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Eigenvectors and Eigenvalues

Explained Visually

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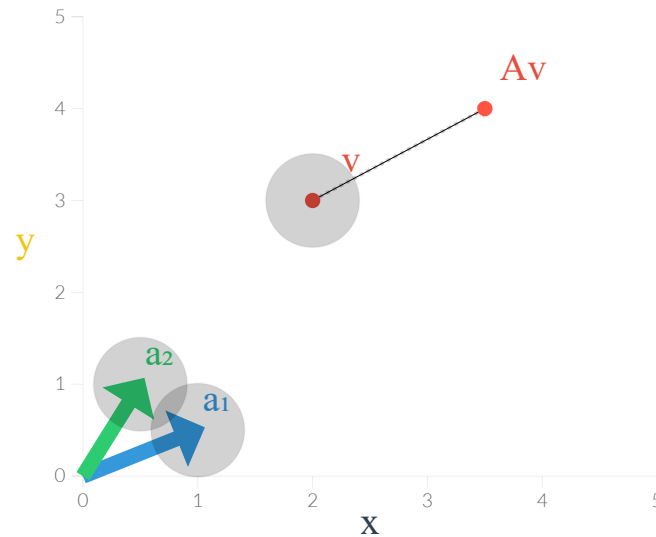
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By [Victor Powell](#) and [Lewis Lehe](#)

Eigenvalues/vectors are instrumental to understanding electrical circuits, mechanical systems, ecology and even Google's [PageRank](#) algorithm. Let's see if visualization can make these ideas more intuitive.

To begin, let \mathbf{v} be a vector (shown as a point) and \mathbf{A} be a matrix with columns \mathbf{a}_1 and \mathbf{a}_2 (shown as arrows). If we multiply \mathbf{v} by \mathbf{A} , then \mathbf{A} sends \mathbf{v} to a new vector \mathbf{Av} .



$$A = \begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \end{bmatrix} = \begin{bmatrix} 1.00 & 0.50 \\ 0.50 & 1.00 \end{bmatrix}$$

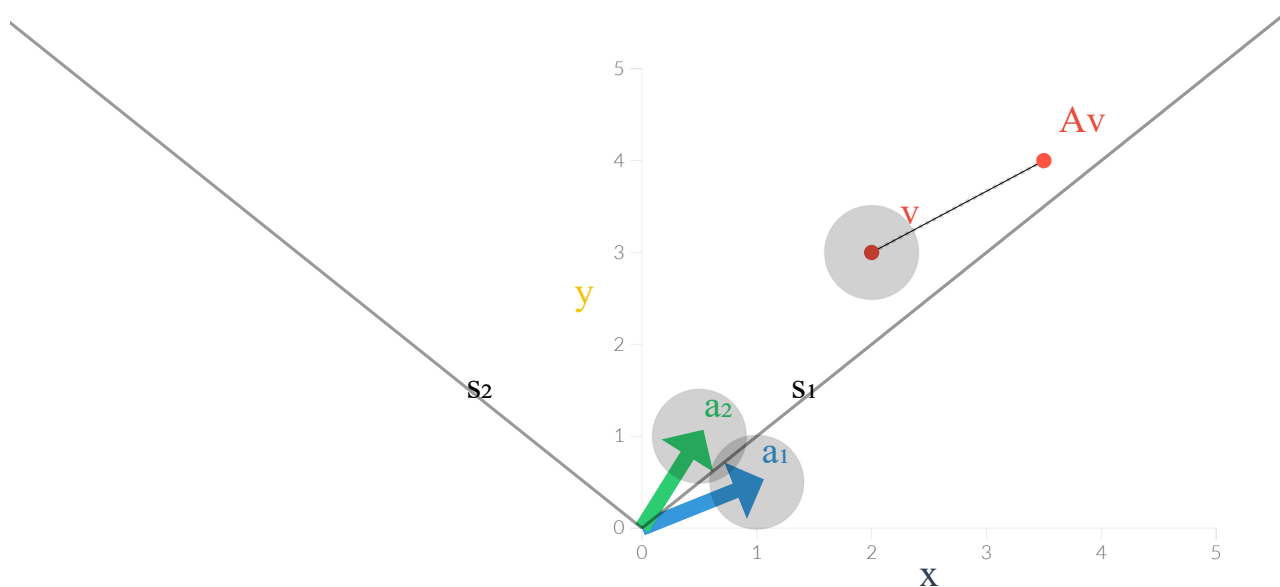
$$v = \begin{bmatrix} 2.00 \\ 3.00 \end{bmatrix}$$

$$Av = \begin{bmatrix} 3.50 \\ 4.00 \end{bmatrix}$$

If you can draw a line through the three points $(0,0)$, v and Av , then Av is just v multiplied by a number λ ; that is, $Av = \lambda v$. In this case, we call λ an **eigenvalue** and v an **eigenvector**. For example, here $(1, 2)$ is an eigenvector and 5 an eigenvalue.

