Lec 1 : March 28

· Disease Transmission

\# infected units at time 0

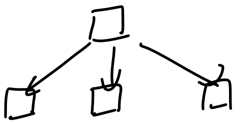
Time 0  $Z_0 = 1$ p_0

Host 0

Time 1 : multiple possibilities

 death → $Z_1 = 0$, $p_1 = p_0$

 alive w/ no transmission → $Z_1 = 1$, p_1

 transmitted to multiple individuals → $Z_1 = k+1$ $p_{k+1}, k = 1, 2, \ldots$

time 0 for these guys

· what is the expected # of individuals at time n?

· Probability of the disease dying out?

· Not an iid process

- Gambler's Ruin

Person A: capital \$a

Person B: capital \$b

$\varepsilon_i \overset{iid}{\sim}$ coin tosses, i.e. $\varepsilon_i = +1$ or -1 w/ prob. $1/2$

$\varepsilon_i = +1 \rightarrow$ Person A gets \$1 from person B

$\varepsilon_i = -1 \rightarrow$ Person A gives \$1 to Person B

Expected length of game before one goes broke?

- Markov Chain Monte Carlo

D: Data

Likelihood model: $p(D|\theta)$

want to infer about θ

Bayesian inference: prior $p(\theta)$

posterior: $p(\theta|D) \propto p(D|\theta)\,p(\theta)$

MCMC: iteratively generates samples to approximate posterior

Martingales (Durrett 5.2)

- Gambling on iid coin tosses (bet \$1 on each toss)

$\varepsilon_i = +1 \rightarrow$ guessed correct (get \$2 back) — \$1 invested, \$1 reward (profit)

$\varepsilon_i = -1 \rightarrow$ guessed incorrect (lose \$1)

Total profit/loss after n tosses: $\sum_{i=1}^{n} \varepsilon_i$

Bet H_i amount for i^{th} toss: $\sum_{i=1}^{n} H_i \varepsilon_i$

Martingale Betting Strategy:

$$H_1 = 1, \quad H_i = 2H_{i-1} \text{ if } \varepsilon_i = -1$$
$$= 1 \quad \text{if } \varepsilon_i = +1$$

$+1$

$-1 + 2 = +1$

$-1 - 2 + 4 = +1$

If you lose the first k tosses, you bet 2^k on the next toss

Filtration:

History of all coin tosses before time n

$= \varepsilon_1, \ldots, \varepsilon_n$

$\mathcal{F}_{n-1} = \sigma(\varepsilon_1, \ldots, \varepsilon_n)$

$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ increasing sequence of σ-algebras

- We say a sequence of RVs X_n is adapted to $\mathcal{F}_n$ if

X_n is $\mathcal{F}_n$-measureable and $\mathcal{F}_n$ is increasing (filtration)

$X_1, \dots, X_{n-1}$ is $\mathcal{F}_n$-measureable b/c

X_1 is $\mathcal{F}_1$-measurable $\Rightarrow$ $\mathcal{F}_n$-measurable

Ex:

X_n = profit of gambler at time $n = \varepsilon_1 + \dots + \varepsilon_n$

$$\mathcal{F}_{n-1} = \sigma(\varepsilon_1, \dots, \varepsilon_{n-1}), \quad \mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$$

$X_1 = \varepsilon_1$

$X_2 = \varepsilon_1 + \varepsilon_2$

$\varepsilon_1 = X_1$

$\varepsilon_2 = X_2 - X_1$

$\vdots$

$\varepsilon_n = X_n - X_{n-1}$

ε's and X's are 1-1 functions of each other, so

$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ (defaut on smallest filtration)

$$E(X_{n+1} \mid \mathcal{F}_n) = E(X_n + \varepsilon_{n+1} \mid \mathcal{F}_n)$$

$$= E(X_n \mid \mathcal{F}_n) + E(\varepsilon_{n+1} \mid \mathcal{F}_n)$$

$$= X_n + \underbrace{E(\varepsilon_{n+1})}_{0 \text{ if fair coin toss}}$$

$$= X_n$$

Martingales: A sequence of RVs is called a Martingale if the following conditions are met:

i) $E|X_n| < \infty \quad \forall n$

ii) X_n is adapted to $\mathcal{F}_n$ (Default: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$)

iii) $E(X_{n+1} \mid \mathcal{F}_n) = X_n \quad \forall n$

given what you know at time n, on average you don't make or lose money at time n+1

Exercise 5.2.1:

X_n is adapted to $\mathcal{G}_n$ and X_n is a Mart. wrt $\mathcal{G}_n$. Then $\mathcal{F}_n = \sigma(X_1, \ldots, X_n) \subseteq \mathcal{G}_n$ and X_n is a Mart. wrt $\mathcal{F}_n$.

Pf:/ $X_1, \ldots, X_n$ is $\mathcal{G}_n$ measurable

$\Rightarrow \quad \mathcal{F}_n = \sigma(X_1, \ldots, X_n) \subseteq \mathcal{G}_n$

Need to show $E(X_{n+1} \mid \mathcal{F}_n) = X_n$

we know $E(X_{n+1} \mid \mathcal{G}_n) = X_n$

$$E(X_{n+1} \mid \mathcal{F}_n) = E(E(X_{n+1} \mid \mathcal{G}_n) \mid \mathcal{F}_n)$$ nested sigma algebra property
$$= E(X_n \mid \mathcal{F}_n) = X_n$$

unfair game:

$$\varepsilon_i = +1 \quad \text{w.p. } p \neq 1/2$$
$$= -1 \quad \text{w.p. } 1-p \quad , \quad \varepsilon_i\text{'s are iid}$$

$$E(X_{n+1} \mid \mathcal{F}_n) = X_n + E(\varepsilon_{n+1})$$
$$= X_n + (2p-1)$$

$p < 1/2$: $E(X_{n+1} \mid \mathcal{F}_n) \le X_n \ \forall n \Rightarrow X_n =$ super Martingale

$p > 1/2$: $E(X_{n+1} \mid \mathcal{F}_n) \geq X_n \ \forall n \Rightarrow X_n =$ sub Martingale

X_n is a super-M $\Leftrightarrow$ $-X_n$ is a sub-M

<u>Thm. 5.2.1</u> X_n is a super-M. Then for $n > m$,
sub-M $\geq$
Mart. $=$

$$E(X_n \mid \mathcal{F}_m) \le X_m$$

pf:/ $E(X_n \mid \mathcal{F}_m) = E(E(X_n \mid \mathcal{F}_{n-1}) \mid \mathcal{F}_m)$

$$\le E(X_{n-1} \mid \mathcal{F}_m)$$ by monotonicity of conditional expectations

$\leq$

$\vdots$

$\leq E(X_{m+1} \mid \mathcal{F}_m)$

$\leq X_m$

<u>Thm. 5.2.3</u> X_n Mart. φ is a convex function

s.t. $E|\varphi(X_n)| < \infty \ \forall n$

$\Rightarrow \varphi(X_n)$ is a sub-M.

