ESSC4520 L06 Exercise

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Exercise 1. Answer:

$\overrightarrow{c_g} = \frac{\partial\omega}{\partial k}\hat{\imath} + \frac{\partial\omega}{\partial l}\hat{\jmath} + \frac{\partial\omega}{\partial m}\hat{k}$

$\overrightarrow{c_g} = \frac{\partial}{\partial k}(\pm\sqrt{gHK^2+f^2})\hat{\imath} + \frac{\partial}{\partial l}(\pm\sqrt{gHK^2+f^2})\hat{\jmath} + \frac{\partial}{\partial m}(\pm\sqrt{gHK^2+f^2})\hat{k}$

For $\frac{\partial\omega}{\partial k}$:

$\frac{\partial}{\partial k}(\pm\sqrt{gHK^2+f^2})$

$= \frac{\partial}{\partial k}[\pm\sqrt{gH(k^2+l^2+m^2)+f^2}]$

$= \pm\frac{\partial[gH(k^2+l^2+m^2)+f^2]^{\frac{1}{2}}}{\partial(gH(k^2+l^2+m^2)+f^2)}\frac{\partial(gH(k^2+l^2+m^2)+f^2)}{\partial k}$

$= \pm\frac{1}{2}\frac{1}{\sqrt{gH(k^2+l^2+m^2)+f^2}}\frac{\partial gHk^2}{\partial k}$

$= \pm\frac{2gHk}{2\sqrt{gH(k^2+l^2+m^2)+f^2}}$

$= \pm\frac{gHk}{\sqrt{gHK^2+f^2}}$

Simulaly, for $\frac{\partial\omega}{\partial l}$ and $\frac{\pm\partial\omega}{\partial m}$:

$\frac{\partial}{\partial l}(\pm\sqrt{gHK^2+f^2})$

$= \pm\frac{gHl}{\sqrt{gHK^2+f^2}}$

$\frac{\partial}{\partial m}(\pm\sqrt{gHK^2+f^2})$

$= \pm\frac{gHm}{\sqrt{gHK^2+f^2}}$

Therefore, we can get:

$\overrightarrow{c_g} = \pm\frac{gHk}{\sqrt{gHK^2+f^2}}\hat{\imath} \pm \frac{gHl}{\sqrt{gHK^2+f^2}}\hat{\jmath} \pm \frac{gHm}{\sqrt{gHK^2+f^2}}\hat{k}$

$\overrightarrow{c_g} = \frac{gH}{\sqrt{gHK^2+f^2}}(\pm k\hat{\imath} \pm l\hat{\jmath} + \pm m\hat{k})$

Considering a barotrophic 2D model,

$\overrightarrow{c_g} = \frac{gH}{\sqrt{gHK^2+f^2}}(\pm k\hat{\imath} \pm l\hat{\jmath})$

Exercise 2. Answer:

Substituting

$u(x,y,t) = Ae^{\iota(kx+ly-\omega t)}$
$v(x,y,t) = Be^{\iota(kx+ly-\omega t)}$
$\eta(x,y,t) = Ce^{\iota(kx+ly-\omega t)}$

into

$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + \bar{v}\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} + fv$ ...(1)
$\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} + \bar{v}\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y} + fu$ ...(2)
$\frac{\partial \eta}{\partial t} + \bar{u}\frac{\partial \eta}{\partial x} + \bar{v}\frac{\partial \eta}{\partial y} = -H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$ ...(3)

(1) becomes

$\frac{\partial}{\partial t}(Ae^{\iota(kx+ly-\omega t)})+\bar{u}\frac{\partial}{\partial x}(Ae^{\iota(kx+ly-\omega t)})+\bar{v}\frac{\partial}{\partial y}(Ae^{\iota(kx+ly-\omega t)}) = -g\frac{\partial}{\partial x}(Ce^{\iota(kx+ly-\omega t)})+fBe^{\iota(kx+ly-\omega t)}$

$-\iota\omega Ae^{\iota(kx+ly-\omega t)} + \iota k\bar{u}Ae^{\iota(kx+ly-\omega t)} + \iota l\bar{v}Ae^{\iota(kx+ly-\omega t)} = -\iota kgCe^{\iota(kx+ly-\omega t)} + fBe^{\iota(kx+ly-\omega t)}$

$-\iota\omega A + \iota k\bar{u}A + \iota l\bar{v}A = -\iota kgC + fB$

$\iota A(k\bar{u} + l\bar{v} - \omega) - fB + \iota kgC = 0$

(2) becomes

$\frac{\partial}{\partial t}(Be^{\iota(kx+ly-\omega t)})+\bar{u}\frac{\partial}{\partial x}(Be^{\iota(kx+ly-\omega t)})+\bar{v}\frac{\partial}{\partial y}(Be^{\iota(kx+ly-\omega t)}) = -g\frac{\partial}{\partial y}(Ce^{\iota(kx+ly-\omega t)})-fAe^{\iota(kx+ly-\omega t)}$

