Homework 1

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1.1 a

Probability distribution:

- $\mathbf{Pr}(Y=0)=\frac{1}{4}$
- $\mathbf{Pr}(Y=1)=\frac{1}{2}$
- $\mathbf{Pr}(Y=2)=\frac{1}{4}$

1.2 b

Cumulative distribution:

- $\mathbf{Pr}(Y\leq 0)=\frac{1}{4}$
- $\mathbf{Pr}(Y\leq 1)=\frac{3}{4}$
- $\mathbf{Pr}(Y\leq 2)=1$

1.3 c

Mean and variance of Y:

$$\mathbb{E}[Y]=\sum_{i=1}^{3} y_i \mathbf{Pr}(Y=y_i)=0(1/4)+1(1/2)+2(1/4)=1$$

$$\begin{aligned}
Var(Y)=\sigma_Y^2 &= \sum_{i=1}^{3}(y_i-\mathbb{E}[Y])^2\mathbf{Pr}(Y=y_i)\\
&=(0-1)^2(1/4)+(1-1)^2(1/2)+(2-1)^2(1/4)\\
&=\frac{1}{2}
\end{aligned}$$

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Given information in Fahrenheit: $\sigma_Y=Std(Y)=7 \implies Var(Y)=\sigma_Y^2=49$, $\mathbb{E}[Y]=70$.

Now define a new variable $X=(Y-32)\frac{5}{9}$.

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}\left[\frac{5}{9}(Y-32)\right]\\
&=\frac{5}{9}\left(\mathbb{E}[Y]-32\right)\\
&=\frac{5}{9}(70-32)=\frac{5}{9}(38)
\end{aligned}$$

$$\begin{aligned}
Var(X) &= Var\left(\frac{5}{9}(Y-32)\right) \\
&= \frac{25}{81}Var(Y) \\
&= \frac{25}{81}(49)
\end{aligned}$$

$$Std(X) = \sqrt{Var(X)} = \frac{5}{9}(7)$$

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3.1 a

Probability distribution, $\mathbb{E}[Y]$, $\text{Var}(Y)$

$$\mathbf{Pr}(Y=y) = \sum_x \mathbf{Pr}(Y=y, X=x)$$

- $\mathbf{Pr}(Y=14) = \mathbf{Pr}(Y=14|X=1) + \mathbf{Pr}(Y=14|X=5) + \mathbf{Pr}(Y=14|X=8) = 0.02+0.17+0.02 = 0.21$
- $\mathbf{Pr}(Y=22) = 0.05+0.15+0.03 = 0.23$
- $\mathbf{Pr}(Y=40) = 0.1+0.05+0.15 = 0.3$
- $\mathbf{Pr}(Y=40) = 0.03+0.02+0.1 = 0.15$
- $\mathbf{Pr}(Y=65) = 0.01+0.01+0.09 = 0.11$

$$\mathbb{E}[Y] = 14(0.21) + 22(0.23) + 30(0.3) + 40(0.15) + 65(0.11) = 30.15$$

$$\begin{aligned}
\text{Var}(Y) &= (14-30.15)^2(0.21) + (22-30.15)^2(0.23) \\
&+ (30-30.15)^2(0.3) + (40-30.15)^2(0.15) + (65-30.15)^2(0.11) \\
&= 218.2075
\end{aligned}$$

Verify law of total probability:

$$0.21+0.23+0.3+0.15+0.11 = 1$$

Verify variance:

$$14^2(0.21) + 22^2(0.23) + 900(0.3) + 1600(0.15) + 65^2(0.11) - 30.15^2 = 218.2075$$

3.2 b

Probability distribution, conditional expectation and variance given $X=8$

$$\mathbf{Pr}(Y=y|X=8) = \frac{\mathbf{Pr}(Y=y, X=8)}{\mathbf{Pr}(X=8)}$$

Begin by finding with law of total probability:

$$\begin{aligned}
Pr(X=8) &= \sum_y \mathbf{Pr}(X=8, Y=y) \\
&= \mathbf{Pr}(X=8, Y=14) + \mathbf{Pr}(X=8, Y=22) + \mathbf{Pr}(X=8, Y=30) \\
&+ \mathbf{Pr}(X=8, Y=40) + \mathbf{Pr}(X=8, Y=65) \\
&= 0.02+0.03+0.15+0.1+0.09 \\
&= 0.39
\end{aligned}$$