Pf:/ WTS: $E(\varphi(X_{n+1}) \mid \mathcal{F}_n) \geq \varphi(X_n)$

use Jensen's b/c φ is convex:

$$LHS \geq \varphi(E(X_{n+1} \mid \mathcal{F}_n)) = \varphi(X_n)$$

Corollary: X_n Mart. $E|X_n|^p < \infty \ \forall n$ when $p > 1$

$|X_n|^p$ is a sub-M

Pf:/ $\varphi(x) = |x|^p$

$|\,|$ is a convex function.

Lec 2: March 30

Midterm: lecture 8

Thm. 5.2.4: X_n sub-M, φ is a non-decreasing convex function such that $E|\varphi(X_n)| < \infty \ \forall n$

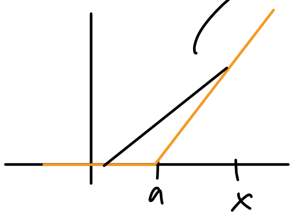
$\Rightarrow \varphi(X_n)$ is sub-M.

Pf:/ $E(\varphi(X_{n+1}) \mid \mathcal{F}_n) \overset{\text{Jensen's}}{\geq} \varphi(E(X_{n+1} \mid \mathcal{F}_n)) \geq \varphi(X_n)$

i.) X_n sub-M $\Rightarrow (X_n - a)^+$ is sub-M

$\varphi(x) = (x-a)^+$

line never goes below the curve



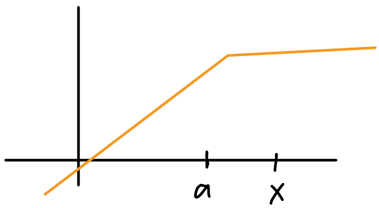
$\varphi(x) \leq |x| + |a|$

$E|\varphi(X_n)| \leq E|X_n| + |a| < \infty$

ii) X_n is a super-M $\Rightarrow$ $\underbrace{X_n \wedge a}_{\text{minimum}}$ is a super-M

pf:/ $\varphi(x) = x \wedge a$

concave function



$|\varphi(x)| \leq |x| + |a|$

$E|\varphi(X_n)| < \infty$

$$E(\varphi(X_{n+1}) \mid \mathcal{F}_n) \leq \varphi(E(X_{n+1} \mid \mathcal{F}_n))$$

$$\leq \varphi(X_n)$$ b/c super-M, and φ is monotonic

Gambling Example:

Predictable Process:

H_n is predictable if H_n is $\mathcal{F}_{n-1}$-measurable $\forall\, n$.

Total profit $= \sum_{i=1}^{n} H_i \varepsilon_i = \sum_{i=1}^{n} H_i (X_i - X_{i-1}) = (H \cdot X)_n$

$X_n = \sum_{i=1}^{n} \varepsilon_i$, $(H \cdot X)_0 = 0$

Thm. 5.2.5: X_n super-M. $H_n \geq 0$ predictable and H_n bounded. Then $(H \cdot X)_n$ is super-M.

Pf:/ $E|X_n| < \infty \ \forall n$, H_n is bounded

$$\Rightarrow E\left|\sum_{i=1}^{n} H_i (X_i - X_{i-1})\right| < \infty \ \forall n.$$

$$E((H \cdot X)_{n+1} \mid \mathcal{F}_n) = E((H \cdot X)_n + H_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n)$$
$$= (H \cdot X)_n + E(H_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n)$$

$X \in \mathcal{F} \Rightarrow$
$E(XY \mid \mathcal{F}) = X E(Y \mid \mathcal{F})$

$$= (H \cdot X)_n + H_{n+1}(E(X_{n+1} \mid \mathcal{F}_n) - X_n)$$
$$\leq (H \cdot X)_n$$

corollary: X_n sub-M $\Rightarrow (H \cdot X)_n$ sub-M

Stopping Time: τ wrt filtration $\mathcal{F}_n$
if $\{\tau = n\} \in \mathcal{F}_n \ \forall n$.

$\{\tau \geq n\} \in \mathcal{F}_{n-1} \ \forall n$

$$\hookrightarrow = \Big(\bigcup_{i=1}^{n-1} \underbrace{\{\tau = i\}}_{\in \mathcal{F}_i \subseteq \mathcal{F}_{n-1}}\Big)^c$$

$\in \mathcal{F}_i \subseteq \mathcal{F}_{n-1} \Rightarrow \{\tau \geq n\} \in \mathcal{F}_{n-1}$

Thm. 5.2.6: X_n super-M, τ is a stopping time

$$\Rightarrow \underbrace{X_{\tau \wedge n}}_{} \text{ is super-M}$$

random b/c τ is RV

Pf:/ $H_n = I(\tau \geq n)$ is predictable.

$\Rightarrow (H \cdot X)_n$ is super-M

$$(H\cdot X)_n = \sum_{i=1}^{n} H_i (X_i - X_{i-1}) = \sum_{i=1}^{n} I(\tau \geq n)(X_i - X_{i-1})$$

$$\left\{\begin{array}{ll} \tau < n & = \sum_{i=1}^{\tau} (X_i - X_{i-1}) \\ \tau \geq n & = \sum_{i=1}^{n} (X_i - X_{i-1}) \end{array}\right\} = \sum_{i=1}^{\tau\wedge n} (X_i - X_{i-1})$$

$$= X_{\tau\wedge n} - X_0$$

$X_{\tau\wedge n} - X_0$ is a super-M $\Rightarrow X_{\tau\wedge n}$ is a super-M

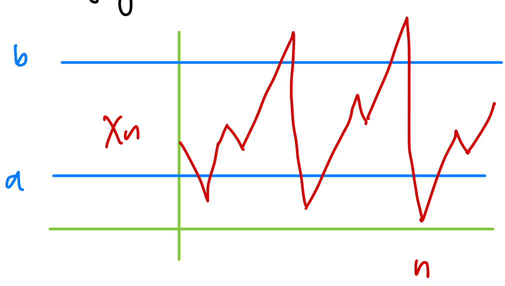
· same for sub-M

Thm. 5.2.8 Martingale convergence:

X_n is a sub-M with $\sup EX_n^+ < \infty$. Then

$X_n \xrightarrow{a.s.}$ to some X and $E|X| < \infty$.

