Comp3121 assignment 4 Question 4
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Question 4 *Antoni the Merchant*

[30 marks] Antoni wants to earn as much money as possible over the summer holidays as a travelling merchant in the DynaProg continent. He will begin his journey with \$0 wealth, and he only has D days before term starts and he is pulled back into answering forum questions, so he needs to plan very carefully. After thorough research, he has found n countries where he can turn an easy profit; in fact, he knows that if he is in country $i \in [1..n]$ on day $d \in [1..D]$, he is guaranteed to earn $P[i][d] > 0$ dollars.

All n countries have airports, and a direct flight to every other country. Each day Antoni can either travel to another country, or stay where he is. There is no restriction on how many times he can visit a certain country, nor on the number of days he can stay per visit.

However, the tax laws in DynaProg are vicious - every day he is taxed a *percentage of his current wealth*. Specifically, you are given a Tax Table $T[1..n][1..n]$ where $0 \leq T[j][i] \leq 100$ is the percentage Antoni will be taxed if he is currently in country j and was in country i the previous day. He is not taxed anything on the first day as he has \$0 wealth.

Help Antoni prepare a full itinerary (which country to be in on every day) that maximises his wealth by the end of D days.

> Tax is deducted before the earnings are added for the day.
>
> On Day 1, Antoni will not be taxed since his wealth is \$0 and there is nothing to deduct. He will then earn $\$P[i][1]$ for whichever country i he has chosen to start at.
>
> On Day 2, he can either stay in country i, or travel to another country j. If he chooses to stay, he will be taxed $T[i][i]\%$ of his current wealth, and then earn $\$P[i][2]$. If he chooses to travel, he will be taxed $T[j][i]\%$ of his current wealth, and then earn $\$P[j][2]$.

4.1 [12 marks, 4 marks per part] Antoni has prepared a few Greedy approaches to solving this problem, but is not convinced they will always earn him the most money.

Provide a counter example for each of the following approaches. Your counter example must state the values for n, D, P, T, the wealth Greedy obtains and its itinerary, and finally, a more optimal wealth and the itinerary used to attain it. The more optimal solution does not have to be the best, it just has to beat Greedy.

[A] On each day d, travel to the country where Antoni will earn the most money (i.e. travel to the country i such that $P[i][d]$ is maximised, for all days $1 \leq d \leq D$). If there is a tie, choose the country with the lower tax rate.

[B] First, travel to the country with the highest Day 1 profit (i.e. travel to the country i such that $P[i][1]$ is maximised). Then, on each subsequent day d, travel to the country that minimises the amount of tax (in dollars) that Antoni will be charged (i.e. travel to the country j such that $T[j][i]\% \times \texttt{current\_net\_wealth}$ is minimised, for all days $2 \leq d \leq D$). If there is a tie, choose the country with the higher profit.

[C] First, travel to the country with the highest Day 1 profit. Then, on each subsequent day d, travel to the country that maximises Antoni's wealth at the end of the day (i.e. travel to the country j such that $\texttt{current\_net\_wealth} \times (100 - T[j][i])\% + P[j][d]$ is maximised, for all days $2 \leq d \leq D$). Break ties arbitrarily.

4.2 [18 marks] Design an $O(n^2D)$ algorithm that finds the maximum possible wealth Antoni can attain, and also provides the itinerary he needs to follow.

4.1

(a)

We assume P is around the same and T is a lot different.

n = 2 (countries), D = 2 (days)

P (price to earn $):

	D1	D2
n1	$200	$199
n2	$199	$200

T (tax percentage):

i \ j	1	2
1	1%	90%
2	1%	1%

Case 1: (n1→n1)
Stay in countries n1 on both days,
the wealth: (200+199)*0.99 = 395.01

Case 2: (n2→n2)
Stay in countries n2 on both days,
the wealth: (199+200)*0.99 = 395.01

Case 3: Itinerary: n1→n2
Stay in countries n1 at first day then travels to n2 on the second day,
the wealth: (200+199)*0.1=39.9

Case 4: (optimal way) Itinerary: n2→n1
Stay in countries n2 at first day then travels to n1 on the second day,
the wealth: (200+200)*0.99=396

By applying greedy, Antoni always stay at the country he can earns the most (Case 3), so he travels from n1 (Day1) to n2(Day2). However, the wealth is the minimum compares to other 3 cases. Either case 4 is the optimal solution.

