Graduate Seminar on Advanced Algebra Jendrik Stelzner December 12, 2019

Jordan Quiver, Part I

Talk 10 on Hall Algebras and Quantum Groups

1 The Jordan Quiver and its Nilpotent Representations

Definition 1. The *Jordan quiver* is the quiver that consists of a single vertex and a single edge, which is necessarily a loop.

Throughout this talk the Jordan quiver is denoted by Q. See Figure 1 for a visualization. In the following we write $\Bbbk$ to mean a field or $\mathbb{F}_1$.

A representation of the Jordan quiver over $\Bbbk$ is the same as a pair (V, f) consisting of a $\Bbbk$-vector space V together with an endomorphism f of V. A homomorphism of representations $\varphi\colon (V, f) \to (W, g)$ is then precisely a homomorphism of vector spaces $\varphi\colon V \to W$ that makes the following square diagram commute:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

If $(V, f) \cong (W, g)$ then in particular $V \cong W$ as vector spaces. So to understand the isomorphism classes of Q-representations over $\Bbbk$ we may assume that $V = W$. The commutativity of the above square diagram, together with the requirement that φ is an isomorphism, means precisely that the endomorphisms f, g of V are similar. We hence find that that the isomorphism classes of Q-representations over $\Bbbk$ correspond one-to-one to conjugacy classes of endomorphisms of $\Bbbk$-vector spaces.

Suppose that V is finite-dimensional. If $\Bbbk$ is a field then these conjugacy classes are can be understood via the rational canonical form. In the case that $\Bbbk$ is also algebraically closed, or that it is $\mathbb{F}_1$, or that we are interested only in nilpotent endomorphisms, one can use the usual Jordan normal form.

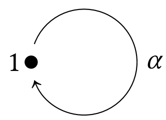

Figure 1: The Jordan quiver Q.

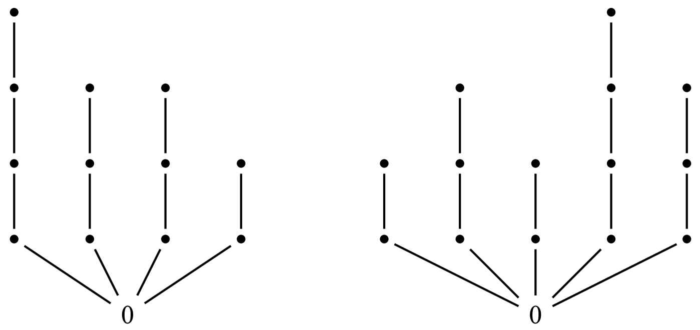

Figure 2: The representations $\mathrm{N}_{(4,3,3,2)}$ and $\mathrm{N}_{(2,3,2,4,3)}$ over $\mathbb{F}_1$.

Recall 2. A representation $V = ((V_i)_{i\in\Gamma_0}, (f_\alpha)_{\alpha\in\Gamma_1})$ of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is *nilpotent* if there exists some $N \geq 0$ such that for every path $\alpha_n, \dots, \alpha_1$ in Γ of length $n \geq N$ we have $f_{\alpha_n} \circ \cdots \circ f_{\alpha_1} = 0$. (See [Szc11, Definition 4.4].)

If Γ is finite and has no oriented cycles then every representation of Γ is nilpotent.

A representation (V, f) of the Jordan quiver is nilpotent if and only if the endomorphism f is nilpotent. We will in the rest of this talk restrict our attention to finite-dimensional, nilpotent representations of the Jordan quiver.

Definition 3. The category $\mathbf{Rep}^{\mathrm{nil}}(Q, \Bbbk)$ is the full subcategory of $\mathbf{Rep}(Q, \Bbbk)$ whose objects are the finite-dimensional, nilpotent representations of Q over $\Bbbk$. The set of isomorphism classes in $\mathbf{Rep}^{\mathrm{nil}}(Q, \Bbbk)$ is denoted by $\mathrm{Iso}(Q, \Bbbk)$.

