Computational Astrophysics HW1

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- **Derive the stability criterion of the Lax-Wendroff scheme for solving the advection equation; demonstrate that it is second-order accurate.**

I. Stability Criterion of the Lax-Wendroff scheme

Step1: $u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_{j+1}^{n} + u_{j}^{n}\right) - \frac{v\Delta t}{2\Delta x}(u_{j+1}^{n} - u_{j}^{n})$

Step2: $u_{j}^{n+1} = u_{j}^{n} - \frac{v\Delta t}{\Delta x}(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2})$

Let $r \equiv \frac{v\Delta t}{\Delta x}$, and bring step1 inside to step2, we have

$$u_j^{n+1} = u_j^n - r\left[\frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{r}{2}\left(u_{j+1}^n - u_j^n\right) - \frac{1}{2}\left(u_j^n + u_{j-1}^n\right) + \frac{r}{2}\left(u_j^n - u_{j-1}^n\right)\right]$$

$$\Rightarrow u_j^{n+1} = u_j^n - \frac{r}{2}\left[\left(u_{j+1}^n - u_{j-1}^n\right) - r(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\right]$$

Insert $u_j^n = \xi^n \cdot e^{i(kj\Delta x)}$,

$$\Rightarrow \xi^{n+1} \cdot e^{i(kj\Delta x)} = \xi^n \cdot e^{i(kj\Delta x)}$$
$$-\frac{r}{2}\left[\xi^n\left(e^{ik(j+1)\Delta x} - e^{ik(j-1)\Delta x}\right)\right.$$
$$\left.- r\xi^n\left(e^{ik(j+1)\Delta x} - 2e^{i(kj\Delta x)} + e^{ik(j-1)\Delta x}\right)\right]$$

$$\Rightarrow \xi = 1 - \frac{r}{2}\left[e^{ik\Delta x} - e^{-ik\Delta x} - r\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right]$$

$$\Rightarrow \xi = 1 - ir\sin(k\cdot\Delta x) - r^2(1 - \cos(k\cdot\Delta x))$$

So

$$|\xi|^2 = [1 - r^2(1-\cos(k\cdot\Delta x))]^2 + r^2\sin^2(k\cdot\Delta x)$$
$$= [1 - r^2(1-\cos(k\cdot\Delta x))]^2 + r^2(1-\cos^2(k\cdot\Delta x))$$
$$= [1 - r^2(1-\cos(k\cdot\Delta x))]^2$$
$$+ r^2(1-\cos(k\cdot\Delta x))(1+\cos(k\cdot\Delta x))$$
$$\therefore |\xi|^2 = 1 - r^2(1-r^2)(1-\cos(k\cdot\Delta x))^2$$

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II. Second order accurate]

From step2, we get

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\mathrm{v}\frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{\Delta x}$$

On LHS, write $u_j^{n+1} \to u(x, t+\Delta t)$ and $u_j^n \to u(x,t)$, and do the Taylor expansion with the center $\left(x, t + \frac{\Delta t}{2}\right)$.

$$u(x,t+\Delta t) \approx u\left(x,t+\frac{\Delta t}{2}\right) + \frac{\Delta t}{2}\frac{\partial u\left(x,t+\frac{\Delta t}{2}\right)}{\partial t} + \frac{1}{2!}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial u^2\left(x,t+\frac{\Delta t}{2}\right)}{\partial t^2} + \cdots$$

$$u(x,t) \approx u\left(x,t+\frac{\Delta t}{2}\right) - \frac{\Delta t}{2}\frac{\partial u\left(x,t+\frac{\Delta t}{2}\right)}{\partial t} + \frac{1}{2!}\left(\frac{\Delta t}{2}\right)^2 \frac{\partial u^2\left(x,t+\frac{\Delta t}{2}\right)}{\partial t^2} - \cdots$$

$$\Rightarrow u(x,t+\Delta t) - u(x,t) \approx \Delta t \cdot \frac{\partial u\left(x,t+\frac{\Delta t}{2}\right)}{\partial t} + \frac{2}{3!}\left(\frac{\Delta t}{2}\right)^3 \cdot \frac{\partial u^3\left(x,t+\frac{\Delta t}{2}\right)}{\partial t^3} + \cdots$$

$$\Rightarrow \frac{\partial u\left(x,t+\frac{\Delta t}{2}\right)}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t^2)$$

Same method and argument goes with expanding the RHS. Write $u_{j+1/2}^{n+1/2} \rightarrow$ $u\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)$ and $u_{j-1/2}^{n+1/2} \rightarrow u\left(x-\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)$, expand with the center $\left(x, t+\frac{\Delta t}{2}\right)$. We have,

$$\frac{\partial u\left(x,t+\frac{\Delta t}{2}\right)}{\partial x} \approx \frac{u\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) - u\left(x-\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)}{\Delta x} + O(\Delta x^2)$$

We can see that Lax-Wendroff Scheme is 2nd order accurate.

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By testing with the code directly, we can also see that it is 2nd order accurate. If there are 2 times of the previous sample points, Δx is cut to half, then the error decreases by a factor of 4.

N sample points	Error
100	9.471×10^{-4}
200	2.368×10^{-4}
400	5.922×10^{-5}
800	1.480×10^{-5}

- **Demonstrate that the Crank-Nicolson scheme is unconditionally stable for solving the diffusion equation.**

  Crank-Nicolson Scheme:

$$u_j^{n+1} = u_j^n + \frac{D \cdot \Delta t}{2\Delta x^2}\left[\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) + \left(u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}\right)\right]$$

  Let $\alpha \equiv \frac{D\cdot\Delta t}{\Delta x^2}$, and insert $u_j^n = \xi^n \cdot e^{i(kj\Delta x)}$,

$$\Rightarrow \xi^{n+1}e^{ikj\Delta x} = \xi^{n}e^{ikj\Delta x}$$
$$+\frac{\alpha}{2}\left[\xi^{n+1}\cdot e^{ik(j+1)\Delta x} - 2\xi^{n+1}\cdot e^{ikj\Delta x} + \xi^{n+1}e^{ik(j-1)\Delta x}\right.$$
$$\left.+\xi^{n}e^{ik(j+1)\Delta x} - 2\xi^{n}e^{ikj\Delta x} + \xi^{n}e^{ik(j-1)\Delta x}\right]$$
$$\Rightarrow \xi = 1 + \frac{\alpha}{2}\left[\xi\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right) + \left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)\right]$$
$$\Rightarrow \xi = \frac{1+\alpha[\cos(k\Delta x) - 1]}{1-\alpha[\cos(k\Delta x) - 1]}$$

Since $-1 \leq \cos\theta \leq 1 \Rightarrow \cos\theta - 1 \leq 0$,
$1-\alpha[\cos(k\Delta x) - 1] \geq 1+\alpha[\cos(k\Delta x) - 1]$ will always satisfy. So Crank Nicolson Scheme is unconditionally stable.

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