$-\iota\omega Be^{\iota(kx+ly-\omega t)} + \iota k\bar{u}Be^{\iota(kx+ly-\omega t)} + \iota l\bar{v}Be^{\iota(kx+ly-\omega t)} = -\iota lgCe^{\iota(kx+ly-\omega t)} - fAe^{\iota(kx+ly-\omega t)}$

$-\iota\omega B + \iota k\bar{u}B + \iota l\bar{v}B = -\iota lgC - fA$

$fA + \iota B(k\bar{u} + l\bar{v} - \omega) + \iota lgC = 0$

(3) becomes

$\frac{\partial}{\partial t}(Be^{\iota(kx+ly-\omega t)})+\bar{u}\frac{\partial}{\partial x}(Ce^{\iota(kx+ly-\omega t)})+\bar{v}\frac{\partial}{\partial y}(Ce^{\iota(kx+ly-\omega t)}) = -H(\frac{\partial}{\partial x}[Ae^{\iota(kx+ly-\omega t)})+\frac{\partial}{\partial y}(Be^{\iota(kx+ly-\omega t)})]$

$-\iota\omega Ce^{\iota(kx+ly-\omega t)}+\iota k\bar{u}Ce^{\iota(kx+ly-\omega t)}+\iota l\bar{v}Ce^{\iota(kx+ly-\omega t)} = -\iota kHAe^{\iota(kx+ly-\omega t)}-\iota lHBe^{\iota(kx+ly-\omega t)}$

$-\omega C + k\bar{u}C + l\bar{v}C = -kHA - lHB$

$kHA + lHB + C(k\bar{u} + l\bar{v} - \omega) = 0$

We can write them in matrix form

$$\begin{pmatrix} \iota(k\bar{u}+l\bar{v}-\omega) & -f & \iota kg \\ f & \iota(k\bar{u}+l\bar{v}-\omega) & \iota lg \\ kH & lH & (k\bar{u}+l\bar{v}-\omega) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

To get non-trivial solutions of A, B and C, we solve for the determinant of the matrix = 0

$$\begin{vmatrix} \iota(k\bar{u}+l\bar{v}-\omega) & -f & \iota kg \\ f & \iota(k\bar{u}+l\bar{v}-\omega) & \iota lg \\ kH & lH & (k\bar{u}+l\bar{v}-\omega) \end{vmatrix} = 0$$

$\iota(k\bar{u}+l\bar{v}-\omega)[\iota(k\bar{u}+l\bar{v}-\omega)^2-\iota l^2gH]+f[f(k\bar{u}+l\bar{v}-\omega)-\iota klgH]+\iota kg[flH-\iota kH(k\bar{u}+l\bar{v}-\omega)]$
$= 0$

$-(k\bar{u}+l\bar{v}-\omega)^3+l^2gH(k\bar{u}+l\bar{v}-\omega)+f^2(k\bar{u}+l\bar{v}-\omega)-\iota fklgH+\iota fklgH+k^2gH(k\bar{u}+l\bar{v}-\omega) = 0$

$-(k\bar{u}+l\bar{v}-\omega)^2+l^2gH+f^2+k^2gH = 0$

$(k\bar{u}+l\bar{v}-\omega)^2 = l^2gH+f^2+k^2gH$

$k\bar{u}+l\bar{v}-\omega = \pm\sqrt{gH(k^2+l^2)+f^2}$

$\omega = k\bar{u}+l\bar{v} \pm \sqrt{gH(k^2+l^2)+f^2}$

Since $K^2 = k^2+l^2$ for 2D model, the dispersion relationship becomes

$\omega = k\bar{u}+l\bar{v} \pm \sqrt{gHK^2+f^2}$

Exercise 3. Answer:

$$
\begin{aligned}
(\lambda^2-1)^2+\sigma^2\lambda^2 &= 0\\
(\lambda^2)^2-2\lambda^2+1+\sigma^2\lambda^2 &= 0\\
(\lambda^2)^2+(\sigma^2-2)\lambda^2+1 &= 0\\
\lambda^2 &= \frac{-(\sigma^2-2)\pm\sqrt{(\sigma^2-2)^2-4}}{2}\\
\lambda^2 &= \frac{-(\sigma^2-2)\pm\sqrt{\sigma^4-4\sigma^2+4-4}}{2}\\
\lambda^2 &= \frac{-(\sigma^2-2)\pm\sqrt{\sigma^4-4\sigma^2}}{2}\\
\lambda^2 &= \frac{2-\sigma^2}{2}\pm\frac{\sqrt{\sigma^4-4\sigma^2}}{2}
\end{aligned}
$$