Every nilpotent endomorphism on a finite-dimensional $\Bbbk$-vector space admits a Jordan normal form. We can therefore classify the isomorphism classes of $\mathbf{Rep}^{\mathrm{nil}}(Q, \Bbbk)$: For every dimension $d \geq 0$ let

$$\mathrm{N}_d \coloneqq \left(\Bbbk^d, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \right)$$

if $\Bbbk$ is a field, and let

$$\mathrm{N}_d \coloneqq (\{0, 1, \dots, d\}, [d \mapsto (d-1) \mapsto (d-2) \mapsto \cdots \mapsto 1 \mapsto 0 \mapsto 0])$$

if $\Bbbk = \mathbb{F}_1$. For every tupel $(d_1, \dots d_n)$ of dimensions $d_i \geq 0$ let

$$\mathrm{N}_{(d_1,\dots,d_n)} \coloneqq \mathrm{N}_{d_1} \oplus \cdots \oplus \mathrm{N}_{d_n}\,.$$

See Figure 2 for a visualization of $\mathrm{N}_{(d_1,\dots,d_n)}$.

Proposition 4 (Jordan normal form for nilpotent endomorphisms)**.** Let $\Bbbk$ be any field.

1. Every finite-dimensional, nilpotent representation of Q over $\Bbbk$ is isomorphic to a representation of the form $\mathrm{N}_{(d_1,\dots,d_n)}$ for some $n \geq 0$ and $d_1, \dots, d_n \geq 1$.

2. Two such representations $\mathrm{N}_{(d_1,\dots,d_n)}$ and $\mathrm{N}_{(d'_1,\dots,d'_m)}$ are isomorphic if and only if $n = m$ and the tuples $(d_1, \dots, d_n)$ and $(d'_1, \dots, d'_m)$ are the same up to permutation.

We can reformulate the above proposition in terms of partitions:

Definition 5. For every $n \geq 0$ let $\mathrm{Par}(n)$ be the set of partition of the number n, i.e.

$$\mathrm{Par}(n) := \{(\lambda_1, \dots, \lambda_l) \mid \lambda_1 \geq \cdots \geq \lambda_l \geq 1, \lambda_1 + \cdots + \lambda_l = n\}.$$ [1]

The set of all partitions is denoted by

$$\mathrm{Par} := \coprod_{n \geq 0} \mathrm{Par}(n).$$

Corollary 6. The representations N_λ with $\lambda \in \mathrm{Par}$ form a set of representatives for $\mathrm{Iso}(Q, \Bbbk)$.

We find in particular that the category $\mathbf{Rep}^{\mathrm{nil}}(Q, \Bbbk)$ admits only finitely many isomorphism classes of objects.

2 The Hall Algebra of the Jordan Quiver over $\mathbb{F}_q$

We consider for $\Bbbk$ the finite field $\mathbb{F}_q$.

Proposition 7. The category $\mathbf{Rep}^{\mathrm{nil}}(Q)$ is hereditary.

Proof. ????? □

Lemma 8. Let $S := \mathrm{N}_1$.

1. The representation S is the unique simple object of $\mathbf{Rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ (up to isomorphism).
2. The Groethendieck group $\mathrm{K}_0(\mathbf{Rep}^{\mathrm{nil}}(Q, \mathbb{F}_q))$ is freely generated by the class $[S]$. Thus

   $$\mathrm{K}_0(\mathbf{Rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)) \cong \mathbb{Z}$$

   via the map $[M] \mapsto \dim(M)$.
3. We have both $\mathrm{Hom}(S, S) = \mathbb{F}_q$ and $\mathrm{Ext}^1(S, S) = \mathbb{F}_q$.

Proof.

1. The indecomposable objects of $\mathbf{Rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ are precisely N_i with $i \geq 1$. The representation N_i has (up to isomorphism) the subrepresentations N_j with $j = 0, \dots, i$. Thus only N_1 is simple.
2. This follows from the previous assertion since each objects of $\mathbf{Rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ admits a composition series, whose composition factors are necessarily S.

