2. Linear Electrodynamics of Superconductors

2.1 The London Equations

In order to understand the behavior of a superconductor in an external electromagnetic field, let us use the so-called two-fluid model. We assume that all free electrons of the superconductor are divided into two groups: superconducting electrons of density n_s and normal electrons of density n_n. The total density of free electrons is $n = n_s + n_n$. As the temperature increases from 0 to T_c, the density n_s decreases from n to 0.

Let us start our systematic study of the superconductor in an electromagnetic field with the simplest case. Assume that both the electric and magnetic fields are so weak that they do not have any appreciable influence on the superconducting electron density. Assume, in addition, that the density n_s is the same everywhere, i.e., spatial variations of n_s are disregarded. The relation between the current, electric field, and magnetic field in this case is linear and described by the London equations [17].

2.1.1 The First London Equation

The equation of motion for superconducting electrons in an electric field is

$$n_s m \frac{\mathrm{d}\boldsymbol{v}_s}{\mathrm{d}t} = n_s e\boldsymbol{E} , \tag{2.1}$$

where m is the electron mass, e is the electron charge, $\boldsymbol{v}_s$ is the superfluid velocity, and n_s is the number density of the superfluid. Taking into account that the supercurrent density is $\boldsymbol{j}_s = n_s e\boldsymbol{v}_s$, we have

$$\boldsymbol{E} = \frac{\mathrm{d}}{\mathrm{d}t}(\Lambda\boldsymbol{j}_s) \tag{2.2}$$

with

$$\Lambda = m/n_s e^2 . \tag{2.3}$$

Equation (2.2) is simply Newton's second law for the superconducting electrons. It follows from this equation that in the stationary state, that is, when $\mathrm{d}\boldsymbol{j}_s/\mathrm{d}t = 0$, there is no electric field inside the superconductor. Note that here we disregard all possible spatial variations of the chemical potential of the superconducting electrons. Such variations become important, for example, when one considers the interface between a superconducting and a normal region, with a non-zero current passing through the interface. More details about this situation can be found in Chap. 7.

2.1.2 The Second London Equation

Let us now work out the relation between the supercurrent and the magnetic field in a superconductor. We denote the true microscopic magnetic field at a given point of a superconductor by $\boldsymbol{H}(\boldsymbol{r})$. Here an additional explanation is needed. We remember from Chap. 1 that magnetic field does not penetrate the interior of a type-I superconductor. Now we shall find that this is true only to a certain extent. Magnetic field does in fact penetrate to a shallow depth (of the order of 500–1000 Å) at the surface of a body. Our task at the moment is to work out how this field, $\boldsymbol{H}(\boldsymbol{r})$, varies in space.

Assume that the free energy of a superconductor in zero magnetic field is $\mathcal{F}_{s0}$. The kinetic energy density of the supercurrent is

$$W_{\text{kin}} = \frac{n_s m v_s^2}{2} = \frac{m j_s^2}{2 n_s e^2} . \tag{2.4}$$

Taking into account Maxwell's equation

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c} \boldsymbol{j}_s , \tag{2.5}$$

expression (2.4) for W_{kin} becomes

$$W_{\text{kin}} = \frac{\lambda^2}{8\pi} (\operatorname{curl} \boldsymbol{H})^2 , \tag{2.6}$$

where we have defined

$$\lambda^2 = \frac{mc^2}{4\pi n_s e^2} . \tag{2.7}$$

As we already know, the magnetic energy density at some point of the superconductor is $H^2/8\pi$. Therefore, the free energy of the superconductor as a whole, including both the kinetic energy of the supercurrent and the magnetic field energy, is

$$\mathcal{F}_{sH} = \mathcal{F}_{s0} + \frac{1}{8\pi} \int [\boldsymbol{H}^2 + \lambda^2 (\operatorname{curl} \boldsymbol{H})^2] \, \mathrm{d}V . \tag{2.8}$$

The integration is carried out over the entire volume of the superconductor.

Let us now solve a variational problem, namely, let us find the identity of the function $\boldsymbol{H}(\boldsymbol{r})$ corresponding to the minimum value of $\mathcal{F}_{sH}$. We note that a more accurate approach would be to minimize the Gibbs free energy. We shall indeed do that when deriving the Ginzburg–Landau equations (see Sect. 3.2). However, the result is independent of which particular function is examined for a minimum, $\mathcal{F}_{sH}$ or $\mathcal{G}_{sH}$. Therefore, we examine $\mathcal{F}_{sH}$ as it is easier to do.

Let us ascribe a small variation $\delta\boldsymbol{H}(\boldsymbol{r})$ to the function $\boldsymbol{H}(\boldsymbol{r})$. The resulting change in $\mathcal{F}_{sH}$ will be $\delta\mathcal{F}_{sH}$:

$$\delta\mathcal{F}_{sH} = \frac{1}{8\pi}\int(2\boldsymbol{H}\cdot\delta\boldsymbol{H} + 2\lambda^2\mathrm{curl}\,\boldsymbol{H}\cdot\mathrm{curl}\,\delta\boldsymbol{H})\,\mathrm{d}V\,. \tag{2.9}$$

The function $\boldsymbol{H}(\boldsymbol{r})$ to be found is the function that makes $\mathcal{F}_{sH}$ a minimum, that is,

$$\delta\mathcal{F}_{sH} = 0\,. \tag{2.10}$$

Using the identity

$$\boldsymbol{a}\cdot\mathrm{curl}\,\boldsymbol{b} = \boldsymbol{b}\cdot\mathrm{curl}\,\boldsymbol{a} - \mathrm{div}\,[\boldsymbol{a}\times\boldsymbol{b}] \tag{2.11}$$

and combining (2.9) and (2.10), we obtain

$$\int[\boldsymbol{H} + \lambda^2(\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{H})]\cdot\delta\boldsymbol{H}\,\mathrm{d}V - \int\mathrm{div}\,[\mathrm{curl}\,\boldsymbol{H}\times\delta\boldsymbol{H}]\,\mathrm{d}V = 0\,. \tag{2.12}$$

The second integral in (2.12) is zero, as one can see from the following. By Gauss's theorem, the second term in (2.12) can be written in the form $\oint[\mathrm{curl}\,\boldsymbol{H}\times\delta\boldsymbol{H}]\cdot\mathrm{d}\boldsymbol{S}$, where the integration is carried out over the surface of the superconductor. But the field at the surface is fixed, it is the external field; therefore $\delta\boldsymbol{H}(\boldsymbol{r}) = 0$ there.

We have arrived at the equation $\int(\boldsymbol{H} + \lambda^2\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{H})\cdot\delta\boldsymbol{H}\,\mathrm{d}V = 0$. For an arbitrary variation $\delta\boldsymbol{H}(\boldsymbol{r})$, this equation can be satisfied only if the sum in parentheses is zero. Thus we have obtained the equation for the magnetic field in a superconductor:

$$\boldsymbol{H} + \lambda^2\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{H} = 0\,. \tag{2.13}$$

This is the second London equation. It can also be written in a different form. Using Maxwell's equation (2.5) and the equality $\boldsymbol{H} = \mathrm{curl}\,\boldsymbol{A}$, we obtain from (2.13)

$$\boldsymbol{j}_s = -\frac{c}{4\pi\lambda^2}\boldsymbol{A}\,. \tag{2.14}$$

One can go from (2.13) to (2.14) only on the condition that the so-called London gauge is chosen for the vector potential:

$$\mathrm{div}\,\boldsymbol{A} = 0\,, \tag{2.15}$$

$$\boldsymbol{A}\cdot\boldsymbol{n} = 0\,, \tag{2.16}$$

where $\boldsymbol{n}$ is the unit vector normal to the superconductor's surface.

