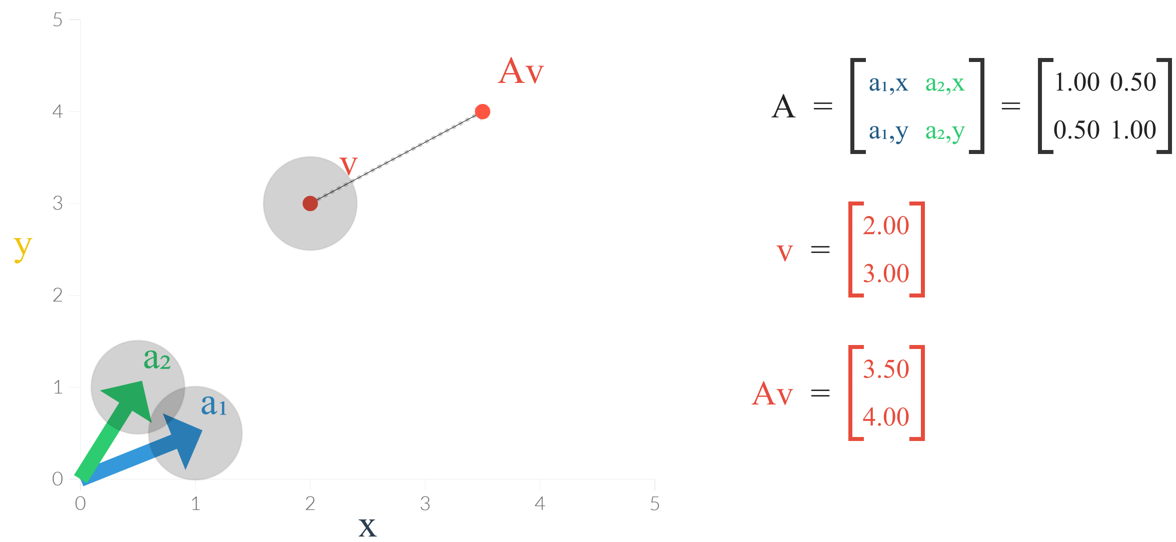
Eigenvectors and Eigenvalues

Explained Visually

By Victor Powell and Lewis Lehe

Eigenvalues/vectors are instrumental to understanding electrical circuits, mechanical systems, ecology and even Google's PageRank algorithm. Let's