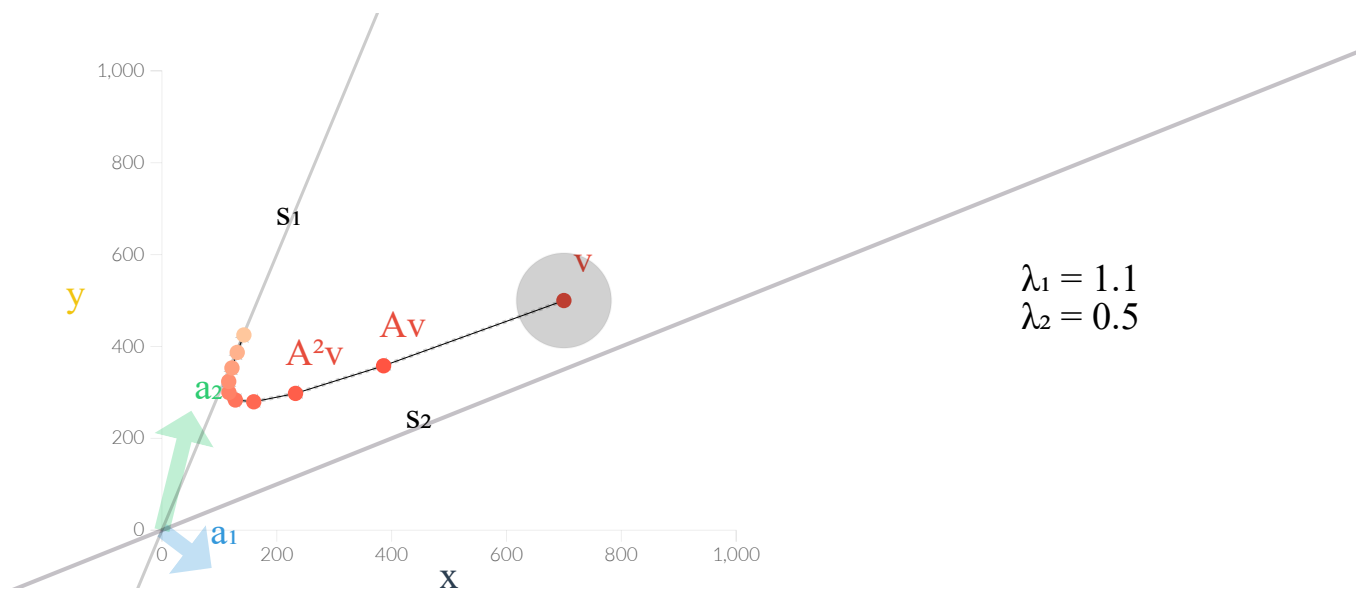
$$Av = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda v.$$

Below, change the columns of A and drag v to be an eigenvector. Note three facts: First, every point on the same line as an eigenvector is an eigenvector. Those lines are **eigenspaces**, and each has an associated eigenvalue. Second, if you place v on an eigenspace (either s_1 or s_2) with associated eigenvalue $\lambda < 1$, then Av is closer to $(0,0)$ than v ; but when $\lambda > 1$, it's farther. Third, both eigenspaces depend on both columns of A : it is not as though a_1 only affects s_1 .



What are eigenvalues/vectors good for?

If you keep multiplying v by A , you get a sequence v, Av, A^2v , etc. Eigenspaces attract that sequence and eigenvalues tell you whether it ends up at $(0,0)$ or far away. Therefore, eigenvectors/values tell us about systems that evolve step-by-step.



Let's explore some applications and properties of these sequences.

