SPRING 2023

Information Technology Research

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Lecture slides are based on the supplemental materials of the textbook: https://algs4.cs.princeton.edu

http://algs4.cs.princeton.edu

1.5 Union-Find

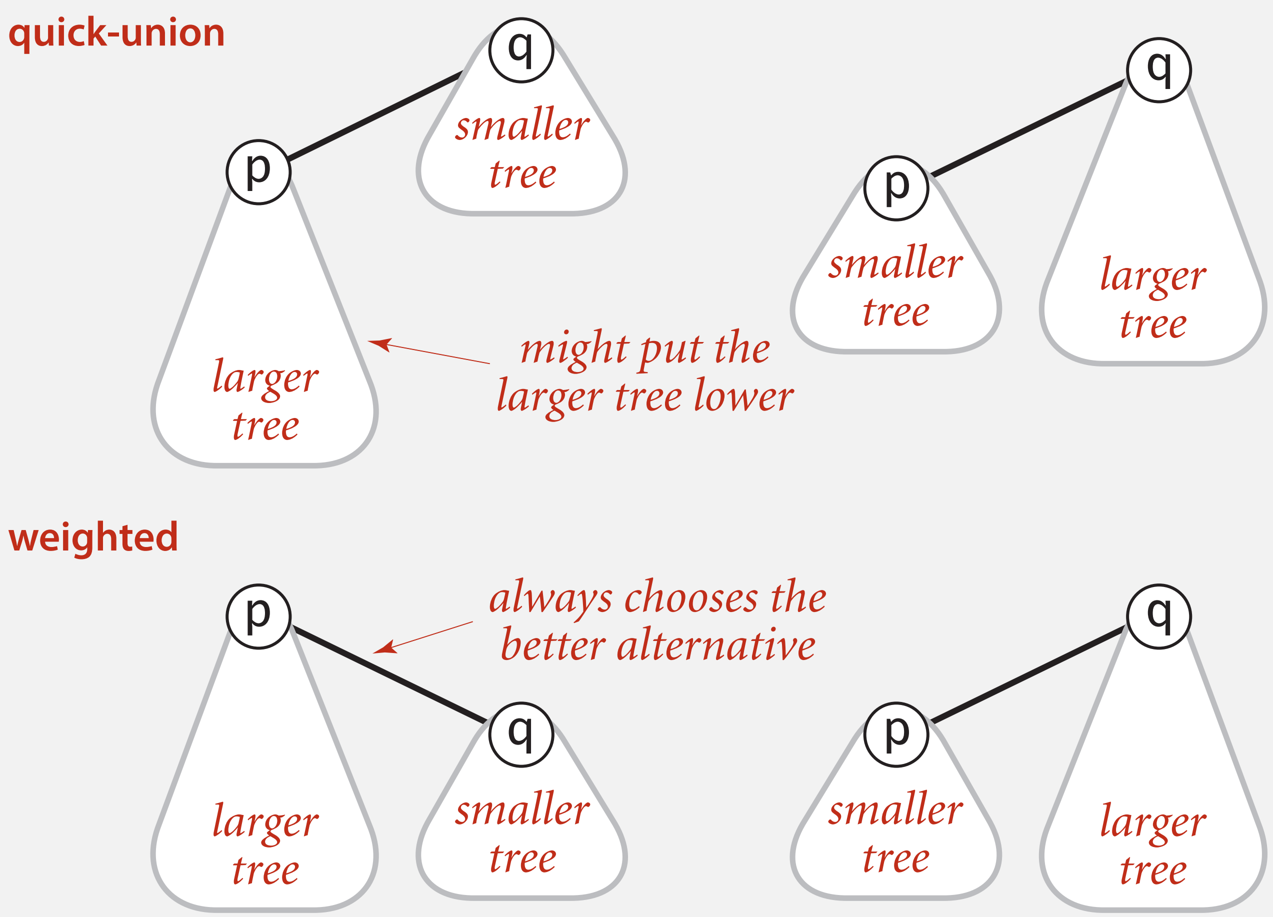
- dynamic connectivity
- quick find
- quick union
- *improvements*
- applications

Improvement 1: weighting

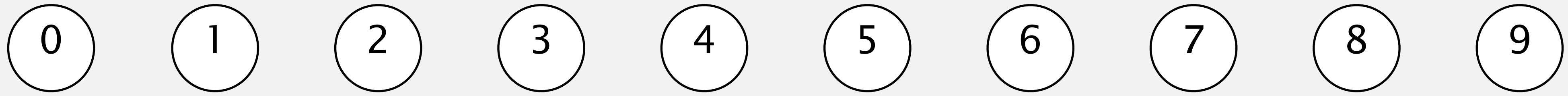
Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternative:
union by height



Weighted quick-union demo

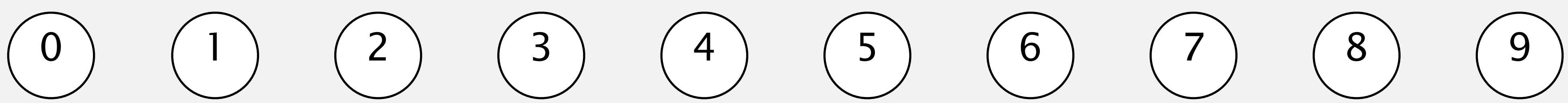


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Weighted quick-union demo

後面掛到前面

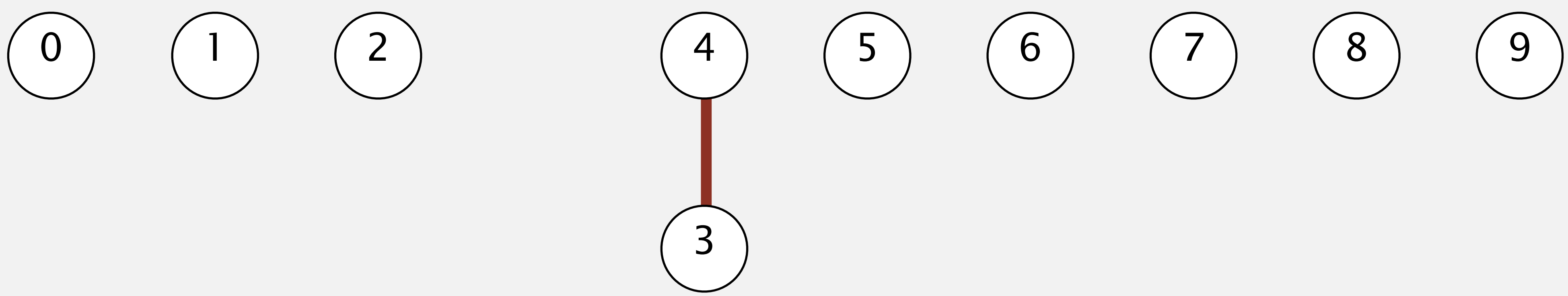
union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

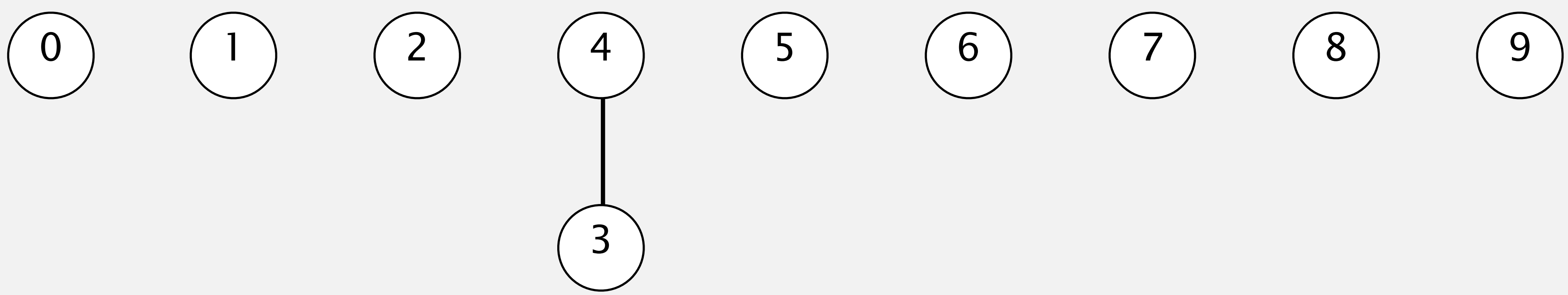
Weighted quick-union demo

union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

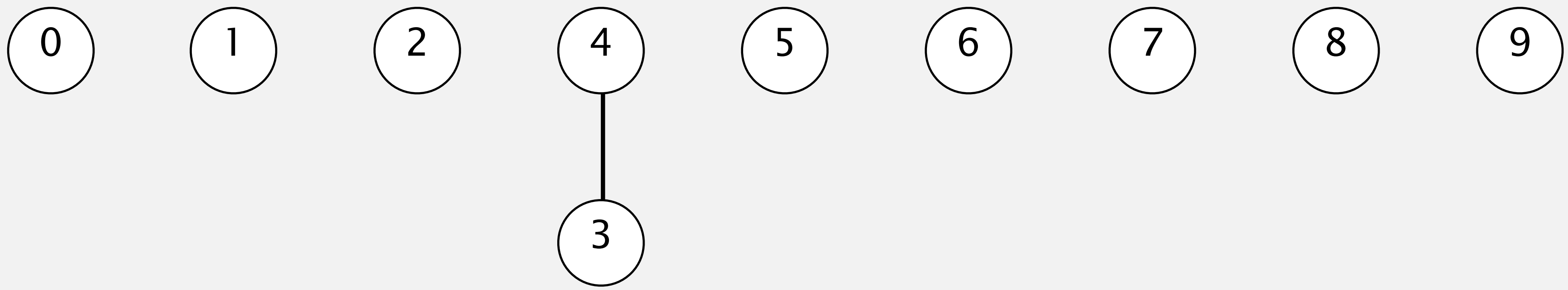
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

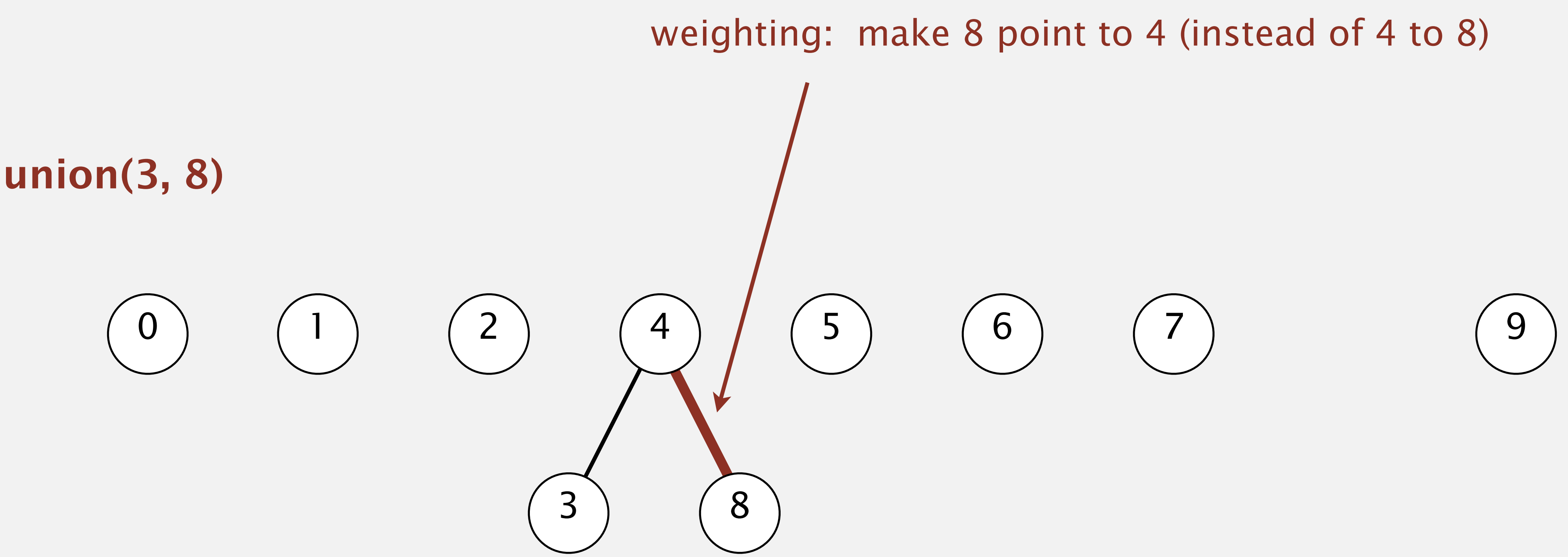
Weighted quick-union demo

union(3, 8)



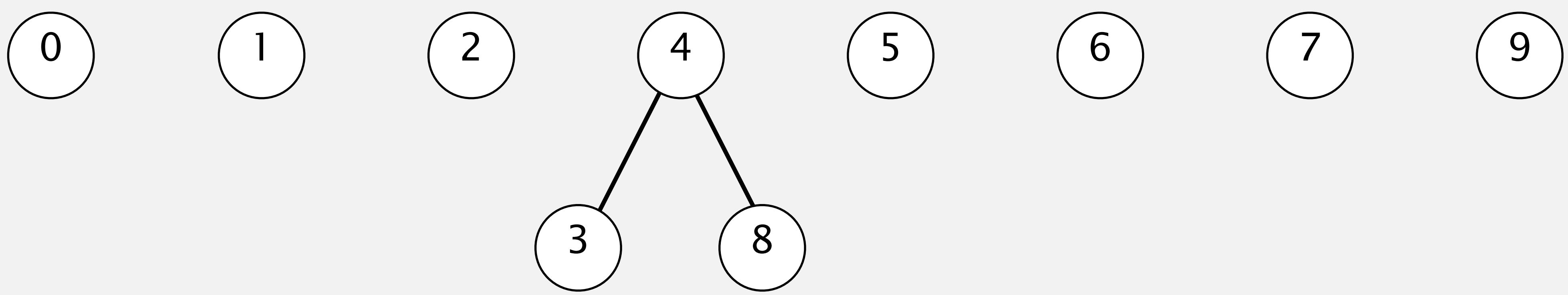
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

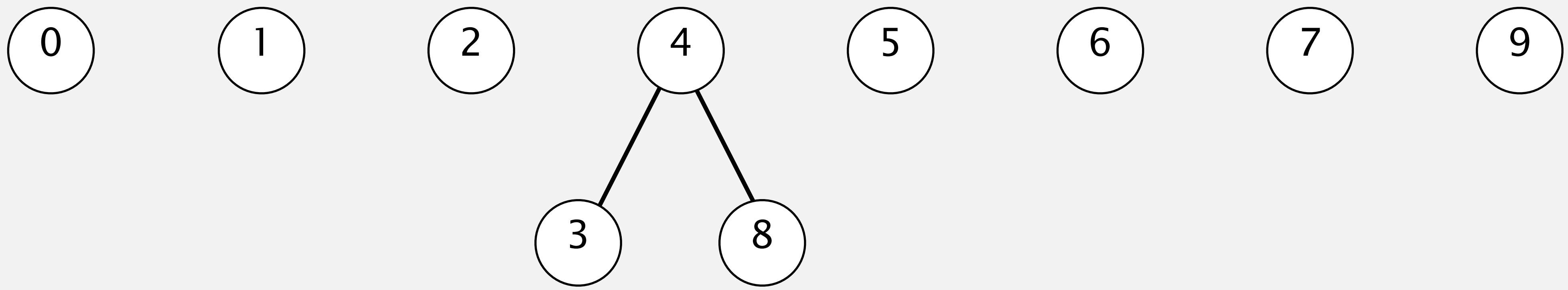
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

Weighted quick-union demo

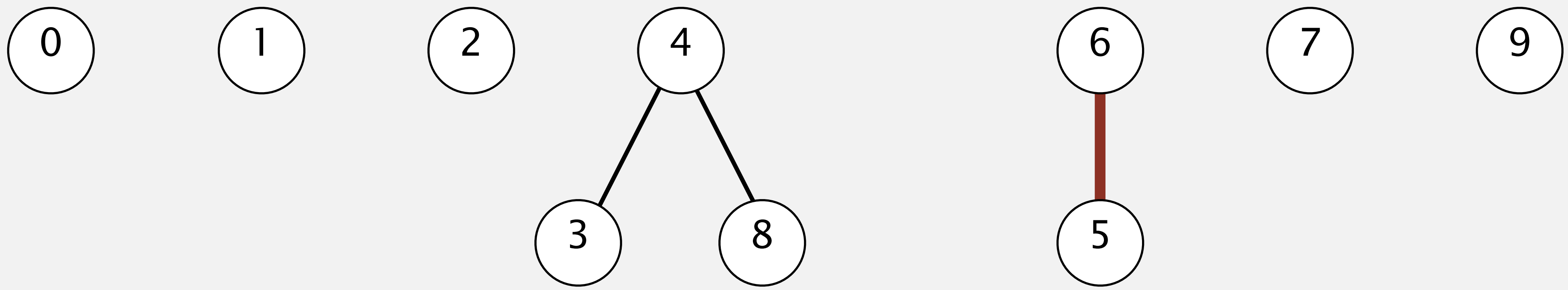
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