see if visualization can make these ideas more intuitive.

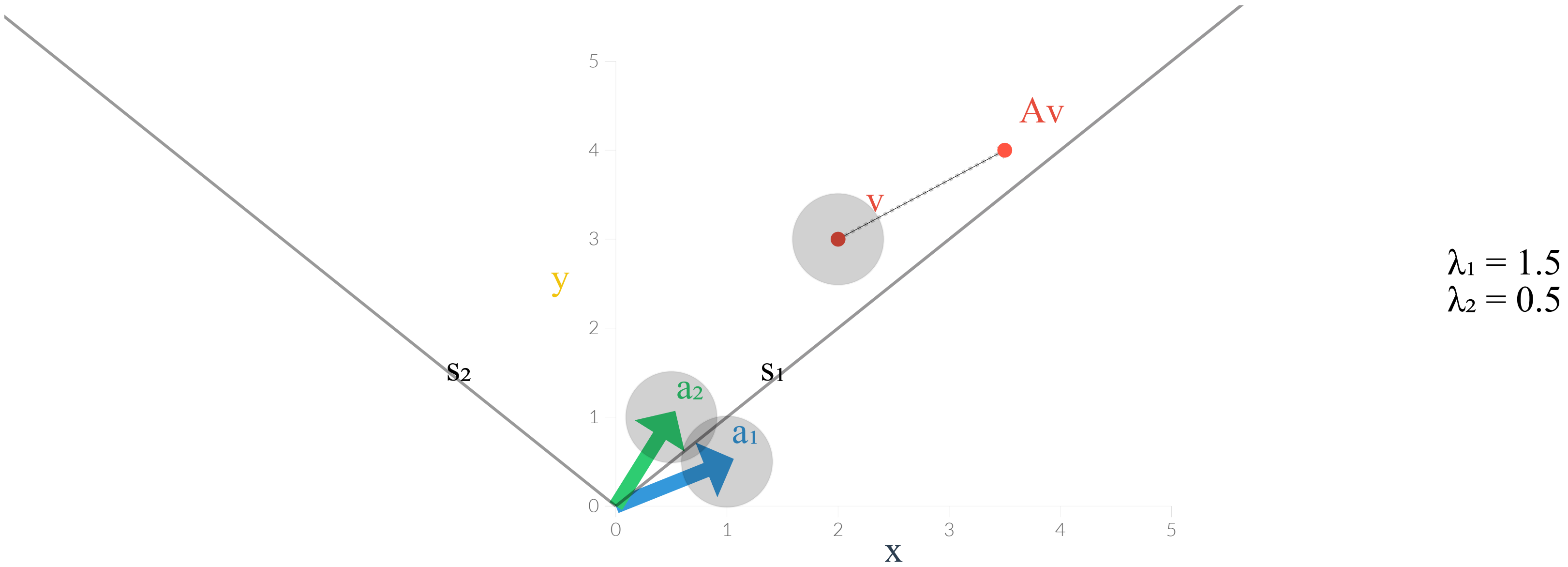
To begin, let v be a vector (shown as a point) and A be a matrix with columns a_1 and a_2 (shown as arrows). If we multiply v by A, then A sends v to a new vector Av.



