SOLUTIONS TO FINAL EXAM (5/17/18)

#1. a.) periodic if $x(t+T) = x(t)$ for all t
or $x[n+N] = x[n]$ for all n

(i.) $x[n] = (-j)^n$

n	$x[n]$
0	1
1	$-j$
2	-1
3	j
4	1
5	$-j$
⋮	⋮

Clearly, we have $x[n] = x[n+4]$ for all n. Thus, $x[n]$ is periodic. The fundamental period is $N_0 = 4$, and the fundamental radian frequency is

$$\omega_0 = 2\pi/N_0 = \pi/2$$

– Alternatively, $-j = e^{-j\pi/2}$ (Euler's)

$$x[n] = e^{-j\frac{\pi}{2}n} = e^{j\omega_0 n}$$

This is periodic if $|\omega_0|$ is rational. In this case, $|\omega_0|/2\pi \overset{2\pi}{=} 1/4$. So, this function is periodic.

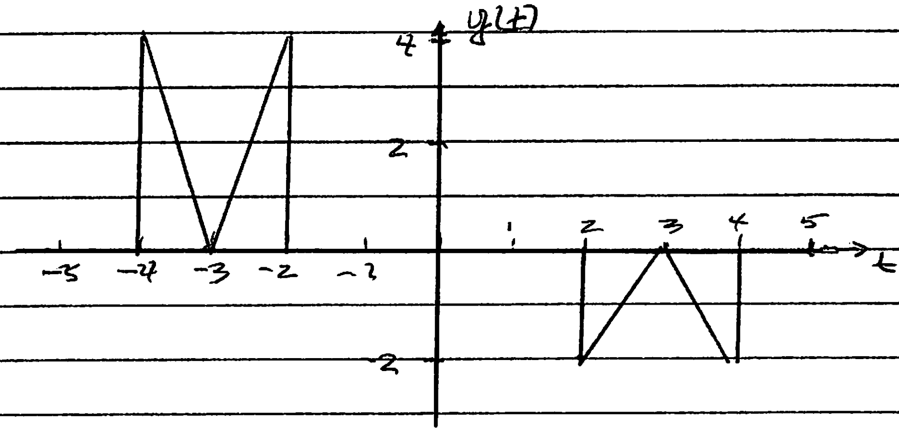
(ii) $x(t) = \dfrac{\sin \pi t}{\pi t}$ periodic / decaying amplitude

To be periodic, $x(t+T) = x(t)$ for all t. Clearly, this function is not periodic since the amplitude is decaying with time.

F1. cont'd) b.) (i) $x(t)\delta(t-0.5) = x(0.5)\delta(t-0.5)$
$= 1 \quad \delta(t-0.5)$
$= \delta(t-0.5)$

(ii) $\int_{-\infty}^{\infty} x(t)\delta(t-0.5)\,dt = \int_{-\infty}^{\infty} \overset{1}{\cancel{x(0.5)}}\delta(t-0.5)\,dt$
$= \int_{-\infty}^{\infty} \delta(t-0.5)\,dt$
$= \underline{1}$

(iii) The output $y(t)$ is
$y(t) = x(t) * h(t)$
$= x(t) * [2\delta(t+3) - \delta(t-3)]$
$= 2x(t+3) - x(t-3)$



c.) impulse response = response of the system to an input which is an impulse

$h[n] = y[n]\Big|_{x[n]=\delta[n]}$
$= \alpha\,\delta[n-1] + \beta\,\delta[n-2]$

#1. cont'd) d] (i) causal ⇒ "no output before input applied"
↳ this is true for all three test cases

So, the system could be causal.

(ii) linear ⇒ "linear combination of inputs produces the same linear combination of the individual outputs"

$$x_3(t) = x_1(t) + x_1(t-a)$$

But

$$y_3(t) \neq y_1(t) + y_1(t-a)$$

So, the system cannot be linear.