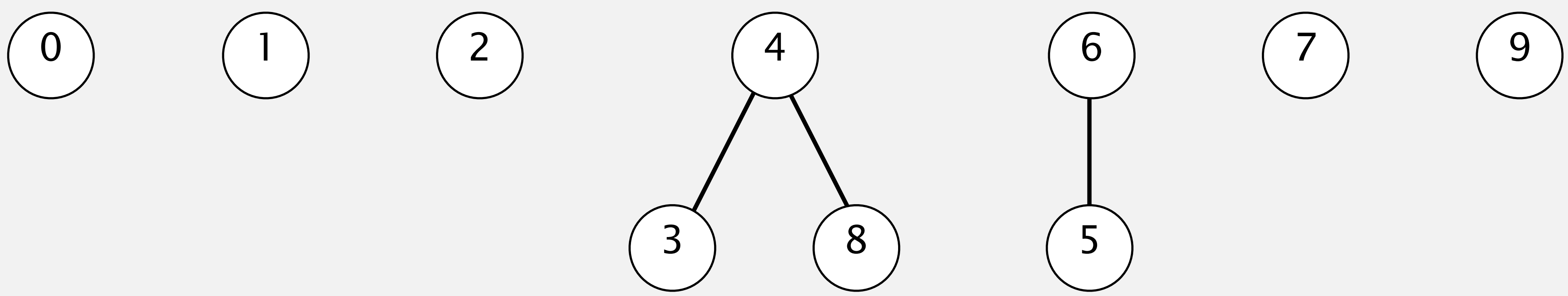
Weighted quick-union demo

union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

Weighted quick-union demo

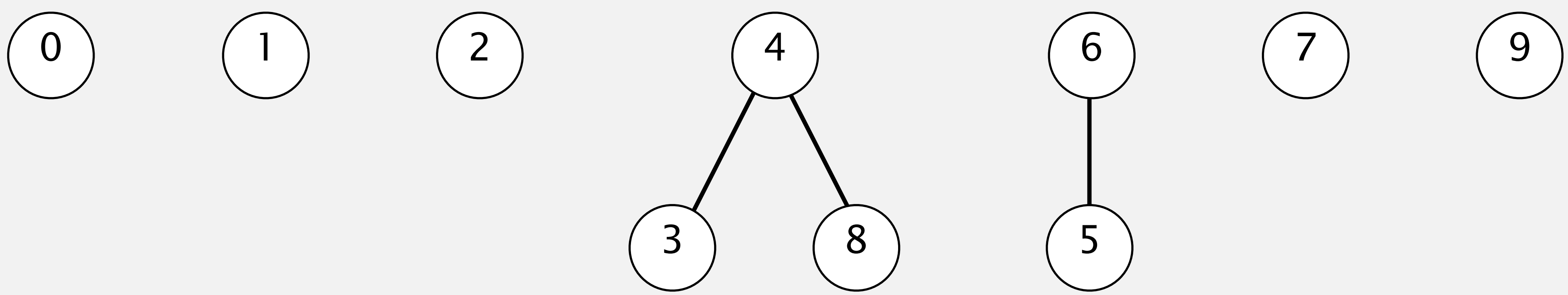


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

Weighted quick-union demo

大小樹如何看 -> 看節點數
所以是9掛到4下面

union(9, 4)

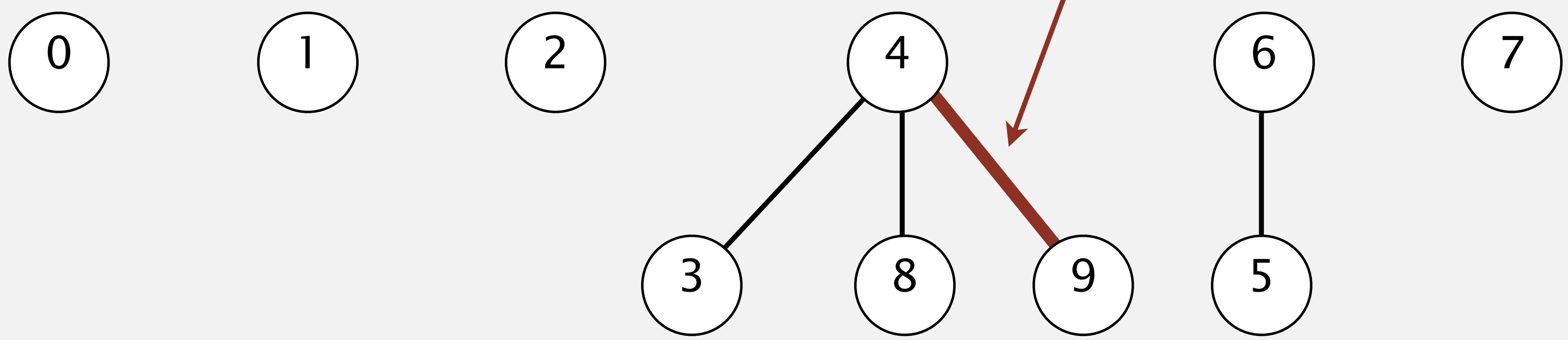


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

Weighted quick-union demo

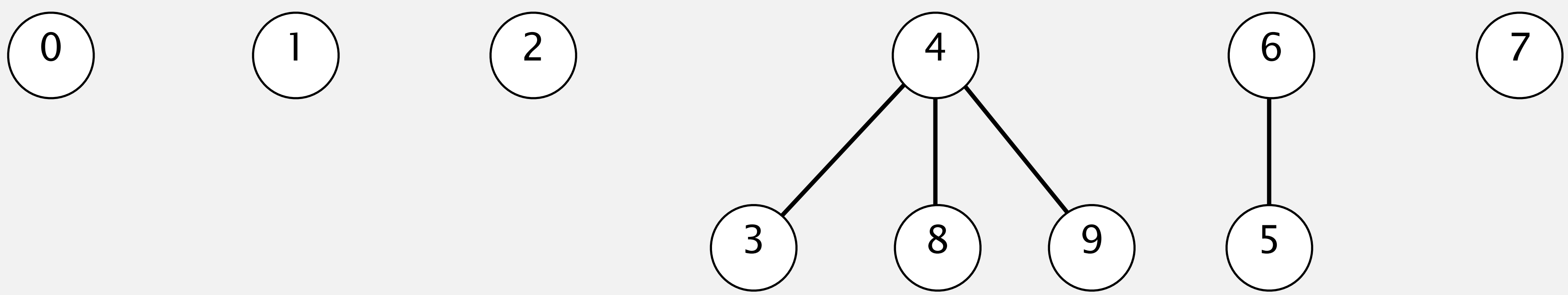
weighting: make 9 point to 4

union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

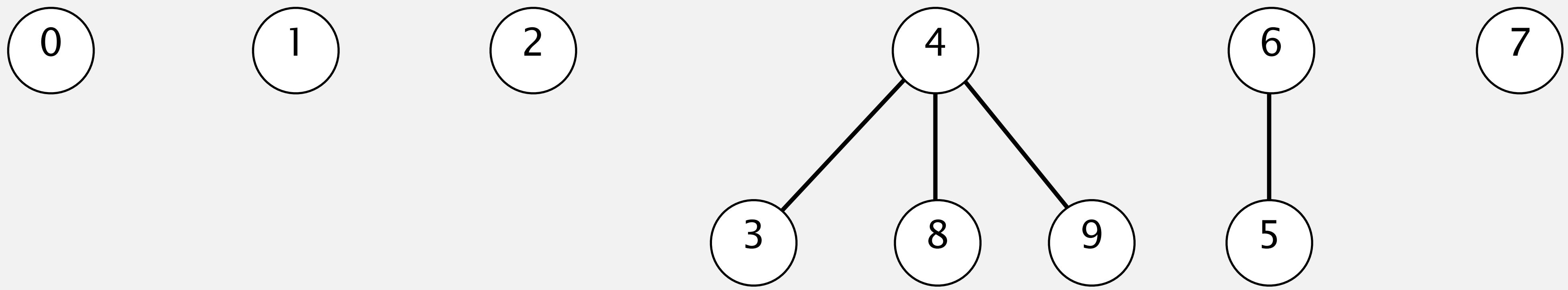
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

Weighted quick-union demo

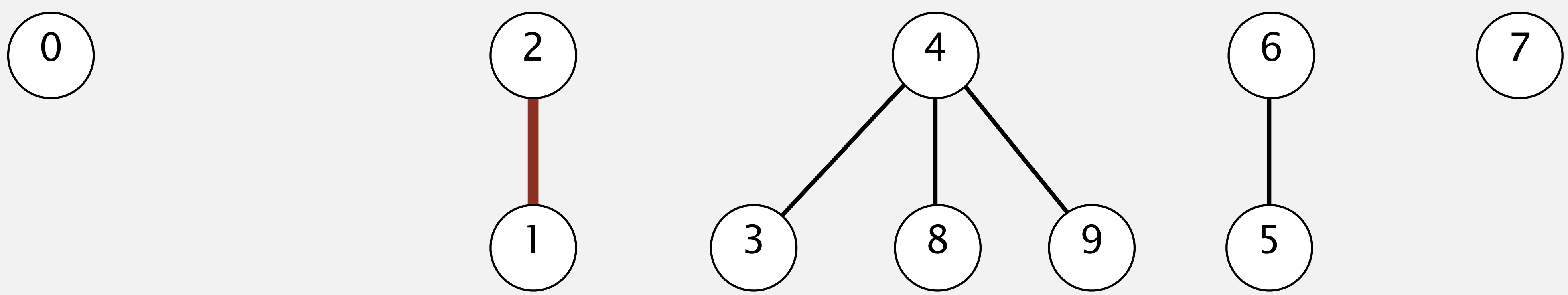
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

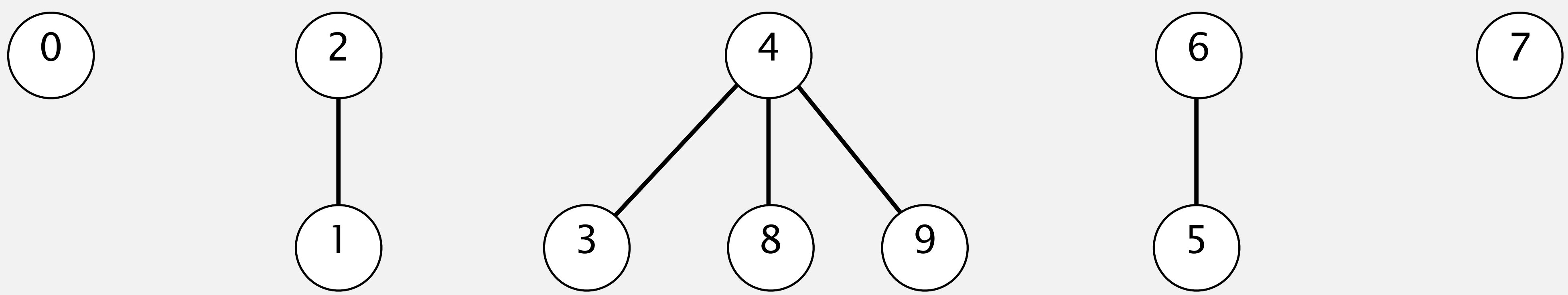
Weighted quick-union demo

union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

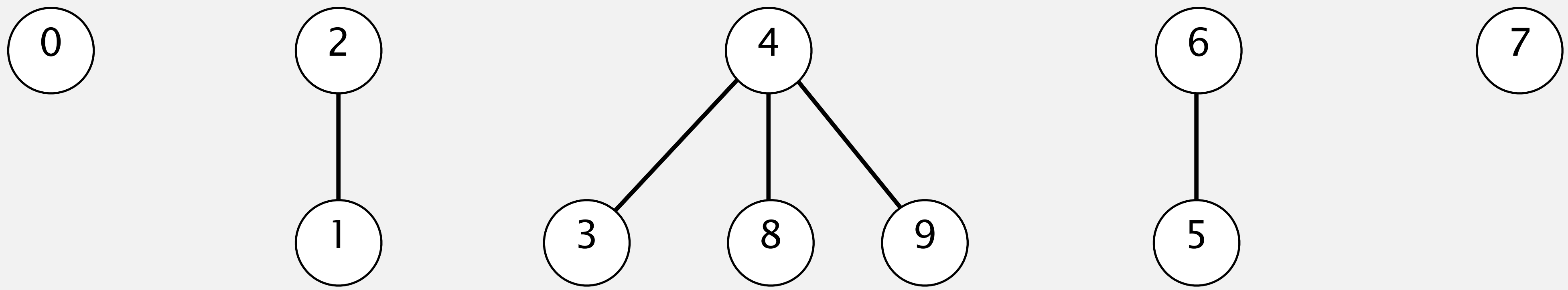
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

union(5, 0)

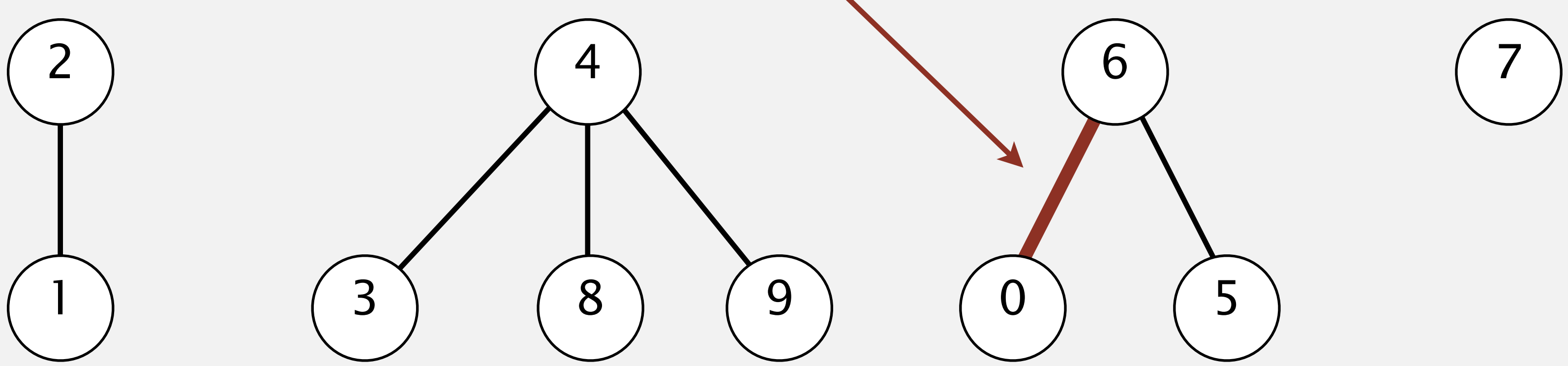


	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

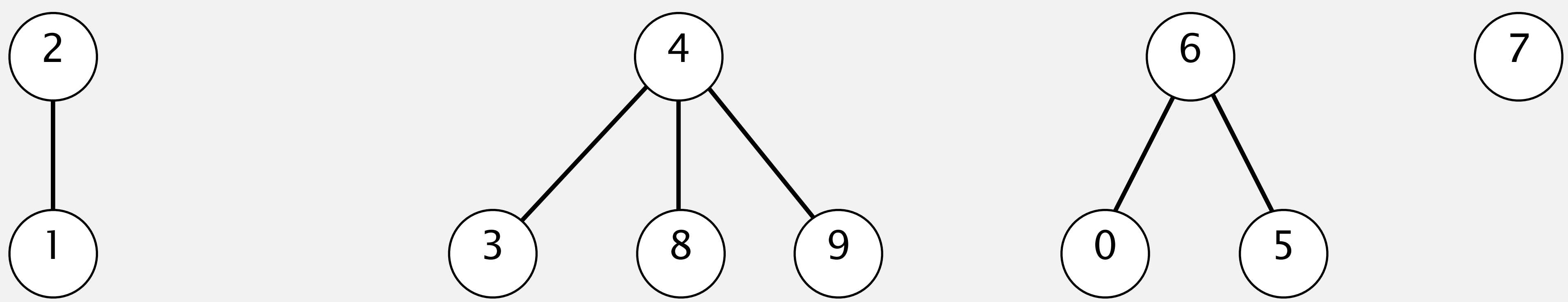
weighting: make 0 point to 6 (instead of 6 to 0)

union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

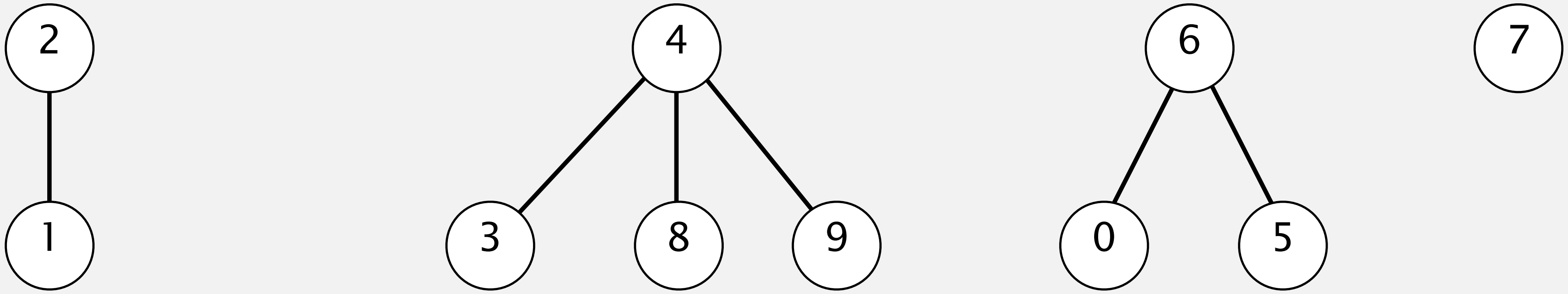
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

union(7, 2)