If you can draw a line through the three points $(0, 0)$, $\boldsymbol{v}$ and $\boldsymbol{Av}$, then $\boldsymbol{Av}$ is just $\boldsymbol{v}$ multiplied by a number $\boldsymbol{\lambda}$; that is, $\boldsymbol{Av} = \boldsymbol{\lambda v}$. In this case, we call $\boldsymbol{\lambda}$ an **eigenvalue** and $\boldsymbol{v}$ an **eigenvector**. For example, here $(1, 2)$ is an eigvector and $\mathbf{5}$ an eigenvalue.

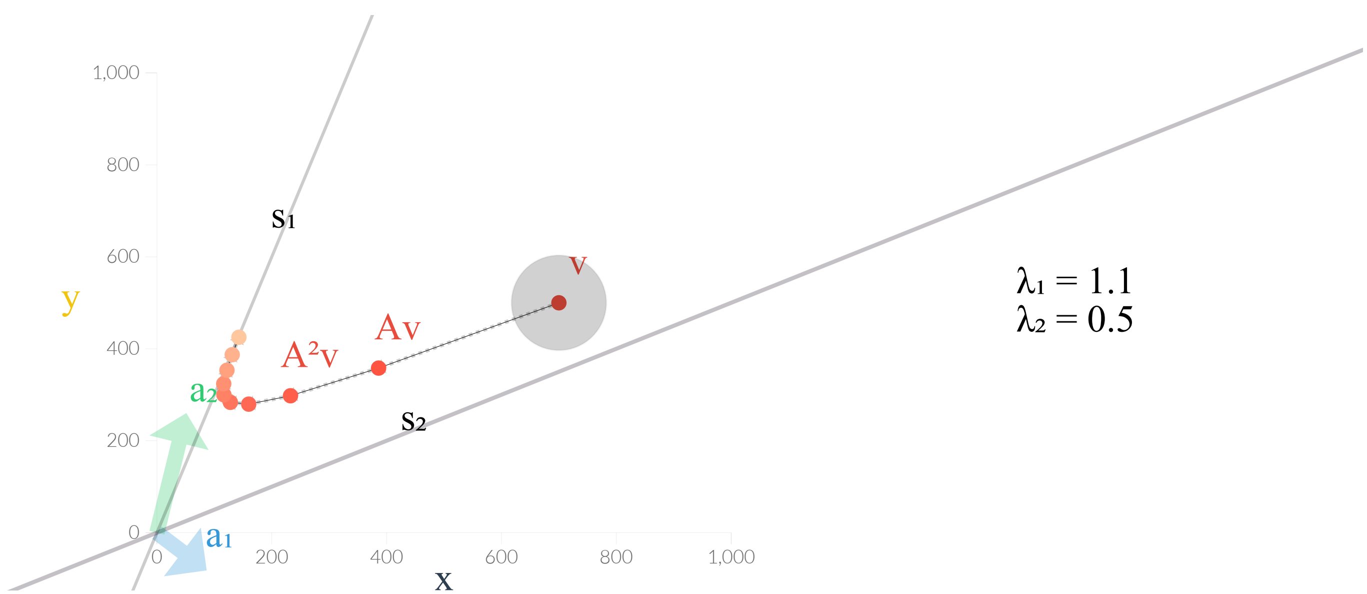
$$Av = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda v.$$

