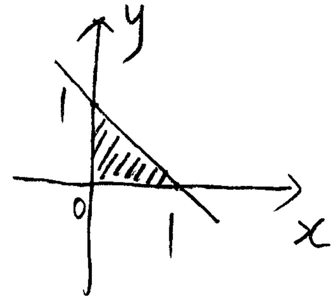
Solution to HW8

8.2 (a). Since $f_{XY}(x,y)$ is a joint distribution,

(1 point)

$$\iint f_{XY}(x,y)\,dx\,dy = 1$$




$$\iint c x^2 y \, dx\, dy = 1$$

$$\int_0^1 \int_0^{1-x} c x^2 y \, dy\, dx = 1$$

$$\int_0^1 \left[\tfrac{1}{2} c x^2 y^2\right]_0^{1-x} dx = 1 \Rightarrow \int_0^1 \left\{\tfrac{1}{2} c x^2 (1-x)^2\right\} dx = 1$$

$$\Rightarrow \frac{c}{2}\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5\right]_0^1 = 1$$

$$\Rightarrow C = 60$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$$= \int_0^{1-x} c x^2 y \cdot dy = \left[\tfrac{1}{2} c x^2 y^2\right]_0^{1-x} = \frac{c}{2} x^2 (1-x)^2 = 30 x^2 (1-x)^2$$

$0 \le x \le 1$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$$= \int_0^{1-y} c x^2 y \, dx = \left[\tfrac{1}{3} c \cdot x^3 \cdot y\right]_0^{1-y} = \frac{c}{3} y \cdot (1-y)^3 = 20 \cdot y (1-y)^3$$

$0 \le y \le 1$

(c) $E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx = \int_0^1 30 x^3 (1-x)^2 dx = \frac{1}{2}$

$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 20 y^2 (1-y)^3 dy = \frac{1}{3}$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \frac{2}{7}$$

$$E(Y^2) = \int_0^1 y^2 \cdot f_Y(y)\,dy = \frac{1}{7}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{7} - \left(\frac{1}{2}\right)^2 = \frac{1}{28}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{7} - \left(\frac{1}{3}\right)^2 = \frac{2}{63}$$

(c). $E(XY) = \int_0^1 \int_0^{1-x} xy \cdot f_{XY}(x, y)\,dx\,dy = \frac{1}{7}$

Since $E(XY) \neq E(X) \cdot E(Y)$, $X$ and $Y$ are dependent.

8.3. Since $X$ and $Y$ are IID uniform random variables, we have

(1 point)

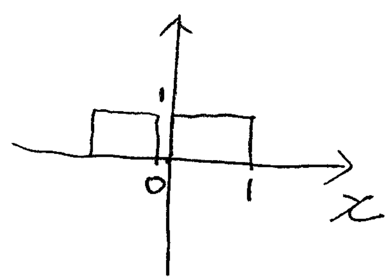
$$f_Z(z) = f_X * f_Y$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z - x) \cdot dx.$$

There are 4 cases for calculating $f_Z(z)$

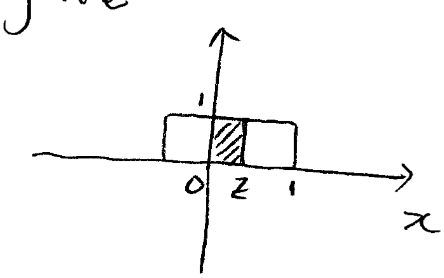
(i) $Z < 0$, there is no overlap between $f_X(x)$ and $f_Y(z-x)$

$f_Z(z) = 0$

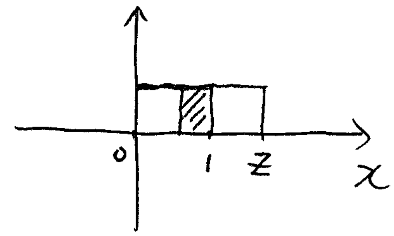



(ii) $0 < Z \leq 1$ The overlap is depicted by the right figure, so

$f_Z(z) = \int_0^z 1 \cdot 1 \cdot dx = z$



(iii) $1 < z \le 2$ The overlap is depicted by the right figure, so

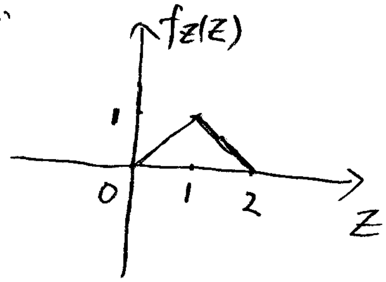



$$f_Z(z) = \int_{z-1}^{1} 1 \cdot 1 \, dx = 1-(z-1) = 2-z$$

(iv) $z > 2$. There is no overlap, so

$$f_Z(z) = 0$$

Therefore $f_Z(z)$ can be illustrated by the figure below




$$E(Z) = \int_0^2 z \cdot f_Z(z)\, dz = \int_0^1 z \, z \cdot dz + \int_1^2 z(2-z) \cdot dz$$