Since $|\lambda| = |\sqrt{(real)^2+(imaginary)^2}|$, for $\sigma^2 \leq 4$:

$$
\begin{aligned}
|\lambda^2| &= \left|\tfrac{(2-\sigma^2)^2}{4}+\tfrac{\sigma^4-4\sigma^2}{4}\right|\\
&= \left|\tfrac{(2-\sigma^2)^2+\sigma^4-4\sigma^2}{4}\right|\\
&= \left|\tfrac{4-4\sigma^2+\sigma^4+\sigma^4-4\sigma^2}{4}\right|\\
&= \left|\tfrac{2-4\sigma^2+\sigma^4}{2}\right|\\
&= \left|1-2\sigma^2+\tfrac{1}{2}\sigma^4\right|\\
&= \left|1+\tfrac{1}{2}(-4\sigma^2+\sigma^4)\right|
\end{aligned}
$$

Considering minimum σ^2

$$
\begin{aligned}
\frac{d}{d\sigma^2}(-2\sigma^2+\frac{1}{2}\sigma^4) &= 0\\
\sigma^2-2 &= 0\\
\sigma^2 &= 2
\end{aligned}
$$

Therefore, $2 \leq \sigma^2 \leq 4$.

When $\sigma^2 = 2, \frac{1}{2}[-4(2)+(2)^2] = -4+2 = -2$.

When $\sigma^2 = 4, \frac{1}{2}[-4(4)+(4)^2] = -8+8 = 0$.

We can get,

$$
\begin{gathered}
-2 \le \frac{1}{2}(-4\sigma^2 + \sigma^4) \le 0 \\
-1 \le \frac{1}{2}(-4\sigma^2 + \sigma^4) \le 1 \\
-1 \le \lambda^2 \le 1
\end{gathered}
$$

Therefore, $|\lambda^2| \le 1$, thus $|\lambda| \le 1$.

So, when $\sigma^2 \le 4$, no λ will have an amplitude greater than 1.

Considering $\sigma^2 > 4$, for the negative roots:

$$
|\lambda^2| = \left| \frac{2-\sigma^2}{2} - \frac{\sqrt{\sigma^4 - 4\sigma^2}}{2} \right|
$$

For $\frac{2-\sigma^2}{2}$,

$$
\begin{gathered}
\sigma^2 > 4 \\
-\sigma^2 < -4 \\
2 - \sigma^2 < -2 \\
\frac{2-\sigma^2}{2} < -1
\end{gathered}
$$

and

$$
\frac{\sqrt{\sigma^4 - 4\sigma^2}}{2} > 0
$$

Therefore,

$$
|\lambda^2| = \left| \frac{2-\sigma^2}{2} - \frac{\sqrt{\sigma^4 - 4\sigma^2}}{2} \right| > 1
$$

Thus, $|\lambda| > 1$.

So, when $\sigma^2 > 4$ at least one root will have an amplitude greater than 1.

Exercise 4. Answers:

For,
$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} + fv$ ...(1)
$\frac{\partial v}{\partial t} = -fu$ ...(2)
$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}$ ...(3)

Considering Leapfrog Scheme on the 1D staggered grid given, for (1),

$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} = -\frac{g}{d}(\eta_i^n - \eta_{i-1}^n) + fv_i^n$

$u_i^{n+1} = u_i^{n-1} - \frac{2g\Delta t}{d}(\eta_i^n - \eta_{i-1}^n) + 2\Delta t f v_i^n$

for (2),

$\frac{v_i^{n+1}-v_i^{n-1}}{2\Delta t} = -fu_i^n$

$v_i^{n+1} = v_i^{n-1} - 2\Delta t f u_i^n$

for (3),

$\frac{\eta_i^{n+1}-\eta_i^{n-1}}{2\Delta t} = -\frac{H}{d}(u_{i+1}^n - u_i^n)$

$\eta_i^{n+1} = \eta_i^{n-1} - \frac{2\Delta t H}{d}(u_{i+1}^n - u_i^n)$

Substituting,

$u_i^n = \lambda^n A e^{\iota kid}$
$v_i^n = \lambda^n B e^{\iota kid}$
$\eta_i^n = \lambda^n C e^{\iota kid}$

