Chapter 2 Capacitor circuits

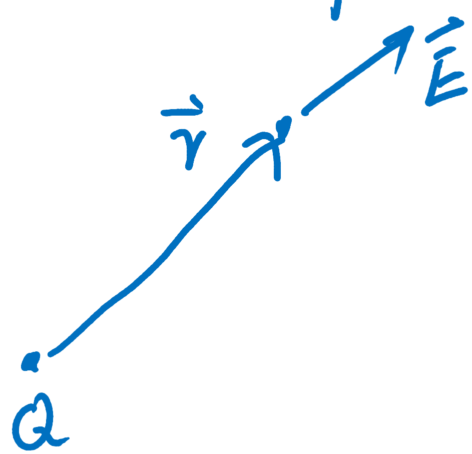
* Electric field and potentials.

Electric force: acting at a distance

An electric field extends outward from every charge and permeates all of space.

Q : a point charge.

Electric field: $\vec{E} = k\dfrac{Q}{r^2}\hat{r} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{r}$

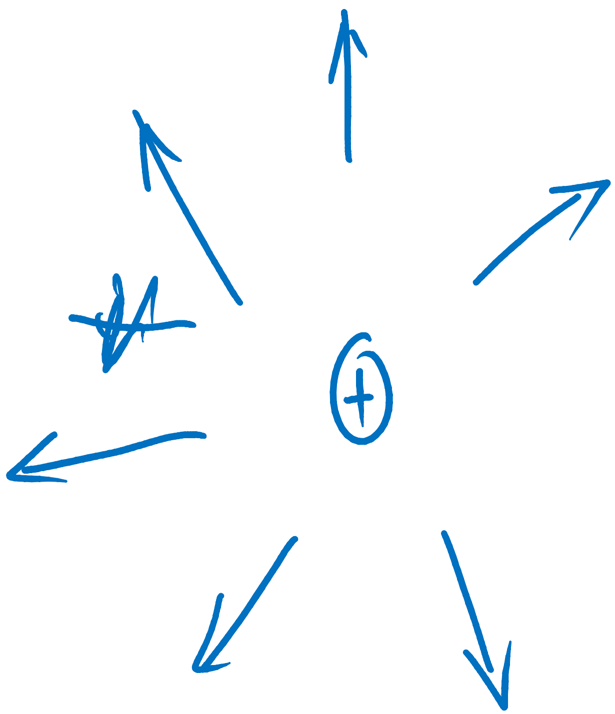


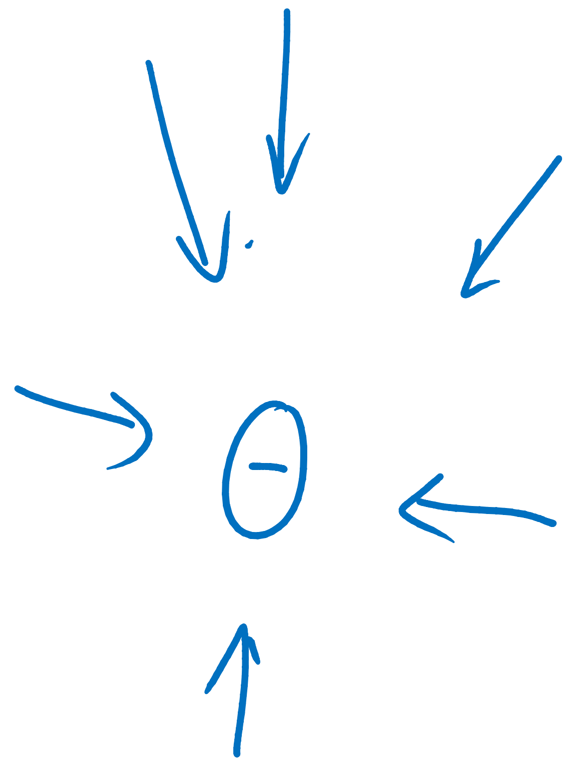
$\hat{r}$: unit vector along $\vec{r}$

$$k = 8.99 \times 10^{9}\ N\ m^2/C^2$$

$$\varepsilon_0 = \frac{1}{4\pi k} = 8.85 \times 10^{-12}\ C^2/N.m^2$$

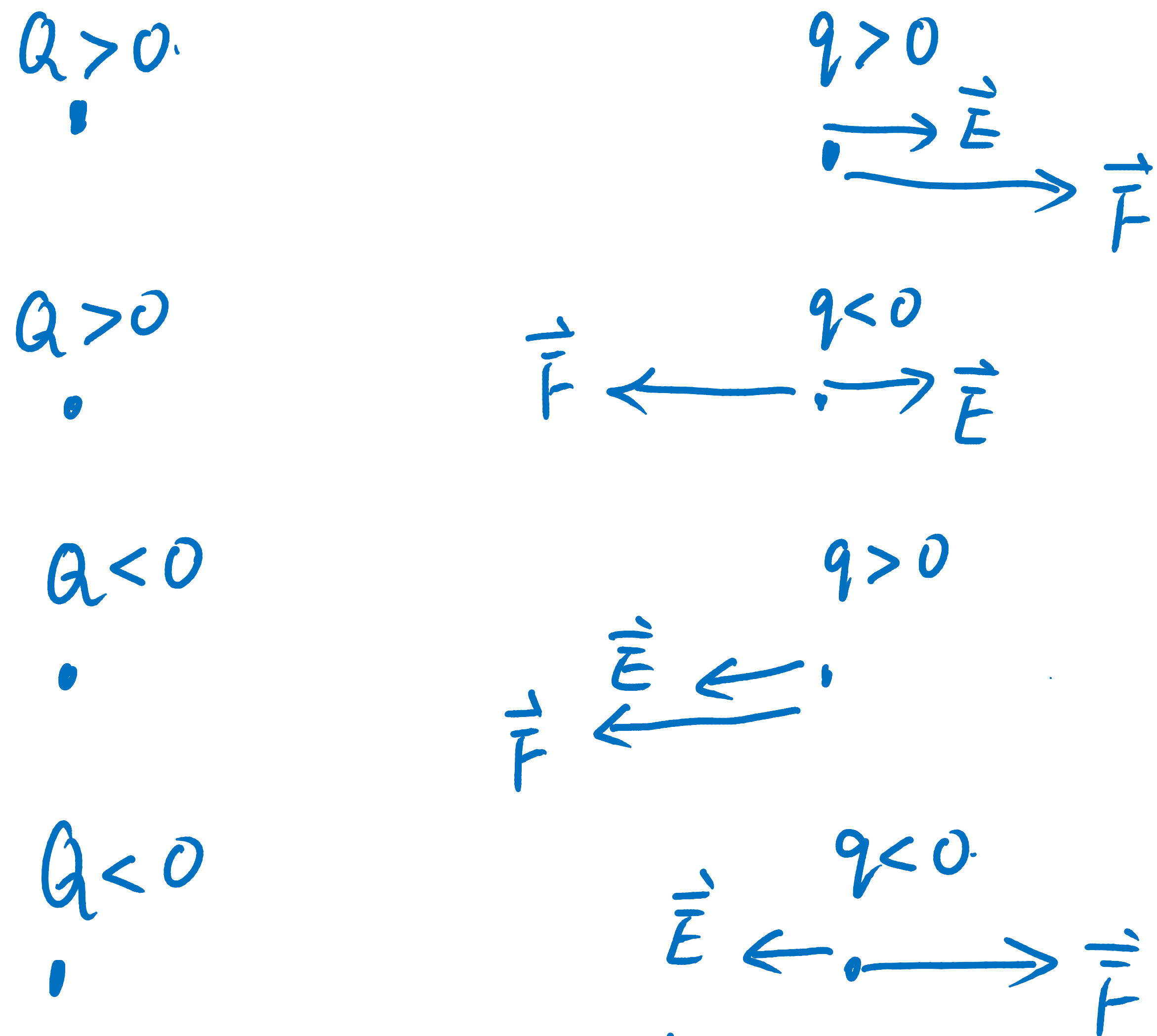
$Q > 0$ $\qquad\qquad$ $Q < 0$





$\vec{E}$ outward $\qquad\qquad$ $\vec{E}$ inward.

Electric force $\vec{F} = q\vec{E}$



Like charges repel; Unlike charges attract.

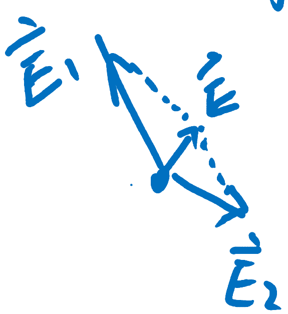
Superposition principle.