Below, change the columns of $\boldsymbol{A}$ and drag $\boldsymbol{v}$ to be an eigenvector. Note three facts: First, every point on the same line as an eigenvector is an eigenvector. Those lines are **eigenspaces**, and each has an associated eigenvalue. Second, if you place $\boldsymbol{v}$ on an eigenspace (either $\boldsymbol{s_1}$ or $\boldsymbol{s_2}$) with associated eigenvalue $\boldsymbol{\lambda} < 1$, then $\boldsymbol{Av}$ is closer to $(0, 0)$ than $\boldsymbol{v}$; but when $\boldsymbol{\lambda} > 1$, it's farther. Third, both eigenspaces depend on both columns of $\boldsymbol{A}$: it is not as though $\boldsymbol{a_1}$ only affects $\boldsymbol{s_1}$.



What are eigenvalues/vectors good for?

If you keep multiplying v by A, you get a sequence $v, Av, A^2v,$ etc. Eigenspaces attract that sequence and eigenvalues tell you whether it ends up at $(0, 0)$ or far away. Therefore, eigenvectors/values tell us about systems that evolve step-by-step.



Let's explore some applications and properties of these sequences.

Fibonacci Sequence

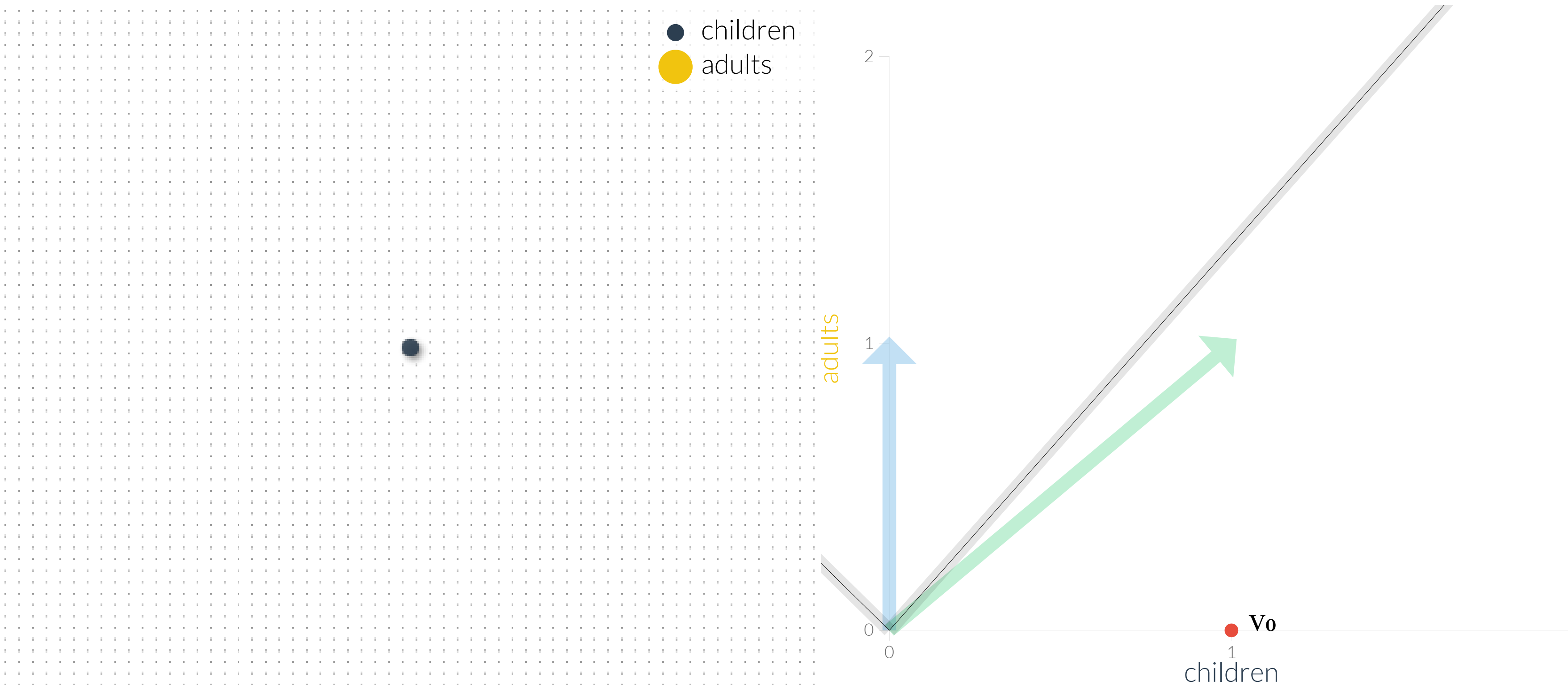
Suppose you have some amoebas in a petri dish. Every minute, all adult amoebas produce one child amoeba, and all child amoebas grow into adults (Note: this is not really how amoebas reproduce.). So if t is a minute, the equation of this system is