Fibonacci Sequence

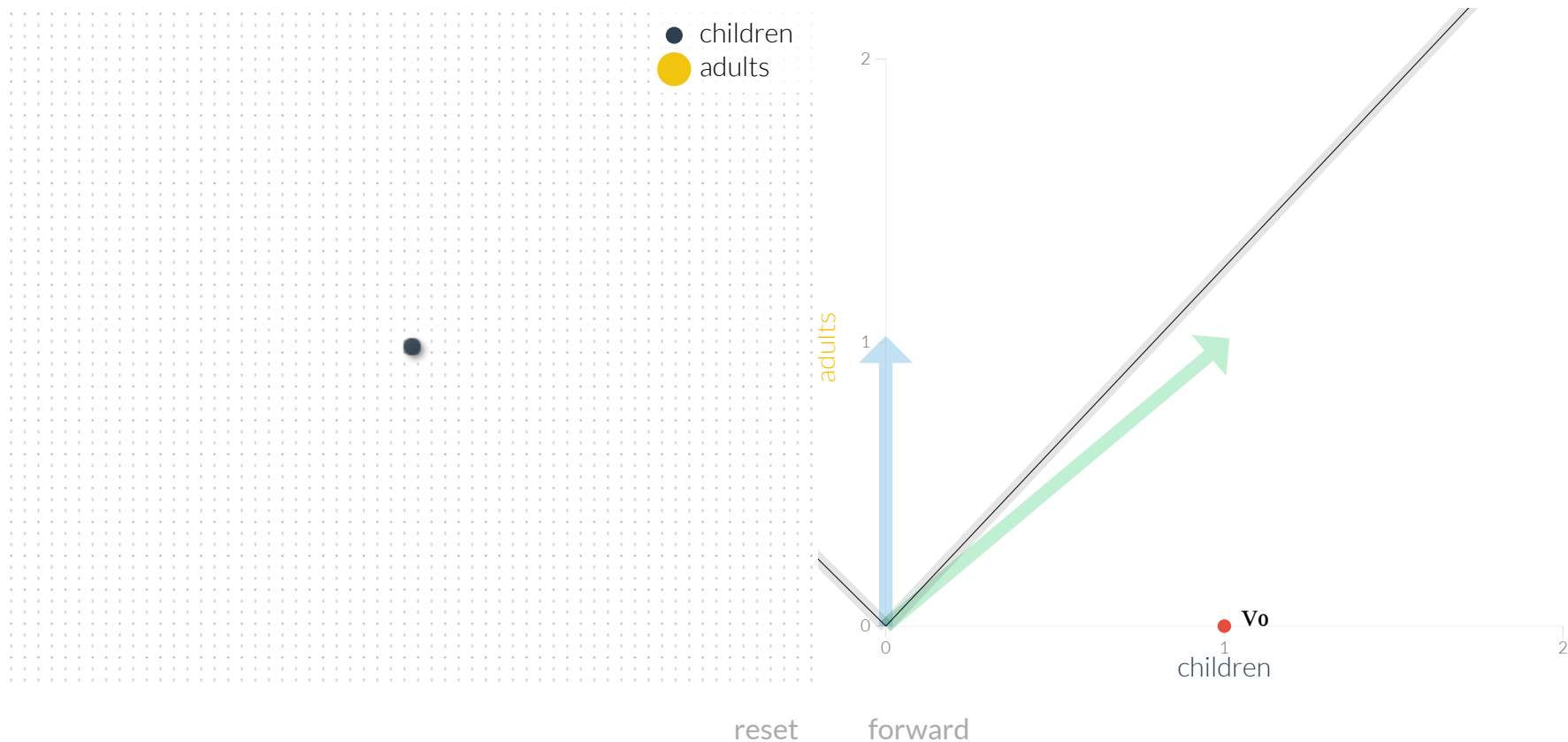
Suppose you have some amoebas in a petri dish. Every minute, all adult amoebas produce one child amoeba, and all child amoebas grow into adults (Note: this is not really how amoebas reproduce.). So if t is a minute, the equation of this system is

$$\begin{aligned}\text{adults}_{t+1} &= \text{adults}_t + \text{children}_t \\ \text{children}_{t+1} &= \text{adults}_t\end{aligned}$$

which we can rewrite in matrix form like

$$v_{t+1} = A \cdot v_t$$
$$\begin{pmatrix} \text{adults}_{t+1} \\ \text{children}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{adults}_t \\ \text{children}_t \end{pmatrix}$$

Below, press "Forward" to step ahead a minute. The total population is the [Fibonacci Sequence](#).