#2. impulse $\delta(t)$ → $\boxed{\text{system}}$ → impulse response $h(t)$

LTI

causal ⇒ $h(t) = 0,\ t < 0$

$$\therefore\ h(t) = e^{-t}u(t)$$

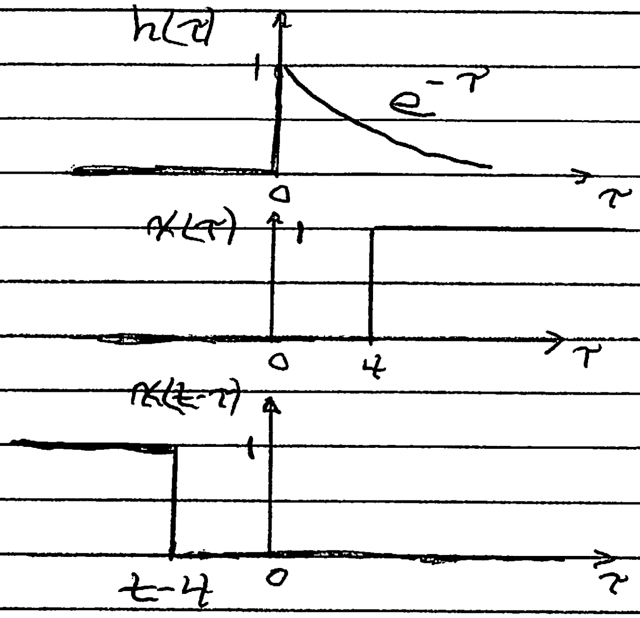
What is output $y(t)$ for input $x(t) = u(t-4)$?

↳ convolution $y(t) = x(t) * h(t)$

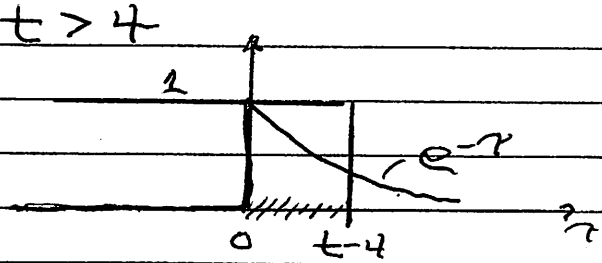
$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

(of course, you could flip and shift $x(t)$ or $h(t)$)

(2. cont'd)



- $t-4<0 \Rightarrow t<4$ no overlap $y(t)=0$
- $t>4$



$$y(t) = \int_0^{t-4} 1 \cdot e^{-\tau}\, d\tau$$
$$= -e^{-\tau}\Big|_0^{t-4}$$
$$= 1 - e^{-(t-4)}$$

$$\therefore \quad y(t) = \left[1 - e^{-(t-4)}\right] u(t-4)$$

#3. a.) dc value = average value

$= \int_{-\infty}^{\infty} x(t)\,dt$

$\omega = 0$

$= \int x(t) e^{-j\omega t}\,dt \Big|_{\omega=0}$

$= X(j0) = 2$

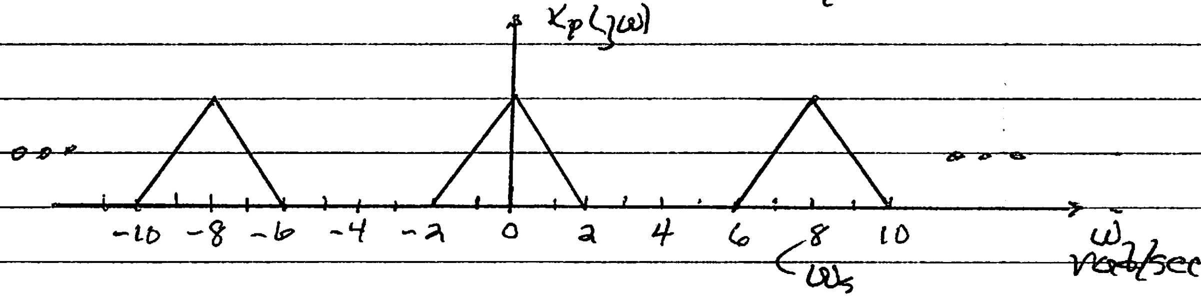
b.) • Nyquist rate = $2 \times \underbrace{\text{bandwidth of } x(t)}_{\text{max freq in } X(j\omega)}$