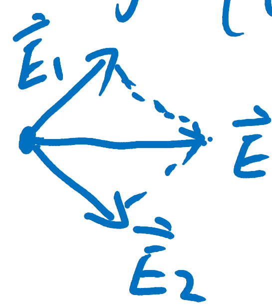
$\vec{E}$ at a given point is due to more than one charge: $Q_1, Q_2, Q_3, \dots$

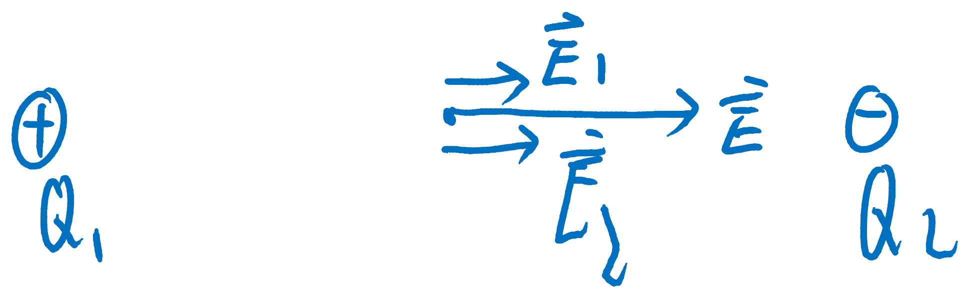
* $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots$

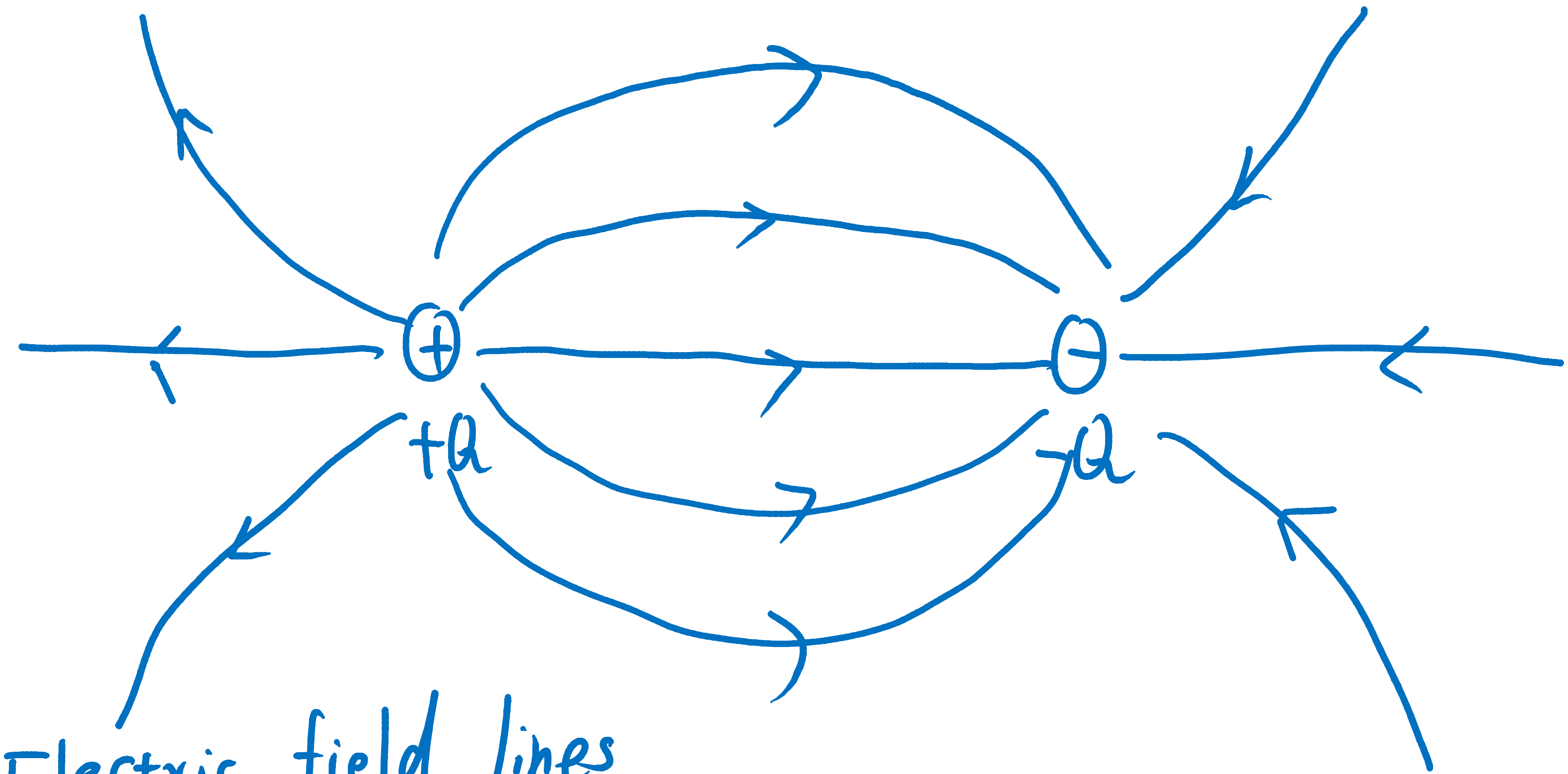
where $\vec{E}_1$ is due to Q_1, $\vec{E}_2$ to Q_2, ...

$\vec{E}$ from two opposite charges ($Q_1 > 0$, $Q_2 < 0$, $|Q_1| = |Q_2|$)



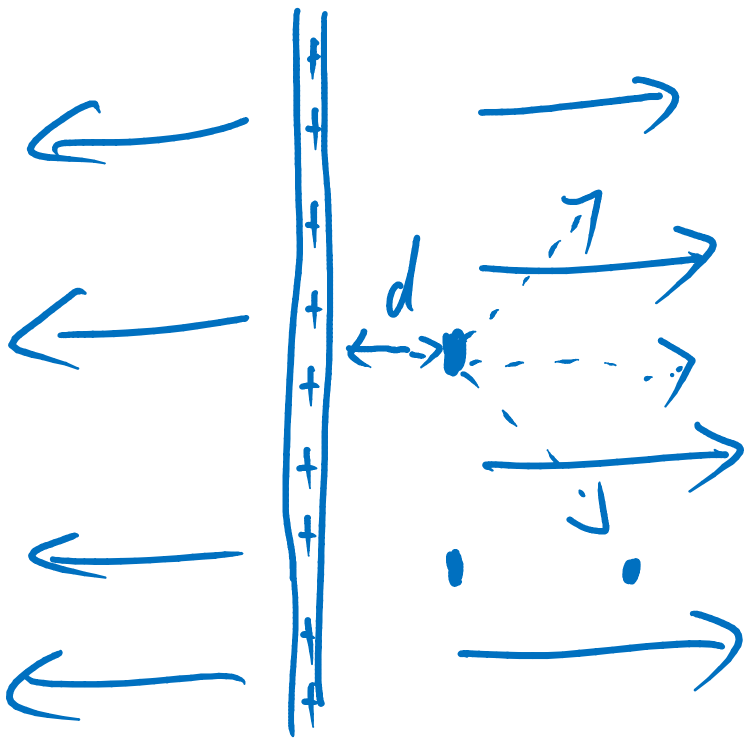






Electric field lines

1. The electric field points in the direction tangent to the field line at any point.
2. Magnitude of field is proportional to the number of lines crossing unit area perpendicular to the lines.
3. Electric field lines start at positive charge and end on negative charge.



$\vec{E}$

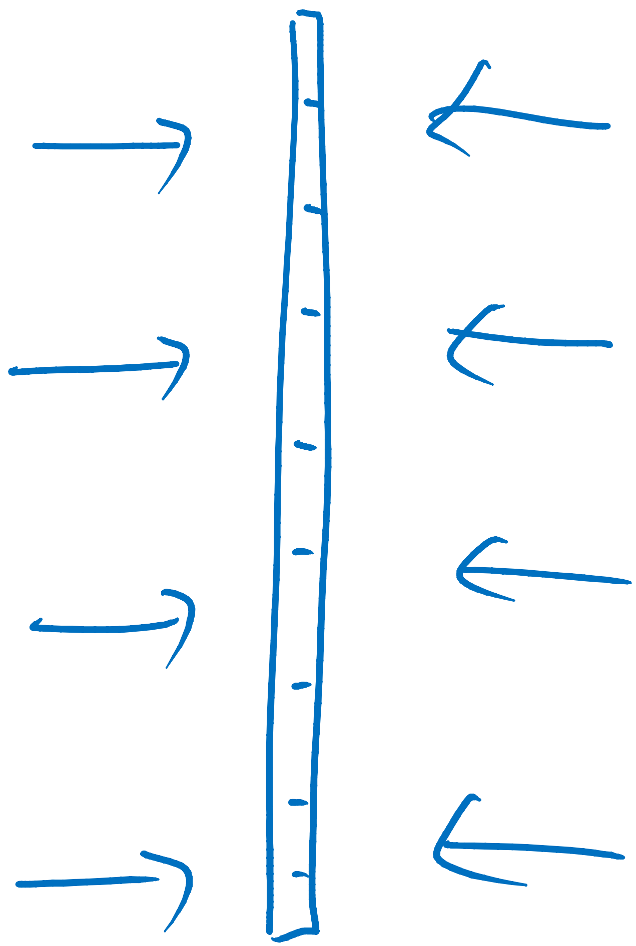
$$\sigma = \frac{Q}{A}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

