Multigrids Method

Goal: Solve $\mathcal{L} u = f$

$$\mathcal{L}_h u_h = f_h$$

$\tilde{u}_h$: approximate solution

u_h: exact solution.

$$v_h = u_h - \tilde{u}_h \qquad \text{(Error, Correction)}$$

$$d_h = \mathcal{L}_h \tilde{u}_h - f_h = -\mathcal{L}_h v_h \qquad \text{(Residual)}$$

↑ $\mathcal{L}_h$ is linear

> Idea: Find $\hat{v}_h$ to correct $\tilde{u}_h$, using $\mathcal{L}_h u_h = -d_h$

(1)

$$\hat{\mathcal{L}}_h \hat{v}_h = -d_h \longrightarrow \tilde{u}_h^{new} = \tilde{u}_h + \hat{v}_h$$

A simpler operator than $\mathcal{L}_h$

(2) We "coarsify" rather than "simplify". Coarse-Grid Correction

$$\mathcal{L}_h v_h = -d_h$$

↓ $\mathcal{R}$: restriction

$$\mathcal{L}_H \tilde{v}_H = -d_H \qquad H = 2h$$

↓ P: prolongation

$$\tilde{v}_h \longrightarrow \tilde{u}_h^{new} = \tilde{u}_h + \tilde{v}_h$$

Two-Grid Iteration

Initial guess $\tilde{u}_h^{old}$

S: Smoothing Operator

Pre-smoothing $\tilde{u}_h^{old,smooth} = S\,\tilde{u}_h^{old}$ ← Apply some steps only!

Residual $d_h = \mathcal{L}_h \tilde{u}_h^{old,smooth} - f_h = -\mathcal{L}_h v_h$

$v_h = u_h - \tilde{u}_h$

Coarse Grid Correction:

$\Rightarrow \mathcal{L}_h v_h = -d_h$

↓ R: Restriction

$\mathcal{L}_{2h} \tilde{v}_{2h} = -d_{2h}$, solve $\tilde{v}_{2h}$ ← Exactly!

(Through inverse matrix, or by relaxation)

↓ P: Prolongation

$\tilde{v}_h$

Obtain $\tilde{u}_h^{new} = \tilde{u}_h^{old,smooth} + \tilde{v}_h$

Post-Smoothing $\tilde{u}_h^{new,smooth} = S\,\tilde{u}_h^{new}$ ← Apply some steps only!

Test if it reaches the target error

def error $= \mathcal{L}_h \tilde{u}_h^{new,smooth} - f_h$

No.

Yes!

Result: $\tilde{u}_h^{new,smooth}$

Details:

(1) S: Smoothing Operator

Gauss-Seidel, with even/odd method

Numerical Recipe P.1069 (20.6.12)

(2) P: Prolongation, R: Restriction

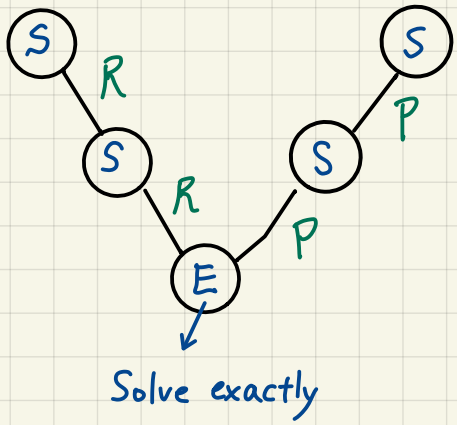
$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \qquad R = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

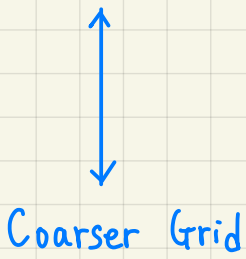
Works well with Poisson eq.