Equation (2.15), together with (2.14), specifies the conditions of continuity of the current and absence of a supercurrent source, while (2.16) assures that no supercurrent can pass through the boundary of a superconducting body.

It is assumed, of course, that there are no external circuits or contacts to current leads.

Using (2.3) and (2.7), we can write (2.14) in the form

$$\boldsymbol{j}_s = -\frac{1}{c\Lambda}\boldsymbol{A}\,, \tag{2.17}$$

$$\Lambda = 4\pi\lambda^2/c^2\,. \tag{2.18}$$

In the rest of the book, we shall often use the second London equation in the form (2.17).

2.2 Magnetic Field Penetration Depth

Let us apply the London equations to examine how a magnetic field penetrates a superconductor. Consider a superconducting semispace $x > 0$. The surface of the superconductor coincides with the plane $x = 0$. An external magnetic field H_0 is oriented along the z axis. To solve this problem, we shall use (2.13). Taking into account the symmetry of the problem and that $\operatorname{curl}\operatorname{curl}\boldsymbol{H} = -\nabla^2\boldsymbol{H}$, we can rewrite (2.13) in the form

$$\mathrm{d}^2H/\mathrm{d}x^2 - \lambda^{-2}H = 0\,. \tag{2.19}$$

The boundary conditions are: $H(0) = H_0$, $H(\infty) = 0$. The second boundary condition takes account of the Meissner–Ochsenfeld effect. The solution of (2.19) is

$$H = H_0\,\mathrm{e}^{-x/\lambda}\,. \tag{2.20}$$

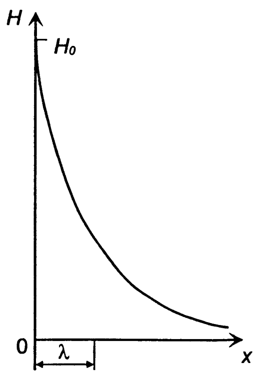

Fig. 2.1. Magnetic field penetration into a bulk superconductor. The field at the surface is H_0

It means that the magnetic field falls off with increasing distance from the surface of the superconductor. The characteristic decay length is λ (see

Fig. 2.1). This length clarifies the physical significance of the quantity λ formally defined by (2.7) and is called the London magnetic field penetration depth:

$$\lambda = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2} . \tag{2.21}$$

It follows that the screening (Meissner) supercurrent at the surface falls off over the same length. Indeed, the supercurrent is $\boldsymbol{j}_s = (c/4\pi)\,\mathrm{curl}\,\boldsymbol{H}$, which in our simple geometry reduces to $j_s = (c/4\pi)\,\mathrm{d}H/\mathrm{d}x$. Substituting (2.20) in the last expression, we get

$$j_s = \frac{cH_0}{4\pi\lambda}\,\mathrm{e}^{-x/\lambda} . \tag{2.22}$$

One can see from (2.21) that λ is temperature-dependent because it depends on n_s. A good approximation for the temperature dependence of λ is given by the empirical formula

$$\lambda(T) = \frac{\lambda(0)}{\left[1 - (T/T_c)^4\right]^{1/2}} . \tag{2.23}$$

Let us now evaluate $\lambda(0)$. At $T = 0$ all electrons are superconducting, that is, $n_s = n \approx 10^{22}\ \mathrm{cm}^{-3}$. Substituting this in (2.21), together with $m \sim 10^{-27}$ g, $c \sim 3 \times 10^{-10}\ \mathrm{cm\,s^{-1}}$, $e = 4.8 \times 10^{-10}$ esu, we obtain $\lambda(0) \sim 600$ Å.

The values of $\lambda(0)$ for some superconductors are given in Table 2.1.

Table 2.1. London penetration depths for some superconductors [2]

Element	Al	Cd	Hg	In	Nb	Pb	Sn	Tl	$YBa_2Cu_3O_7$
$\lambda(0)$ /Å	500	1300	380–450 (anisotropy)	640	470	390	510	920	1700

2.3 Nonlocal Electrodynamics of Superconductors

Everything said so far about the electrodynamics of superconductors falls into the category of the so-called local electrodynamics. For instance, London's equation (2.17) relates the supercurrent density $\boldsymbol{j}_s$ (that is, the velocity of the supercurrent carriers, $\boldsymbol{v}_s$) to the vector potential $\boldsymbol{A}$ at the *same point.* Therefore, strictly speaking, (2.17) is applicable only if the size of the current carriers is much smaller than the characteristic length over which the vector potential changes, that is, smaller than the penetration depth λ. We know that the superconducting current carriers are pairs of electrons. Let us denote the size of a pair by ξ_0. For pure metals, ξ_0 is of the order $\xi_0 \sim 10^{-4}$ cm, as we shall find out later on, in Chap. 6. On the other hand, the penetration depth is $\lambda \sim 10^{-5}$–10^{-6} cm. Therefore, the local London electrodynamics is not applicable to pure superconductors, because the magnetic field changes appreciably over the length ξ_0.

Hence, the local equation (2.17) must be replaced by a nonlocal one relating the velocity of a particle to the magnetic field which is allowed to change substantially over the size of the particle, ξ_0. Such a nonlocal relation was proposed by A.B. Pippard [18] several years before the microscopic theory of superconductivity appeared.

In its general form, the nonlocal relation between $\boldsymbol{j}_s$ and $\boldsymbol{A}$ can be written as

$$\boldsymbol{j}_s(\boldsymbol{r}) = \int \hat{Q}(\boldsymbol{r} - \boldsymbol{r}')\, \boldsymbol{A}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}' \ , \tag{2.24}$$

where $\hat{Q}$ is an operator which, operating on the vector $\boldsymbol{A}$, converts it into the vector $\hat{Q}\boldsymbol{A}$. The operation range of the operator $\hat{Q}(\boldsymbol{r} - \boldsymbol{r}')$ is taken to be ξ_0, that is, $\hat{Q}(\boldsymbol{r} - \boldsymbol{r}')$ is nonzero only for $|\boldsymbol{r} - \boldsymbol{r}'| \leq \xi_0$. This is how the effect of the vector $\boldsymbol{A}$ on a large particle (a supercurrent carrier) is averaged out. If we now let the operation range go asymptotically to zero, $\hat{Q}$ turns into the δ-function, and we come back to local electrodynamics.

Pippard's proposal was to choose $\hat{Q}\boldsymbol{A}$ in the form

$$\hat{Q}(\boldsymbol{r}-\boldsymbol{r}')\, \boldsymbol{A}(\boldsymbol{r}') = -\frac{3n_s e^2}{4\pi m c \xi_0} \frac{(\boldsymbol{r} - \boldsymbol{r}')}{(\boldsymbol{r} - \boldsymbol{r}')^4} \left[\boldsymbol{A}(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')\right] \mathrm{e}^{-|\boldsymbol{r}-\boldsymbol{r}'|/\xi_0} \ . \tag{2.25}$$

The exact form of the magnetic field penetration into a superconductor in the nonlocal case differs from the exponential dependence. However, in this case one can also speak of a magnetic field penetration depth defined as

$$\lambda = \frac{1}{H_0} \int_0^\infty H \,\mathrm{d}x \ . \tag{2.26}$$

Here H_0 is the field at the surface of a semi-infinite superconductor. If the field near the surface falls off exponentially, all three definitions of λ [(2.26), (2.19), and (2.20)] coincide.