$$\begin{aligned} \text{adults}_{t+1} &= \text{adults}_t + \text{children}_t \\ \text{children}_{t+1} &= \text{adults}_t \end{aligned}$$

which we can rewrite in matrix form like

$$\begin{aligned} v_{t+1} &= A \qquad\qquad \cdot\, v_t \\ \begin{pmatrix} \text{adults}_{t+1} \\ \text{children}_{t+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{adults}_t \\ \text{children}_t \end{pmatrix} \end{aligned}$$

Below, press "Forward" to step ahead a minute. The total population is the Fibonacci Sequence.



reset forward

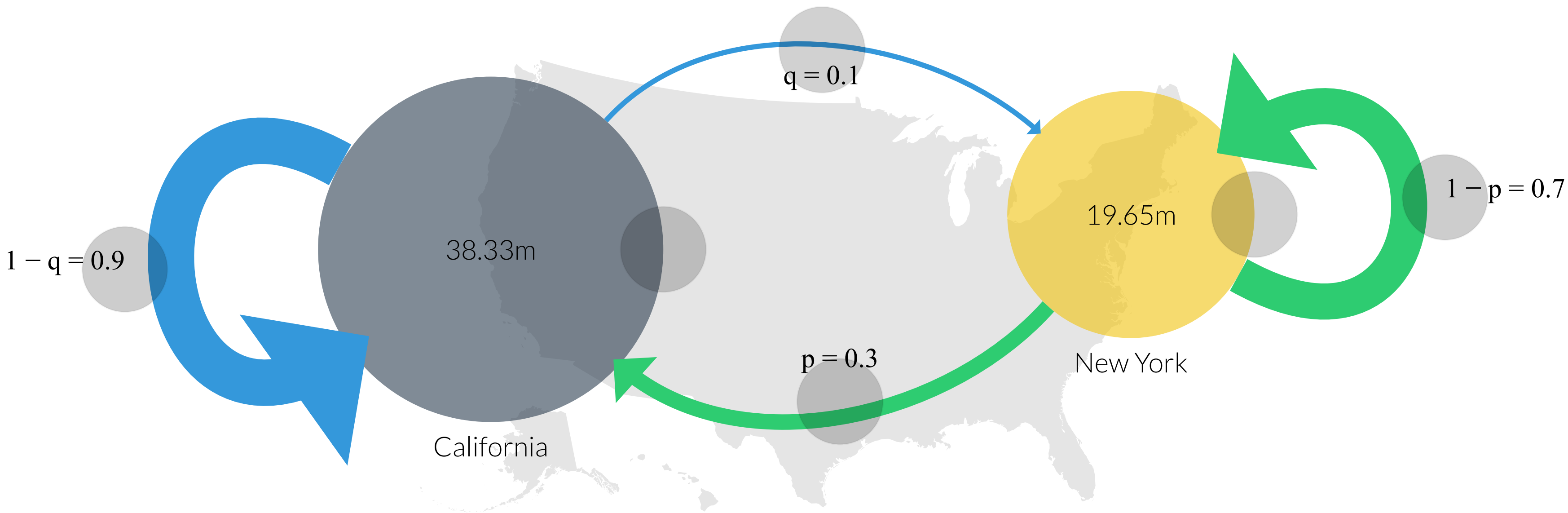
1 child + 0 adults = 1

1 1 2 3 5 8 13 21 34 55 89 144 233

As you can see, the system goes toward the grey line, which is an eigenspace with $\lambda = (1+\sqrt{5})/2 > 1$.

Steady States

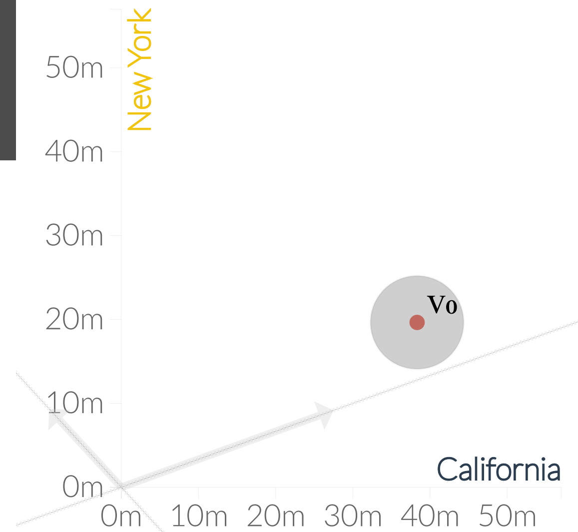
Suppose that, every year, a fraction p of New Yorkers move to California and a fraction q of Californians move to New York. Drag the circles to decide these fractions and the number starting in each state.



To understand the system better, we can start by writing it in matrix terms like:

$$v_{t+1} = Av_t$$
$$\begin{pmatrix} \text{New York}_{t+1} \\ \text{California}_{t+1} \end{pmatrix} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \cdot \begin{pmatrix} \text{New York}_t \\ \text{California}_t \end{pmatrix}$$

It turns out that a matrix like A, whose entries are positive and whose columns add up to one (try it!), is called a Markov matrix, and it always has $\lambda = 1$ as its largest eigenvalue. That means there's a value of v_t for which $Av_t = \lambda v_t = 1v_t = v_t$. At this "steady state," the same number of people move in each direction, and the populations stay the same forever. Hover over the animation to see the system go to the steady state.



For more on Markov matrices, check out our explanation of Markov Chains.

