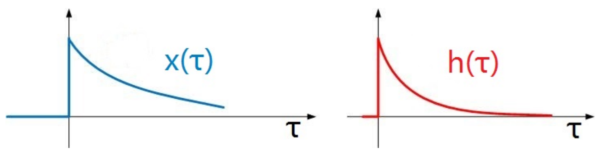
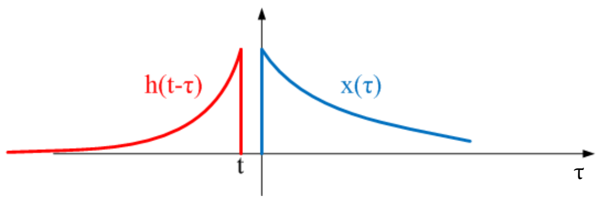
SOLUTION TO HOMEWORK #3

2.22

(a) Evaluate the convolution using the graphical approach used in class. Here we have

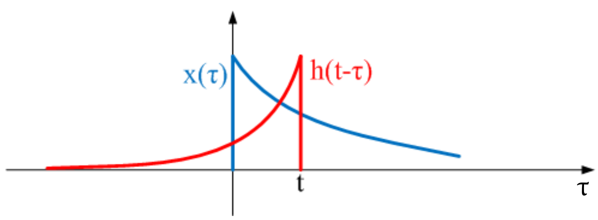


If $t < 0$,



and there is no overlap. So the product $x(\tau)h(t-\tau) = 0$ and $y(t) = 0$.

If $t \geq 0$,



then, when $\alpha \neq \beta$

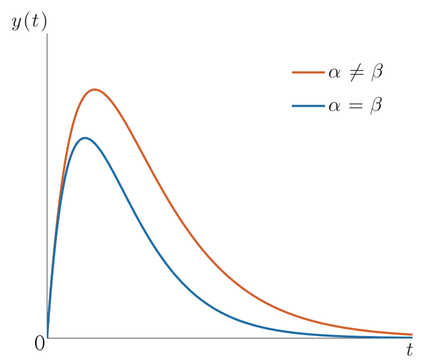
$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau \\
&= \int_0^t e^{(-\alpha+\beta)\tau}e^{-\beta t}d\tau = e^{-\beta t}\int_0^t e^{(-\alpha+\beta)\tau}d\tau \\
&= e^{-\beta t}\left[-\frac{1}{\alpha-\beta}e^{-(\alpha-\beta)\tau}\right]\Bigg|_0^t = e^{-\beta t}\left(\frac{1}{\beta-\alpha}\right)\left[e^{-(\alpha-\beta)t}-1\right] \\
&= \frac{e^{-\alpha t}-e^{-\beta t}}{\beta-\alpha}
\end{aligned}
$$

When $\alpha = \beta$,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau \\ &= \int_0^t e^{(-\alpha+\beta)\tau}e^{-\beta t}d\tau = e^{-\beta t}\int_0^t 1 d\tau \\ &= e^{-\beta t}\left[\tau\right]\Big|_0^t = e^{-\beta t}\left[t-0\right] \\ &= te^{-\beta t} \end{aligned}$$

Therefore

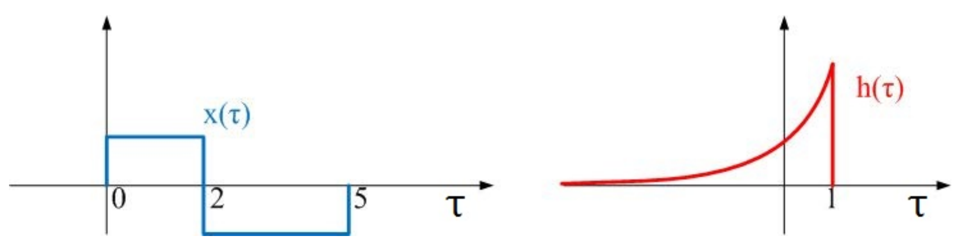
$$y(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}u(t), & \alpha \neq \beta \\ te^{-\beta t}u(t), & \alpha = \beta \end{cases}$$



