ELEG 310

3/13/2018

PMF $\quad p_X(k) = P(X = k) \qquad k = \ldots$

$$\mu_X = E(X) = \sum_{k=0}^{\infty} k\, p_X(k) \qquad E(X^2) = \sum k^2 p_X(k)$$

mean

MGF $\quad M(u) = E(e^{uX}) = \sum_{k=0}^{\infty} e^{uk} p_X(k) \Leftrightarrow M(0) = 1$

$$\left.\frac{dM}{du}\right|_{u=0} = E(Xe^{uX})\Big|_{u=0} = E(X)$$

2 or more variables

$$p_{XY}(k,l) = P(X = k \cap Y = l)$$

$$E(XY) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} kl\, p_{XY}(k,l) \leftarrow$$

Covariance $E\big((X-\mu_X)(Y-\mu_Y)\big) \overset{\text{def}}{=} \sigma_{XY} \overset{\text{thm}}{=} \underbrace{E(XY)}_{\text{Correlation}} - \mu_X \mu_Y$

Variance $\sigma_X^2 \overset{\text{def}}{=} E\big((X-\mu_X)^2\big) \overset{\text{thm}}{=} E(X^2) - \mu_X^2$

X & Y are independent if $p_{XY}(k,l) = p_X(k)\, p_Y(l)$ for all k, l

X & Y are uncorrelated if $E(XY) = \mu_X \mu_Y$

ind $\Rightarrow$ unc

unc $\not\Rightarrow$ ind

X & Y ind $\Rightarrow$ $Z = X + Y$

$P_Z = P_X * P_Y$

Ex. $p_X(k) = \left[\frac{1}{2}, 0, \frac{1}{2}\right] = \frac{1}{2}\left[1, 0, 1\right]$ $\quad k = 1, 2, 3$

$p_Y(\ell) = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{3}\right] = \frac{1}{12}\left[3, 3, 2, 4\right]$ $\quad \ell = 1, 2, 3, 4$

$$
\begin{array}{c|cccccc}
 & 3 & 3 & 2 & 4 & & \\
\hline
1 & 3 & 3 & 2 & 4 & & \\
0 & & 0 & 0 & 0 & 0 & \\
1 & & & 3 & 3 & 2 & 4 \\
\hline
 & 3 & 3 & 5 & 7 & 2 & 4
\end{array}
$$

$\underbrace{3\ 3\ 5\ 7\ 2\ 4}_{\cancel{12}\ 24}$

$m = 2$

$P(Z = 2) = \cancel{\frac{3}{12}}\ \frac{3}{24}$

Sums of RVs

$Z = X + Y$ $\quad$ X & Y may not be ind.

$$E(Z) = E(X+Y) = E(X) + E(Y) = \mu_x + \mu_y$$

$$Var(Z) = E\left((X+Y-\mu_x-\mu_y)^2\right)$$

$$= E\left(\left((X-\mu_x) + (Y-\mu_y)\right)^2\right)$$

$$= E\left((X-\mu_x)^2 + 2(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2\right)$$

$$= \underbrace{E\left((X-\mu_x)^2\right)}_{\sigma_x^2} + 2\underbrace{E\left((X-\mu_x)(Y-\mu_y)\right)}_{\sigma_{xy}} + \underbrace{E\left((Y-\mu_y)^2\right)}_{\sigma_y^2}$$

$$\sigma_x^2 + 2\sigma_{xy} + \sigma_y^2$$

Ex. $Z = V + X + Y$

$$E(Z) = E(V) + E(X) + E(Y)$$

$$\sigma_Z^2 = \sigma_V^2 + \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} + 2\sigma_{VX} + 2\sigma_{VY}$$

