# Representation Theory I

Jendrik Stelzner

April 12, 2015

## Contents

# 1 Basic Definitions

**Definition.** A *$k$-Lie algebra* is a vector space $\mathfrak{g}$ (over some field $k$) together with a $k$-bilinear map

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

satisfying the following:

1. $[\cdot,\cdot]$ is *alternating*, i.e. $[x,x]=0$ for every $x\in\mathfrak{g}$.
2. The *Jacobi identity*

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \text{for all } x,y,z\in\mathfrak{g}.$$

$[\cdot,\cdot]$ is called a *Lie bracket.*

**Remark.** $[\cdot,\cdot]$ is antisymmetric, i.e. $[y,x]=-[x,y]$ for all $x,y\in\mathfrak{g}$, because

$$0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]=[x,y]+[y,x].$$

**Definition.** Let $A$ be a $k$-algebra. A *derivation of* $A$ is a $k$-linear map $d\colon A\to A$ such that

$$d(ab)=d(a)b+ad(b) \quad \text{for al l} a,b\in A.$$

We set

$$\mathrm{Der}(A):=\{d\colon A\to A \mid d \text{ is a derivation of } A\}.$$

**Remark.** $\mathrm{Der}(A)$ is clearly a $k$-vector space.

**Examples.** 1. Any vector space $V$ becomes a Lie algebra via

$$[x,y]=0 \quad \text{for all } x,y\in V.$$

2. Any associative $k$-algebra $A$ becomes a Lie algebra via

$$[a,b]=ab-ba \quad \text{for all } a,b\in A.$$

It is clear that $[\cdot,\cdot]$ is alternating and the Jacobi identity can be verified by some easy calculation.

In particular $\mathrm{M}_n(k)$ is a Lie algebra via

$$[A,B]=AB-BA \quad \text{for all } A,B\in\mathrm{M}_n(k).$$

This is called the *general linear Lie algebra* and is denoted by $\mathfrak{gl}_n(k)$ or $\mathfrak{gl}(n,k)$.

More generally for any $k$-vector space the space $\mathrm{End}_k(V)$ becomes a Lie algebra via

$$[\varphi_1,\varphi_2]:=\varphi_1\circ\varphi_2-\varphi_2\circ\varphi_1 \quad \text{for all } \varphi_1,\varphi_2\in\mathrm{End}_k.$$

This is called the *general linear Lie algebra for* $V$ and is denoted by $\mathfrak{gl}(V)$.

A Lie algebra $\mathfrak{g}$ is called linear if $\mathfrak{g}\subseteq\mathfrak{gl}(V)$ for some finite dimensional vector space $V$.

3. Let $A$ be a $k$-algebra. $\mathrm{Der}(A)$ is a Lie algebra via

$$[d, d'] := d \circ d' - d' \circ d \quad \text{for all } d, d' \in \mathrm{Der}(A).$$

Is is an easy calculation to show that $[d, d']$ is again a derivation. Notice that the Jacobi identity for $\mathrm{Der}(A)$ follows from the Jacobi identity for $\mathfrak{gl}(A)$.

We will now look at how to construct new Lie algebras from old ones.

**Definition.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over the same field $k$. Then the *product* of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is defined as the $k$-vector space $\mathfrak{g}_1 \times \mathfrak{g}_2$ together with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]) \quad \text{for all } (x_1, y_1), (x_2, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2.$$

Let $\mathfrak{g}$ be a Lie algebra over $k$ and $A$ an associative, commutative $k$-algebra. Then $\mathfrak{g} \otimes_k A$ is a Lie algebra via

$$[x \otimes a, y \otimes b] = [x, y] \otimes (ab) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } a, b \in A.$$

**Definition.** Let $\mathfrak{g}$ be a Lie algebra and $A = k[t, t^{-1}]$ be the algebra of Laurent polynomials over $k$. Then

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes_k A$$

with the Lie bracket as above is called the *loop (Lie) algebra* of $\mathfrak{g}$.

