$\boxed{\text{Math 3F - 2/2/26}}$

- HW 1.2, 2.1-2 is due tonight.

#1.2.7: Does $y = \dfrac{1+2\ln x}{x^2} + \dfrac{1}{2}$ satisfy $y' = \dfrac{x^2 - 2x^2 y + 2}{x^3}$, $y(1) = \dfrac{3}{2}$

$$y' = \frac{x^2 \cdot \frac{2}{x} - (1+2\ln x)\cdot 2x}{x^4} = \frac{-4\ \ln x}{x^3}$$

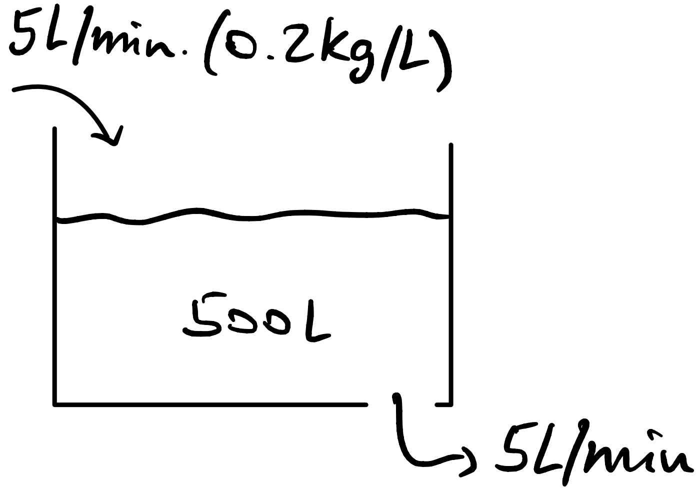
$$\frac{x^2 - 2x^2 y + 2}{x^3} = \frac{x^2 - 2x^2\left(\frac{1+2\ln x}{x^2} + \frac{1}{2}\right) + 2}{x^3}$$

$$= \frac{x^2 - 2 - 4\ln x - x^2 + 2}{x^3} = -\frac{4\ln x}{x^3}$$

✓

$$y(1) = \frac{1+2\ln 1}{1^2} + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$ ✓

#2.2.6:



$A(t)$ = mass of salt in the tank at time t

$A(0) = 5$ kg.

$$\frac{dA}{dt} = [\text{rate in}] - [\text{rate out}]$$

$$= \frac{5 \; \cancel{L}}{\text{min.}} \cdot \frac{0.2 \; \text{kg}}{\cancel{L}} - \frac{5 \; L}{\text{min}} \cdot \frac{A \; \text{kg}}{500 \; L}$$

$$\boxed{\frac{dA}{dt} = 1 - 0.01A \,, \quad A(0) = 5}$$

(both separable and linear)

Solve the IVP to get $A(t)$.

Find $A(10)$.

Concentration: $\dfrac{A(10)}{500}$.

#1.2.12: $y'' = xe^{2x}, \quad y(0)=7, \quad y'(0)=1$

$+$	x	e^{2x}
$-$	1	$\frac{1}{2}e^{2x}$
$+$	0	$\frac{1}{4}e^{2x}$

$$y' = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C_1$$

$$y'(0) = 0 - \frac{1}{4} + C_1 = 1 \longrightarrow C_1 = \frac{5}{4}$$

$$y' = \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + \frac{5}{4}$$

$+$	$\frac{x}{2}-\frac{1}{4}$	e^{2x}
$-$	$\frac{1}{2}$	$\frac{1}{2}e^{2x}$
$+$	0	$\frac{1}{4}e^{2x}$

$$y = \frac{1}{2}\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} - \frac{1}{8}e^{2x} + \frac{5}{4}x + C_2$$

$$= \frac{1}{4}(x-1)e^{2x} + \frac{5}{4}x + C_2$$

$$y(0) = \frac{1}{4}(-1) + C_2 = 7 \longrightarrow C_2 = \frac{29}{4}$$

$$\boxed{y = \frac{1}{4}(x-1)e^{2x} + \frac{5}{4}x + \frac{29}{4}}$$

2.3: Exact Equations

Def: A first-order ODE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is called exact if there exists a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

Ex: $(3x^2y-2)\,dx + x^3\,dy = 0$

$F(x,y) = x^3y - 2x$ is such that

$$\frac{\partial F}{\partial x} = 3x^2y - 2 \text{ and } \frac{\partial F}{\partial y} = x^3$$

$\rightarrow$ The equation is exact.