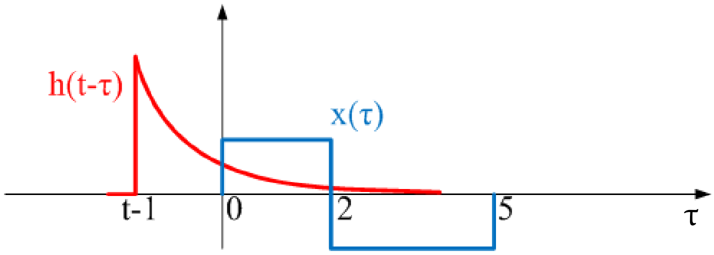
(b) Convolve

$$\begin{aligned} x(t) &= u(t) - 2u(t-2) + u(t-5) \\ &\text{and} \\ h(t) &= e^{2t}u(1-t) \end{aligned}$$

First, plot $x(\tau)$ and $h(t-\tau)$



Then, the convolution is performed as follows.

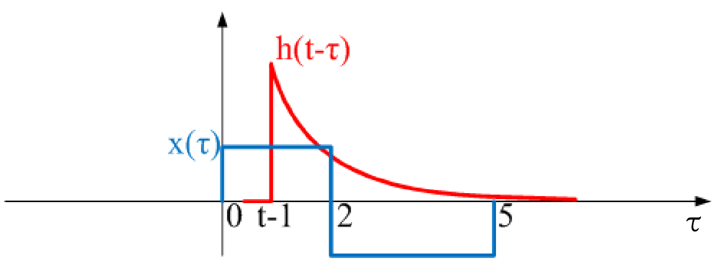


If $t \leq 1$,

then,

$$
\begin{aligned}
y(t) &= \int_0^2 e^{2(t-\tau)} d\tau - \int_2^5 e^{2(t-\tau)} d\tau \\
&= e^{2t} \int_0^2 e^{-2\tau} d\tau - e^{2t} \int_2^5 e^{-2\tau} d\tau \\
&= e^{2t} \left(-\frac{1}{2}\right) \left[e^{-2\tau}\right] \Big|_0^2 - e^{2t} \left(-\frac{1}{2}\right) \left[e^{-2\tau}\right] \Big|_2^5 \\
&= e^{2t} \left(-\frac{1}{2}\right) \left[e^{-4} - 1\right] - e^{2t} \left(-\frac{1}{2}\right) \left[e^{-10} - e^{-4}\right] \\
&= e^{2t} \left(-\frac{1}{2}\right) \left[2e^{-4} - e^{-10} - 1\right] \\
&= \frac{e^{2t}}{2} \left[1 - 2e^{-4} + e^{-10}\right]
\end{aligned}
$$

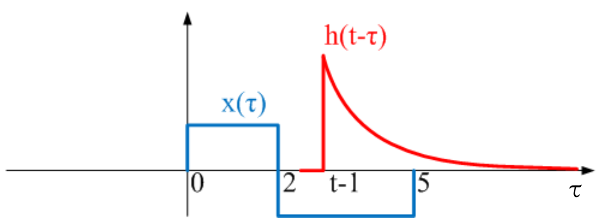
If $1 \leq t \leq 3$,



then,

$$
\begin{aligned}
y(t) &= \int_{t-1}^{2} e^{2(t-\tau)} d\tau - \int_{2}^{5} e^{2(t-\tau)} d\tau \\
&= e^{2t} \int_{t-1}^{2} e^{-2\tau} d\tau - e^{2t} \int_{2}^{5} e^{-2\tau} d\tau \\
&= e^{2t} \left(-\frac{1}{2}\right) \left[e^{-2\tau}\right] \Big|_{t-1}^{2} - e^{2t} \left(-\frac{1}{2}\right) \left[e^{-2\tau}\right] \Big|_{2}^{5} \\
&= e^{2t} \left(-\frac{1}{2}\right) \left[e^{-4} - e^{-2t+2}\right] - e^{2t} \left(-\frac{1}{2}\right) \left[e^{-10} - e^{-4}\right] \\
&= e^{2t} \left(-\frac{1}{2}\right) \left[2e^{-4} - e^{-10} - e^{-2t+2}\right] \\
&= \frac{1}{2}[e^{2} + e^{2t-10} - 2e^{2t-4}]
\end{aligned}
$$

