Exam#2 (Math 342)
Nov 7 2018

STUDENT NAME: ______________________________

Instructions:

The duration of the test is 50 minutes. The test consists of 4 questions and the points are specified next to each question. Total grade = 50. Detail your calculations and simplify your answers as much as possible. For problems on differential equations, whenever possible, look for a power series solution about $x_0 = 0$. No aid sheet, class notes or calculator allowed.

1) [14] Consider the orthonormal set $S = \left\{\frac{1}{\sqrt{2}}, \sin(x), \sin(2x), \sin(3x), \ldots, \sin(nx)\right\}$ in $C[-\pi, \pi]$ with inner product defined by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

(a) Check that $\|\sin(kx)\|^2 = 1$ for $k > 0$ and $\langle \sin(kx), \sin(px)\rangle = 0$ for $k \neq p$.

(b) Compute the inner products

$$\langle x, \frac{1}{\sqrt{2}}\rangle\,, \qquad \langle x, \sin(kx)\rangle \quad \text{for} \quad k > 0$$

(c) Use part (b) to express the function $h(x) = x$ in series form as a linear combination of elements of S.

Note: $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$, $\quad \sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$

Solution:

(a)

$$\begin{aligned}
\|\sin(kx)\|^2 = \langle \sin(kx), \sin(kx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 kx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big[1-\cos(2kx)\big]\,dx \\
&= \frac{1}{\pi}\left[x\big|_0^{\pi} - \frac{1}{2k}\sin(2kx)\big|_0^{\pi}\right] = 1
\end{aligned}$$

$$\begin{aligned}
\langle \sin(kx), \sin(px)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx)\ \sin(px)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big[\cos((k-p)x) - \cos((k+p)x)\big]\,dx \\
&= \frac{1}{\pi}\left[\frac{1}{k-p}\sin((k-p)x)\big|_0^{\pi} - \frac{1}{k+p}\sin((k+p)x)\big|_0^{\pi}\right] = 0
\end{aligned}$$

(b)

$$\langle x, \frac{1}{\sqrt{2}}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}x\,dx = \frac{1}{\pi\sqrt{2}}\frac{1}{2}x^2\big|_{-\pi}^{\pi} = 0$$

$$\langle x, \sin(kx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{2}{\pi}\left[-\frac{x}{k}\cos(kx)\big|_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\right] = -\frac{2}{k}\cos(k\pi) = \frac{2}{k}(-1)^{k+1}$$

(c)

$$x = \langle x, \frac{1}{\sqrt{2}}\rangle\,\frac{1}{\sqrt{2}} + \sum_{k=1}^{n}\langle x, \sin(kx)\rangle\ \sin(kx) = 2\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\ \sin(kx)$$

2) [11] Consider the orthonormal set $S = \left\{ \frac{1}{\sqrt{2}}, \cos(x), \cos(2x), \cos(3x), \cos(4x) \right\}$ in $C[-\pi, \pi]$ with inner product defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

(a) Use trigonometric identities to write $h(x) = \left[\cos^2(x) - \sin^2(x) \right]^2$ as a linear combination of elements of S.

(b) Use part (a) to find the values of the following integrals:

$$(i) \int_{-\pi}^{\pi} h(x)\, \cos(x)\, dx\,, \quad (ii) \int_{-\pi}^{\pi} h(x)\, \cos(2x)\, dx\,, \quad (iii) \int_{-\pi}^{\pi} h(x)\, \cos(3x)\, dx\,, \quad (iv) \int_{-\pi}^{\pi} h(x)\, \cos(4x)\, dx$$

Solution:

(a)

$$\begin{aligned}
h(x) = \left[\cos^2(x) - \sin^2(x) \right]^2 &= \cos^4(x) - 2\cos^2(x)\sin^2(x) + \sin^4(x) \\
&= \frac{1}{4}\left[1 + \cos(2x)\right]^2 - \frac{1}{2}\left[1 - \cos^2(2x)\right] + \frac{1}{4}\left[1 - \cos(2x)\right]^2 \\
&= \cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x) \\
&= \langle h(x), \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} + \langle h(x), \cos(4x) \rangle\, \cos(4x)
\end{aligned}$$

