Comp 3121 Algorithms & Programming Tech - Question 1

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Question 1 *Arc Competition*

[**20 marks**] Arc is hosting a training session for the members of various societies at UNSW. There are n students who are Arc members, and $m < n$ registered societies. Each of the n students is a member of at least 1 and at most 4 of the m societies. Each of the m societies must send exactly 1 student to represent them at the training session.

In order to keep the crowd diverse, Arc would like to avoid all of the student representatives studying degrees from the same faculty. Each of the n students is enrolled in a single degree offered by 1 of the k faculties at UNSW. At most u_i students from faculty i will be allowed to attend the event.

Design an $O(nm)$ algorithm that determines if it is possible to find a selection of students to attend the event such that all of these criteria are met. If it is possible, also identify which of the m students will attend.

We construct a flow network with:

1. Source S and Sink T.
2. m vertices for societies, for each society S_p that $1 \le p \le m$.
3. n vertices for students, for each student T_j that $1 \le j \le n$.
4. k vertices for faculties, for each faculty F_i that $1 \le i \le k$.
5. For each society S_p to student T_j, there is an edge with capacity of one if the student j is a member from society i. For each student T_j, there are at least one at most four edges to connect with societies. There are at most 4n edges when all students n are in 4 societies.
6. For each faculty F_i to student T_j, there is an edge with capacity 1 if the student j enrols the degree from that faculty, and each student T_j can be only connected with 1 faculty. There are at most n edges.
7. From source to each society S_p, there is an edge with capacity 1 since each society can only assign one student to attend the event. There are total m edges.
8. From each faculty F_i to the sink, there is an edge with capacity ui since there are at most ui students from faculty i are allowed to join. There are total k edges.

Consider the flow in this graph:

1. The path from source S to sink T represents who is the representative of each society and which faculty is the student study in.
2. If the edge between S_p and T_j receives the flow which means student j represents society m to attend the event.
3. If the edge between T_j and F_i receives the flow which means the student j is from faculty i.
4. The maximum flow F ($F \le n$) presents the number of representatives that have successfully been chosen.

To determine whether it is possible for a selection of students to attend the event, we find the maximum flow by applying Ford-Fulkerson's Algorithm to the flow network.

case1: If the maximum flow $F = m$, which means all m students are successfully selected so the assignment is possible.

case2: If the maximum flow $F < m$, which means only F students are selected.

Time complexity

We apply the Ford-Fulkerson algorithm in this problem. There are at most $(k + 4n + m + n) < 7n$ edges since $n > m$ and $n \ge k$ (Every student n must choose 1 faculty, there are at most n faculties be chosen), and the max flow is bounded by m. So the time complexity is $O(E|f|) = O(m * (k + 4n + m + n)) = O(mn)$.