Final, May 26, 2009

NAME:

1. You are playing a five card poker game and the first two cards dealt to you are $(A\diamond, K\diamond)$. Three more cards are dealt to you, giving you five in total.

a) What is the Pr[four Aces and one King]?

b) What is the Pr[any four of a kind]?

c) What is the probability you have a full house (three cards of one rank and two of another rank)?

d) Given you have a full house, what is the probability the full house is three Aces and two Kings?

~~2~~ cards have been dealt, so 50 cards remain

a) $P[4 \text{ Aces } \& \ 1K] = P[\text{next 3 cards are Aces}]$

$$= \frac{\binom{3}{3}}{\binom{50}{3}} = \frac{1}{\left(\frac{50 \cdot 49 \cdot 48}{3 \cdot 2 \cdot 1}\right)} = \frac{1}{400 \cdot 49} = \frac{1}{19600}$$

b) $P_r[\text{any 4 of a kind}] = P[\text{next 3 cards are Aces or 3K's}]$

$$= \frac{\binom{3}{3} + \binom{3}{3}}{\binom{50}{3}} = \frac{2}{19600} = \frac{1}{9800}$$

c) $P(\text{full house}) = P(3A\ 2K \text{ or } 2A\ 3K)$

$$= P(3A\ 2K) + P(2A\ 3K)$$

$$= \frac{\binom{3}{2}\binom{3}{1} + \binom{3}{1}\binom{3}{2}}{\binom{50}{3}} = \frac{18}{19600}$$

d) $$P(3A\ 2K \mid \text{full house}) = \frac{\frac{\binom{3}{2}\binom{3}{1}}{\binom{50}{3}}}{\frac{\binom{3}{2}\binom{3}{1} + \binom{3}{1}\binom{3}{2}}{\binom{50}{3}}} = \frac{1}{2}$$

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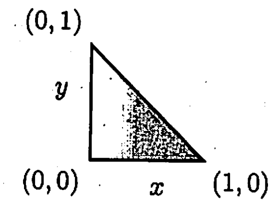
2. X and Y have density $f_{XY}(x,y) = cx^2$ in the triangle below and $f_{XY}(x,y) = 0$ elsewhere.



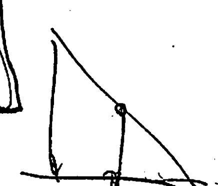
a) What is $f_X(x)$?

b) What is $f_Y(y)$?

c) What is $f_{X|Y}(x|Y=y)$? Show the integral of $f_{X|Y}(x|Y=y)$ over the appropriate range of x is 1.

d) Write one and two dimensional integrals for $E[X]$. (You don't have to do the integrals, though doing them is a good check.)

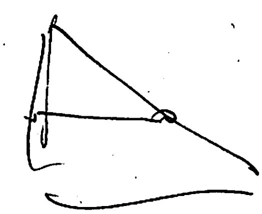
a) $f_X(x) = \int_0^{1-x} cx^2\,dy = \boxed{cx^2(1-x) \quad 0<x<1}$



$1 = c\int_0^1 x^2(1-x)\,dx = c\int_0^1 (x^2-x^3)\,dx = c\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{c}{12}$

$\Rightarrow \boxed{c=12}$

b) $f_Y(y) = \int_0^{1-y} 12x^2\,dx = \left.\frac{12x^3}{3}\right|_0^{1-y} = 4(1-y)^3$



$0<y<1$

c) $f_{X|Y}(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{12x^2}{4(1-y)^3} = \boxed{\frac{3x^2}{(1-y)^3} \quad 0<x<1-y}$

$1 \overset{?}{=} \int_0^{1-y} \frac{3x^2}{(1-y)^3}\,dx = \int_0^{1-y} \frac{1}{(1-y)^3} \left.\frac{3x^3}{3}\right|_{x=0}^{1-y} = \frac{3}{3}\frac{(1-y)^3}{(1-y)^3} = 1$

d) $E[X] = \int_0^1 x\,12x^2(1-x)\,dx = \int_0^1\int_0^{1-x} x\,12x^2\,dy\,dx$

3. Let S be binomial with parameters $n = 3$ and $p = 2/3$.

a) What are the PMF values $\Pr[S = k]$ for all values of k?

b) Sketch the distribution function of S.