Another example of construction new Lie algebras from old ones are *central extensions*: Let $\mathfrak{g}$ be any $k$-Lie algebra.

$$\tilde{\mathfrak{g}} := \mathfrak{g} \otimes k = \{x + \lambda c \mid x \in \mathfrak{g}, \lambda \in k\},$$

where we understand $c$ as a formal variable. Suppose that $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to k$ is a $k$-bilinear map satisfying the following properties:

1. $\kappa$ is antisymmetric, i.e. $\kappa(x, y) = -\kappa(y, x)$ for all $x, y \in \mathfrak{g}$.

2. The 2-cocycle condition

$$\kappa([x, y], z) + \kappa([y, z], x) + \kappa([z, x], y) = 0 \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Then $\tilde{\mathfrak{g}}$ becomes a Lie algebra via

$$[x + \lambda c, y + \mu c] := [x, y] + \kappa(x, y)\lambda\mu c.$$

Note that $c$ is central in $\tilde{\mathfrak{g}}$ in the sense that $[x, c] = 0$ for all $x \in \mathfrak{g}$.

**Example.** Let $\mathfrak{g} = \mathfrak{gl}_n(k)$. We define a symmetric bilinear form on $\mathfrak{g}$ via

$$(A, B)_{\mathrm{tr}} = \mathrm{tr}(AB).$$

We define a bilinear form

$$\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to k[t, t^{-1}], (x \otimes p, y \otimes q) \mapsto (x, y)_{\mathrm{tr}}\, pq$$

We now get a 2-cocycle $\kappa\colon \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to k$ via

$$\kappa(a, b) := \mathrm{Res}\left(\frac{\partial a}{\partial t}, b\right).$$

$\kappa$ is also antisymmetric: Let $a = x \otimes t^i$ and $b = y \otimes t^j$ with $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$. Then

$$\begin{aligned}\kappa(x \otimes t^i, y \otimes t^j) = \operatorname{Res}(ix \otimes t^{i-1}, y \otimes t^j) &= \operatorname{Res}(it^{i+j-1}(x,y)_{\mathrm{tr}}) \\ &= \begin{cases} i(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

In the same way we find that

$$\kappa(y \otimes t^j, x \otimes t^i) = \begin{cases} j(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(\cdot,\cdot)_{\mathrm{tr}}$ is symmetric we find that

$$\begin{aligned}\kappa(x \otimes t^i, y \otimes t^j) &= \begin{cases} i(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -j(x,y)_{\mathrm{tr}} & \text{if } i+j=0, \\ 0 & \text{otherwise,} \end{cases} \\ &= -\kappa(y \otimes t^j, x \otimes t^i).\end{aligned}$$

As for all algebraic structures morphisms are of interest.

**Definition.** Given $k$-Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ a *homomorphism of Lie algebras* $\mathfrak{g}_1 \to \mathfrak{g}_2$ is a $k$-linear map $f \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$f([x,y]) = [f(x), f(y)] \quad \text{for all } x, y \in \mathfrak{g}_1.$$

**Examples.** 1. For any Lie algebra $\mathfrak{g}$ the identity $\mathrm{id}_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.

2. Given Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ and Lie algebra homomorphisms $f_1 \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $f_2 \colon \mathfrak{g}_2 \to \mathfrak{g}_3$ the composition $f_2 \circ f_1 \colon \mathfrak{g}_1 \to \mathfrak{g}_3$ is also a homomorphism of Lie algebras.

**Remark.** We find that we have a category of $k$-Lie algebras. An *isomorphism of $k$-Lie algebras* is an isomorphism in the category of $k$-Lie algebras. Clearly every isomorphism is an homomorphism and bijective. The converse also holds: If $f \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a bijective homomorphism of $k$-Lie algebras, then the $k$-linear map $f^{-1} \colon \mathfrak{g}_2 \to \mathfrak{g}_1$ is also an homomorphism of $k$-Lie algebras, because for all $x, y \in \mathfrak{g}_2$

$$\begin{aligned}f^{-1}([x,y]) &= f^{-1}([f(f^{-1}(x)), f(f^{-1}(y))]) \\ &= f^{-1}(f([f^{-1}(x), f^{-1}(y)])) \\ &= [f^{-1}(x), f^{-1}(y)].\end{aligned}$$

So an isomorphism of Lie algebras is just a homomorphism of Lie algebras which is bijective.