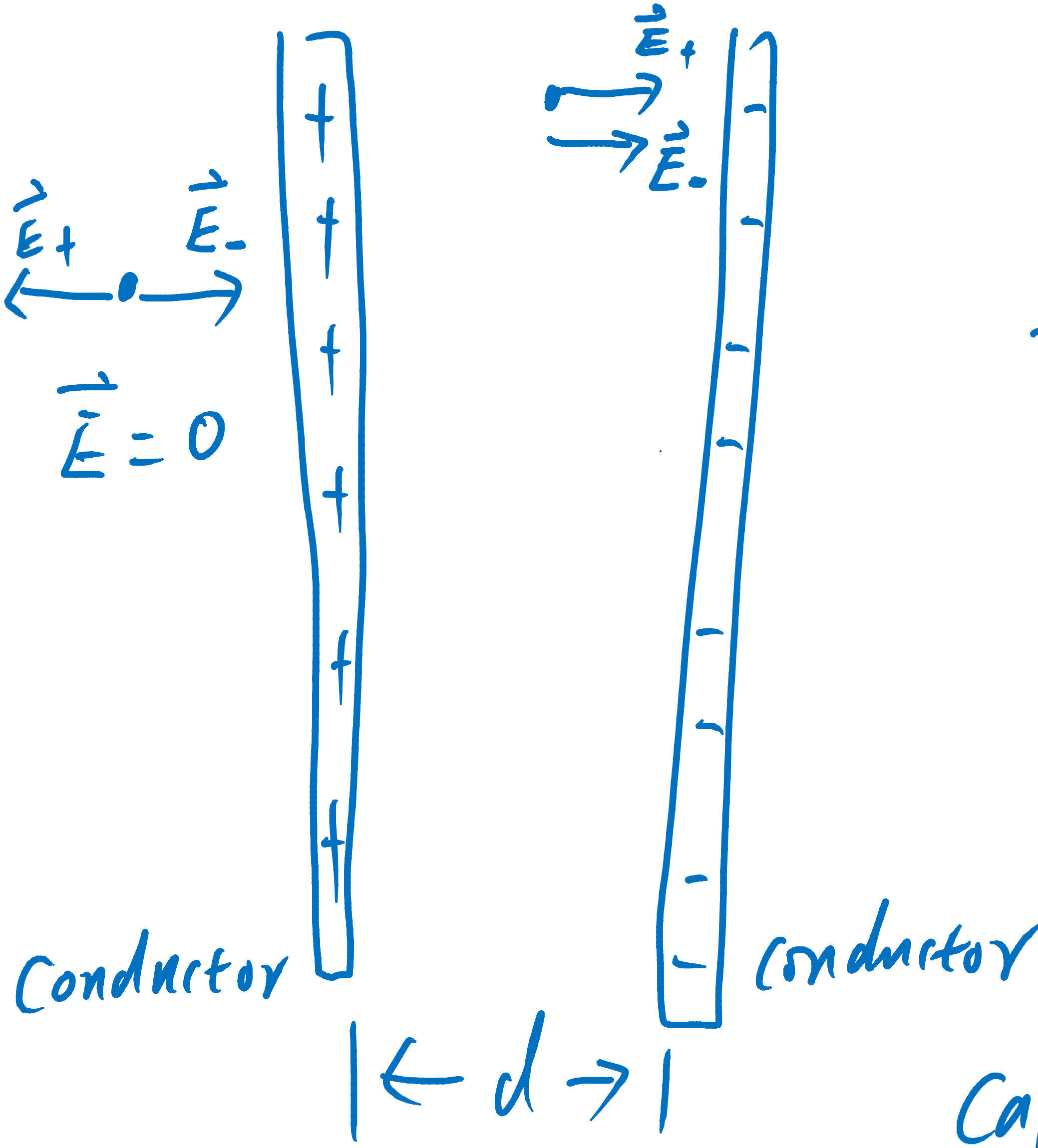
$$\varepsilon_0 = 8.85 \times 10^{-12}\, C^2/Nm^2$$

Infinitely large conductor plate.

$d \ll \sqrt{A}$



$$E = \frac{\sigma}{2\varepsilon_0}$$



Outside $E = 0$

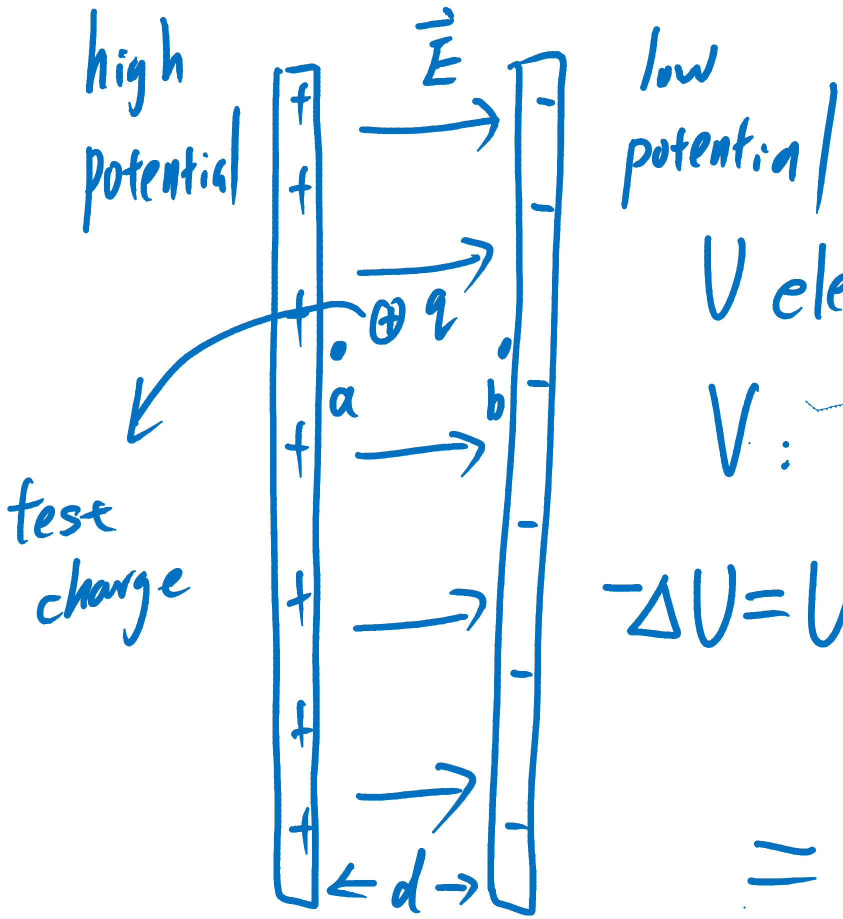
&

Inside $E = 2\left(\frac{\sigma}{2\varepsilon_0}\right)$

$$E = \frac{\sigma}{\varepsilon_0}$$

Capacitor

$d \ll \sqrt{A}$



$\vec{F} = q\vec{E}$

U electric potential energy

V: electric potential (voltage) $V = \frac{U}{q}$

$$-\Delta U = U_a - U_b = W\Big|_a^b = \int_a^b \vec{F} \cdot d\vec{l}$$

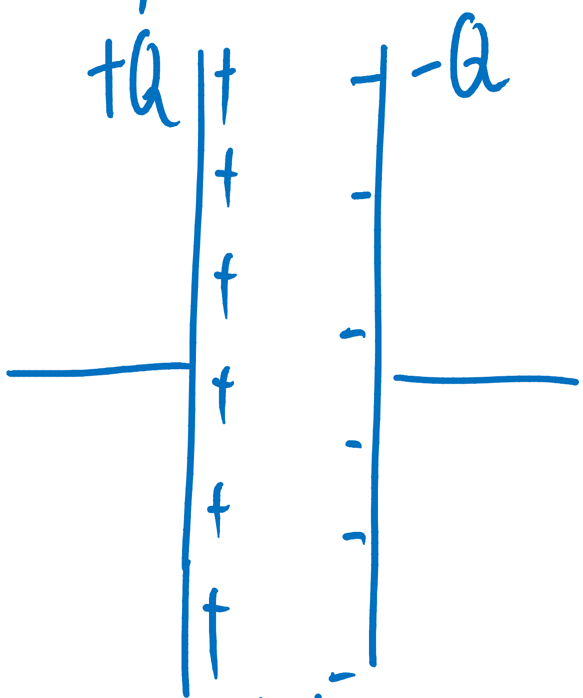
$$= qEd$$

$$\Delta V = V_a - V_b = \frac{U_a - U_b}{q} = qEd/q = E \cdot d$$

$$\Delta V = E \cdot d$$

Along the field lines, V decreases.

* Capacitors



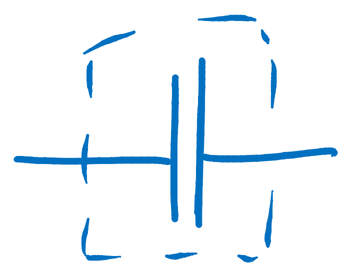
two parallel conducting plates

Capacitor is a device that can store electric charge.

