SOLUTIONS TO EXAM #2 (4/26/19)

#1. a.) The Fourier series coefficients of a discrete-time signal are <u>periodic</u>. This is only true for (II). So, this is the only one that could correspond to the spectrum of a discrete-time signal.

b.) Using the symmetry properties, when a time-domain signal is real and even, its spectrum must be real and even. This is only true for (III).

c.) I $\rightarrow \omega_0 = \pi \rightarrow$ contains frequencies
$\omega = k\omega_0 = k\pi$ where
$k = -4, -3, 3, 4$
$\therefore \omega$: $\pm 4\pi$ rad/sec
$\pm 3\pi$ rad/sec

II $\rightarrow \omega_0 = \pi/3 \rightarrow$ periodic, in one period contains frequencies
$\omega = k\omega_0 = k\pi/3$ where
$k = -1, 1$
$\therefore \omega$: $\pm \pi/3$ rad/sec
and its periodic repetitions

III $\rightarrow \omega_0 = \pi \rightarrow$ contains frequencies
$\omega = k\omega_0 = k\pi$ where
$k = -4, -3, -2, -1, 1, 2, 3, 4$
$\therefore \omega$: $\pm 4\pi$ rad/sec
$\pm 3\pi$ rad/sec
$\pm 2\pi$ rad/sec
$\pm \pi$ rad/sec

#1 (cont'd) d.) Assume signal with spectrum (III) is passed through a lowpass filter that will not pass $\omega > 2\omega_0$. So, only frequencies $\omega = \pm\pi$ rad/sec and $\pm 2\pi$ rad/sec will pass

∴ The spectrum of the output signal has coefficients, b_k where

$$b_k = \begin{cases} 2, & k=-2 \\ 1, & k=-1 \\ 1, & k=1 \\ 2, & k=2 \end{cases} \quad \& \ 0 \text{ otherwise}$$

$$y(t) = \text{output signal} = \sum_{k=-\infty}^{\infty} b_k e^{jk\overset{\pi}{\omega_0} t}$$

$$= \underbrace{2e^{-j2\pi t} + 2e^{j2\pi t}}_{4\cos 2\pi t} + \underbrace{e^{-j\pi t} + e^{j\pi t}}_{2\cos \pi t}$$

$$\therefore\ y(t) = 2\cos \pi t + 4\cos 2\pi t$$

#2. a.) i.) $x(t) = \cos 6\pi t$ (can find Fourier series coefficients by inspection)

$$= \frac{1}{2}\left(e^{j6\pi t} + e^{-j6\pi t}\right) \qquad \omega_0 = 6\pi$$

equate terms

But, $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk6\pi t}$

$a_1 = \frac{1}{2}$

$a_{-1} = \frac{1}{2}$

$a_k = 0$, otherwise

#2a) cont'd) ii) $x[n] = \sin\frac{\pi}{4}n \longrightarrow N = \frac{2\pi}{\omega_0} = 8$

↳ discrete-time signal, so spectrum is periodic with period 8.

We can again do this by inspection.

$$x[n] = \sin\frac{\pi}{4}n = \frac{1}{2j}\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)$$

equate terms

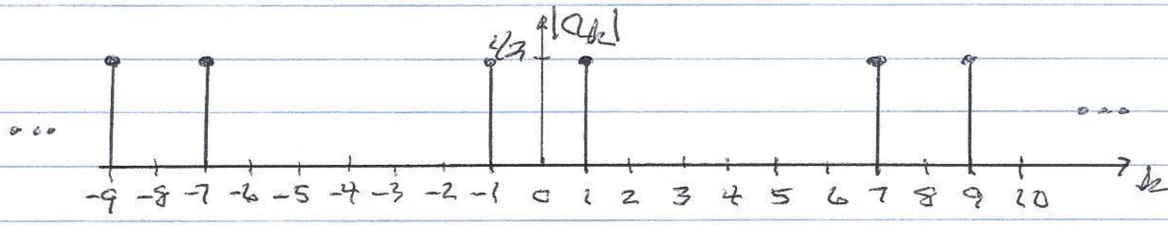
But $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{\pi}{4}n}$

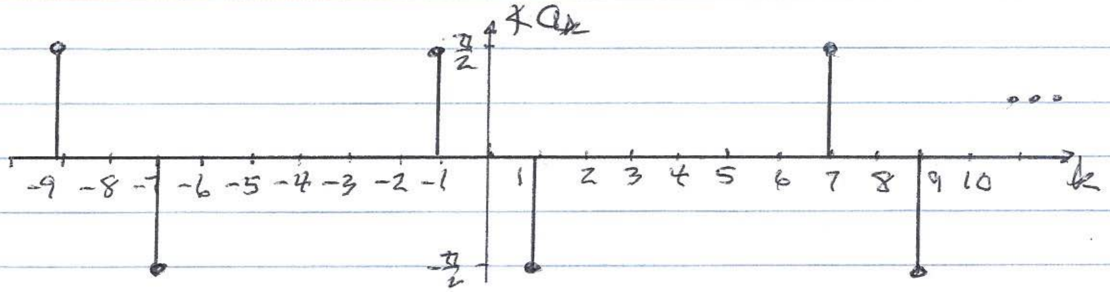
$$a_1 = \frac{1}{2j} = -\frac{j}{2} \Rightarrow |a_1| = \frac{1}{2},\ \measuredangle a_1 = -\frac{\pi}{2}$$