**Definition.** Let $\mathfrak{g}$ be a $k$-Lie algebra. A representation of $\mathfrak{g}$ is a $k$-vector space $V$ together with a homomorphism of Lie algebras $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$.

**Remark.** Equivalently a representation of $\mathfrak{g}$ is a $k$-vector space $V$ together with a $k$-bilinear map $\mathfrak{g} \times V \to V, (x, v) \mapsto x.v$ such that

$$x.y.v - y.x.v = [x, y].v \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V.$$

**Definition.** Let $\mathfrak{g}$ be a Lie algebra. Then

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto [x, \cdot]$$

is called the *adjoint representation* of $\mathfrak{g}$.

**Remark.** That ad is a homomorphism of Lie algebras is equivalent to

$$\mathrm{ad}([x, y])(z) = [\mathrm{ad}(x), \mathrm{ad}(y)](z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Because

$$\mathrm{ad}([x, y])(z) = [[x, y], z] = -[z, [x, y]]$$

and

$$\begin{aligned}[\mathrm{ad}(x), \mathrm{ad}(y)](z) &= (\mathrm{ad}(x) \circ \mathrm{ad}(y))(z) - (\mathrm{ad}(y) \circ \mathrm{ad}(x))(z) \\ &= [x, [y, z]] - [y, [x, z]] = [x, [y, z]] + [y, [z, x]]\end{aligned}$$

this is equivalent to the Jacobi identity.

**Definition.** Let $\mathfrak{g}$ be a $k$-Lie algebra. A *Lie subalgebra* of a $\mathfrak{g}$ is a $k$-linear subspace $L \subseteq \mathfrak{g}$ such that

$$[x, y] \in L \quad \text{for all } x, y \in L.$$

An *ideal* inside $\mathfrak{g}$ is a $k$-linear subspace $I \subseteq \mathfrak{g}$ such that

$$[x, y] \in I \quad \text{for all } x \in \mathfrak{g} \text{ and } y \in I.$$

We denote ideals by $I \lhd \mathfrak{g}$.

For a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ it is clear that $\mathfrak{h}$ becomes a Lie algebra by restricting the Lie bracket of $\mathfrak{g}$ to $\mathfrak{h}$. The inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is then clearly a homomorphism of Lie algebras.

Also notice that every ideal inside $\mathfrak{g}$ is also a subalgebra of $\mathfrak{g}$.

**Example.** Let $\mathfrak{g} = \mathfrak{gl}_n(k)$. Then

$$\mathfrak{sl}_n(k) = \{A \in \mathfrak{g} \mid \mathrm{tr}\, A = 0\}$$

is a subalgebra. Even more

$$\mathfrak{sl}_n(k) = [\mathfrak{g}, \mathfrak{g}].$$

To see this first notice that on the one hand $A, B \in \mathfrak{g}$

$$\mathrm{tr}[A, B] = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(AB) - \mathrm{tr}(AB) = 0.$$

Therefore $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{sl}_n(k)$. On the other hand notice that $\mathfrak{sl}_n(k)$ has a basis given by the elementary matrices $e_{ij}$ with $1 \le i \ne j \le n$ and $e_{11} - e_{ii}$ with $1 < i \le n$. Each of these matrices is given as a commutator, namely $e_{ij} = [e_{ii}, e_{ij}]$ for $1 \le i \ne j \le n$ and $e_{11} - e_{ii} = [e_{1i}, e_{i1}]$ for $1 < i \le n$. Therefore $\mathfrak{sl}_n(k) \subseteq [\mathfrak{g}, \mathfrak{g}]$.