- $\mathbf{Pr}(Y = 14|X = 8) = \frac{\mathbf{Pr}(Y=14, X=8)}{\mathbf{Pr}(X=8)} = \frac{0.02}{0.39}$
- $\mathbf{Pr}(Y = 22|X = 8) = \frac{0.03}{0.39}$
- $\mathbf{Pr}(Y = 30|X = 8) = \frac{0.15}{0.39}$
- $\mathbf{Pr}(Y = 40|X = 8) = \frac{0.1}{0.39}$
- $\mathbf{Pr}(Y = 65|X = 8) = \frac{0.09}{0.39}$

Verify the conditional distribution with law of total probability:

$$\frac{0.02}{0.39} + \frac{0.03}{0.39} + \frac{0.15}{0.39} + \frac{0.1}{0.39} + \frac{0.09}{0.39} = 1$$

$$\begin{aligned}
\mathbb{E}[Y|X = 8] &= \sum_{i=1}^{5} y_i \mathbf{Pr}(Y = y_i|X = 8) \\
&= 14(0.02) + 22(0.03) + 30(0.15) + 40(0.1) + 65(0.09) \\
&= 15.29
\end{aligned}$$

$$\begin{aligned}
\text{Var}[Y|X = 8] &= \sum_{i=1}^{5} (y_i - \mathbb{E}[Y|X = 8])^2 \mathbf{Pr}(Y = y_i|X = 8) \\
&= (14 - 15.29)^2 \frac{0.02}{0.39} + (22 - 15.29)^2 \frac{0.03}{0.39} + (30 - 15.29)^2 \frac{0.15}{0.39} \\
&+ (40 - 15.29)^2 \frac{0.1}{0.39} + (65 - 15.29)^2 \frac{0.09}{0.39} \\
&\approx 813.5835871794
\end{aligned}$$

3.3 c

Covariance and correlation

First compute

$$\mathbb{E}[X] = 1(0.21) + 5(0.4) + 8(0.39) = 5.33$$

$$Var(X) = 1(0.21) + 25(0.4) + 64(0.39) - 5.33^2 = 6.7611$$

$$\begin{aligned}
Cov(X, Y) &= \sum_i \sum_j (x_j - \mu_X)(y_i - \mu_Y) \mathbf{Pr}(X = x_j, Y = y_i) \\
&= 0.439495 + 1.1319 + 9.429105 \\
&= 11.0005
\end{aligned}$$

$$\begin{aligned}
Corr(X, Y) &= \frac{Cov(X, Y)}{\sigma_x \sigma_Y} \\
&= \frac{Cov(X, Y)}{\sqrt{6.7611}\sqrt{218.2075}} \\
&= \frac{11.0005}{\sqrt{6.7611}\sqrt{218.2075}} \\
&\approx 0.28639729306
\end{aligned}$$

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4.1 a

$Y \sim N(1,4)$ find $\mathbf{Pr}(Y \leq 3)$

$$\mathbf{Pr}(Y \leq 3) = \mathbf{Pr}\left(\frac{Y-1}{2} = Z \leq \frac{3-1}{2}\right) = \Phi(1) \approx 0.8413$$

4.2 b

$Y \sim N(3,9)$ find $\mathbf{Pr}(Y > 0)$

$$\mathbf{Pr}(Y > 0) = 1 - \mathbf{Pr}(Y \leq 0) = 1 - (\mathbf{Pr}\left(Z \leq \frac{0-3}{3}\right) = 1 - \Phi(-1) \approx 1 - 0.1587 \approx 0.8413$$

4.3 c

$Y \sim N(50,25)$, find $\mathbf{Pr}(40 \leq Y \leq 52)$

$$\mathbf{Pr}\left(\frac{40-50}{5} \leq Z \leq \frac{52-50}{5}\right) = \mathbf{Pr}(-2 \leq Z \leq 2/5) = \Phi(0.4) - \Phi(-2) \approx 0.6326$$

4.4 d

$Y \sim N(5,2)$, find $\mathbf{Pr}(6 \leq Y \leq 8)$

$$\mathbf{Pr}\left(\frac{6-5}{\sqrt{2}} \leq Z \leq \frac{3}{\sqrt{2}}\right) = \Phi\left(\frac{3}{\sqrt{2}}\right) - \Phi\left(\frac{1}{\sqrt{2}}\right) \approx 0.98305 - 0.76025 \approx 0.2228$$