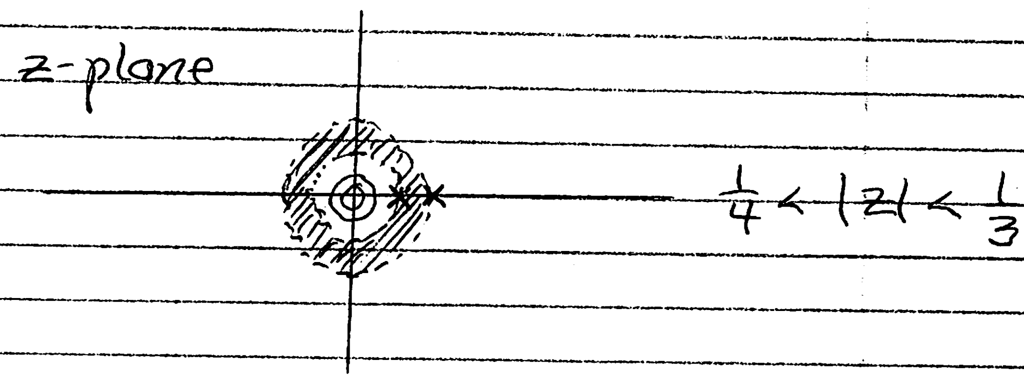
$$H(z) = \frac{-1/3}{1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2}}$$

$$= \frac{-\frac{1}{3} z^2}{z^2 - \frac{7}{12} z + \frac{1}{12}}$$

$$= \frac{-\frac{1}{3} z^2}{(z - \frac{1}{4})(z - \frac{1}{3})}$$

poles: $H(z) \to \infty \qquad z = \frac{1}{4},\ z = \frac{1}{3}$

zeros: $H(z) \to 0 \qquad z = 0,\ z = 0$



c.)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-1/3}{1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2}}$$

$$Y(z)\left(1 - \frac{7}{12} z^{-1} + \frac{1}{12} z^{-2}\right) = -\frac{1}{3} X(z)$$

$$\Downarrow \mathcal{Z}^{-1}$$

$$y[n] - \frac{7}{12} y[n-1] + \frac{1}{12} y[n-2] = -\frac{1}{3} x[n]$$

#2. cont'd) d.) The Fourier transform of $h[n]$ does not exist because the ROC does not contain the unit circle.

e.) The system is not causal because the ROC is not outside the outermost pole. We can also see this directly by looking at $h[n]$. For the system to be causal, $h[n]$ must be 0 for $n < 0$, which is not the case here.

#3. a) (i) $h(t) = e^{t}u(t)$

$$H(s) = \int_0^\infty e^{t} e^{-st}\,dt$$

$$= \int_0^\infty e^{-(s-1)t}\,dt$$

$$= \frac{1}{s-1}, \quad \mathrm{Re}\{s\} > 1$$

This system is not stable

- $\int_{-\infty}^{\infty} |h(t)|\,dt \to \infty$
- ROC does not contain $j\omega$-axis

(ii) From the feedback structure,

$$Y(s) = H(s)\left(X(s) - A\,Y(s)\right)$$

$$Y(s)\left(1 + A H(s)\right) = H(s)\,X(s)$$

$$\therefore H_{feed}(s) = \frac{H(s)}{1 + A H(s)}$$

$$= \frac{1/(s-1)}{1 + A/(s-1)}$$

$$= \frac{1}{s + (A-1)}$$

#3a. cont'd) To make this system stable, the pole must be moved to the left-half plane

$$\Downarrow$$

$$A-1 > 0$$

$$\therefore \quad A > 1$$

b.) $h(t) = e^{-3t} \sin 4t \, u(t)$

$$= e^{-3t} \frac{1}{2j}\left(e^{4jt} - e^{-4jt}\right) u(t)$$

$$= \frac{1}{2j}\left(e^{-3t+4jt} - e^{-3t} e^{-4jt}\right) u(t)$$

$$\Downarrow \mathcal{L}$$

$$H(s) = \frac{1}{2j}\left[\frac{1}{s+3-4j} + \frac{1}{s+3+4j}\right]$$

poles: $s = -3-4j$, $s = -3+4j$

ROC: $Re\{s\} > -3$

c.) $x(t) = e^{-t} \frac{d}{dt}\left(e^{-(t+1)} u(t+1)\right)$

Let $x_1(t) = e^{-t} u(t)$.
Then, $X_1(s) = \frac{1}{s+1} \quad Re\{s\} > -1$

Let $x_2(t) = e^{-(t+1)} u(t+1) = x_1(t+1)$
Then, $X_2(s) = e^{s} \cdot \frac{1}{s+1}$, $Re\{s\} > -1$

(property 9.5.2)

#3.c. cont'd) Let $x_3(t) = \frac{d}{dt} x_2(t)$. Then,

$$X_3(s) = sX_2(s) = e^{s} \frac{s}{s+1}, \; Re\{s\} > -1$$

(property 9.5.7)