[1] We want to point out that in this talk we do not allow a partition to contain zero as an entry. This is done purely for technical reasons.

3. We have $\operatorname{Hom}(S,S) = \mathbb{F}_q$ because S is one-dimensional.

   Computation of $\operatorname{Ext}^1(S,S)$: Can be done via homological algebra or by explicit counting of Yoneda extensions. (It still needs to be decided which one to use.) □

Corollary 9. The Euler form of $\mathbf{Rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ is trivial.

Proof. Let $K := \mathrm{K}_0(\mathbf{Rep}^{\mathrm{nil}}(Q,\mathbb{F}_q))$. We can regard the Euler form of $\mathbf{Rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ as a bilinear form

$$\langle -,- \rangle \colon\ K \times K \to \mathbb{Q}^{\times}\,.$$

Since S is a generator of K is sufficies to show that $\langle S,S\rangle = 1$. This holds true because

$$\langle S,S\rangle = \Bigl(\#\operatorname{Hom}(S,S)\Bigr) \cdot \Bigl(\#\operatorname{Ext}^1(S,S)\Bigr)^{-1} = q \cdot q^{-1} = 1\,.$$

This proves the assertion. □

We find from the above that $\mathbf{Rep}^{\mathrm{nil}}(Q,\mathbb{F})$ is a abelian, finitary, hereditary category, which admits only finitely many isomorphism classes of objects (i.e. it is essentially finite). We are thus well-prepared to consider the Hall algebra $\mathbf{H}(Q,\mathbb{F}_q)$.

1. The underlying vector space of $\mathbf{H}(Q,\mathbb{F}_q)$ is free on the set of isomorphism classes, $\operatorname{Iso}(Q,\mathbb{F}_q)$. This basis is indexed by the set of partitions Par.
2. The multiplication on $\mathbf{H}(Q,\mathbb{F}_q)$ is given by

   $$[M]\cdot[N] = \sum_{[R]\in\operatorname{Iso}(Q,\mathbb{F}_q)} C^R_{M,N}[R]$$

   where

   $$C^R_{M,N} = \text{number of subrepresentations } L \text{ of } R \text{ with } L \cong N \text{ and } R/L \cong M\,.$$

   The multiplicative neutral element of $\mathbf{H}(Q,\mathbb{F}_q)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_q)} = [0]$.
3. Since the Euler form of $\mathbf{Rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ vanishes and $\operatorname{Iso}(Q,\mathbb{F}_q)$ is finite we find that Green's product makes the Hall algebra $\mathbf{H}(Q,\mathbb{F}_q)$ into a bialgebra. Its comultiplication is given by

   $$\Delta([M]) = \sum_{[M],[N]\in\operatorname{Iso}(Q,\mathbb{F}_q)} \frac{1}{a_R} P^R_{M,N}[M]\otimes[N]$$

   where a_R is the size of the automorphism group $\operatorname{Aut}(R)$, and $P^R_{M,N}$ is the number of short exact sequences $0 \to N \to R \to M \to 0$. The counit $\varepsilon\colon\ \mathbf{H}(Q,\mathbb{F}_q) \to \mathbb{C}$ is given by

   $$\varepsilon([M]) = \begin{cases} 1 & \text{if } M = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. We have a grading on $\mathbf{H}(Q,\mathbb{F}_q)$ over the Grothendieck group $\mathrm{K}(\mathbf{Rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)) \cong \mathbb{Z}$, given by

   $$\deg([M]) = \dim(M)\,.$$

   This grading makes $\mathbf{H}(Q,\mathbb{F}_q)$ into a graded bialgebra.

5. The graded bialgebra $\mathbf{H}(Q, \mathbb{F}_q)$ is connected (i.e. its degree zero part is the ground field). It is therefore already a graded Hopf algebra.

We will in the rest of this talk be mostly concerned with the upcoming Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$. But we will here compute at least some of the structure constants of $\mathbf{H}(Q, \mathbb{F}_q)$. For this we follow [Sch09, Example 2.2].