Capacitance: $C = \frac{Q}{V}$

V: voltage difference

Symbol for C



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

$$V = E \cdot d = \left(\frac{Q}{\varepsilon_0 A}\right) \cdot d = V$$

$$C = \frac{Q}{V} = \varepsilon_0 \frac{A}{d}$$

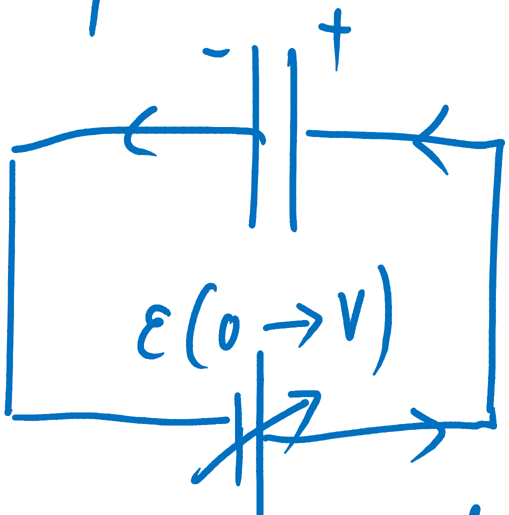
$\hookrightarrow$ vaccum dielectric constant.

$\varepsilon_0 \rightarrow \varepsilon$

unit of C

$$\frac{\text{Coulomb}}{\text{Volt}} = \text{Faraday}$$

A capacitor stores electric energy



variable emf

$$W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

electric energy

$(Q = CV)$

static process dq emf = capacitor voltage

$$dW = V \cdot dq$$

$$W = \int_0^Q V \cdot dq$$

$$= \int_0^Q \frac{q}{C} dq = \left.\frac{q^2}{2C}\right|_0^Q = \frac{Q^2}{2C}$$

Review for Exam 1.

Electric current $I = \frac{dQ}{dt}$

Constant current $I = \frac{\Delta Q}{\Delta t}$

$$\Delta Q = I \cdot \Delta t$$

ΔQ in Coulomb (C)

I in Ampere (A)

Number of electrons $\Delta N = \frac{\Delta Q}{e}$ $e = 1.6 \times 10^{-19}$ C

Ohm's law $R = \frac{V}{I}$, $I = \frac{V}{R}$, $V = IR$

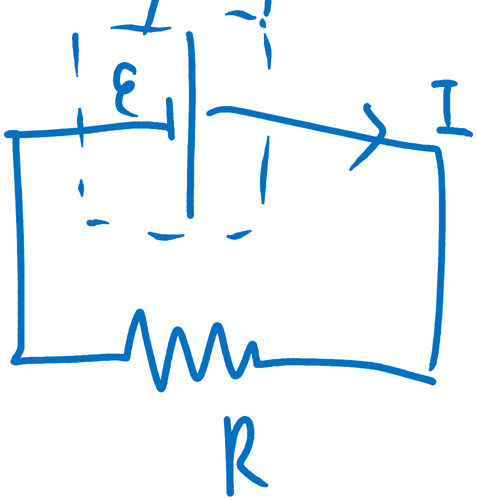
$$R = \rho \frac{L}{A}$$

Electric power $P = IV$

Joule heating on resistor

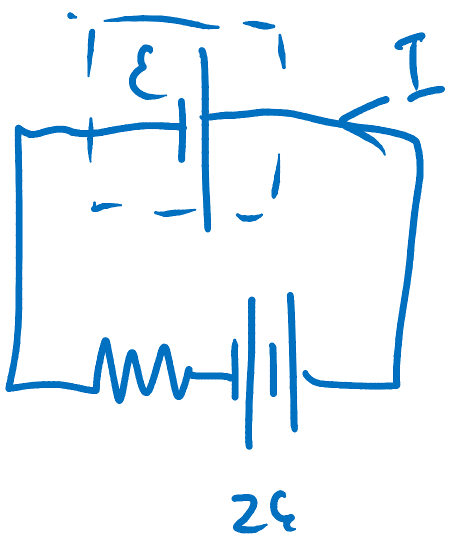
$$P = IV = I^2 R = \frac{V^2}{R}$$

Battery power.



discharging

$$P = I\varepsilon$$



charging

$$P = I\varepsilon$$

Resistors in series.

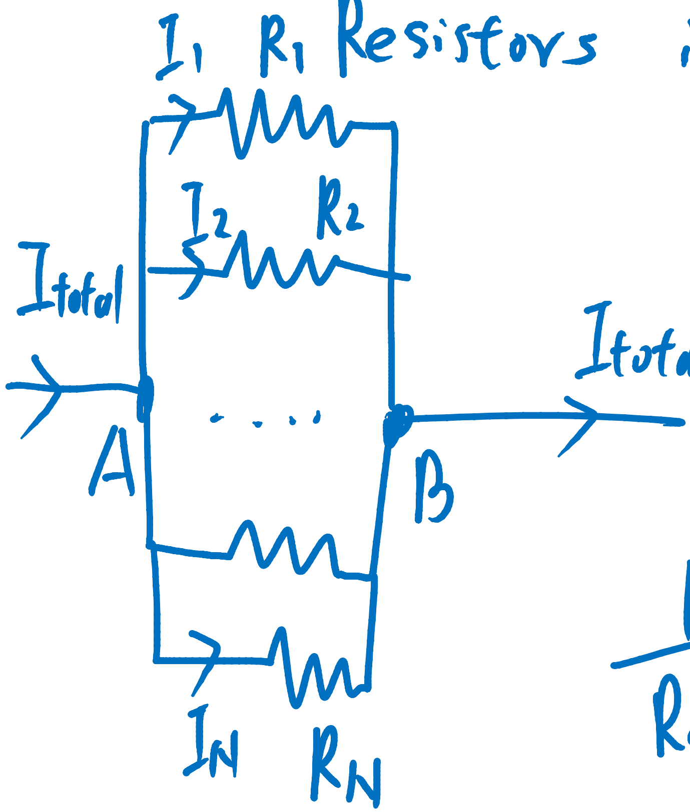
$$V_{total} = V_1 + V_2 + \cdots + V_N$$

$$R_{eq} = R_1 + R_2 + \cdots + R_N$$

Current : Same

$$V_m = V_{total} \cdot \frac{R_m}{R_1 + R_2 + \cdots + R_m + \cdots + R_N} \qquad m = 1, 2, \ldots, N$$

Resistors in parallel



V: same on each resistor.

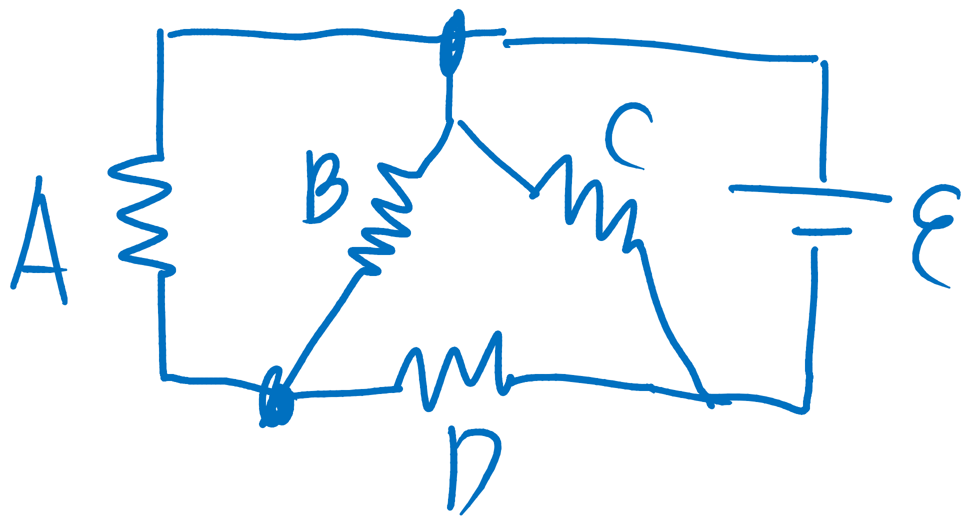
$$I_{total} = I_1 + I_2 + \cdots + I_N$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

current divider rule

$$I_m = I_{total} \cdot \frac{\frac{1}{R_m}}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_m} + \cdots + \frac{1}{R_N}}$$

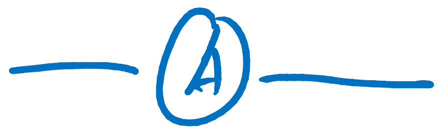
$m = 1, 2, \cdots, N$



$$(A \parallel B + D) \parallel C$$

Meters

Ammeter



ideal meter $r = 0$

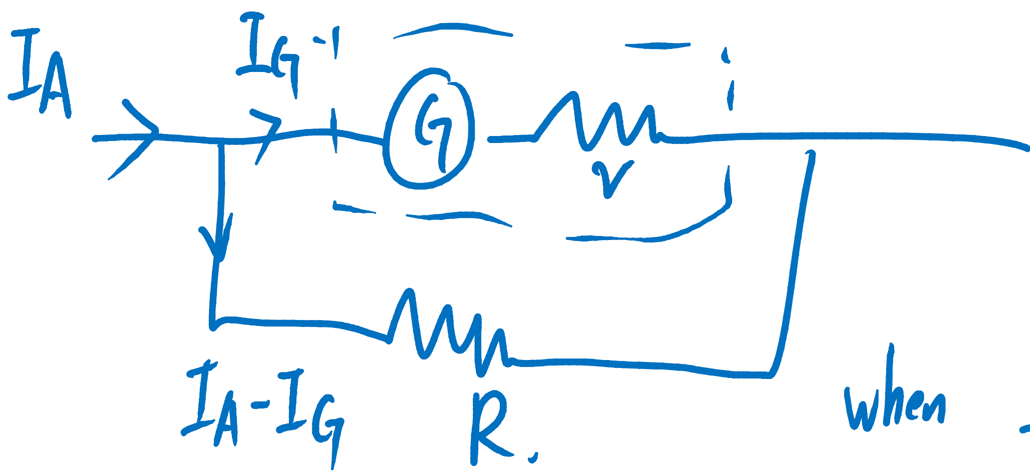
practical $r > 0$

basic ammeter : galvanometer

full scale deflection

I_G

$I_A \gg I_G$. Change range.