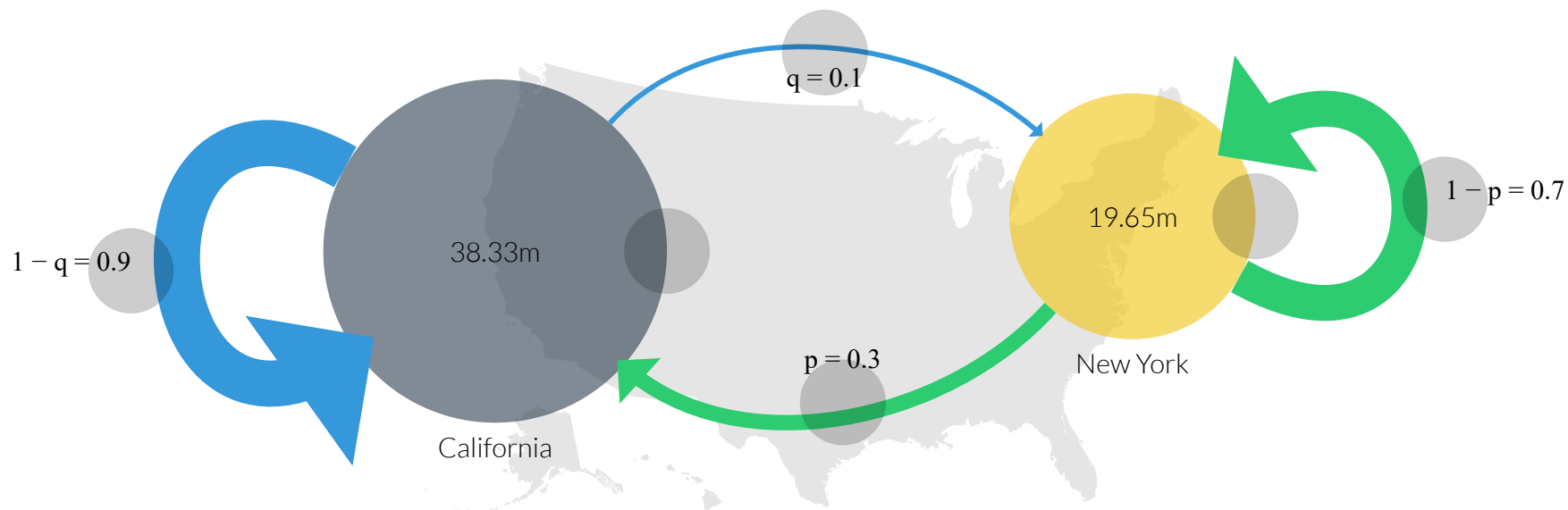
1 child + 0 adults = 1



As you can see, the system goes toward the grey line, which is an eigenspace with $\lambda = (1 + \sqrt{5})/2 > 1$.

Steady States

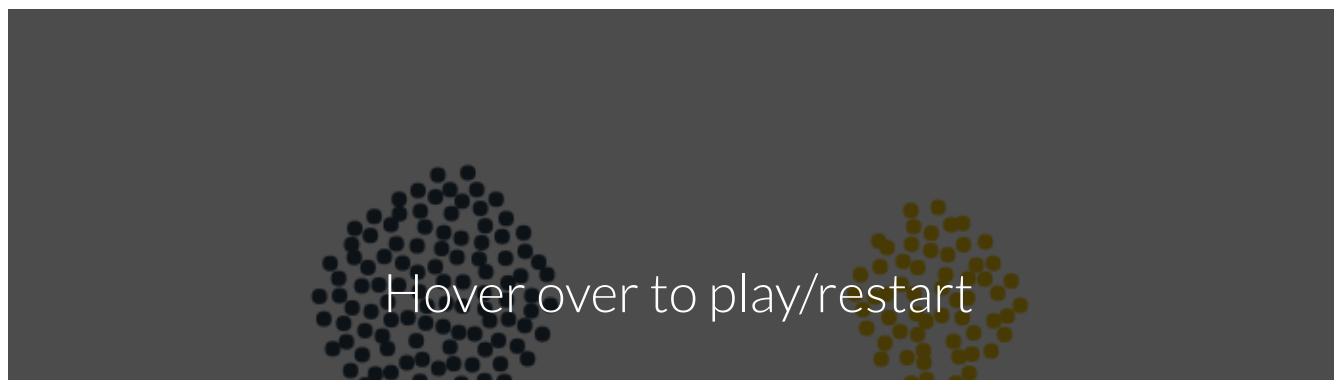
Suppose that, every year, a fraction p of New Yorkers move to California and a fraction q of Californians move to New York. Drag the circles to decide these fractions and the number starting in each state.

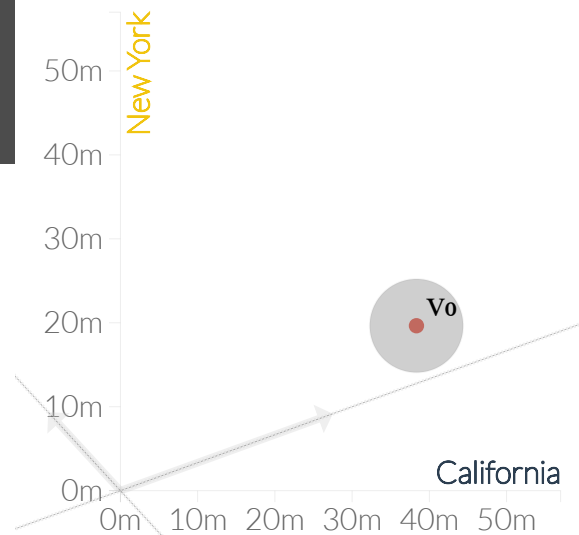


To understand the system better, we can start by writing it in matrix terms like:

$$\begin{pmatrix} \text{New York}_{t+1} \\ \text{California}_{t+1} \end{pmatrix} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \cdot \begin{pmatrix} \text{New York}_t \\ \text{California}_t \end{pmatrix}$$