Recall 10. For $k \in \mathbb{N}$ the *quantum integer* $[k]_q$ is given by

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1}.$$

We have $[0]_q = 0$ and $[1]_q = 1$. The *quantum factorial* is given by

$$[k]_q! = [k]_q [k-1]_q \cdots [1]_q.$$

For $k, l \in \mathbb{N}$ the *quantum binomial* is given by

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q \cdots [k-l+1]_q}{[l]_q!}.$$

If $l > k$ then this is zero, and if $l \leq k$ then the quantum binomial can also be expressed as

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q! \, [k-l]_q!}.$$

The quantum binomial satsfies the recursive relation

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = q^l \begin{bmatrix} k-1 \\ l \end{bmatrix}_q + \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q$$

for all $k, l \in \mathbb{N}$. It hence follows by induction that the quantum binomial $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ is a polynomial in q with natural coefficients, i.e.

$$\begin{bmatrix} k \\ l \end{bmatrix}_q \in \mathbb{N}[q].$$

By taking the limit $q \to 1$ (i.e. by setting q equal to 1) the quantum integer $[k]$ becomes the usual integer k, the quantum factorial $[k]_q!$ becomes the usual factorial $k!$ and the quantum binomial coefficient $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ becomes the usual binomial $\binom{k}{l}$.

Lemma 11. For all dimensions $n, d \geq 0$ we have

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \begin{bmatrix} n \\ d \end{bmatrix}_q.$$

Proof. If $d > n$ then both numbers are zero, so suppose that $d \leq n$. Let

$$F_d(n) := (q^n - 1) \cdots (q^n - q^{d-1}) = (q-1)^d \, q^{d(d-1)/2} \, [n]_q \cdots [n-d+1]_q$$

This is the number of linear independent tupels $(v_1, \dots, v_d)$ of vectors in $\mathbb{F}_q^n$. We find that

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{\#\mathrm{GL}(d, \mathbb{F}_q)} \,.$$

We have $\#\mathrm{GL}(d, \mathbb{F}_q) = F_d(d)$ and thus

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{F_d(d)} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q \cdots [1]_q} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q!} = \begin{bmatrix} n \\ d \end{bmatrix}_q ,$$

as claimed. □

Example 12. For any three partition $\lambda, \mu, \kappa \in \mathrm{Par}$ we abbreviate the structure constant

$$C^{\mathrm{N}_\kappa}_{\mathrm{N}_\lambda, \mathrm{N}_\mu}$$

as $C^\kappa_{\lambda,\mu}$.

1. Let $\lambda = (1^n)$ and $\mu = (1^m)$. We consider the partition $\kappa := (1^{n+m})$. The action of the edge of the Jordan quiver Q on the representations N_λ, N_μ and N_κ is trivial. We thus find that every m-dimensional linear subspace L of N_κ satisfies the conditions $L \cong \mathrm{N}_\mu$ and $\mathrm{N}_\kappa / L \cong \mathrm{N}_\lambda$. The structure constant $C^\kappa_{\lambda,\mu}$ is therefore given by

$$\begin{aligned} C^\kappa_{\lambda,\mu} &= \text{number of } m\text{-dimensional linear subspaces of } \mathrm{N}_\kappa \\ &= \text{number of } m\text{-dimensional linear subspaces of } \mathbb{F}_q^{n+m} \\ &= \#\mathrm{Gr}(m, n+m, \mathbb{F}_q) \\ &= \begin{bmatrix} n+m \\ m \end{bmatrix}_q . \end{aligned}$$