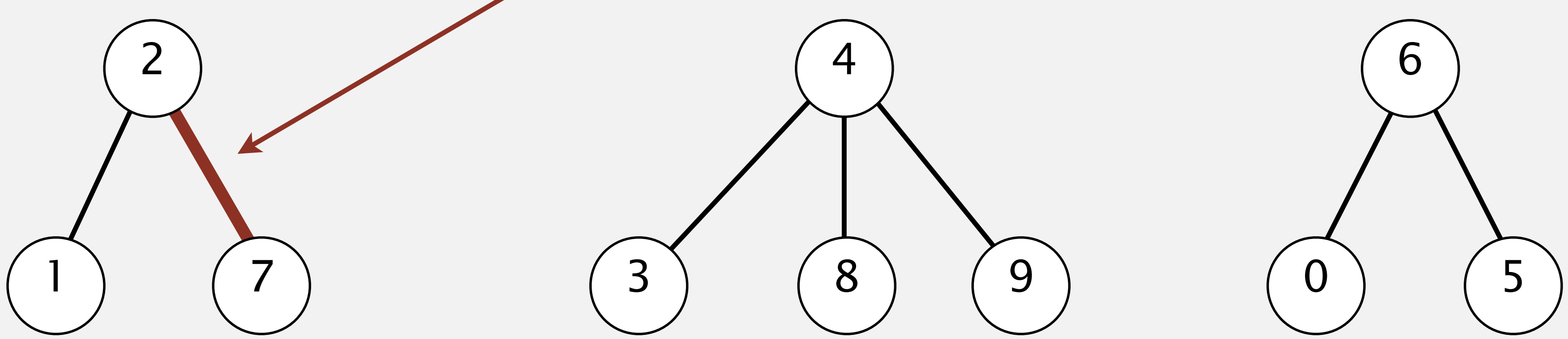


	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

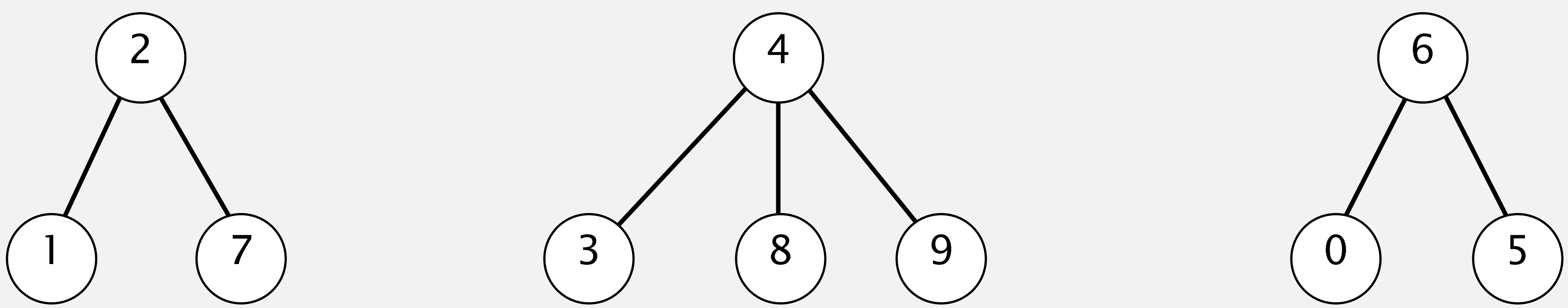
weighting: make 7 point to 2

union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

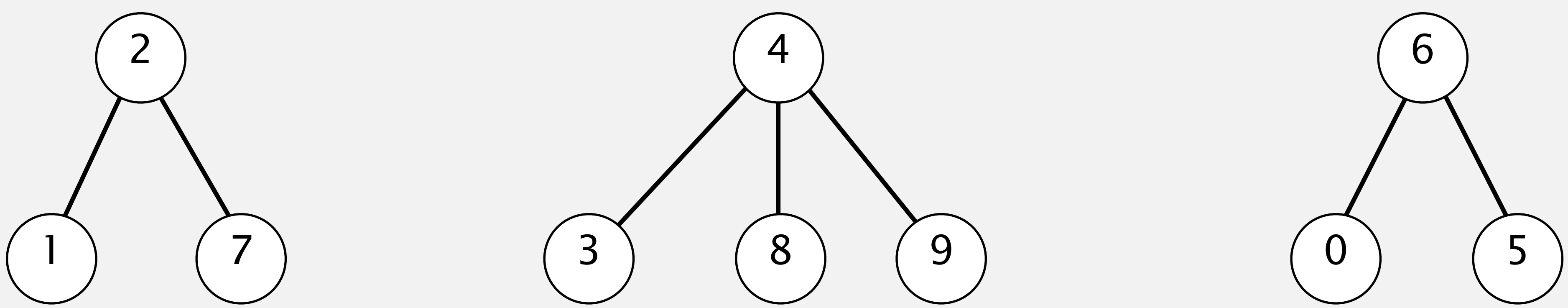
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

Weighted quick-union demo

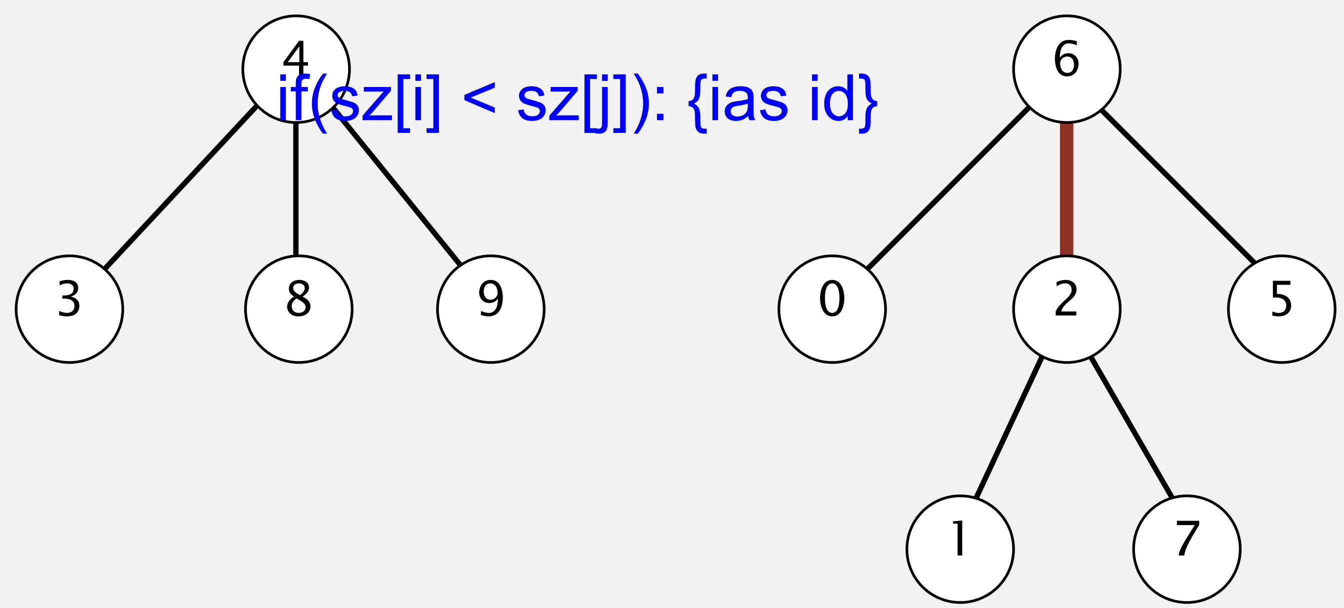
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

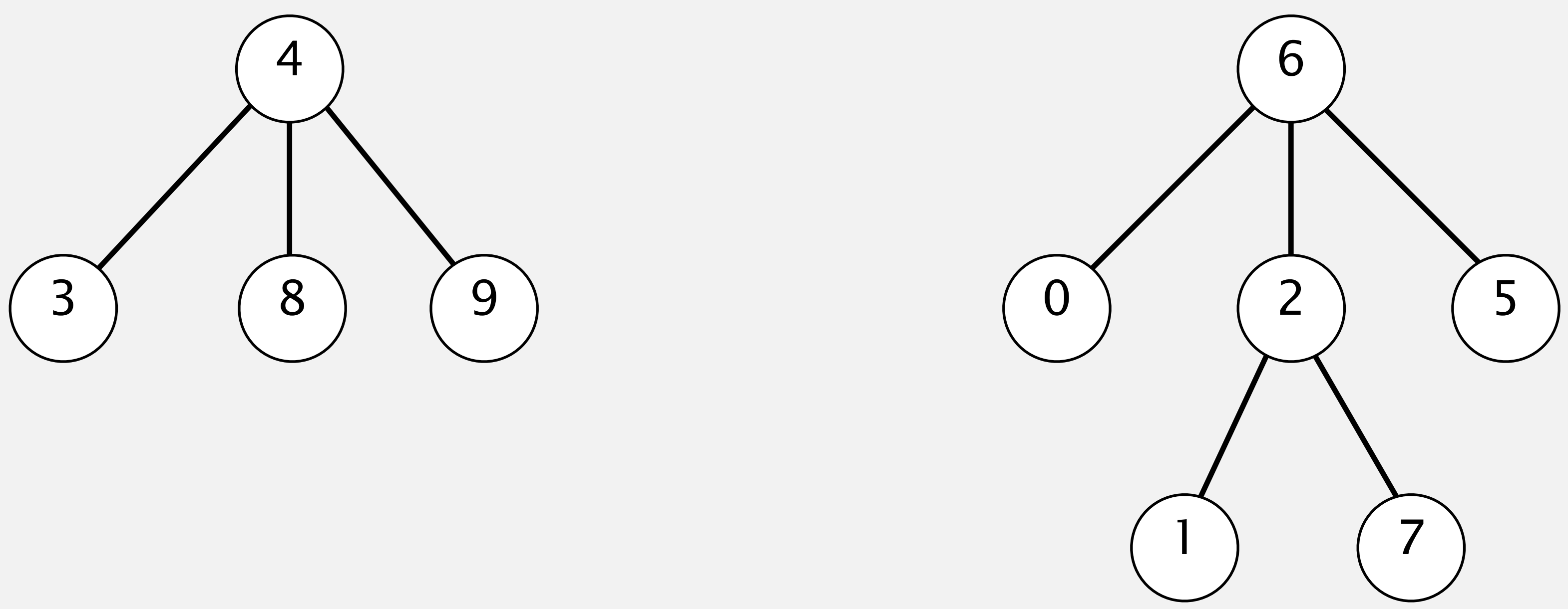
Weighted quick-union demo

union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

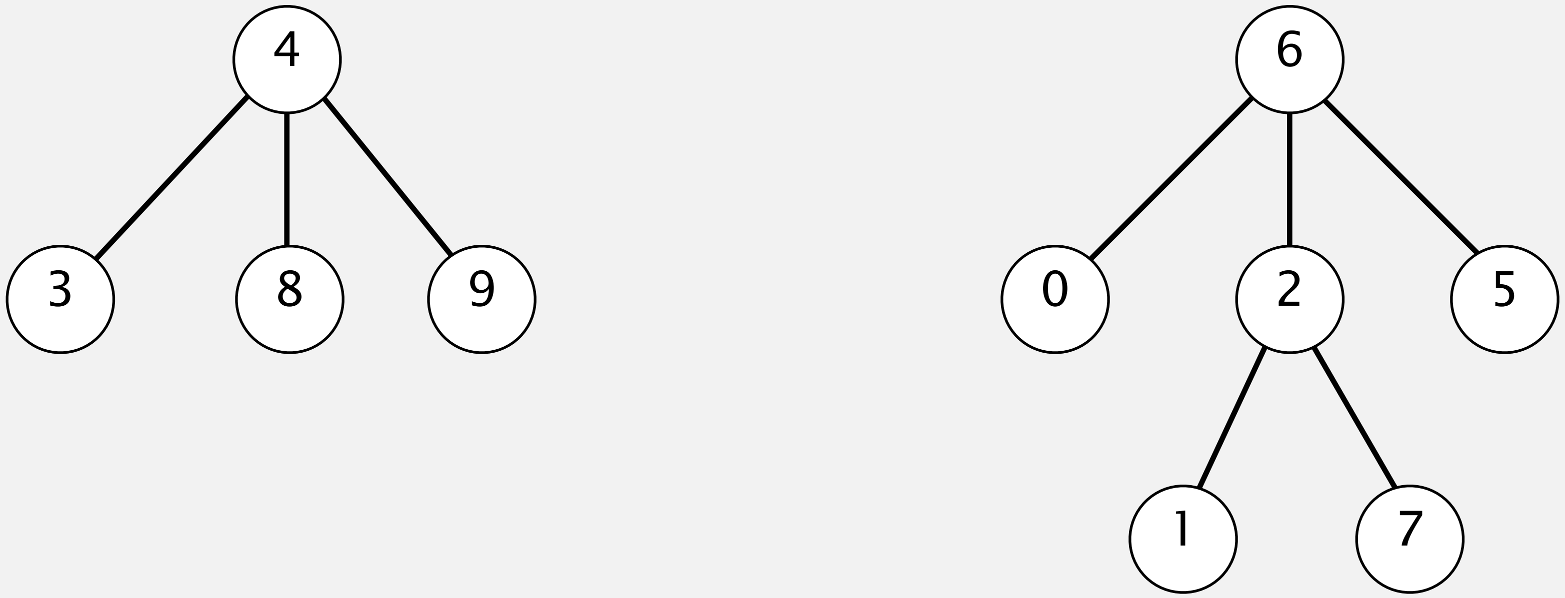
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

Weighted quick-union demo

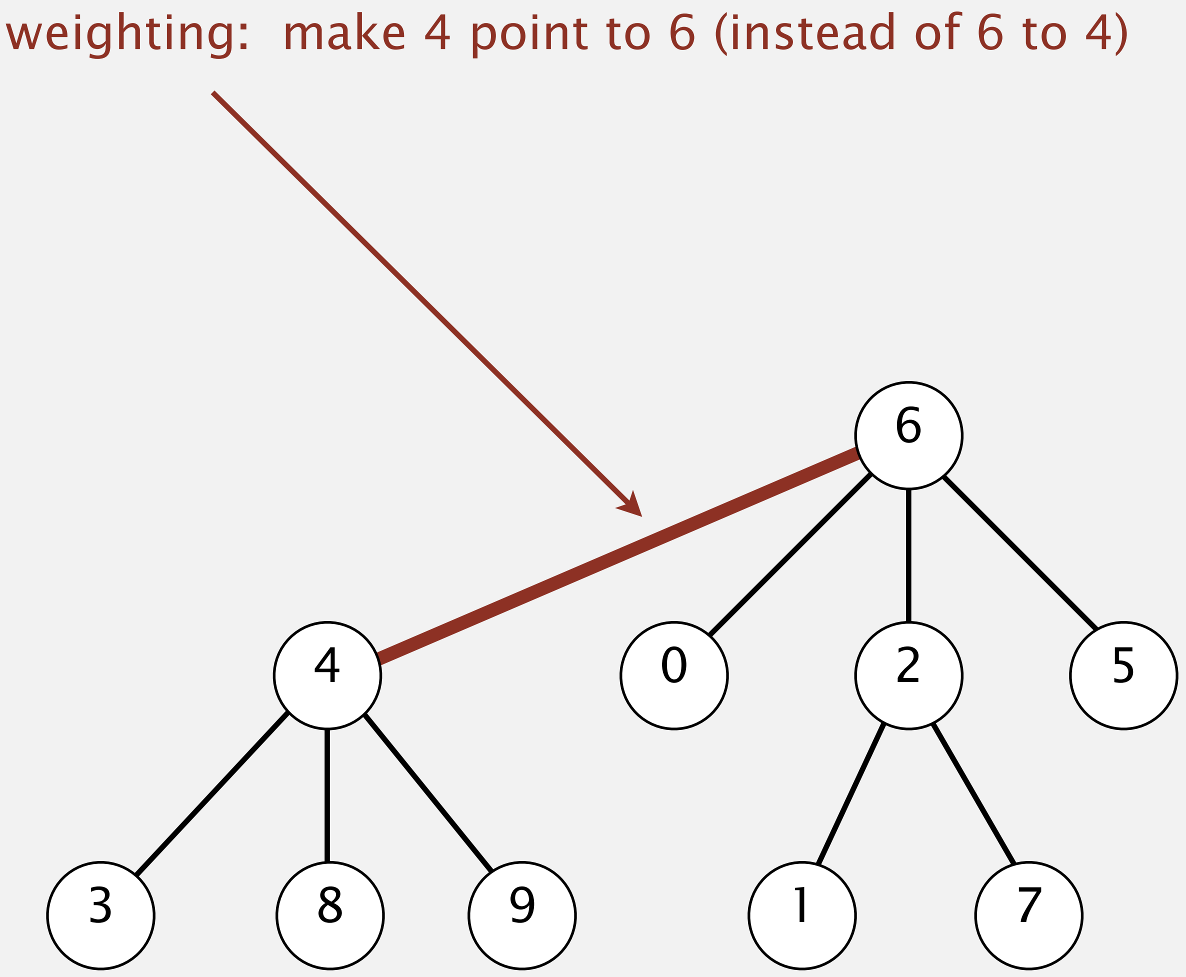
union(7, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

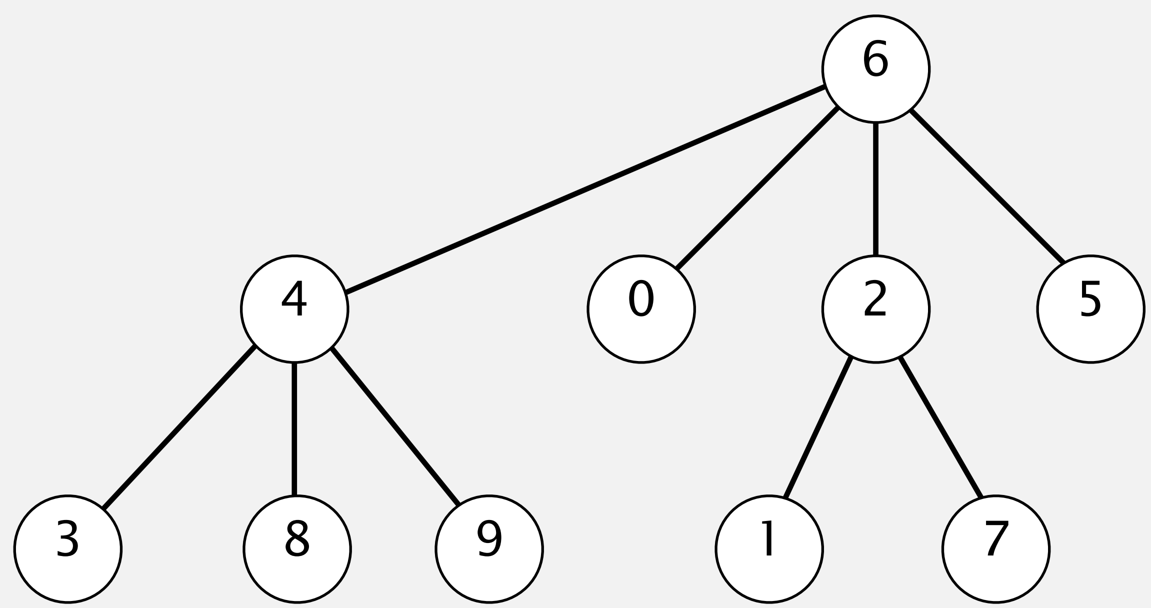
Weighted quick-union demo

union(7, 3)



