ELEC 310

May 10, 2018

$X_1\ X_2 \cdots X_n$ IID unknown μ, σ^2

$$\hat{\mu} = \overline{X}_n = \frac{\sum_{i=1}^{n} X_i}{n} = \text{sample average}$$

$E\hat{\mu} = \mu \Rightarrow$ unbiased

$\operatorname{Var} \hat{\mu} = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$ consistency

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \qquad E\hat{\sigma}^2 = \sigma^2$$

Order statistics $X_1 \cdots X_n \to \boxed{\text{sort}}\ X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$

$X_{(1)}$: min

$X_{(N/2)}$: median

$X_{(n)}$: max

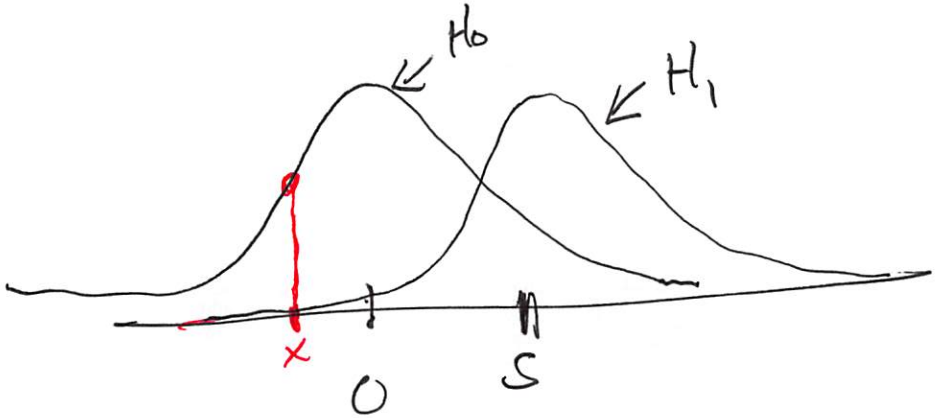
Hypothesis Testing

null? hypothesis H_0: $X \sim N(0, \sigma^2)$

alternative hypothesis H_1: $X \sim N(s, \sigma^2)$

s = strength of reflection

$s > 0$



$$\hat{H} = \begin{cases} 1 & \text{guess } H_0 \text{ is True} \\ 0 & \text{guess } H_1 \text{ is True} \end{cases}$$

Define:

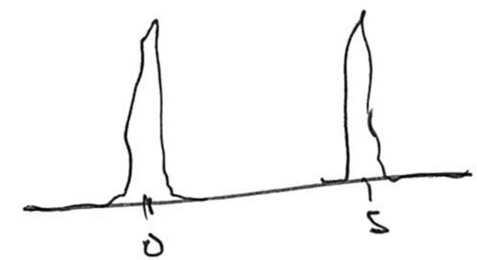
True Positive rate $\quad P(\hat{H}=1 \mid H_1 = \text{True})$

True Neg $\quad \hat{H}=0$ and H_0 is True

False Alarm – False Pos $\quad \hat{H}=1$ and H_0 is TRUE

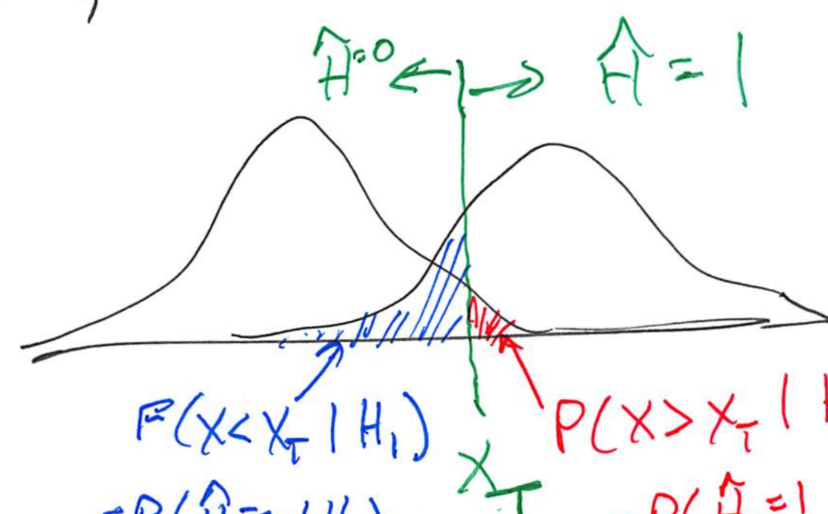
Miss – False Neg $\quad \hat{H}=0$ and H_1 is TRUE