$\binom{n}{k} p^k (1-p)^{n-k}$

a) $P[S=0] = \binom{3}{0} \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

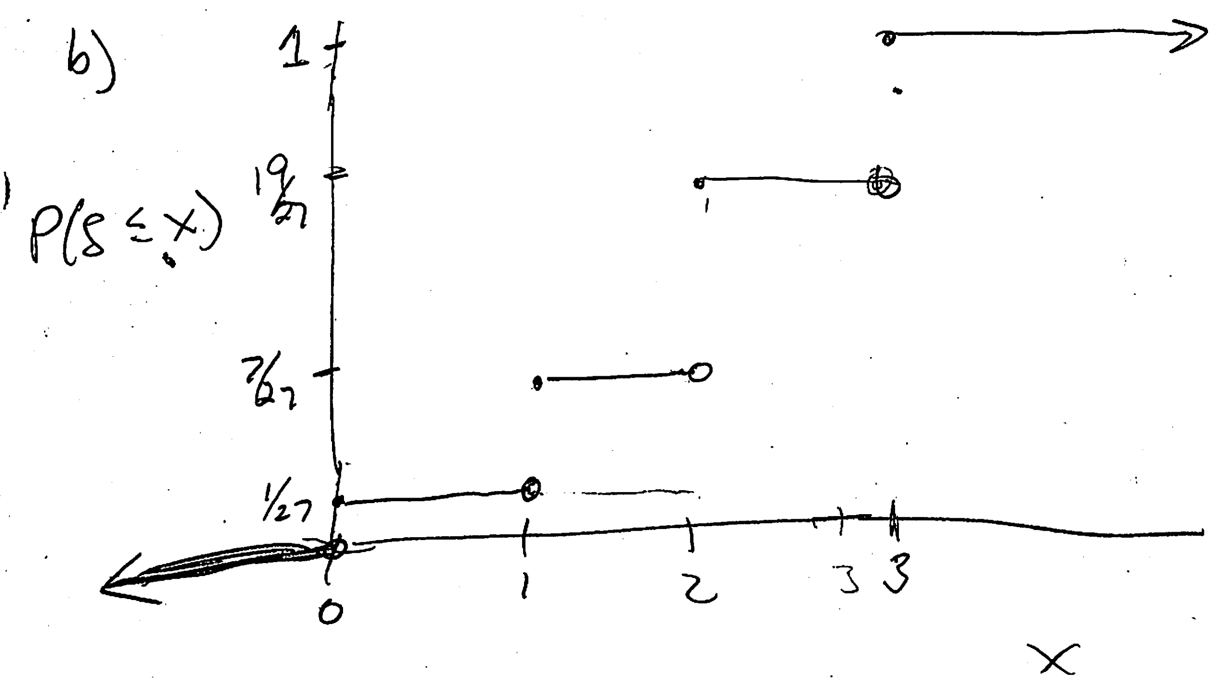
$P[S=1] = \binom{3}{1} \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^2 = \frac{6}{27}$

$P[S=2] = \binom{3}{2} \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) = \frac{12}{27}$

$P[S=3] = \binom{3}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right) = \frac{8}{27}$

Check $\frac{1+6+12+8}{27} = \frac{27}{27} = 1$

b)



4. Let X_1 and X_2 be independent rolls of an ordinary six-sided die (all sides equally likely) and $S = X_1 + X_2$.

a) What is the PMF of S, $\Pr[S = k]$? List the PMF for all values of k.

b) What is the conditional probability, $\Pr[X_1 = k | S = 4]$?

c) What is the conditional expected value, $E[X_1 | S = 4]$?

a) Convolve the P pmf's

$P[S=2] = \frac{1}{36}$

$P[S=3] = 2/36$

$P[S=4] = 3/36$

$P[S=5] = 4/36$

$P[S=6] = 5/36$

$P[S=7] = 6/36$

$P[S=8] = 5/36$

$P[S=9] = 4/36$

$P[S=10] = 3/36$

$P[S=11] = 2/36$

$P[S=12] = 1/36$

$P[S=k] = 0$ for $k < 2$ and $k > 12$

	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$					
$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$					
$\frac{1}{6}$		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$				
$\frac{1}{6}$			$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$			
$\frac{1}{6}$				$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		
$\frac{1}{6}$					$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	
$\frac{1}{6}$						$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
	S=2	S=3				S=7					S=1

b) $P[X_1 = k | S=4] = \frac{P[X_1 = k \cap X_1 + X_2 = 4]}{P[X_1 + X_2 = 4]} = \frac{P[X_1 = k \cap X_2 = 4-k]}{3/36}$

$= \frac{P[X_1 = k] P[X_2 = 4-k]}{3/36} = \begin{cases} \frac{\frac{1}{6}\cdot\frac{1}{6}}{3/36} = \frac{1}{3} & k = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$

c) $E[X_1 | S=4] = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 3 = \boxed{2}$

5.

Let $X \sim N(2,9)$, use the table on the right to find,

a) $\Pr[X \le 0]$.

b) $\Pr[-1 \le X \le 1]$.

c) $\Pr[X \ge 1]$.