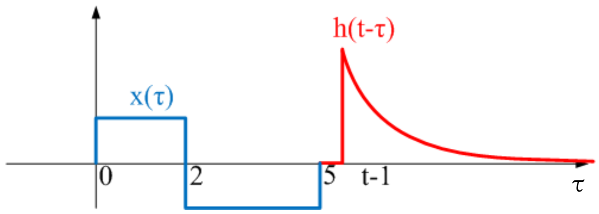
If $3 \leq t \leq 6$,



then,

$$
\begin{aligned}
y(t) &= -\int_{t-1}^{5} e^{2(t-\tau)} d\tau \\
&= -e^{2t} \int_{t-1}^{5} e^{-2\tau} d\tau \\
&= -e^{2t} \left(-\frac{1}{2}\right) \left[e^{-2\tau}\right] \Big|_{t-1}^{5} \\
&= -e^{2t} \left(-\frac{1}{2}\right) \left[e^{-10} - e^{-2t+2}\right] \\
&= \frac{1}{2}[e^{2t-10} - e^{2}]
\end{aligned}
$$

If $t > 6$,

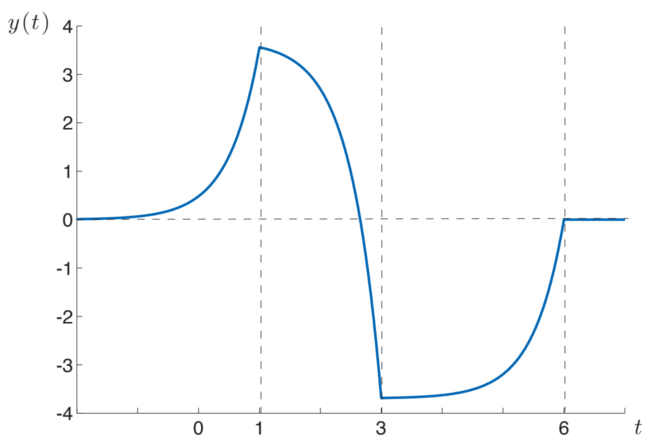


and then,

$$y(t) = 0$$

Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \begin{cases} \frac{e^{2t}}{2}[1 - 2e^{-4} + e^{-10}], & t \leq 1 \\ \frac{1}{2}[e^2 + e^{2t-10} - 2e^{2t-4}], & 1 \leq t \leq 3 \\ \frac{1}{2}[e^{2t-10} - e^2], & 3 \leq t \leq 6 \\ 0, & t > 6 \end{cases}$$



2.28

(a) The impulse response

$$h[n] = (\frac{1}{5})^n u[n]$$

When $n < 0$, $h[n] = 0$. Therefore the system is **causal**.

Now consider

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{5}\right)^n u[n] \right| = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1-\frac{1}{5}} = \frac{5}{4} < \infty$$

Therefore, the system is **stable**.

(c) The impulse response

$$h[n] = (\frac{1}{2})^n u[-n]$$

When $n < 0$, $h[n] \neq 0$, e.g. $h[-1] = 2$. Therefore, the system is **not causal**. Now, consider

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |(\frac{1}{2})^n u[-n]| = \sum_{n=-\infty}^{0} (\frac{1}{2})^n = \sum_{n=0}^{\infty} 2^n \to \infty$$

Therefore, the system is **not stable**.

(d) The impulse response

$$h[n] = (5)^n u[3-n]$$

When $n < 0$, $h[n] \neq 0$, e.g. $h[-1] = 5^{-1}$. Therefore, the system is **not causal**. Now, consider

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |5^n u[3-n]| = \sum_{n=-\infty}^{3} 5^n$$

Let $m = -n$, we have

$$\sum_{n=-\infty}^{3} 5^n = \sum_{m=-3}^{\infty} 5^{-m} = \sum_{m=-3}^{\infty} \left(\frac{1}{5}\right)^m = \left(\frac{1}{5}\right)^{-3} \times \frac{1}{1-\frac{1}{5}} = \frac{625}{4} < \infty$$

Therefore, the system is **stable**.

