# Representation Theory I

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# 1 Basic Definitions

**Definition.** A *$k$-Lie algebra* is a vector space $\mathfrak{g}$ (over some field $k$) together with a $k$-bilinear map

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

satisfying the following:

1. $[\cdot,\cdot]$ is *alternating*, i.e. $[x,x]=0$ for every $x\in\mathfrak{g}$.

2. The *Jacobi identity*

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \text{for all } x,y,z\in\mathfrak{g}.$$

$[\cdot,\cdot]$ is called a *Lie bracket.*

**Remark.** $[\cdot,\cdot]$ is antisymmetric, i.e. $[y,x]=-[x,y]$ for all $x,y\in\mathfrak{g}$, because

$$0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]=[x,y]+[y,x].$$

**Definition.** Let $A$ be a $k$-algebra. A *derivation of* $A$ is a $k$-linear map $d\colon A\to A$ such that

$$d(ab)=d(a)b+ad(b) \quad \text{for al l} a,b\in A.$$

We set

$$\mathrm{Der}(A)\coloneqq\{d\colon A\to A \mid d \text{ is a derivation of } A\}.$$

**Remark.** $\mathrm{Der}(A)$ is clearly a $k$-vector space.

**Examples.** 1. Any vector space $V$ becomes a Lie algebra via

$$[x,y]=0 \quad \text{for all } x,y\in V.$$

2. Any associative $k$-algebra $A$ becomes a Lie algebra via

$$[a,b]=ab-ba \quad \text{for all } a,b\in A.$$

   It is clear that $[\cdot,\cdot]$ is alternating and the Jacobi identity can be verified by some easy calculation.

   In particular $\mathrm{M}_n(k)$ is a Lie algebra via

$$[A,B]=AB-BA \quad \text{for all } A,B\in\mathrm{M}_n(k).$$

   This is called the *general linear Lie algebra* and is denoted by $\mathfrak{gl}_n(k)$ or $\mathfrak{gl}(n,k)$.

   More generally for any $k$-vector space the space $\mathrm{End}_k(V)$ becomes a Lie algebra via

$$[\varphi_1,\varphi_2]\coloneqq\varphi_1\circ\varphi_2-\varphi_2\colon\varphi_1 \quad \text{for all } \varphi_1,\varphi_2\in\mathrm{End}_k.$$

   This is called the *general linear Lie algebra for* $V$ and is denoted by $\mathfrak{gl}(V)$.

3. Let $A$ be a $k$-algebra. $\mathrm{Der}(A)$ is a Lie algebra via

$$[d,d']\coloneqq d\circ d'-d'\circ d \quad \text{for all } d,d'\in\mathrm{Der}(A).$$

   Is is an easy calculation to show that $[d,d']$ is again a derivation. Notice that the Jacobi identity for $\mathrm{Der}(A)$ follows from the Jacobi identity for $\mathfrak{gl}(A)$.

We will now look at how to construct new Lie algebras from old ones.

**Definition.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the same field $k$. Then the *product* of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is defined as the $k$-vector space $\mathfrak{g}_1 \times \mathfrak{g}_2$ together with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]) \quad \text{for all } (x_1, y_2), (x_2, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

Let $\mathfrak{g}$ be a Lie algebra over $k$ and $A$ an associative, commutative $k$-algebra. Then $\mathfrak{g} \otimes_k A$ is a Lie algebra via

$$[x \otimes a, y \otimes b] = [x, y] \otimes (ab) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

**Definition.** Let $\mathfrak{g}$ be a Lie algebra and $A = k[t, t^{-1}]$ be the algebra of Laurent polynomials over $k$. Then

$$\mathcal{L}(\mathfrak{g}) \coloneqq \mathfrak{g} \otimes_k A$$

with the Lie bracket as above is called the *loop (Lie) algebra* of $\mathfrak{g}$.

Another example of construction new Lie algebras from old ones are *central extensions*: Let $\mathfrak{g}$ be any $k$-Lie algebra.

$$\tilde{\mathfrak{g}} \coloneqq \mathfrak{g} \otimes k = \{x + \lambda c \mid x \in \mathfrak{g}, \lambda \in k\},$$

where we understand $c$ as a formal variable. Suppose that $\kappa \colon \mathfrak{g} \times \mathfrak{g} \to k$ is a $k$-bilinear map satisfying the following properties:

1. $\kappa$ is antisymmetric, i.e. $\kappa(x, y) = -\kappa(y, x)$ for all $x, y \in \mathfrak{g}$.
2. The 2-cocycle condition

$$\kappa([x, y], z) + \kappa([y, z], x) + \kappa([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then $\tilde{\mathfrak{g}}$ becomes a Lie algebra via

$$[x + \lambda c, y + \mu c] \coloneqq [x, y] + \kappa(x, y)\lambda\mu c.$$

Note that $c$ is central in $\tilde{\mathfrak{g}}$ in the sense that $[x, c] = 0$ for all $x \in \mathfrak{g}$.