$\sigma \ll S$



$$P(TP) + P(FN) = 1$$

$$P(FP) + P(TN) = 1$$



$P(X < X_T \mid H_1)$
$= P(\hat{H}=0 \mid H_1)$
$=$ miss prob

$P(X > X_T \mid H_0)$
$= P(\hat{H}_0 = 1 \mid H_0)$
$=$ false alarm prob

Easy: Communications

$$P(H_0) = P(H_1) = \frac{1}{2}$$

$\uparrow$ bit = 0 $\qquad$ $\uparrow$ bit = 1

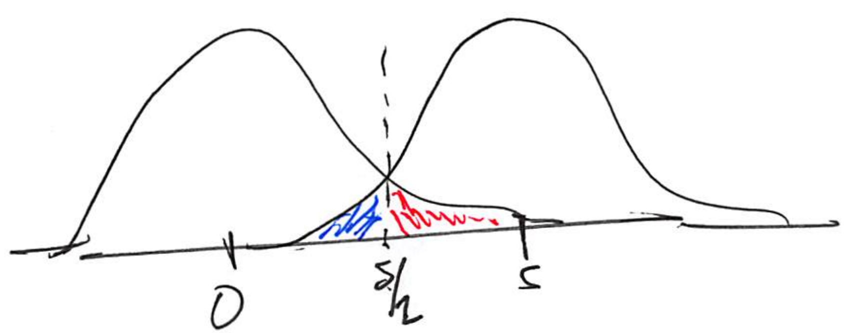
minimize $\quad P(FP) + P(FN)$

$$P(\text{error}) = P(\hat{H}=1 \mid H_0)P(H_0) + P(\hat{H}=0 \mid H_1)P(H_1)$$

min $P(\text{error})$

in this example $\quad H_0 : X \sim N(0, \sigma^2) \qquad X_T = \frac{S}{2}$

$\qquad H_1 : X \sim N(S, \sigma^2)$



Neyman - Pearson Test

Likelihood Ratio $L(X) = \dfrac{f_1(x)}{f_0(x)} \underset{H_0}{\overset{H_1}{\gtrless}} L_0$

if $L(x) > L_0$ decide $\hat{H} = 1$

$L(x) < L_0$ decide $\hat{H} = 0$

Ex. $$L(x) = \frac{\cancel{\frac{1}{\sqrt{2\pi}\sigma}}\, e^{-\frac{1}{2}\left(\frac{x-\delta}{\sigma}\right)^2}}{\cancel{\frac{1}{\sqrt{2\pi}\sigma}}\, e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}} = \frac{e^{-\frac{1}{2\sigma^2}(x^2-2\delta x+\delta^2)}}{e^{-\frac{1}{2}\sigma^2(x^2)}}$$

$$= e^{+\frac{\delta x}{\sigma^2}}\, e^{-\delta^2/2\sigma^2} \overset{?}{\gtrless} L_0$$

x_T gives equality

$$l(x) = \log(L(x)) \overset{?}{\gtrless} l_0 = \log(L_0)$$

$$+\frac{sx}{\sigma^2} - \frac{\delta^2}{2\sigma^2} \overset{?}{\gtrless} l_0$$

$$\cancel{x = l_0 + \frac{s^2}{2\sigma^2}}$$

$$+\frac{sx}{\sigma^2} \gtrless l_0 + \frac{s^2}{2\sigma^2}$$

$$x \gtrless \frac{\sigma^2}{s}\left(l_0 + \frac{s^2}{2\sigma^2}\right) = x_\tau$$

$$\boxed{\begin{array}{l} \text{if } x > x_\tau,\ \hat{H} = 1 \\ \text{else},\ \hat{H} = 0 \end{array}}$$

FP: $\hat{H} = 1 \mid H_0$ true

$$P(FP) = P(X > X_T \mid H_0)$$

$$= 1 - P(X < X_T \mid H_0)$$

$$= 1 - P\left(\frac{X}{\sigma} < \frac{X_T}{\sigma} \mid H_0\right)$$

$$= 1 - \Phi\left(\frac{X_T}{\sigma}\right)$$

solve for X_T

subs into

$$P(FN) = P(X < X_T \mid H_1)$$

$$= P\left(\frac{X - s}{\sigma} < \frac{X_T - s}{\sigma} \mid H_1\right)$$

$$= \Phi\left(\frac{X_T - s}{\sigma}\right)$$

$$P(TP) = 1 - \Phi\left(\Phi^{-1}\left(1 - Pr(FP)\right) - \frac{s}{\sigma}\right)$$

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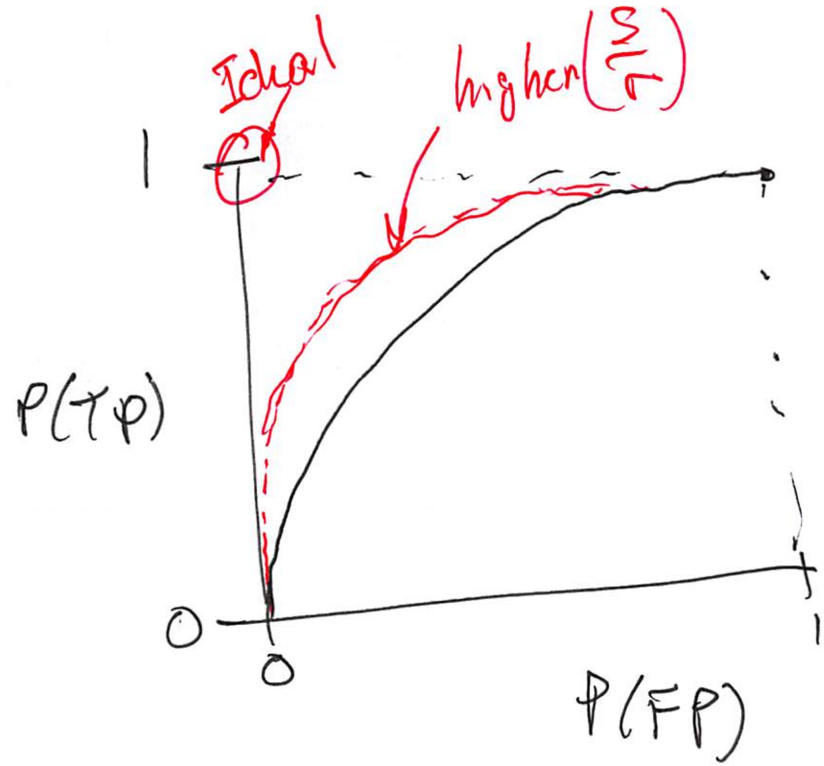

ROC curve

(Reciever Operating Characteristic)