Ex: $2x\,dx + 3y^2\,dy = 0$ is exact.

$$F(x,y) = x^2 + y^3 + \pi$$

Ex: $2x\,dx + 2x\,dy = 0$

Suppose there exists $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = 2x \longrightarrow F = \int 2x\,dx = x^2 + g(y)$$

and $$\frac{\partial F}{\partial y} = 2x \longrightarrow F = \int 2x\,dy = 2xy + h(x)$$

These cannot equal

$\rightarrow$ no such F.

$\rightarrow$ The equation is <u>not</u> exact.

Theorem: The equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Note: If the eqn. is exact, then there is F s.t. $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. But

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

from Calc III

Ex: $\underbrace{(3x^2 - 2xy + 2)}_{M}\,dx + \underbrace{(6y^2 - x^2 + 3)}_{N}\,dy = 0$ exact?

yes

$$\frac{\partial M}{\partial y} = -2x = -2x = \frac{\partial N}{\partial x}$$

Ex: $(e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0$?

$$\frac{\partial M}{\partial y} = e^x \cos y - 2\sin x = \frac{\partial N}{\partial x}$$

exact!

Ex: $(x\ln y + xy)\,dx + (y\ln x + xy)\,dy = 0$?

not exact

$$\frac{\partial M}{\partial y} = \frac{x}{y} + x$$

$$\frac{\partial N}{\partial x} = \frac{y}{x} + y$$

$\rightarrow$ not equal

Solving an exact equation $M(x,y)\,dx + N(x,y)\,dy = 0$:

① Find $F(x,y)$ from the definition.

② Rewrite the equation as:

$$\underbrace{\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy}_{\text{the total differential of } F(x,y)} = 0$$

$\Rightarrow F(x,y) = c$

③ The solution of the eqn. is given implicitly by $F(x,y) = c$. (You should try to solve for y explicitly.)

Ex: $(3x^2-2xy+2)\,dx+(6y^2-x^2+3)\,dy=0$

(a) Exact? Yes (see above.)

(b) Solve it.

$$\frac{\partial F}{\partial x}=3x^2-2xy+2 \rightarrow F=x^3-x^2y+2x+g(y)$$

$$\frac{\partial F}{\partial y}=6y^2-x^2+3 \rightarrow F=2y^3-x^2y+3y+h(x)$$

$$\rightarrow F(x,y)=x^3-x^2y+2x+2y^3+3y$$

$$\boxed{x^3-x^2y+2x+2y^3+3y=c}$$

Ex: $(e^x\sin y-2y\sin x)\,dx+(e^x\cos y+2\cos x)\,dy=0$

It's exact. (See above.)

Solve it!

$$\frac{\partial F}{\partial x}=e^x\sin y-2y\sin x \rightarrow F=e^x\sin y+2y\cos x+g(y)$$

$$\frac{\partial F}{\partial y} = e^x \cos y + 2\cos x \longrightarrow F = e^x \sin y + 2y\cos x + h(x)$$

$$F(x,y) = e^x \sin y + 2y\cos x$$

$$\boxed{e^x \sin y + 2y\cos x = c}$$

Ex: $y\,dx + (2x - ye^y)\,dy = 0$ is not exact, but we can it make it so by multiplying by $\mu = y^m$.

$$y^{m+1}dx + (2xy^m - y^{m+1}e^y)\,dy = 0$$

exact iff $\frac{\partial}{\partial y}\left[y^{m+1}\right] = \frac{\partial}{\partial x}\left[2xy^m - y^{m+1}e^y\right]$

$$(m+1)y^m = 2y^m$$

$$m = 1$$

$$y^2dx + (2xy - y^2e^y)\,dy = 0 \quad \text{exact}$$