we gets,

$$\begin{cases} \lambda^{n+1}Ae^{\iota kid} = \lambda^{n-1}Ae^{\iota kid} - \frac{2g\Delta t}{d}[\lambda^n Ce^{\iota kid} - \lambda^n Ce^{\iota k(i-1)d}] + 2\Delta t f\lambda^n Be^{\iota kid} \\ \lambda^{n+1}Be^{\iota kid} = \lambda^{n-1}Be^{\iota kid} - 2\Delta t f\lambda^n Ae^{\iota kid} \\ \lambda^{n+1}Ce^{\iota kid} = \lambda^{n-1}Ce^{\iota kid} - \frac{2\Delta tH}{d}[\lambda^n Ae^{\iota k(i+1)d} - \lambda^n Ae^{\iota kid}] \end{cases}$$

Simplified the above equations we gets,

$$\begin{cases} \lambda A = \lambda^{-1}A - \frac{2g\Delta t}{d}[C(1-e^{-\iota kd})] + 2\Delta t f B \\ \lambda B = \lambda^{-1}B - 2\Delta t f A \\ \lambda C = \lambda^{-1}C - \frac{2\Delta tH}{d}[A(e^{\iota kd}-1)] \end{cases}$$

$(\lambda - \lambda^{-1})A - 2\Delta t f B + \frac{2\Delta t g}{d}(1-e^{-\iota kd})C = 0$

$2\Delta t f A + (\lambda - \lambda^{-1})B = 0$

$\frac{2\Delta t H}{d}(e^{\iota kd} - 1)A + (\lambda - \lambda^{-1})C = 0$

Write it as a matrix

$$\begin{pmatrix} (\lambda - \lambda^{-1}) & -2\Delta t f & \frac{2\Delta t g}{d}(1 - e^{-\iota kd}) \\ 2\Delta t f & (\lambda - \lambda^{-1}) & 0 \\ \frac{2\Delta t H}{d}(e^{\iota kd} - 1) & 0 & (\lambda - \lambda^{-1}) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

To get non trivial solutions

$$\begin{vmatrix} (\lambda^2 - 1) & -2\Delta t f\lambda & \frac{2\Delta t g}{d}(1 - e^{-\iota kd})\lambda \\ 2\Delta t f\lambda & (\lambda^2 - 1) & 0 \\ \frac{2\Delta t H}{d}(e^{\iota kd} - 1)\lambda & 0 & (\lambda^2 - 1) \end{vmatrix} = 0$$

$$0 = (\lambda^2 - 1)^3 + 4(\Delta t)^2 f^2 \lambda^2 (\lambda^2 - 1) - \frac{4(\Delta t)^2 gH}{d^2}(1 - e^{-\iota kd})(e^{\iota kd} - 1)(\lambda^2 - 1)\lambda^2$$
$$0 = (\lambda^2 - 1)^2 + 4(\Delta t)^2 f^2 \lambda^2 + \frac{4(\Delta t)^2 gH}{d^2}(1 - e^{-\iota kd})(1 - e^{\iota kd})\lambda^2$$
$$0 = (\lambda^2 - 1)^2 + 4(\Delta t)^2 f^2 \lambda^2 + \frac{8(\Delta t)^2 gH}{d^2}(1 - coskd)\lambda^2$$

Since $\sqrt{gH} = c$,

$$0 = (\lambda^2 - 1)^2 + 4(\Delta t)^2 f^2 \lambda^2 + \frac{8(\Delta t)^2 c^2}{d^2}(1 - coskd)\lambda^2$$

Let $\sigma^2 = 4(\Delta t)^2 f^2 + \frac{8(\Delta t)^2 c^2}{d^2}(1 - coskd)$

$$0 = (\lambda^2 - 1)^2 + \sigma^2 \lambda^2$$

Since the CFL condition of $|\lambda| \leq 1$ is met when $\sigma^2 \leq 4$, and for the most limiting case to blow up, $coskd = -1$.

$$\sigma^2 \leq 4$$
$$4(\Delta t)^2 f^2 + \frac{8(\Delta t)^2 c^2}{d^2}(1 - coskd) \leq 4$$
$$4(\Delta t)^2 f^2 + \frac{8(\Delta t)^2 c^2}{d^2}(1 - 1) \leq 4$$
$$4(\Delta t)^2 f^2 + \frac{16(\Delta t)^2 c^2}{d^2} \leq 4$$
$$\frac{(\Delta t)^2 f^2}{4} + \frac{(\Delta t)^2 c^2}{d^2} \leq \frac{1}{4}$$
$$\left| \sqrt{\frac{(\Delta t)^2 c^2}{d^2} + \frac{(\Delta t)^2 f^2}{4}} \right| \leq \frac{1}{2}$$