Pf strategy:



upcrossing: going from below a to above b (happens twice here)

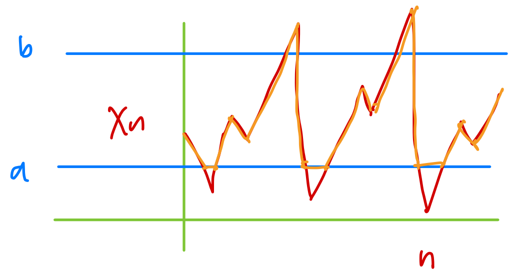
· for any a, b, there are a finite number of upcrossings.

Doob's upcrossing inequality (Thm. 5.2.7):

U_n : # upcrossings till time n

$H_n = I$(upcrossing happening at time n)

follows X_n but is flat at a



$Y_n = (X_n - a)^+ + a$

Y_n sub-M.

$(H \cdot Y)_n \geq (b-a) U_n$

Lec 3: April 4

Martingale convergence:

we have shown $(b-a)U_n \le (H\cdot Y)_n$

$K = 1 - H$, $K \geq 0$, predictable.

$$E(K\cdot Y)_n \geq E(K\cdot Y)_0 = 0 \quad (\text{Thm. } 5.2.5)$$

$$Y_n - Y_0 = (1\cdot Y)_n = \big((H+1-H)\cdot Y\big)_n$$

$$= (H\cdot Y)_n + (K\cdot Y)_n$$

$$\geq (b-a)U_n$$

$$E(Y_n - Y_0) \geq (b-a)EU_n$$

$$EU_n \le \frac{1}{(b-a)}\left(E(X_n - a)^+ - E(X_0 - a)^+\right)$$

Doob's upcrossing inequality

(Thm. 5.2.7)

Thm. 5.2.8 (Martingale convergence):

$M = \sup E X_n^+$

$$EU_n \le \frac{1}{b-a} E(X_n - a)^+ \le \frac{EX_n^+ + |a|}{b-a} \le \frac{M + |a|}{b-a}$$

took out negative term from Doob's inequality

$$\underbrace{U_n}_{} \uparrow U(a,b) = \#\text{ of } \underbrace{\{X_n\}}_{} \text{ from } a \text{ to } b$$

non-negative RV
bounded expectation
increases pointwise to $U(a,b)$

entire sequence

$$EU(a,b) < \infty \quad (\text{MCT})$$

$$P(U(a,b) < \infty) = 1$$

$$P\Big(\bigcap_{\substack{a,b \in \mathbb{Q} \\ a<b}} \{U(a,b) < \infty\}\Big) = 1$$

There exists no (a,b) s.t. there are infinite oscillations from below a to above b.

$$\Rightarrow \quad P(X_n \xrightarrow{a.s.} X) = 1$$

Thm. 5.2.9:

$X_n \geq 0$, super-M. Then $X_n \xrightarrow{a.s.} X$ with $EX \leq EX_0$

Pf:/ $Y_n = -X_n$ is a sub-M.

conditions: $\sup E Y_n^+ < \infty$ ✓ b/c $E Y_n^+ = 0$

$\Rightarrow Y_n \to Y$

$\therefore X_n \to X = -Y$.

$E \underline{\lim} X_n \leq \underline{\lim} E X_n$ (Fatou's lemma)

$EX \leq EX_0$ $\quad$ $EX_n \leq EX_0 \ \forall n$ (super-M)

Thm. 5.2.10: Doob Decomposition

Any sub-M X_n can be expressed uniquely as

$X_n = M_n + A_n$, M_n is Mart, A_n is predictable, $A_n \geq A_{n-1} \ \forall n$, $A_0 = 0$.

Pf:/ $X_n = M_n + A_n$

$E(X_n \mid \mathcal{F}_{n-1}) = M_{n-1} + A_n$ $\quad$ M_n is Mart $\quad$ $A_n \in \mathcal{F}_{n-1}$

$= X_{n-1} - A_{n-1} + A_n$

$$A_n = E(X_n | \mathcal{F}_{n-1}) - X_{n-1} + A_{n-1}$$

$\in \mathcal{F}_{n-1}$ $\quad \in \mathcal{F}_{n-1}$ $\quad \in \mathcal{F}_{n-2}$

$\mathcal{F}_{n-2} \subseteq \mathcal{F}_{n-1}$

≥ 0 b/c sub-M

$$\Rightarrow A_n \in \mathcal{F}_{n-1}$$

$$\Rightarrow A_n \geq A_{n-1}$$

$$M_n = X_n - A_n$$

EX: $X_n = \sum_{i=1}^{n} \varepsilon_i$, $\varepsilon_i \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$, $p > 1/2$

$$\begin{aligned} A_n &= E(X_n | \mathcal{F}_{n-1}) - X_{n-1} + A_{n-1} \\ &= E(X_{n-1} + \varepsilon_n | \mathcal{F}_{n-1}) - X_{n-1} + A_{n-1} \\ &= E(\varepsilon_n) + A_{n-1} \\ &= 2p-1 + A_{n-1} \\ &= n(2p-1) \end{aligned}$$

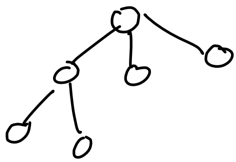
$$M_n = X_n - n(2p-1) = \sum_{i=1}^{n} (\varepsilon_i - (2p-1))$$

Ex: Branching process

$n=0$

$n=1$

$n=2$



offspring distribution D

$P(D=k) = p_k$, $p_k \geq 0$, $\sum_0^\infty p_k = 1$

ε_i^m = # offspring of ith unit of mth generation (time)

Z_n = pop. size at time n $\qquad \varepsilon_i^m \overset{iid}{\sim} D$

$Z_0 = 1$

$$Z_n = \sum_{i=1}^{Z_{n-1}} \varepsilon_i^{n-1}$$

$$\mathcal{F}_n = \{\varepsilon_i^m : m<n, i \geq 1\}$$

$$Z_n = \sum_k \underbrace{I(Z_{n-1}=k)}_{\in \mathcal{F}_{n-1}} \underbrace{\left(\sum_{i=1}^k \varepsilon_i^{n-1}\right)}_{\in \mathcal{F}_n} \quad \in \mathcal{F}_n$$

$\mu = \sum_k k p_k = E(D)$ = mean of offspring dist.

case I : $\mu < 1$

case II : $\mu = 1$ $\quad$ a) $p_1 = 1 \Rightarrow Z_n = 1 \; \forall n$ (no extinction)

$\qquad\qquad\qquad\qquad$ b) $p_1 \neq 1$

case III : $\mu > 1$

case I: $\mu < 1$:

lemma 5.3.6: $\frac{Z_n}{\mu^n}$ is Mart.

