ELEG 310 Midterm 2, May 10, 2010

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1. Let S be binomial with parameters $n = 4$ and $p = 0.5$.

   a) What is $\Pr[3 \le S \le 4]$?

   b) Give a good Gaussian approximation to the probability above. How close are the two answers? (A table of Gaussian probabilities is included on the last page.)

a) $P(3 \le S \le 4) = P(S=3) + P(S=4)$

$$= \binom{4}{3}(0.5)^3(0.5)^1 + \binom{4}{4}(0.5)^4(0.5)^0 = \frac{4}{16} + \frac{1}{16} = \boxed{\frac{5}{16}}$$

b) $E[S] = np = 4 \cdot 0.5 = 2$

$Var[S] = np(1-p) = 4(0.5)(0.5) = 1$

$$P[3 \le S \le 4] = P\left[\frac{3-2}{1} \le \frac{S-2}{1} \le \frac{4-2}{1}\right]$$

$$\approx \Phi\left(\frac{4-2+0.5}{1}\right) - \Phi\left(\frac{3-2-0.5}{1}\right)$$

$$= \Phi(2.5) - \Phi(0.5)$$

$$= 0.994 - 0.692 = \boxed{0.302}$$

$$\begin{array}{r} 0.312 \\ 16 \overline{)5} \\ 48 \\ \hline 20 \\ 16 \\ \hline 40 \end{array}$$

Exact = 0.312 Approx = 0.302 Pretty close!

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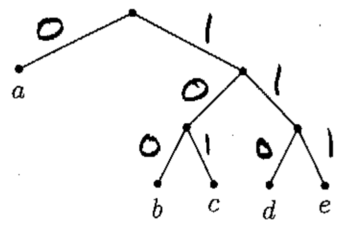
2. A simple coding tree for a five letter alphabet is shown below. Assume the probabilities of each letter from a to e are 0.3, 0.2, 0.2, 0.2 and 0.1, respectively.

a) Assign a codeword (sequence of 0s and 1s) to each codeword according to the tree above. Let $L(\boldsymbol{X})$ be the length in bits of the codeword. What is $E[L(\boldsymbol{X})]$?

b) Find the optimal tree for the same alphabet. What is that tree and what is its $E[L(\boldsymbol{X})]$?

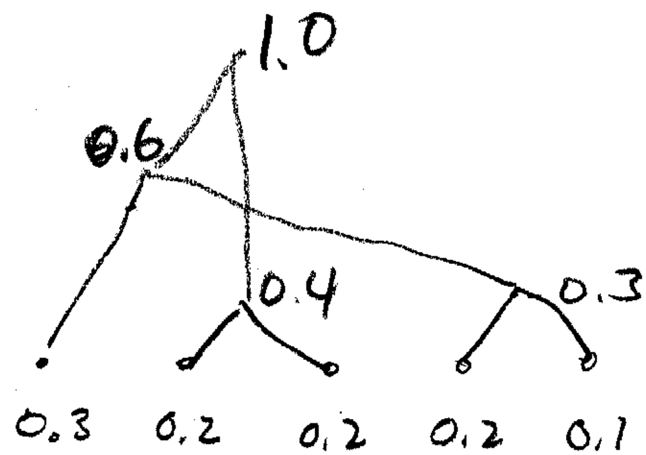
c) Give an expression for the entropy of this source. What is the relationship between the entropy and $E[L(\boldsymbol{X})]$?



	a	b	c	d	e
Code	0	100	101	110	111
length	1	3	3	3	3

a) $E[L(X)] = 1\cdot 0.3 + 3\times 0.2 + 3\times 0.2 + 3\times 0.2 + 3\times 0.1 = 2.4$ bits/letter

b) optimal tree = Huffman



$$EL(x) = 2\cdot 0.3 + 2\cdot 0.2 + 2\cdot 0.2 + 3\cdot 0.2 + 3\cdot 0.1 = 2.3 \text{ bits/letter}$$

c) entropy $= -0.3\log_2 0.3 - 0.2\log_2 0.2 - 0.2\log_2 0.2 - 0.2\log_2 0.2 - 0.1\log_2 0.1$ (= 2.246 bits/letter)

$E[L(x)] \geq$ entropy

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3. Let $f_X(x) = x/2$ for $0 < x < 2$ and $f(x) = 0$ otherwise. Let $Y = X^2$. What are the following?

a. $E[X]$

b. $\mathrm{Var}[X]$

c. $F_X(x)$

d. $f_Y(y)$

e. $E[Y]$

a) $E[X] = \int_0^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_0^2 = \frac{8}{6} = \frac{4}{3}$

b) $E[x^2] = \int_0^2 \frac{x^3}{2}\,dx = \frac{x^4}{8}\Big|_0^2 = \frac{16}{8} = 2$

