EXERCISE SHEET 23 (SELECTED QUESTIONS)

1) Mean $= E(X) = 1\times0.1 + 2\times0.2 + 3\times0.3 + 4\times0.2 + 5\times0.1 + 6\times0.1$
$= 3.3$ //

Variance $= Var(X) = (1-3.3)^2\times0.1 + (2-3.3)^2\times0.2 + (3-3.3)^2\times0.3 + (4-3.3)^2\times0.2 + (5-3.3)^2\times0.1 + (6-3.3)^2\times0.1$
$= 2.01$ //

2) $Pr(X=x) = 0.1(x+1)$

$Pr(0) = 0.1$

$Pr(1) = 0.2$

$Pr(2) = 0.3$

$Pr(3) = 0.4$

Mean $= E(X) = 0\times0.1 + 1\times0.2 + 2\times0.3 + 3\times0.4$
$= 2$

Variance $= Var(X) = (0-2)^2\times0.1 + (1-2)^2\times0.2 + (2-2)^2\times0.3 + (3-2)^2\times0.4$
$= 1$

3) a) Mean $= E(X) = 1\times0.6 + 2\times0.3 + 3\times0.03 + 4\times0.05 + 5\times0.02$
$= 1.59$ //

Variance $= Var(X) = (1-1.59)^2\times0.6 + (2-1.59)^2\times0.3 + (3-1.59)^2\times0.3 + (4-1.59)^2\times0.05 + (5-1.59)^2\times0.02$
$= 0.8419$ //

4) a) $E(X) = 0\times0.2 + 1\times0.15 + 2\times0.25 + 3\times0.4$
$= 1.85$ //

b) $E(X^2) = 0^2\times0.2 + 1^2\times0.15 + 2^2\times0.25 + 3^2\times0.4$
$= 4.75$ //

c) $E(X^2) - [E(X)]^2 = 4.75 - (1.85)^2$
$= 1.3275$ //

d) $E(2X-1) = (2(0)-1)\times0.2 + (2(1)-1)\times0.15 + (2(2)-1)\times0.25 + (2(3)-1)\times0.4$

$= 2.7$ //

⑤ a)

x	$Pr(X=x)$
1	k
2	$4k$
3	$9k$
4	$16k$

b) $k + 4k + 9k + 16k = 1$

$k = \frac{1}{30}$ //

c) $E(X) = 1 \times \frac{1}{30} + 2 \times 4\left(\frac{1}{30}\right) + 3 \times 9\left(\frac{1}{30}\right) + 4 \times 16\left(\frac{1}{30}\right)$

$= \frac{10}{3}$ //

d) $Var(X) = \left(1 - \frac{10}{3}\right)^2 \times \frac{1}{30} + \left(2 - \frac{10}{3}\right)^2 \times 4\left(\frac{1}{30}\right) + \left(3 - \frac{10}{3}\right)^2 \times 9\left(\frac{1}{30}\right) + \left(4 - \frac{10}{3}\right)^2 \times 16\left(\frac{1}{30}\right)$

$= \frac{31}{45}$ //

⑥ a) $E(X) = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.3 + 4 \times 0.1 + 5 \times 0.1 + 6 \times 0.2$

$= 3.5$ //

b) $E(Y) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= 3.5$ //

c) i)

z	$Pr(Z=z)$
2	$\frac{1}{60}$
3	$\frac{3}{60}$
4	$\frac{6}{60}$
5	$\frac{7}{60}$
6	$\frac{8}{60}$
7	$\frac{10}{60}$
8	$\frac{9}{60}$
9	$\frac{7}{60}$
10	$\frac{4}{60}$

11	$\frac{3}{60}$
12	$\frac{2}{60}$

vi) $E(Z) = 2\times\frac{1}{60} + 3\times\frac{3}{60} + 4\times\frac{6}{60} + 5\times\frac{7}{60} + 6\times\frac{8}{60} + 7\times\frac{10}{60} + 8\times\frac{9}{60} + 9\times\frac{7}{60} + 10\times\frac{4}{60} + 11\times\frac{3}{60} + 12\times\frac{2}{60}$