Finally, $x(t) = e^{-t} x_3(t)$. Using property 9.5.3, with a shift $s_0 = 1$, we get

$$X(s) = X_3(s+1) = e^{s+1} \frac{s+1}{s+2}, \; Re\{s\} > -2$$

d.) Use the Initial and Final Value Theorems

(i) $h(t)$ at $t=0$

Initial Value Theorem $\quad h(0) = \lim_{s\to\infty} sH(s)$

$$= \lim_{s\to\infty} \frac{-3s^3+2s}{s^3+s^2+3s+2}$$

$$= \lim_{s\to\infty} \frac{-3+2/s^2}{1+1/s+3/s^2+2/s^3}$$

$$= -3$$

(ii) $h(t)$ as $t\to\infty$ $\quad \lim_{t\to\infty} h(t) = \lim_{s\to 0} sH(s)$

Final Value Theorem

$$= \lim_{s\to 0} \frac{-3s^3+2s}{s^3+s^2+3s+2}$$

$$= 0$$

#4. Discrete-time LTI system with

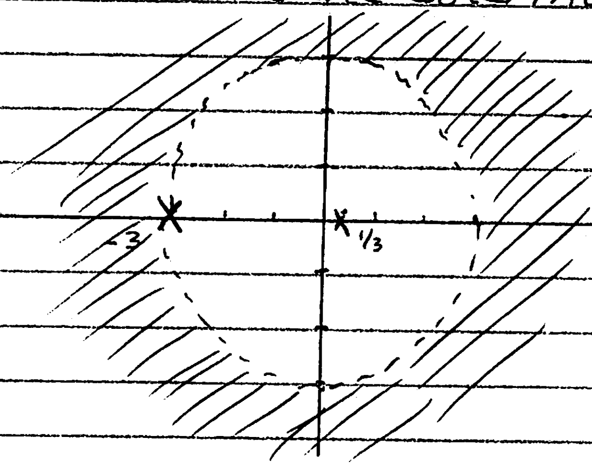
$$H(z) = \frac{z^2}{z^2 + \frac{8}{3}z - 1}$$

$$= \frac{1}{1 + \frac{8}{3}z^{-1} - z^{-2}}$$

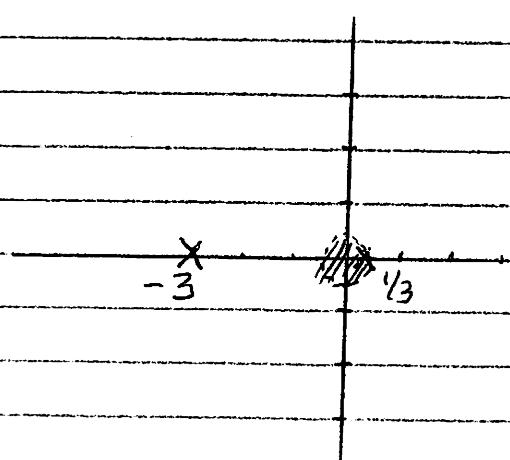
$$= \frac{1}{(1-\frac{1}{3}z^{-1})(1+3z^{-1})}$$

poles: $z = 1/3$, $z = -3$

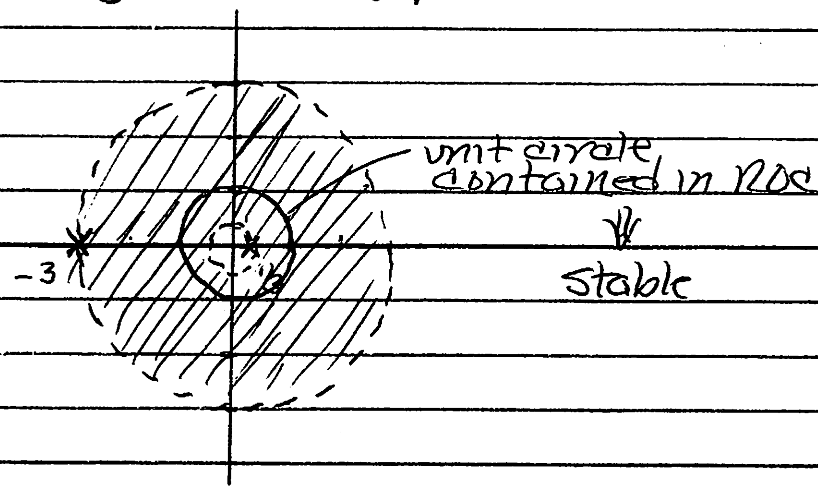
- ROC #1: outside the outermost pole $|z| > 3$



- ROC #2: inside the innermost pole $|z| < \frac{1}{3}$



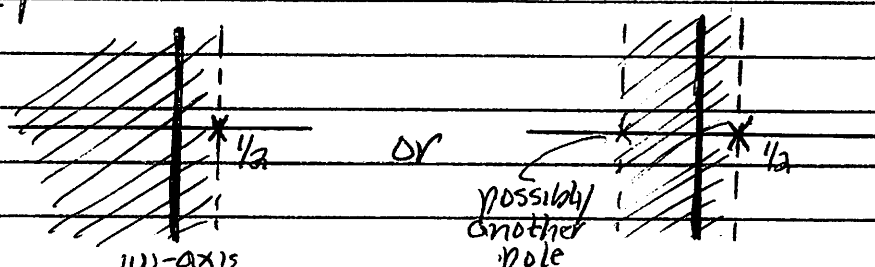
#4 cont'd) • ROC # 3: ring bounded by poles $\frac{1}{3} < |z| < 3$



Extra Credit • Fourier transform exists ⇒ ROC contains $j\omega$-axis

• $H(s)$ has a pole at $s = 1/2$

∴ ROC is either



~~left half plane~~ ~~strip in s plane~~

a.) left-sided signal? Yes. This is one possibility because the ROC must contain the $j\omega$-axis.

b.) right-sided signal? No. The ROC must be to the left of $s = 1/2$ to contain the $j\omega$-axis.

c.) two-sided signal? Yes. See the ROC on the right above.