Pf:/ $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^n = \sum_k I(Z_n = k)\left(\sum_{i=1}^{k} \xi_i^n\right)$

$$E(Z_{n+1} \mid \mathcal{F}_n) = \sum_k I(Z_n = k)\, E\left(\sum_{i=1}^{k} \xi_i^n \mid \mathcal{F}_n\right)$$

indep.

$$= \sum_k I(Z_n = k)\, E\left(\sum_{i=1}^{k} \xi_i^n\right)$$

$$= \sum_k k\mu\, I(Z_n = k) = Z_n \mu$$

$$E\left(\frac{Z_{n+1}}{\mu^{n+1}} \mid \mathcal{F}_n\right) = \frac{Z_n}{\mu^n}$$ true for any μ.

Thm. 5.3.7: If $\mu < 1$ then $Z_n = 0$ $\forall n$

and $\frac{Z_n}{\mu^n} \xrightarrow{a.s.} 0$.

Pf:/ $E\left(\frac{Z_n}{\mu^n}\right) = E\left(\frac{Z_0}{\mu^0}\right) = 1 \Rightarrow E(Z_n) = \mu^n$

$$P(Z_n > 0) = \sum_{k=1}^{\infty} P(Z_n = k) \le \sum_{k=1}^{\infty} k P(Z_n = k)$$

$$= EZ_n = \mu^n$$

$$\sum P(Z_n > 0) \le \sum_n \mu^n < \infty \quad (\text{as } \mu < 1)$$

$$P(Z_n > 0 \text{ i.o.}) = 0$$

Lec 4: April 6

$$X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$
$$\{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall\, B \in \mathcal{B}(\mathbb{R})$$

Branching Processes

(certain extinction)

Thm. 5.3.8: $\mu = 1$ and $p_1 < 1 \Rightarrow p_0 > 0$

Pf:/ Z_n is a Martingale, $Z_n \geq 0$

$\frac{Z_n}{\mu^n}$ mart, $\mu = 1$ $\Rightarrow Z_n$ mart

$$Z_n \xrightarrow{a.s.} Z_\infty \quad \text{(by super-M convergence)}$$

$$E Z_\infty \leq E Z_0 = 1$$

$$P(Z_\infty < \infty) = 1$$

Z_n is a sequence of integers. So if it converges, it becomes constant

$$A_k = \{Z_\infty = k\}$$

$$\subseteq \bigcup_{N=1}^{\infty} \{Z_n = k,\ \forall\, n \geq N\}, \quad k \neq 0$$

$$P(Z_{n+1} = k \mid Z_n = k) \leq P(Z_{n+1} \neq 0 \mid Z_n = k)$$
$$= 1 - p_0^k$$
$$= \delta < 1$$

$\mu = 1 \cdot p_1 + 2 \cdot p_2 + \cdots + \cdots$

If $p_0 = 0$, $\mu > p_1 + p_2 + \cdots = 1$

but $\mu = 1$ so $p_0 > 0$

$$P(Z_n = k, \forall n \geq N) = P(Z_n = k) P(Z_{n+1} = k \mid Z_n = k)$$
$$\leq P(Z_n = k) \underbrace{\times \delta \times \delta \cdots}_{\text{b/c } \delta < 1} = 0$$

· population cannot $\underbrace{\text{stay}}_{\text{means happening i.o.}}$ stable at any non-zero k

Probability generating functions (PGF):

$$\varphi(s) = E s^X, \quad s \in [0, 1]$$

Discrete RVs:

$$\varphi(s) = \sum_{k=0}^{\infty} p_k s^k, \quad p_k = P(X = k), \ p_k \geq 0,$$

$$\sum_{k=1}^{\infty} p_k = 1$$

ex: $X \sim Ber(p)$

$$\varphi(s) = \sum_{k=0}^{1} p_k s^k \qquad p_0 = 1-p, \quad p_1 = p$$

$$= (1-p) + ps$$

$X \sim Pois(\lambda)$

$$\varphi(s) = \sum_{k=0}^{\infty} p_k s^k$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k s^k}{k!} = e^{-\lambda} e^{\lambda s}$$

Properties of PGFs:

(1) $\varphi(0) = p_0 = P(X=0)$

(2) $\varphi(1) = 1$

(3) $\varphi(s) = \sum_{k=0}^{\infty} p_k s^k$, continuous, non-decreasing

(4) $\varphi'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}$, $s<1$

$\varphi'(1) = \mu$

$\varphi''(s) = \sum_{k=2}^{\infty} k(k-1) p_k s^{k-2}$, $s<1$

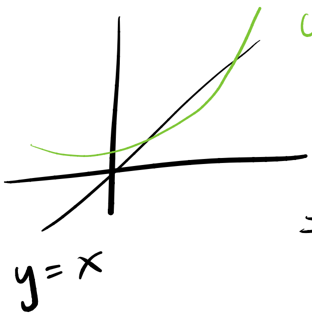
$\varphi''(s) > 0 \Rightarrow \varphi$ is convex.

Fixed Point:

s_0 is a fixed point of a function φ if $\varphi(s_0) = s_0$

Ex: $\varphi(s) = s^2$

$\varphi(s_0) = s_0 \Rightarrow s_0^2 = s_0 \Rightarrow s_0 = 0$ or 1



convex curve cannot cut the line at more than 2 points

$\Rightarrow$ has max 2 fixed points.