$$= \left[\tfrac{1}{3}z^3\right]_0^1 + \left[z^2 - \tfrac{1}{3}z^3\right]_1^2$$

$$= \tfrac{1}{3} + \left[\left(4-\tfrac{8}{3}\right) - \left(1-\tfrac{1}{3}\right)\right]$$

$$= \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{2}{3} = 1$$

$$E(Z^2) = \int_0^2 z^2 f_Z(z)\, dz = \int_0^1 z^2 \cdot z \, dz + \int_1^2 z^2(2-z)\, dz$$

$$= \left[\tfrac{1}{4}z^4\right]_0^1 + \left[\tfrac{2}{3}z^3 - \tfrac{1}{4}z^4\right]_1^2$$

$$= \tfrac{1}{4} + \left[\left(\tfrac{2}{3}\cdot 8 - \tfrac{1}{4}\cdot 16\right) - \left(\tfrac{2}{3} - \tfrac{1}{4}\right)\right]$$

$$= \tfrac{1}{4} + \tfrac{11}{12} = \tfrac{7}{6}$$

$$Var(Z) = E(Z^2) - E(Z)^2 = \tfrac{7}{6} - 1 = \tfrac{1}{6}$$

8.6 Since $X$ and $Y$ are independent,

(2 point) $$f_Z(z) = f_X(x) * f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(z-y)\, dy$$

Besides $f_X(z-y) \geq 0$ for $0 \leq z-y \leq 1 \Rightarrow z-1 \leq y \leq z$

$f_Y(y) \geq 0$ for $y \geq 0$

Therefore, we have 3 cases.

(i) $z<0$, $(z-1 \leq y \leq z) \cap (y \geq 0) = \phi$,

Thus $f_Z(z) = 0$

(ii) $0 \leq z < 1$ $(z-1 \leq y \leq z) \cap (y \geq 0) = 0 \leq y \leq z$

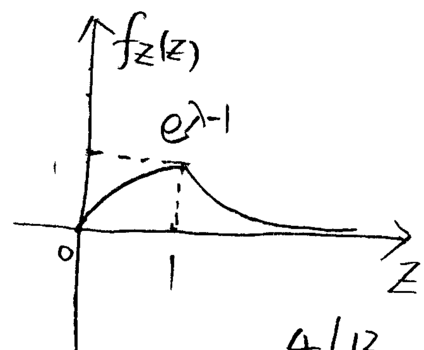
Thus $f_Z(z) = \int_0^z \lambda e^{-\lambda y} dy = \left[-e^{-\lambda y}\right]_0^z = 1 - e^{-\lambda z}$

(iii) $z > 1$ $(z-1 \leq y \leq z) \cap (y \geq 0) = z-1 \leq y \leq z$

Thus $f_Z(z) = \int_{z-1}^{z} \lambda e^{-\lambda y} dy = \left[-e^{-\lambda y}\right]_{z-1}^{z}$

$$= \left[e^{-\lambda(z-1)} - e^{-\lambda z}\right]$$

$$= e^{-z}(e^{\lambda} - 1)$$




Overall, the density of $Z$ is similar as the right figure.

$$E(Z) = \int_{-\infty}^{\infty} z \cdot f_Z(z) \, dz$$

$$= \int_0^1 z(1-e^{-\lambda z})dz + \int_1^{\infty} z\{e^{-\lambda z}(e^{\lambda}-1)\} \cdot dz$$

$$= \int_0^1 z \cdot dz - \int_0^1 z e^{-\lambda z} dz + (e^{\lambda}-1) \cdot \int_1^{\infty} z e^{-\lambda z} \cdot dz$$

part (i) $\int_0^1 z\,dz = \frac{1}{2}z^2\Big|_0^1 = \frac{1}{2}$

part (ii) $\int_0^1 z e^{-\lambda z} dz = (-\frac{1}{\lambda}) \cdot \int_0^1 z\, de^{-\lambda z}$

$$= (-\frac{1}{\lambda}) \cdot \left\{ z \cdot e^{-\lambda z}\Big|_0^1 - \int_0^1 e^{-\lambda z} dz \right\}$$

$$= (-\frac{1}{\lambda}) \cdot \left\{ e^{-\lambda} - \int_0^1 e^{-\lambda z} dz \right\}$$