It turns out that a matrix like A , whose entries are positive and whose columns add up to one (try it!), is called a [Markov matrix](#), and it always has $\lambda = 1$ as its largest eigenvalue. That means there's a value of v_t for which $Av_t = \lambda v_t = 1v_t = v_t$. At this "steady state," the same number of people move in each direction, and the populations stay the same forever. Hover over the animation to see the system go to the steady state.





For more on Markov matrices, check out our explanation of [Markov Chains](#).

Complex eigenvalues

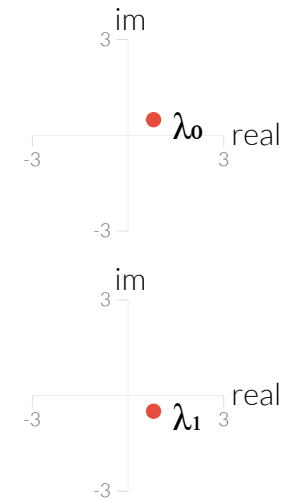
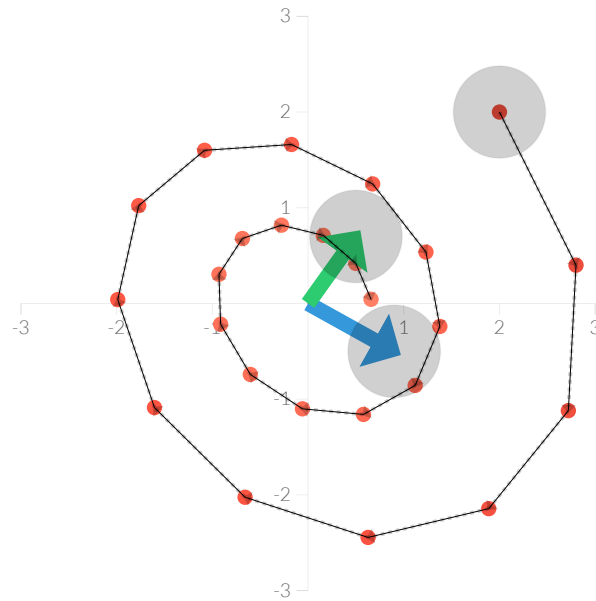
So far we've only looked at systems with real eigenvalues. But looking at the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, who's to say λ and \mathbf{v} can't have some imaginary part? That it can't be a [complex](#) number? For example,

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = (1 + i) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Here, $1 + i$ is an eigenvalue and $(1, i)$ is an eigenvector.

If a matrix has complex eigenvalues, its sequence spirals around $(0, 0)$. To see this, drag \mathbf{A} 's columns (the arrows) around until you get a spiral. The eigenvalues are plotted in the real/imaginary plane to the right. You'll see that whenever the eigenvalues have an imaginary part, the system spirals, no matter where you start things off.

steps:



Learning more

We've really only scratched the surface of what linear algebra is all about. To learn more, check out the legendary Gilbert Strang's [Linear Algebra](#) course at MIT's Open Courseware site. To get more practice with applications of eigenvalues/vectors, also check out the excellent [Differential Equations](#) course.

For more explanations, visit the Explained Visually [project homepage](#).

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Name



Harry Moreno • 3 years ago

the `complex eigenvalues` animation was not visible to me. Firefox osx


28 ^ | • Reply • Share ›



Leonardo Donelli → Harry Moreno • 3 years ago

Same. Firefox on Ubuntu 14.04


6 ^ | • Reply • Share ›



S.Nkm → Harry Moreno • 3 years ago

Same. Firefox / CentOS

1 ^ | • Reply • Share ›



evandrix → Harry Moreno • 3 years ago



use Chrome or similar WebKit-powered web browser.

1 ^ | v • Reply • Share ›



Artemiy ➔ Harry Moreno • 2 years ago

Same. Firefox 48, OpenSUSE Leap 42.1

^ | v • Reply • Share ›



Karol Mieczysław Marcjan ➔ Harry Moreno • 3 years ago

Same, Firefox on Arch Linux.