We have similar properties for ideals inside of Lie algebras als for ideals inside of a ring: For a collection of ideals $I_\lambda$, $\lambda \in \Lambda$, their intersection $\bigcap_{\lambda\in\Lambda} I_\lambda$ and their sum $\sum_{\lambda\in\Lambda} I_\lambda$ are also ideals inside $\mathfrak{g}$. For ideals $I, J \lhd \mathfrak{g}$ the subspace $[I, J]$ is also an ideal inside $\mathfrak{g}$. (For any two subsets $X, Y \subseteq \mathfrak{g}$ we denote by $[X, Y]$ the linear subspace generated by the commutators $[x, y]$ with $x \in X$, $y \in Y$.) To see that $[I, J]$ is an ideal notice that for any $x \in I$, $y \in J$ and $z \in \mathfrak{g}$

$$[z, [x, y]] = -[x, \underbrace{[y, z]}_{\in J}] - [y, \underbrace{[z, x]}_{\in I}].$$

**Remark.** If $\mathfrak{g}$ is a Lie algebra and $I \lhd \mathfrak{g}$ then the quotient vector space $\mathfrak{g}/I$ is also a Lie algebra via

$$[x + I, y + I] = [x, y] + I.$$

It is easy to see that this bilinear map is well-defined. The properties for the Lie bracket on $\mathfrak{g}/I$ follow from the ones for the Lie bracket on $\mathfrak{g}$.

From the definition of the Lie bracket on $\mathfrak{g}/I$ it is clear that the canonical projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$ is a homomorphism of Lie algebras.

We have the usual theorems about homomorphisms and ideals.

**Proposition 1.** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras and $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ a homomorphism of Lie algebras.*

1. $\ker f \lhd \mathfrak{g}_1$ *is an ideal.*

2. $\operatorname{im} f \subseteq \mathfrak{g}_2$ *is a Lie subalgebra.*

3. *If $I \lhd \mathfrak{g}_1$ is an ideal with $\ker f \subseteq I$ then there exists a unique homomorphism of Lie algebras $\tilde{f}\colon \mathfrak{g}_1/I \to \mathfrak{g}_2$ with $f = \tilde{f} \circ \pi$ where $\pi\colon \mathfrak{g}_1 \to \mathfrak{g}_1/I$ is the canonical projection.*

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
\mathfrak{g}_1/I \arrow[r, "\exists!\tilde{f}"'] & \mathfrak{g}_2
\end{tikzcd}$$

4. *If $I, J \lhd \mathfrak{g}$ are subideals with $I \subseteq J$ then $J/I$ is an ideal inside $\mathfrak{g}/I$ and the map*

$$(\mathfrak{g}/I)/(J/I) \to \mathfrak{g}/I, (x + I) + (J/I) \mapsto x + I$$

*is a natural isomorphism of Lie algebras.*

5. *If $I, J \lhd \mathfrak{g}$ are subideals then there is a natural isomorphism*

$$(I + J)/J \to I/(I \cap J)$$

*defined by*

$$(x + J) + I \mapsto x + (I \cap J) \quad \text{for every } x \in I$$

*is a natural isomorphism of Lie algebras.*

**Definition.** Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is

$$Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for every } y \in \mathfrak{g}\} = \ker \operatorname{ad}.$$

$\mathfrak{g}$ is called *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Clearly $\mathfrak{g}$ is abelian if and only if $Z(\mathfrak{g}) = \mathfrak{g}$. Notice that $[\mathfrak{g}, \mathfrak{g}]$ and $Z(\mathfrak{g})$ are ideals inside $\mathfrak{g}$.

**Definition.** A Lie algebra $\mathfrak{g}$ is *simple* if $0$ and $\mathfrak{g}$ are the only ideals inside $\mathfrak{g}$ and $\mathfrak{g}$ is not abelian.

