

REPRESENTATION THEORY I

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Definition. A k -Lie algebra is a vector space \mathfrak{g} (over some field k) together with a k -bilinear map

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying the following:

1. $[\cdot, \cdot]$ is *alternating*, i.e. $[x, x] = 0$ for every $x \in \mathfrak{g}$.
2. The *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

$[\cdot, \cdot]$ is called a *Lie bracket*.

Remark. $[\cdot, \cdot]$ is antisymmetric, i.e. $[y, x] = -[x, y]$ for all $x, y \in \mathfrak{g}$, because

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Definition. Let A be a k -algebra. A *derivation of A* is a k -linear map $d: A \rightarrow A$ such that

$$d(ab) = d(a)b + ad(b) \quad \text{for all } a, b \in A.$$

We set

$$\text{Der}(A) := \{d: A \rightarrow A \mid d \text{ is a derivation of } A\}.$$

Remark. $\text{Der}(A)$ is clearly a k -vector space.

Examples. 1. Any vector space V becomes a Lie algebra via

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

2. Any associative k -algebra A becomes a Lie algebra via

$$[a, b] = ab - ba \quad \text{for all } a, b \in A.$$

It is clear that $[\cdot, \cdot]$ is alternating and the Jacobi identity can be verified by some easy calculation.

In particular $M_n(k)$ is a Lie algebra via

$$[A, B] = AB - BA \quad \text{for all } A, B \in M_n(k).$$

This is called the *general linear Lie algebra* and is denoted by $\mathfrak{gl}_n(k)$ or $\mathfrak{gl}(n, k)$.

More generally for any k -vector space the space $\text{End}_k(V)$ becomes a Lie algebra via

$$[\varphi_1, \varphi_2] := \varphi_1 \circ \varphi_2 - \varphi_2 \circ \varphi_1 \quad \text{for all } \varphi_1, \varphi_2 \in \text{End}_k(V).$$

This is called the *general linear Lie algebra for V* and is denoted by $\mathfrak{gl}(V)$.

3. Let A be a k -algebra. $\text{Der}(A)$ is a Lie algebra via

$$[d, d'] := d \circ d' - d' \circ d \quad \text{for all } d, d' \in \text{Der}(A).$$

It is an easy calculation to show that $[d, d']$ is again a derivation. Notice that the Jacobi identity for $\text{Der}(A)$ follows from the Jacobi identity for $\mathfrak{gl}(A)$.

We will now look at how to construct new Lie algebras from old ones.

Definition. Let \mathfrak{g}_1 and \mathfrak{g}_2 be Lie algebras over the same field k . Then the *product* of \mathfrak{g}_1 and \mathfrak{g}_2 is defined as the k -vector space $\mathfrak{g}_1 \times \mathfrak{g}_2$ together with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]) \quad \text{for all } (x_1, y_1), (x_2, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

Let \mathfrak{g} be a Lie algebra over k and A an associative, commutative k -algebra. Then $\mathfrak{g} \otimes_k A$ is a Lie algebra via

$$[x \otimes a, y \otimes b] = [x, y] \otimes (ab) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

Definition. Let \mathfrak{g} be a Lie algebra and $A = k[t, t^{-1}]$ be the algebra of Laurent polynomials over k . Then

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_k A$$

with the Lie bracket as above is called the *loop (Lie) algebra* of \mathfrak{g} .

Another example of construction new Lie algebras from old ones are *central extensions*: Let \mathfrak{g} be any k -Lie algebra.

$$\tilde{\mathfrak{g}} := \mathfrak{g} \otimes k = \{x + \lambda c \mid x \in \mathfrak{g}, \lambda \in k\},$$

where we understand c as a formal variable. Suppose that $\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow k$ is a k -bilinear map satisfying the following properties:

1. κ is antisymmetric, i.e. $\kappa(x, y) = -\kappa(y, x)$ for all $x, y \in \mathfrak{g}$.
2. The 2-cocycle condition

$$\kappa([x, y], z) + \kappa([y, z], x) + \kappa([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then $\tilde{\mathfrak{g}}$ becomes a Lie algebra via

$$[x + \lambda c, y + \mu c] := [x, y] + \kappa(x, y)\lambda\mu c.$$

Note that c is central in $\tilde{\mathfrak{g}}$ in the sense that $[x, c] = 0$ for all $x \in \mathfrak{g}$.