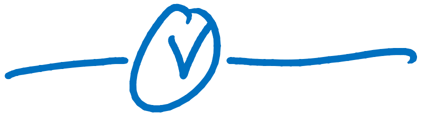


$$I_G \cdot r = (I_A - I_G) R$$

$$R = \frac{I_G \cdot r}{I_A - I_G}$$

when $I_A \gg I_G$, $R = \frac{I_G}{I_A} r$

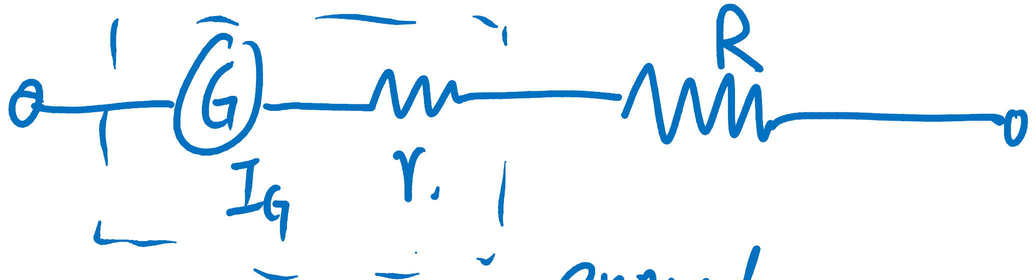
Voltmeter



$r = \infty$ for ideal meter.

$r < \infty$ for practical meter

Construct a voltmeter from a galvonometer.



expand range to V_v

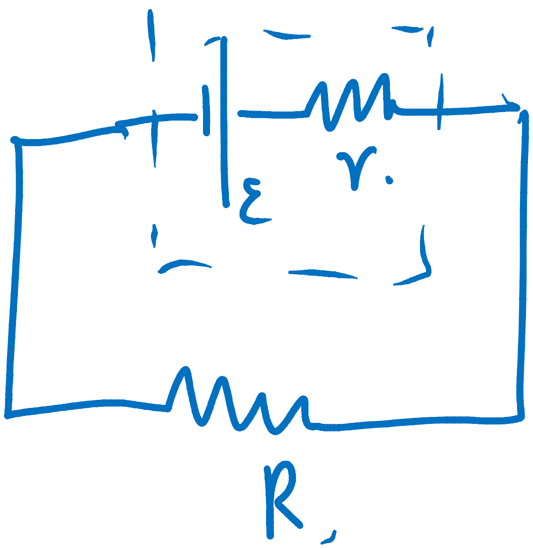
$V_G = I_G \cdot r$

$$\frac{V_v}{R+r} = I_G$$

$V_v \gg V_G$

$$\frac{V_v}{I_G} \gg r \qquad R = \frac{V_v}{I_G} - r \approx \frac{V_v}{I_G} = \frac{V_v}{V_G} r$$

practical voltage source.



ideal battery $r=0$

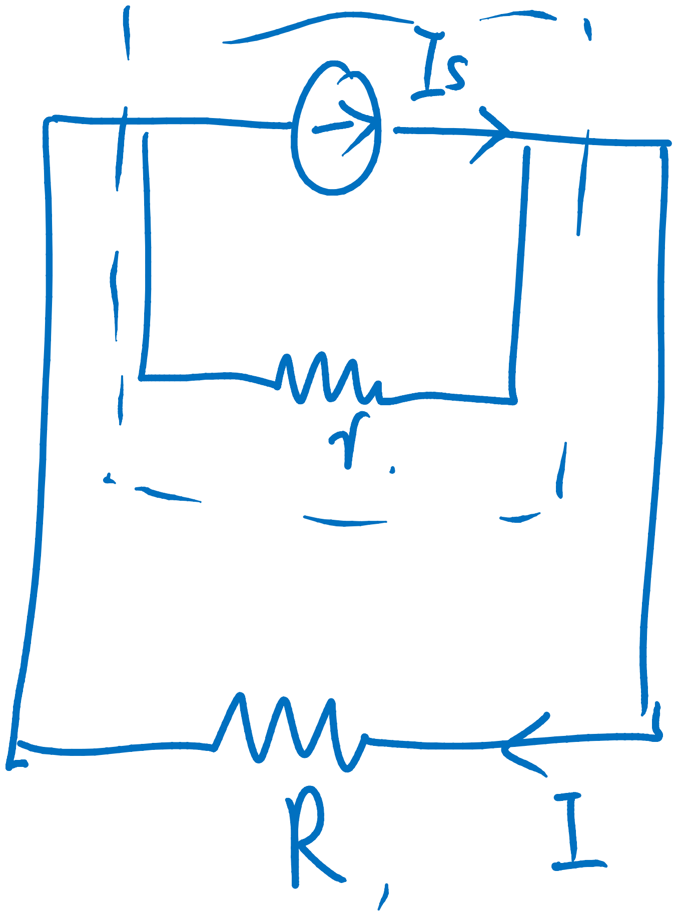
practical battery $r>0$

terminal voltage.

$$V = \varepsilon \cdot \frac{R}{R+r} < \varepsilon$$

$$R \to \infty \qquad V \to \varepsilon$$

practical current source



ideal $r = \infty$

practical $r < \infty$

$$I = I_s \cdot \frac{\frac{1}{R}}{\frac{1}{R} + \frac{1}{r}}$$

$$I = I_s \frac{r}{R+r} < I_s$$

$r \to \infty \quad I \to I_s$

(short circuit) $R \to 0 \quad I \to I_s$

Kirchhoff's laws

KCL $\sum I_{in} = \sum I_{out}$

KVL $\sum \Delta V = 0$ around closed loop

along current across resistor $\Delta V < 0$

against current across resistor $\Delta V > 0$

Mesh analysis

KCL implicit

KVL explicit

Norton and Thevenin

$V_{TH} \quad R_{TH} = R_N \quad I_N$

$V_{TH} = V_{oc}$

$I_N = I_{sc}$

$R_{TH} = R_N = R_{eq}$ (after killing sources)

Only two are independent

$V_{oc} = I_{sc} \cdot R_{eq}$

$$C = \frac{q}{V} \qquad C = \varepsilon_0 \frac{A}{d}$$

i–v characteristics of C

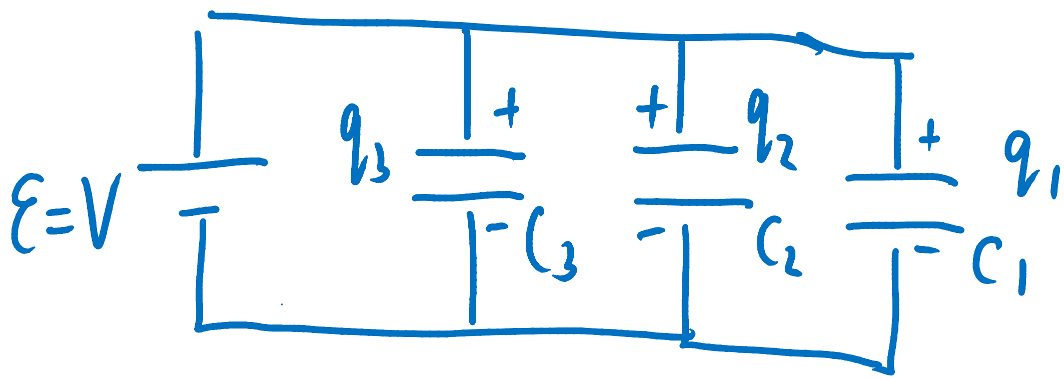
$$i = \frac{dq}{dt} = \frac{d(Cv)}{dt} = C\frac{dv}{dt}$$