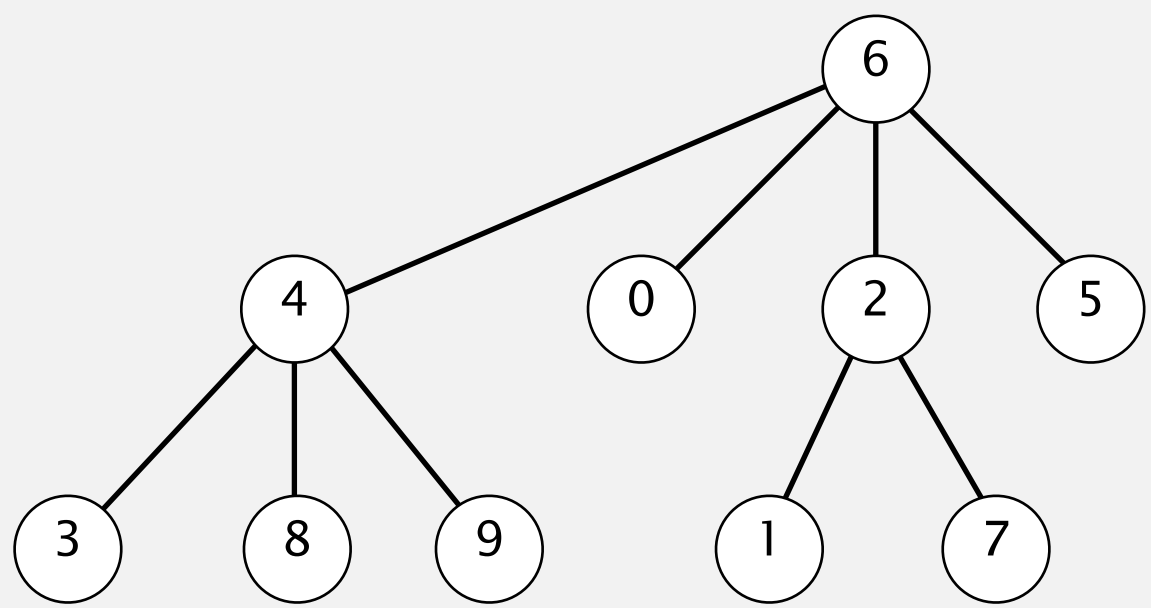
	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

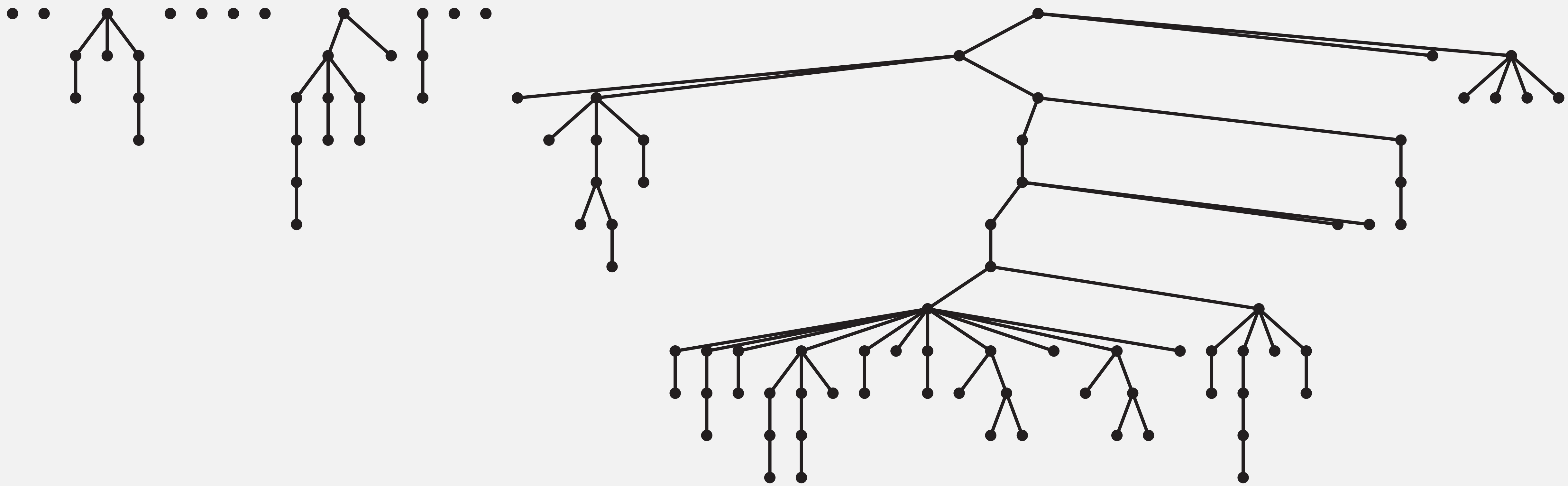
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

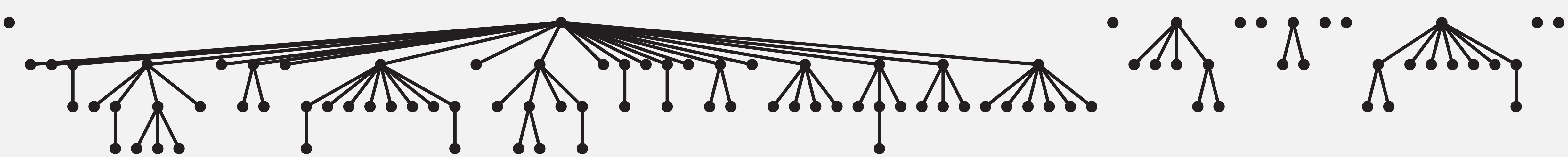
Quick-union and weighted quick-union example

quick-union



average distance to root: 5.11

weighted



average distance to root: 1.52

Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

if(sz[i] < sz[j]): { id[i] = j; sz[j]}
else :{ id[j] = i; }

```
int i = find(p);
int j = find(q);
if (i == j) return;
What to write here? 3 mins.
sz[i] = size of tree for node i.
```

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

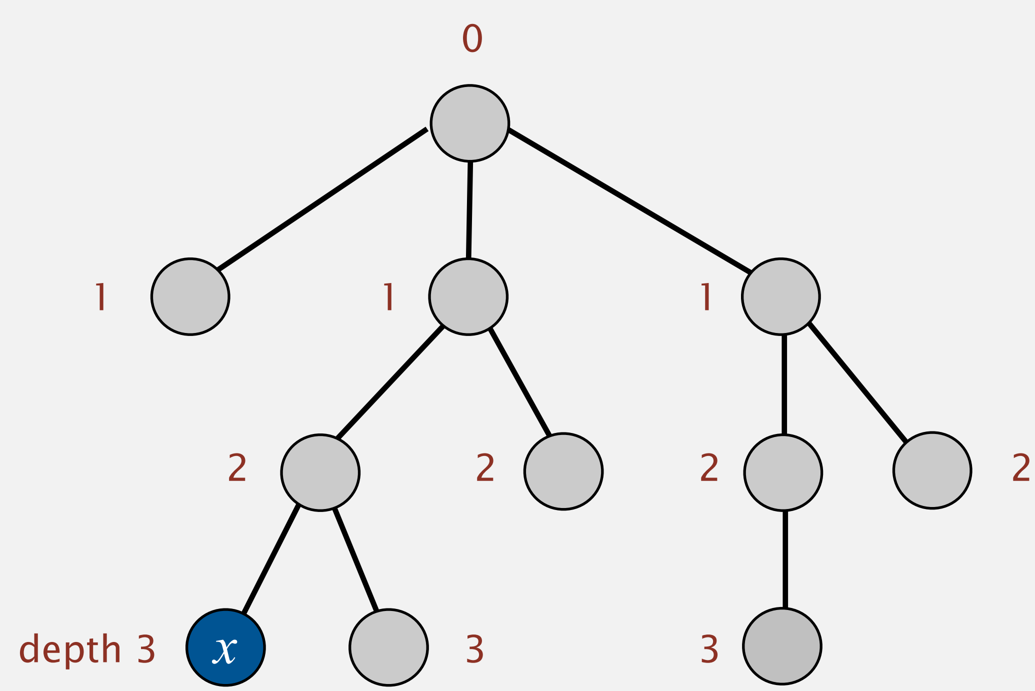
Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.



N = 11
depth(x) = 3 ≤ lg N

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

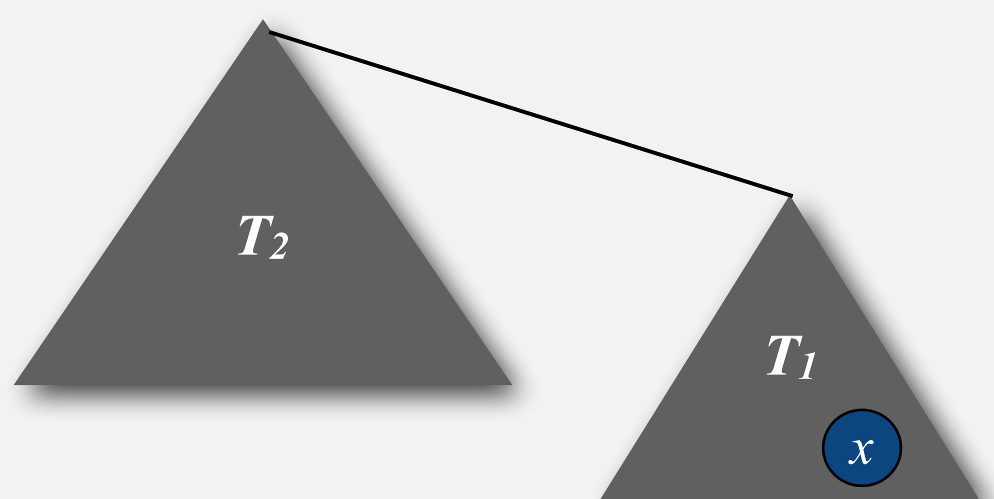
lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.

Pf. What causes the depth of object x to increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\lg N$ times. Why?



$$\left.\begin{array}{c} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ \vdots \\ N \end{array}\right\} \lg N$$

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

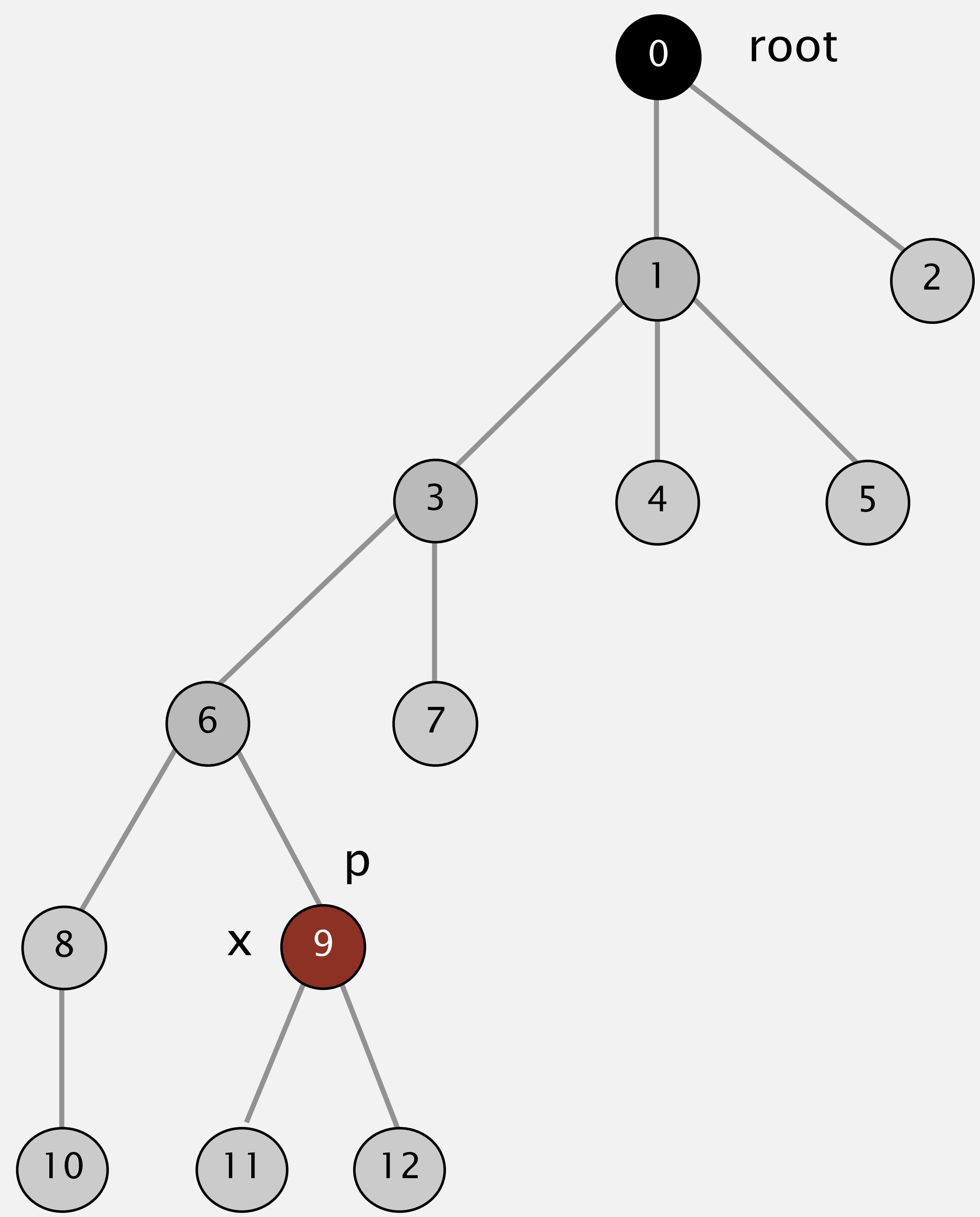
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Improvement 2: path compression

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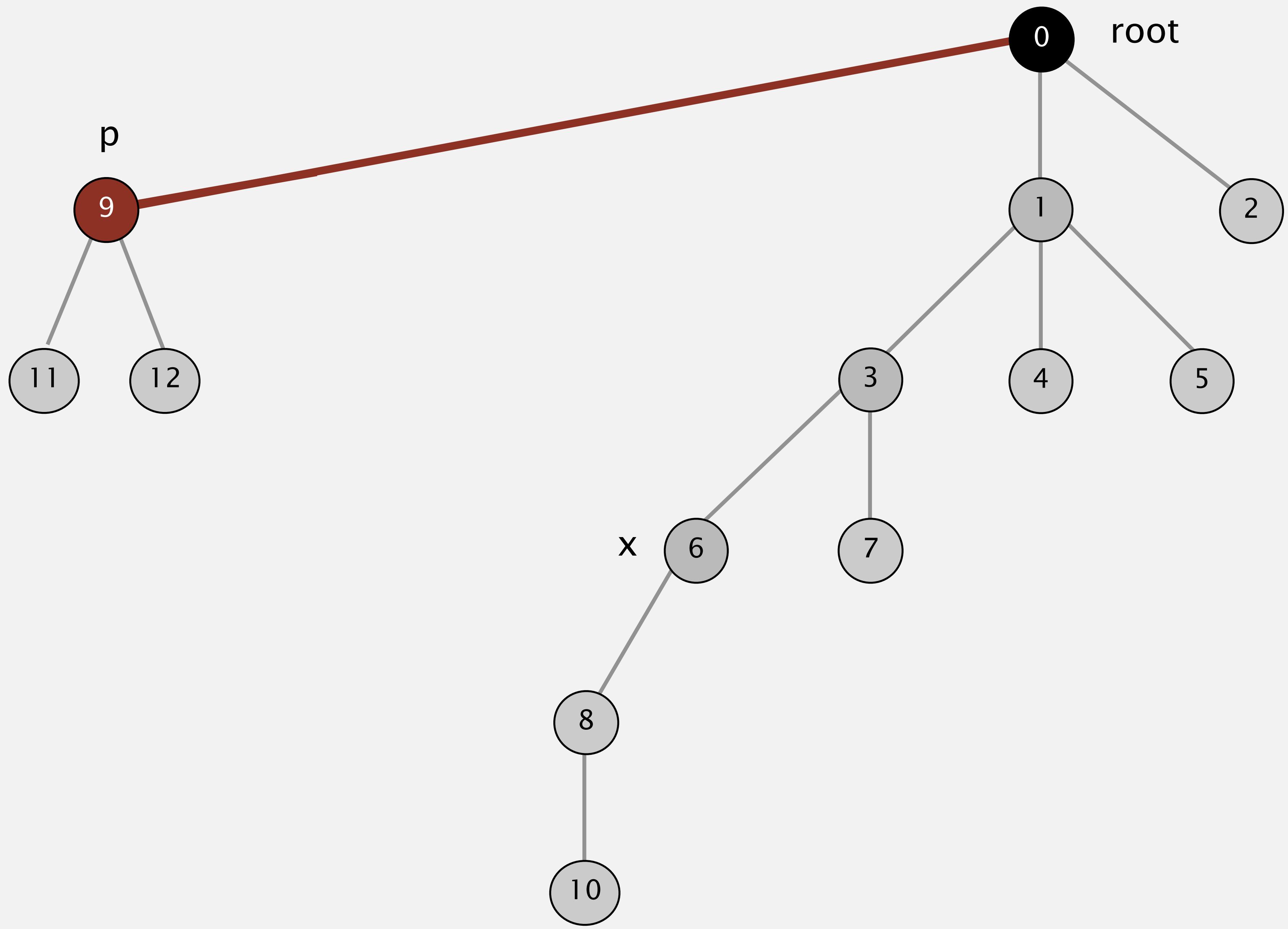