$$= (-\frac{1}{\lambda}) \cdot \left\{ e^{-\lambda} - (-\frac{1}{\lambda}) \cdot e^{-\lambda z}\Big|_0^1 \right\}$$

$$= (-\frac{1}{\lambda}) \left\{ e^{-\lambda} + \frac{1}{\lambda} \cdot (e^{-\lambda}-1) \right\}$$

$$= -\frac{1}{\lambda}e^{-\lambda} - \frac{1}{\lambda^2}e^{-\lambda} + \frac{1}{\lambda^2}$$

part (iii) $\int_1^{\infty} z e^{-\lambda z} dz = \int_1^{\infty} (-\frac{1}{\lambda}) z\, de^{-\lambda z}$

$$= (-\frac{1}{\lambda}) \cdot \left\{ z e^{-\lambda z}\Big|_1^{\infty} - \int_1^{\infty} e^{-\lambda z} dz \right\}$$

$$= (-\frac{1}{\lambda}) \left\{ z \cdot e^{-\lambda z}\Big|_1^{\infty} - (-\frac{1}{\lambda}) e^{-\lambda z}\Big|_1^{\infty} \right\}$$

$$= (-\frac{1}{\lambda}) \left\{ -e^{-\lambda} + \frac{1}{\lambda} \cdot (-e^{-\lambda}) \right\}$$

$$= \frac{1}{\lambda} \left\{ (1+\frac{1}{\lambda}) \cdot e^{-\lambda} \right\}$$

$$= (\frac{1}{\lambda} + \frac{1}{\lambda^2}) e^{-\lambda}$$

$$E(Z) = \frac{1}{2} + \frac{1}{\lambda}e^{-\lambda} + \frac{1}{\lambda^2}e^{-\lambda} - \frac{1}{\lambda^2} + (e^{\lambda}-1)\cdot\left(\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)e^{-\lambda}$$

$$= \frac{1}{2} + \frac{1}{\lambda}e^{-\lambda} + \frac{1}{\lambda^2}e^{-\lambda} - \frac{1}{\lambda^2} + (1-e^{-\lambda})\left(\frac{1}{\lambda}+\frac{1}{\lambda^2}\right)$$

$$= \frac{1}{2} + \underline{\frac{1}{\lambda}e^{-\lambda}} + \frac{1}{\lambda^2}e^{-\lambda} - \frac{1}{\lambda^2} + \frac{1}{\lambda} + \frac{1}{\lambda^2} - \underline{\frac{1}{\lambda}e^{-\lambda}} - \frac{1}{\lambda^2}e^{-\lambda}$$

$$= \frac{1}{2} + \frac{1}{\lambda}$$

To direct calculate the variance of $Z$ is not easily. Let's consider an alternative method

Assuming $X_1 \cdots X_n$ be random variables, then

$$Var\left[\sum_{i=1}^{n} X_i\right] = E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left[E\left(\sum_{i=1}^{n} X_i\right)\right]^2$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right] - \left(\sum_{i=1}^{n} E(X_i)\right)^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i X_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i)\,E(X_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(E(X_i X_j) - E(X_i)\cdot E(X_j)\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} conv(X_i\, X_j)$$

We can find that if the variables are uncorrelated, namely that $con(X_i\, X_j) = 0$ for $i \neq j$. Therefore

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var(X_i)$$

Since X and Y are independent, we have

$$Var(Z) = Var(X+Y) = Var(X) + Var(Y)$$

Since $X \sim U(0,1)$ $\quad Var(X) = \frac{1}{12}(1-0)^2 = \frac{1}{12}$

Since Y is exponential distribution

$$Var(Y) = \frac{1}{\lambda^2}$$

Therefore $Var(Z) = \frac{1}{12} + \frac{1}{\lambda^2}$

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z)\,dz$$

$$= \begin{cases} \int_0^z f_Z(z)dz = \int_0^z (1-e^{-\lambda z})dz & 0 < z \leq 1 \\ \int_0^z f_Z(z)dz = \int_0^1 (1-e^{-\lambda z})dz + \int_1^z e^{-\lambda z}(e^{\lambda}-1)\,dz & z > 1 \end{cases}$$

$$= \begin{cases} z + \frac{1}{\lambda}e^{-\lambda z} - \frac{1}{\lambda}\,; & 0 < z \leq 1 \\ 1 - \frac{1}{\lambda}e^{-\lambda(z-1)} + \frac{1}{\lambda}e^{-\lambda z}\,; & z > 1 \end{cases}$$

8.9 Based on the conclusion we derived from 8.6 we have
(1 point)

$$E(S) = n \cdot \mu$$

$$Var(S) = n \cdot \sigma^2$$

8.17 Since the density is uniform $f_{XY}(x,y) = C$, Besides, we have
(1 point) (a)

$$F_{XY}(\infty\ \infty) = \iint_D f_{XY}(x,y)\,dx\,dy = 1$$

Where D represents the shaded semicircle.