$= 2 \times 2$ radians/sec

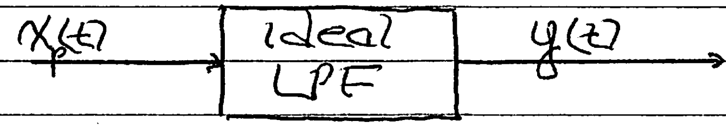
$= 4$ radians/sec

• Sampled at $2 \times$ Nyquist rate $= 8$ rads/sec

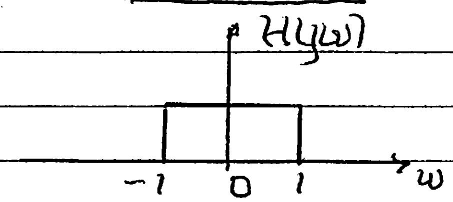
$\omega_s = 8$ radians/sec

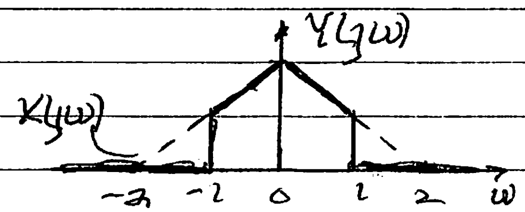


c.)



$Y(j\omega) = X(j\omega)H(j\omega)$





ω_M for $Y(j\omega)$ is 1 rad/sec

$\therefore \omega_s = 2\omega_M$

$= 2$ rad/sec

#4. a)

$$X(j\omega) = \underbrace{\frac{2\sin(\omega-2)}{\omega-2}}_{X_1(j\omega)} * \underbrace{\frac{e^{-2j\omega}\sin 2\omega}{\omega}}_{X_2(j\omega)}$$

Since convolution in frequency is multiplication in time (see Property 4.5), $x(t) = 2\pi x_1(t) x_2(t)$

Let $Y_1(j\omega) = \frac{2\sin\omega}{\omega}$, then $X_1(j\omega) = Y_1(j(\omega-2))$, and, using Property 4.3.6, $x_1(t) = y_1(t)e^{j2t}$.
Inverting $Y_1(j\omega)$, we get

$$y_1(t) = \begin{cases} 1, & |t|<1 \\ 0, & |t|>1 \end{cases}$$

Similarly, let $Y_2(j\omega) = \frac{2\sin 2\omega}{\omega}$; then $X_2(j\omega) = \frac{1}{2}e^{-2j\omega}Y_2(j\omega)$, and using Property 4.3.2, $x_2(t) = \frac{1}{2}y_2(t-2)$.
Inverting $Y_2(j\omega)$, we get

$$y_2(t) = \begin{cases} 1, & |t|<2 \\ 0, & |t|>2 \end{cases}$$

Therefore,

$$x(t) = 2\pi x_1(t) x_2(t)$$

$$= \left[2\pi y_1(t)e^{j2t}\right]\cdot\left[\frac{1}{2}y_2(t-2)\right]$$

$$= \begin{cases} \pi e^{j2t}, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$$

4 (cont'd) b) (i) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$X(e^{j0}) = X(e^{j\omega})\Big|_{\omega=0} = \sum_{n=-\infty}^{\infty} x[n]$$

$$= (1) + \cancel{(-2)} + (2) + \cancel{(1)} + \cancel{(-2)} + \cancel{(2)} + (2) + \cancel{(-1)} + \cancel{(2)}$$

$$= 5$$

(ii) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \overset{\text{Parseval}}{=} 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$

$$= 2\pi\left[1+4+4+1+4+4+4+1+4\right]$$

$$= 54\pi$$

c) To solve this problem, use the Initial and Final Value Theorems for Laplace Transforms