$= 7$ //

$E(X)+E(Y) = 3.5+3.5 = 7$

$\therefore E(Z) = E(X)+E(Y)$ <verified>

7 a)

x	$Pr(X=x)$
0	0.08
1	0.44
2	0.48

b) $E(X) = 0\times0.08 + 1\times0.44 + 2\times0.48$

$= 1.4$ //

c) $Var(X) = (0-1.4)^2\times0.08 + (1-1.4)^2\times0.44 + (2-1.4)^2\times0.48$

$= 0.4$ //

8 a) $\boxed{Var(Y) = E(Y^2) - (E(Y))^2}$

Substitute $Y = aX$

$Var(aX) = E((aX)^2) - (E(aX))^2$

$= E(a^2X^2) - (aE(X))^2$

$= a^2E(X^2) - a^2(E(X))^2$

$= a^2(E(X^2) - (E(X))^2)$

$= a^2 Var(X)$ //

b) $E(X) = (-4)\times0.05 + (-2)\times0.15 + (-1)\times0.2 + 0\times0.1 + 3\times0.1 + 5\times0.4$

$= 1.6$

$Var(X) = (-4-1.6)\times0.05 + (-2-1.6)\times0.15 + (-1-1.6)\times0.2 + (0-1.6)\times0.1 + (3-1.6)\times0.1 + (5-1.6)\times0.4$

$= \frac{497}{50}$

$$\begin{aligned} Var(7X) &= 7^2 \, Var(X) \\ &= 49 \times \frac{497}{50} \\ &= \frac{24353}{50} \\ &= 487.06 \; // \end{aligned}$$

③ b) $E(1.50 + 0.25X) = 1.50 + 0.25\,E(X)$

$= \$1.8975 \; //$

EXERCISE SHEET 23

① – ⑧ ⟨Selected Questions⟩

⑨ $$\Pr(\text{exactly 2 women}) = \frac{{}^{6}C_{1} \times {}^{4}C_{2}}{{}^{10}C_{3}}$$
$$= 0.3$$

⑩ $$\Pr(\text{1 cracked egg chosen}) = \frac{{}^{9}C_{2} \times {}^{3}C_{1}}{{}^{12}C_{3}}$$
$$= \frac{27}{55}$$
$$= 0.4909090909\ldots$$
$$\approx 0.4909 \ \langle 4\text{ d.p.} \rangle$$

⑪ a) $$\Pr(\text{no defective articles}) = \frac{{}^{16}C_{5}}{{}^{20}C_{5}}$$
$$= \frac{91}{323}$$
$$= 0.2817337461$$
$$\approx 0.2817 \ \langle 4\text{dp} \rangle$$

b) $$\Pr(\text{at least 3 defective}) = \Pr(\text{3 defective}) + \Pr(\text{4 defective})$$
$$= \frac{{}^{16}C_{2} \times {}^{4}C_{3}}{{}^{20}C_{5}} + \frac{{}^{16}C_{1} \times {}^{4}C_{4}}{{}^{20}C_{5}}$$
$$= \frac{31}{969}$$
$$= 0.03199174407$$
$$\approx 0.0320 \ \langle 4\text{dp} \rangle$$