(b)

We assume T is around the same and P is a lot different.

n = 2 (countries), D = 2 (days)

P (price to earn $) :

	D1	D2
n1	$200	$1
n2	$1	$200

T(tax percentage) :

i \ j	1	2
1	50%	49%
2	49%	50%

Case 1: (n1→n1)
    Stay in countries n1 on both days,
    the wealth: (200+1)*0.5 = 100.5

Case 2: (n2→n2)
    Stay in countries n2 on both days,
    the wealth: (1+200)*0.5 = 100.5

Case 3: Itinerary :n1→n2
    Stay in countries n1 at first day then travels to n2 on the second day,
    the wealth: (1+1)*0.51=1.02

Case 4: (optimal way) Itinerary: n2→n1
    Stay in countries n2 at first day then travels to n1 on the second day,
    the wealth: (200+200)*0.51=204

As Antoni starts from the country with the highest profit. Then travel to the country with the minimum tax (Case 3: as the profit earned in n1 is the max on the first day and traveling to n2 has less tax.) However, case4 is the more optimal way.

(c)
n = 3 (countries), D = 3 (days)
P (price to earn$) :

	D1	D2	D3
n1	**100**	100	100
n2	99	**100**	100
n3	99	100	100

T(tax percentage) :

i \ j	1	2	3
1	2%	**1%**	2%
2	**99%**	**99%**	**99%**
3	1%	1%	1%

Case 1:
n1→n1→…
Day1 to Day2: 100*0.98+100=198

Case 2:
Itinerary: n1→n2→ (n1 or n2 or n3)
Stay at n1 at the first day, n2 at the second day, either n1, n2, or n3 at the third day
Day1 to Day2: 100+100*0.99=199
Day2 to Day3: 199*0.01+100=101.99

Case 3:
n1→n3→…
Day1 to Day2: 100*0.98+100=198

Case 4 (optimal way):
Itinerary: n1→n3→n3 or n1→n1→n3
Solution 1: Stay at n1 at the first day, n3 at the second day, n3 at the third day
Solution 2: Stay at n1 at the first day, n1 at the second day, n3 at the third day
Day1 to Day2: 100+100*0.98=198
Day2 to Day3: 198*0.99+100=296.02

Given by the question, Antoni start with the country with highest profit (n1), then travel to the country that maximise his wealth at the end of the day. We compare case1, case2 and case3 to see which case meet the maximum wealth of traveling from n1 to n1, n2, or n3. From the above n1→n2 has the max value, so then we travel to other countries. As n2 travel to n1, n2, n3 all generate the same wealth, the greedy approach is the case 2 above. However, the maximum wealth we generate from Case 4 is the optimal solution compared to Case2.

4.2

Subproblem: For each $1 \leq i \leq n$ and $0 \leq \mathrm{d} \leq \mathrm{D}$ let $Q(i,\mathrm{d})$ be the problem of determining $Opt(i,\mathrm{d})$, which is the maximum wealth that country i is the last country Antoni visited during day d .

Base case: $d = 1, that\ Opt(j, 1) = P[j][1]$
Recursion: $Opt(j, d) = max(Opt(i, d-1) * (1 - T[j][i]) + P[j][d])\ for\ i = 1 \dots n$

We create a n*D table T to store the maximum wealth from different countries on each day. We consider i as the country we visited on day d-1, and during day d Antoni travels to country j. As every day Antoni has n options for choosing which countries he is traveling for the next day, we iterate through i from 1…n and calculate all the possible wealth traveling from country i to country j by the the formula

$$wealth = Opt(i, d-1) * (1 - T[j][i]) + P[j][d]).$$

The array then stores the maximum one from the n values that we calculate by the above formula. Since we want to know the itinerary to follow, we create another n*D array C to backtrack the countries that Antoni has gone to. We store each element from the index of the maximum wealth that we store in table T. That is, if T stores the maximum wealth of traveling from country i to country j, i will be stored into the array C.

Justification
We prove the optimal subproblem by contradiction. For each $i = 1 \dots n$ is the previous country before visiting country j.
$Opt(j, d)$ is the maximum wealth when the j is the last country he visited during day d. As the recursion is determined above, that $Opt(j, d) = max(Opt(i, d-1) * (1 - T[j][i]) + P[j][d])$.
Suppose there exists another maximum cost $Opt'(i, d-1)$ that is larger than $Opt(i, d-1)$. That is, we can find another maximum wealth by $Opt'(i, d-1)$ which contradicts our previous assumption that $Opt(j, d)$ is the maximum wealth.

Answer:
We can find the maximum wealth from $T[1 \dots n][D]$ and the itinerary from $C[1 \dots n][D]$.

Time complexity:
We create table T to store the wealth. As we store the value inside the table has to iterate through n elements that takes $O(n^2D)$, and search up which is the maximum value takes $O(1)$. search up takes $O(1)$. And another n*D array C to store the record which Antoni has gone to only takes linear time to store the value, which the time complexity is $O(nD)$. Thus, the total time complexity is $O(n^2D) + O(nD) = O(n^2D)$.