   We see in particular that $C^\kappa_{\lambda,\mu}$ depends is a polynomial way on q. We have for example

$$C^{(1^{n+1})}_{(1^n),(1)} = \#\mathrm{Gr}(1, n+1, \mathbb{F}_q) = \#\mathbb{P}^n(\mathbb{F}_q) = \frac{q^{n+1} - 1}{q - 1} = [n+1]_q = 1 + q + \cdots + q^n ,$$

   and also

$$\begin{aligned} C^{(1^{n+2})}_{(1^n),(1,1)} = \begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q &= \frac{[n+2]_q [n+1]_q}{[2]_q} \\ &= \frac{(1 + q + \cdots + q^n)(1 + q + \cdots + q^{n+1})}{1+q} \\ &= \begin{cases} (1 + q + \cdots + q^n)(1 + q^2 + \cdots + q^n) & \text{if } n \text{ is even,} \\ (1 + q^2 + \cdots + q^{n-1})(1 + q + \cdots + q^{n+1}) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

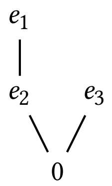

Figure 3: The representations $\mathrm{N}_{(2,1)}$ over $\mathbb{F}_q$.

2. Let now $\lambda = (n)$ and $\mu = (m)$. We consider the partition $\kappa = (n+m)$. The representation N_κ has the standard basis $e_1, \dots, e_{n+m}$, and the subrepresentations of N_κ are given by $\langle e_1, \dots, e_i \rangle$ for $i = 0, \dots, n+m$. The subrepresentations $L \coloneqq \langle e_1, \dots, e_m \rangle$ is the unique one that is isomorphic to N_μ, and its quotient N_κ / L is isomorphic to N_λ. Thus
$$C^{(n+m)}_{(n),(m)} = 1\,.$$
3. Let us compute the coefficients $C^{(2,1)}_{(1),(2)}$ and $C^{(2,1)}_{(2),(1)}$. We use for $\mathrm{N}_{(2,1)}$ the basis e_1, e_2, e_3 with $\alpha e_1 = e_2$ and $\alpha e_2 = \alpha e_3 = 0$ where α denotes the loop of Q. See Figure 3.

   The coefficient $C^{(2,1)}_{(1),(2)}$ is the number of subrepresentations L of $\mathrm{N}_{(2,1)}$ with $L \cong \mathrm{N}_2$ and $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_1$. The condition $L \cong \mathrm{N}_2$ means that L is cyclically generated by a vector $v = ae_1 + be_2 + ce_3$ with $a \neq 0$. We may assume that $a = 1$. Then
$$\langle v \rangle_{\mathbb{F}_q Q} = \langle v, \alpha v \rangle_{\mathbb{F}_q} = \langle e_1 + be_2 + ce_3, e_2 \rangle_{\mathbb{F}_q} = \langle e_1 + ce_3, e_2 \rangle\,.$$
   For any such subrepresentation L the quotient $\mathrm{N}_{(2,1)}/L$ is one-dimensional and thus isomorphic to N_1. We get for every coefficient $c \in \mathbb{F}_q$ a different representation. Hence
$$C^{(2,1)}_{(1),(2)} = \#\mathbb{F}_q = q\,.$$
   The coefficient $C^{(2,1)}_{(1),(2)}$ is the number of subrepresentations L of $\mathrm{N}_{(2,1)}$ with $L \cong \mathrm{N}_1$ and $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_2$. The condition $L \cong \mathrm{N}_1$ means that L is cyclically generated by a nonzero vector $v = be_2 + ce_3$ with $a \neq 0$.

   If $b \neq 0$ then we may assume that $b = 1$, so that $v = e_2 + ce_3$. Then $\mathrm{N}_{(2,1)}/L$ has the basis vectors $[e_1]$, $[e_3]$ with $\alpha[e_1] = -c[e_3]$ and $\alpha[e_3] = 0$. Thus $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_2$ if $c \neq 0$ and $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_{(1,1)}$ if $c = 0$. In the case $b \neq 0$ we thus have $q-1$ choices for L.

   If $b = 0$ then $c \neq 0$ and we may assume that $c = 1$. Then $v = e_3$ and thus Then $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_2$.

   We thus find that there are q choices for L, i.e
$$C^{(2,1)}_{(2),(1)} = q\,.$$
4. One finds in the same way as above that more generally
$$C^{(n,1)}_{(n),(1)} = q = C^{(n,1)}_{(1),(n)}$$
   for every $n \geq 2$.