⑫ $$\Pr(\text{at least 2 red}) = \Pr(\text{2 red}) + \Pr(\text{3 red})$$
$$= \frac{{}^{15}C_{1} \times {}^{5}C_{2}}{{}^{20}C_{3}} + \frac{{}^{5}C_{3}}{{}^{20}C_{3}}$$

$$= \frac{8}{57}$$

$$= 0.1403508772\ldots$$

$$\approx 0.1404 \langle 4\text{dp} \rangle //$$

13 $$\Pr(\text{2 faulty on display}) = \frac{{}^{7}C_{3} \times {}^{3}C_{2}}{{}^{10}C_{5}}$$

$$= \frac{5}{12}$$

$$= 0.416666666\ldots$$

$$\approx 0.4167 \langle 4\text{dp} \rangle //$$

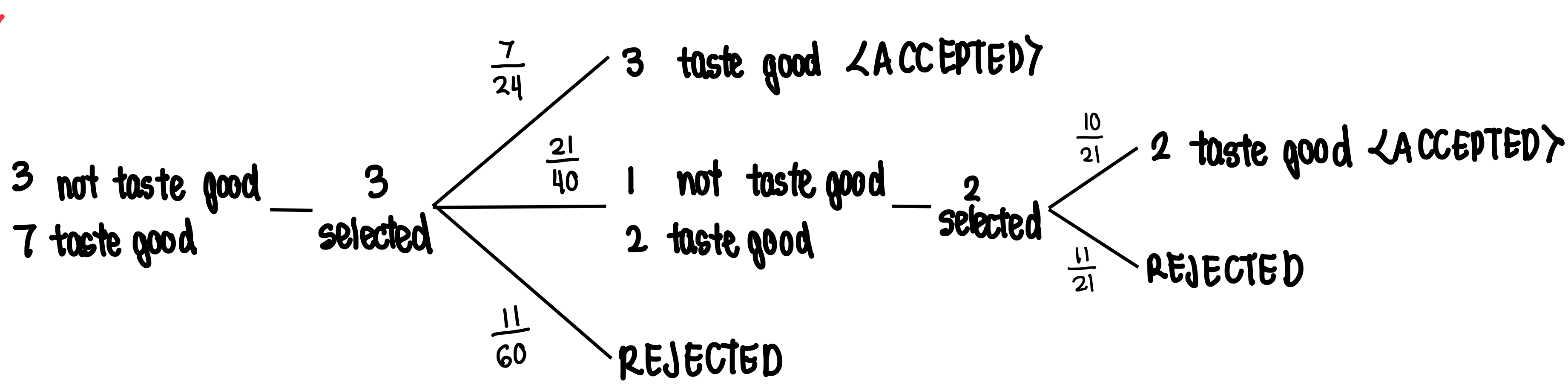
14 $$\Pr(\text{2 men in committee}) = \frac{{}^{5}C_{2} \times {}^{7}C_{1}}{{}^{12}C_{3}}$$

$$= \frac{7}{22}$$

$$= 0.3181818182$$

$$\approx 0.3182 \langle 4\text{dp} \rangle //$$

15



Working :

$$\frac{{}^{7}C_{3}}{{}^{10}C_{3}} = \frac{7}{24} \qquad \frac{{}^{5}C_{2}}{{}^{7}C_{2}} = \frac{10}{21}$$

$$\frac{{}^{3}C_{1} \times {}^{7}C_{2}}{{}^{10}C_{3}} = \frac{21}{40} \qquad 1 - \frac{10}{21} = \frac{11}{21}$$

$$1 - \frac{7}{24} - \frac{21}{40} = \frac{11}{60}$$

$$\Pr(\text{ACCEPTED}) = \frac{7}{24} + \frac{21}{40} \times \frac{10}{21}$$

$$= \frac{13}{24}$$

$= 0.541666666...$
$\approx 0.54167 \ \langle 4dp \rangle$ //

16. Pr (2 blue balls chosen) $= \frac{1}{15}$

Let x represent the number of blue balls

$$\frac{1}{15} = \frac{{}^{x}C_2}{{}^{10}C_2}$$

$$ {}^{x}C_2 = 3$$

$$\frac{x!}{(x-2)\,2!} = 3$$

$$\frac{x(x-1)\cancel{(x-2)}}{\cancel{(x-2)}\,2} = 3$$

$$x(x-1) = 6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$x = 3$ $\quad$ $x = -2$