Thm. 5-2.9:

$\mu > 1$, Z_n = pop. size at time n

φ_n = pgf of Z_n

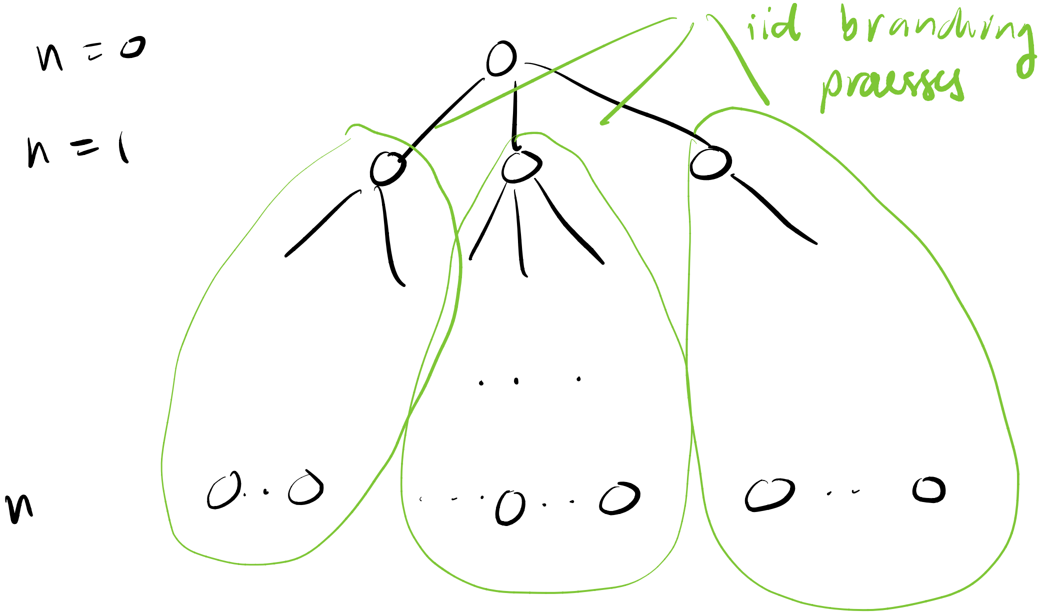
$\varphi_n(0) = P(Z_n = 0) \uparrow$ (φ_n is an increasing function)

$\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$

θ = extinction probability

prob. is cont. under increasing lims

$= P(\lim\{Z_n = 0\}) = \lim P(Z_n = 0) = \lim \varphi_n(0)$



can group descendants based on ancestry at n=1

$$\varphi_n(0) = \sum_{k=0}^{\infty} p_k (\varphi_{n-1}(0))^k$$

· have pop. k at n=1

· so all k "sub-trees" must have population 0.

$$= \sum p_k t^k = \varphi(t), \quad t = \varphi_{n-1}(0)$$

$$= \varphi(\varphi_{n-1}(0))$$

$$\varphi_n(0) = \varphi(\varphi_{n-1}(0)) = \dots = \underbrace{\varphi(\varphi(\cdots\varphi(0)))}_{n \text{ times}}$$

$$\theta = \lim_n \varphi_n(0) \underbrace{=}_{\varphi \text{ is cont.}} \varphi(\lim_n \varphi_{n-1}(0))$$

$$= \varphi(\theta)$$

$\Rightarrow$ extinction probability is a fixed point of PGF

$\rightarrow$ can solve for fixed points of dists. to get extinction probabilities.

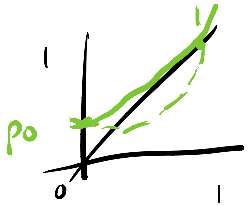
$\varphi(1) = 1$ is a fixed point.

First show that $\exists$ another fixed point of φ in $[0,1)$:

case I: $p_0 = 0$

$\varphi(0) = 0, \ \theta = 0$

case II: $p_0 > 0$



both points must be connected by φ.

$\varphi'(1) = \mu > 1.$

$\Rightarrow$ slope of tangent at $1 > 45°$

so φ must be the dotted line.

$\Rightarrow \exists$ a fixed point $\rho < 1$.

But is $\theta = \rho$ or $\theta = 1$?

$0 \leq \rho$

$\varphi(0) \leq \varphi(\rho) = \rho$

$\vdots$

$\theta = \lim_n \varphi_n(0) \leq \rho$

$\therefore \theta = \rho$

EX: exercise 5.3.13

$p_0 = 1/8, \quad p_1 = 3/8, \quad p_2 = 3/8, \quad p_3 = 1/8$

what's the probability this branching process will die out?

$$\varphi(s) = \frac{1}{8} + \frac{3s}{8} + \frac{3s^2}{8} + \frac{s^3}{8}$$

$\varphi(s) = s$ (solve for a fixed point)

$$s^3 + 3s^2 - 5s + 1 = 0$$

$$s_0 \approx 0.236$$

T = time to extinction.

$$P(T=n) = P(Z_n = 0) - P(Z_{n-1} = 0)$$

$$= \varphi_n(0) - \varphi_{n-1}(0)$$

$ET = \infty$ if $P(T<\infty) < 1$

if $P(T<\infty) = 1$

$$= \sum_{n=0}^{\infty} P(T>n) = \sum_{n=0}^{\infty} P(Z_n > 0) = \sum_{n=0}^{\infty} (1 - \varphi_n(0))$$

Lec 5: April 11

Optional Stopping Theorem:

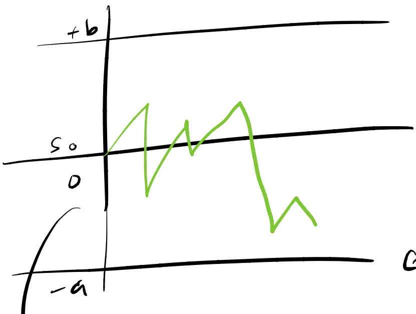
If X_n is a Martingale, and τ is a stopping time, then is $E X_\tau = E X_0$?

Gambler's Ruin Problem:

Gambler's capital = \$a

House capital = \$b

Gambler's profit $S_n = S_0 + \sum_{i=1}^{n} \varepsilon_i$, $\varepsilon_i \overset{iid}{=} \begin{matrix} +1 \\ -1 \end{matrix}$ w.p. $1/2$



House goes broke

Gambler goes broke

can only go up or down by 1 unit at a time.

(symmetric random walk)

τ_{-a} = time to reach $-a$ for the first time

τ_b = time to reach b for the first time.

$$\underbrace{\tau = \tau_{-a} \wedge \tau_b}_{\text{duration of the game.}}$$

minimum

$$P(\tau > n+a+b \mid \mathcal{F}_n) \leq 1-\left(\tfrac{1}{2}\right)^{a+b}$$

HW1 Q3: τ is a stopping time. $P(\tau > n+k \mid \mathcal{F}_n) \leq \delta$, $\delta < 1$ and $k > 0$. Then $E\tau < \infty$.

