Multigrids Method

Goal: Solve $\mathcal{L} u = f$

$$\mathcal{L}_h u_h = f_h$$

$\tilde{u}_h$: approximate solution

u_h: exact solution.

$$v_h = u_h - \tilde{u}_h \qquad \text{(Error, Correction)}$$

$$d_h = \mathcal{L}_h \tilde{u}_h - f_h = -\mathcal{L}_h v_h \qquad \text{(Residual)}$$

$\uparrow$ $\mathcal{L}_h$ is linear

$\boxed{\text{Idea: Find } \hat{v}_h \text{ to correct } \tilde{u}_h, \text{ using } \mathcal{L}_h u_h = -d_h}$

(1) $\underbrace{\hat{\mathcal{L}}_h}_{\text{A simpler operator than } \mathcal{L}_h} \hat{v}_h = -d_h \longrightarrow \tilde{u}_h^{new} = \tilde{u}_h + \hat{v}_h$

(2) We "coarsify" rather than "simplify". <u>Coarse-Grid Correction</u>

$$\mathcal{L}_h v_h = -d_h$$

$\downarrow$ R: restriction

$$\mathcal{L}_H \tilde{v}_H = -d_H \qquad H = 2h$$

$\downarrow$ P: prolongation

$$\tilde{v}_h \longrightarrow \tilde{u}_h^{new} = \tilde{u}_h + \tilde{v}_h$$

Two-Grid Iteration

Initial guess $\tilde{u}_h^{old}$

S: Smoothing Operator

Pre-smoothing $\tilde{u}_h^{old,\,smooth} = S\,\tilde{u}_h^{old}$ ← Apply some steps only!

Residual $d_h = \mathcal{L}_h \tilde{u}_h^{old,\,smooth} - f_h = -\mathcal{L}_h \underset{\sim}{v_h}$

$v_h = u_h - \tilde{u}_h$

Coarse Grid Correction

$\Rightarrow \mathcal{L}_h v_h = -d_h$

$\downarrow$ R: Restriction

$\mathcal{L}_{2h} \tilde{v}_{2h} = -d_{2h}$, solve $\tilde{v}_{2h}$ ← Exactly!

(Through inverse matrix, or by relaxation)

$\downarrow$ P: Prolongation

$\tilde{v}_h$

Obtain $\tilde{u}_h^{new} = \tilde{u}_h^{old} + \tilde{v}_h$

Post-Smoothing $\tilde{u}_h^{new,\,smooth} = S\,\tilde{u}_h^{new}$ ← Apply some steps only!

Test if it reaches the target error

def error $= \mathcal{L}_h \tilde{u}_h^{new,\,smooth} - f_h$

No.

Yes!

Result: $\tilde{u}_h^{new,\,smooth}$

Details:

(1) S: Smoothing Operator

Gauss-Seidel, with even/odd method

Numerical Recipe
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(2) P: Prolongation , R: Restriction

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \qquad R = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

Works well with Poisson eq.

Numerical Recipe
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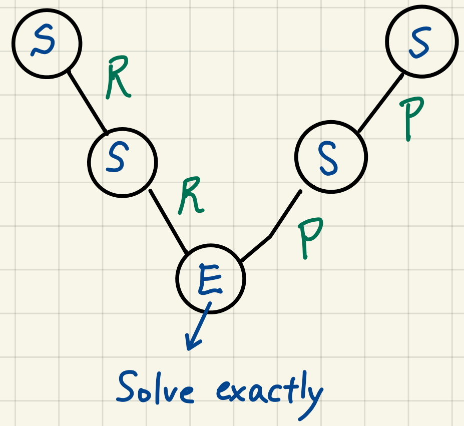
Multigrid Method

Ex:

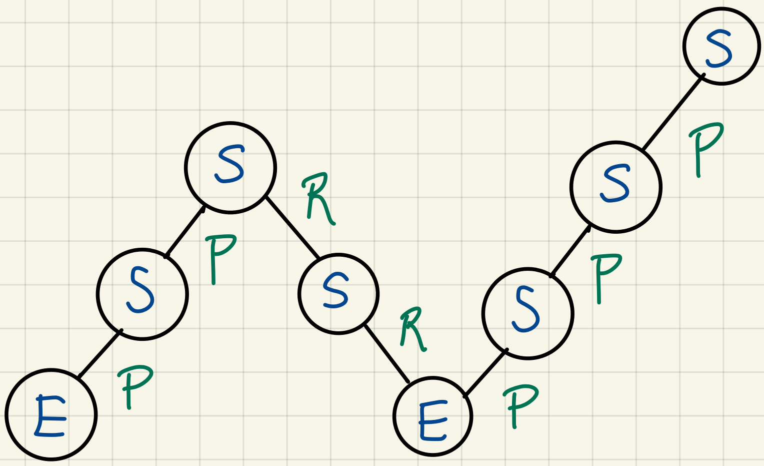


Finer Grid

↕

Coarser Grid

Full Multigrid Algorithm



Produces solution at all levels!