**Lemma 2.** *Let $\mathfrak{g}$ be a simple Lie algebra. Then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$.*

*Proof.* Because $\mathfrak{g}$ is simple it is not abelian. Therefore $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and $Z(\mathfrak{g}) \neq \mathfrak{g}$. Since $[\mathfrak{g}, \mathfrak{g}]$ and $Z(\mathfrak{g})$ are ideals inside $\mathfrak{g}$ it follows that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $Z(\mathfrak{g}) = 0$. ☐

**Corollary 3.** *Let $\mathfrak{g}$ be simple. Then $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is injective. In particular $\mathfrak{g}$ is linear.*

*Proof.* This directly follows from $\ker \operatorname{ad} = Z(\mathfrak{g}) = 0$. ☐

**Theorem 4** (Ado). *Any finite dimensional Lie algebra is linear.*

A proof of Ado's theorem will (maybe) be given later.

**Remark.** For a Lie algebra $\mathfrak{g}$ the ideal $[\mathfrak{g}, \mathfrak{g}]$ is the minimal ideal inside $\mathfrak{g}$ such that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian. Furthermore given any abelian Lie algebra $\mathfrak{h}$ any homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{h}$ factorizes through a unique homomorphism of Lie algebras $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathfrak{h}$.

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr] \arrow[dr] & & \mathfrak{h} \\
& \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \arrow[ur, "\exists!"'] &
\end{tikzcd}$$

**Examples.** 1. Since $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] = \mathfrak{sl}_n(k) \neq \mathfrak{gl}_n(k)$ we find that $\mathfrak{gl}_n(k)$ is not simple.

2. Let $\mathfrak{g} = \mathfrak{sl}_2(k)$. Then $\mathfrak{g}$ is simple if and only if $\operatorname{char} k \neq 2$. To see this consider the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

of $\mathfrak{sl}_2(k)$. Then

$$[e, h] = -2e, [e, f] = h, [h, f] = -2f.$$

If $\operatorname{char} k = 2$ then $h$ spans a 1-dimensional ideal, thus $\mathfrak{sl}_2(k)$ is not simple. Suppose that $\operatorname{char} k \neq 2$ and let $I \subseteq \mathfrak{sl}_2(k)$ be an ideal with $I \neq 0$. It is clear that if $I$ contains one of the basis vectors $e$, $h$ or $f$ it follows that $I = \mathfrak{sl}_2(k)$. Let $x \in I$ with $x \neq 0$ and write $x = \alpha e + \beta h + \gamma f$. Then

$$[e, x] = -2\beta e + \gamma h \quad \text{and} \quad [e, [e, x]] = -2\gamma e.$$

Since $\gamma = 0$ or $\gamma \neq 0$ we find that $e \in I$.

**Remark.** $\mathfrak{sl}_n(\mathbb{C})$ is simple for all $n \geq 2$.

## 2 Nilpotent and solvable Lie algebras

**Definition.** Let $A$ be a $k$-algebra. An element $a \in A$ is called *nilpotent* if $a^n = 0$ for some $n \geq 1$. Given a Lie algebra $\mathfrak{g}$ an element $x \in \mathfrak{g}$ is called ad-*nilpotent* if $\mathrm{ad}(x) \in \mathrm{End}_k(\mathfrak{g})$ is nilpotent.

**Definition.** Let $g$ be a Lie algebra. Define $\mathfrak{g}^0 := \mathfrak{g}$ and $\mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i]$ for all $i \geq 0$. Then

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

is called the *central series* of $\mathfrak{g}$. Also define $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, g^{(i)}]$ for all $i \geq 0$. Then

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$$

is called the *derived series* of $\mathfrak{g}$. $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{g}^i = 0$ for some $i$ and *solvable* if $g^{(i)} = 0$ for some $i$.

It is clear that every nilpotent Lie algebra is also solvable.

**Examples.** 1. Because $[\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$ and $[\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$ for $n \geq 2$ (since $\mathfrak{sl}_n(\mathbb{C})$ is simple) $\mathfrak{gl}_n(\mathbb{C})$ is not simple.

2. If $\mathfrak{g}$ is abelian then $\mathfrak{g}$ is nilpotent.

3. A Heisenberge Lie algebras consists of a real vector space with basis $P_1, \dots, P_n$, $Q_1, \dots, Q_n, C$ together with the Lie bracket satisfying the following conditions:

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_i, C] = 0 \quad \text{and} \quad [P_i, Q_j] = \delta_{ij} C.$$

This defines a nilpotent Lie algebra.