Improvement 2: path compression

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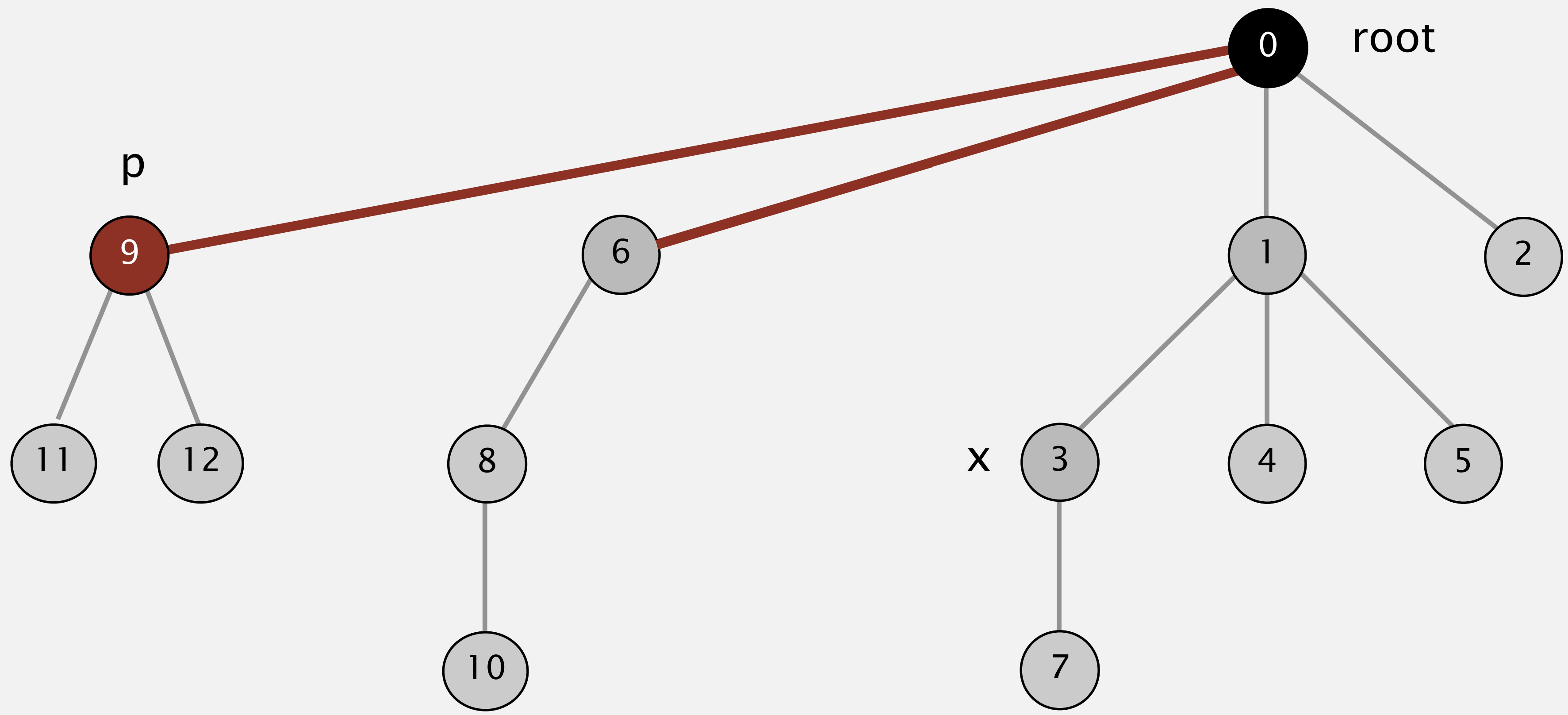

Improvement 2: path compression

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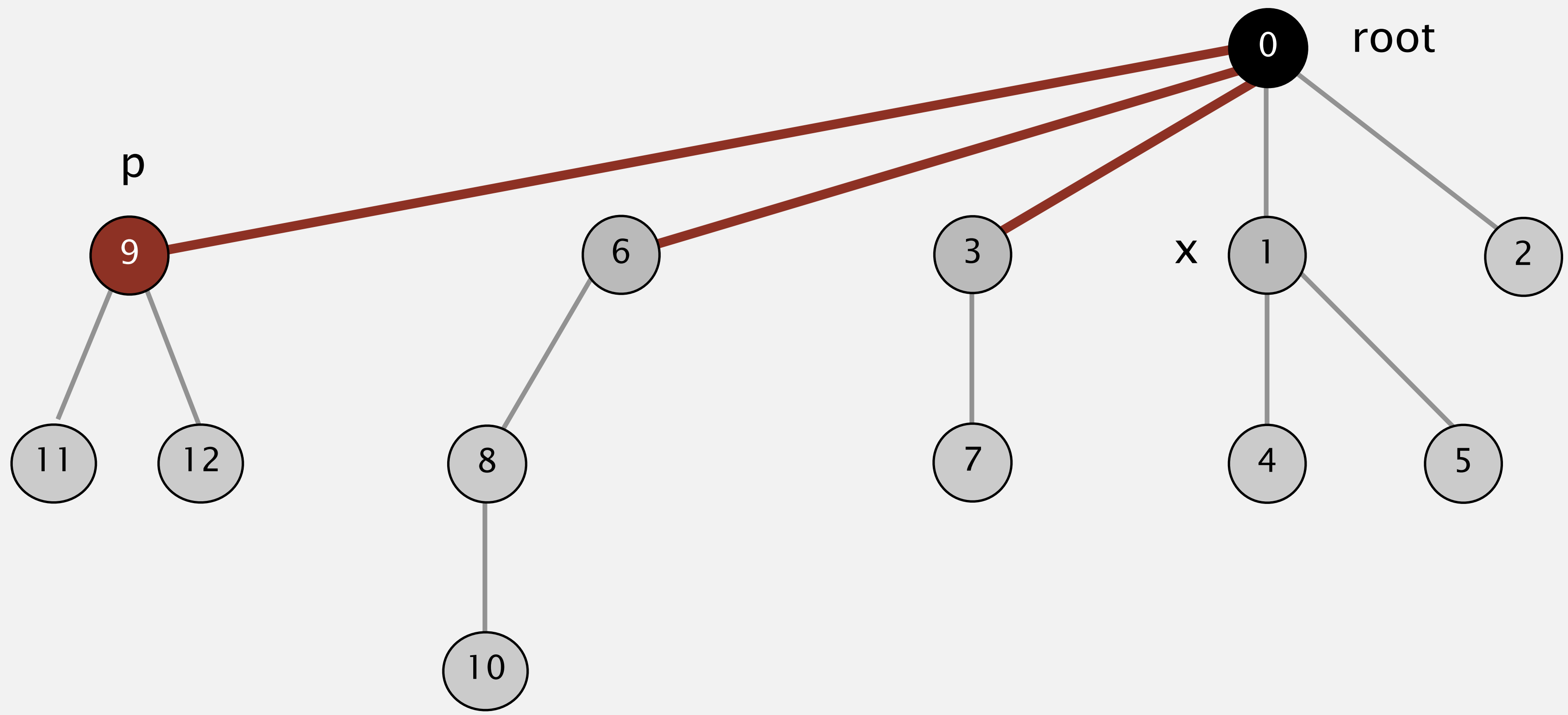
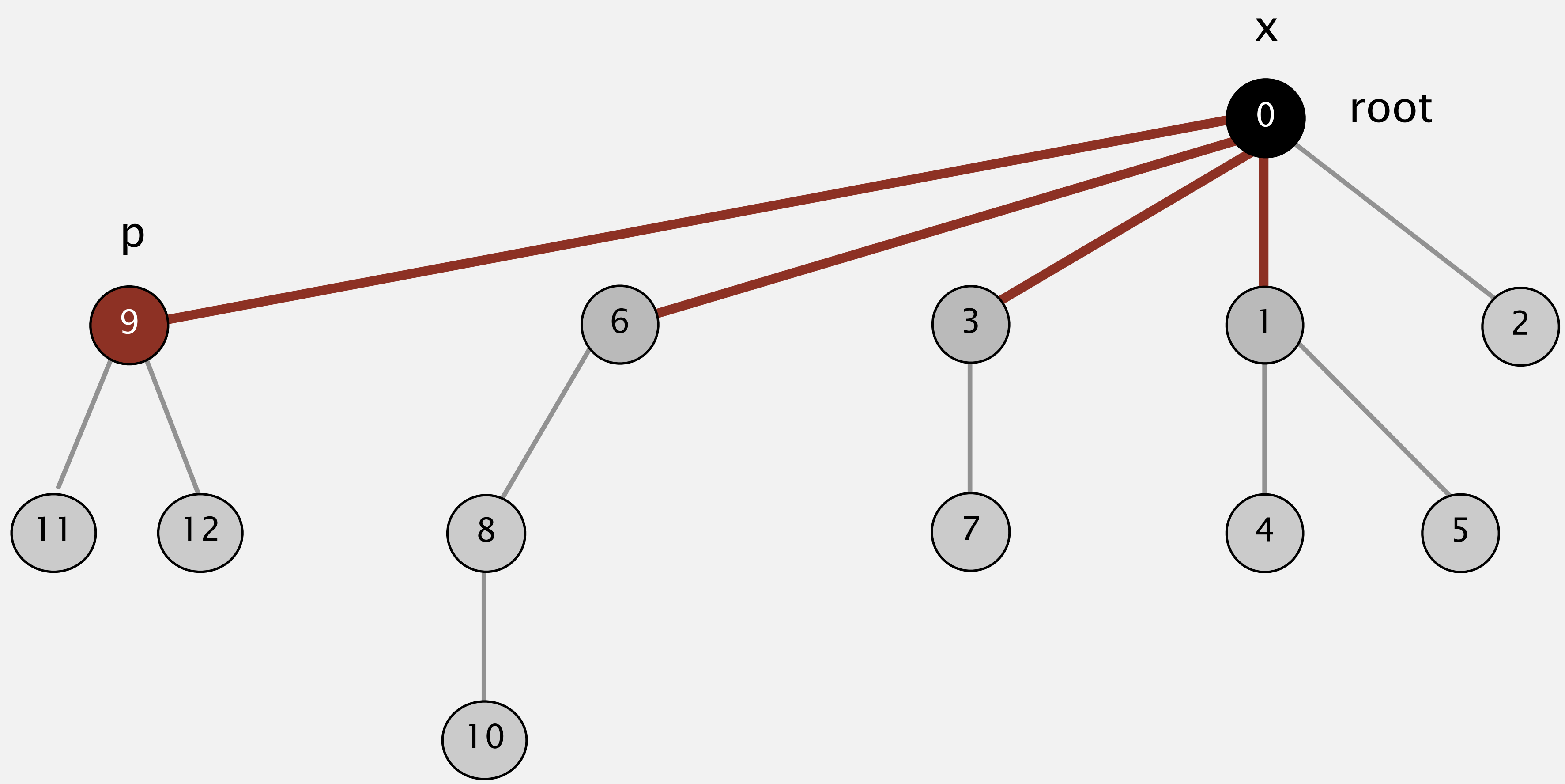


Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Bottom line. Now, `find()` has the side effect of compressing the tree.

