ECON236/FIN637: HW 3

Cindy Chung and Emma Rockall

November 29, 2021

Question a

The optimization of the banker is as follows. Per the email exchange, we assume the wealth term of the banker does not include the deposits of the household as the households own the deposits. We also assume the banker gets to keep all excess return from any investments of the household's deposits (and per the prompt, we assume bankers issue interest r_t on deposits to the household).

$$\max_{c_t,\alpha_t,\beta_t} E_0\left[\int_{t=0}^{\infty} e^{-\rho t}\ln c_t dt\right]$$

with the controls being how much to consume in period t, how much of their own wealth to invest in the risky asset (α_t), and how much of the household's wealth (in the form of the deposits) to invest in the risky asset vs the bond holdings (β_t). The dynamic budget constraint of the banker is:

$$dw_t = \epsilon_t(dR_t - r_t dt) + r_t w_t d_t + \cancel{r_t D_t d_t} - \cancel{r_t D_t d_t} - c_t dt$$
$$\iff \boxed{dw_t = \epsilon_t(dR_t - r_t dt) + r_t w_t d_t - c_t dt}$$

where $\boxed{dR_t = \frac{Y_t}{P_t}dt + \frac{dP_t}{P_t}}$ and $\boxed{\epsilon_t = \alpha_t w_t + \beta_t D_t}$. The household's dynamic budget constraint is:

$$\boxed{dw_t^h = r_t D_t dt - c_t^h dt + dL_t}$$

Question b

The consumption goods market clearing condition is:

$$\boxed{c_t + c_t^h = Y_t + L_t}$$

There are 2 asset market clearing conditions here. Following Basak and Cuoco, we assume the bond is in zero net supply, so the bond market clearing conditions are:

$$\alpha_t = \beta_t = 1$$

We also normalise the number of stocks to 1, so the risky asset clearing condition is:

$$\alpha_t w_t + \beta_t w_t^h = P_t$$

Taking these conditions together gives us a single asset market clearing condition:

$$\boxed{w_t + w_t^h = P_t}$$

Question c

Since the banker's utility flow is in log utility form, we can guess and verify a value function of the form $v(w, Y) = A \ln w + Bf(Y)$ as in the notes and from lecture, we derived that optimal consumption for the log utility case is: $\boxed{c_t = \rho w_t}$.

The consumption rule of the household is provided in the prompt: $\boxed{c_t^h = \rho^h w_t^h}$.

We then use the market clearing conditions to solve for P_t as a function of the state variables. We have the asset market clearing condition and from the goods market clearing condition, we know that $c_t^h = Y_t + L_t - c_t$:

$$\begin{aligned} P_t = w_t^h + w_t = \frac{c_t^h}{\rho^h} + w_t &= \frac{Y_t + L_t - c_t}{\rho^h} + w_t \\ &= \frac{Y_t + L_t - c_t}{\rho^h} + w_t \\ &= \frac{Y_t + L_t - \rho w_t}{\rho^h} + w_t \\ &= \frac{Y_t + L_t}{\rho^h} + w_t \left(1 - \frac{\rho}{\rho^h}\right) \end{aligned}$$

Therefore we get:

$$P_t = \frac{D_t}{\rho^{\mathrm{h}}} + \left(1 - \frac{\rho}{\rho^{\mathrm{h}}}\right) w_t$$

and we use that $\eta_t = \frac{P_t}{w_t}$ from the prompt to ensure we are only using state variables:

$$\begin{aligned} P_t &= \frac{Y_t + L_t}{\rho^{\mathrm{h}}} + \left(1 - \frac{\rho}{\rho^{\mathrm{h}}}\right) \frac{P_t}{\eta_t} \\ \iff \eta_t P_t &= \eta_t \frac{Y_t + L_t}{\rho^{\mathrm{h}}} + \left(1 - \frac{\rho}{\rho^{\mathrm{h}}}\right) P_t \\ \iff & \boxed{P_t = \frac{\frac{Y_t + L_t}{\rho^{\mathrm{h}}} \eta_t}{\eta_t - 1 + \frac{\rho}{\rho^h}}} \end{aligned}$$