Entropy & Information Theory

$$H(X) = -\sum_{k=0}^{\infty} p_X(k) \log_2 p_X(k)$$

$\log_2 8 = 3$

$\Rightarrow 2^3 = 8$

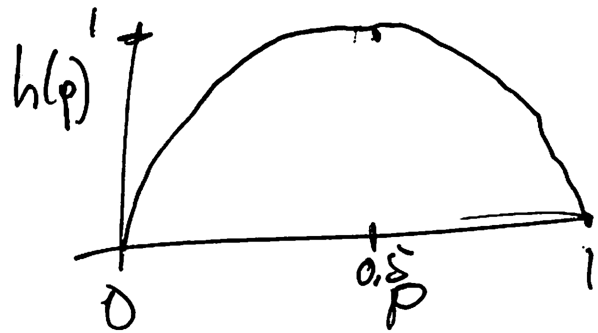
$=$ bits

$=$ measure of how random X is

Ex. X is Bernoulli RV $\quad P(X=1)=p \quad P(X=0)=1-p=q$

$$H(X) = h(p) = -\sum_{k=0}^{1} p(k)\log_2 p(k) \qquad H(X)\geq 0$$

$$= -q\log_2 q - p\log_2 p = -(1-p)\log_2(1-p) - p\log_2(p)$$



Ex. X uniform $\quad k = 1, 2, \dots, m$

$$H(X) = -\sum_{k=1}^{m} p(k)\log_2 p(k) = -\sum_{k=1}^{M} \frac{1}{m}\log_2\left(\frac{1}{m}\right) = \log(m)$$

Theorem if $P(X=k) = p(k)$ $k=1,\dots,m$,

$$0 \le H(X) \le \log_2 m$$

Calculus of Variations

$$\max_{p(k)} H(X) = -\sum p(k) \log_2 p(k)$$

$$\text{s.t. } p(k) \geq 0 \text{ and } \sum_{k=1}^{m} p(k) = 1$$

$$\Rightarrow \max_{p(k),\lambda} \underbrace{H(X) + \lambda\left(1 - \sum_{k=1}^{m} p(k)\right)}$$