^ | v • Reply • Share ›



Pawnda ➔ Harry Moreno • 3 years ago

Same for me, Firefox on Manjaro.

^ | v • Reply • Share ›



Mark ➔ Harry Moreno • 3 years ago

Same here. FF 35.0.1 on Linux Mint 17.1.

^ | v • Reply • Share ›



cattly ➔ Mark • 2 years ago

Works for me - Chromium on Linux mint

^ | v • Reply • Share ›



LionessLover ➔ Harry Moreno • 3 years ago

It worked for me initially, but after a few seconds of playing with the complex values diagram the page crashed. Chrome Version 42.0.2311.135 m (64-bit) on Windows 8.1

EDIT: I think I've found that this happens when I drag the green and blue arrow end points in a way that makes the entire diagram HUGE. Possibly the x and y values become so large they lead to an overflow. It should not crash, but maybe guard against large x and/or y values in the code of the diagram?

^ | v • Reply • Share ›



kamil ➔ Harry Moreno • 2 years ago

Upgrade to Chome

^ | v • Reply • Share ›



cpsievert • 3 years ago

Great post! I think "a matrix like A, whose rows add up to *zero*" should be "a matrix like A, whose rows add up to *one*"

18 ^ | v • Reply • Share ›



Lewis → cpsievert • 3 years ago

yep good catch. we are going to push a batch of corrections.

1 ^ | v • Reply • Share ›



JohnThackr → Lewis • 3 years ago

Also, it should be "columns add up to 1," and the matrix A should be transposed so that the first row is $(1-p \ q)$, since $NY_{t+1} = (1-p)NY_t + qCA_t$

Alternatively, you can leave A like it is, but apply it via right-multiply to the row vector $(NY_t \ CA_t)$. But as written it is incorrect.

Love the work and animation, though.

11 ^ | v • Reply • Share ›



Dan Warren → JohnThackr • 3 years ago

Thanks for this, that was confusing the hell out of me. Absolutely love the demonstration otherwise!

2 ^ | v • Reply • Share ›



evansenter → JohnThackr • 3 years ago

Yep, came to point out both. Great article.

^ | v • Reply • Share ›



Donny V. • 3 years ago

I love the way this is put together! If you know anything about neural networks in computing I would love to see it explained in this format. Especially the "Whats this good for" section.

8 ^ | v • Reply • Share ›

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Lewis • 3 years ago

I am trying to get better at writing concise text. Is there any part of the text that is very unclear to any of you?

8 ^ | v • Reply • Share ›



Eyelet → Lewis • 3 years ago

Great post. It would be nice if you explained what λ_1 and λ_2 is. It first shows up in the 2nd graph.

3 ^ | v • Reply • Share ›



yachris → Lewis • 3 years ago

Excellent start -- but the third paragraph (starting with "If you can draw a line through...") is too terse... as the visualization is set up, you *can't* draw a line between $0,0$, v and Av (and it's not until later that it's made clear that you can drag points). The next paragraph (starting with "Below, change the bases of A ...") is not clear at all. You're assuming we know what "change the bases of A " means, so it'd be nice to either explain that or link to another page explaining that. Thanks for doing this!

3 ^ | v • Reply • Share ›



Im So Meta Even This Acronym → Lewis • 3 years ago

Intro is a bit confusing. What's λ , and why is it there? Why do I need it? The example equation below it is also confusing (I understand little math notation).

It's still not clear why is it great that I know two coinciding vectors are eigenvectors. They have a name, that's great, but anything more?

In the "What are eigenvalues/vectors good for?" section the first graph is nice, but I don't understand the curved thing. I thought we are talking about straight lines :)

The examples you gave are really pretty, but they made me less sure if I understand eigenvector/spaces/numbers :)

In one word, these tools are only good for visualization?

^ | v • Reply • Share ›



diego898 → Lewis • 3 years ago

Hello! I posted a separate comment but figured I'd just reply to this one. First and foremost, fantastic job both of you! We had some

people in the lab that requested clarification on the role of the "S" arrows, the "gray" lines. Thanks again!

^ | v • Reply • Share ›



Lewis → diego898 • 3 years ago

okay those are supposed to represent the eigenspaces. I'll update the text with a reference in a bit. thanks.

1 ^ | v • Reply • Share ›



dont get it • 3 years ago

Don't get it.

in steady states that would mean: new york $t+1$ = new york t - (fraction of new yourkers moving to california * new yourk) + (fraction of new yourkers moving to california * california)

That does not make sence to me. I would have expected the last part beeing (fraction of calironiars moving to new york * california)

5 ^ | v • Reply • Share ›



Iwan Satria • 3 years ago

Apologies if my understanding is wrong, but for the New Yorker - Californian matrix, shouldn't it be like this?