We shall not attempt here to solve the nonlocal problem explicitly. Rather, we shall show how to find an estimate of the correct answer [4]. Assume that the true dependence $H(x)$ is approximated by an exponential dependence with a new penetration depth denoted by λ_P (Pippard's penetration depth). Then, the vector potential $\boldsymbol{A}$ acts on a particle of size ξ_0 only within the depth $\lambda_P \ll \xi_0$ near the surface of the particle. The result is that, although the particle still participates in creating the current density $\boldsymbol{j}_s$, the effect of $\boldsymbol{A}$ on it is reduced, because only the part of it λ_P/ξ_0 'feels' the presence of the vector potential. Accordingly, the current density is reduced by a factor of ξ_0/λ_P in comparison with the local case. Inserting this coefficient in (2.14) we obtain

$$\boldsymbol{j}_s = -\frac{c}{4\pi\lambda^2}\frac{\lambda_P}{\xi_0}\boldsymbol{A}\,. \tag{2.27}$$

Rewriting (2.27) in the form

$$\boldsymbol{j}_s = -\frac{c}{4\pi\lambda_P^2}\boldsymbol{A}\,, \tag{2.28}$$

we obtain, just as we wished, the exponential penetration of the magnetic field to the depth λ_P. Comparing (2.27) and (2.28), we get $\lambda_P^2 = \lambda^2\xi_0/\lambda_P$, which leads to the following estimate for λ_P:

$$\lambda_P \approx (\lambda^2\xi_0)^{1/3}\,. \tag{2.29}$$

The quantity λ in (2.29) is defined by (2.7). Then it follows from (2.29) that, in the case $\lambda \ll \xi_0$, we have $\lambda_P \gg \lambda$, i.e., the nonlocal electrodynamics predicts deeper penetration of the magnetic field than would be expected from the London equations. This statement assumes, of course, that λ_P also satisfies the inequality $\lambda_P \ll \xi_0$, which is not always the case even for pure metals.

A typical representative of superconductors which are well-described by the nonlocal relations (the so-called Pippard superconductors) is Al. In contrast, Pb, even of high purity, is a London superconductor. When the temperature T approaches T_c, all superconductors become local (London superconductors) because λ diverges at $T \to T_c$ while ξ_0 is independent of temperature.

Everything said so far applies to pure metals, that is, those characterized by a mean free path $l \gg \xi_0$. If a metal contains a large number of impurities, $l \ll \xi_0$ can occur. We shall refer to such metals as dirty superconductors. Alloys also fall into this category. In very dirty metals, the role of the coherence length is played by the mean free path l. With the help of the microscopic theory, one can show that the magnetic field penetration depth for dirty superconductors is $\lambda_d \approx \lambda\,(\xi_0/l)^{1/2}$ at $l \ll \xi_0$. Thus, superconducting alloys are described well by the local London equations. In the rest of the book we shall use local equations only.

2.4 Quantum Generalization of the London Equations. Magnetic Flux Quantization

2.4.1 Generalized London Equation

It was already mentioned in Chap. 1 that the elementary carrier of the supercurrent is a pair of electrons called the Cooper pair. All pairs occupy the same energy level, or the same quantum state, i.e., they form a condensate. The wavefunction of a particle in the condensate can be written in the form

$\Psi(\boldsymbol{r}) = (n_s/2)^{1/2}\,\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{r})}$, where θ is the phase of the wavefunction. The normalization of $\Psi(\boldsymbol{r})$ has taken into account that the density of electron pairs is $n_s/2$, where n_s is the density of the superconducting electrons.

Consider a particle of mass $2m$ and charge $2e$ moving in a magnetic field. Let us show that its momentum can be written as

$$\hbar\nabla\theta = 2m\boldsymbol{v}_s + \frac{2e}{c}\boldsymbol{A}\,, \tag{2.30}$$

where $\hbar$ is Planck's constant. In the absence of magnetic field, the particle flow density $n_s\boldsymbol{v}_s/2$ can be written in the form $(\mathrm{i}\hbar/4m)(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$. Substituting here the expression $\Psi(\boldsymbol{r}) = (n_s/2)^{1/2}\,\mathrm{e}^{\mathrm{i}\theta}$, we get $\hbar\nabla\theta = 2m\boldsymbol{v}_s$. Thus the total momentum of a particle moving in a magnetic field, $\hbar\nabla\theta$, is a sum of the momentum $2m\boldsymbol{v}$ and the momentum $(2e/c)\,\boldsymbol{A}$ due to the magnetic field.

Taking the expression for the supercurrent density in the form

$$\boldsymbol{j}_s = n_s e\boldsymbol{v}_s \tag{2.31}$$

and using (2.7) and (2.18), it is easy to obtain from (2.30) the generalized second London equation:

$$\boldsymbol{j}_s = \frac{1}{c\Lambda}\left(\frac{\Phi_0}{2\pi}\nabla\theta - \boldsymbol{A}\right)\,. \tag{2.32}$$

Here we use the notation $\Phi_0 = \pi\hbar c/e$. The quantity Φ_0 has the dimensions of magnetic flux. Its physical significance will become clear in the following section.

2.4.2 Magnetic Flux Quantization

We now proceed to a remarkably interesting phenomenon: magnetic flux quantization in superconductors.

Consider a hole through a bulk superconductor, such as shown in Fig. 2.2. At first, the temperature is $T > T_c$ and the superconductor is in the normal state. Then we apply an external field H_0 parallel to the axis of the cylinder, and lower the temperature so that the specimen goes into the superconducting state. The field is now pushed out of the interior of the superconductor, while in the hole some frozen magnetic flux remains. This flux is produced by the supercurrent generated at the internal surface of the hole. Let us find this frozen magnetic flux.

Consider the contour C inside the superconductor, as in Fig. 2.2, enclosing the hole so that the distance between the contour and the internal surface of the hole is everywhere well in excess of λ. Then at any point of the contour, the supercurrent is $\boldsymbol{j}_s = 0$ and the path integral of (2.32) along the contour reduces to

$$\frac{\Phi_0}{2\pi}\oint_C \nabla\theta\cdot\mathrm{d}\boldsymbol{l} = \oint_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}\,. \tag{2.33}$$

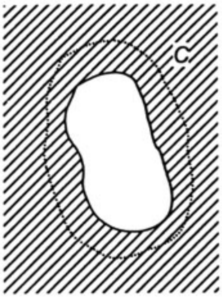

Fig. 2.2. Solid superconductor (*dashed area*) contains a hole. The contour C goes around the hole through the interior of the superconductor

Taking into account that

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Phi \,, \tag{2.34}$$

we have

$$\Phi = \frac{\Phi_0}{2\pi} \oint_C \nabla\theta \cdot \mathrm{d}\boldsymbol{l} \,. \tag{2.35}$$

Here Φ is the total magnetic flux through the contour C. From (2.35), one can see immediately that θ is a multiple-valued function; it changes by a certain value after every full circle around the hole. But the wavefunction Ψ must be single-valued. Therefore, we have to stipulate that the change in θ after a full circle around the hole containing the magnetic flux must be an integral multiple of $2\pi n$, where $n = 0, 1, 2 \ldots$. Indeed, the addition of $2\pi n$ to $\theta(\boldsymbol{r})$ does not change the function $\Psi(\boldsymbol{r}) = (n_s/2)^{1/2}\,\mathrm{e}^{\mathrm{i}\theta}$, because $\mathrm{e}^{\mathrm{i}2\pi n} = 1$. Therefore, $\oint_C \nabla\theta \cdot \mathrm{d}\boldsymbol{l} = 2\pi n$ and (2.35) can finally be written as

$$\Phi = n\Phi_0 \,, \tag{2.36}$$

where

$$\Phi_0 = \frac{\pi\hbar c}{e} = \frac{hc}{2e} \,. \tag{2.37}$$

It follows from (2.36) that the magnetic flux enclosed in the hole (or, more accurately, the magnetic flux through the contour C) can only assume values that are integral multiples of the minimum possible value of magnetic flux, the magnetic flux quantum Φ_0. The value of Φ_0 is defined by (2.37):

$$\Phi_0 = 2.07 \times 10^{-7}\ \mathrm{G\,cm^2} \,.$$

Physically, the origin of the magnetic flux quantization is the same as the quantization of electron orbits in atom. The wavefunction of electrons moving along a closed orbit must contain an integral number of wavelengths over the length of the orbit.