We observe that in the above examples we always have $C^\kappa_{\lambda,\mu} = C^\kappa_{\mu,\lambda}$. We will see in next week's talk that the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ is indeed commutative, which means precisely that $C^\kappa_{\lambda,\mu} = C^\kappa_{\mu,\lambda}$ for any three partitions $\lambda, \mu, \kappa \in \mathrm{Par}$.

3 The Hall Algebra of the Jordan Quiver over $\mathbb{F}_1$

We will now consider the case that $\Bbbk$ is $\mathbb{F}_1$. We have seen in last week's talk how to construct the Hall algebra of Q over $\mathbb{F}_1$:

Recall 13. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is a graded, cocommutative Hopf algebra (over the ground field $\mathbb{C}$). Its structure is given as follows:

- The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_1)$ is the free $\mathbb{C}$-vector space on the set $\mathrm{Iso}(Q, \mathbb{F}_1)$. The set $\mathrm{Iso}(Q, \mathbb{F}_1)$ is indexed by the set of partitions Par.
- The grading of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $\mathbf{H}(Q, \mathbb{F}_1)_d = \langle [M] \mid \dim(M) = d \rangle_{\mathbb{C}}$.
- The multiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

  $$[M] \cdot [N] := \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)} C^R_{M,N} [R]$$

  where the structure coefficients $C^R_{M,N}$ are given by

  $$C^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \mid L \cong N, R/L \cong M\}.$$

- The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_1)} = [0]$.
- The comultiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

  $$\Delta([M]) = \sum_{\substack{[R],[L] \in \mathrm{Iso}(Q,\mathbb{F}_1) \\ M \cong R \oplus L}} [R] \otimes [L].$$

We see in particular that an isomorphism class $[M]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$ if and only if the representation M is indecomposable. We have seen that more generally the Lie algebra of primitive elements of $\mathbf{H}(Q, \mathbb{F}_1)$ has a basis consisting of all such $[M]$.

Example 14. We can again compute some structure constants:

1. Let again $\lambda = (1^n)$ and $\mu = (1^m)$, and consider $\kappa = (1^{n+m})$. We find as before that

   $$C^\kappa_{\lambda,\mu} = \text{number of } m\text{-dimensional subspaces of } \mathrm{N}_{n+m} = \binom{n+m}{m}.$$

   This is the same result as before by taking the limit $q \to 1$.
2. Let again $\lambda = (n)$ and $\mu = (m)$ and consider $\kappa = (n+m)$. We find as before that

   $$C^\kappa_{\lambda,\mu} = 1.$$

3. Let us compute the product $[\mathrm{N}_i] \cdot [\mathrm{N}_j]$. We observe that if $[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)$ and L is a subrepresentation of R that is isomorphic to N_j then the quotient R/L results from R by contracting one of the Jordan chains by j elements. If $R/L \cong \mathrm{N}_i$ then this means that R

consists of a single Jordan chain of length $i + j$, or of two Jordan chains of length i and j respectively. Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i,j)}] . + b[\mathrm{N}_{i+j}]$$

We have seen above that $b = C^{(i+j)}_{(i),(j)} = 1$. The coffient a is the number of entries of (i, j) that are of length j. Thus

$$a = \begin{cases} 1 & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = \begin{cases} [\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i \neq j, \\ 2[\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i = j. \end{cases}$$

We see in particular that $[\mathrm{N}_i]$ and $[\mathrm{N}_j]$ commute.