$\langle$rejected because ${}^{n}C_r$, $n > 0 \rangle$

$\therefore$ There are 3 blue balls in the box //

17. a) $\frac{1}{3}x^3 + 3x - 2\ln|x| + c$

b) $-\frac{1}{2}e^{-2x} - \frac{3}{4}\cos 4x + C$

c) $\cos 2x - \frac{1}{3}\sin 3x + C$

18. a) $\int_1^2 \frac{1}{2x}\,dx = \left[\frac{1}{2}\ln|x|\right]_1^2$

$= \frac{1}{2}\ln|2| - \frac{1}{2}\ln|1|$

$= \ln\sqrt{2}$ //

b) $\int_1^9 \frac{1}{2x}\,dx = \left[\frac{1}{2}\ln|x|\right]_1^9$

$= \frac{1}{2}\ln|9| - \frac{1}{2}\ln|1|$

$= \ln\sqrt{9}$

$= \ln 3$ //

c) $\int_0^{\frac{\pi}{2}} (5x+\sin 2x)\,dx = \left[\frac{5}{2}x^2 - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{5}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\cos\left(2\left(\frac{\pi}{2}\right)\right)\right) - \left(\frac{5}{2}(0)^2 - \frac{1}{2}\cos(2(0))\right)$

$= \frac{5\pi^2}{8} + \frac{1}{2} + \frac{1}{2}$

$= \frac{5}{8}\pi^2 + 1$ //

d) $\int_1^2 \left(2+\frac{1}{x}\right)^2 dx = \left[4x + 4\ln|x| - \frac{1}{x}\right]_1^2$

$= \left(4(2) + 4\ln|2| - \frac{1}{2}\right) - \left(4(1) + 4\ln|1| - \frac{1}{1}\right)$

$= 8 + 4\ln 2 - \frac{1}{2} - 3$

$= \frac{9}{2} + 4\ln 2$ //

e) $\int_1^3 (x^3+1)\,dx = \left[\frac{1}{4}x^4 + x\right]_1^3$

$= \left(\frac{1}{4}(3)^4 + (3)\right) - \left(\frac{1}{4}(1)^4 + (1)\right)$

$= 22$ //

f) $\int_1^4 \frac{x+1}{\sqrt{x}}\,dx = \left[\frac{2\sqrt{x}\,(x+3)}{3}\right]_1^4$

$= \left(\frac{2\sqrt{4}\,(4+3)}{3}\right) - \left(\frac{2\sqrt{1}\,(1+3)}{3}\right)$

$$= \frac{28}{3} - \frac{8}{3}$$

$$= \frac{20}{3} //$$

(19) a) $\int_1^2 \left(e^{2x} + \frac{4}{x}\right) dx = \left[\frac{1}{2}e^{2x} + 4\ln|x|\right]_1^2$

$$= \left(\frac{1}{2}e^{2(2)} + 4\ln|2|\right) - \left(\frac{1}{2}e^{2(1)} + 4\ln|1|\right)$$

$$= \frac{1}{2}e^4 + 4\ln 2 - \frac{1}{2}e^2$$

$$= 26.37713569$$

$$\approx 26.377 \langle 3dp \rangle //$$

b) $\int_1^2 \frac{e^{2x} - e^{-x}}{e^x} dx = \left[e^x + \frac{e^{-2x}}{2}\right]_1^2$

$$= \left(e^{(2)} + \frac{e^{-2(2)}}{2}\right) - \left(e^{(1)} + \frac{e^{-2(1)}}{2}\right)$$