Example. Let $\mathfrak{g} = \mathfrak{gl}_n(k)$. We define a symmetric bilinear form on \mathfrak{g} via

$$(A, B)_{\text{tr}} = \text{tr}(AB).$$

We define a bilinear form

$$\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow k[t, t^{-1}], (x \otimes p, y \otimes q) \mapsto (x, y)_{\text{tr}} pq$$

We now get a 2-cocycle $\kappa: \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \rightarrow k$ via

$$\kappa(a, b) := \text{Res} \left(\frac{\partial a}{\partial t}, b \right).$$

κ is also antisymmetric: Let $a = x \otimes t^i$ and $b = y \otimes t^j$ with $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$. Then

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \text{Res}(ix \otimes t^{i-1}, y \otimes t^j) = \text{Res}(it^{i+j-1}(x, y)_{\text{tr}}) \\ &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In the same way we find that

$$\kappa(y \otimes t^j, x \otimes t^i) = \begin{cases} j(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(\cdot, \cdot)_{\text{tr}}$ is symmetric we find that

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -j(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= -\kappa(y \otimes t^j, x \otimes t^i). \end{aligned}$$

As for all algebraic structures morphisms are of interest.

Definition. Given k -Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 a homomorphism of Lie algebras $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a k -linear map $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that

$$f([x, y]) = [f(x), f(y)] \quad \text{for all } x, y \in \mathfrak{g}_1.$$

Examples. 1. For any Lie algebra \mathfrak{g} the identity $\text{id}_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra homomorphism.

2. Given Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{g}_3 and Lie algebra homomorphisms $f_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ and $f_2: \mathfrak{g}_2 \rightarrow \mathfrak{g}_3$ the composition $f_2 \circ f_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_3$ is also a homomorphism of Lie algebras.

Remark. We find that we have a category of k -Lie algebras.

Definition. Let \mathfrak{g} be a k -Lie algebra. A representation of \mathfrak{g} is a k -vector space V together with a homomorphism of Lie algebras $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$.

Remark. Equivalently a representation of \mathfrak{g} is a k -vector space V together with a k -bilinear map $\mathfrak{g} \times V \rightarrow V, (x, v) \mapsto x.v$ such that

$$x.y.v - y.x.v = [x, y].v \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V.$$

Definition. Let \mathfrak{g} be a Lie algebra. Then

$$\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}), x \mapsto [x, \cdot]$$

is called the *adjoint representation* of \mathfrak{g} .

Remark. That ad is a homomorphism of Lie algebras is equivalent to

$$\text{ad}([x, y])(z) = [\text{ad}(x), \text{ad}(y)](z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Because

$$\text{ad}([x, y])(z) = [[x, y], z] = -[z, [x, y]]$$

and

$$\begin{aligned} [\text{ad}(x), \text{ad}(y)](z) &= (\text{ad}(x) \circ \text{ad}(y))(z) - (\text{ad}(y) \circ \text{ad}(x))(z) \\ &= [x, [y, z]] - [y, [x, z]] = [x, [y, z]] + [y, [z, x]] \end{aligned}$$

this is equivalent to the Jacobi identity.

Definition. Let \mathfrak{g} be a k -Lie algebra.

A *Lie subalgebra* of a \mathfrak{g} is a k -linear subspace $L \subseteq \mathfrak{g}$ such that $[x, y] \in L$ for all $x, y \in L$.

An *ideal* inside \mathfrak{g} is a k -linear subspace $I \subseteq \mathfrak{g}$ such that $[x, y] \in I$ for all $x \in \mathfrak{g}$ and $y \in I$. We denote ideals by $I \triangleleft \mathfrak{g}$.

Notice that any ideal is also a Lie subalgebra.

Remark. If \mathfrak{g} is a Lie algebra and $I \triangleleft \mathfrak{g}$ then the quotient vector space \mathfrak{g}/I is also a Lie algebra via

$$[x + I, y + I] = [x, y] + I.$$

It is easy to see that this bilinear map is well-defined. The properties for the Lie bracket on \mathfrak{g}/I follow from the one for the Lie bracket on \mathfrak{g} .

From the definition of the Lie bracket on \mathfrak{g}/I it is clear that the canonical projection $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/I$ is a homomorphism of Lie algebras.

We have the usual theorems about homomorphisms and ideals.

Proposition 1. Let \mathfrak{g}_1 and \mathfrak{g}_2 be Lie algebras and $f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ a homomorphism of Lie algebras.

1. $\ker f \triangleleft \mathfrak{g}_1$.
2. $\operatorname{im} f$ is a Lie subalgebra.
3. If $I \triangleleft \mathfrak{g}_1$ such that $\ker f \subseteq I$ then there exists a unique Lie algebra homomorphism $\hat{f}: \mathfrak{g}_1/I \rightarrow \mathfrak{g}_2$ with $f = \hat{f} \circ \pi$ where $\pi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1/I$ is the canonical projection.

$$\begin{array}{ccc} \mathfrak{g}_1 & & \\ \pi \downarrow & \searrow f & \\ \mathfrak{g}_1/I & \xrightarrow{\hat{f}} & \mathfrak{g}_2 \end{array}$$