ECON236/FIN637: HW 3

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Question a

The optimization of the banker is as follows. Per the email exchange, we assume the wealth term of the banker does not include the deposits of the household as the households own the deposits. We also assume the banker gets to keep all excess return from any investments of the household's deposits (and per the prompt, we assume bankers issue interest r_t on deposits to the household).

$$\max_{c_t,\epsilon_t} E_0\left[\int_{t=0}^{\infty} e^{-\rho t}\ln c_t dt\right]$$

with the controls being how much the banker should choose to consume in period t and what proportion of the combined wealth (that of both the banker's and the household's) the banker should choose to invest in the risky asset. Specifically, we let $\epsilon_t(w_t + D_t) = \alpha_t w_t + \beta_t D_t$ where α_t is the share of the banker's wealth to invest in the risky asset and β_t is the share of the deposits (equivalent to the household's wealth) to invest in the risky asset. The dynamic budget constraint of the banker is:

$$dw_t = \epsilon_t(w_t + D_t)(dR_t - r_t dt) + r_t w_t d_t + \cancel{r_t D_t d_t} - \cancel{r_t D_t d_t} - c_t dt$$
$$\iff \boxed{dw_t = \epsilon_t(w_t + D_t)(dR_t - r_t dt) + r_t w_t d_t - c_t dt}$$

where $\boxed{dR_t = \frac{Y_t}{P_t}dt + \frac{dP_t}{P_t}}$ and $\boxed{\epsilon_t(w_t + D_t) = \alpha_t w_t + \beta_t D_t}$. The household's dynamic budget constraint is:

$$\boxed{dw_t^h = r_t D_t dt - c_t^h dt + dL_t}$$

Question b

The consumption goods market clearing condition is:

$$\boxed{c_t + c_t^h = Y_t + L_t}$$

There are 2 asset market clearing conditions here. Following Basak and Cuoco, we assume the bond is in zero net supply, so the bond market clearing conditions are:

$$\alpha_t = \beta_t = \epsilon_t = 1$$

We also normalise the number of stocks to 1, so the risky asset clearing condition is:

$$\boxed{\alpha_t w_t + \beta_t w_t^h = P_t}$$

Taking these conditions together gives us a single asset market clearing condition:

$$\boxed{w_t + w_t^h = P_t}$$

Question c

Since the banker's utility flow is in log utility form, we can guess and verify a value function of the form $v(w, Y) = A \ln w + Bf(Y)$ as in the notes and from lecture, we derived that optimal consumption for the log utility case is: $\boxed{c_t = \rho w_t}$.

The consumption rule of the household is provided in the prompt: $\boxed{c_t^h = \rho^h w_t^h}$.

We then use the market clearing conditions to solve for P_t as a function of the state variables. We have the asset market clearing condition and from the goods market clearing condition, we know that $c_t^h = Y_t + L_t - c_t$:

$$
\begin{aligned}
P_t = w_t^h + w_t = \frac{c_t^h}{\rho^h} + w_t &= \frac{Y_t + L_t - c_t}{\rho^h} + w_t \\
&= \frac{Y_t + L_t - c_t}{\rho^h} + w_t \\
&= \frac{Y_t + L_t - \rho w_t}{\rho^h} + w_t \\
&= \frac{Y_t + L_t}{\rho^h} + w_t \left(1 - \frac{\rho}{\rho^h}\right)
\end{aligned}
$$

Therefore we get:

$$P_t = \frac{D_t}{\rho^{\mathrm{h}}} + \left(1 - \frac{\rho}{\rho^{\mathrm{h}}}\right) w_t$$

and we use that $\eta_t = \frac{P_t}{w_t}$ from the prompt to ensure we are only using state variables:

$$
\begin{aligned}
P_t &= \frac{Y_t + L_t}{\rho^{\mathrm{h}}} + \left(1 - \frac{\rho}{\rho^{\mathrm{h}}}\right) \frac{P_t}{\eta_t} \\
\iff \eta_t P_t &= \eta_t \frac{Y_t + L_t}{\rho^{\mathrm{h}}} + \left(1 - \frac{\rho}{\rho^{\mathrm{h}}}\right) P_t \\
\iff & \boxed{P_t = \frac{Y_t + L_t}{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}}
\end{aligned}
$$