Path compression: Java implementation

Two-pass implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

```
public int find(int i)
{
   while (i != id[i])
   {
What to write here? 2 mins.
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

Path compression: Java implementation

Two-pass implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

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```
public int find(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

http://algs4.cs.princeton.edu

Class Rules Update

- In-class assignments: No late assignments will be accepted unless the student obtained specific permission from the instructor in advance.
- Updated in syllabus and course introduction slide

CLASS RULES

Class meeting

- Be on time, I will start class on time.

Exams

- No make up exams unless you obtained the instructor's specific permission in advance.

Assignments

- In-class: no late assignments, due by the end of class, unless you obtained specific permission from the instructor in advance.
- obtained the instructor's permission in advance.
  - Paired programming.
  - Please bring your laptop to the classroom.
- Take-home:
  - Individual work.
  - 10% off for every day late & will not be accepted 5 days after the due date.
  - You can discuss ideas with classmates and TAs if you cannot do it on

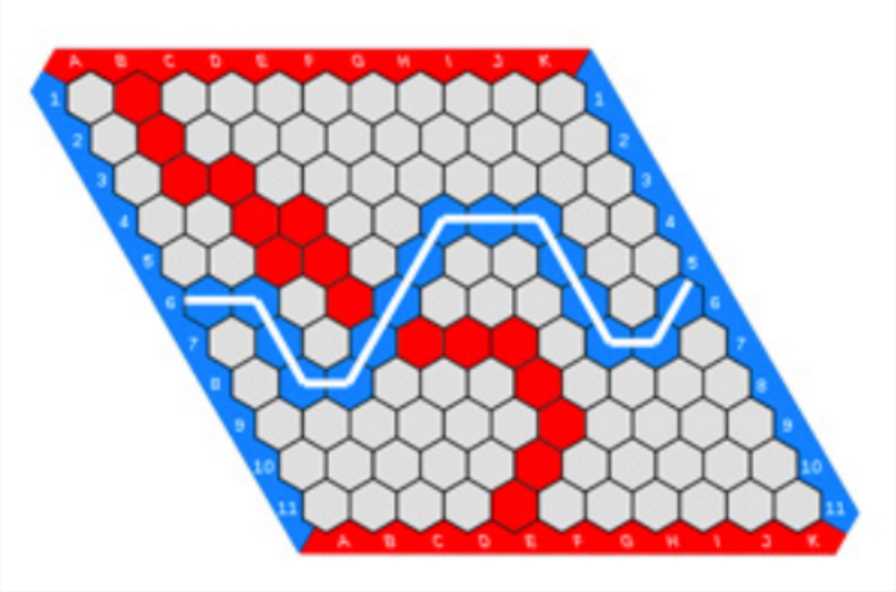
http://algs4.cs.princeton.edu

1.5 UNION-FIND

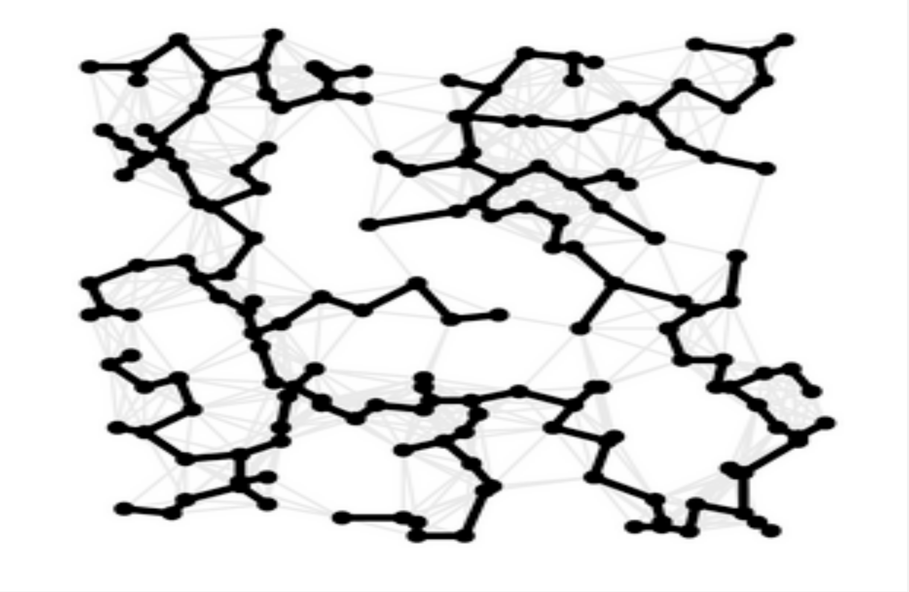
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Union-find applications

✓ Dynamic connectivity.
- Percolation.
- Games (Go, Hex).
- Least common ancestor.
- Kruskal's minimum spanning tree algorithm.
