

# REPRESENTATION THEORY I

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# 1 Basic Definitions

**Definition.** A  $k$ -Lie algebra is a vector space  $\mathfrak{g}$  (over some field  $k$ ) together with a  $k$ -bilinear map

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying the following:

1.  $[\cdot, \cdot]$  is *alternating*, i.e.  $[x, x] = 0$  for every  $x \in \mathfrak{g}$ .
2. The *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

$[\cdot, \cdot]$  is called a *Lie bracket*.

**Remark.**  $[\cdot, \cdot]$  is antisymmetric, i.e.  $[y, x] = -[x, y]$  for all  $x, y \in \mathfrak{g}$ , because

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

**Definition.** Let  $A$  be a  $k$ -algebra. A *derivation of  $A$*  is a  $k$ -linear map  $d: A \rightarrow A$  such that

$$d(ab) = d(a)b + ad(b) \quad \text{für alle } a, b \in A.$$

We set

$$\text{Der}(A) := \{d: A \rightarrow A \mid d \text{ is a derivation of } A\}.$$

**Remark.**  $\text{Der}(A)$  is clearly a  $k$ -vector space.

**Examples.** 1. Any vector space  $V$  becomes a Lie algebra via

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

2. Any associative  $k$ -algebra  $A$  becomes a Lie algebra via

$$[a, b] = ab - ba \quad \text{for all } a, b \in A.$$

It is clear that  $[\cdot, \cdot]$  is alternating and the Jacobi identity can be verified by some easy calculation.

In particular  $M_n(k)$  is a Lie algebra via

$$[A, B] = AB - BA \quad \text{for all } A, B \in M_n(k).$$

This is called the *general linear Lie algebra* and is denoted by  $\mathfrak{gl}_n(k)$  or  $\mathfrak{gl}(n, k)$ .

More generally for any  $k$ -vector space the space  $\text{End}_k(V)$  becomes a Lie algebra via

$$[\varphi_1, \varphi_2] := \varphi_1 \circ \varphi_2 - \varphi_2 \circ \varphi_1 \quad \text{for all } \varphi_1, \varphi_2 \in \text{End}_k(V).$$

This is called the *general linear Lie algebra for  $V$*  and is denoted by  $\mathfrak{gl}(V)$ .

3. Let  $A$  be a  $k$ -algebra.  $\text{Der}(A)$  is a Lie algebra via

$$[d, d'] := d \circ d' - d' \circ d \quad \text{for all } d, d' \in \text{Der}(A).$$

It is an easy calculation to show that  $[d, d']$  is again a derivation. Notice that the Jacobi identity for  $\text{Der}(A)$  follows from the Jacobi identity for  $\mathfrak{gl}(A)$ .

We will now look at how to construct new Lie algebras from old ones.

**Definition.** Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be Lie algebras over the same field  $k$ . Then the *product* of  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  is defined as the  $k$ -vector space  $\mathfrak{g}_1 \times \mathfrak{g}_2$  together with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]) \quad \text{for all } (x_1, y_1), (x_2, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

Let  $\mathfrak{g}$  be a Lie algebra over  $k$  and  $A$  an associative, commutative  $k$ -algebra. Then  $\mathfrak{g} \otimes_k A$  is a Lie algebra via

$$[x \otimes a, y \otimes b] = [x, y] \otimes (ab) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

**Definition.** Let  $\mathfrak{g}$  be a Lie algebra and  $A = k[t, t^{-1}]$  be the algebra of Laurent polynomials over  $k$ . Then

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_k A$$

with the Lie bracket as above is called the *loop (Lie) algebra* of  $\mathfrak{g}$ .

Another example of construction new Lie algebras from old ones are *central extensions*: Let  $\mathfrak{g}$  be any  $k$ -Lie algebra.

$$\tilde{\mathfrak{g}} := \mathfrak{g} \otimes k = \{x + \lambda c \mid x \in \mathfrak{g}, \lambda \in k\},$$

where we understand  $c$  as a formal variable. Suppose that  $\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow k$  is a  $k$ -bilinear map satisfying the following properties:

1.  $\kappa$  is antisymmetric, i.e.  $\kappa(x, y) = -\kappa(y, x)$  for all  $x, y \in \mathfrak{g}$ .
2. The 2-cocycle condition

$$\kappa([x, y], z) + \kappa([y, z], x) + \kappa([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then  $\tilde{\mathfrak{g}}$  becomes a Lie algebra via

$$[x + \lambda c, y + \mu c] := [x, y] + \kappa(x, y)\lambda\mu c.$$

Note that  $c$  is central in  $\tilde{\mathfrak{g}}$  in the sense that  $[x, c] = 0$  for all  $x \in \mathfrak{g}$ .

