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1. Starting with the Binomial PMF

$$P(K = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where:

- n is the number of trials (potential edges),
- p is the probability of success (edge inclusion),
- k is the number of successes (degree of the vertex).

2. Approximating the Binomial Coefficient

For large n and small k, we use Stirling's approximation:

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

The binomial coefficient can be expressed as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

When k is small compared to n, we have:

$$(n-k)! \approx (n-k)^{n-k} e^{-(n-k)} \sqrt{2\pi(n-k)}$$

Thus, the binomial coefficient simplifies to:

$$\binom{n}{k} \approx \frac{n^k}{k!}$$

3. Approximating $(1 - p)^{n-k}$

For small p and large n:

$$(1-p)^{n-k} \approx e^{(n-k)\ln(1-p)}$$

Using the first-order approximation for the natural logarithm:

$$\ln(1-p) \approx -p \quad \text{when } p \text{ is small,}$$

we can substitute:

$$e^{(n-k)\ln(1-p)} \approx e^{(n-k)(-p)}$$

which simplifies to:

$$(1-p)^{n-k} \approx e^{-p(n-k)}$$

4. Combining the Approximations

Substitute the approximations into the binomial PMF:

$$P(K=k) \approx \frac{n^k}{k!} p^k e^{-np}$$

Since $\langle k \rangle = np$ (mean degree), we express $n^k p^k$ in terms of $\langle k \rangle$:

$$n^k p^k = \langle k \rangle^k$$

So:

$$P(K=k) \approx \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

Which is Poissonian approximation.