$P(\tau < \infty) = 1 \Rightarrow$ game ends a.s.

$$P(\text{gamblers' ruin}) = \overbrace{P(\tau_{-a} < \tau_b)}^{P(S_\tau = -a)}$$

$$= 1 - \underbrace{P(\tau_b < \tau_{-a})}_{P(S_\tau = b)}$$

one of these two have to happen b/c $P(\tau < \infty) = 1$

$ES_n = ES_0 = 0$, if $ES_\tau = ES_0$ (OST):

$$ES_\tau = (-a)P(\tau_{-a} < \tau_b) + b(1 - P(\tau_{-a} < \tau_b)) = 0$$

$$P(\tau_{-a} < \tau_b) = \frac{b}{a+b}$$

Counterexample to OST:

τ_b = first time to reach a profit of b.

$$S_{\tau_b} = b$$

$$ES_{\tau_b} \neq ES_0 = 0$$

OST (Williams Thm 10.10):

a.) X_n is a super-M, τ is a stopping time. Then X_τ is integrable and $EX_\tau \leq EX_0$ if any of the following holds:

i) τ is bounded ($\tau \leq k$ a.s. for some k)

ii) $|X_n| \leq k \ \forall n$ and $P(\tau < \infty) = 1$

iii) $E(\tau) < \infty$ and $|X_n - X_{n-1}| \leq k \ \forall n$

b.) If X_n is Mart. then $EX_\tau = EX_0$ if any of (i)–(iii) hold.

(i) $Y_n = X_{\tau \wedge n}$. From thm. 5.2.6 (Durrett), Y_n is super-M.

$\therefore EY_n \leq EY_0 \ \forall n.$

$$\tau \le k \text{ a.s.} \Rightarrow Y_n \xrightarrow[\text{a.s.}]{n\to\infty} X_\tau$$

Fatou's lemma

$$E(X_\tau) = E(\lim Y_n) \le \underline{\lim}\, E(Y_n)$$

$$\le E Y_0$$

$$= E X_0$$

(ii) $P(\tau < \infty) = 1 \Rightarrow X_{\tau\wedge n} \xrightarrow{\text{a.s.}} X_\tau$

$$\underbrace{|X_{\tau\wedge n}| \le K}_{\text{has pointwise limit + boundedness}} \overset{\text{DCT}}{\Rightarrow} E X_{\tau\wedge n} \to E X_\tau$$

$$E X_\tau = \underbrace{E X_{\tau\wedge n} \le}_{\text{b/c super-M}} E X_{\tau\wedge 0} = E X_0$$

at most τ dominator

(iii) $$|X_{\tau\wedge n} - X_0| = \left|\sum_{i=1}^{\tau\wedge n} \underbrace{(X_i - X_{i-1})}_{\text{bounded by } K}\right| \le K\tau$$

$$E(\tau) < \infty \Rightarrow P(\tau < \infty) = 1 \Rightarrow X_{\tau \wedge n} \to X_\tau$$

$$X_{\tau \wedge n} - X_0 \to X_\tau - X_0$$ pointwise limit

$$0 \geq E(X_{\tau \wedge n} - X_0) \xrightarrow{DCT} E(X_\tau) - E(X_0)$$

$$\therefore E(X_\tau) \leq E(X_0)$$

Back to Gamblers' ruin:

$$|S_n - S_{n-1}| = 1, \; E\tau < \infty$$

$\Rightarrow$ OST holds

Expected duration $E(\tau)$

$$X_n = S_n^2 = (S_{n-1} + \varepsilon_n)^2 = S_{n-1}^2 + 2\varepsilon_n S_{n-1} + 1$$

$$E(S_n^2 | \mathcal{F}_{n-1}) = S_{n-1}^2 + 0 + 1$$

$$= S_{n-1}^2 + 1 \quad (\text{sub-}M)$$

(convex transformation of M is sub-M)

$\therefore E(S_n^2 - n \mid \mathcal{F}_{n-1}) = S_{n-1}^2 - (n-1)$

$\therefore X_n = S_n^2 - n$

$E X_\tau = E X_0 = 0$

$E(S_\tau^2) = E(\tau)$

$$a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = ab = E(\tau)$$

$X_n = S_n^2 - n$

$X_n - X_{n-1} = (S_n + S_{n-1})(S_n - S_{n-1}) - 1$

none of the three conditions hold $\Rightarrow$ can't apply OST.

consider: $Y_n = X_{\tau \wedge n}$

$S_{\tau \wedge n}$ is bounded

$|Y_n - Y_{n-1}|$ is bounded

$$E X_0 = E Y_0 = E Y_\tau = E X_\tau$$

Asymmetric Random walk:

$$S_n = \sum_{i=1}^{n} \varepsilon_i, \quad \varepsilon_i = \begin{cases} +1 & \text{wp } p \\ -1 & \text{wp } (1-p) \end{cases} \quad p \neq 1/2$$

$$E(S_n \mid \mathcal{F}_{n-1}) = S_{n-1} + E(\varepsilon_n)$$

$$= S_{n-1} + (2p-1)$$

$$X_n = \varphi(S_n), \quad \varphi(x) = \left(\frac{1-p}{p}\right)^x$$

$$E(X_n \mid \mathcal{F}_{n-1}) = E(\varphi(S_n) \mid \mathcal{F}_{n-1})$$

$$= E\left(\left(\frac{1-p}{p}\right)^{S_{n-1}+\varepsilon_n} \Big| \mathcal{F}_{n-1}\right)$$

$$= \left(\frac{1-p}{p}\right)^{S_n} E\left(\left(\frac{1-p}{p}\right)^{\varepsilon_n}\right)$$

$$= \varphi(S_{n-1})$$

$$E(\varepsilon_n) = \frac{1-p}{p} \cdot p + \frac{p}{1-p} \cdot (1-p)$$

$$= 1$$

$$= X_{n-1}$$

$\Rightarrow X_n$ is a Martingale

$X_n = \left(\frac{1-p}{p}\right)^{S_n}$ is Mart.

$Y_n = X_{\tau \wedge n}$, $|Y_n| \le \varphi(-a) \vee \varphi(b)$

$\therefore E X_\tau = E Y_\tau = E Y_0 = E X_0$

$\therefore E X_\tau = \varphi(-a) P(\tau_{-a} < \tau_b)$

$+ \varphi(b) P(\tau_b < \tau_{-a})$

$\therefore P(\tau_{-a} < \tau_b) = \dfrac{1 - \varphi(b)}{\varphi(-a) - \varphi(b)}$

Lec 6: April 13

$$S_n = S_0 + \sum_{i=1}^{n} \varepsilon_i$$

$$E(S_n \mid \mathcal{F}_{n-1}) = S_{n-1} + (2p-1)$$

$$E(S_n - n(2p-1) \mid \mathcal{F}_{n-1}) = S_{n-1} - (n-1)(2p-1)$$

increments bounded by $2p-1$

$$Z_n = S_n - \underbrace{n(2p-1)}_{\text{linear in } n} \text{ is Mart.}$$

Applying OST:

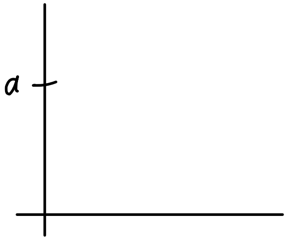
$$E Z_\tau = E Z_0 = 0 \quad \therefore E S_\tau = E\tau(2p-1)$$

$$E\tau = \frac{1}{2p-1}\left(-a P(\tau_{-a} < \tau_b) + b P(\tau_b < \tau_a)\right)$$

If you have a martingale w/ n in it as a linear term, you can use OST to get the expected value of the stopping time.

