Problem Statement

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Given:

The preferential attachment probability:

$$\Pi(k_i) = \frac{1}{t + m_0} \approx \frac{1}{t}$$

The differential equation:

$$\frac{dk_i}{dt} = \frac{m}{t}$$

Initial condition: At time $t = t_i$, node i has degree $k_i(t_i) = m$.
Objective: Find the degree distribution $P(k)$.

Step 1: Solve the Differential Equation for $k_i(t)$

We have the differential equation:

$$\frac{dk_i}{dt} = \frac{m}{t}$$

This is a separable differential equation. Integrate both sides with respect to t:

$$\int dk_i = m \int \frac{1}{t}\, dt$$

Performing the integration:

$$k_i(t) = m \ln t + C$$

Determine the constant of integration C:
Using the initial condition $k_i(t_i) = m$:

$$m = m \ln t_i + C$$

Solve for C:

$$C = m - m \ln t_i$$

Substitute C back into $k_i(t)$:

$$k_i(t) = m \ln t + m - m \ln t_i = m(\ln t - \ln t_i) + m$$

$$k_i(t) = m \ln \left(\frac{t}{t_i} \right) + m$$

Step 2: Express t in Terms of k

Rewriting the equation:

$$k = k_i(t) = m \ln \left(\frac{t}{t_i} \right) + m$$

Subtract m from both sides:

$$k - m = m \ln \left(\frac{t}{t_i} \right)$$

Divide both sides by m:

$$\frac{k - m}{m} = \ln \left(\frac{t}{t_i} \right)$$

Exponentiate both sides to solve for t:

$$\exp \left(\frac{k - m}{m} \right) = \frac{t}{t_i}$$

So,

$$t = t_i \exp \left(\frac{k - m}{m} \right)$$

Step 3: Find the Degree Distribution $P(k)$

Assumptions:

- Nodes are added to the network uniformly over time from 0 to t.
- The probability density function of a node's introduction time t_i is:

$$P(t_i) = \frac{1}{t}, \quad 0 \leq t_i \leq t$$

Compute the Cumulative Distribution Function $P(k_i \geq k)$:

The probability that a node has degree $k_i(t) \geq k$ at time t is equal to the probability that its introduction time t_i is less than or equal to t', where t' satisfies:

$$t' = t \exp \left(-\frac{k - m}{m} \right)$$

Therefore,

$$P(k_i \geq k) = P(t_i \leq t') = \int_0^{t'} P(t_i)\, dt_i = \int_0^{t'} \frac{1}{t}\, dt_i = \frac{t'}{t}$$

Substitute t':

$$P(k_i \geq k) = \frac{t \exp\left(-\frac{k-m}{m}\right)}{t} = \exp\left(-\frac{k-m}{m}\right)$$

Compute the Degree Distribution $P(k)$:

The degree distribution $P(k)$ is the derivative of the cumulative distribution $P(k_i \geq k)$:

$$P(k) = -\frac{d}{dk} P(k_i \geq k)$$

Compute the derivative:

$$P(k) = -\frac{d}{dk}\left[\exp\left(-\frac{k-m}{m}\right)\right] = -\left(-\frac{1}{m}\right)\exp\left(-\frac{k-m}{m}\right) = \frac{1}{m}\exp\left(-\frac{k-m}{m}\right)$$

Simplify the exponent:

$$\exp\left(-\frac{k-m}{m}\right) = \exp\left(-\frac{k}{m}+1\right) = e \cdot \exp\left(-\frac{k}{m}\right)$$

Therefore,

$$P(k) = \frac{e}{m}\exp\left(-\frac{k}{m}\right)$$

Final Result

The degree distribution is:

$$P(k) = \frac{e}{m}\exp\left(-\frac{k}{m}\right)$$