Experimentally, magnetic flux quantization was discovered in 1961 almost simultaneously in the USA (B. Deaver and W. Fairbank) [8] and in Germany (R. Doll and M. Näbauer) [9]. It is interesting to note that F. London, who predicted the flux quantization, believed that the flux quantum should be hc/e, that is, he predicted a value twice as large as Φ_0. But that is not surprising because he believed that the charge of an elementary supercurrent carrier was equal to the electron charge e. Experiment confirmed the validity of (2.37). Thus the experiments that detected the quantization of magnetic flux also provided direct evidence that the supercurrent is carried by pairs of electrons.

2.5 Magnetic Field and Current Distributions in Simple Configurations of Superconductors

2.5.1 Thin Slab in a Parallel Magnetic Field

In this section we shall analyze magnetic field and current distributions in simple configurations of superconductors. We start with the case of an infinite slab of thickness d placed in a uniform magnetic field H_0 parallel to its surface. Assume that the plane $x = 0$ is in the center of the slab and that its surfaces coincide with the planes $x = \pm d/2$. The magnetic field is along the z axis.

In the interior of the slab, the magnetic field H must satisfy (2.13). Then, from symmetry arguments, H is along the z axis and depends only on the x coordinate. Therefore (2.13) can be rewritten as:

$$\mathrm{d}^2 H/\mathrm{d}x^2 - \lambda^{-2} H = 0 \tag{2.38}$$

with the boundary conditions $H(\pm d/2) = H_0$. The general solution of (2.38) is

$$H = H_1 \cosh(x/\lambda) + H_2 \sinh(x/\lambda) , \tag{2.39}$$

where H_1 and H_2 are integration constants. Substituting the boundary conditions into (2.39), we get two algebraic equations with two unknowns that can be solved easily. The final result is

$$H(x) = H_0 \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)} . \tag{2.40}$$

The supercurrent density in the slab can be found by combining (2.40) and Maxwell's equation $\operatorname{curl} \boldsymbol{H} = (4\pi/c)\boldsymbol{j}_s$:

$$j_s = -\frac{c}{4\pi}\frac{\mathrm{d}H}{\mathrm{d}x} . \tag{2.41}$$

As a result, we get

$$j_s = -\frac{cH_0}{4\pi\lambda}\frac{\sinh(x/\lambda)}{\cosh(d/2\lambda)} . \tag{2.42}$$

If the slab is thick ($d \gg \lambda$), it follows from (2.40) and (2.42) that both the magnetic field and the current penetrate into it only to a certain depth of the order λ. In the other limiting case, a thin film ($d \ll \lambda$), we get, in linear approximation and after expanding the hyperbolic functions in terms of the small parameters x/λ and $d/2\lambda$:

$$H = H_0 \, , \qquad j_s = \frac{cH_0 x}{4\pi\lambda^2} \, .$$

This means that the magnetic field penetrates the entire film, while the supercurrent density varies linearly with depth.

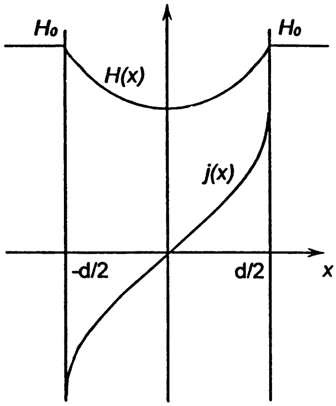

Fig. 2.3. Distribution of current and magnetic field across a thin film placed in a parallel uniform magnetic field

The current and field distributions obtained above are shown in Fig. 2.3. The current circulates at the edges of the slab so that the induced magnetic field cancels out the external field H_0 in its interior.

2.5.2 Thin Slab Carrying a Current

In this section we consider an infinite thin slab carrying a current in the absence of magnetic field. We assume that the slab is the same as in Sect. 2.5.1 and the current flows in the direction of the y axis. Further we assume that the current is uniform along the z axis, that is, edge effects are disregarded.

Consider a part of the slab in the form of a strip of unit width parallel to the z axis. The current I flows along the z axis and generates a magnetic field $\pm H_I$ at the surfaces of the slab ($x = \pm d/2$). Substituting these boundary conditions into the general solution (2.39), we find the field distribution in the slab:

$$H(x) = -H_I \frac{\sinh(x/\lambda)}{\sinh(d/2\lambda)} \, , \tag{2.43}$$

where $H_I = 2\pi I/c$.

Using Maxwell's equation (2.41), we find the distribution of current:

$$j_s(x) = \frac{cH_I}{4\pi\lambda} \frac{\cosh(x/\lambda)}{\sinh(d/2\lambda)} . \tag{2.44}$$

It follows from (2.43) and (2.44) that, similarly to the previous case, for $d \gg \lambda$ both the magnetic field and the current are present only within a surface layer of thickness λ. If the film is thin ($d \ll \lambda$), the pattern is different: the current flows through its entire cross-section and the field is a linear function of the coordinate:

$$H = -H_I \frac{2x}{d} , \qquad j_s = \frac{cH_I}{2\pi d} = \frac{I}{d} .$$

Recall that a uniform current in an infinite thin slab generates a uniform magnetic field outside the slab, independent of the coordinates. The field and current distributions are shown in Fig. 2.4.

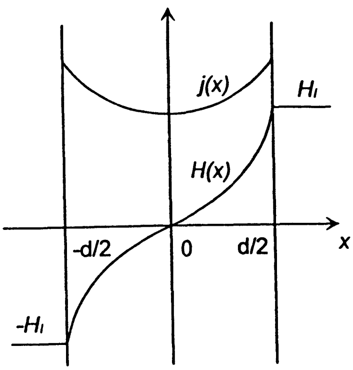

Fig. 2.4. Distribution of current and magnetic field across a thin film carrying a current

2.5.3 Thin Slab Carrying a Current in a Transverse Uniform Field

Consider a thin slab in a uniform external field H_0 parallel to the z axis. The slab carries a current along the y direction which is uniform along the z axis, as in Sect. 2.5.2. The total current through the cross-section of unit height is I. It generates a field $\pm H_I$ at the surfaces of the slab ($x = \pm d/2$). Thus, in this problem we have a superposition of the conditions from the two previous problems. Due to the linearity of the London equations, this should lead to a superposition of their solutions.

Consider a particular case $H_I = H_0$. The external field H_0 in this case cancels out the field generated by the current on one side of the slab and doubles it on the other side. As a result, the current I flows on one side of the slab only. Such a situation can be realized if the external field, H_0, is generated by a second slab carrying the same current I but in the opposite direction,

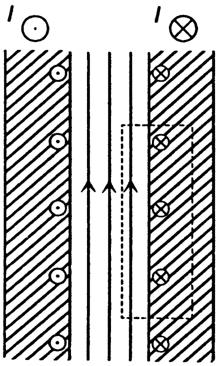

Fig. 2.5. If two parallel superconducting slabs carry currents which are equal in value and opposite in direction, the resulting magnetic field is 'locked' in the gap between the slabs

as in Fig. 2.5. The field in the gap between the slabs[1] is $H = (4\pi/c)\,I$. One can assume that this field is generated by both currents. Due to the current I flowing in the left-hand slab, there is a uniform field $H_I = 2\pi I/c$ directed upwards everywhere to the right of the slab and downwards everywhere to the left of it. Vice versa, the current from the right-hand slab generates a field H_I directed downwards everywhere to the right of the slab and upwards everywhere to the left of it. The result is that both to the right and to the left of the pair of slabs the fields cancel each other out while in the gap the field is doubled.