Lagrange multiplier

$$H(X) = -\sum_{k=1}^{m} p(k) \log p(k)$$

$$\frac{d\, p\log p}{dp} = \log p + p \cdot \frac{1}{p} = (\log p + 1)$$

$$\frac{\partial(\ )}{\partial p(l)} = -\log p(l) - 1 - \lambda = 0$$

$l = 1, 2, \dots, m$

$$\frac{d(\ )}{d\lambda} = 1 - \sum_{k=1}^{m} p(k)$$

1 eqn

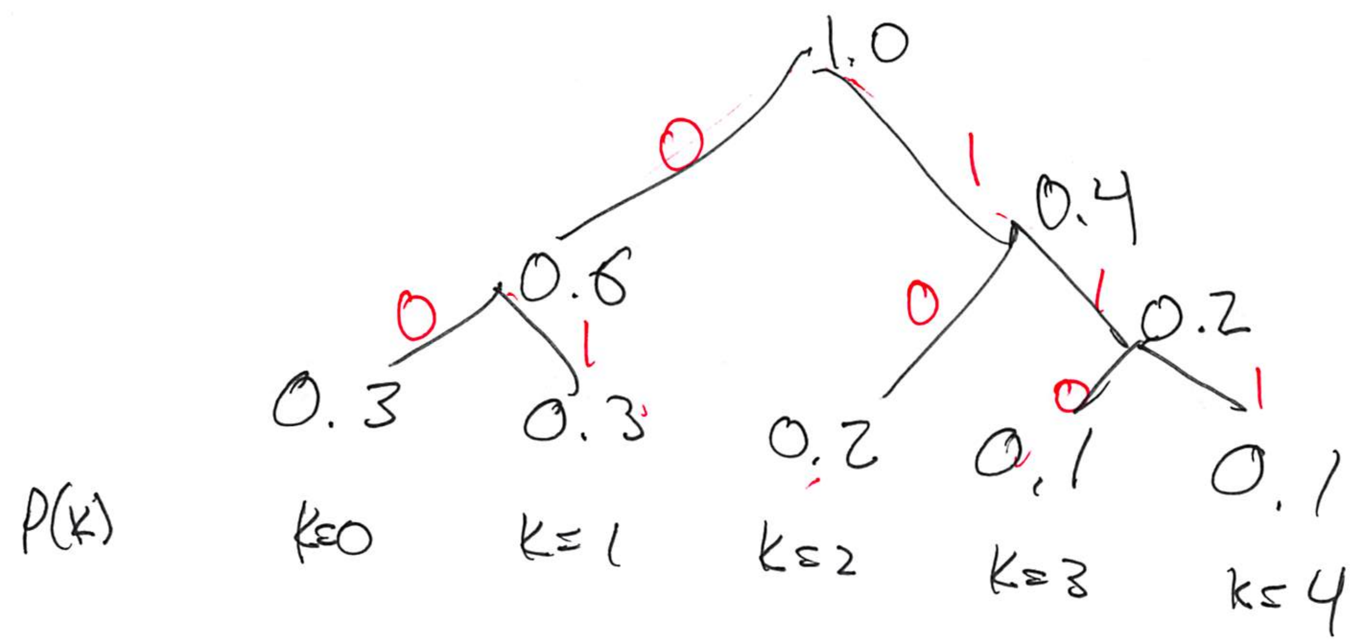
$$0 = -\log p(\ell) - 1 - \lambda$$

$$\Rightarrow \log_2 p(\ell) = -1 - \lambda \qquad \ell = 1, 2, \dots m$$

$$\Rightarrow \log_2 p(\ell) = \text{const}$$

$$\Rightarrow p(\ell) = \text{const} \quad \Rightarrow p(\ell) = \frac{1}{m}$$

Theorem: transmitting X requires at least $H(X)$ bits on average.



Codebook

K	
0	00
1	01
2	10
3	110
4	111

Huffman algorithm

$$L(X) = \#\text{bits required to transmit } X$$

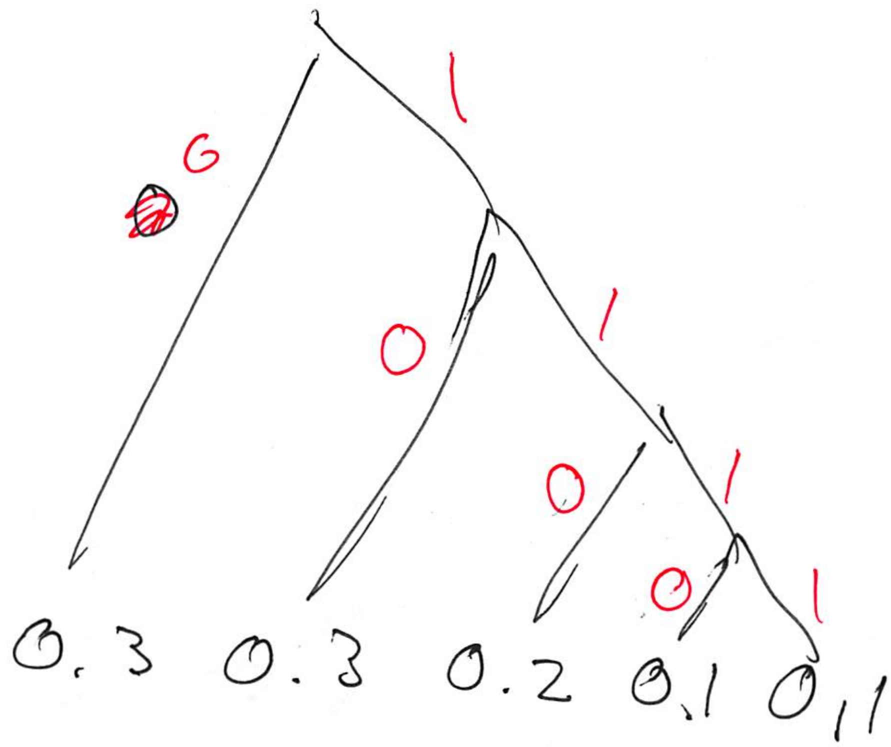
$$E(L(X)) = \sum_{k=0}^{4} L(k)\,P(k) = 2\times0.3 + 2\times0.3 + 2\times0.2 + 3\times0.1 + 3\times0.1$$

$$= 2.2 \text{ bits}$$

$$E(L(x)) = 1\times 0.3 + 2\times 0.3 + 3\times 0.2 + 4\times 0.1 + \cancel{3}\,4\times 0.1$$

$$= \cancel{2.4}\ \text{bits}$$

$$2.3$$



FIX

0	~~0~~	0
1	~~01~~	10
2	~~001~~	110
3	~~0001~~	1110
4	~~0000~~	1111