Question d

Now we conjecture that P_t follows the process:

$$\frac{dP_t}{P_t} = \mu_{P,t} dt + \sigma_{P,t} dZ_t$$

Then we can solve for the equilibrium ϵ_t total amount invested in the risky asset. This is very similar to what we had in lecture in that the HJB is:

$$0 = \max_{c_t, \epsilon_t} E_t[e^{-\rho t} u(c_t) + \mathcal{D}V(w_t, Y_t, t) dt]$$

where $u(c_t) = \ln(c_t)$, w_t is the endogenous state, and Y_t is the exogenous state. The form of the value function is similarly:

$$V(w_t, Y_t, t) = e^{-\rho t} v(w_t, Y_t)$$

Following the slides, for log utility we can guess and verify the following form for $v(w, Y)$:

$$v(w, Y) = Alnw + Bf(Y)$$

So that:

$$\mathcal{D}V(w_t, Y_t, t) = -\rho V dt + V_Y E_t[dY_t] + \frac{1}{2}V_{YY}[dY_t.dY_t] + V_w E_t[dw_t] + \frac{1}{2}V_{ww}[dw_t.dw_t]$$

Note that due to the log utility form, we can separate out w and Y and do not have the usual cross terms in Ito's lemma. Taking FOCs we therefore have:

$$\epsilon_t^* = \frac{-V_w}{w_t V_{ww}} \frac{\frac{Y_t}{P_t} + \mu_{P,t} - r_t}{(\sigma_{P,t})^2}$$

where $v_w = \frac{A}{w_t}$ and $v_{ww} = -\frac{A}{w_t^2}$, so the optimal ϵ_t:

$$\epsilon_t^* = \frac{\frac{Y_t}{P_t} + \mu_{P,t} - r_t}{(\sigma_{P,t})^2}$$

In equilibrium, we previously argued that $\epsilon_t = 1$ so we can therefore write the relation between $\mu_{P,t}$ and $\sigma_{P,t}$ as:

$$\begin{aligned}
&\epsilon_t = 1 \\
\iff &\frac{\mu_{P,t} + \frac{Y_t}{P_t} - r_t}{(\sigma_{P,t})^2} = 1 \\
\iff &\boxed{\mu_{P,t} + \frac{Y_t}{P_t} - r_t = (\sigma_{P,t})^2}
\end{aligned}$$

Question e

We use the goods market clearing condition and differentiate; then we plug in the dynamic budget constraints. Note that with the above conjectured process for P_t, dR_t can be expressed as:

$$dR_t = \frac{Y_t}{P_t}dt + \frac{dP_t}{P_t} = \frac{Y_t}{P_t}dt + \mu_{P,t}dt + \sigma_{P,t}dZ_t$$

Under the goods market clearing condition we have (where per the email exchange we work with the case where $g_Y = g_L = g$):

$$\begin{aligned}
&c_t + c_t^h = Y_t + L_t \\
\iff &\rho w_t + \rho^h w^h = Y_t + L_t \\
\implies &\rho dw_t + \rho^h dw^h \\
\iff &\rho(\epsilon_t(w_t + D_t)(dR_t - r_t dt) + r_t w_t d_t - c_t dt) + \rho^h(r_t D_t dt - c_t^h dt + dL_t) = dY_t + dL_t \\
\iff &\rho(\epsilon_t(w_t + D_t)(dR_t - r_t dt) + r_t w_t d_t - c_t dt) + \rho^h(r_t D_t dt - c_t^h dt) = dY_t + (1-\rho^h)dL_t \\
\iff &\rho(\epsilon_t(w_t + D_t)(dR_t - r_t dt) + r_t w_t d_t - c_t dt) + \rho^h(r_t D_t dt - c_t^h dt) = Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)(L_t g dt + \sigma_L L_t dZ_t) \\
\iff &\rho\epsilon_t(w_t + D_t)dR_t - \rho\epsilon_t(w_t + D_t)r_t dt + \rho r_t w_t d_t - \rho c_t dt + \rho^h r_t D_t dt - \rho^h c_t^h dt \\
&= Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)L_t g dt + (1-\rho^h)\sigma_L L_t dZ_t \\
\iff &\rho\epsilon_t(w_t + D_t)dR_t = \rho\epsilon_t(w_t + D_t)r_t dt - \rho r_t w_t d_t + \rho c_t dt - \rho^h r_t D_t dt + \rho^h c_t^h dt \\
&+ Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)L_t g dt + (1-\rho^h)\sigma_L L_t dZ_t \\
\iff &\rho\epsilon_t(w_t + D_t)(\frac{Y_t}{P_t}dt + \mu_{P,t}dt + \sigma_{P,t}dZ_t) = \rho\epsilon_t(w_t + D_t)r_t dt - \rho r_t w_t d_t + \rho c_t dt - \rho^h r_t D_t dt + \rho^h c_t^h dt \\
&+ Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)L_t g dt + (1-\rho^h)\sigma_L L_t dZ_t
\end{aligned}$$