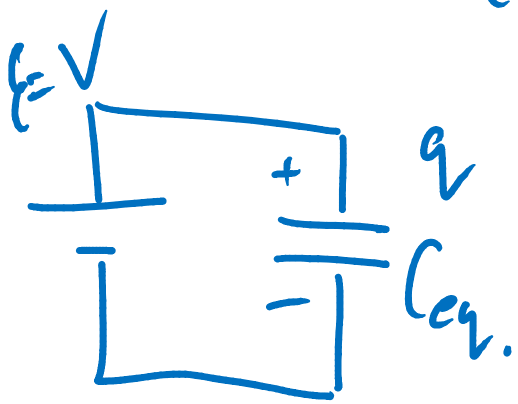
$$i = C\frac{dv}{dt}$$

i–v need initial conditions,

* Capacitors in series and in parallel

Parallel





$$q = q_1 + q_2 + q_3$$

$$V = V_1 = V_2 = V_3$$

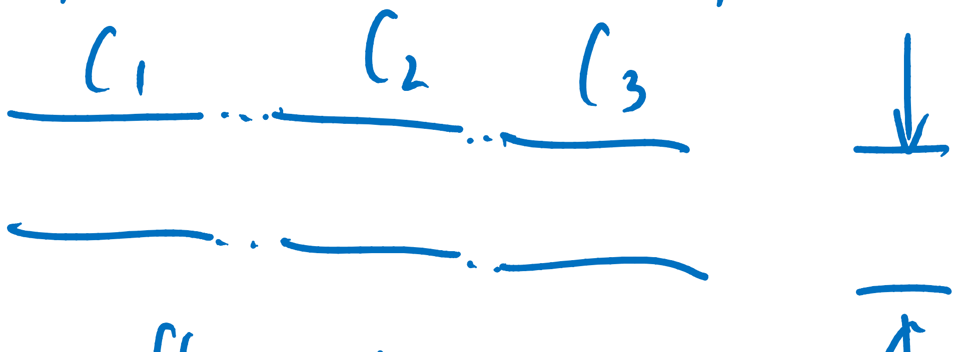
$$C_{eq} = \frac{q}{V} \qquad C_1 = \frac{q_1}{V} \qquad C_2 = \frac{q_2}{V} \qquad C_3 = \frac{q_3}{V}$$

$$C_{eq} = \frac{q}{V} = \frac{q_1+q_2+q_3}{V} = \frac{q_1}{V} + \frac{q_2}{V} + \frac{q_3}{V}$$

$$= C_1 + C_2 + C_3$$

$$\boxed{C_{eq} = C_1 + C_2 + C_3}$$

Imagine putting C_1, C_2, C_3 parallel like this:



assume same d.

parallel: effectively increase area.

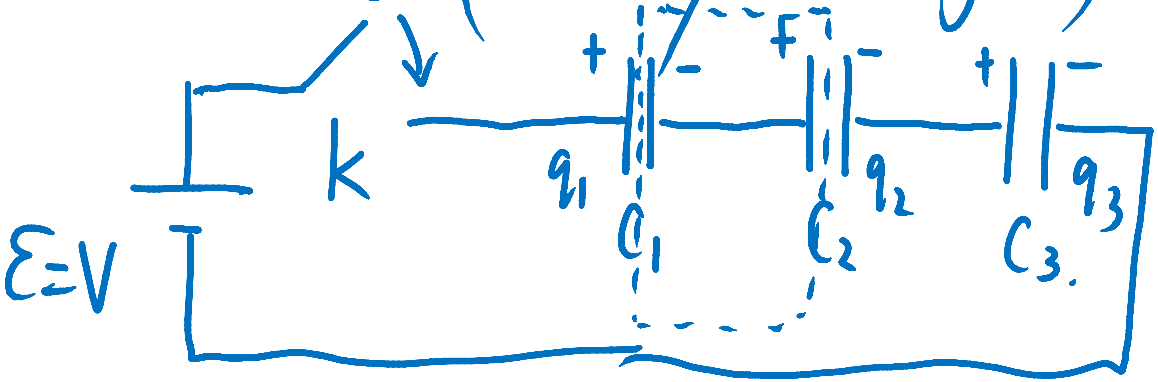
$$A = A_1 + A_2 + A_3 \quad \text{same } d.$$

$$C_{eq} = \varepsilon_0 \frac{A}{d} = \varepsilon_0 \frac{(A_1 + A_2 + A_3)}{d}$$

$$= \varepsilon_0 \frac{A_1}{d} + \varepsilon_0 \frac{A_2}{d} + \varepsilon_0 \frac{A_3}{d}$$

$$= C_1 + C_2 + C_3$$

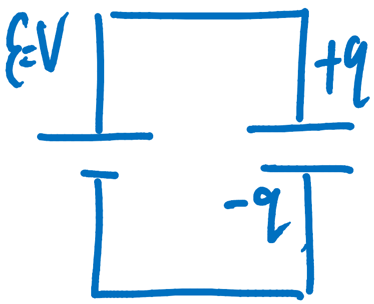
In series (initially uncharged)



$$\frac{1}{C_{eq}}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}$$

$$-q_1+q_2=0$$

$$q=q_1=q_2=q_3$$



$C_{eq}=?$

$$V=V_1+V_2+V_3$$

$$\frac{\cancel{q}}{C_{eq}}=\frac{\cancel{q_1}}{C_1}+\frac{\cancel{q_2}}{C_2}+\frac{\cancel{q_3}}{C_3}$$

$$q = q_1 = q_2 = q_3 \qquad V_2 = ?$$

$$C_{eq} V = C_1 \underline{V_1} = C_2 V_2 = C_3 V_3 \qquad V_3 = ?$$

$$V_1 = \frac{C_{eq}}{C_1} V$$

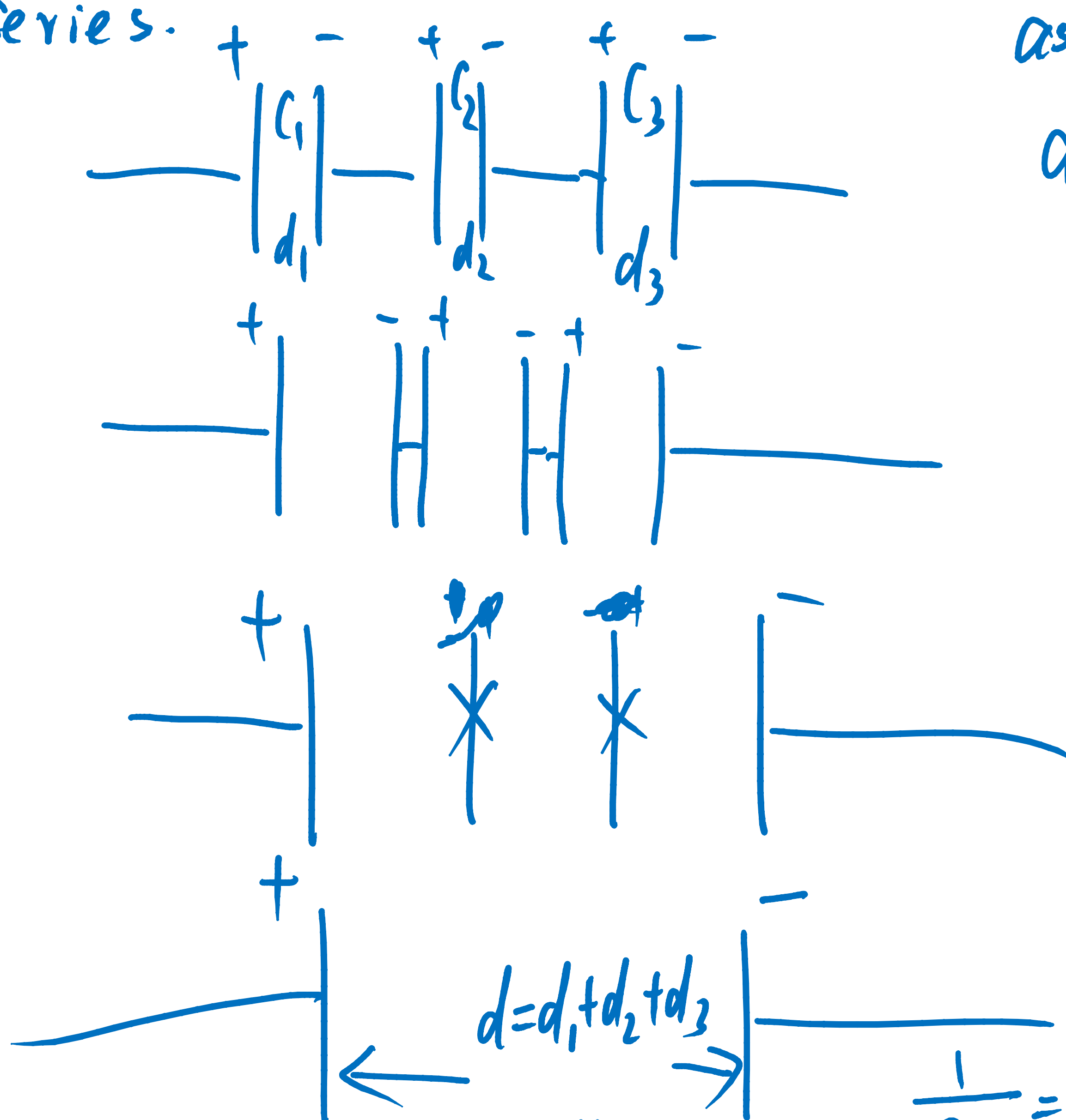
$$= \left(\frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \right) \frac{1}{C_1} V.$$

$$V_1 = \frac{\left(\frac{1}{C_1}\right)}{\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)} V$$

voltage divider rule for capacitor,

$$V_1 = \frac{(1/C_1)}{(1/C_{eq})} V$$

In series.