(i) $h(0^+) = \lim_{s\to\infty} sH(s)$

$$= \lim_{s\to\infty} \frac{\cancel{s}\cdot 10(2s+3)}{\cancel{s}(s^2+2s+5)}$$

$$= 0$$

(ii) $\lim_{t\to\infty} h(t) = \lim_{s\to 0} sH(s)$

$$= \lim_{s\to 0} \frac{\cancel{s}\cdot 10(2s+3)}{\cancel{s}(s^2+2s+5)}$$

$$= \frac{10(3)}{5} = \frac{30}{5}$$

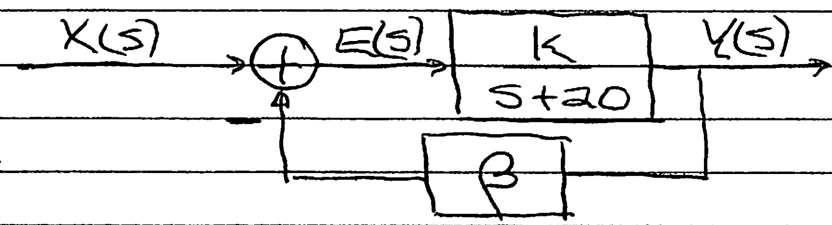
$$= 6$$

#4 cont'd) d.) For the Fourier transform to exist, the $j\omega$-axis must be included in the ROC; also, since the Laplace transform is rational, the ROC does not contain any poles.

(i) left-sided signal?
If $x(t)$ is a left-sided signal, the ROC must be a left-half plane (LHP) and to the left of the leftmost pole, i.e. $Re\{s\} < -1/2$. So, this ROC would not include the $j\omega$-axis. Since we know it does, $x(t)$ cannot be a left-sided signal.

(ii) right-sided signal?
If $x(t)$ is a right-sided signal, the ROC must be a RHP and to the right of the rightmost pole, i.e., if this is the only pole, $Re\{s\} > -1/2$. So, this ROC could include the $j\omega$-axis and $x(t)$ can be a right-sided signal. (Any other pole would either be negative and greater than $-1/2$, or on the RHP and then the function would be two-sided.)

e.)



$$Y(s) = \frac{K}{s+20} E(s)$$

$$E(s) = X(s) - \beta Y(s)$$

e. cont'd)

$$Y(s)\left(1+\frac{K\beta}{s+20}\right)=\frac{K}{s+20}X(s)$$

$$H(s)=\frac{K}{s+20+\beta K}$$

To be stable (system is causal and $H(s)$ is rational), all poles must be in the LHP.

$$\therefore \text{pole} = s_0 = -20-\beta K < 0$$

$$-\beta K < 20$$

$$K > -\frac{20}{\beta}$$

f.) We know

$$\frac{\sin Wt}{\pi t} \xleftrightarrow{\mathcal{F}} \begin{cases} 1, & |\omega|<W \\ 0, & |\omega|>W \end{cases}$$

Therefore,

$$\left(\frac{\sin Wt}{\pi t}\right)^4 \longleftrightarrow \cancel{X(j\omega)*X(j\omega)*X(j\omega)*X(j\omega)}$$

The bandwidth of $\frac{\sin Wt}{\pi t}$ is $W = 100\pi$ rad/sec. So, the bandwidth of $\left(\frac{\sin Wt}{\pi t}\right)^4$ is $\omega_M = 400\pi$ rad/sec. According to the Sampling Theorem, the Nyquist rate is twice the highest frequency in the signal, $2\omega_M$. Therefore, the Nyquist rate is 800π rad/sec or in Hz, 400 Hz $(f_s = \omega_s/2\pi)$.

#5.

$$H(j\omega) = \frac{j\omega}{(j\omega+1)(j\omega-4)}$$

a.) $H(j\omega) = \dfrac{A}{j\omega+1} + \dfrac{B}{j\omega-4}$

$$A = H(j\omega)(j\omega+1)\Big|_{j\omega=-1} = \frac{j\omega}{j\omega-4} = \frac{-1}{-5} = \frac{1}{5}$$

$$B = H(j\omega)(j\omega-4)\Big|_{j\omega=4} = \frac{j\omega}{j\omega+1} = \frac{4}{5} = \frac{4}{5}$$

$$\therefore\ h(t) = \frac{1}{5}e^{-t}u(t) + \frac{4}{5}e^{4t}u(t)$$

b.) Only memoryless if $h(t) = K\delta(t)$. This system clearly has memory.

c.) Causal $\Rightarrow$ $h(t)=0$, $t<0$. This is true. So, this system is causal.