Therefore $\iint_D C \cdot dx\,dy = 1$

Sine the area of the semicircle is equal to $\frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2}$

Therefore $\frac{\pi}{2} C = 1 \Rightarrow C = \frac{2}{\pi}$

$$f_{XY}(x,y) = \begin{cases} \frac{2}{\pi} & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

(b). $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$$= \int_0^{\sqrt{1-x^2}} \frac{2}{\pi} \cdot dy = \frac{2}{\pi} \cdot \sqrt{1-x^2} \qquad -1 \le x \le 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{XY}(x,y)\,dx = \frac{4}{\pi} \cdot \sqrt{1-y_2} \qquad 0 \le y \le 1$$

(c) $f_{XY}(0,0) = \frac{2}{\pi}$

$f_X(0) = \frac{2}{\pi}$

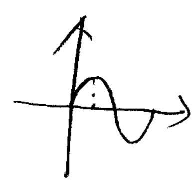
$f_Y(0) = \frac{4}{\pi}$

$f_{XY}(0,0) \neq f_X(0) f_Y(0)$ so X and Y are dependent

(d) $E(X) = \int_{-1}^{1} x \cdot \frac{2}{\pi} \sqrt{1-x^2}\, dx$

Since $g(x) = x \cdot \frac{2}{\pi} \cdot \sqrt{1-x^2}$ is an odd function. i.e. $g(x) = -g(-x)$

Therefore $E(X) = 0$



$$E(Y) = \int_0^1 y \cdot \frac{4}{\pi} \cdot \sqrt{1-y^2}\, dy$$

$$= \frac{4}{\pi} \int_0^1 y \cdot \sqrt{1-y^2}\, dy$$

$$= \frac{2}{\pi} \int_0^1 \sqrt{1-y^2}\, dy^2$$

$$= \frac{2}{\pi} \int_0^1 (1-y^2)^{1/2}\, dy^2$$

$$= \frac{2}{\pi} \int_0^1 (1-y')^{1/2}\, dy', \quad y' = y^2$$

$$= \frac{2}{\pi} \cdot \left(-\frac{2}{3}\right) \cdot (1-y')^{\frac{3}{2}} \Big|_0^1$$

$$= \left(-\frac{4}{3\pi}\right)(-1)$$

$$= \frac{4}{3\pi}$$

8.18. (a) $f_{X|Y}(x|Y=y) = \dfrac{f_{XY}(x, Y=y)}{f_Y(Y=y)}$

(1 point)

$$= \frac{2}{\pi} \cdot \frac{\pi}{4} \cdot \frac{1}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}} \quad 0 \le y \le 1$$

$$f_{Y|X}(y|X=x) = \frac{f_{XY}(X=x, y)}{f_X(X=x)}$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \quad -1 \le x \le 1$$

(b) $E_{X|Y}(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)dx$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \, f_{X|Y}(x|y)\,dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \cdot dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \cdot \frac{1}{2}x^2 \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$= 0$$

$E_{Y|X}(Y|X=x) = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x)\,dy$

$$= \int_0^{\sqrt{1-x^2}} y \cdot \frac{1}{\sqrt{1-x^2}}\,dy$$

$$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{2}y^2 \Big|_0^{\sqrt{1-x^2}}$$

$$= \frac{1}{2}\sqrt{1-x^2}$$

8.22

$$Pr\{X \geq 3\} = 1 - Pr\{X < 3\}$$

$$= 1 - (1 - e^{-\frac{1}{3}\cdot 3})$$

$$= 1 - 1 + e^{-1} = e^{-1}$$

$$Pr\{X \geq 4 | X \geq 3\} = \frac{Pr\{X \geq 4 \cap X \geq 3\}}{Pr\{X \geq 3\}}$$

$$= \frac{Pr\{X \geq 4\}}{Pr\{X \geq 3\}}$$

$$= \frac{1 - Pr\{X < 4\}}{1 - Pr\{X < 3\}} = \frac{1 - (1 - e^{-\frac{4}{3}})}{1 - (1 - e^{-1})} = \frac{e^{-\frac{4}{3}}}{e^{-1}} = e^{-\frac{1}{3}}$$