**Example.** Let $\mathfrak{g} = \mathfrak{gl}_n(k)$. We define a symmetric bilinear form on $\mathfrak{g}$ via

$$(A, B)_{\text{tr}} = \text{tr}(AB).$$

We define a bilinear form

$$\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to k[t, t^{-1}], (x \otimes p, y \otimes q) \mapsto (x, y)_{\text{tr}}\, pq$$

We now get a 2-cocycle $\kappa \colon \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to k$ via

$$\kappa(a, b) \coloneqq \text{Res}\left(\frac{\partial a}{\partial t}, b\right).$$

$\kappa$ is also antisymmetric: Let $a = x \otimes t^i$ and $b = y \otimes t^j$ with $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$. Then

$$\begin{aligned}\kappa(x \otimes t^i, y \otimes t^j) = \text{Res}(ix \otimes t^{i-1}, y \otimes t^j) &= \text{Res}(it^{i+j-1}(x, y)_{\text{tr}}) \\ &= \begin{cases} i(x, y)_{\text{tr}} & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

In the same way we find that

$$\kappa(y \otimes t^j, x \otimes t^i) = \begin{cases} j(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(\cdot,\cdot)_{\mathrm{tr}}$ is symmetric we find that

$$\begin{aligned} \kappa(x \otimes t^i, y \otimes t^j) &= \begin{cases} i(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -j(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise,} \end{cases} \\ &= -\kappa(y \otimes t^j, x \otimes t^i). \end{aligned}$$

As for all algebraic structures morphisms are of interest.

**Definition.** Given $k$-Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ a homomorphism of Lie algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$ is a $k$-linear map $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$f([x,y]) = [f(x), f(y)] \quad \text{for all } x, y \in \mathfrak{g}_1.$$

**Examples.** 1. For any Lie algebra $\mathfrak{g}$ the identity $\mathrm{id}_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.

2. Given Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ and Lie algebra homomorphisms $f_1\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $f_2\colon \mathfrak{g}_2 \to \mathfrak{g}_3$ the composition $f_2 \circ f_1\colon \mathfrak{g}_1 \to \mathfrak{g}_3$ is also a homomorphism of Lie algebras.

**Remark.** We find that we have a category of $k$-Lie algebras.

**Definition.** Let $\mathfrak{g}$ be a $k$-Lie algebra. A representation of $\mathfrak{g}$ is a $k$-vector space $V$ together with a homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

**Remark.** Equivalently a representation of $\mathfrak{g}$ is a $k$-vector space $V$ together with a $k$-bilinear map $\mathfrak{g} \times V \to V$, $(x,v) \mapsto x.v$ such that

$$x.y.v - y.x.v = [x,y].v \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V.$$

**Definition.** Let $\mathfrak{g}$ be a Lie algebra. Then

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto [x, \cdot]$$

is called the *adjoint representation* of $\mathfrak{g}$.

**Remark.** That ad is a homomorphism of Lie algebras is equivalent to

$$\mathrm{ad}([x,y])(z) = [\mathrm{ad}(x), \mathrm{ad}(y)](z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Because

$$\mathrm{ad}([x,y])(z) = [[x,y],z] = -[z,[x,y]]$$

and

$$\begin{aligned} [\mathrm{ad}(x), \mathrm{ad}(y)](z) &= (\mathrm{ad}(x) \circ \mathrm{ad}(y))(z) - (\mathrm{ad}(y) \circ \mathrm{ad}(x))(z) \\ &= [x,[y,z]] - [y,[x,z]] = [x,[y,z]] + [y,[z,x]] \end{aligned}$$

this is equivalent to the Jacobi identity.

**Definition.** Let $\mathfrak{g}$ be a $k$-Lie algebra.

A *Lie subalgebra* of a $\mathfrak{g}$ is a $k$-linear subspace $L \subseteq \mathfrak{g}$ such that $[x, y] \in L$ for all $x, y \in L$.

An ideal inside $\mathfrak{g}$ is a $k$-linear subspace $I \subseteq \mathfrak{g}$ such that $[x, y] \in I$ for all $x \in \mathfrak{g}$ and $y \in I$. We denote ideals by $I \lhd \mathfrak{g}$.

Notice that any ideal is also a Lie subalgebra.

**Remark.** If $\mathfrak{g}$ is a Lie algebra and $I \lhd \mathfrak{g}$ then the quotient vector space $\mathfrak{g}/I$ is also a Lie algebra via

$$[x + I, y + I] = [x, y] + I.$$

It is easy to see that this bilinear map is well-defined. The properties for the Lie bracket on $\mathfrak{g}/I$ follow from the one for the Lie bracket on $\mathfrak{g}$.

From the definition of the Lie bracket on $\mathfrak{g}/I$ it is clear that the canonical projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$ is a homomorphism of Lie algebras.

We have the usual theorems about homomorphisms and ideals.

**Proposition 1.** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ a homomorphism of Lie algebras.*

1. $\ker f \lhd \mathfrak{g}$.
2. $\operatorname{im} f$ *is a Lie subalgebra.*
3. *If $I \lhd \mathfrak{g}$ is an ideal with* $\ker f \subseteq I$ *then there exists a unique homomorphism of Lie algebras $\tilde{f}\colon \mathfrak{g}_1/I \to \mathfrak{g}_2$ with $f = \tilde{f} \circ \pi$ where $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$ is the canonical projection.*

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
\mathfrak{g}/I \arrow[r, "\exists! \tilde{f}"'] & \mathfrak{g}_2
\end{tikzcd}$$