- ...



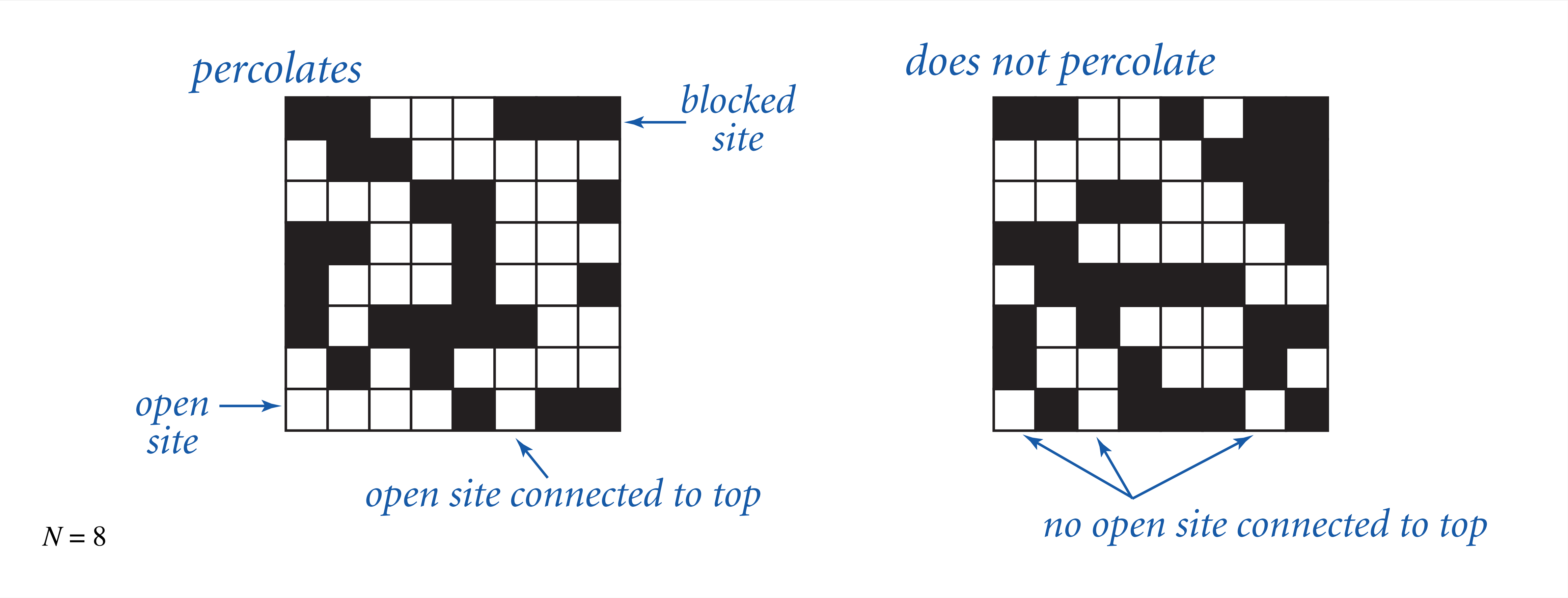
https://en.wikipedia.org/wiki/Hex_(board_game)



Percolation

An abstract model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.



Percolation

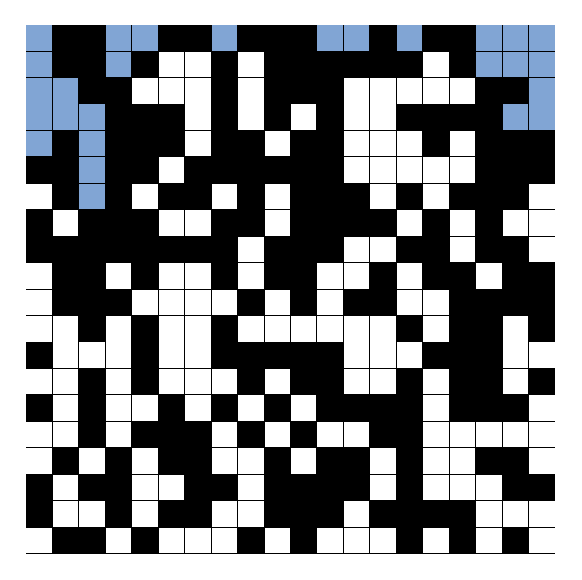
An abstract model for many physical systems:

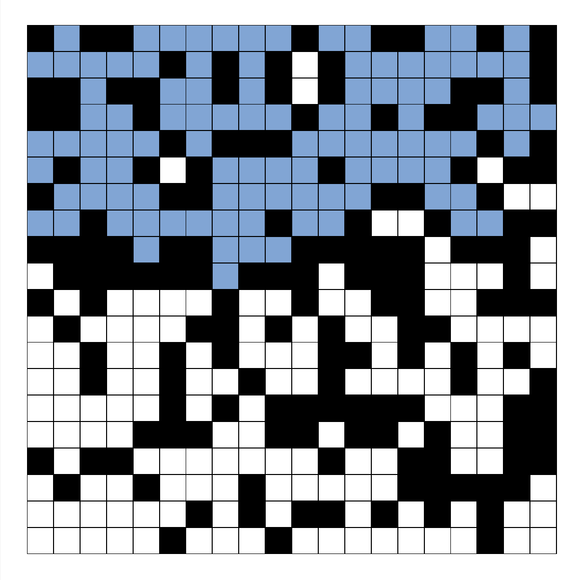
- N-by-N grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

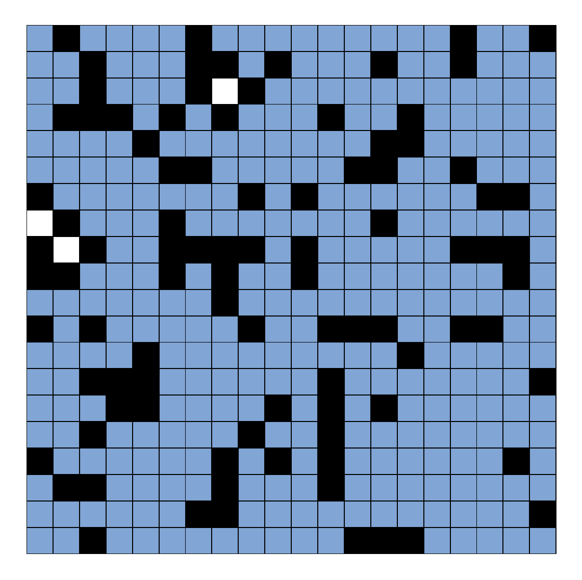
model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

Depends on grid size N and site vacancy probability p.



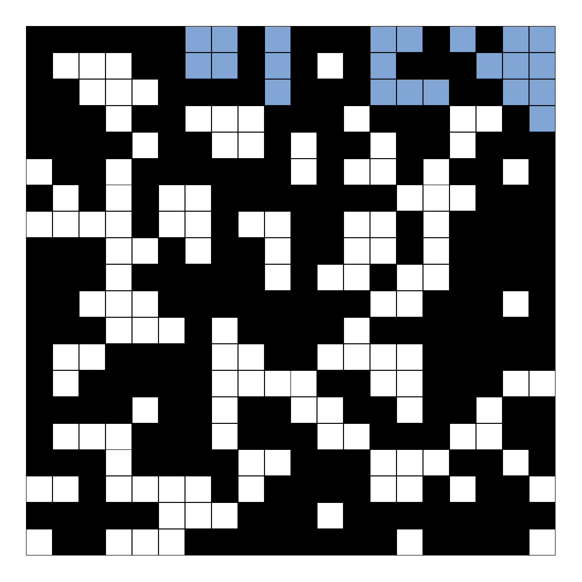


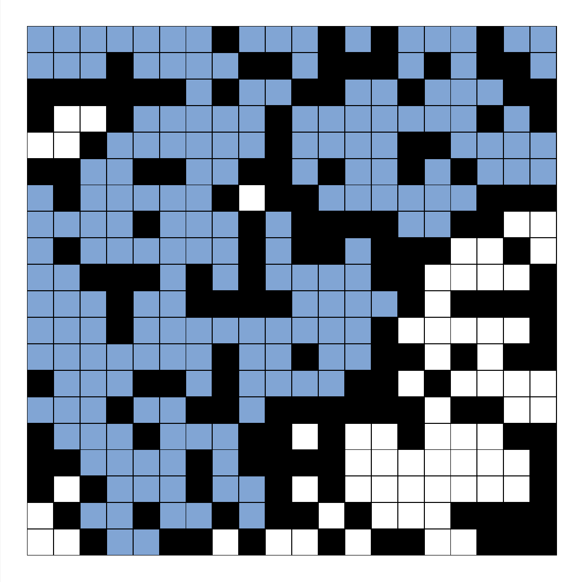


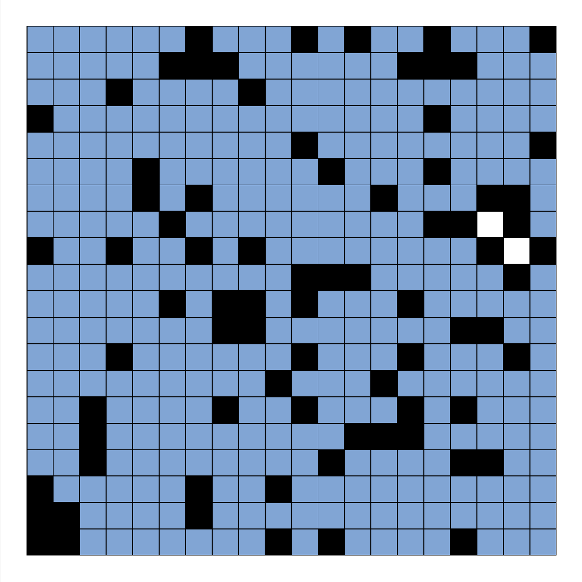
p low (0.4)
does not percolate

p medium (0.6)
percolates?

p high (0.8)
percolates





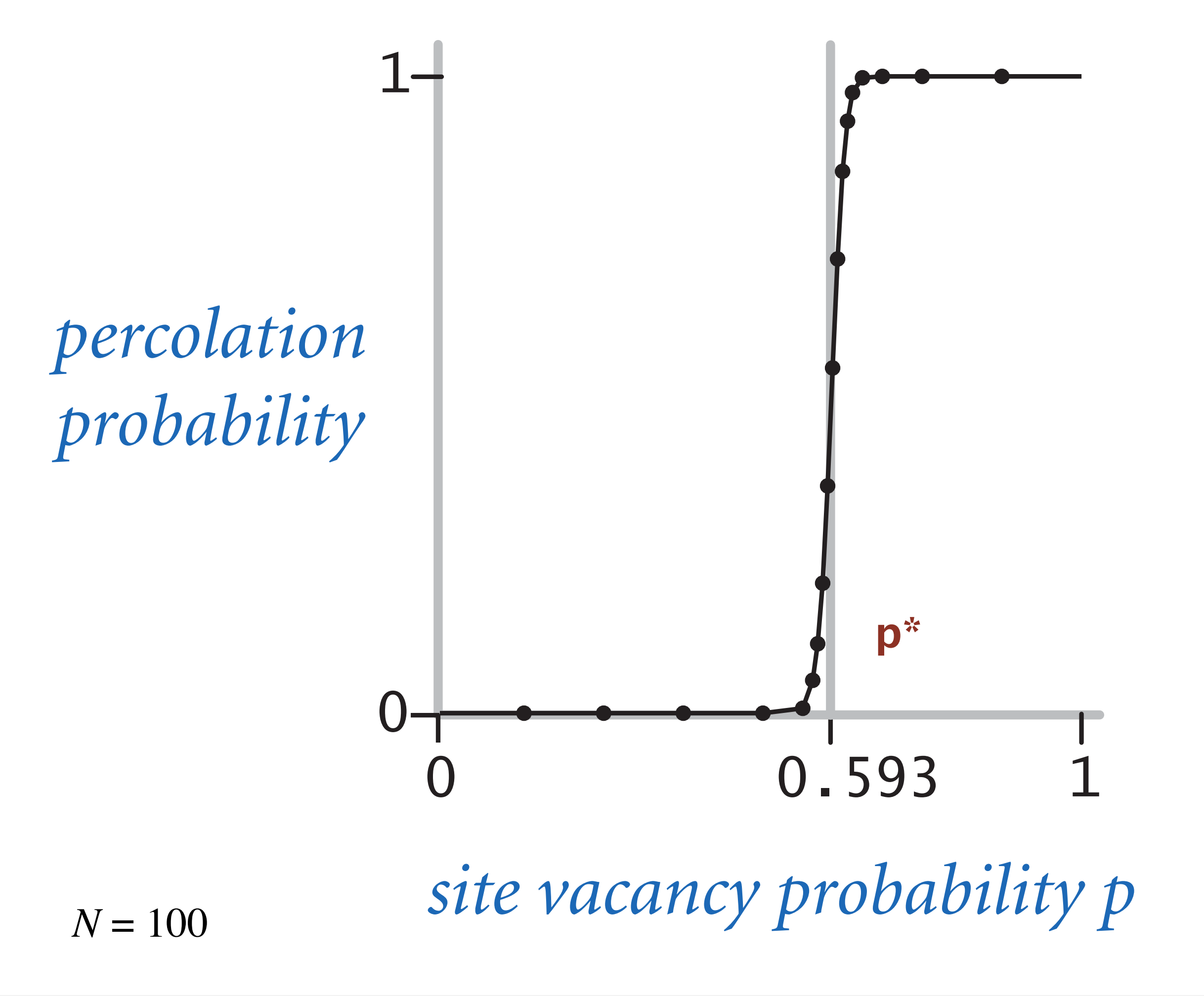


Percolation phase transition

When N is large, theory guarantees a sharp threshold p^*.

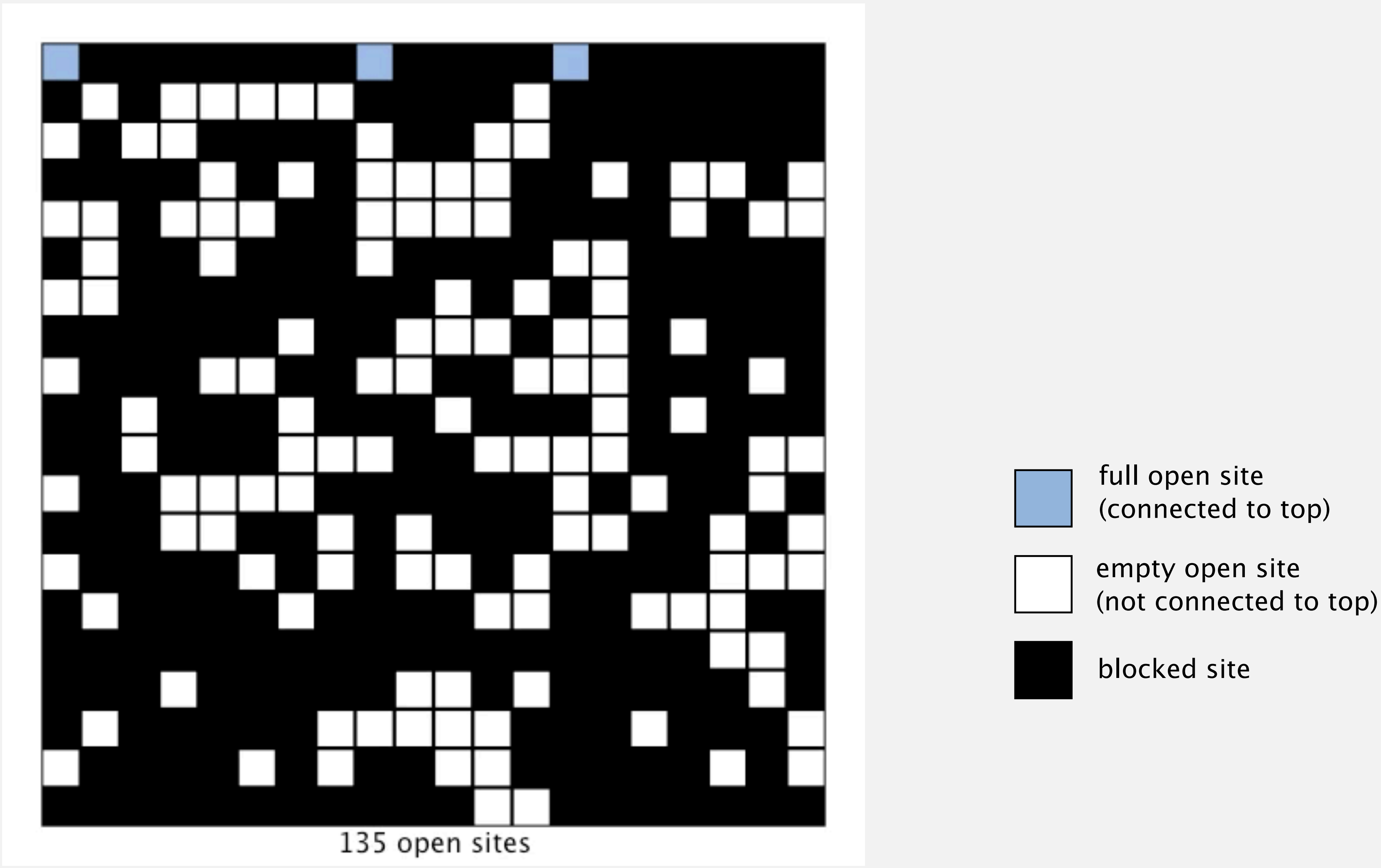
- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

Q. What is the value of p^* ?



Monte Carlo simulation

- Initialize all sites in an N-by-N grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.



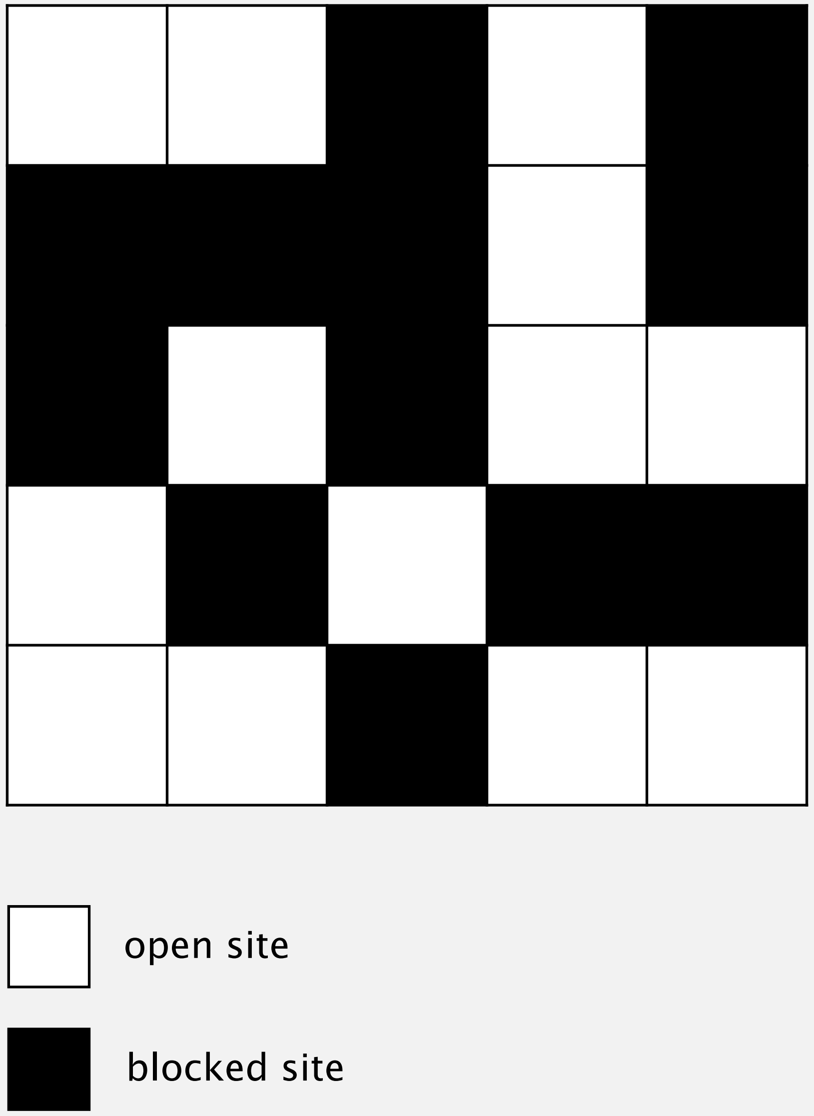
$N = 20$

Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

A. Model as a dynamic connectivity problem and use union-find.

$N = 5$

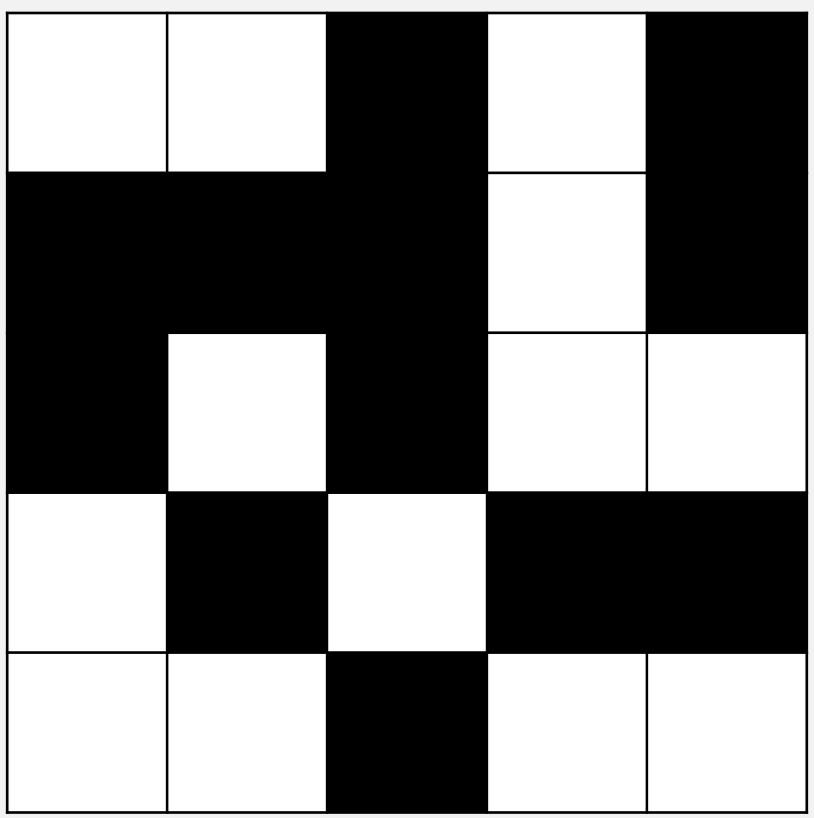


Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

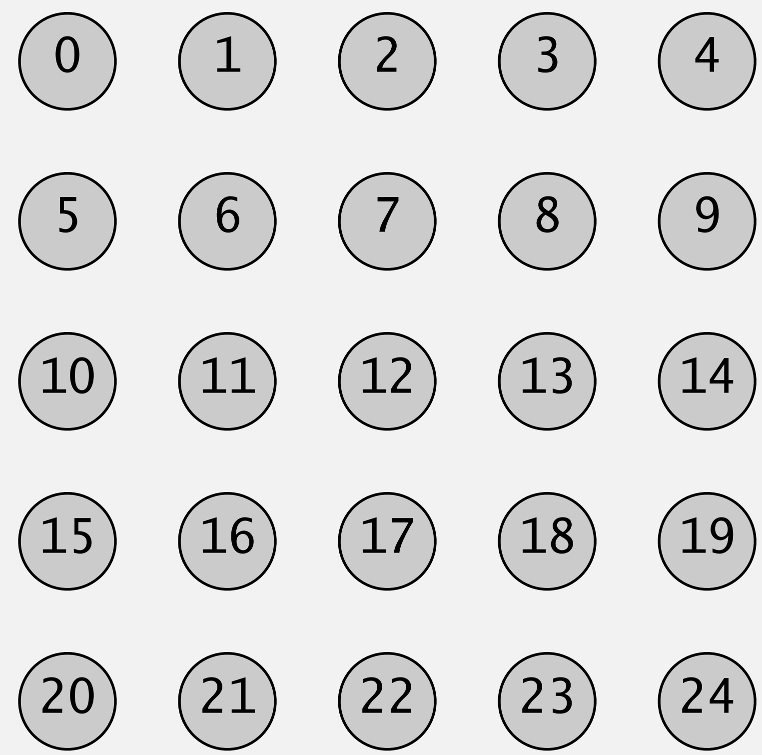
- Create an object for each site and name them 0 to $N^2 - 1$.

$N = 5$



open site

blocked site

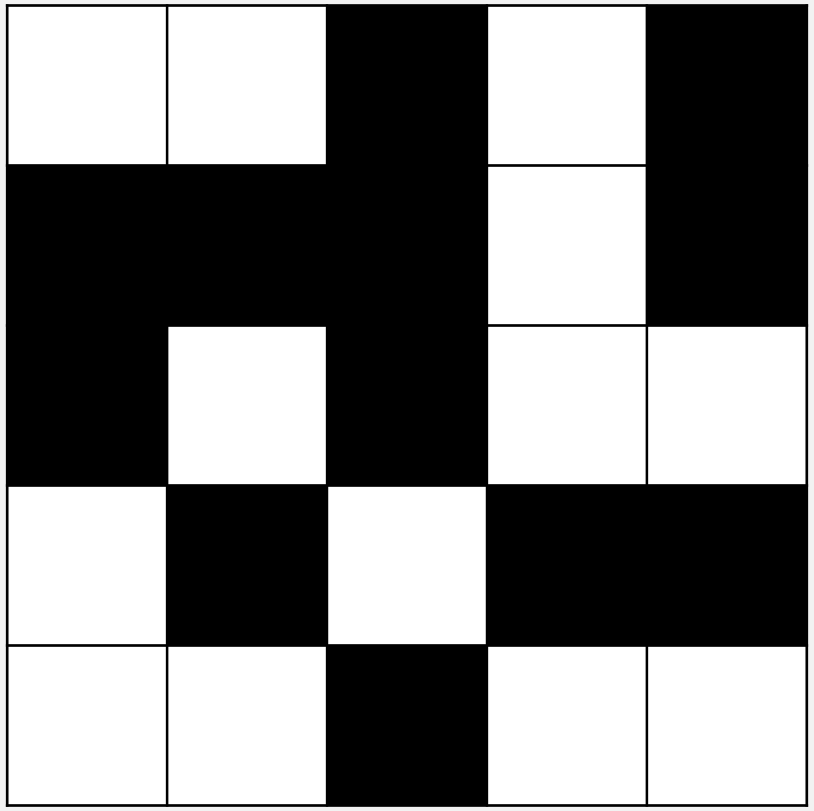


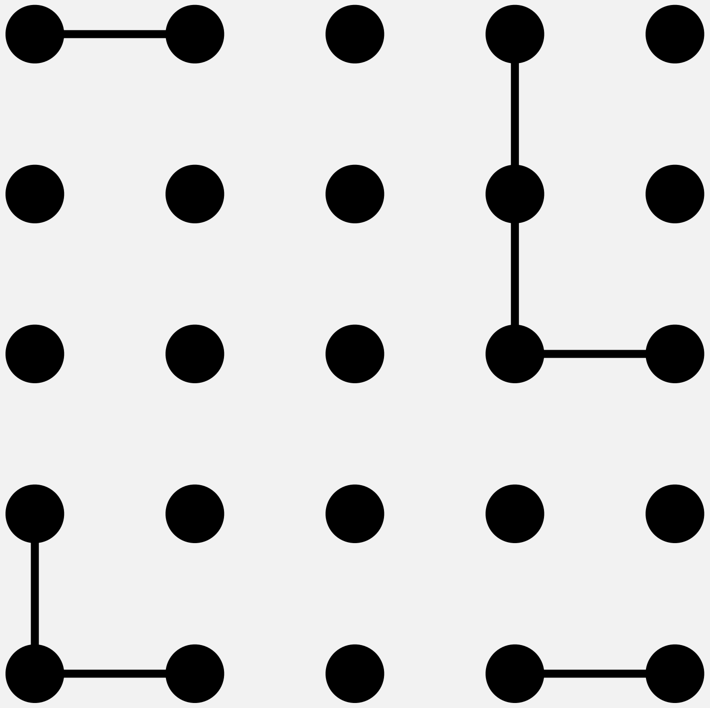
Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component iff connected by open sites.

$N = 5$





open site

blocked site

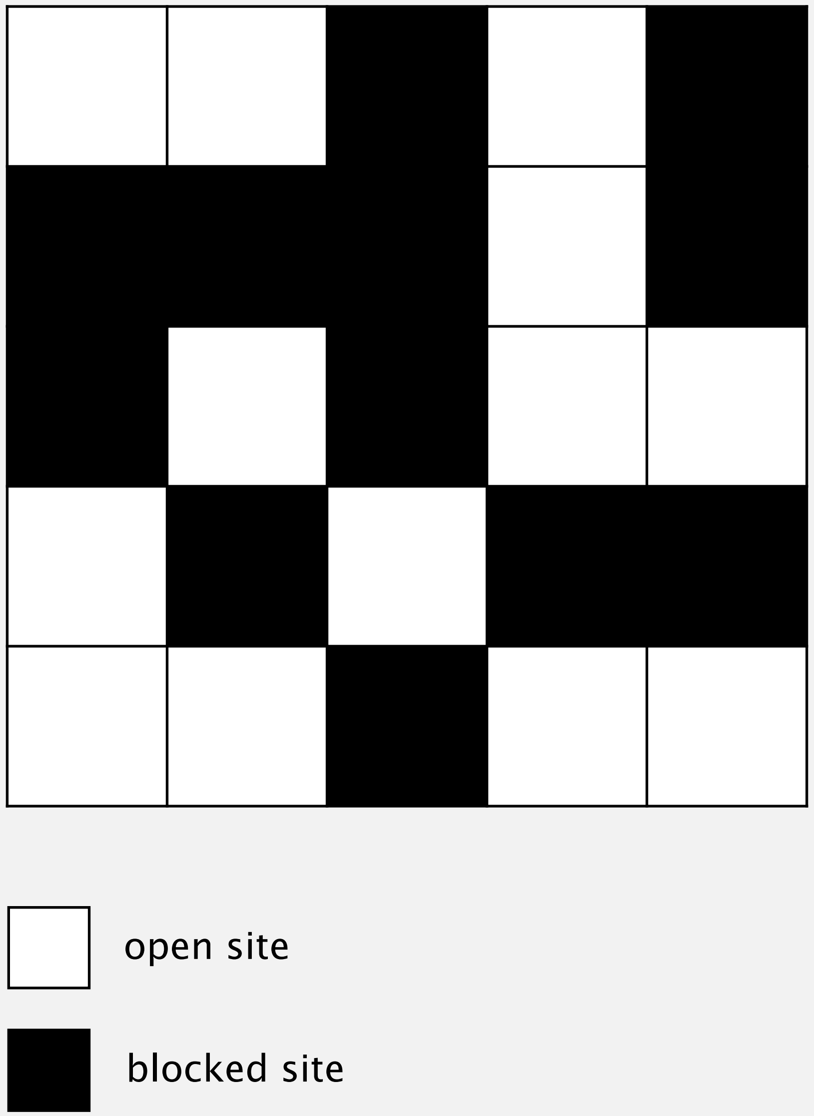
Dynamic connectivity solution to estimate percolation threshold

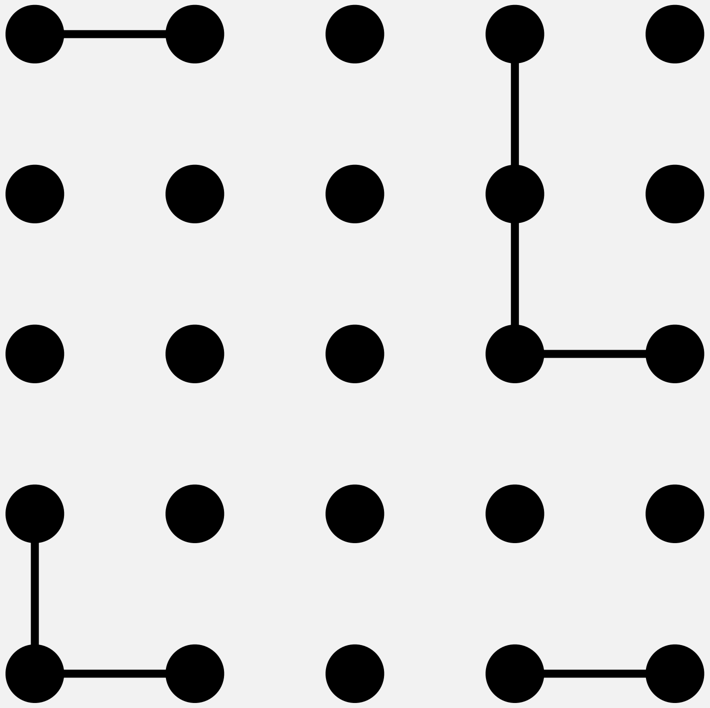
Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component iff connected by open sites.
- Percolates iff any site on bottom row is connected to any site on top row.

brute-force algorithm: N ² calls to connected()

$N = 5$





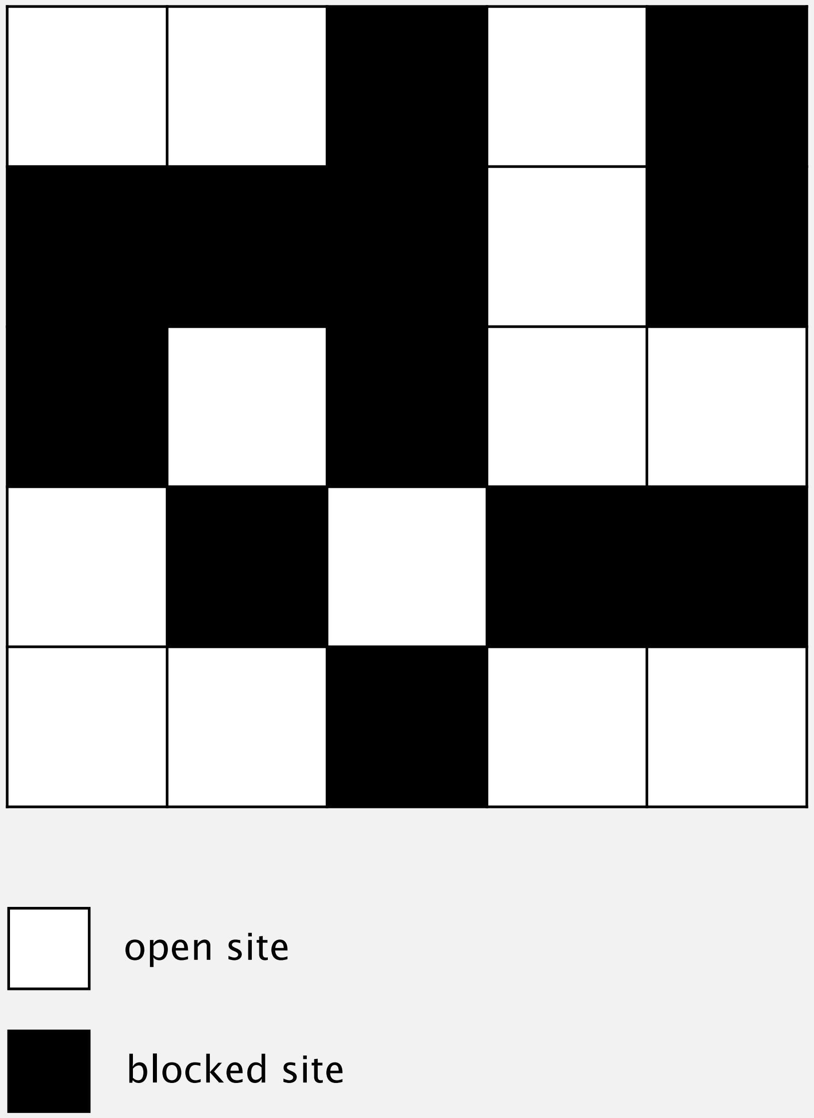
Dynamic connectivity solution to estimate percolation threshold

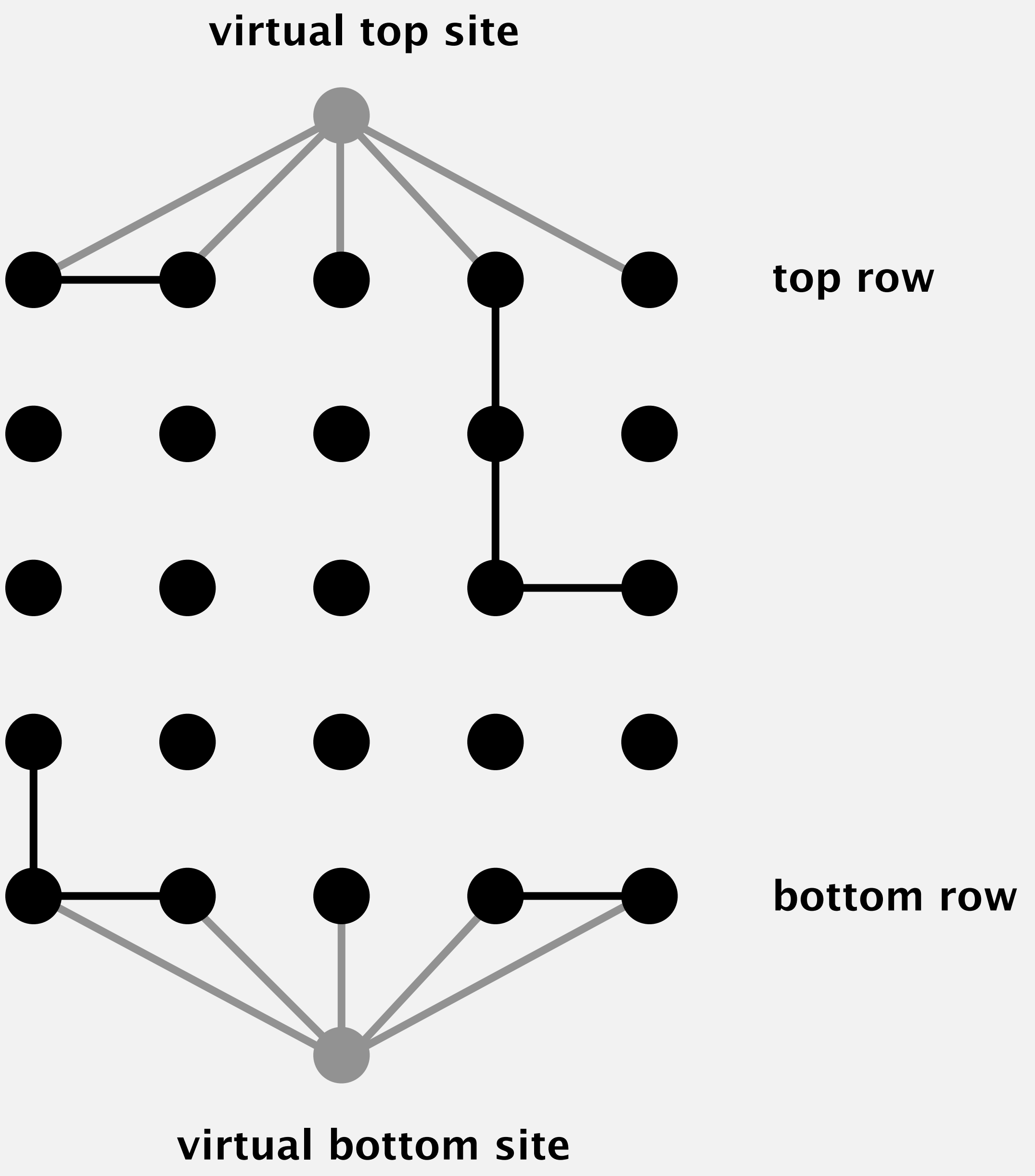
Clever trick. Introduce 2 virtual sites (and connections to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.

more efficient algorithm: only 1 call to connected()

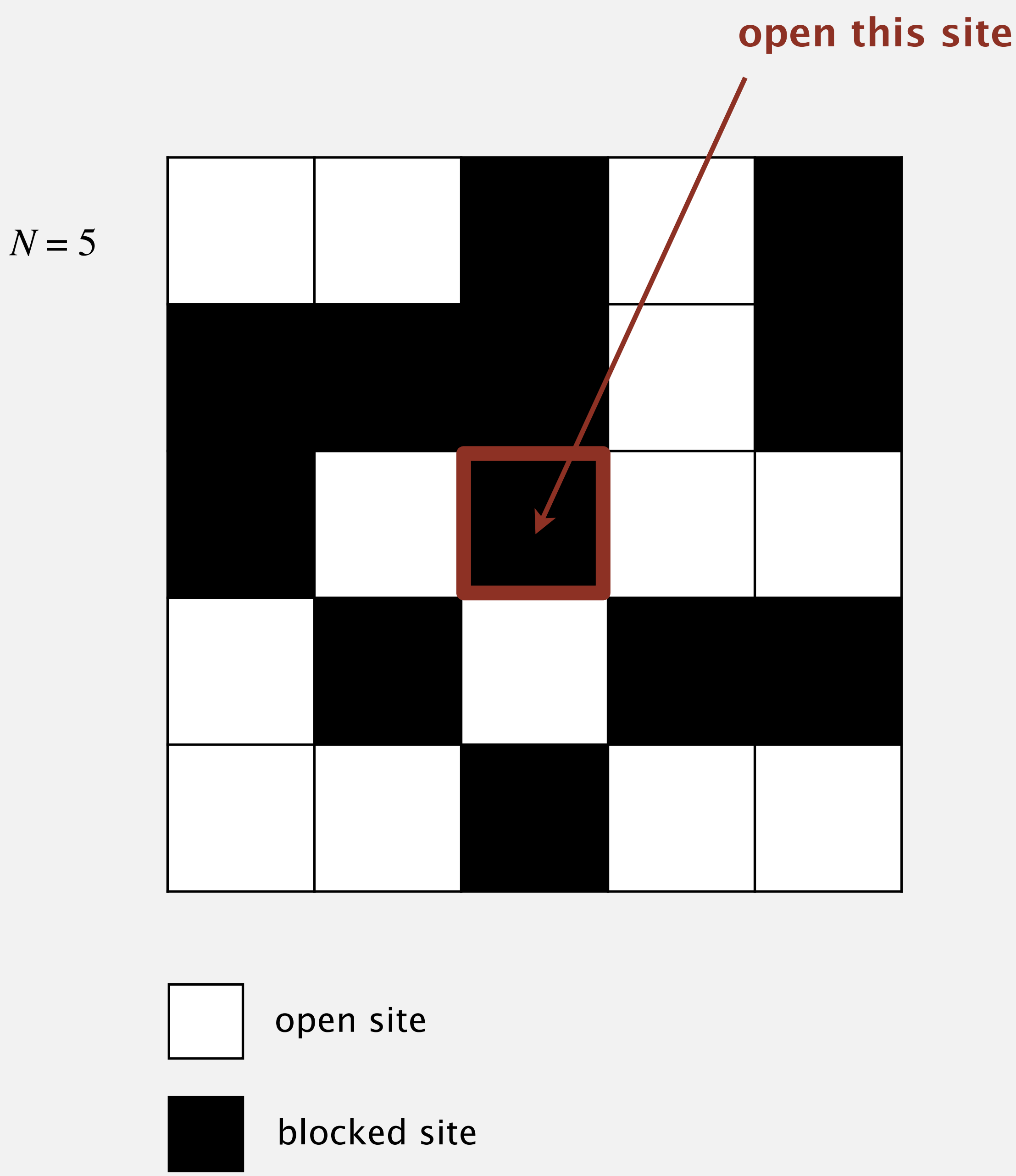
$N = 5$

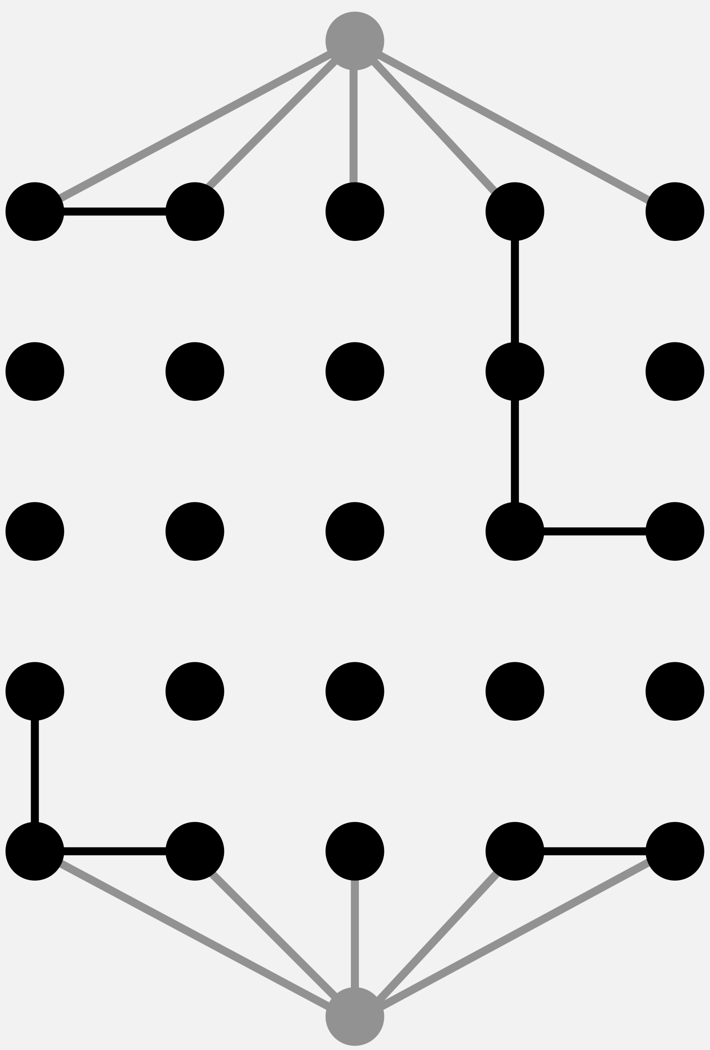




Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?





Dynamic connectivity solution to estimate percolation threshold

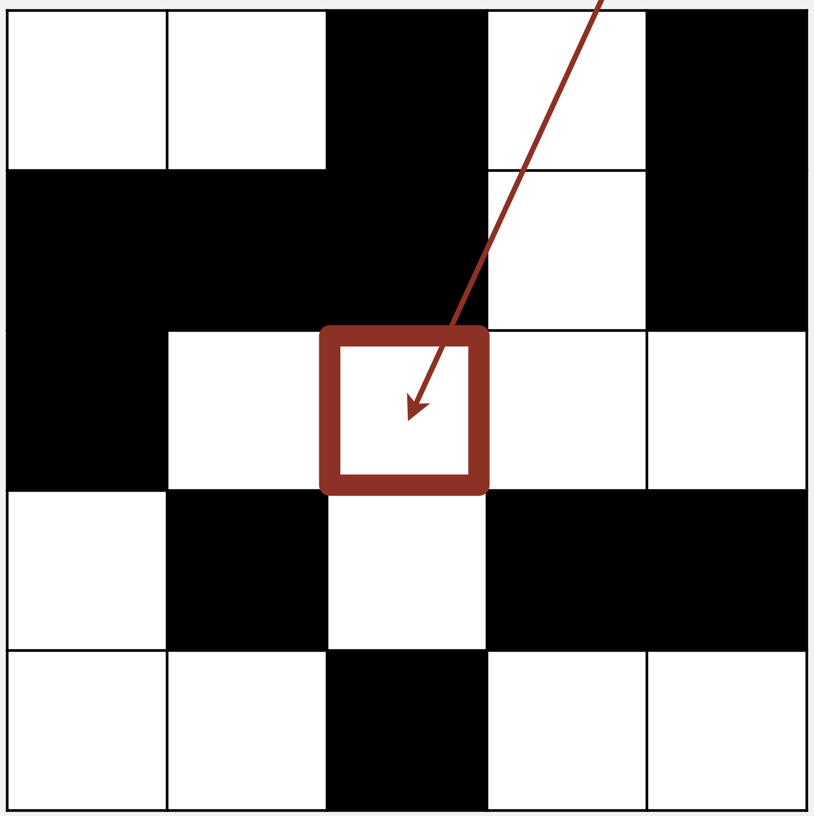
Q. How to model opening a new site?

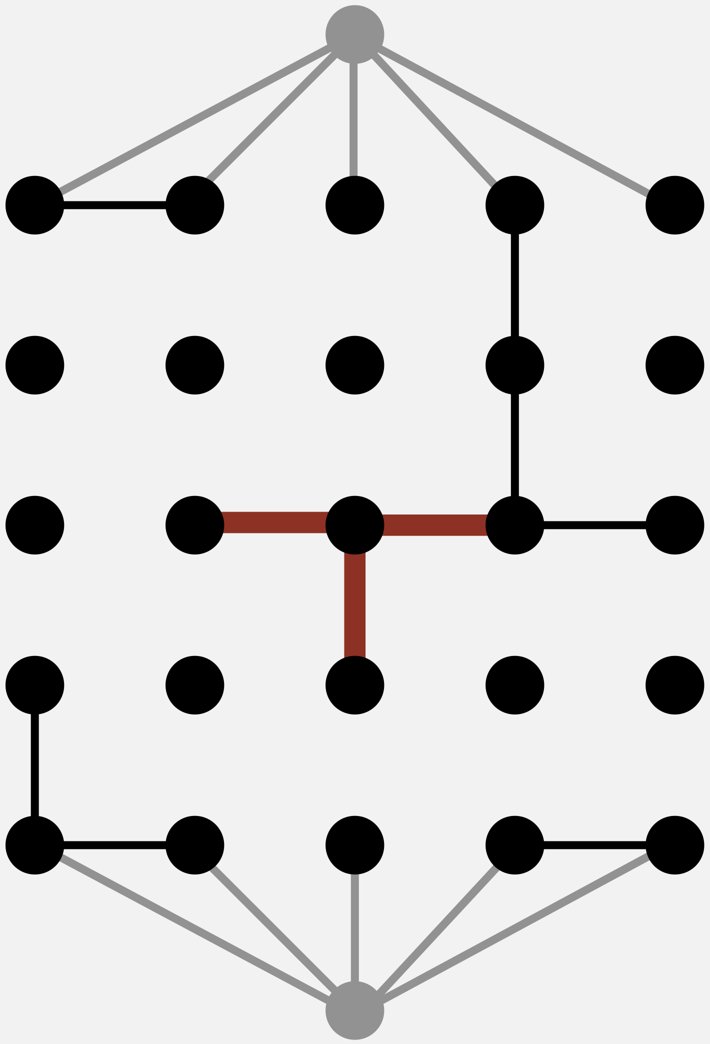
A. Mark new site as open; connect it to all of its adjacent open sites.

up to 4 calls to `union()`

open this site

$N = 5$





open site

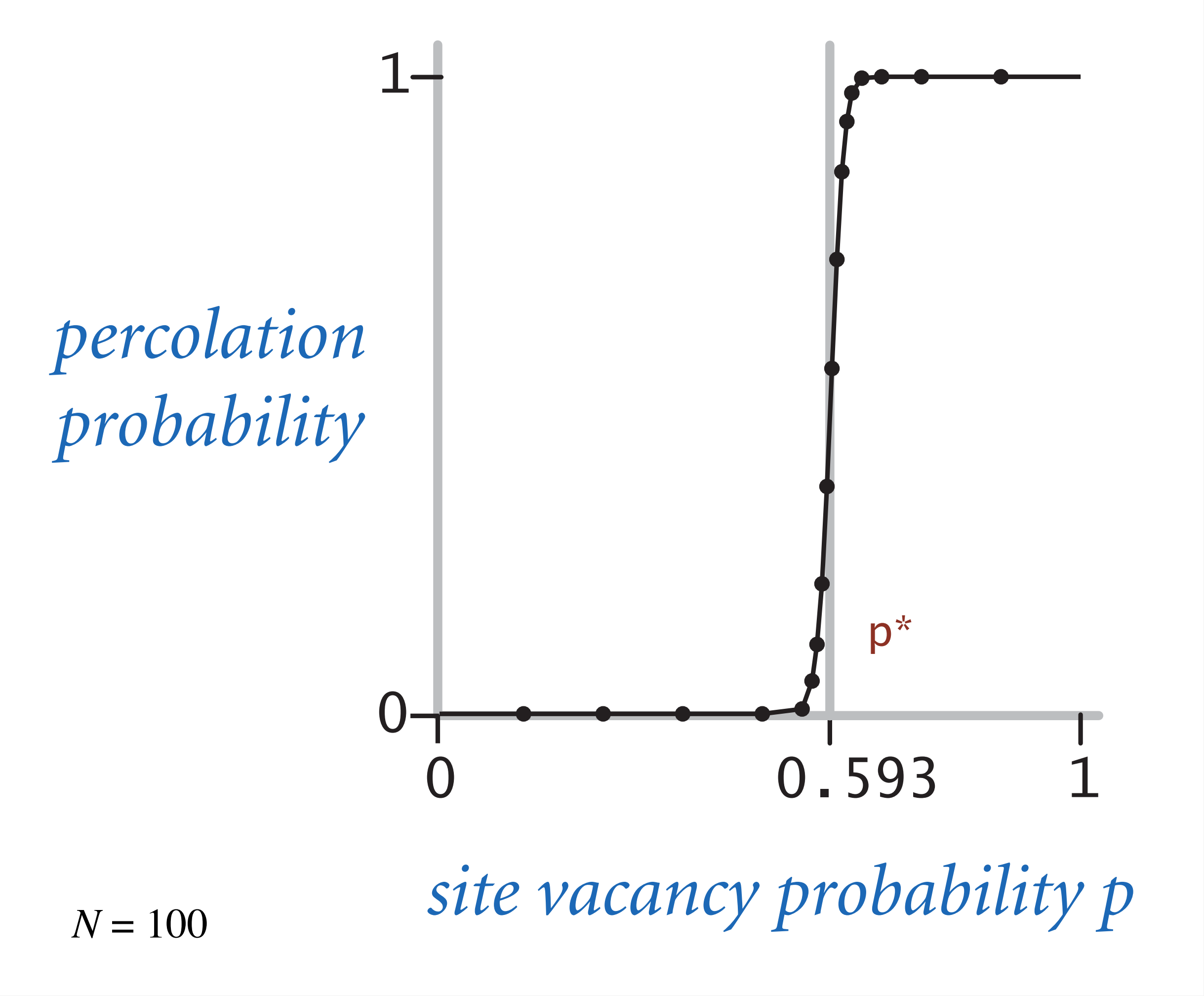
blocked site

Percolation threshold

Q. What is percolation threshold p^* ?

A. About 0.592746 for large square lattices.

constant known only via simulation



Fast algorithm enables accurate answer to scientific question.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

A little mathematical analysis.