$$E\{X | X \geq 3\} = \int_3^{\infty} x \cdot f(X | x \geq 3) \, dx$$

$$F_X(X | x \geq 3) = \frac{Pr\{X \leq x \cap X \geq 3\}}{Pr\{X \geq 3\}} = \frac{Pr\{3 \leq X \leq x\}}{Pr\{X \geq 3\}} = \frac{e^{-3\lambda} - e^{-x\lambda}}{e^{-3\lambda}}$$

$$= 1 - e^{-(x-3)\lambda}$$

$$f_X(X | X \geq 3) = \lambda e^{-(x-3)\lambda} \quad \text{for } x \geq 3$$

$$E\{X | X \geq 3\} = \int_3^{\infty} x \, \lambda e^{-(x-3)\lambda} \, dx$$

$$= \lambda \cdot e^{3\lambda} \int_3^{\infty} x \cdot e^{-\lambda x} \, dx$$

$$= \lambda e^{3\lambda} \left\{ (-3e^{-3\lambda}) + \frac{1}{\lambda}(-e^{-3\lambda}) \right\} \cdot \left(-\frac{1}{\lambda}\right)$$

$$E[X|X\geq 3] = e^{3\lambda}\left\{ 3e^{-3\lambda} + \frac{1}{\lambda}e^{-3\lambda}\right\}$$

$$= 3 + \frac{1}{\lambda} = 6$$

8.28 (a) $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ $\quad E_X(x) = np \quad Var(x) = np(1-p)$

(2 points) $f_Y(Y) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad E_Y(Y) = \frac{1}{2} \quad Var(Y) = \frac{1}{12}$

Since X and Y are independent

$$E(Z) = E(X) + E(Y) = np + \frac{1}{2}$$

$$Var(Z) = Var(X) + Var(Y) = np(1-p) + \frac{1}{12}$$

(b) $f_Z(Z) = f_X(x) * f_Y(y)$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx$$

Since X is discrete, namely $X = X_k$, $k \in \{0, 1, \cdots n\}$

$$f_Z(z) = \int_{-\infty}^{\infty} \sum_k f_X \delta(x - x_k) \cdot f_Y(z - x_k)\,dx$$

$$= \sum_k f_X(x_k) \cdot f_Y(z - x_k)$$

We need to guarantee $0 \leq Z - X_k \leq 1$. so $X_k \leq Z \leq X_k + 1$

Therefore $0 \leq Z \leq n+1$

$$f_Z(z) = \binom{n}{k} p^k (1-p)^{n-k} \quad k \leq z \leq k+1, \quad k = \{0, 1, \cdots n\}$$

$$F_Z(z) = \sum_{k=0}^{k-1} \binom{n}{k} p^k (1-p)^{n-k} + (z-k)\binom{n}{k} p^k (1-p)^{n-k}$$

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\,dz$$

$$= \sum_{k=0}^{n} \int_{k}^{k+1} z \cdot f_Z(z)\,dz$$

$$= \sum_{k=0}^{n} \int_{k}^{k+1} z \cdot \binom{n}{k} p^k (1-p)^{n-k}\,dz$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \int_{k}^{k+1} z\,dz$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{1}{2}\left\{(k+1)^2 - k^2\right\}$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \frac{1}{2}(2k+1)$$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{n} \frac{1}{2}\binom{n}{k} p^k (1-p)^{n-k}$$

$$= E_X(X) + \frac{1}{2} = np + \frac{1}{2}$$

$$E(Z^2) = \int_{-\infty}^{\infty} z^2 f_Z(z)\,dz$$

$$= \sum_{k=0}^{n} \int_{k}^{k+1} z^2 \binom{n}{k} p^k (1-p)^{n-k}\,dz = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \int_{k}^{k+1} z^2\,dz$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{1}{3} z^3 \Big|_{k}^{k+1} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \left(k^2 + k + \frac{1}{3}\right)$$

$$= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} + \frac{1}{3} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= E_X(X^2) + E_X(X) + \frac{1}{3}$$

Given $E(Z^2)$ and $E(Z)$

We have $Var(Z) = E(Z^2) - E(Z)^2$

$$= E_X(X^2) + E_X(X) + \frac{1}{3} - \left(E_X(X) + \frac{1}{2}\right)^2$$

$$= E_X(X^2) + E_X(X) + \frac{1}{3} - \left(E_X^2(X) + E_X(X) + \frac{1}{4}\right)$$

$$= \left\{E_X(X^2) - E_X^2(X)\right\} + \frac{1}{3} - \frac{1}{4}$$

$$= Var(X) + \frac{1}{12}$$

$$= np(1-p) + \frac{1}{12}$$

13/13