$\mathrm{Var}[x] = E[x^2] - E[x]^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{18-16}{9} = \frac{2}{9}$

c) $F_x(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ \int_0^x \frac{y}{2}\,dy = \frac{x^2}{4} & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$

d) $f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{4} & 0 < y < 4 \\ 0 & y > 4 \end{cases}$

$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$= P(0 \le X \le \sqrt{y}) = F_X(\sqrt{y}) = \begin{cases} 0 & \sqrt{y} < 0 \\ y/4 & 0 < y < 4 \\ 1 & y > 4 \end{cases}$

e) $E[Y] = E[X^2] = 2$ (from above)

$= \int_0^4 y \frac{1}{4}\,dy = \frac{y^2}{2 \cdot 4}\Big|_0^4 = \frac{16}{8} = 2$

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4. Let $\boldsymbol{X} \sim N(-1, 2)$ and $\boldsymbol{Y} \sim N(2, 4)$ be independent Gaussian random variables. Let $\boldsymbol{W} = \boldsymbol{X} + \boldsymbol{Y}$ and $\boldsymbol{V} = \boldsymbol{X} - \boldsymbol{Y}$. What are the following?

a. $E[\boldsymbol{W}]$

b. $\mathrm{Var}[\boldsymbol{W}]$

c. $\mathrm{Cov}[\boldsymbol{W}, \boldsymbol{V}]$

d. $f_{\boldsymbol{W}}(w)$

a) $E[W] = E[X] + E[Y] = -1 + 2 = \boxed{1}$

b) $\mathrm{Var}[W] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ since X & Y ind.

$= 2 + 4 = \boxed{6}$

c) $\mathrm{Cov}[W, V] = E\left[(W - E[W])(V - E[V])\right]$

$= E\left[\left((X+1) + (Y-2)\right)\left((X+1) - (Y-2)\right)\right]$

$= E\left[(X+1)^2\right] - E\left[(X+1)(Y-2)\right] + E\left[(X+1)(Y-2)\right]$

$- E\left[(Y-2)^2\right]$

$= \mathrm{Var}[X] - \mathrm{Var}[Y] = 2 - 4 = \boxed{-2}$

d) $W \sim N(1, 6)$

$f_W(w) = \dfrac{1}{\sqrt{2\pi \cdot 6}} e^{-\frac{1}{2 \cdot 6}(w-1)^2} \quad -\infty < w < \infty$

$= \boxed{\dfrac{1}{\sqrt{12\pi}} e^{-(w-1)^2/12}} \quad -\infty < w < \infty$

Table 1: Values of the Standard Normal Distribution Function

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	1.00	0.8413	2.00	0.9772	3.00	0.9987
0.05	0.5199	1.05	0.8531	2.05	0.9798	3.05	0.9989
0.10	0.5398	1.10	0.8643	2.10	0.9821	3.10	0.9990
0.15	0.5596	1.15	0.8749	2.15	0.9842	3.15	0.9992
0.20	0.5793	1.20	0.8849	2.20	0.9861	3.20	0.9993
0.25	0.5987	1.25	0.8944	2.25	0.9878	3.25	0.9994
0.30	0.6179	1.30	0.9032	2.30	0.9893	3.30	0.9995
0.35	0.6368	1.35	0.9115	2.35	0.9906	3.35	0.9996
0.40	0.6554	1.40	0.9192	2.40	0.9918	3.40	0.9997
0.45	0.6736	1.45	0.9265	2.45	0.9929	3.45	0.9997
0.50	0.6915	1.50	0.9332	2.50	0.9938	3.50	0.9998
0.55	0.7088	1.55	0.9394	2.55	0.9946	3.55	0.9998
0.60	0.7257	1.60	0.9452	2.60	0.9953	3.60	0.9998
0.65	0.7422	1.65	0.9505	2.65	0.9960	3.65	0.9999
0.70	0.7580	1.70	0.9554	2.70	0.9965	3.70	0.9999
0.75	0.7734	1.75	0.9599	2.75	0.9970	3.75	0.9999
0.80	0.7881	1.80	0.9641	2.80	0.9974	3.80	0.9999
0.85	0.8023	1.85	0.9678	2.85	0.9978	3.85	0.9999
0.90	0.8159	1.90	0.9713	2.90	0.9981	3.90	1.0000
0.95	0.8289	1.95	0.9744	2.95	0.9984	3.95	1.0000