2.5.4 Thin Film Above a Superconducting Semispace

In this section we discuss a situation which is particularly important from a practical point of view: a thin film above a semi-infinite superconducting space (we shall refer to the latter as a superconducting screen). In order to find the distribution of the magnetic field when the film carries a certain current, we start with a very simple case.

Consider a rectilinear current-carrying conductor, which is placed above a superconducting screen and oriented along the y axis. Its distance from the screen is a. Let us find the magnetic field above the screen. If there were no screen, the magnetic field lines would form concentric circles centered on the conductor. But because the field lines cannot penetrate a superconductor (due to the Meissner–Ochsenfeld effect), it is obvious that, with the superconducting screen in place, the magnetic field around it will be distorted. Let us find this field.

In the semispace $z > 0$ and outside the conductor, there are no electric currents and $\operatorname{curl} \boldsymbol{H} = 0$. Therefore, we can introduce a magnetic field potential satisfying the Laplace equation. On the other hand, we know that the magnetic field at the surface of a superconductor is always tangential to the

[1] Stokes' theorem states that $\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = (4\pi/c)\,I$, where I is the total current through the surface bounded by the contour along which the path integral is taken. Carrying out the integration along the dashed contour (Fig. 2.5), we obtain $H = (4\pi/c)\,I$.

surface, that is, $H_z\,(z = 0) = 0$. This boundary condition assures that the solution of the Laplace equation is single-valued.

The correct field in the region $z > 0$ can be found very simply by means of the method of images. It is easy to see that it will be the field generated by two rectilinear currents, equal in value and opposite in direction, without the superconducting screen. For one of the currents, the distance from the plane $z = 0$ is a, i.e., the coordinates of its cross-section are $(0, a)$, while for the other (the image of the first) the coordinates are $(0, -a)$. Then in the semispace $z > 0$ and outside the conductor, the magnetic field still satisfies the equation $\operatorname{curl} \boldsymbol{H} = 0$, and the boundary condition $H_z\,(z = 0) = 0$ is satisfied automatically due to symmetry of the problem.

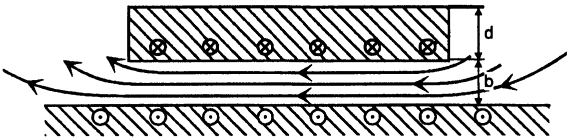

Fig. 2.6. Magnetic field generated by a current-carrying film placed above a superconducting semi-space

Thus we can apply the method of images to derive the field generated by a superconducting film placed above a semi-infinite superconducting screen. Let the thickness of the film be $d > \lambda$ and its width $w \gg \lambda$. The film is placed at a distance b from the screen and carries a current I. Let us find the current distributions in the film and in the screen and the magnetic field in the gap between the film and the screen.

Using the method of images, we replace the effect of the screen with that of an image film placed at a distance $2b$ from our film. The image also carries a current, but the direction of this current is opposite to that in the real film. Thus we have arrived at the problem of field and current distributions for two parallel films with opposite currents. This problem has just been solved in Sect. 2.5.3 (Fig. 2.5) and we already know the answer: in the gap between the film and the screen, there is a uniform magnetic field $H_I = \dfrac{4\pi}{c}\dfrac{I}{w}$. The current I in the film flows only near the surface adjacent to the screen, within a layer $\sim \lambda$. The current in the screen also flows within a layer $\sim \lambda$ near the surface and its surface density is I/w. The direction of this current is opposite to that of the current in the film. The relation between the field H_I and the current in the screen is given by the law of the total current. This situation is shown schematically in Fig. 2.6. The edge effects are disregarded in the above consideration. The magnetic field distribution incorporating the edge effects is shown in Fig. 2.7 [19].

The results that we have just obtained agree with reality if the real film playing the role of the screen can indeed be approximated by a semi-infinite screen. Below there is an example of a situation in which such an approximation is not valid.

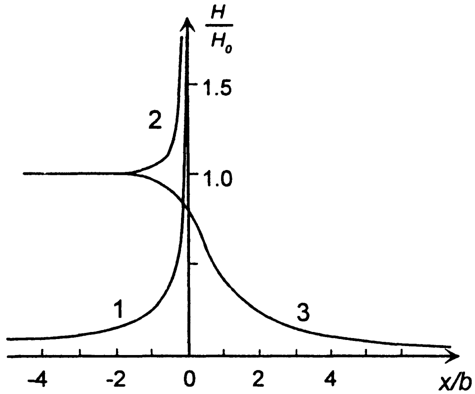

Fig. 2.7. Magnetic field distribution near the edge of a current-carrying superconducting film placed in the vicinity of a superconducting screen: *1* – field at the upper surface of the film; *2* – field at the lower surface of the screen; *3* – field at the surface of the screen

Consider a thick superconducting film (of thickness $d \geq \lambda$) of width w and length l deposited onto a glass substrate; $w \ll \lambda$. The film has been covered with a thin insulating layer of thickness b, such that $b \ll w$, and then another superconducting film has been deposited on top, also of thickness w and of length well in excess of l. Then a current from an external source is applied to the upper superconducting film. It is easy to realize that the

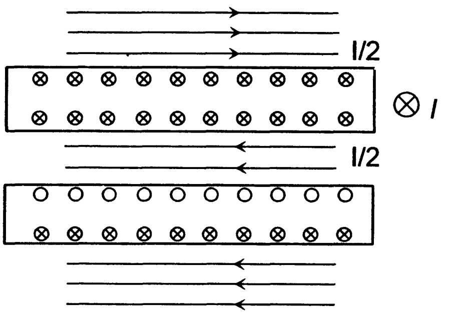

Fig. 2.8. Field and current distributions for two parallel closely spaced films. A current I is applied to the upper film. The lower film is not supplied with current

lower film cannot produce any screening effect in this situation. Indeed, if the upper film carries a certain current I, this current generates a magnetic field in the gap between the films. This means that the currents at the lower surface of the upper film and at the upper surface of the lower film must be equal in value. Furthermore, because the length of the lower film is finite, the current at its upper surface has to close at its lower surface, as in Fig. 2.8. As a result, the magnetic field at this surface is exactly the same as that in the gap between the films. Moreover, the current in the upper film is now split equally between its upper and lower surfaces, as illustrated in Fig. 2.8. Thus, the magnetic fields above and below the pair of films and in the gap are exactly the same as those in the absence of the lower film. That is, the latter does not produce any screening effect.

2.5.5 Short-Circuit Principle

Quite often in modern superconducting electronics, when complex multilayer systems are being designed, it can be rather difficult to have an instant grasp of how the magnetic fields and electric currents will be distributed in such a system. Great help can be provided here by the so-called short-circuit principle [20]. It can be applied to thick films, of thicknesses larger than the penetration depth, provided their widths and lengths are well in excess of the distances between the films.

Consider two superconducting films separated by an insulating layer. If the films carry electric currents, the currents along the neighboring surfaces must be equal in value and opposite in direction. Indeed, the surface current is determined by the magnetic field at the surface of a superconductor, and the magnetic field in the gap between the two films is common to both of them. If the thickness of the insulating layer is small, the magnetic flux through this layer is also small and the magnetic field at the edges of the films, associated with this flux, is negligible. Hence we can ignore the magnetic field at the edges and argue that the other parts of the films 'are not aware' of the existence of this flux and 'will not notice' if it disappears. And it can disappear as a result of short-circuiting, that is, bringing into contact two neighboring superconducting surfaces carrying electric currents that are equal in value and opposite in direction.