#6

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}}$$

a.) $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}}$

$$Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}\right) = X(e^{j\omega})$$

$$\Downarrow \mathcal{F}^{-1}$$

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$

#6 cont'd) b.) impulse response, $h[n]$?

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}}$$

$$= \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{4}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{4}e^{-j\omega}}$$

$$A = H(e^{j\omega})(1-\frac{1}{2}\underbrace{e^{-j\omega}}_{v})\Big|_{v=2} = \frac{1}{1+\frac{1}{4}v}\Big|_{v=2} = \frac{2}{3}$$

$$B = H(e^{j\omega})(1+\frac{1}{4}\underbrace{e^{-j\omega}}_{v})\Big|_{v=-4} = \frac{1}{1-\frac{1}{2}v}\Big|_{v=-4} = \frac{1}{3}$$

$$H(e^{j\omega}) = \frac{2/3}{1-\frac{1}{2}e^{-j\omega}} + \frac{1/3}{1+\frac{1}{4}e^{-j\omega}}$$

$$\Big\Downarrow \mathcal{F}^{-1}$$

$$\therefore h[n] = \frac{2}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(-\frac{1}{4}\right)^n u[n]$$

#7.

$$H(s) = \frac{-s-4}{s^2+3s+2}$$

$$= \frac{-s-4}{(s+2)(s+1)}$$

a) poles $\Rightarrow H(s) \to \infty$ $\quad s=-1,\ s=-2$

zeros $\Rightarrow H(s) \to 0$ $\quad s=-4,\ s=\infty$

#7. cont'd) b.) (i) left of leftmost pole $\text{Re}\{s\} < -2$

(ii) right of rightmost pole $\text{Re}\{s\} > -1$

(iii) bounded by poles $-2 < \text{Re}\{s\} < -1$

c.) $$H(s) = \frac{Y(s)}{X(s)} = \frac{-s-4}{s^2+3s+2}$$

$$Y(s)(s^2+3s+2) = X(s)(-s-4)$$

$$\Downarrow \mathcal{L}^{-1}$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)$$
$$= -\frac{dx(t)}{dt} - 4x(t)$$

#8. (The first part of this problem was part of Problem #2 on Exam #2.)

(a.) $$y(t) = x(t)\cos\omega_0 t$$
$$= x(t)\left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right)$$
$$= \frac{x(t)}{2}e^{j\omega_0 t} + \frac{x(t)}{2}e^{-j\omega_0 t}$$

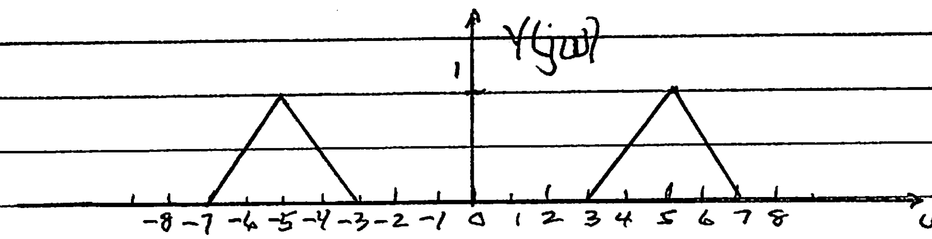
Use the frequency shift property (4.3.6)

$$e^{j\omega_0 t}x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

$$\therefore Y(j\omega) = \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$$