**Proposition 5.** *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If $\mathfrak{h}$ is a Lie algebra and $f \colon \mathfrak{g} \to \mathfrak{h}$ aLie algebras homomorphism then $f(\mathfrak{g})^i = f(\mathfrak{g}^i)$ and $f(\mathfrak{g})^{(i)} = f(\mathfrak{g}^{(i)})$ for all $i \geq 0$.*

2. *If $\mathfrak{g}$ is nilpotent (resp. solvable) then any Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and any quotient of $\mathfrak{g}$ (by an ideal) is nilpotent (resp. nilpotent).*

3. *If $I \lhd \mathfrak{g}$ with $I \subseteq Z(\mathfrak{g})$ and $\mathfrak{g}/I$ is nilpotent then $\mathfrak{g}$ is nilpotent.*

4. *If $\mathfrak{g} \neq 0$ is nilpotent then $Z(\mathfrak{g}) \neq 0$.*

5. *If $\mathfrak{g}$ is nilpotent and $x \in \mathfrak{g}$ then $x$ is* ad-*nilpotent.*

6. *If $I \lhd \mathfrak{g}$ then $I^i$ and $I^{(i)}$ are ideals inside $\mathfrak{g}$ for all $i \geq 0$.*

*Proof.* 1. It suficies to show that for any two subsets $X, Y \subseteq \mathfrak{g}$

$$f([X,Y]) = [f(X), f(Y)]$$

the statement then follows inductively. It holds because $f$ is a Lie algebra homomorphism and therefore

$$\begin{aligned} f([X,Y]) &= f(\mathrm{span}_k\{[x,y] \mid x \in X, y \in Y\}) \\ &= \mathrm{span}_k\{f([x,y]) \mid x \in X, y \in Y\} \\ &= \mathrm{span}_k\{[f(x), f(y)] \mid x \in X, y \in Y\} \\ &= \mathrm{span}_k\{[x', y'] \mid x' \in f(X), y' \in f(Y)\} \\ &= [f(X), f(Y)]. \end{aligned}$$

2. The statement about subalgebras is clear since $\mathfrak{h}^i \subseteq \mathfrak{g}^i$ and $\mathfrak{h}^{(i)} \subseteq \mathfrak{g}^{(i)}$ for all $i$. The statement about quotient follow using the canonical projection $\pi\colon \mathfrak{g}/I$ where $I \lhd \mathfrak{g}$. Because $\pi$ is a Lie algebra homomorphism we have

   $$(\mathfrak{g}/I)^i = \pi(\mathfrak{g})^i = \pi(\mathfrak{g}^i) = 0$$

   for $i$ big enough. For solvable $\mathfrak{g}$ the corresponding statements follow in the same way.

3. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/I$ be the canonical projection. Because $\mathfrak{g}/I$ is nilpotent there exists some $i \geq 0$ with $(\mathfrak{g}/I)^i = 0$ and therefore

   $$0 = (\mathfrak{g}/I)^i = \pi(\mathfrak{g})^i = \pi(\mathfrak{g}^i).$$

   Thus $\mathfrak{g}^i \subseteq I \subseteq Z(G)$ and hence $\mathfrak{g}^{i+1} = 0$.

4. Let $i \geq 1$ be minimal such that $\mathfrak{g}^{i-1} \neq 0$ but $\mathfrak{g}^i = 0$. Then $\mathfrak{g}^{i-1} \subseteq Z(\mathfrak{g})$ and therefore $Z(\mathfrak{g}) \neq 0$.

5. Since $\mathfrak{g}$ is nilpotent $\mathfrak{g}^i = 0$ for some $i \geq 0$. Then

   $$(\mathrm{ad}(x))^i(\mathfrak{g}) \subseteq \mathfrak{g}^i = 0,$$

   so $(\mathrm{ad}(x))^i = 0$.

6. This follows inductively by using that $[I, J]$ is an ideal inside $\mathfrak{g}$ for any $I, J \lhd \mathfrak{g}$.

□