$(1-p \ q)$

$(p \ 1-q)$

meaning:

$NewYorkers(t+1) = (1-p)NewYorkers(t) + (q)Californians(t)$

$Californians(t+1) = (p)NewYorkers(t) + (1-q)Californians(t)$

4 ^ | v • Reply • Share ›



Karl Frost → Iwan Satria • 3 years ago

i was about to say the same thing

1 ^ | v • Reply • Share ›



Kenny SF • 3 years ago

What frameworks do you use to create these interactive visualizations?

4 ^ | v • Reply • Share ›



Koen Boncquet → Kenny SF • 3 years ago

Based on a quick inspection of the source code you can see that they are using the library d3.js for interactive visualisations

<http://d3js.org/>

1 ^ | v • Reply • Share ›



newguy • 3 years ago

isn't the nyc and ca matrix supposed to be $[1-p, q]$? (not $[1-p, p]$?) NYC keeps $1-p$, but then gets q times CA. Currently it's say that the probability p of someone leaving nyc, times the pop of CA, becomes a NYC resident. which is nonsensical.

3 ^ | v • Reply • Share ›



rickasaurus • 3 years ago

Crashing my browser with Eigenvalues (That last visualization did it when I dragged λ_1 to have a negative y component). Great post though, it may be the best explanation of eigenvectors that I've ever seen.

2 ^ | v • Reply • Share ›

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Ellie Kesselman • 3 years ago

Thank you for that PDF, FinalVersionFixed. It is great!

1 ^ | v • Reply • Share ›



Jose F Rodrigues Jr • 3 years ago

Nice!

1 ^ | v • Reply • Share ›



Santosh • 3 years ago

Amazing page, I wish I had this in college.

1 ^ | v • Reply • Share ›



Sushant Srivastava • 3 years ago



Excellent.

1 ^ | v • Reply • Share ›



biomedicalblog • 3 years ago

Please make more! Interesting topics for this format may be: PCA, Singular Value Decomposition, Neural Networks

1 ^ | v • Reply • Share ›



Peter • 3 years ago

This is fantastic. Each time I jump disciplines, I see a new explanation for eigenvectors, eigenvalues, and singular values. Each time, I learn a new way to think about it. I wish there was something a little more advanced than Strang's course -- something with a dozen explanations like this one that I could work through. From dynamics to statistics to image compression to machine learning to all the other applications....

1 ^ | v • Reply • Share ›

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Neven Sajko • 3 years ago

The last interactive graph causes Chromium to crash when the eigenvector goes too far offscreen. To reproduce, move the green base to negative-x (left), and increase the number of steps.

1 ^ | v • Reply • Share ›



TheSisb • 3 years ago

This is really excellent. Thank you.

1 ^ | v • Reply • Share ›



einsteinisgaurav • 3 years ago

I would love to see algorithms like simulated annealing, genetic algorithm explained visually. Awesome work!