Complex eigenvalues

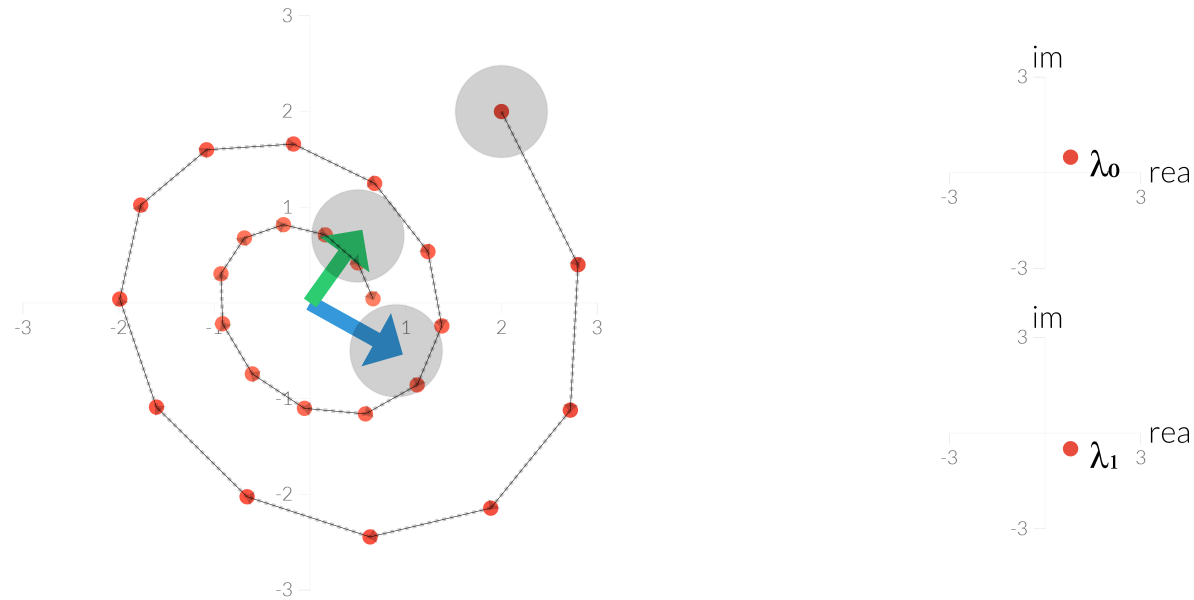
So far we've only looked at systems with real eigenvalues. But looking at the equation $Av = \lambda v$, who's to say λ and v can't have some imaginary part? That it can't be a complex number? For example,

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = (1+i) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Here, $1+i$ is an eigenvalue and $(1, i)$ is an eigenvector.

If a matrix has complex eigenvalues, its sequence spirals around $(0, 0)$. To see this, drag A's columns (the arrows) around until you get a spiral. The eigenvalues are plotted in the real/imaginary plane to the right. You'll see that whenever the eigenvalues have an imaginary part, the system spirals, no matter where you start things off.

steps:



Learning more

We've really only scratched the surface of what linear algebra is all about. To learn more, check out the legendary Gilbert Strang's Linear Algebra course at MIT's Open Courseware site. To get more practice with applications of eigenvalues/vectors, also ceck out the excellent Differential Equations course.

For more explanations, visit the Explained Visually project homepage.

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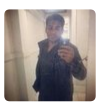


Harry Moreno • 3 years ago

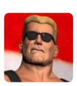
the `complex eigenvalues` animation was not visible to me. Firefox osx

28 • Reply • Share ›

Leonardo Donelli → Harry Moreno • 3 years ago

Same. Firefox on Ubuntu 14.04

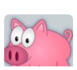
6 • Reply • Share ›



S.Nkm → Harry Moreno • 3 years ago

Same. Firefox / CentOS

1 • Reply • Share ›



evandrix → Harry Moreno • 3 years ago use Chrome or similar WebKit-powered web browser.

1 • Reply • Share ›

Artemiy Harry Moreno • 2 years ago

Same. Firefox 48, OpenSUSE Leap 42.1

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Karol Mieczysław Marcjan Harry Moreno • 3 years ago

Same, Firefox on Arch Linux.

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Pawnda Harry Moreno • 3 years ago

Same for me, Firefox on Manjaro.

• Reply • Share ›

Mark Harry Moreno • 3 years ago

Same here. FF 35.0.1 on Linux Mint 17.1.

• Reply • Share ›

cattly Mark • 2 years ago

Works for me - Chromium on Linux mint

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LionessLover Harry Moreno • 3 years ago

It worked for me initially, but after a few seconds of playing with the complex values diagram the page crashed. Chrome Version 42.0.2311.135 m (64-bit) on Windows 8.1

EDIT: I think I've found that this happens when I drag the green and blue arrow end points in a way that makes the entire diagram HUGE. Possibly the x and y values become so large they lead to an overflow. It should not crash, but maybe guard against large x and/or y values in the code of the diagram?