Question d

Now we conjecture that P_t follows the process:

$$\frac{dP_t}{P_t} = \mu_{P,t} dt + \sigma_{P,t} dZ_t$$

Then we can solve for the equilibrium ϵ_t total amount invested in the risky asset. This is very similar to what we had in lecture in that the HJB is:

$$0 = \max_{c_t, \epsilon_t} E_t[e^{-\rho t} u(c_t) + \mathcal{D}V(w_t, Y_t, t) dt]$$

where $u(c_t) = \ln(c_t)$, w_t is the endogenous state, and Y_t is the exogenous state. The form of the value function is similarly:

$$V(w_t, Y_t, t) = e^{-\rho t} v(w_t, Y_t)$$

so that: **insert equation here on slide 18** Per the slides, one can conjecture and verify the following form for $v(w, Y)$: **insert conjectured v eqn on slide 19** so following similar steps to the lecture, we conclude the following optimal ϵ_t:

$$\epsilon_t = \frac{\frac{Y_t}{P_t} + \mu_{P,t} - r_t}{(\sigma_{P,t})^2}$$

Under the asset market clearing condition we have:

$$\begin{aligned} \epsilon_t &= w_t + w_t^h = P_t \\ \iff \frac{\frac{Y_t}{P_t} + \mu_{P,t} - r_t}{(\sigma_{P,t})^2} &= P_t \end{aligned}$$

Using our expression for P_t from (c), we can then write the relation between $\mu_{P,t}$ and $\sigma_{P,t}$ in state variables as:

$$\mu_{P,t}dt + \sigma_{P,t}dZ_t =$$

Option 2 starts here:

$$\begin{aligned}
& w_t + w_t^h = P_t \\
\iff & dw_t + dw_t^h = dP_t \\
\iff & (\eta_t(dR_t - r_t dt) + r_t w_t dt + \beta_t D_t(dR_t - r_t dt) - \beta_t r_t D_t dt - c_t dt) + (r_t D_t dt - c_t^h dt) = P_t\mu_{P,t}dt + P_t\sigma_{P,t}dZ_t \\
\iff & (\eta_t dR_t - \eta_t r_t dt + r_t w_t dt + \beta_t D_t dR_t - \beta_t D_t r_t dt - \beta_t r_t D_t dt - c_t dt) + (r_t D_t dt - c_t^h dt) \\
& = P_t\mu_{P,t}dt + P_t\sigma_{P,t}dZ_t \\
\iff & ((\eta_t + \beta_t D_t)dR_t - \eta_t r_t dt + r_t w_t dt - \beta_t D_t r_t dt - \beta_t r_t D_t dt - c_t dt) + (r_t D_t dt - c_t^h dt) \\
& = P_t\mu_{P,t}dt + P_t\sigma_{P,t}dZ_t \\
\iff & ((\eta_t + \beta_t D_t)(\frac{Y_t}{P_t}dt + \mu_{P,t}dt + \sigma_{P,t}dZ_t) - \eta_t r_t dt + r_t w_t dt - \beta_t D_t r_t dt - \beta_t r_t D_t dt - c_t dt) + (r_t D_t dt - c_t^h dt) \\
& = P_t\mu_{P,t}dt + P_t\sigma_{P,t}dZ_t \\
\iff & (\eta_t + \beta_t D_t)(\frac{Y_t}{P_t}dt + \mu_{P,t}dt + \sigma_{P,t}dZ_t) + K_t dt = P_t\mu_{P,t}dt + P_t\sigma_{P,t}dZ_t \quad \text{(condensing terms)} \\
\iff & (\eta_t + \beta_t D_t)\frac{Y_t}{P_t}dt + (\eta_t + \beta_t D_t)(\mu_{P,t}dt + \sigma_{P,t}dZ_t) + K_t dt = P_t(\mu_{P,t}dt + \sigma_{P,t}dZ_t) \\
\iff & (\eta_t + \beta_t D_t)\frac{Y_t}{P_t}dt + K_t dt = (P_t - \eta_t - \beta_t D_t)(\mu_{P,t}dt + \sigma_{P,t}dZ_t)
\end{aligned}$$

where $K_t dt = -\eta_t r_t dt + r_t w_t dt - \beta_t D_t r_t dt - \beta_t r_t D_t dt - c_t dt) + (r_t D_t dt - c_t^h dt)$. So the relation we get between $\mu_{P,t}$ and $\sigma_{P,t}$ is:

$$\mu_{P,t}dt + \sigma_{P,t}dZ_t = \frac{(\eta_t + \beta_t D_t)\frac{Y_t}{P_t}dt + K_t dt}{(P_t - \eta_t - \beta_t D_t)}$$

and where $\beta_t = \frac{Y_t + L_t + \mu_{P,t} - r_t}{\sigma_{P,t}^2}$.