**Example.** Let  $\mathfrak{g} = \mathfrak{gl}_n(k)$ . We define a symmetric bilinear form on  $\mathfrak{g}$  via

$$(A, B)_{\text{tr}} = \text{tr}(AB).$$

We define a bilinear form

$$\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow k[t, t^{-1}], (x \otimes p, y \otimes q) \mapsto (x, y)_{\text{tr}} pq$$

We now get a 2-cocycle  $\kappa: \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow k$  via

$$\kappa(a, b) := \text{Res} \left( \frac{\partial a}{\partial t}, b \right).$$

$\kappa$  is also antisymmetric: Let  $a = x \otimes t^i$  and  $b = y \otimes t^j$  with  $x, y \in \mathfrak{g}$  and  $i, j \in \mathbb{Z}$ . Then

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \text{Res}(ix \otimes t^{i-1}, y \otimes t^j) = \text{Res}(it^{i+j-1}(x, y)_{\text{tr}}) \\ &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the same way we find that

$$\kappa(y \otimes t^j, x \otimes t^i) = \begin{cases} j(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $(\cdot, \cdot)_{\text{tr}}$  is symmetric we find that

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -j(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} = -\kappa(y \otimes t^j, x \otimes t^i). \end{aligned}$$

As for all algebraic structures morphisms are of interest.

**Definition.** Given  $k$ -Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  a homomorphism of Lie algebras  $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a  $k$ -linear map  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  such that

$$f([x, y]) = [f(x), f(y)] \quad \text{for all } x, y \in \mathfrak{g}_1.$$

**Examples.** 1. For any Lie algebra  $\mathfrak{g}$  the identity  $\text{id}_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathfrak{g}$  is a Lie algebra homomorphism.

2. Given Lie algebras  $\mathfrak{g}_1, \mathfrak{g}_2$  and  $\mathfrak{g}_3$  and Lie algebra homomorphisms  $f_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  and  $f_2: \mathfrak{g}_2 \rightarrow \mathfrak{g}_3$  the composition  $f_2 \circ f_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_3$  is also a homomorphism of Lie algebras.

**Remark.** We find that we have a category of  $k$ -Lie algebras.

**Definition.** Let  $\mathfrak{g}$  be a  $k$ -Lie algebra. A representation of  $\mathfrak{g}$  is a  $k$ -vector space  $V$  together with a homomorphism of Lie algebras  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ .

**Remark.** Equivalently a representation of  $\mathfrak{g}$  is a  $k$ -vector space  $V$  together with a  $k$ -bilinear map  $\mathfrak{g} \times V \rightarrow V, (x, v) \mapsto x.v$  such that

$$x.y.v - y.x.v = [x, y].v \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V.$$

**Definition.** Let  $\mathfrak{g}$  be a Lie algebra. Then

$$\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), x \mapsto [x, \cdot]$$

is called the *adjoint representation* of  $\mathfrak{g}$ .

**Remark.** That  $\text{ad}$  is a homomorphism of Lie algebras is equivalent to

$$\text{ad}([x, y])(z) = [\text{ad}(x), \text{ad}(y)](z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Because

$$\text{ad}([x, y])(z) = [[x, y], z] = -[z, [x, y]]$$

and

$$\begin{aligned} [\text{ad}(x), \text{ad}(y)](z) &= (\text{ad}(x) \circ \text{ad}(y))(z) - (\text{ad}(y) \circ \text{ad}(x))(z) \\ &= [x, [y, z]] - [y, [x, z]] = [x, [y, z]] + [y, [z, x]] \end{aligned}$$

this is equivalent to the Jacobi identity.

**Definition.** Let  $\mathfrak{g}$  be a  $k$ -Lie algebra.

A *Lie subalgebra* of a  $\mathfrak{g}$  is a  $k$ -linear subspace  $L \subseteq \mathfrak{g}$  such that  $[x, y] \in L$  for all  $x, y \in L$ .

An *ideal* inside  $\mathfrak{g}$  is a  $k$ -linear subspace  $I \subseteq \mathfrak{g}$  such that  $[x, y] \in I$  for all  $x \in \mathfrak{g}$  and  $y \in I$ . We denote ideals by  $I \triangleleft \mathfrak{g}$ .

Notice that any ideal is also a Lie subalgebra.

**Remark.** If  $\mathfrak{g}$  is a Lie algebra and  $I \triangleleft \mathfrak{g}$  then the quotient vector space  $\mathfrak{g}/I$  is also a Lie algebra via

$$[x + I, y + I] = [x, y] + I.$$

It is easy to see that this bilinear map is well-defined. The properties for the Lie bracket on  $\mathfrak{g}/I$  follow from the one for the Lie bracket on  $\mathfrak{g}$ .

From the definition of the Lie bracket on  $\mathfrak{g}/I$  it is clear that the canonical projection  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I$  is a homomorphism of Lie algebras.

We have the usual theorems about homomorphisms and ideals.

**Proposition 1.** Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be Lie algebras and  $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  a homomorphism of Lie algebras.

1.  $\ker f \triangleleft \mathfrak{g}_1$ .
2.  $\text{im } f$  is a Lie subalgebra.
3. If  $I \triangleleft \mathfrak{g}_1$  such that  $\ker f \subseteq I$  then there exists a unique Lie algebra homomorphism  $\hat{f}: \mathfrak{g}_1/I \rightarrow \mathfrak{g}_2$  with  $f = \hat{f} \circ \pi$  where  $\pi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/I$  is the canonical projection.

$$\begin{array}{ccc} \mathfrak{g}_1 & & \\ \pi \downarrow & \searrow f & \\ \mathfrak{g}_1/I & \xrightarrow{\hat{f}} & \mathfrak{g}_2 \end{array}$$