Math 342

Homework#6 solutions

Sec. 4.3 (Z):

40)

$$\mathcal{L}\big\{(3t+1)\,U(t-1)\big\} = 3\mathcal{L}\big\{(t-1)U(t-1)\big\} + 4\mathcal{L}\big\{U(t-1)\big\} = \frac{3e^{-s}}{s^2} + \frac{4e^{-s}}{s}$$

42)

$$\mathcal{L}\big\{\sin(t)\,U(t-\pi/2)\big\} = \mathcal{L}\big\{\cos(t-\pi/2)\,U(t-\pi/2)\big\} = \frac{s\,e^{-\pi s/2}}{s^2+1}$$

60)

$$\mathcal{L}\big\{\sin(t) - \sin(t)\,U(t-2\pi)\big\} = \mathcal{L}\big\{\sin(t) - \sin(t-2\pi)\,U(t-2\pi)\big\} = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$$

68) Taking the Laplace transform

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 5\big[s\mathcal{L}\{y\} - y(0)\big] + 6\mathcal{L}\{y\} = \frac{e^{-s}}{s}$$

which implies

$$\begin{aligned}\mathcal{L}\{y\} &= e^{-s}\frac{1}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)} \\ &= e^{-s}\left[\frac{1}{6s} - \frac{1}{2(s-2)} + \frac{1}{3(s-3)}\right] - \frac{1}{s-2} + \frac{1}{s-3}\end{aligned}$$

so

$$y(t) = \left[\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right]U(t-1) - e^{2t} + e^{3t}$$

Additional problems:

1)

$$\begin{aligned}\sin t * \sin t &= \int_0^t \sin\tau\,\sin(t-\tau)\,d\tau = \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos t]\,d\tau \\ &= \frac{1}{2}\left[\frac{1}{2}\sin(2\tau - t) - \tau\cos t\right]_0^t = \frac{1}{2}(\sin t - t\cos t)\end{aligned}$$

2)

(a)

$$\text{If}\quad F(s) = \frac{1}{s^2(s^2+k^2)}$$

$$
\text{so} \quad f(t) = \frac{1}{k}\, t * \sin kt \;=\; \frac{1}{k}\int_0^t (t-\tau)\, \sin k\tau \, d\tau
$$

$$
= \frac{t}{k}\int_0^t \sin k\tau \, d\tau - \frac{1}{k}\int_0^t \tau\, \sin k\tau\, d\tau
$$

$$
= \frac{kt - \sin kt}{k^3}
$$

(b)

$$
\text{If} \quad F(s) = \frac{s}{(s-3)(s^2+1)}
$$

$$
\text{so} \quad f(t) = e^{3t} * \cos t \;=\; \int_0^t \cos\tau\, e^{3(t-\tau)}\, d\tau = e^{3t}\int_0^t e^{-3\tau}\, \cos\tau\, d\tau
$$

$$
= e^{3t}\left[\frac{e^{-3\tau}}{10}(-3\cos\tau + \sin\tau)\right]_0^t = \frac{1}{10}(3e^{3t} - 3\cos t + \sin t)
$$

3)

$$
\mathcal{L}\{e^{2t}\cos 3t\} = \frac{s-2}{s^2-4s+13}
$$

so

$$
\mathcal{L}\{t\, e^{2t}\cos 3t\} = -\frac{d}{ds}\left(\frac{s-2}{s^2-4s+13}\right) = \frac{s^2-4s-5}{(s^2-4s+13)^2}
$$

4)

$$
\text{If} \quad F(s) = \arctan\left(\frac{3}{s+2}\right)
$$

$$
\text{so} \quad f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{-\frac{3}{(s+2)^2+9}\} = \frac{e^{-2t}\, \sin 3t}{t}
$$

5)

(a) Taking the Laplace transform

$$
-\frac{d}{ds}\left[s^2 Y(s)\right] + 4\frac{d}{ds}\left[sY(s)\right] - sY(s) - 4\frac{d}{ds}Y(s) + 2Y(s) = 0
$$

$$
(s^2-4s+4)Y'(s) + (3s-6)Y(s) = 0
$$

$$
(s-2)Y'(s) + 3Y(s) = 0
$$

$$
Y(s) = \frac{C}{(s-2)^3}
$$

$$
\text{so} \quad y(t) = Ct^2 e^{2t} \quad \text{with} \quad C \neq 0
$$

(b) Taking the Laplace transform

$$
\begin{aligned}
&-\frac{d}{ds}\left[s^2 Y(s)\right] - 2\frac{d}{ds}\left[sY(s)\right] - 2sY(s) - 2Y(s) = 0 \\
&-(s^2+2s)Y'(s) - (4s+4)Y(s) = 0 \\
&Y(s) = \frac{C}{(s^2+2s)^2} = \frac{C}{s^2(s+2)^2} \\
&Y(s) = C\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right] \quad \text{(method of partial fractions)} \\
&\text{so} \quad y(t) = C\left(1 - t - e^{-2t} - t\,e^{-2t}\right) \quad \text{with} \quad C \neq 0
\end{aligned}
$$

or

$$
\begin{aligned}
&y(t) = C\,t * (te^{-2t}) = C\int_0^t (t-\tau)\,\tau\,e^{-2\tau}\,d\tau \quad \text{(convolution method)} \\
&y(t) = C\left(1 - t - e^{-2t} - t\,e^{-2t}\right) \quad \text{with} \quad C \neq 0
\end{aligned}
$$