shift up in frequency — shift down in frequency

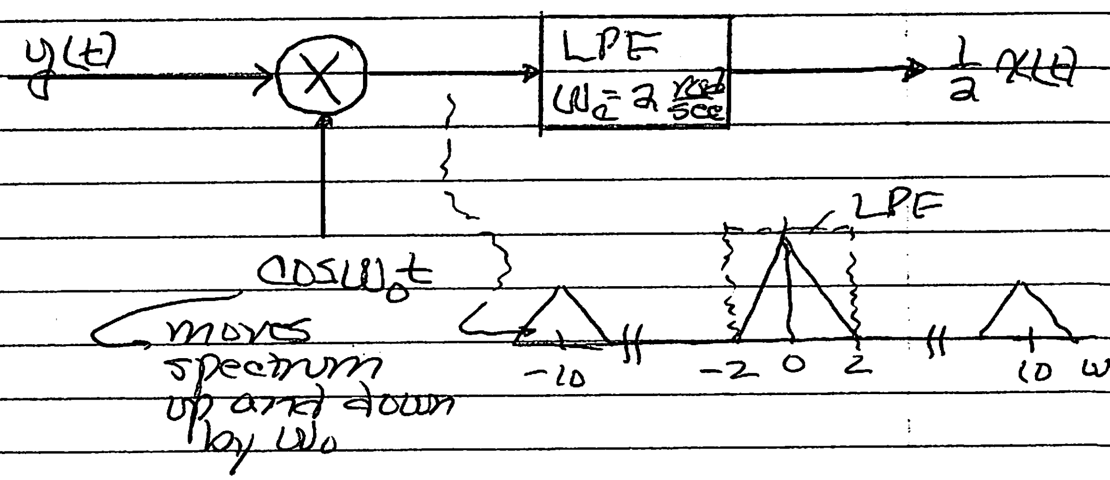
8a. cont'd)



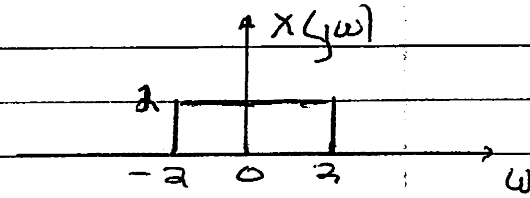
b.) At the receiver side, we can recover $x(t)$ simply by multiplying $y(t)$ by $\cos\omega_0 t$ (moving the spectrum up and down again). Then, a lowpass filter with the cutoff frequency bigger than 2 rad/sec but smaller than ω_0 is used.

In the time domain,

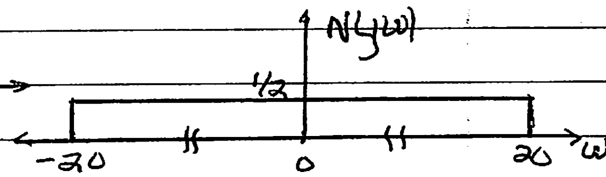
$$
\begin{aligned}
s(t) &= y(t)\cos\omega_0 t\Big|_{LPF} \\
&= \left[x(t)\cos^2\omega_0 t\right]\Big|_{LPF} \\
&= \left(\frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_0 t\right)\Big|_{LPF} \\
&= \frac{1}{2}x(t)
\end{aligned}
$$



#9 desired signal $x(t) \overset{\mathcal{F}}{\longleftrightarrow}$



noise signal $n(t) \overset{\mathcal{F}}{\longleftrightarrow}$



a.) average noise power $= \int_{-\infty}^{\infty} |n(t)|^2 dt \overset{\text{Parseval}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} |N(j\omega)|^2 d\omega$

$$= \frac{1}{2\pi}\left(\frac{1}{4} \cdot 40\right)$$

$$= \frac{5}{\pi}$$

b.) measured signal $y(t) = x(t) + n(t)$

$x(t)$: has bandwidth from $\omega = -2$ to 2

$n(t)$: has bandwidth from $\omega = -20$ to 20

So, if we LPF $y(t)$ with a cutoff at $\omega_c = 2$ rad/sec, the signal will pass unaffected, while the noise power will be reduced to

$$\frac{1}{2\pi} \int_{-2}^{2} \left(\frac{1}{4}\right) d\omega = \frac{1}{2\pi} \cdot 1 = \frac{1}{2\pi}$$

∴ The noise power has been reduced by a factor of 10 (10 dB)