assume same A

$$d=d_1+d_2+d_3$$

$$C=\varepsilon_0\frac{A}{d}.$$

$$d=\varepsilon_0\frac{A}{C}$$

$$d_1=\varepsilon_0\frac{A}{C_1} \quad d_2=\varepsilon_0\frac{A}{C_2}$$

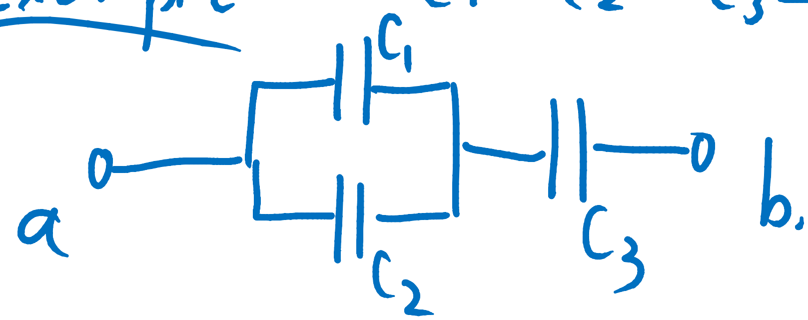
$$d_3=\varepsilon_0\frac{A}{C_3}.$$

$$d=\varepsilon_0\frac{A}{C_{eq}}$$

$$\frac{1}{C_{eq}}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}$$

C_{eq} is smaller than any of C_1, C_2, C_3

Example. $C_1 = C_2 = C_3 = C$



(1) C_{eq} between a and b ?

$$C_{1//2} = C_1 + C_2 = 2C$$

$$\frac{1}{C_{eq}} = \frac{1}{2C} + \frac{1}{C} \quad C_{eq} = \frac{2}{3}C$$

(2) If $\mathcal{E} = V$ is connected between a and b, what is the charge on each capacitor ?

$$V_3 = V \cdot \frac{\frac{1}{C_3}}{\frac{1}{C_3} + \frac{1}{C_{1//2}}} = \frac{2}{3}V$$

$$q_3 = V_3 \cdot C_3 = \frac{2}{3}VC$$

$$V_1 = V_2 = V - V_3 = \frac{1}{3}V$$

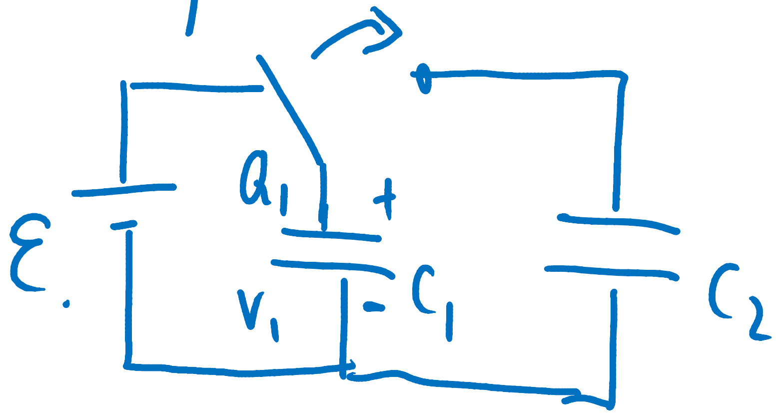
$$q_1 = C_1 V_1 = \frac{1}{3}VC$$

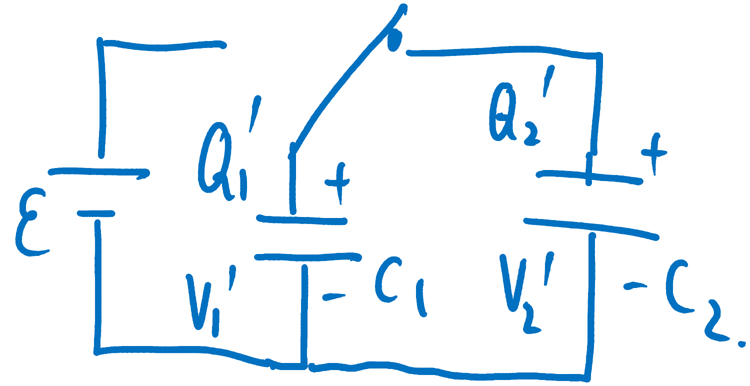
$$q_2 = C_2 V_2 = \frac{1}{3}VC$$

Initially charged capacitors

- charge conservation
- KVL
- ~~Ca~~ voltage – charge relation.

Example.





$$V_1 = \varepsilon$$

$$Q_1 = C_1 V_1 = C_1 \varepsilon \quad (1)$$

$$Q_1' + Q_2' = Q_1 \quad (2)$$

$$V_1' = V_2'$$

$$\frac{Q_1'}{C_1} = \frac{Q_2'}{C_2} \quad (3)$$

(1)(2) $\Rightarrow$ $Q_1' + Q_2' = C_1 \varepsilon$

~~into (1)~~ $Q_1' = C_1 \varepsilon - Q_2'$

into (3) $\dfrac{C_1 \varepsilon - Q_2'}{C_1} = \dfrac{Q_2'}{C_2} \Rightarrow Q_2' = \dfrac{\varepsilon C_1 C_2}{C_1 + C_2}$

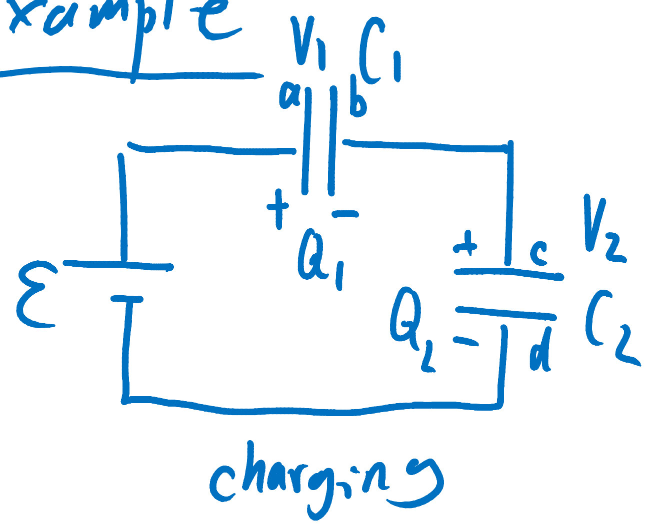
$$Q_1' = C_1 \varepsilon - Q_2' = \frac{C_1^2 \varepsilon}{C_1 + C_2}$$

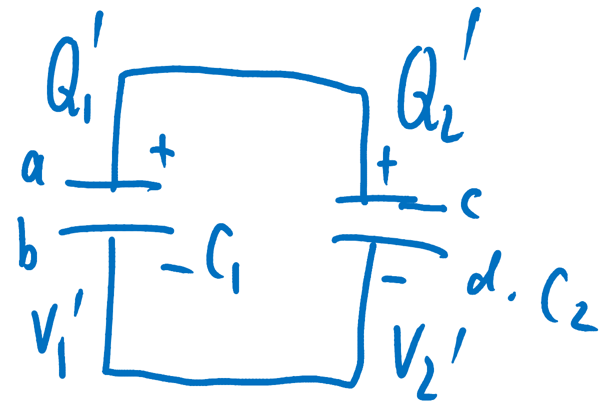
$$V_1' = V_2' = \frac{Q_2'}{C_2} = \frac{Q_1'}{C_1} = \frac{\varepsilon C_1}{C_1 + C_2}$$

If $C_1 = C_2 = C$, $V_1' = V_2' = \frac{\varepsilon}{2}$

$$Q_1' = Q_2' = \frac{C\varepsilon}{2}$$

Example





$Q_1' = ?$

$Q_2' = ?$

$$Q_1 = Q_2 = \mathcal{E} \cdot C_{eq} = \mathcal{E}\,\frac{C_1 C_2}{C_1 + C_2}$$

$$V_1 = \frac{Q_1}{C_1} = \frac{\mathcal{E} C_2}{C_1 + C_2}$$

$$V_2 = \frac{Q_2}{C_2} = \frac{\mathcal{E} C_1}{C_1 + C_2}$$

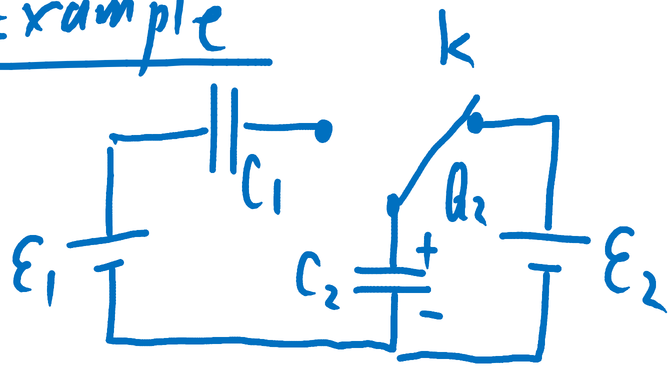
$$Q_1' + Q_2' = Q_1 + Q_2 = \frac{2\mathcal{E} C_1 C_2}{C_1 + C_2} \qquad (1)$$

$$V_1' = V_2'$$

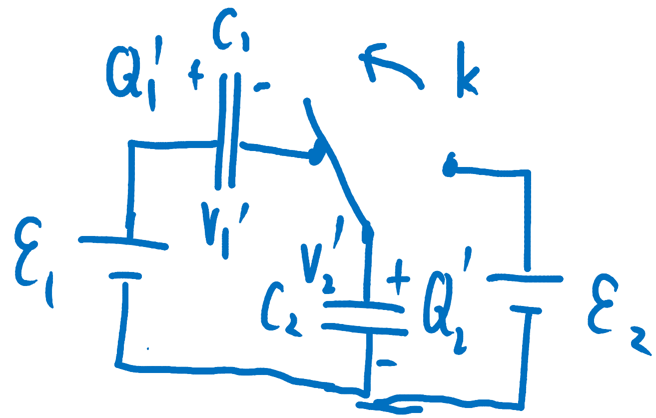
$$\frac{Q_1'}{C_1} = \frac{Q_2'}{C_2} \qquad (2)$$

$$(1)(2) \Longrightarrow \quad Q_1' = \frac{2\mathcal{E} C_1^2 C_2}{(C_1 + C_2)^2} \qquad Q_2' = \frac{2\mathcal{E} C_1 C_2^2}{(C_1 + C_2)^2}$$