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kamil Harry Moreno • 2 years ago

Upgrade to Chome

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cpsievert • 3 years ago

Great post! I think "a matrix like A, whose rows add up to *zero*" should be "a matrix like A, whose rows add up to *one*"

18 ^ | v • Reply • Share ›

Lewis ➦ cpsievert • 3 years ago

yep good catch. we are going to push a batch of corrections.

1 ^ | v • Reply • Share ›

JohnThackr ➦ Lewis • 3 years ago

Also, it should be "columns add up to 1," and the matrix A should be transposed so that the first row is (1-p q), since NY_{t+1} = (1-p)NY_t + qCA_t

Alternatively, you can leave A like it is, but apply it via right-multiply to the row vector (NY_t CA_t). But as written it is incorrect.

Love the work and animation, though.

11 ^ | v • Reply • Share ›

Dan Warren ➦ JohnThackr • 3 years ago

Thanks for this, that was confusing the hell out of me. Absolutely love the demonstration otherwise!

2 ^ | v • Reply • Share ›

evansenter ➦ JohnThackr • 3 years ago

Yep, came to point out both. Great article.

^ | v • Reply • Share ›

Donny V. • 3 years ago

I love the way this is put together! If you know anything about neural networks in computing I would love to see it explained in this format. Especially the "Whats this good for" section.

8 ^ | v • Reply • Share ›

Lewis • 3 years ago

I am trying to get better at writing concise text. Is there any part of the text that is very unclear to any of you?

8 ^ | v • Reply • Share ›

Eyelet ➦ Lewis • 3 years ago

Great post. It would be nice if you explained what lambda1 and lambda2 is. It first shows up in the 2nd graph.

3 ^ | v • Reply • Share ›

yachris ➦ Lewis • 3 years ago

Excellent start -- but the third paragraph (starting with "If you can draw a line through...") is too terse... as the visualization is set up, you *can't* draw a line between 0,0, v and Av (and it's not until later that it's made clear that you can drag points). The next paragraph (starting with "Below, change the bases of A...") is not clear at all. You're assuming we know what "change the bases of A" means, so it'd be nice to either explain that or link to another page explaining that. Thanks for doing this!

3 ^ | v • Reply • Share ›

Im So Meta Even This Acronym ➦ Lewis • 3 years ago

Intro is a bit confusing. What's lambda, and why is it there? Why do I need it? The example equation below it is also confusing (I understand little math notation).

It's still not clear why is it great that I know two coinciding vectors are eigenvectors. They have a name, that's great, but anything more?

In the "What are eigenvalues/vectors good for?" section the first graph is nice, but I don't understand the curved thing. I thought we are talking about straight lines :)

The examples you gave are really pretty, but they made me less sure if I understand eigenvector/spaces/numbers :)

In one word, these tools are only good for visualization?

^ | v • Reply • Share ›

diego898 ➦ Lewis • 3 years ago

Hello! I posted a separate comment but figured I'd just reply to this one. First and foremost, fantastic job both of you! We had some people in the lab that requested clarification on the role of the "S" arrows, the "gray" lines. Thanks again!

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Lewis ↪ diego898 • 3 years ago

okay those are supposed to represent the eigenspaces. I'll update the text with a reference in a bit. thanks.

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dont get it • 3 years ago

Don't get it.

in steady states that would mean: new york t+1 = new york t - (fraction of new yourkers moving to california * new yourk) + (fraction of new yourkers moving to california * california)

That does not make sence to me. I would have expected the last part beeing (fraction of calironiars moving to new york * california)

5 • Reply • Share ›

Iwan Satria • 3 years ago

Apologies if my understanding is wrong, but for the New Yorker - Californian matrix, shouldn't it be like this?

(1-p q)

(p 1-q)

meaning:

NewYorkers(t+1) = (1-p)NewYorkers(t) + (q)Californians(t)

Californians(t+1) = (p)NewYorkers(t) + (1-q)Californians(t)

4 • Reply • Share ›

Karl Frost ↪ Iwan Satria • 3 years ago

i was about to say the same thing

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Kenny SF • 3 years ago

What frameworks do you use to create these interactive visualizations?

