ESSC 4520
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L03 Exercise

<u>Ex1</u>

$$F(x) = ax^3 + bx^2 + cx + d$$

a) Choosing a = 1, b = 2, c = 3 and d = 4, x = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and Δx = 1.
The absolute error between numerical method and analytical method:

	x_1=1	x_2=2	x_3=3	x_4=4	x_5=5	x_6=6	x_7=7	x_8=8	x_9=9	x_10=10
Forward	6	9	12	15	18	21	24	27	30	33
Backward	-4	-7	-10	-13	-16	-19	-22	-25	-28	-31
Centered	1	1	1	1	1	1	1	1	1	1

Therefore, we can see that the centered finite differencing comes closest to the analytical values.

b) Choosing a = 1, b = 2, c = 3 and d = 4, x = 1 and Δx = [1, 0.1, 0.01].
The absolute error between centered finite differencing scheme and analytical method:

	(Δx)_1	(Δx)_2	(Δx)_3
Δx	1	0.1	0.01
$\frac{dF}{dx}$	1	0.01	0.0001

Therefore, we can see that when Δx becomes smaller, the absolute error is also smaller.

Ex2

Taking a = 1, b = 2, c = 3, x = [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and Δx = [3, 2, 1].

	x_1=11	x_2=12	x_3=13	x_4=14	x_5=15	x_6=16	x_7=17	x_8=18	x_9=19	x_10=20
analytical	24	26	28	30	32	34	36	38	40	42
Δx=3	24	26	28	30	32	34	36	38	40	42
Δx=2	24	26	28	30	32	34	36	38	40	42
Δx=1	24	26	28	30	32	34	36	38	40	42
Absolute error (Δx=3)	0	0	0	0	0	0	0	0	0	0
Absolute error (Δx=2)	0	0	0	0	0	0	0	0	0	0
Absolute error (Δx=1)	0	0	0	0	0	0	0	0	0	0

From the result, we can see that the absolute error for $\frac{dF}{dx}$ is very small that ~0 with all cases of x and Δx.

$$F(x) = ax^2 + bx + c$$

The analytical result:

$$\frac{dF}{dx} = \frac{d}{dx}(ax^2 + bx + c) = 2ax + b$$

By centered finite differencing:

$$\frac{dF}{dx} \sim \frac{F(x+\Delta x) - F(x-\Delta x)}{2\Delta x}$$
$$= \frac{[a(x+\Delta x)^2 + b(x+\Delta x) + c] - [a(x-\Delta x)^2 + b(x-\Delta x) + c]}{2\Delta x}$$
$$= \frac{a(x^2 + 2x\Delta x + \Delta x^2) + b(x+\Delta x) + c - a(x^2 - 2x\Delta x + \Delta x^2) - b(x-\Delta x) - c}{2\Delta x}$$
$$= \frac{4ax\Delta x + 2b\Delta x}{2\Delta x}$$
$$= \frac{2\Delta x(2ax + b)}{2\Delta x}$$
$$= 2ax + b$$

Therefore, the centered finite differencing gives same result with the analytical method in this situation. That’s why the absolute error ~ 0.

Ex3

$$F(x) = 400cos\left(\frac{\pi x}{16}\right)$$

a)

	analytical	forward	Backward	centered	4th-order
$\frac{dF}{dx}$	-43.63	-49.75	-36.96	-43.35	-43.63
Absolute error	0	-6.111	6.670	0.2798	0.002152

From the table above, we can figure out that the 4th-order differencing scheme gives closest result with the analytical method.

b)

	analytical	2nd-order	4th-order
$\frac{d^2F}{dx^2}$	-12.82	-12.78	-12.82
Absolute error	0	0.04114	0.0002110

From the table above, we can see that the 4th-order differencing scheme gives closest result with the analytical method.

Ex4

$$F(x+\Delta x) = F(x) + F'(x)\Delta x + \frac{1}{2!}F^{(2)}(x)\Delta x^2 + \frac{1}{3!}F^{(3)}(x)\Delta x^3 + \frac{1}{4!}F^{(4)}(x)\Delta x^4 + \frac{1}{5!}F^{(5)}(x)\Delta x^5 + \frac{1}{6!}F^{(6)}(x)\Delta x^6 + \cdots$$

$$F(x-\Delta x) = F(x) - F'(x)\Delta x + \frac{1}{2!}F^{(2)}(x)\Delta x^2 - \frac{1}{3!}F^{(3)}(x)\Delta x^3 + \frac{1}{4!}F^{(4)}(x)\Delta x^4 - \frac{1}{5!}F^{(5)}(x)\Delta x^5 + \frac{1}{6!}F^{(6)}(x)\Delta x^6 + \cdots$$

$$F(x+\Delta x) + F(x-\Delta x)$$
$$= 2F(x) + \frac{2}{2!}F^{(2)}\Delta x^2 + \frac{2}{4!}F^{(4)}\Delta x^4 + \frac{2}{6!}F^{(6)}\Delta x^6 + \cdots$$
$$\frac{d^2F}{dx^2} = \frac{F(x+\Delta x) + F(x-\Delta x) - 2F(x)}{\Delta x^2} + \frac{2}{4!}F^{(4)}\Delta x^2 + \frac{2}{6!}F^{(6)}\Delta x^4 + \cdots$$

$$F(x+2\Delta x) = F(x) + F'(x)(2\Delta x) + \frac{1}{2!}F^{(2)}(x)(2\Delta x)^2 + \frac{1}{3!}F^{(3)}(x)(2\Delta x)^3 + \frac{1}{4!}F^{(4)}(x)(2\Delta x)^4 + \frac{1}{5!}F^{(5)}(x)(2\Delta x)^5 + \cdots$$