(b) By identification with the above result, we have

$$\int_{-\pi}^{\pi} h(x)\, \cos(x)\, dx = \pi \langle h(x), \cos(x) \rangle = 0\,, \quad \int_{-\pi}^{\pi} h(x)\, \cos(2x)\, dx = \pi \langle h(x), \cos(2x) \rangle = 0$$

$$\int_{-\pi}^{\pi} h(x)\, \cos(3x)\, dx = \pi \langle h(x), \cos(3x) \rangle = 0\,, \quad \int_{-\pi}^{\pi} h(x)\, \cos(4x)\, dx = \pi \langle h(x), \cos(4x) \rangle = \frac{\pi}{2}$$

3) [11] Find a power series solution of the differential equation

$$(x-1)\,y' + 2y = 0$$

Determine the recurrence relation for the coefficients in the power series as well as the radius of convergence and the interval of convergence for the power series.

Solution:

$$y(x) = \sum_{n=0}^{\infty} c_n\, x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n\, c_n\, x^{n-1}$$

Insert in the equation:

$$\sum_{n=1}^{\infty} n\, c_n\, x^n - \sum_{n=1}^{\infty} n\, c_n\, x^{n-1} + 2\sum_{n=0}^{\infty} c_n\, x^n = 0$$

$$\sum_{n=0}^{\infty} n\, c_n\, x^n - \sum_{n=0}^{\infty} (n+1)\, c_{n+1}\, x^n + 2\sum_{n=0}^{\infty} c_n\, x^n = 0$$

$$\Rightarrow n\, c_n - (n+1)\, c_{n+1} + 2\, c_n = 0 \Rightarrow c_{n+1} = \frac{n+2}{n+1} c_n \quad \text{(recurrence formula)}$$

The radius of convergence is $R = 1$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_{n+1}\, x^{n+1}}{c_n\, x^n} \right| = |x| \lim_{n\to\infty} \frac{n+2}{n+1} = |x| < 1$$

The power series converges for $-1 < x < 1$.

4) [14] Find the recurrence relation and the radius of convergence for a power series solution of the initial value problem

$$y'' - x^2 y' - 3xy = 0\,, \quad y(0) = 0\,, \quad y'(0) = 1$$

Give the first two nonzero terms in the solution.

Solution:

$$y(x) = \sum_{n=0}^{\infty} c_n\, x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n\, c_n\, x^{n-1} \quad \Rightarrow \quad y''(x) = \sum_{n=2}^{\infty} n\,(n-1)\, c_n\, x^{n-2}$$

Insert in the equation:

$$\sum_{n=2}^{\infty} n\,(n-1)\, c_n\, x^{n-2} - \sum_{n=1}^{\infty} n\, c_n\, x^{n+1} - 3\sum_{n=0}^{\infty} c_n\, x^{n+1} = 0$$

$$\sum_{n=-1}^{\infty} (n+3)\,(n+2)\, c_{n+3}\, x^{n+1} - \sum_{n=0}^{\infty} n\, c_n\, x^{n+1} - 3\sum_{n=0}^{\infty} c_n\, x^{n+1} = 0$$

$$2c_2 + \sum_{n=0}^{\infty}(n+3)\,(n+2)\, c_{n+3}\, x^{n+1} - \sum_{n=0}^{\infty} n\, c_n\, x^{n+1} - 3\sum_{n=0}^{\infty} c_n\, x^{n+1} = 0$$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow (n+3)\,(n+2)\, c_{n+3} - n\, c_n - 3\, c_n = 0 \Rightarrow c_{n+3} = \frac{c_n}{n+2} \quad \text{(recurrence formula)}$$

The radius of convergence is $R = \infty$ as implied by the ratio test

$$\lim_{n\to\infty} \left| \frac{c_{n+3}\, x^{n+3}}{c_n\, x^n} \right| = |x|^3 \lim_{n\to\infty} \frac{1}{n+2} < 1$$

The initial conditions imply $y(0) = 0 \Rightarrow c_0 = 0$ and $y'(0) = 1 \Rightarrow c_1 = 1$.

Therefore the first two nonzero terms are $y(x) = x + \frac{1}{3}x^4 + \ldots$