We can now formulate the short-circuit principle [20]: *In a complex system of superconducting films, if two neighboring film surfaces are short-circuited, it will not affect the current distribution in any other part of the system other than the two short-circuited surfaces.*

Consider several examples illustrating this principle.

(1) Let us find the distribution of currents in two parallel and identical thick films separated by a thin gap. The first film carries a current I_1 and the second one carries a current I_2, in the same direction. The two films are short-circuited. As a result, we have one film carrying a total current $I_1 + I_2$ distributed uniformly over the outer surfaces of the short-circuited films. Thus, the current at each of those surfaces is $(I_1 + I_2)/2$. We now know the currents at the outer surfaces of our system of films. Let i denote the current at the inner surfaces of the films. Because the total current in the first film, I_1, is fixed, we have $(I_1 + I_2)/2 + i = I_1$, i.e., $i = (I_1 - I_2)/2$. All surface currents in our system are now defined.

Note that if $I_2 = 0$, we come straight to the result shown in Fig. 2.8, that is, to the absence of screening by a superconducting film.

(2) If an insulated current-carrying wire is covered with a superconducting sheath, the latter will not be capable of screening the magnetic field generated by the current in the wire. To elucidate this statement, let us short-circuit the inner surface of the sheath and the surface of the wire. Then the entire current from the wire flows at the outer surface of the sheath and generates a magnetic field in the surrounding space.

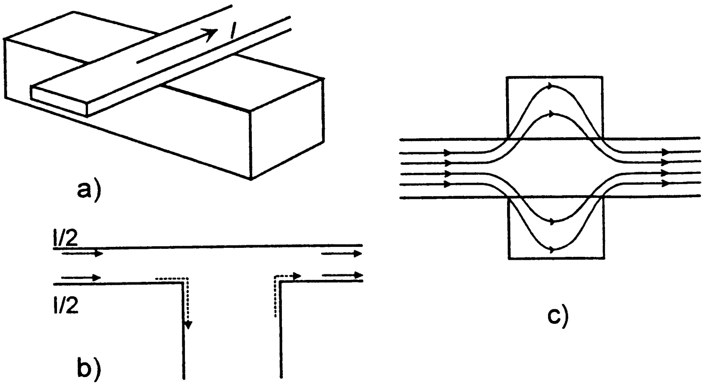

Fig. 2.9. Current-carrying film separated from the flat surface of a bulk superconductor by a thin gap: (**a**) general view; (**b**) side view after short-circuiting; (**c**) top view

(3) Let us find the distribution of currents in a system consisting of a thick superconducting film carrying a current which is placed above a massive superconducting bar (see Fig. 2.9). If the lower surface of the film and the upper surface of the bar are short-circuited, the resulting distribution of currents is that sketched in Fig. 2.9 (b). Indeed, at first the current from the film spreads over the upper surface of the bar, but a small fraction of it goes around the bar, along its lower surface. As a result, the current density at the upper surface of the film decreases sharply. Taking this into account, one can argue that in the initial system, without the short-circuit, the current in the part of the film immediately above the bar flows predominantly at the lower surface of the film. A current of the same value and opposite in direction must then appear at the upper surface of the bar, in the part situated immediately below the film. Further, this current spreads over the upper surface of the bar and closes there, while a small fraction of it closes over a path around the bar.

One can conclude that the role of the bar in this system approaches the role of the semi-infinite screen (see Fig. 2.7).

2.6 Kinetic Inductance

The inductance of a section of an electric circuit is usually defined by the energy $\mathcal{F}^{\mathrm{M}}$ of the magnetic field generated by a current I through the circuit:

$$\mathcal{F}^{\mathrm{M}} = \frac{1}{8\pi}\int H^2\,\mathrm{d}V = \frac{1}{2c^2}\,L^{\mathrm{M}} I^2\,. \tag{2.45}$$

The integral is taken over the entire space. We shall refer to this inductance as a magnetic, or geometrical, inductance. In addition, the process of generating the current I in the circuit requires a part of the energy to be converted into the kinetic energy $\mathcal{F}^{\mathrm{K}}$ of the current carriers (electrons). With this part of the energy can be associated the so-called kinetic inductance L^{K}:

$$\mathcal{F}^{\mathrm{K}} = \int n \frac{mv^2}{2} \, \mathrm{d}V = \frac{1}{2c^2} L^{\mathrm{K}} I^2 , \tag{2.46}$$

where n is the number density of the current carriers, m is the mass of one carrier and v is the velocity. The integration is carried out over the volume of the conductor.

If the section of the circuit is normal, the contribution of the kinetic inductance to the total resistivity is very small compared to its active resistivity and usually neglected. The contribution of the kinetic inductance in normal conductors can only be significant at very high frequencies (over 10^{13} Hz). In superconductors, however, the kinetic inductance sometimes plays an important role.

Recall that the supercurrent density is $\boldsymbol{j}_{\mathrm{s}} = n_{\mathrm{s}} e \boldsymbol{v}_{\mathrm{s}}$. Then, from (2.46), we obtain the following definition of the kinetic inductance of a superconductor:

$$L^{\mathrm{K}} = c^2 \Lambda \int \frac{j_{\mathrm{s}}^2}{I^2} \, \mathrm{d}V , \tag{2.47}$$

where the integral is taken over the volume of the superconductor and I is the total current through the superconductor.

Let us illustrate the concept of kinetic inductance with specific examples.

(1) Consider a superconducting wire of length l and radius R. Assume that $R \gg \lambda$. If the wire carries a current I, this current must circulate near its surface. The current density j_{s} at a distance r from the center of the wire is $j_{\mathrm{s}}(x) = j_{\mathrm{s0}} \, \mathrm{e}^{-x/\lambda}$, where $x = R - r$, $j_{\mathrm{s0}} = j_{\mathrm{s}}(0)$. The total current is $I = 2\pi R \lambda j_{\mathrm{s0}}$. Substituting these into (2.47) and carrying out the integration, we get $L^{\mathrm{K}} = l\lambda/R$.

Let us now introduce another quantity which is particularly useful in many applications: the inductance per square $L_{\square}$. For a flat superconductor, the greater its length, the larger its inductance and resistivity. Furthermore, the greater its width, the smaller the inductance. Therefore, the inductance of a square is always the same for a given superconductor, whether one considers 1 km^2 or 1 mm^2 of it. For our wire, since the circumference of the cross-section is $2\pi R$, the kinetic inductance per square is

$$L_{\square}^{\mathrm{K}} = 2\pi\lambda . \tag{2.48}$$

If we use (2.45) and calculate the part of the magnetic inductance associated with the magnetic field penetrating the superconductor (more accurately, penetrating the layer of the order λ near the surface) we obtain the same result:

$$L_{\square}^{\mathrm{M}} = 2\pi\lambda . \tag{2.49}$$

This result is also valid for the flat surface of a superconducting semispace.

The total inductance per square of the layer λ at the surface of a bulk superconductor is equal to the sum of (2.48) and (2.49):

$$L_\square = 4\pi\lambda \, . \tag{2.50}$$

In the International System of Units (SI), equation (2.50) takes the form: $L_\square = \mu_0\lambda$, with $\mu_0 = 4\pi \times 10^{-7}$ H/m. As follows from (2.48–50), in CGS $L_\square$ is measured in cm, and in SI in H, where 1 cm corresponds to 10^{-9} H $= 1$ nH.

One can say that λ is a characteristic of the inertia of the current carriers since, for a semispace, $L_\square^{\mathrm{K}}$ depends only on the penetration depth. If $\lambda \approx 5 \times 10^{-6}$ cm, then $L_\square = 4\pi\lambda = 6.3 \times 10^{-5}$ cm $= 6.3 \times 10^{-14}$ H.