Example



$$Q_2 = C_2 \mathcal{E}_2$$



$Q_1' = ?$

$Q_2' = ?$

$V_1' = ?$

$V_2' = ?$

$$-Q_1' + Q_2' = Q_2 = C_2 \mathcal{E}_2 \quad (1)$$

$$\mathcal{E}_1 = V_1' + V_2'$$

$$\mathcal{E}_1 = \frac{Q_1'}{C_1} + \frac{Q_2'}{C_2} \quad (2)$$

$(1)(2) \Rightarrow$

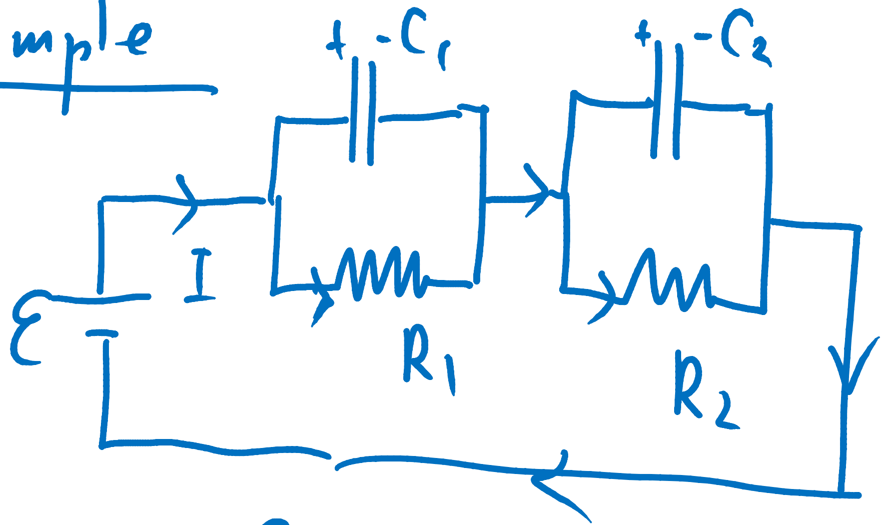
$$Q_1' = \frac{(\mathcal{E}_1 - \mathcal{E}_2) C_1 C_2}{C_1 + C_2}$$

$$Q_2' = \frac{C_2 (\mathcal{E}_1 C_1 + \mathcal{E}_2 C_2)}{C_1 + C_2}$$

$$V_1' = \frac{Q_1'}{C_1} = \frac{(\mathcal{E}_1 - \mathcal{E}_2) C_2}{C_1 + C_2}$$

$$V_2' = \frac{\mathcal{E}_1 C_1 + \mathcal{E}_2 C_2}{C_1 + C_2}$$

Example



$Q_1 = ?$

$Q_2 = ?$

$V_1 = ?$

$V_2 = ?$

$$I = \frac{\varepsilon}{R_1 + R_2}$$

$$V_1 = IR_1 = \frac{\varepsilon R_1}{R_1 + R_2} \qquad V_2 = IR_2 = \frac{\varepsilon R_2}{R_1 + R_2}$$

$$Q_1 = C_1 V_1 = \frac{\varepsilon C_1 R_1}{R_1 + R_2}$$

$$Q_2 = C_2 V_2 = \frac{\varepsilon C_2 R_2}{R_1 + R_2}$$

Summary of Capacitor

— Insulating material between conducting plates

$i = 0$ in d.c. circuit in <u>steady states</u>

equivalent to open circuit.

— charge can be stored on plates of capacitor.

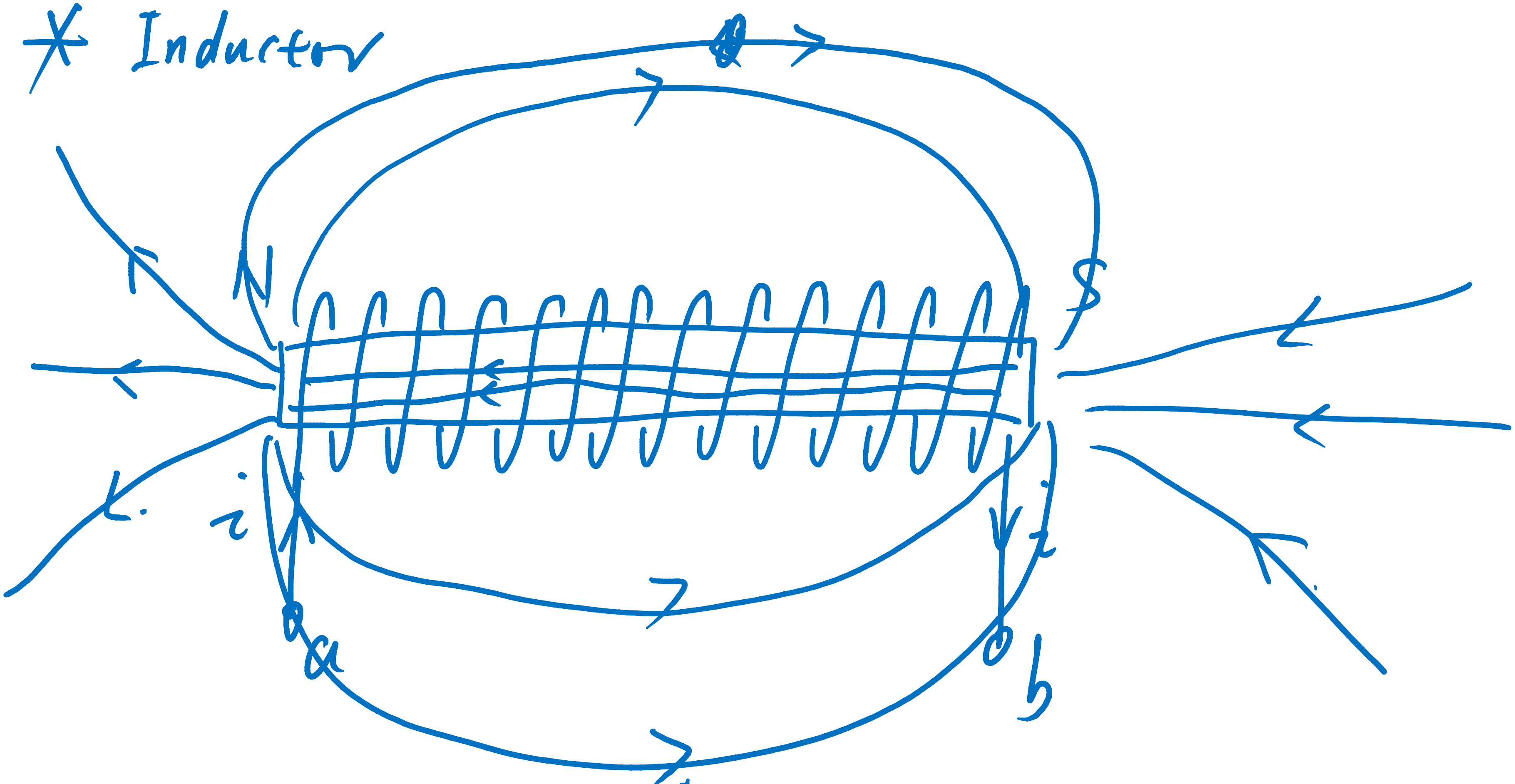
$$q = CV$$

— $i - v$ relationship $\quad i = \frac{dq}{dt} = C\frac{dv}{dt}$

— energy stored.

$$W = \frac{1}{2}CV^2$$

* Inductor



A current in Solenoid generates a magnetic field

Magnetic field lines:

field direction (tangential direction of lines)

magnitude (density of lines)

$i-v$ relation inductor

$$v_L = L\frac{di_L}{dt} \qquad \left(\text{compare to } C:\ i = C\frac{dv_c}{dt}\right)$$

L : inductance. unit : Henry (H)

Energy Stored: $W_L = \frac{1}{2}L\,i_L^2(t)$

$$\left(\text{Compare to } C:\quad W_c = \frac{1}{2}CV_c^2\right)$$

In d.c. ~~[illegible]~~ circuit in steady states, L can be replaced by short circuit.

$$v_L = L\underbrace{\frac{di_L}{dt}}_{=0} = 0 \quad \text{in d.c. circuits}$$

symbol 

inductors in series

$$L_{eq} = L_1 + L_2 + L_3 + \cdots$$

inductors in parallel

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots$$