$$a_{-1} = -\frac{1}{2j} = \frac{j}{2} \Rightarrow |a_{-1}| = \frac{1}{2},\ \measuredangle a_1 = \frac{\pi}{2}$$

$$a_k = 0,\quad k = -3, -2, 0, 2, 3, 4$$

periodic
N = 8





#2 cont'd) b.) $\omega_0 = \pi$

If we are given the Fourier series coefficients (a_k's), the continuous-time, periodic signal can be reconstructed as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t}$$
$$= \underbrace{2e^{-j3\pi t} + 2e^{j3\pi t}}_{4\cos 3\pi t} + \underbrace{j e^{-j2\pi t} - j e^{j2\pi t}}_{\frac{1}{j}(e^{j2\pi t} - e^{-j2\pi t}) = 2\sin 2\pi t}$$

$$\therefore \quad x(t) = 2\sin 2\pi t + 4\cos 3\pi t$$

#3. a.)

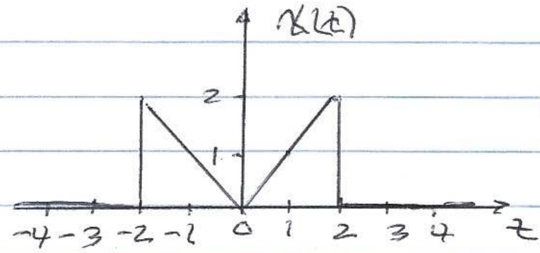
$$x(t) = \sum_{m=0}^{\infty} \alpha^m \delta(t-m) \qquad |\alpha| < 1$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \alpha^m \delta(t-m) e^{-j\omega t} dt$$
$$= \sum_{m=0}^{\infty} \alpha^m \underbrace{\int_{-\infty}^{\infty} \underbrace{\delta(t-m)}_{\text{impulse at } t=m} e^{-j\omega t} dt}_{\text{sifting} \quad e^{-j\omega m}}$$

$$X(j\omega) = \sum_{m=0}^{\infty} \alpha^m e^{-j\omega m} = \underbrace{\sum_{m=0}^{\infty} (\alpha e^{-j\omega})^m}_{\text{geometric series}}$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

#3 cont'd) b.)



$$x(t) = \begin{cases} |t|, & -2 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

i) Use Parseval's relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$\therefore \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= 2\pi (2) \int_0^2 t^2 \, dt$$

$$= 4\pi \left. \frac{t^3}{3} \right|_0^2 = \frac{32}{3}\pi$$

ii) Use $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega$

$$\therefore \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega} \, d\omega = 2\pi \left. x(t) \right|_{t=-1}$$

$$= 2\pi (1) = 2\pi$$

c) $s(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{F}} S(j\omega) = \frac{1}{1+j\omega}$

$$X(j\omega) = j\frac{d}{d\omega}\left(e^{j2\omega} \underbrace{S\left(j\frac{\omega}{3}\right)}\right)$$

$$\hookrightarrow Y(j\omega) \rightarrow y(t) = 3s(3t)$$

$$= 3e^{-3t}u(3t)$$

$$= 3e^{-3t}u(t)$$

#3. a) cont'd)

$$W(j\omega) = Y(j\omega)\, e^{j2\omega}$$
$$w(t) = y(t+2) = 3e^{-3(t+2)}\,u(t+2)$$

$$X(j\omega) = j\frac{d}{d\omega} W(j\omega)$$

$$\therefore\ x(t) = t\,w(t) = 3t\,e^{-3(t+2)}\,u(t+2)$$

#4. a.) $$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx(t)}{dt} + 3x(t)$$

$$\Downarrow \mathcal{F}$$

$$(j\omega)^2 Y(j\omega) + 6j\omega Y(j\omega) + 8Y(j\omega) = j\omega X(j\omega) + 3X(j\omega)$$
$$Y(j\omega)\Big((j\omega)^2 + 6(j\omega) + 8\Big) = X(j\omega)(j\omega+3)$$
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+3}{(j\omega)^2+6j\omega+8}$$
$$= \frac{j\omega+3}{(j\omega+4)(j\omega+2)}$$

b.) Impulse response $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$. Use partial fraction expansion and the fact that

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega+a}$$

$$H(j\omega) = \frac{j\omega+3}{(j\omega+4)(j\omega+2)} = \frac{A}{j\omega+4} + \frac{B}{j\omega+2}$$

$$A = H(j\omega)(j\omega+4)\Big|_{j\omega=-4} = \frac{j\omega+3}{j\omega+2}\Big|_{j\omega=-4} = \frac{-1}{-2} = \frac{1}{2}$$
$$B = H(j\omega)(j\omega+2)\Big|_{j\omega=-2} = \frac{j\omega+3}{j\omega+4}\Big|_{j\omega=-2} = \frac{1}{2} = \frac{1}{2}$$

