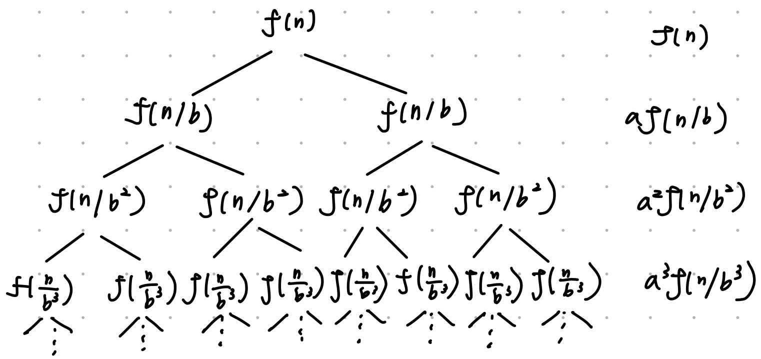
Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■



The tree has depth $\log_b n$ and branching factor a. From the recurrence tree we can get

$$T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a})$$

<case1>

If $f(n) = O(n^{\log_b a - \epsilon})$

then $T(n) = \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a - \epsilon} + O(n\log_b a)$

$$\Rightarrow \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a - \epsilon} = n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n} a^i b^{-i\log_b a} b^{i\epsilon}$$

$$= n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n} a^i a^{-i} b^{i\epsilon}$$

$$= n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n} b^{i\epsilon} = n^{\log_b a - \epsilon} \frac{b^{\epsilon(\log_b n + 1)} - 1}{b^\epsilon - 1}$$

$$= n^{\log_b a - \epsilon} \frac{n^\epsilon b^\epsilon - 1}{b^\epsilon - 1} \leq n^{\log_b a - \epsilon} \frac{n^\epsilon b^\epsilon}{b^\epsilon - 1}$$

$$= n^{\log_b a} \cancelto{1}{\frac{b^\epsilon}{b^\epsilon - 1}} = n^{\log_b a}$$

$\therefore$ $T(n) = O(n^{\log_b a})$

<case2>

If $f(n) = \Theta(n^{\log_b a})$

then $T(n) = \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a} + O(n^{\log_b a})$

$$\Rightarrow \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a} = n^{\log_b a} \sum_{i=0}^{\log_b n} a^i b^{-i \log_b a}$$

$$= n^{\log_b a} \sum_{i=0}^{\log_b n} a^i a^{-i}$$

$$= n^{\log_b a} \sum_{i=0}^{\log_b n} 1$$

$$= n^{\log_b a} (\log_b n + 1) = \Theta(n^{\log_b a} \log n)$$

$$\Rightarrow T(n) = \Theta(n^{\log_b a} \log n) + O(n^{\log_b a}) = \Theta(n^{\log_b a} \log n)$$

<case3>

If $f(n) = \Omega(n^{\log_b a + \epsilon})$

then $T(n) = \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a + \epsilon} + O(n^{\log_b a})$

$$\Rightarrow \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a + \epsilon} \leq \sum_{i=0}^{\log_b n} c^i f(n) = f(n) \sum_{i=0}^{\log_b n} c^i$$

$$= f(n) \frac{1}{1-c}$$

$$\Rightarrow T(n) \leq \sum_{i=0}^{\log_b n} c^i f(n) + O(n^{\log_b a}) = f(n) \frac{1}{1-c} + O(n^{\log_b a})$$

$$= \Theta(f(n))$$