Computational Astrophysics HW3

R08244002 蔡欣蓉

- **Demonstrate numerically that (i)Jacobi (ii)Gauss-Seidel and (iii)SOR are second order accurate.**

  A. Analytic solution

  Reference: https://math.stackexchange.com/questions/1251117/analytic-solution-to-poisson-equation

  $\nabla^2 u(x,y) = 2x(y-1)(y-2x+xy+2)e^{x-y},\ (x,y) \in (0,1) \times (0,1)$

  With boundary conditions,

  $$u(x,0) = u(x,1) = 0, x \in [0,1]$$
  $$u(0,y) = u(1,y) = 0, y \in [0,1]$$

  Has solution

  $$u(x,y) = e^{x-y} \cdot x \cdot (1-x) \cdot y \cdot (1-y)$$

  B. Compare with the result

  Define the error as $error = \frac{\sum_{i,j=1}^{N} |\phi_{i,j} - (analytic\ \ solution)|}{N^2}$

	Jacobi, Gauss-Seidel, SOR		
	16×16	32×32	64×64
Error	5.73×10^{-5}	1.44×10^{-5}	3.61×10^{-6}

  Calculate the laplace equation in the region $[0,1] \times [0,1]$ with different grid size, we can see that as Δ turned half (which is grid size times 4), the error decreased by a factor of 4. Three of them are second order accurate.

- **Determine the optimum overrelaxation parameter ω in SOR**

  Using Scipy.optimize.minimize to find the optimal ω. As grid size getting large, ω becomes bigger. Because the more correction it can make in its reasonable range, the faster it converges.

	SOR		
	16×16	32×32	64×64
ω	1.697	1.828	1.909

- **How do (1)wall-clock time and (2)the number of iterations required to reach convergence scale with grid size?**

	Jacobi			Gauss-Seidel			SOR		
	16×16	32×32	64×64	16×16	32×32	64×64	16×16	32×32	64×64
Iterations	1313	4956	19223	659	2482	9618	71	137	269
Time Used (sec)	2.25	33.15	419.13	1.07	15.36	205.33	0.20	1.32	10.02

1. We can clearly see that the numbers of iteration for Jacobi and Gauss-Seidel to reach the stopping criteria is proportional to N^2 (size of the grid is $N \times N$), and SOR is proportional to N, as predicted.
2. Gauss-Seidel converges faster than Jacobi, even though both of them scales as N^2.
3. SOR method is the best of all, though we have to determine the optimum overrelaxation parameter first, which takes a lot of time.
4. Even though the time scale is not that precise, we can see that it grows with a factor of 10 ~ 15, which is roughly proportional to N^4. Not expected this.

- **Final result**