Team Math

Welcome to the Team Math Test. You will have 75 minutes to complete the following test. No calculators are allowed. Rulers and compasses are allowed but are not necessary for any question, and unfortunately will not be provided. This does mean that no answer will be directly determinable by exact pictures and fine measurements, so just sketch a picture if you need one. Questions are not necessarily in increasing order of difficulty. Split up the questions as you will, only one set of answers will be collected. Do not submit multiple solutions per question. Write solutions to proofs on scratch paper, which should be provided. Numerical questions should be answered in the answer spaces provided. Though your score is the fundamental deciding factor in placement, ties will be broken based on speed. Numerical questions will be worth 2 points while a proof can be awarded up to 3. Best of luck and have fun!
Numerical Answers:

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

Numerical/Answer Only Questions:

1. I am a coin manufacturer and I tell you that I can make the next coin as weighted as you want. i.e. Heads can come up 2/3 of the time if you want. How should you weight the coin so that you have the greatest chance of getting 1 head and 1 tail after 2 flips (not necessarily in that order)?

2. From a regular deck of cards, I draw 5 cards and get an Ace-high Flush. Then, I put my cards back in the deck and you draw 5 cards. What is the probability that your hand is better (ties don't count)? Hint: The hands that are better are Full house, 4-of a kind and Straight-flush. You can write the answer as a product of numbers, no need to find the actual numerical answer. i.e. if your answer is $\frac{1209390*10923890}{120938102}$, don't worry about multiplying these out.

3. How many 0's are at the end of the expansion of 2013! ?

4. How many ways are there to label the 10 seats at a round table with the numbers 1-3, 3 identical pictures of a turtle, 2 identical cardboard cutouts of Bobby Wang, and 2 different Batman stickers if each seat gets exactly 1 item. (Numbers are distinguishable, pictures are not, cutouts are not and batman stickers are not).

5. Take a tetrahedron of side length $6\sqrt{2}$. Consider a cube whose surface area is equal to the volume of this tetrahedron. then consider a sphere whose radius is equal to this cube's side length. Divide the volume of this sphere by π and set this number as x.

   Now consider a cube of side length $2\sqrt{6}$. Let the surface area of this cube be the quantity A. Each angle of a regular polygon with y sides has angle measure A.

   What is the area of a triangle with height x and base y?

6. How many ways are there to get from $(0,0)$ to $(12,16)$ if only moving directly to the right or directly upward is allowed? i.e. no diagonal moves and no moving backward.

7. Square $TAMS$ has sides of length 2. Let the center of the square be point E and let the midpoint of segment TA be the point F. Construct the line l, which bisects the angle FEA, and let G be the point at which this line intersects TA. Find the area of triangle SEG.

8. What is the number of $8by8$ matrices, in which every entry is either 0 or 1 and every row and column has an odd number of 1's.

9. Factor the following: $3(a+b+c)(a^2+b^2+c^2)+b^3+8a^2(b+c)+8c^2(a+b)+8b^2(a+c)+a(b^2+c^2)+c(a^2+b^2)+ac(a+c)$

10. What is the largest amount of elements possible in a subset of the set (1,2,...2013) such that no two elements differ by exactly 4 or 7?

11. What is the maximum amount of elements that can be in a set S, if each element of S is a 10-digit "number" and no two elements can be exactly the same or the same with just 1 digit that's different. (A "number" is just a string of digits, i.e. it can begin with 0's if you want).

Proofs

1. Why can't there be a finite amount of prime numbers?
2. Write up 3 proofs of the Pythagorean theorem (partial credit will be given for 1 or 2 proofs).
3. Show that $1^{2013} + 2^{3+4+5+6} + 3^{4^5} + 4^{5^6} + 5^{6^7}$ is composite
4. Prove that in a room with 2013 people, if nobody shakes the same person's hand twice, there must be two people in the room that have shaken the same number of hands. (A person can shake hands with as many or as few people as he wants out of the 2013, and a handshake is mutual).
5. You are given 17 distinct points on a circle. Connect every possible pair of these points. For each of these connections, color the line segment either red, blue or green. Show that there much be a triangle of one color (it doesn't matter what color). i.e. there are 3 points, among which every connection is the same color.
6. 2013 TAMS students walk into a room. Each boy knows an odd number of boys in the room (0 counts as an even number so no boys can know 0 other boys). Show that there is an even number of boys in the room.
7. Show that for any positive integers x and y, the product $(20a+13\text{b})(20b+13\text{a})$ cannot be a power of 2.
8. Prove that the product 1!*2!*...*100! can be made into a perfect square by taking out one of the original 100 factors.
9. Prove Fermat's Little Theorem, which states that $a^p - a$ is divisible by p for any prime p.

Bonus: Prove that there are an infinite amount of pairs of primes which differ by exactly 2.