4. We find in the same way that for all $i_1, \dots, i_r \geq 1$ and $j \geq 1$,

$$[\mathrm{N}_{(i_1,\dots,i_r)}] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i_1,\dots,i_r,j)}] + \sum_\lambda b_\lambda [\mathrm{N}_\lambda] ,$$

where λ runs through all distinct tupels of the form $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ with $1 \leq k \leq r$. The coeffiecient a is given by

$$a = \text{how often } j \text{ occurs in } (i_1, \dots, i_r, j)$$

and the coefficient of b_λ for $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ are given by

$$b_\lambda = \text{how often } i_k + j \text{ occurs in } \lambda .$$

We have for example

$$\begin{aligned} [\mathrm{N}_{(5,3,3,2,1)}] \cdot [\mathrm{N}_2] = {} & 2[\mathrm{N}_{(5,3,3,2,2,1)}] \\ & + [\mathrm{N}_{(7,3,3,2,1)}] + 2[\mathrm{N}_{(5,5,3,2,1)}] + [\mathrm{N}_{(5,4,3,3,1)}] + 3[\mathrm{N}_{(5,3,3,3,2)}] . \end{aligned}$$

We see by induction that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated as an algebra by the N_i with $i \geq 1$.

Corollary 15. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is commutative.

Remark 16. We have seen last week that the Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is the universal enveloping algebra of its Lie algebra of primitive elements, which in turn is spanned (as a vector space) by the N_i. We have thus already seen last week that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated by the N_i as an algebra.

We have hence shown that $\mathbf{H}(Q, \mathbb{F}_1)$ is a commutative, cocommutative, graded Hopf algebra We will in the following show that it is actually the ring of symmetric functions.

4 The Ring of Symmetric Functions

4.1 Definition

For every $k \geq 0$ we denote by

$$\Lambda^{(k)} := \mathbb{C}[x_1, \dots, x_k]^{\mathrm{S}_k}$$

the algebra of symmetric polynomials in k variables. We have for every $r \geq 0$ a homomorphism of graded algebras

$$\Lambda^{(k+1)} \to \Lambda^{(k)}, \quad f(x_1, \dots, x_r, x_{k+1}) \mapsto f(x_1, \dots, x_r, 0).$$

Definition 17. The *ring of symmetric functions* Λ is the limit

$$\Lambda := \lim_{k \geq 0} \Bigl(\Lambda^{(k+1)} \to \Lambda^{(k)} \Bigr)$$

in the category of graded rings. The elements of Λ are *symmetric functions*.

Warning 18. A symmetric function is – contrary to its name – not a function.

Let us make the above definition more explicit: For every $n \geq 0$ we have

$$\begin{aligned} \Lambda_n &= \lim_{k \geq 0} \Bigl(\Lambda_n^{(k+1)} \to \Lambda_n^{(k)} \Bigr) \\ &= \left\{ \bigl(f_k(x_1, \dots, x_n) \bigr)_{k \geq 0} \;\middle|\; \begin{array}{l} f_k(x_1, \dots, x_k) \in \Lambda_n^{(k)} \text{ for every } k \geq 0 \text{ such that} \\ f_{k+1}(x_1, \dots, x_k, 0) = f_k(x_1, \dots, x_k) \text{ for every } k \geq 0 \end{array} \right\}, \end{aligned}$$

and we have overall

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n$$

as vector spaces. The multiplication on Λ is given by $(f_k)_{k \geq 0} \cdot (g_k)_{k \geq 0} = (f_k \cdot g_k)_{k \geq 0}$ for all $(f_k)_{k \geq 0} \in \Lambda_n$ and $(g_k)_{k \geq 0} \in \Lambda_m$.

A homogeneous symmetric function f, say of degree n, is thus the same as a “consistent choice” of homogeneous symmetric polynomials $f_k \in \Lambda^{(k)}$ of degree n for every $k \geq 0$. We have for every number of variables $k \geq 0$ a homomorphism of graded algebras

$$\Lambda \to \Lambda^{(k)}, \quad f \mapsto f(x_1, \dots, x_k)$$

that is given by projection onto the k-th component. For any two symmetric functions f, g we have by construction of Λ that

$$f = g \iff f(x_1, \dots, x_k) = g(x_1, \dots, x_k) \text{ for every } k \geq 0.$$

