ELEG 310 Chap 4 14 March 2019

RV - random variable

pmf → $p_X(k) = P(X = k)$

outcome

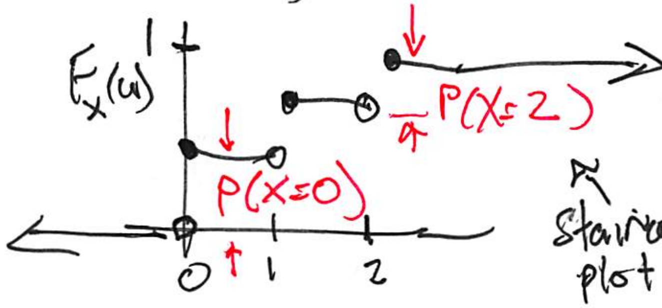
RV

1. $p(k) \geq 0$ for all k

2. $\sum_k p(k) = 1$ $\quad \sum_{k=0}^{\infty} p(k) = 1$

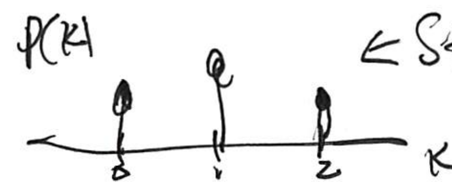
CDF = cumulative distribution function

$F_X(u) = P(X \leq u)$ $\quad -\infty < u < \infty$

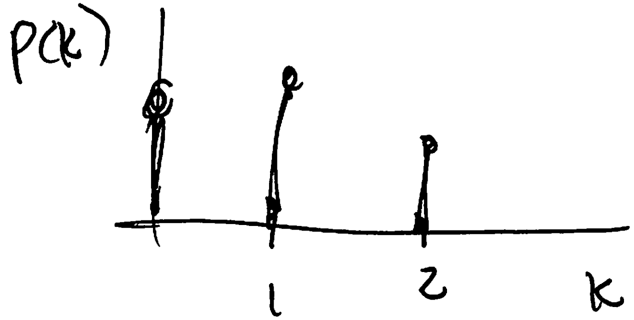


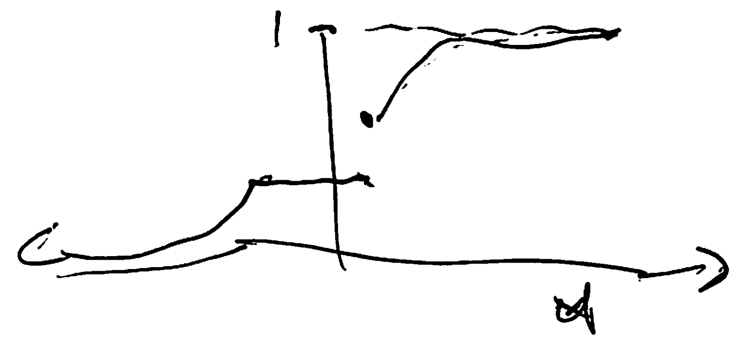
u continuous parameter

staircase plot



← stem plot





Expected Value – probabilistic average

$$EX = E(X) = \sum_k k\, p(k)$$ prob weighted average

$$E(X^2) = \sum_k k^2 p(k) = \sum_{k=0}^{\infty} k^2 p(k)$$

$$E(e^{uX}) = \sum_k e^{uk} p(k)$$

$$E(g(X)) = \sum_k g(k)\, p(k)$$ g = function

Bernoulli RV

$p(0) = 1-p \qquad p(1) = p \qquad 0 \leq p \leq 1$

$EX = 0 \cdot (1-p) + 1(p) = p$

$EX^2 = 0^2 (1-p) + 1^2 p = p$

Uniform $\quad p(k) = \begin{cases} \frac{1}{m} & k = 1, 2, \dots, m \\ 0 & \text{o.w.} \end{cases}$

$EX = 1 \cdot \frac{1}{m} + 2 \cdot \frac{1}{m} + 3 \cdot \frac{1}{m} + \dots m \cdot \frac{1}{m} = \frac{1}{m} \sum_{k=1}^{m} k = \frac{1}{\cancel{m}} \frac{\cancel{m}(m+1)}{2}$

$EX^2 = 1^2 \cdot \frac{1}{m}\ 2^2 \frac{1}{m} \cdot \cdots\ m^2 \frac{1}{m} = \frac{1}{m} \sum_{k=1}^{m} k^2 = \frac{1}{m} \frac{(2m+1) \sim}{6}$?

Poisson

$$P(k) = P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, 1, 2, \ldots \qquad \lambda > 0 \text{ is } \cancel{p} \text{ [illegible]}$$

$$EX = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \qquad \begin{array}{l} l = k-1 \\ k = l+1 \end{array}$$

$$= e^{-\lambda} \sum_{l=0}^{\infty} \frac{\lambda^l \lambda}{l!} = \lambda e^{-\lambda} \underbrace{\sum_{l=0}^{\infty} \frac{\lambda^l}{l!}}_{e^{\lambda}} = \lambda$$

EX = mean $= \mu$ = center of mass

$E(X-\mu)^2$ = Variance = moment of inertia $= \sigma^2$

Ex. Bernoulli: $\mu = p$

$$\sigma^2 = E(X-\mu)^2 = (0-p)^2(1-p) + (1-p)^2 p$$

$$= \cancel{p^2} - \cancel{p^3} + p - \cancel{2}p^2 + \cancel{p^3} = p - p^2 = p(1-p)$$

Ex. Uniform

$$\sigma^2 = \sum_{k=1}^{m} \left(k - \frac{m+1}{2}\right)^2 \cdot \frac{1}{m} = \frac{(m-1)(m+1)}{12}$$

Properties of Expected Values

1. $g(X) \geq 0$ for all X (or $g(k) \geq 0$ for all k)

   then $Eg(X) \geq 0$

   Eg. $\sigma^2 = E(X-\mu)^2 \geq 0$

2. $E(g_1(X) + g_2(X)) = Eg_1(X) + Eg_2(X)$

3. $E(g_1(X)g_2(X)) \neq Eg_1(X)\, Eg_2(X)$ in general

   $\Rightarrow E(X^2) \neq (EX)^2$ in general

4. $\boxed{\sigma^2 \underset{\text{def}}{=} E(X-\mu)^2} \underset{\text{theorem}}{=} \boxed{E(X^2) - \mu^2}$

$$\sigma^2 = E(X^2 - 2\mu X + \mu^2) = E(X^2) - E(2\mu X) + E(\mu^2)$$
$$= E(X^2) - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$$

5. $E(a g(X)) = a\, E\, g(X)$

6. $E(\text{const}) = \text{const}$