$p > 1/2$:

$$P(\tau_{-a} < \tau_b) = \frac{1 - \varphi(b)}{\varphi(-a) - \varphi(b)}$$



$P(\tau_{-a} < \infty) \neq 0$ can still go broke even if the game is biased towards you

$\tau_b \xrightarrow{b\to\infty} \infty$, so $P(\tau_{-a} < \infty) = \lim_{b\to\infty} P(\tau_{-a} < \tau_b)$

$$= \lim_{b\to\infty} \frac{1-\varphi(b)}{\varphi(-a)-\varphi(b)}$$

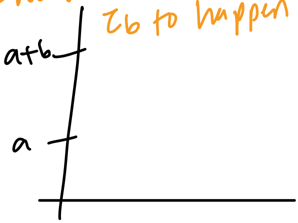
$\varphi(b) = \left(\frac{1-p}{p}\right)^b$

$\to 0 \quad (p > 1/2)$

$$= \frac{1}{\varphi(-a)} = \left(\frac{1-p}{p}\right)^a$$

$P(\tau_b < \infty)$:

need to be here for τ_b to happen



$$P(\tau_b < \infty) = \lim_{a\to\infty} P(\tau_b < \tau_{-a})$$

$$= \lim_{a\to\infty} 1 - \frac{1-\varphi(b)}{\varphi(-a)-\varphi(b)}$$

$\varphi(-a) = \left(\frac{p}{1-p}\right)^a \to \infty$

$(p > 1/2)$

$$= 1 - 0 = 1$$

$E(\tau_{-a}) = \infty$ because $P(\tau_{-a} < \infty) = 1$

$E(\tau_b) = ?$

$Z_n = S_n - n(p-q)$ } not bounded

Can't apply OST b/c $E(\tau_b)$ is not bounded.

Z_n is Mart $\Rightarrow Z_{\tau_b \wedge n}$ is Mart.

$$E\tau_{b \wedge 0} = EZ_0 = 0$$

$$\therefore ES_{\tau_b \wedge n} = E\tau_{b \wedge n}(2p-1)$$

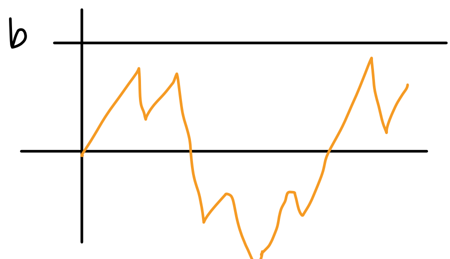
$\tau_b \wedge n \leq \tau_b \wedge (n+1)$

because $P(\tau_b < \infty) = 1$

$\tau_b \wedge n \uparrow \tau_b$

$$\therefore E(\tau_b \wedge n) \xrightarrow{MCT} E(\tau_b)$$

$ES_{\tau_b \wedge n} = E(\tau_b \wedge n)(2p-1)$, $E(\tau_b \wedge n) \uparrow E(\tau_b)$



b/c $S_0 = 0$

$$M = \inf_{0 \leq m < \infty} S_m \leq 0$$

$$M \leq S_{\tau_b \wedge n} \leq b$$

$|S_{\tau_b \wedge n}| \le \underbrace{\underset{\text{random}}{(-M)}}^{\text{dominator}} + \underset{\text{fixed}}{b}$. trying to find a dominator so that we can apply DCT.

need to show its expectation is bounded

$-M \geq 0$, integer RV

tail probabilities formula for integer-valued RVs

$$
\begin{aligned}
E(-M) &= \sum_{a=1}^{\infty} P(-M \geq a) \\
&= \sum_{a=1}^{\infty} P(M \leq -a) \\
&= \sum_{a=1}^{\infty} P(\tau_{-a} < \infty) \\
&= \sum_{a} \left(\frac{1-p}{p}\right)^a < \infty \quad (p > 1/2)
\end{aligned}
$$

$$E S_{\tau_b \wedge n} = E(\tau_b \wedge n)(2p-1)$$

$\downarrow$ DCT $\qquad \downarrow$ MCT

$$b = E S_{\tau_b} = E\tau_b (2p-1)$$

$$\therefore E\tau_b = \frac{b}{2p-1}$$

Monkey writing Shakespeare

ex: Random dice rolls. Keep rolling until you see 1, 2, 3 consecutively.

τ = time for first occurrence of the sequence 123

S_n = total profit of the casino

S_n is a Mart. $ES_n = ES_0 = 0$

Increments of S_n are finite. To apply OST, show

$E(\tau) < \infty$, $P(\tau \leq n+M \mid \mathcal{F}_n) \geq \delta$, $\delta \geq 0$ (HW1Q3)

$(\frac{1}{6})^3$

$$ES_\tau = 0$$

$$S_\tau = \tau - 216$$

$$\Rightarrow E\tau = 216$$
$$= 6^3$$

If we consider the sequence 121 instead:

$$S_\tau = \tau - 216 - 6$$

$$E\tau = 6^3 + 6$$

Monkey Problem:

HAMLET : $E(\tau) = 26^6$

ABRACADABRA : $E(\tau) = 26^{11} + 26 + 26^4$

Lec 7: April 18

Markov chains

Gambler's Ruin:

$$S_n = \sum_{i=1}^{n} \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} \pm 1 \text{ w.p. } p \text{ and } 1-p$$

gambler's capital: \$5, house capital: \$10

τ = either is broke

$X_n = S_{\tau \wedge n}$, X_n can take values in

$$\underbrace{\{-5, -4, \ldots, 9, 10\}}_{\text{State space}}$$

(discrete)

let $\mathcal{F}_n = \{X_1, \ldots, X_n\}$

$$X_{n+1} \mid \mathcal{F}_n = X_n + E(\varepsilon_{n+1} \mid \mathcal{F}_n)$$

X_{n+1} only depends on X_n

$$= \begin{cases} X_n + 1 & \text{w.p. } p \\ X_n - 1 & \text{w.p. } 1-p \end{cases} \quad \text{if } X_n \in \{-4, \ldots, 9\}$$

$$= X_n \text{ if } X_n \in \{-5, 10\}$$

$\Rightarrow X_n$ is a first order Markov chain.