$$F(x-2\Delta x) = F(x) - F'(x)(2\Delta x) + \frac{1}{2!}F^{(2)}(x)(2\Delta x)^2 - \frac{1}{3!}F^{(3)}(x)(2\Delta x)^3 + \frac{1}{4!}F^{(4)}(x)(2\Delta x)^4 - \frac{1}{5!}F^{(5)}(x)(2\Delta x)^5 + \cdots$$

$$F(x+2\Delta x) + F(x-2\Delta x)$$
$$= 2F(x) + \frac{2}{2!}F^{(2)}(2\Delta x)^2 + \frac{2}{4!}F^{(4)}(2\Delta x)^4 + \frac{2}{6!}F^{(6)}(2\Delta x)^6 + \cdots$$
$$\frac{d^2F}{dx^2} = \frac{F(x+2\Delta x) + F(x-2\Delta x) - 2F(x)}{(2\Delta x)^2} + \frac{2}{4!}F^{(4)}(2\Delta x)^2 + \frac{2}{6!}F^{(6)}(2\Delta x)^4 + \cdots$$

Let $a + b = 1$, so that

$$\frac{d^2F}{dx^2}$$
$$= (a+b)\frac{d^2F}{dx^2}$$
$$= a\frac{d^2F}{dx^2} + b\frac{d^2F}{dx^2}$$
$$= \mathrm{a}\frac{F(x+\Delta x) + F(x-\Delta x) - 2F(x)}{\Delta x^2} + \mathrm{b}\frac{F(x+2\Delta x) + F(x-2\Delta x) - 2F(x)}{(2\Delta x)^2} + \mathrm{a}\frac{2}{4!}F^{(4)}\Delta x^2 + 4\mathrm{b}\frac{2}{4!}F^{(4)}\Delta x^2 + \mathrm{a}\frac{2}{6!}F^{(6)}\Delta x^4 + \mathrm{b}\frac{2}{6!}F^{(6)}(2\Delta x)^4 + \cdots$$

To remove the highest order in the error,

$$a\frac{2}{4!}F^{(4)}\Delta x^2 + 4b\frac{2}{4!}F^{(4)}\Delta x^2 = 0.$$

Therefore, $a + 4b = 0$.

$$\begin{cases} a + b = 1 \dots (1) \\ a + 4b = 0 \dots (2) \end{cases}$$

By (2),

$$a + 4b = 0$$
$$a = -4b \dots (3)$$

Sub (3) into (1):

$$(-4b) + b = 1$$
$$-3b = 1$$
$$b = -\frac{1}{3}$$
$$a = -4\left(-\frac{1}{3}\right)$$
$$a = \frac{4}{3}$$

Therefore, $b = -\frac{1}{3}$ and $a = \frac{4}{3}$.

$$\begin{aligned}
\frac{d^2F}{dx^2}
&= \left(\frac{4}{3}\right)\frac{F(x+\Delta x) + F(x-\Delta x) - 2F(x)}{\Delta x^2} \\
&\quad + \left(-\frac{1}{3}\right)\frac{F(x+2\Delta x) + F(x-2\Delta x) - 2F(x)}{(2\Delta x)^2} + \left(\frac{4}{3}\right)\frac{2}{4!}F^{(4)}\Delta x^2 \\
&\quad + 4\left(-\frac{1}{3}\right)\frac{2}{4!}F^{(4)}\Delta x^2 + \left(\frac{4}{3}\right)\frac{2}{6!}F^{(6)}\Delta x^4 + \left(-\frac{1}{3}\right)\frac{2}{6!}F^{(6)}(2\Delta x)^4 + \cdots \\
&= \left(\frac{4}{3}\right)\frac{F(x+\Delta x) + F(x-\Delta x) - 2F(x)}{\Delta x^2} \\
&\quad + \left(-\frac{1}{12}\right)\frac{F(x+2\Delta x) + F(x-2\Delta x) - 2F(x)}{\Delta x^2} + \left(\frac{4}{3}\right)\frac{2}{6!}F^{(6)}\Delta x^4 \\
&\quad + \left(-\frac{1}{3}\right)\frac{2}{6!}F^{(6)}(2\Delta x)^4 + \cdots
\end{aligned}$$

Let $\left(\frac{4}{3}\right)\frac{2}{6!}F^{(6)}\Delta x^4 + \left(-\frac{1}{3}\right)\frac{2}{6!}F^{(6)}(2\Delta x)^4 + \cdots$ be the error ε that we neglect.

Therefore,

$$\begin{aligned}
\frac{d^2F}{dx^2}
&\cong \left(\frac{4}{3}\right)\frac{F(x+\Delta x) + F(x-\Delta x) - 2F(x)}{\Delta x^2} \\
&\quad + \left(-\frac{1}{12}\right)\frac{F(x+2\Delta x) + F(x-2\Delta x) - 2F(x)}{\Delta x^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Delta x^2}\left\{\frac{4}{3}[F(x+\Delta x)+F(x-\Delta x)]-\frac{1}{12}[F(x+2\Delta x)+F(x-2\Delta x)]\right. \\
&\qquad\qquad\qquad \left. -2\left(\frac{4}{3}-\frac{1}{12}\right)\mathrm{F(x)}\right\} \\
&= \frac{1}{\Delta x^2}\left\{\frac{4}{3}[F(x+\Delta x)+F(x-\Delta x)]-\frac{1}{12}[F(x+2\Delta x)+F(x-2\Delta x)]-\frac{5}{2}\mathrm{F(x)}\right\}
\end{aligned}
$$