(2) Consider the kinetic inductance of a thin superconducting film. Assume that the thickness of the film is $d \ll \lambda$ and, therefore, the current in the film is distributed uniformly over its thickness. We shall consider a small section of width w along the width of the film and assume that the current there is also uniform. Restricting the length of the section to the same value w, we obtain for the kinetic energy of the superconducting electrons within this section:

$$\mathcal{F}_\square^{\mathrm{K}} = \frac{\Lambda}{2} {j_s}^2 w^2 d \, .$$

Recalling that the current I through the cross-section is assumed to be uniform, we have $j_s = I/wd$ and

$$\mathcal{F}_\square^{\mathrm{K}} = \frac{1}{2d} \Lambda I^2 = \frac{1}{2d} \frac{4\pi\lambda^2}{c^2} I^2 = \frac{1}{2c^2} L_\square^{\mathrm{K}} I^2 \, .$$

From the last expression we immediately obtain $L_\square^{\mathrm{K}}$ for a thin film:

$$L_\square^{\mathrm{K}} = 4\pi\lambda^2/d \, . \tag{2.51}$$

Now it is obvious that, for $d \ll \lambda$, the kinetic inductance can be significant. For example, for a thin film ($d \sim 10^{-6}$ cm) with the penetration depth $\lambda = 3 \times 10^{-5}$ cm, the kinetic inductance per square is, according to (2.51), $L_\square^{\mathrm{K}} \approx 10^{-2}$ cm $= 10^{-11}$ H.

(3) Finally, consider a thick film above a massive superconducting screen. Assume that the distance between the film and the surface of the screen is b. If the film carries a current, there must be a magnetic field present in the gap between the film and the screen. After calculating the energy of this magnetic field, we find its contribution to the inductance of the film $L_\square = 4\pi b$. In addition, the magnetic field penetrates the film to the depth λ_1 and the screen to the depth λ_2. This penetrating field, from (2.50), makes an additional contribution to the inductance of the system as a whole so that its total inductance is

$$L_\square = 4\pi(b + \lambda_1 + \lambda_2) \, . \tag{2.52}$$

One can see from (2.52) that, in order to reduce the inductance, the film should be placed as close to the screen as possible. However, making b substantially less than λ_1 or λ_2 is not of much use because both the magnetic and the kinetic inductance will remain within a layer of the order of the penetration depth, in the film as well as in the screen.

2.7 Complex Conductivity of a Superconductor

This section deals with the complex conductivity of a superconductor in an electromagnetic field. In what follows we assume that the electron mean free path l is small and, therefore, that the normal skin effect approximation is valid. In other words, l is small enough and the frequencies are low enough that l is less than the penetration depth of the electromagnetic field. The frequency of electron collisions is $\tau^{-1} = v_{\mathrm{F}}/l \gg \omega$, where ω is the frequency of the electromagnetic wave and v_{F} is the electron velocity on the Fermi surface.

In the following analysis we shall partly follow the monograph by Van Duzer and Turner [21].

In order to calculate the conductivity of a superconductor in a high-frequency field, we use the two-fluid model, that is, we assume that there are normal electrons of density n_{n} and superconducting electrons of density n_{s}, and the total density of conduction electrons is $n = n_{\mathrm{s}} + n_{\mathrm{n}}$. The motion of the superconducting electrons is governed by the first London equation (2.2):

$$\boldsymbol{E} = \Lambda\, \mathrm{d}\boldsymbol{j}_{\mathrm{s}}/\mathrm{d}t\,. \tag{2.53}$$

For the normal electrons we can write

$$e\boldsymbol{E} - \frac{m}{n_{\mathrm{n}}e}\frac{\boldsymbol{j}_{\mathrm{n}}}{\tau} = \frac{m}{n_{\mathrm{n}}e}\frac{\mathrm{d}\boldsymbol{j}_{\mathrm{n}}}{\mathrm{d}t}\,. \tag{2.54}$$

The left-hand side of (2.54) describes the forces on the normal electrons: the electric field and the average ‘friction’ due to electron collisions. The right-hand side is the product of the electron mass and acceleration. For one normal electron, this Newton’s second law can be written as

$$\boldsymbol{E} = \frac{n_{\mathrm{s}}}{n_{\mathrm{n}}}\Lambda\frac{\mathrm{d}\boldsymbol{j}_{\mathrm{n}}}{\mathrm{d}t} + \frac{n_{\mathrm{s}}}{n_{\mathrm{n}}}\Lambda\frac{\boldsymbol{j}_{\mathrm{n}}}{\tau}\,. \tag{2.55}$$

Assuming $\boldsymbol{j}_{\mathrm{s}} \propto \mathrm{e}^{\mathrm{i}\omega t}$, we can rewrite (2.53) and (2.55) as

$$\boldsymbol{j}_{\mathrm{s}} = -\mathrm{i}\frac{1}{\Lambda\omega}\boldsymbol{E}\,, \tag{2.56}$$

$$\boldsymbol{j}_{\mathrm{n}} = \frac{n_{\mathrm{n}}}{n_{\mathrm{s}}}\frac{\tau}{\Lambda}\frac{1-\mathrm{i}\omega\tau}{1+(\omega\tau)^2}\boldsymbol{E}\,. \tag{2.57}$$

The total current density is $\boldsymbol{j} = \boldsymbol{j}_{\mathrm{s}} + \boldsymbol{j}_{\mathrm{n}}$, and we finally have

$$\boldsymbol{j} = \sigma\boldsymbol{E}\,, \qquad \sigma = \sigma_1 - \mathrm{i}\sigma_2\,, \tag{2.58}$$

$$\sigma_1 = \frac{n_{\mathrm{n}}}{n_{\mathrm{s}}}\frac{\tau}{\Lambda}\frac{1}{1+(\omega\tau)^2}\,, \tag{2.59}$$

$$\sigma_2 = \frac{1}{\Lambda\omega}\left[1 + \frac{n_\mathrm{n}}{n_\mathrm{s}}\,\frac{(\omega\tau)^2}{1+(\omega\tau)^2}\right] . \tag{2.60}$$

Equations (2.58–60) describe the complex conductivity of a superconductor in a high-frequency electromagnetic field.

2.8 Skin Effect and Surface Impedance

2.8.1 Normal Skin Effect

It is well known that an electromagnetic field penetrates a normal metal to the so-called skin depth, or to the depth of a skin layer. In this section we shall consider how the field penetrates a superconductor. The surface of the superconductor coincides with the plane $x = 0$.

