Homework#3 (Math 342)

(due Wed Oct 17)

P: Linear Algebra: a Modern Introduction, by D. Poole (4th Edition)

Z: Advanced Engineering Mathematics , by D. G. Zill (6th Edition)

Note: Detail your work to receive full credit.

Sec. 7.1 (P): 2, 6, 34, 36

Sec. 5.1 (Z): 1, 2, 3, 4

Additional problems:

1) Given the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ in $C[0,1]$, compute

(a) $\langle e^x, e^{-x}\rangle$

(b) $\langle x, \sin(\pi x)\rangle$

(c) $\langle x^2, x^3\rangle$

2) In $C[0,1]$ with the same inner product as defined in the previous problem, consider the vectors 1 and x.

(a) Find the angle θ between 1 and x

(b) Determine the vector projection p of 1 onto x and verify that $1-p$ is orthogonal to p

(c) Compute $\|1-p\|$, $\|p\|$, $\|1\|$ and verify that the Pythagorean theorem holds

3) Given the inner product $\langle f, g\rangle = (1/\pi)\int_{-\pi}^{\pi} f(x)g(x)dx$ in $C[-\pi,\pi]$, show that $\cos(mx)$ and $\sin(nx)$ are orthogonal and that both are unit vectors (m and n are integers). Determine the distance $\|\cos(mx)-\sin(nx)\|$ between these two vectors.

4) Consider the inner product $\langle p, q\rangle = \sum_{i=1}^{5} p(x_i)q(x_i)$ in $\mathbb{P}_5$, where $x_i = (i-3)/2$ for $i = 1,\ldots,5$.

(a) Compute $\|x\|$

(b) Compute $\|x^2\|$

(c) Compute the distance $\|x - x^2\|$ between x and x^2

(d) Show that x and x^2 are orthogonal.

5) Prove that, for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in a vector space with an inner product,

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 + \|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$$

6) Let θ be a fixed real number and let

$$\boldsymbol{x}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

(a) Show that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are orthogonal and unit vectors (hence they form an orthonormal basis for $\mathbb{R}^2$ because there are two of them).

(b) Given a vector $\boldsymbol{y} = (\alpha, \beta)^\top$ in $\mathbb{R}^2$, express it as a linear combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ where c_1, c_2 depend on α, β and θ.

(c) Verify that $c_1^2 + c_2^2 = \|\boldsymbol{y}\|^2 = \alpha^2 + \beta^2$ (Parseval's theorem).

7) The set

$$S = \left\{ \frac{1}{\sqrt{2}}, \cos(x), \cos(2x), \cos(3x), \cos(4x) \right\}$$

is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product as defined in Problem 3.

(a) Use trigonometric identities to write the function $\sin^4(x)$ as a linear combination of elements of S.

(b) Use the result from Part (a) to deduce the values of the following integrals

$$\int_{-\pi}^{\pi} \sin^4(x)\cos(x)dx\,, \quad \int_{-\pi}^{\pi} \sin^4(x)\cos(2x)dx\,, \quad \int_{-\pi}^{\pi} \sin^4(x)\cos(3x)dx\,, \quad \int_{-\pi}^{\pi} \sin^4(x)\cos(4x)dx$$