$$= 4.612264448$$

$$\approx 4.612 \langle 3dp \rangle //$$

c) $\int_0^{0.5} \sec^2(2x)\, dx = \left[\frac{1}{2}\tan(2x)\right]_0^{0.5}$

$$= \frac{1}{2}\tan(2(0.5)) - \frac{1}{2}\tan(2(0))$$

$$= 0.7787038623$$

$$\approx 0.779 \langle 3dp \rangle //$$

(20) a) Substitute $y = x^2$ into $y = 15 - 2x$

$x^2 = 15 - 2x$

$x^2 + 2x - 15 = 0$

$(x-3)(x+5) = 0$

$x = 3$

$x = -5$ ⟨rejected, $x > 0$⟩

Substitute $x = 3$ into $y = x^2$

$y = (3)^2$
$= 9$
$\therefore P = (3, 9)$
Substitute $y = 0$ into $y = 15 - 2x$:
$0 = 15 - 2x$
$2x = 15$
$x = 7.5$
$\therefore Q = (7.5, 0)$

b) Area $= \int_0^3 (x^2)\,dx + \int_3^{7.5} (15 - 2x)\,dx$

$= \left[\frac{1}{3}x^3\right]_0^3 + \left[15x - x^2\right]_3^{7.5}$

$= \left(\frac{1}{3}(3)^3 - \frac{1}{3}(0)^3\right) + \left(15(7.5) - (7.5)^2\right) - \left(15(3) - (3)^2\right)$

$= 29.25$ units^2

21 Area $= \left(\int_0^{\frac{\pi}{6}} (\cos x)\,dx - \int_0^{\frac{\pi}{6}} (\sin(2x))\,dx\right) + \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin(2x))\,dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos x)\,dx\right)$

$= \left[\sin x - \frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{2}} + \left[-\frac{1}{2}\cos(2x) - \sin(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(\frac{3}{4} - \frac{1}{2}\right) + \left(-\frac{1}{2} + \frac{3}{4}\right)$

$= 0.5$ units^2

22 Substitute $y = 4x$ into $y = 6 + 7x - 3x^2$
$4x = 6 + 7x - 3x^2$
$3x^2 - 3x - 6 = 0$
$x^2 - x - 2 = 0$
$(x - 2)(x + 1) = 0$
$x = 2 \qquad x = -1$