Numerical Recipe P.1072 Top

Multigrid Method

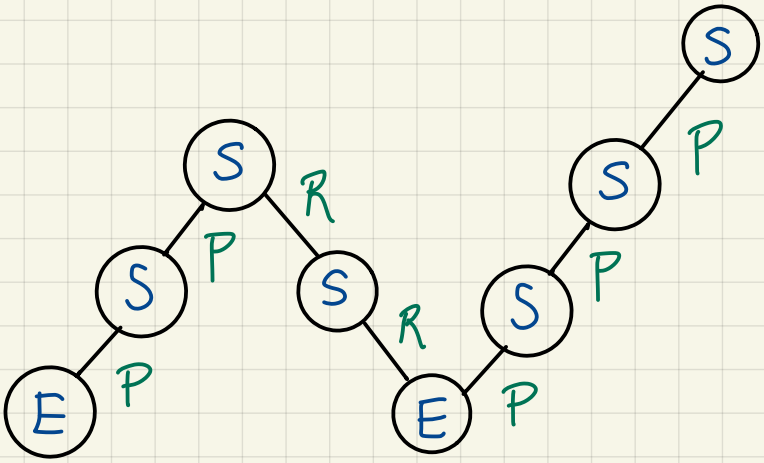
Ex:





想成在 Two-Grid Iteration solve exactly 那邊再用一次 Two-Grid Iteration.

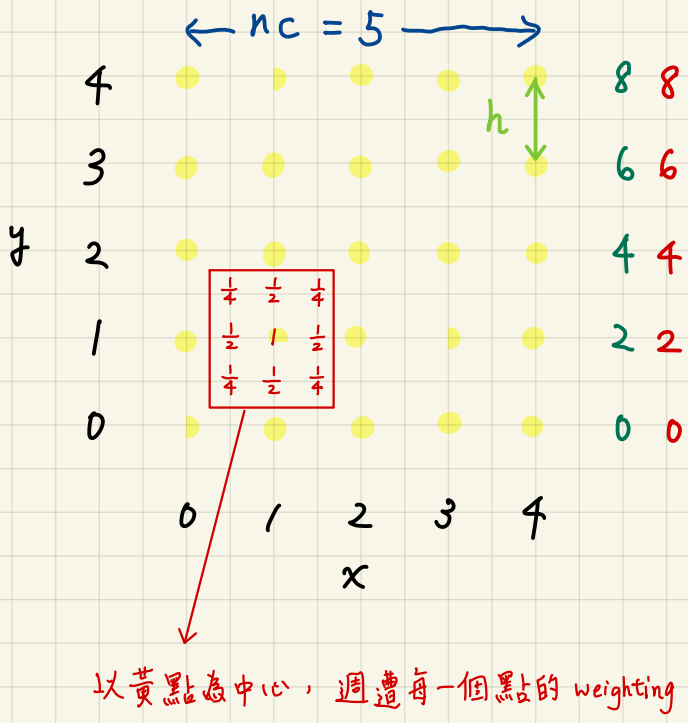
Full Multigrid Algorithm

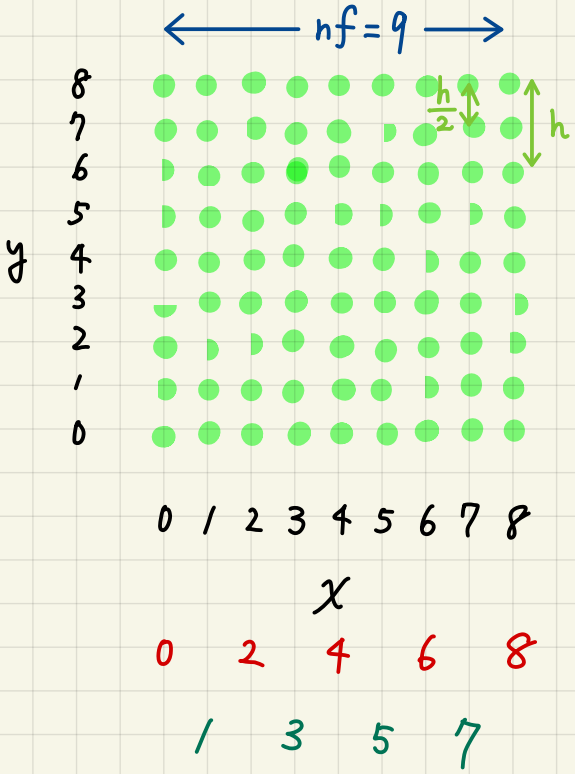


Produces solution at all levels!

Prolongation Details

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$





Things to do:

1. Prolongation / Restriction 有沒有辦法還原
2. 做完 prolongation 後要怎麼加回 Correction.