1 ^ | v • Reply • Share ›

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172 173 174 175 176 177 178 179 180 181 182 183 184 185 186 187 188 189 190 191 192 193 194 195 196 197 198 199 200 201 202 203 204 205 206 207 208 209 210 211 212 213 214 215 216 217 218 219 220 221 222 223 224 225 226 227 228 229 230 231 232 233 234 235 236 237 238 239 240 241 242 243 244 245 246 247 248 249 250 251 252 253 254 255 256 257 258 259 260 261 262 263 264 265 266 267 268 269 270 271 272 273 274 275 276 277 278 279 280 281 282 283 284 285 286 287 288 289 290 291 292 293 294 295 296 297 298 299 300 301 302 303 304 305 306 307 308 309 310 311 312 313 314 315 316 317 318 319 320 321 322 323 324 325 326 327 328 329 330 331 332 333 334 335 336 337 338 339 340 341 342 343 344 345 346 347 348 349 350 351 352 353 354 355 356 357 358 359 360 361 362 363 364 365 366 367 368 369 370 371 372 373 374 375 376 377 378 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 408 409 410 411 412 413 414 415 416 417 418 419 420 421 422 423 424 425 426 427 428 429 430 431 432 433 434 435 436 437 438 439 440 441 442 443 444 445 446 447 448 449 450 451 452 453 454 455 456 457 458 459 460 461 462 463 464 465 466 467 468 469 470 471 472 473 474 475 476 477 478 479 480 481 482 483 484 485 486 487 488 489 490 491 492 493 494 495 496 497 498 499 500 501 502 503 504 505 506 507 508 509 510 511 512 513 514 515 516 517 518 519 520 521 522 523 524 525 526 527 528 529 530 531 532 533 534 535 536 537 538 539 540 541 542 543 544 545 546 547 548 549 550 551 552 553 554 555 556 557 558 559 560 561 562 563 564 565 566 567 568 569 570 571 572 573 574 575 576 577 578 579 580 581 582 583 584 585 586 587 588 589 590 591 592 593 594 595 596 597 598 599 600 601 602 603 604 605 606 607 608 609 610 611 612 613 614 615 616 617 618 619 620 621 622 623 624 625 626 627 628 629 630 631 632 633 634 635 636 637 638 639 640 641 642 643 644 645 646 647 648 649 650 651 652 653 654 655 656 657 658 659 660 661 662 663 664 665 666 667 668 669 670 671 672 673 674 675 676 677 678 679 680 681 682 683 684 685 686 687 688 689 690 691 692 693 694 695 696 697 698 699 700 701 702 703 704 705 706 707 708 709 710 711 712 713 714 715 716 717 718 719 720 721 722 723 724 725 726 727 728 729 730 731 732 733 734 735 736 737 738 739 740 741 742 743 744 745 746 747 748 749 750 751 752 753 754 755 756 757 758 759 760 761 762 763 764 765 766 767 768 769 770 771 772 773 774 775 776 777 778 779 780 781 782 783 784 785 786 787 788 789 790 791 792 793 794 795 796 797 798 799 800 801 802 803 804 805 806 807 808 809 810 811 812 813 814 815 816 817 818 819 820 821 822 823 824 825 826 827 828 829 830 831 832 833 834 835 836 837 838 839 840 841 842 843 844 845 846 847 848 849 850 851 852 853 854 855 856 857 858 859 860 861 862 863 864 865 866 867 868 869 870 871 872 873 874 875 876 877 878 879 880 881 882 883 884 885 886 887 888 889 890 891 892 893 894 895 896 897 898 899 900 901 902 903 904 905 906 907 908 909 910 911 912 913 914 915 916 917 918 919 920 921 922 923 924 925 926 927 928 929 930 931 932 933 934 935 936 937 938 939 940 941 942 943 944 945 946 947 948 949 950 951 952 953 954 955 956 957 958 959 960 961 962 963 964 965 966 967 968 969 970 971 972 973 974 975 976 977 978 979 980 981 982 983 984 985 986 987 988 989 990 991 992 993 994 995 996 997 998 999 1000 1001 1002 1003 1004 1005 1006 1007 1008 1009 1010 1011 1012 1013 1014 1015 1016 1017 1018 1019 1020 1021 1022 1023 1024 1025 1026 1027 1028 1029 1030 1031 1032 1033 1034 1035 1036 1037 1038 1039 1040 1



Flávia Rius • a month ago

Thank you very much for that, it was very didactic!

I found here searching about eigenvalues and eigenvectors to understand better the Principal Component Analysis.
It is clearer for me now :)

^ | v • Reply • Share ›



John Singleton • a month ago

Typo - great write up:

"We've really only scratched...also ceck out the excellent Differential Equations course."

^ | v • Reply • Share ›



Bruce Lester • a month ago

This is beautiful work.

^ | v • Reply • Share ›

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Shivanshu Raj • 2 months ago

for $A = \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}$ i calculated eigen values on hands..there only exist one eigen value for this linear operator but your animation show two eigen values each 1.00 what is this ...what kind of eigen values you are showing here...and also for two lamda there exist two eigen vectors but you show only one what kind of eigen vector are you presenting here???!!!!

^ | v • Reply • Share ›



Ritchie Vink • 5 months ago

Really nice visual explanation!



Really nice visual explanations:

^ | v • Reply • Share ›



Dan Murphy • 6 months ago

Thanks for this! Really well-done and helpful.

^ | v • Reply • Share ›



Sven Yurik • 8 months ago

Those are some AMAZING animations! How did you do them??

^ | v • Reply • Share ›



Subir Das • 9 months ago

Brilliant work. Congratulations.

^ | v • Reply • Share ›



Leonard Ikojo • a year ago

Thank you. Just the crash course I needed.

^ | v • Reply • Share ›

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Pirouz Nourian • a year ago

Thanks for the great explanation!

^ | v • Reply • Share ›



Nanda Kishore Rajanala • a year ago




Reading this post. I couldn't figure out the practical applications of this concept. How does Google page rank leverage this? If we are talking



about population movements in the real world, can we always assume that they move towards the steady state due to the eigen value ~ 1 ? I couldn't figure what to make of the spirals either? can someone help pls?

^ | v • Reply • Share ›

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