2.29

(a) The impulse response

$$h(t) = e^{-4t} u(t-2)$$

When $t < 0$, $h(t) = 0$. Therefore, the system is **causal**.
Now consider

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-4t} u(t-2) dt = \int_{2}^{\infty} e^{-4t} dt = -\frac{1}{4} e^{-4t} \Big|_2^{\infty} = \frac{1}{4} e^{-8} < \infty$$

Therefore, the system is **stable**.

(b) The impulse response

$$h(t) = e^{-6t} u(3-t)$$

When $t < 0$, $h(t) \neq 0$, e.g. $h(-1) = e^6$. Therefore, the system is **not causal**.
Now consider

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-6t} u(3-t) dt = \int_{-\infty}^{3} e^{-6t} dt \to \infty$$

Therefore, the system is **not stable**.

(d) The impulse response

$$h(t) = e^{2t}u(-1-t)$$

When $t < 0$, $h(t) \neq 0$, e.g. $h(-2) = e^{-4}$. Therefore, the system is **not causal**. Now consider

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{-1} e^{2t}dt = \frac{1}{2}e^{2t}\Big|_{-\infty}^{-1} = \frac{1}{2}e^{-2} < \infty$$

Therefore, the system is **stable**.

2.30

To determine the impulse response for this system, we need to find the output of the system when the input is an impulse, i.e., $x[n] = \delta[n]$. Since we are asked to assume initial rest, $y[n] = 0$ for $n < 0$. The difference equation is

$$y[n] = x[n] - 2y[n-1]$$

Because $x[n] = \delta[n]$, then for $n = 0$, $x[0] = 1$ and for $n > 0$, $x[n] = 0$. So, iterating the difference equation, we get

n	x[n]	y[n-1]	y[n] = x[n] - 2y[n-1]
⋮	⋮	⋮	⋮
-1	0	0	0
0	1	0	1
1	0	1	-2
2	0	-2	4
⋮	⋮	⋮	⋮

These values can be represented by the closed-form expression

$$y[n] = (-2)^n u[n]$$

where the $u[n]$ comes from the fact that $y[n] = 0$ for $n < 0$. This is the impulse response of the system.

Review Problem

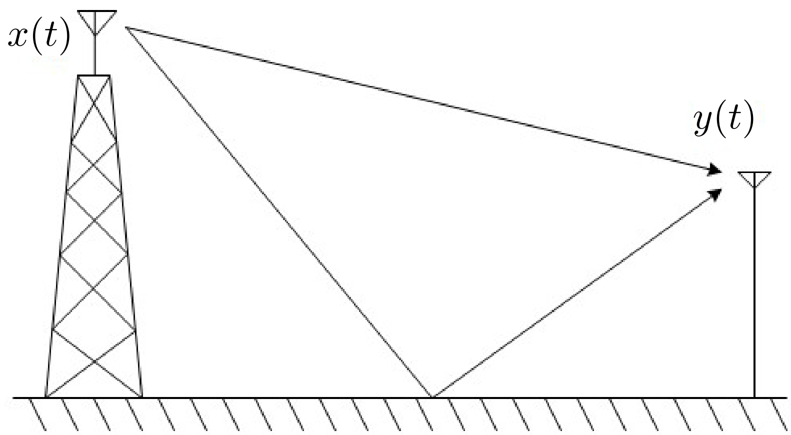
The impulse response is the output of a system when the input is an impulse. Given the input/output relation,

$$y[n] = 2x[n] + x[n-\alpha] + x[n+\beta]$$

simply substitute $x[n] = \delta[n]$ to find the impulse response. Therefore, the impulse response of the system is

$$h[n] = 2\delta[n] + \delta[n-\alpha] + \delta[n+\beta]$$

Conceptual



As shown in the figure, there are two paths: the direct path and the reflected path. The reflected path has a time delay t_0 and an attenuation α with respect to the direct path. Thus, the received signal $y(t)$ is given by

$$y(t) = x(t) + \alpha x(t - t_0),$$

and the impulse response ($y(t)$ for $x(t) = \delta(t)$) is

$$h(t) = \delta(t) + \alpha\delta(t - t_0).$$