Matching coefficients on the dZ_t terms we get:

$$\begin{aligned}\rho\epsilon_t(w_t + D_t)\sigma_{P,t}dZ_t &= Y_t\sigma_Y dZ_t + (1-\rho^h)\sigma_L L_t dZ_t \\ \iff \rho\epsilon_t(w_t + D_t)\sigma_{P,t} &= Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t \\ \iff \sigma_{P,t} &= \frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho\epsilon_t(w_t + D_t)}\end{aligned}$$

where in equilibrium our asset market clearing condition implies $\epsilon_t = 1$ so we can write as:

$$\sigma_{P,t} = \frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho(w_t + D_t)}$$

where $w_t = P_t/\eta_t$ and $D_t = P_t/\epsilon_t - w_t$ so $w_t + D_t = P_t/\epsilon_t = P_t$ under equilibrium of $\epsilon_t = 1$. So substituting that in we get:

$$\sigma_{P,t} = \frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho P_t}$$

From Question c we have:

$$P_t = \frac{Y_t + L_t}{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}$$

and substituting this back into our expression for $\sigma_{P,t}$, we get:

$$\boxed{\sigma_{P,t} = \left(\frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho}\right)\left(\frac{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}{Y_t + L_t}\right)}$$

Question f

We now solve for risk premium which is $\pi_{R,t} = \mu_{R,t} - r_t = \mu_{P,t} + \frac{Y_t}{P_t} - r_t$. Note from Question d, the relation we have between $\mu_{P,t}$ and $\sigma_{P,t}$ is:

$$\begin{aligned}\mu_{P,t} + \frac{Y_t}{P_t} - r_t &= (\sigma_{P,t})^2 \\ \iff \pi_{R,t} &= (\sigma_{P,t})^2\end{aligned}$$

Then using our expression from Questions c and e we get:

$$\boxed{\pi_{R,t} = \left(\left(\frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho}\right)\left(\frac{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}{Y_t + L_t}\right)\right)^2}$$

Question g

From the prompt, we've made the assumption that $\rho^h > \rho$. The intuition here means households are less patient than the bankers. Recall in question c, the expression for price is:

$$P_t = \frac{Y_t + L_t}{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}$$

so we can see that increased leverage decreases price. Intuitively, higher leverage means more of the funds that the banker uses for investments belong to the household. Since the households are more impatient, a greater share of the total wealth stream is consumed, so the price of the asset is lower by the market clearing condition.

In question e, our volatility term is:

$$\sigma_{P,t} = \left(\frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho}\right)\left(\frac{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}{Y_t + L_t}\right)$$

so we can see that as leverage increases, volatility increases.Again, as leverage increases the stream of wealth is more dependent on the wealth process of the household. And so changes in the stream of household wealth are amplified in the dynamics of the total funds for investments. Since households are more impatient, we see more uncertainty and therefore more volatility.

In question f, the term for risk premium is:

$$\pi_{R,t} = \left(\left(\frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho}\right)\left(\frac{\rho^h + \frac{1}{\eta_t}(\rho - \rho^h)}{Y_t + L_t}\right)\right)^2$$

so we can see that as leverage increases, the risk premium increases. This is intuitive as we have seen higher leverage means the volatility in the return on the risky asset is higher, so greater compensation is required to invest in the risky asset.