$$X_{100} \mid X_{99}=3 = \begin{cases} 4 & \text{w.p. } p \\ 2 & \text{w.p. } 1-p \end{cases}$$

time doesn't matter, only value of the previous state

$$= X_{15} \mid X_{14}=3$$

$\rightarrow$ time homogeneous

Lemma: If $X_{n+1} \mid \mathcal{G}_n \sim \varphi(X_n)$,

e.g. $\varphi(X_n) = Poi(X_n)$

then $X_{n+1} \mid X_n \sim \varphi(X_n)$

Pf:/ $P(X_{n+1} \in B \mid X_n) = E[I(X_{n+1} \in B) \mid X_n]$

$$= E\left[E\left(I(X_{n+1} \in B) \mid \mathcal{G}_n\right) \mid X_n\right]$$

$$= E\left[P(X_{n+1} \in B \mid \mathcal{G}_n) \mid X_n\right]$$

$$= E\left[\varphi(X_n)(B) \mid X_n\right]$$

$$= \varphi(X_n)(B)$$

Similarly, $X_{n+1} \mid \mathcal{G}_n \sim \varphi(X_n) \Rightarrow X_{n+1} \mid X_n,$

$$\{X_j : j \in A \subseteq \{1, \dots, n\}\} \sim \varphi(X_n)$$

Transition probability matrix (TPM):

$$TPM = P = (P_{ij}), \quad P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Gambler's ruin:

	-5	-4	$\cdot$	$\cdot$	$\cdot$	10
-5	1	0	$\cdots$			
-4	$1-p$	0	p	$\cdots$		
$\vdots$		$1-p$	0	p	$\cdots$	
10						1

$X_{n+1} \mid X_n$ is the i^{th} row of the matrix $= P[i]$

$$P_{ij} \geq 0, \quad \sum_j P_{ij} = 1$$

Defn: A Markov chain (μ, P) is defined by a discrete distribution μ and a TPM P s.t.:

i) $X_0 \sim \mu$

ii) $X_{n+1} = j \mid X_n = i = P_{ij} = X_{n+1} \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i$

Examples:

1./ 1-dim random walk

$$S_n = \sum_{i=1}^{n} \varepsilon_i, \quad \varepsilon_i = \pm 1 \text{ w.p. } p \text{ and } 1-p$$

State space $= \{\cdots -2, -1, 0, 1, 2, \cdots\}$

$S_{n+1} \mid S_n = i$ is $i \pm 1$ w.p. $p, 1-p$

2./ $X_n = S_{\tau \wedge n}$

3./ Smoking status

X_n = binary smoking status of a person in year n

(1 = smoker, 0 = non-smoker) State space $= \{0, 1\}$

$$P(X_{n+1} = 1 \mid X_n = 1) = \alpha \qquad \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-\beta & \beta \\ 1-\alpha & \alpha \end{bmatrix} \end{array}$$

$$P(X_{n+1} = 1 \mid X_n = 0) = \beta$$

4./ Snakes and ladders

$S = \{1, \ldots, 100\}$

$X_{n+1} \mid X_n = 13 \sim \text{Unif}\{14, 15, \cdots, 19\}$

Thm: X_n is Markov(μ, P) iff $P(X_0 = i_0, \ldots, X_n = i_n)$

$$= \mu_{i_0} P_{i_0 i_1} \cdots P_{i_{n-1} i_n}$$

$\forall\, n \geq 0,\ i_0, i_1, \ldots \in S$ (state space)

Pf:/ $(\Leftarrow)$ $n = 0 \Rightarrow P(X_0 = i_0) = \mu_0 \ \forall\, i_0$

$X_0 \sim \mu$

$$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n)$$

$$= \frac{P(X_0 = i_0, \ldots, X_{n+1} = i_{n+1})}{P(X_0 = i_0, \ldots, X_n = i_n)}$$

$$= P_{i_n i_{n+1}} = \varphi(X_n)(i_{n+1})$$

$$X_{n+1} \mid \mathcal{G}_n \sim \varphi(X_n) \Rightarrow X_{n+1} \mid X_n \sim \varphi(X_n)$$

So X_n is a Markov chain w/ TPM (P_{ij})

$(\Rightarrow)$ $P(X_0 = i_0, \ldots, X_n = i_n)$

$$= P(X_0 = i_0)\, P(X_1 = i_1 \mid X_0 = i_0)\, P(X_2 = i_2 \mid X_0 = i_0, X_1 = i_1) \cdots$$

$$= \mu_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots$$

Notation:

$P_\mu(A)$: Markov chain w/ initial distribution μ

$E_\mu(X)$

$\mu = \delta_i = [0, \ldots 0 \overset{i^{th}\text{ entry}}{1} 0, \ldots 0]$

(special case)

$P_{\delta_i}(A) = P_i(A)$

$E_{\delta_i}(X) = E_i(X)$

Marginal distributions:

$X_0 \sim \mu$

$$P(X_1 = j) = \sum_j P(X_1 = j \mid X_0 = i) P(X_0 = i)$$

$$= \sum_i p_{ij} (\mu_i) = (\mu' P)_j$$

$X_1 \sim \mu' P$ (or $P' \mu$)

$X_2 \sim (\mu' P) P = \mu' P^2$ (or $P'^2 \mu$)

$X_n \sim \mu' P^n$

k-step transition probabilities:

$$P = \begin{array}{c c} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$$P(X_{n+2} = \alpha \mid X_n = \alpha) = 1$$

$$P(X_{n+2} = \beta \mid X_n = \beta) = 1$$

two step transition matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = P^2$$

$$\{X_{2n}\} = \{X_0, X_2, X_4, \dots\} \sim \text{Markov}(\mu, P^2)$$

$$\{X_{kn}\} = \{X_0, X_k, X_{2k} \dots\}, \quad X_k \sim \mu P^k$$

$$\therefore \{X_{kn}\} \sim \text{Markov}(\mu, P^k)$$

Chapman-Kolmogorov Equations:

$$P_x(X_{m+n} = z) = \sum_y P_x(X_m = y)\, P_y(X_n = z)$$

Pf:/ $\text{LHS} = (P^{m+n})_{xz}$

$$\text{RHS} = \sum_y \overbrace{(P^m)}^{B}{}_{xy} \overbrace{(P^n)}^{C}{}_{yz}$$

$$= \sum_y B_{xy} C_{xy} = (BC)_{xz} = (P^{m+n})_{xz} = \text{LHS}$$