#4b) (cont'd) $\therefore H(j\omega) = \dfrac{1/2}{j\omega+4} + \dfrac{1/2}{j\omega+2}$

$$\Downarrow \mathcal{F}^{-1}$$

$$h(t) = \frac{1}{2}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t)$$

#5. $x(t) = \delta(t) \longrightarrow$ [nervous system LTI] $\longrightarrow h(t) =$ impulse response

$$h(t) = \frac{1}{2}e^{-6t}u(t) + \frac{1}{2}e^{-4t}u(t)$$

output for unknown input $= y(t) = \frac{1}{3}e^{-t}u(t) - \frac{1}{3}e^{-4t}u(t)$

a.) $H(j\omega) =$ frequency response $= \mathcal{F}\{h(t)\}$

$$= \frac{1}{2}\frac{1}{j\omega+6} + \frac{1}{2}\frac{1}{j\omega+4}$$

$$= \frac{1/2\,(j\omega+4) + 1/2\,(j\omega+6)}{(j\omega+6)(j\omega+4)}$$

$$\therefore H(j\omega) = \frac{j\omega+5}{(j\omega+6)(j\omega+4)} = \frac{j\omega+5}{(j\omega)^2+10j\omega+24}$$

b.) $\xrightarrow[X(j\omega)]{x(t)}$ [$h(t)$ / $H(j\omega)$] $\xrightarrow[Y(j\omega) = H(j\omega)X(j\omega)]{y(t)}$

$x(t) = ?$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$\therefore X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} \Rightarrow x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$

#5b) cont'd)

$$y(t) = \tfrac{1}{3} e^{-t} u(t) - \tfrac{1}{3} e^{-4t} u(t)$$

$$Y(j\omega) = \frac{1/3}{j\omega+1} - \frac{1/3}{j\omega+4}$$

$$= \frac{1/3(j\omega+4) - 1/3(j\omega+1)}{(j\omega+1)(j\omega+4)}$$

$$= \frac{1}{(j\omega+1)(j\omega+4)}$$

$$\therefore X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = \frac{\dfrac{1}{(j\omega+1)\cancel{(j\omega+4)}}}{\dfrac{j\omega+5}{(j\omega+6)\cancel{(j\omega+4)}}}$$

$$= \frac{j\omega+6}{(j\omega+1)(j\omega+5)}$$

$$\Big\Downarrow \mathcal{F}^{-1} \qquad \text{use partial fraction expansion}$$

$$x(t)$$

$$X(j\omega) = \frac{j\omega+6}{(j\omega+1)(j\omega+5)} = \frac{A}{j\omega+1} + \frac{B}{j\omega+5}$$

$$A = X(j\omega)(j\omega+1)\Big|_{j\omega=-1} = \frac{j\omega+6}{j\omega+5}\Big|_{j\omega=-1} = \frac{5}{4}$$

$$B = X(j\omega)(j\omega+5)\Big|_{j\omega=-5} = \frac{j\omega+6}{j\omega+1}\Big|_{j\omega=-5} = \frac{1}{-4}$$

$$\therefore X(j\omega) = \frac{5/4}{j\omega+1} - \frac{1/4}{j\omega+5}$$

$$\Big\Downarrow \mathcal{F}^{-1}$$

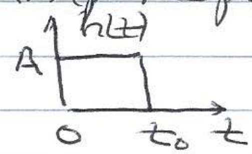
$$x(t) = \frac{5}{4} e^{-t} u(t) - \frac{1}{4} e^{-5t} u(t)$$

Extra Credit #1 The image on the right is blurred compared to the image on the left. This means that the sharp edges are being blurred → high frequency components are being suppressed.

$\therefore$ lowpass filter

Extra Credit #2 This is the same problem I have given in some past years.

$$h(t) = A[u(t) - u(t-t_0)] \rightarrow \text{rectangular imp. response}$$



$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= A \int_0^{t_0} e^{-j\omega t} dt$$

$$= A\left(-\frac{1}{j\omega}\right) e^{-j\omega t}\Big|_0^{t_0} = \frac{A}{j\omega}\left(1 - e^{-j\omega t_0}\right)$$

$$= 2A e^{-j\omega \frac{t_0}{2}} \left[\frac{e^{j\omega t_0/2} - e^{-j\omega t_0/2}}{2j\omega}\right]$$

$$= 2A e^{-j\omega \frac{t_0}{2}} \frac{\sin \omega t_0/2}{\omega}$$

The response will have zeros when $\sin \frac{\omega t_0}{2} = 0$, i.e. at $\omega t_0/2 =$ a multiple of π. We want this to be zero for $f = 60$ Hz and its multiples.

$$\therefore t_0 = \frac{2\pi}{\omega} = \frac{\cancel{2\pi}}{\cancel{2\pi} f} = \frac{1}{f} = \frac{1}{60} \text{ sec}$$

Therefore, if $t_0 = \frac{1}{60}$ sec, we will have zeros at

$$f = \frac{m}{t_0} \Rightarrow 60\text{Hz}, 120\text{Hz}, \ldots$$