Area $= \int_0^2 (4x)\,dx - \int_0^2 (6 + 7x - 3x^2)\,dx$

$= \int_0^2 (-3x^2 - 3x + 6)\,dx$

$$= \left[-x^3 - \frac{3}{2}x^2 + 6x \right]_0^2$$

$$= 10 \text{ units}^2 //$$

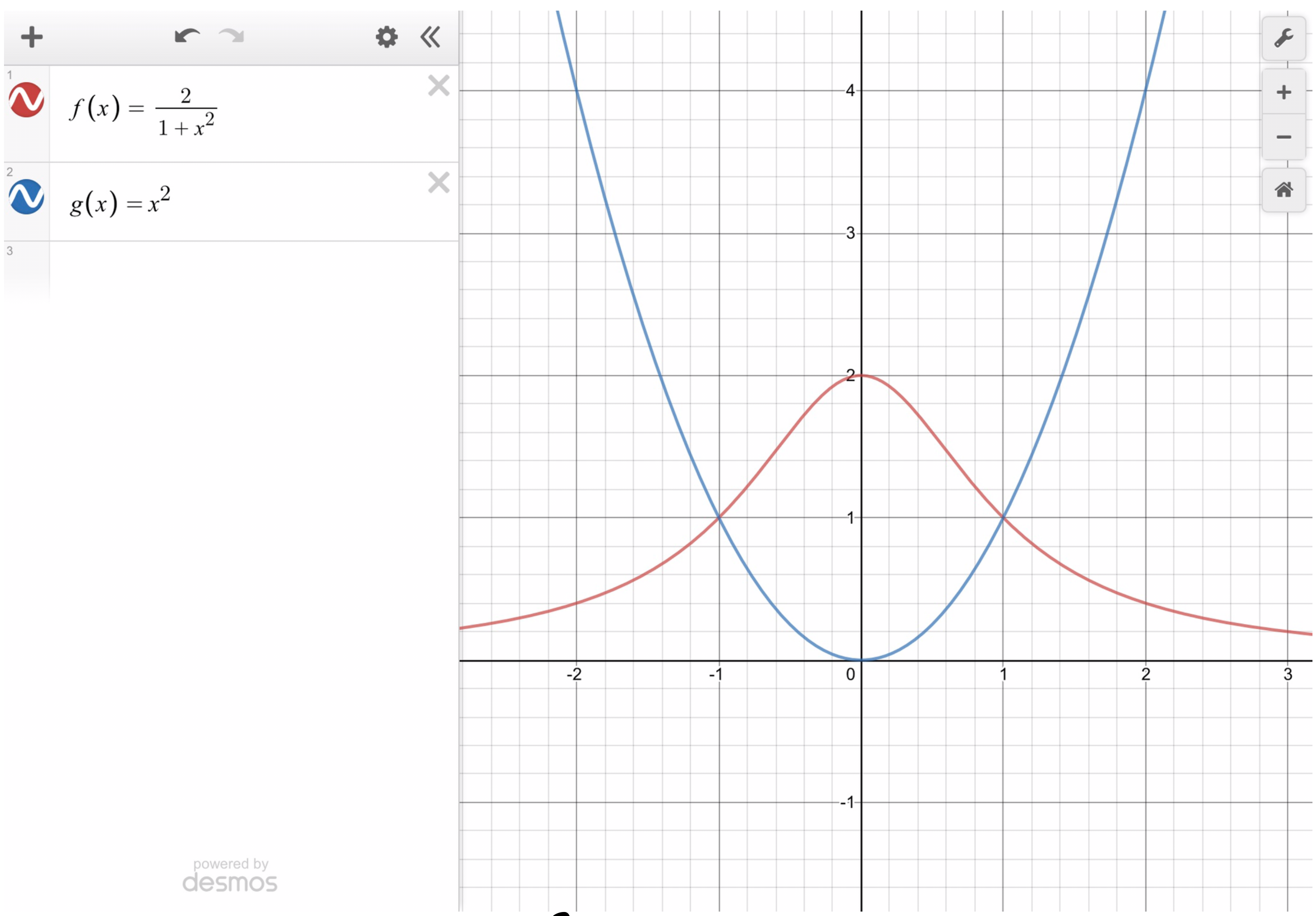
(23) $$\text{Area} = \int_0^{\pi} (\sin x)\,dx + \left(\frac{1}{2}\right)(\pi) - \int_{\pi}^{\frac{7\pi}{6}} (\sin x)\,dx$$

$$= \left[-\cos x \right]_0^{\pi} + \frac{\pi}{2} - \left[-\cos x \right]_{\pi}^{\frac{7\pi}{6}}$$

$$= 3.699 \text{ units}^2 // \ \langle 3dp \rangle$$

(24) a)

b)



c) Substitute $y = x^2$ into $y = \frac{2}{1+x^2}$

$$x^2 = \frac{2}{1+x^2}$$

$$x^2 + x^4 = 2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x^2 = -2 \qquad x^2 = 1$$

$$x = \sqrt{-2} \qquad x = \pm 1$$

$\langle$rejected$\rangle$

Substitute $x=1$ and $x=-1$ into $y=x^2$

$y=1^2$ $\quad$ $y=(-1)^2$

$=1$ $\quad$ $=1$

Points of intersection: $(-1,1)$ and $(1,1)$ //

d) Area $= \int_{-1}^{1} \left(\frac{2}{1+x^2}\right) dx - \int_{-1}^{1} (x^2)\, dx$

$= \left[2\tan^{-1}x\right]_{-1}^{1} - \left[\frac{1}{3}x^3\right]_{-1}^{1}$

$= \pi - \frac{2}{3}$ units^2 //