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Koen Boncquet ↪ Kenny SF • 3 years ago

Based on a quick inspection of the source code you can see that they are using the library d3.js for interactive visualisations

http://d3js.org/

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newguy • 3 years ago

isn't the nyc and ca matrix supposed to be [1-p,q]? (not [1-p,p]?) NYC keeps 1-p, but then gets q times CA. Currently it's say that the probability p of someone leaving nyc, times the pop of CA, becomes a NYC resident. which is nonsensical.

3 • Reply • Share ›

rickasaurus • 3 years ago

Crashing my browser with Eigenvalues (That last visualization did it when I dragged lambda_1 to have a negative y component). Great post though, it may be the best explanation of eigenvectors that I've ever seen.

2 • Reply • Share ›

Ellie Kesselman • 3 years ago

Thank you for that PDF, FinalVersionFixed. It is great!

1 • Reply • Share ›

Jose F Rodrigues Jr • 3 years ago

Nice!

1 • Reply • Share ›

Santosh • 3 years ago

Amazing page, I wish I had this in college.

1 • Reply • Share ›

Sushant Srivastava • 3 years ago

Excellent.

1 • Reply • Share ›

biomedicalblog • 3 years ago

Please make more! Interesting topics for this format may be: PCA, Singular Value Decomposition, Neural Networks

1 • Reply • Share ›

Peter • 3 years ago

This is fantastic. Each time I jump disciplines, I see a new explanation for eigenvectors, eigenvalues, and singular values. Each time, I learn a new way to think about it. I wish there was something a little more advanced than Strang's course -- something with a dozen explanations like this one that I could work through. From dynamics to statistics to image compression to machine learning to all the other applications....

1 • Reply • Share ›

Neven Sajko • 3 years ago

The last interactive graph caueses Chromium to crash when the eigenvector goes too far offscreen. Too reproduce, move the green base to negative-x (left), and increase the number of steps.

1 • Reply • Share ›

TheSisb • 3 years ago

This is really excellent. Thank you.

1 • Reply • Share ›

einsteinisgaurav • 3 years ago

I would love to see algorithms like simulated annealing, genetic algorithm explained visually. Awesome work!

1 • Reply • Share ›

Reply • Share ›

Flávia Rius • a month ago

Thank you very much for that, it was very didactic!
I found here searching about eigenvalues and eigenvectors to understand better the Principal Component Analysis.
It is clearer for me now :)

• Reply • Share ›

John Singleton • a month ago

Typo - great write up:

"We've really only scratched...also ceck out the excellent Differential Equations course."

• Reply • Share ›

Bruce Lester • a month ago

This is beautiful work.

• Reply • Share ›

Shivanshu Raj • 2 months ago

for A=[1 2 0 1] i calculated eigen values on hands..there only exist one eigen value for this linear operator but your animation show two eigen values each 1.00 what is this ...what kind of eigen values you are showing here...and also for two lamda there exist two eigen vectors but you show only one what kind of eigen vector are you presenting here???!!!!

• Reply • Share ›

Ritchie Vink • 5 months ago

Really nice visual explanations!

Really nice visual explanations!

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Dan Murphy • 6 months ago

Thanks for this! Really well-done and helpful.

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Sven Yurik • 8 months ago

Those are some AMAZING animations! How did you do them??

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Subir Das • 9 months ago

Brilliant work. Congratulations.

Reply • Share ›

Leonard Ikojo • a year ago

Thank you. Just the crash course I needed.

Reply • Share ›

Pirouz Nourian • a year ago

Thanks for the great explanation!

Reply • Share ›

Nanda Kishore Rajanala • a year ago

Reading this post, I couldn't figure out the practical applications of this concept. How does Google page rank leverage this? If we are talking about population movements in the real world, can we always assume that they move towards the steady state due to the eigen value ~1? I couldn't figure what to make of the spirals either? can someone help pls?

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