Let us write down Maxwell's equations

$$\operatorname{curl}\boldsymbol{H} = \frac{4\pi}{c}\,\sigma\boldsymbol{E} , \tag{2.61}$$

$$\operatorname{curl}\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial\boldsymbol{H}}{\partial t} . \tag{2.62}$$

Assuming that the magnetic field in the superconductor is $\boldsymbol{H} \propto \mathrm{e}^{-\mathrm{i}(kx-\omega t)}$ and taking the curl of both sides of (2.61), we get

$$-\nabla^2\boldsymbol{H} = -\frac{4\pi}{c}\,\sigma\,\frac{\partial\boldsymbol{H}}{\partial t} . \tag{2.63}$$

Here we have used the equations $\operatorname{div}\boldsymbol{H} = 0$ and $\operatorname{curl}\operatorname{curl}\boldsymbol{H} = -\nabla^2\boldsymbol{H}$. Substituting $\boldsymbol{H} \propto \mathrm{e}^{-\mathrm{i}(kx-\omega t)}$ in (2.63), we obtain

$$k^2 = -\mathrm{i}\,\frac{4\pi}{c^2}\,\sigma\omega , \tag{2.64}$$

and

$$k = (1-\mathrm{i})/\delta , \tag{2.65}$$

where

$$\delta = \left(\frac{c^2}{2\pi\sigma\omega}\right)^{1/2} . \tag{2.66}$$

Our problem is now essentially solved. Indeed, the field penetration is defined by the quantity k which can be expressed through the conductivity σ using (2.65) and (2.66). Let us make some simplifications. Assume that the temperature is not too close to T_c so that $(n_\mathrm{n}/n_\mathrm{s})\,(\omega\tau)^2 \ll 1$. In addition, we assume, as always, that $\omega\tau \ll 1$. Then we obtain from (2.59) and (2.60)

$$\sigma = \frac{n_\mathrm{n}}{n_\mathrm{s}}\,\frac{\tau}{\Lambda} - \mathrm{i}\,\frac{1}{\Lambda\omega} . \tag{2.67}$$

Substituting (2.67) into (2.66), we get

$$\delta = \frac{\sqrt{2}\lambda}{\left(\frac{n_\mathrm{n}}{n_\mathrm{s}}\,\omega\tau - \mathrm{i}\right)^{1/2}} . \tag{2.68}$$

At low frequencies, when $(n_\mathrm{n}/n_\mathrm{s})(\omega\tau) \ll 1$, we have $\delta = \sqrt{2\mathrm{i}}\lambda = \lambda(1+\mathrm{i})$. Substituting this into (2.65), we obtain $k = -\mathrm{i}/\lambda$, i.e., $H \propto \mathrm{e}^{-\mathrm{i}kx} = \mathrm{e}^{-x/\lambda}$. Thus, as one should have expected, a low-frequency magnetic field penetrates a superconductor in the same manner as a dc field, that is, it is exponentially damped over the magnetic field penetration depth. In the general case, the field penetration is described by (2.65) and (2.68).

2.8.2 Surface Impedance

By definition, the surface impedance is

$$Z = \frac{4\pi}{c}\frac{E}{H} . \tag{2.69}$$

This expression has a clear physical meaning. Suppose that there are ac electric and magnetic fields at the surface of a metal, such that the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are mutually orthogonal and tangential to the surface. Then $c\boldsymbol{H}/4\pi$ is the density of the surface current $\boldsymbol{j}_\mathrm{surf}$ and, therefore, Z in (2.69) is E/j_surf, that is, it expresses the surface impedance per square.

Let us now work out the surface impedance of a superconductor. With the magnetic field written as $\boldsymbol{H} \propto \mathrm{e}^{-\mathrm{i}(kx-\omega t)}$, equation (2.61) becomes

$$\mathrm{i}kH = \frac{4\pi}{c}\sigma E . \tag{2.70}$$

Then from (2.69) and (2.65) we have

$$Z\frac{\mathrm{i}k}{\sigma} = \frac{1+\mathrm{i}}{\sigma\delta} . \tag{2.71}$$

Substituting (2.66) and (2.67) into (2.71), we get

$$Z = R_\square + \mathrm{i}X_{L\square} , \tag{2.72}$$

$$R_\square = \frac{2\pi\omega^2\lambda}{c^2}\frac{n_\mathrm{n}}{n_\mathrm{s}}\tau , \tag{2.73}$$

$$X_{L\square} = \frac{4\pi\lambda\omega}{c^2} = \frac{\omega L_\square}{c^2} . \tag{2.74}$$

The real part of the impedance, $R_\square$, reflects the energy dissipation due to heating, while the imaginary part, $X_{L\square}$, is the inductive resistance.

Let us find the temperature dependences of $R_\square$ and $X_{L\square}$. Recalling the empirical relation $\lambda \propto (1-t^4)^{-1/2}$, we have $n_\mathrm{s} \propto 1-t^4$, or $n_\mathrm{s} = n\,(1-t^4)$

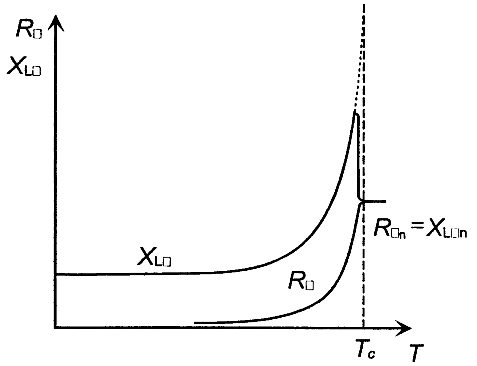

Fig. 2.10. Real and imaginary parts of the surface impedance as functions of temperature [22]

because $\lambda \propto {n_s}^{-1/2}$. Here $t = T/T_c$ and n is the density of free electrons in a metal. Then $n_n = nt^4$ and

$$R_\square \propto t^4/(1-t^4)^{3/2}\ , \qquad X_{L\square} \propto (1-t^4)^{-1/2}\ . \tag{2.75}$$

The above formulas reproduce the temperature dependences of both active and imaginary components of the impedance rather well (at least qualitatively), with an exception of temperatures close to T_c. In this temperature interval, (2.73) and (2.74) are no longer valid. Recall that, while deriving these formulas, we used (2.67) obtained on the assumption that $(n_n/n_s)(\omega\tau)^2 \ll 1$. But no matter how low the frequency is, the superconducting electron density at $T \to T_c$ is $n_s \to 0$ and the inequality is violated. Therefore, at $T \to T_c$ we have from (2.60): $\sigma = (n_n/n_s)\tau/\Lambda - \mathrm{i}\,(n_n/n_s)(\omega\tau)^2/\Lambda\omega$. Neglecting the imaginary part and substituting this expression in (2.65) and (2.71) we obtain

$$Z = \frac{2\pi}{c^2}\left(2\omega\tau\frac{n_s}{n_n}\right)^{1/2}\frac{\lambda}{\tau}(1+\mathrm{i}) = \frac{1+\mathrm{i}}{\sigma_n\delta_n}\ , \tag{2.76}$$

where σ_n and δ_n are the conductivity and the skin depth of the normal metal, respectively.

It follows from (2.76) that, at $T \to T_c$, $R_\square$ and $X_{L\square}$ are equal in value and independent of temperature, since ${n_s}^{1/2}\lambda = \text{const}$.

The temperature dependences of $R_\square$ and $X_{L\square}$ obtained within the two-fluid model are sketched in Fig. 2.10.

As far as high-temperature superconductors are concerned, many experiments indicate that they do not have a ‘clean’ gap. This means that even at zero temperature they have an intrinsic density of quasiparticle states in the gap down to zero energy. As a consequence, the temperature dependences of the two-fluid model are no longer valid for high-T_c materials.

Problems

Problem 2.1. Consider a bulk superconductor containing a cylindrical hole of 0.1 mm diameter. There are 7 magnetic flux quanta trapped in the hole. Find the magnetic field in the hole.

Problem 2.2. Consider a bulk superconductor containing a cylindrical hole of 2 cm diameter. The magnetic field trapped in the hole is $H = 300$ Oe. Find the vector potential $\boldsymbol{A}$ at the distance $R = 2$ cm from the center of the hole. Find the phase gradient $\nabla\theta$ at the same distance.

Problem 2.3. Consider a thin superconducting film of thickness $d \ll \lambda$ deposited onto a dielectric filament. The radius of the filament's cross-section is R. At room temperature, the filament is placed in a longitudinal magnetic field and then cooled down to a temperature below T_c. Then the external field is switched off. Find how the magnetic flux trapped by the filament is quantized.

Problem 2.4. Find the distribution of the magnetic field under the conditions of the previous problem.