Question e

We use the goods market clearing condition and differentiate; then we plug in the dynamic budget constraints. Note that with the above conjectured process for P_t, dR_t can be expressed as:

$$dR_t = \frac{Y_t}{P_t}dt + \frac{dP_t}{P_t} = \frac{Y_t}{P_t}dt + \mu_{P,t}dt + \sigma_{P,t}dZ_t$$

Under the goods market clearing condition we have (where per the email exchange we work with the case where $g_Y = g_L = g$):

$$
\begin{aligned}
& c_t + c_t^h = Y_t + L_t \\
\iff & \rho w_t + \rho^h w^h = Y_t + L_t \\
\implies & \rho dw_t + \rho^h dw^h \\
\iff & \rho(\epsilon_t(dR_t - r_t dt) + r_t w_t d_t - c_t dt) + \rho^h(r_t D_t dt - c_t^h dt + dL_t) = dY_t + dL_t \\
\iff & \rho(\epsilon_t(dR_t - r_t dt) + r_t w_t d_t - c_t dt) + \rho^h(r_t D_t dt - c_t^h dt) = dY_t + (1-\rho^h)dL_t \\
\iff & \rho(\epsilon_t(dR_t - r_t dt) + r_t w_t d_t - c_t dt) + \rho^h(r_t D_t dt - c_t^h dt) = Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)(L_t g dt + \sigma_L L_t dZ_t) \\
\iff & \rho\epsilon_t dR_t - \rho\epsilon_t r_t dt + \rho r_t w_t d_t - \rho c_t dt + \rho^h r_t D_t dt - \rho^h c_t^h dt \\
& = Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)L_t g dt + (1-\rho^h)\sigma_L L_t dZ_t \\
\iff & \rho\epsilon_t dR_t = \rho\epsilon_t r_t dt - \rho r_t w_t d_t + \rho c_t dt - \rho^h r_t D_t dt + \rho^h c_t^h dt \\
& + Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)L_t g dt + (1-\rho^h)\sigma_L L_t dZ_t \\
\iff & \rho\epsilon_t(\frac{Y_t}{P_t}dt + \mu_{P,t}dt + \sigma_{P,t}dZ_t) = \rho\epsilon_t r_t dt - \rho r_t w_t d_t + \rho c_t dt - \rho^h r_t D_t dt + \rho^h c_t^h dt \\
& + Y_t g dt + Y_t \sigma_Y dZ_t + (1-\rho^h)L_t g dt + (1-\rho^h)\sigma_L L_t dZ_t
\end{aligned}
$$

Matching coefficients on the dZ_t terms we get:

$$
\begin{aligned}
\rho\epsilon_t\sigma_{P,t}dZ_t &= Y_t\sigma_Y dZ_t + (1-\rho^h)\sigma_L L_t dZ_t \\
\iff \quad \rho\epsilon_t\sigma_{P,t} &= Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t \\
\iff \quad \sigma_{P,t} &= \frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho\epsilon_t}
\end{aligned}
$$

where in equilibrium our asset market clearing condition implies $\epsilon_t = P_t$ so we can write as:

$$\sigma_{P,t} = \frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho P_t}$$

where from Question c we have:

$$P_t = \frac{\frac{Y_t + L_t}{\rho^{\mathrm{h}}}\eta_t}{\eta - 1 + \frac{\rho}{\rho^h}}$$

so substituting that in we get:

$$\boxed{\sigma_{P,t} = \left(\frac{Y_t\sigma_Y + (1-\rho^h)\sigma_L L_t}{\rho}\right)\left(\frac{\eta_t - 1 + \frac{\rho}{\rho^h}}{\frac{Y_t+L_t}{\rho^{\mathrm{h}}}\eta_t}\right)}$$

Question f

We now solve for risk premium which is $\pi_{R,t} = \mu_{R,t} - r_t = \mu_{P,t} + \frac{Y_t}{P_t} - r_t$. To do so, we first solve for $\mu_{P,t}$. Recall our expression we derived for $\mu_{P,t}$ and $\sigma_{P,t}$ is:

$$
\begin{aligned}
\mu_{P,t}dt + \sigma_{P,t}dZ_t &= \frac{(\eta_t + \beta_t D_t)\frac{Y_t}{P_t}dt + K_t dt}{(P_t - \eta_t - \beta_t D_t)} \\
&\iff
\end{aligned}
$$

Question g