d) $\Pr[2X+1 \le 3]$.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	1.00	0.8413	2.00	0.9772	3.00	0.9987
0.05	0.5199	1.05	0.8531	2.05	0.9798	3.05	0.9989
0.10	0.5398	1.10	0.8643	2.10	0.9821	3.10	0.9990
0.15	0.5596	1.15	0.8749	2.15	0.9842	3.15	0.9992
0.20	0.5793	1.20	0.8849	2.20	0.9861	3.20	0.9993
0.25	0.5987	1.25	0.8944	2.25	0.9878	3.25	0.9994
0.30	0.6179	1.30	0.9032	2.30	0.9893	3.30	0.9995
0.35	0.6368	1.35	0.9115	2.35	0.9906	3.35	0.9996
0.40	0.6554	1.40	0.9192	2.40	0.9918	3.40	0.9997
0.45	0.6736	1.45	0.9265	2.45	0.9929	3.45	0.9997
0.50	0.6915	1.50	0.9332	2.50	0.9938	3.50	0.9998
0.55	0.7088	1.55	0.9394	2.55	0.9946	3.55	0.9998
0.60	0.7257	1.60	0.9452	2.60	0.9953	3.60	0.9998
0.65	0.7422	1.65	0.9505	2.65	0.9960	3.65	0.9999
0.70	0.7580	1.70	0.9554	2.70	0.9965	3.70	0.9999
0.75	0.7734	1.75	0.9599	2.75	0.9970	3.75	0.9999
0.80	0.7881	1.80	0.9641	2.80	0.9974	3.80	0.9999
0.85	0.8023	1.85	0.9678	2.85	0.9978	3.85	0.9999
0.90	0.8159	1.90	0.9713	2.90	0.9981	3.90	1.0000
0.95	0.8289	1.95	0.9744	2.95	0.9984	3.95	1.0000

$9 = 3^2$

a) $P[X \le 0] = P\left(\frac{X-2}{3} \le \frac{0-2}{3}\right)$

$= P(Z \le -\frac{2}{3})$

$= 1 - P(Z \le \frac{2}{3})$

$= 1 - 0.75 = \boxed{0.25}$

b) $P[-1 \le X \le 1] = P\left[\frac{-1-2}{3} \le \frac{X-2}{3} \le \frac{1-2}{3}\right] = P\left[-1 \le Z \le -\frac{1}{3}\right]$

$= P\left[\frac{1}{3} \le Z \le 1\right] = 0.84 - 0.63 = \boxed{0.21}$

c) $P[X \ge 1] = P\left[\frac{X-2}{3} \ge \frac{1-2}{3}\right] = P\left[Z \ge -\frac{1}{3}\right]$

$= P\left[Z \le \frac{1}{3}\right] = \boxed{0.63}$

d) $P[2X+1 \le 3] = P\left[X \le \frac{3-1}{2} = 1\right] = 1 - P[X \ge 1]$

$= 1 - 0.63 = \boxed{0.37}$

6. In the cancer cluster problem, we computed a statistic,

$$S = \sum_{i=1}^{n} \frac{(X_i - \lambda_i)^2}{\lambda_i}$$

where the X_i are IID Poisson random variables each with mean λ_i. We said, if the λ_i are reasonably large, then S is approximately Chi-square. Be as specific as possible with your answers:

a) What are the mean and variance of $Y_i = \dfrac{X_i - \lambda_i}{\sqrt{\lambda_i}}$?

b) If λ_i is reasonably large, what can we say about the distribution of Y_i? Why?

c) Why is S approximately Chi-square?

d) Why can we approximate the Chi-square probability with a Gaussian probability? How would you approximate a Chi-square probability with a Gaussian probability?

a) $E[X_i] = \lambda_i \quad Var\, X_i = \lambda_i$ since $X_i \sim \text{Poisson}(\lambda_i)$

$$E[Y_i] = \frac{E[X_i] - \lambda_i}{\sqrt{\lambda_i}} = \frac{\lambda_i - \lambda_i}{\sqrt{\lambda_i}} = \boxed{0}$$

$$Var[Y_i] = \left(\frac{1}{\sqrt{\lambda_i}}\right)^2 Var[X_i] = \frac{\lambda_i}{\lambda_i} = \boxed{1}$$

b) The CLT says $Y_i \approx N(0,1)$

c) $S = \sum_{i=1}^{n} Y_i^2 \approx$ ~~Sum of~~ (Square Gaussians^2) = Chi-Square

d) The CLT also says $S \sim N(ES, Var S)$

$(ES = n \quad Var\, S = 2n)$

$$P(S \le x) \doteq P\left(\frac{S-n}{\sqrt{n}} \le \frac{x-n}{\sqrt{2n}}\right) = \Phi\left(\frac{x-n}{\sqrt{2n}}\right)$$

7. (Extra credit: credit only for correct work) Let X be Poisson with parameter λ. What is the probability X is even, i.e., $\Pr[X = k]$ for $k = 0, 2, 4, \ldots$? Give a closed form expression.

$$P[X = \text{even}] = P[X=0] + P[X=2] + P[X=4] + \cdots$$

$$= e^{-\lambda} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{\lambda^4}{4!}e^{-\lambda} + \cdots$$

$$= e^{-\lambda}\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right)$$

look at Taylor series for e^{λ} and $e^{-\lambda}$

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots$$

$$+ \left(e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots\right)$$

$$e^{\lambda} + e^{-\lambda} = 2\left(1 + 0 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right)$$

$$\Rightarrow P[X = \text{even}] = \frac{e^{-\lambda}(e^{\lambda} + e^{-\lambda})}{2} = \frac{1}{2}\left(1 + e^{-2\lambda}\right)$$

Check $\lambda = 0$ $\quad P[X = \text{even}] = P[X=0] = e^{-0} = 1$

$\lambda = +\infty$ $\quad P[X = \text{even}] = \frac{1}{2}(1+0) = \frac{1}{2}$ for large numbers of counts, even is as likely as odd.