Example 19. We have for every number of variables $k \geq 0$ and every degree $n \geq 0$ the *elementary symmetric polynomial*

$$e_n^{(k)}(x_1, \dots, x_k) := \sum_{1 \leq i_1 < \dots < i_n \leq k} x_{i_1} \cdots x_{i_n},$$

with $e_n^{(k)} = 0$ whenever $n > k$. These polynomials are homogeneous and satisfy the conditions

$$e_n^{(k+1)}(x_1, \dots, x_k, 0) = e_n^{(k)}(x_1, \dots, x_k)$$

for all $k \geq 0$. These elementary symmetric polynomials $e_n^{(k)}$ with $k \geq 0$ therefore assemble into a single homogeneous symmetric function

$$e_n \in \Lambda_n \,.$$

This is the *n-th elementary symmetric function.*

We find similarly that the *power symmetric polynomials*

$$p_n^{(k)}(x_1, \dots, x_k) := x_1^n + \cdots + x_k^n \,,$$

and the *completely homogenous symmetric polynomials*

$$h_n^{(k)}(x_1, \dots, x_k) := \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq k} x_{i_1} \cdots x_{i_n} = \sum \text{monomials of homogeneous degree } n$$

result in homogeneous symmetric functions

$$p_n, h_n \in \Lambda \,.$$

These are the *power symmetric functions* and *completely homogeneous symmetric functions.*

Warning 20. The ring of symmetric functions Λ is *not* the limit of the rings of symmetric polynomials $\Lambda^{(k)}$ in the category of (commutative) rings. Indeed, the symmetric polyonmials

$$f_k(x_1, \dots, x_k) := x_1 + x_1 x_2 + \cdots + x_1 \cdots x_k$$

satisfy the compatibility condition $f_{k+1}(x_1, \dots, x_k, 0) = f_k(x_1, \dots, x_k)$ for every $k \geq 0$. But there exists no symmetric function $f \in \Lambda$ with $f(x_1, \dots, x_k) = f_k(x_1, \dots, x_k)$ for every $k \geq 0$.

An arbitrary family $(f_k)_{k \geq 0}$ of compatible symmetric polynomials $f_k \in \Lambda^{(k)}$ defines a symmetric function if and only if the degrees $\deg(f_k)$ are bounded, i.e. if there exists some degree $d \geq 0$ with $\deg(f_k) \leq d$ for every $k \geq 0$.

4.2 The Fundamental Theorem on Symmetric Functions

The *fundamental theorem of symmetric polynomials* asserts that for every number of variables $k \geq 0$ the elementary symmetric polynomials $e_1^{(k)}, \dots, e_k^{(k)}$ form an algebraically independent generating set for the algebra of symmetric polynomials $\Lambda^{(k)}$. It follows from this that the completely homogeneous symmetric polynomials $h_1^{(k)}, \dots, h_k^{(k)}$ form an algebraically independent generating set for $\Lambda^{(k)}$, and one can show that the same holds true for the power symmetric polynomials $p_1^{(k)}, \dots, p_k^{(k)}$.[2]

[2]For the elementary symmetric polynomials $e_i^{(k)}$ and homogeneous symmetric polynomials $h_i^{(k)}$ these statements do not only hold over the ground field $\mathbb{C}$, but over every commutative ring. For the power symmetric polynomials $p_i^{(k)}$ we need to work over a field in which the numbers $1, \dots, k$ are invertible.

For every partition $\lambda \in \mathrm{Par}$ we can consider the symmetric polynomials

$$e_\lambda^{(k)} := e_{\lambda_1}^{(k)} \cdots e_{\lambda_l}^{(k)}, \quad h_\lambda^{(k)} := h_{\lambda_1}^{(k)} \cdots h_{\lambda_l}^{(k)}, \quad p_\lambda^{(k)} := p_{\lambda_1}^{(k)} \cdots p_{\lambda_l}^{(k)}.$$

We have just formulated that the symmetric polynomials $e_\lambda^{(k)}$ for $\lambda \in \mathrm{Par}$ with length $\leq k$ form a vector space basis for $\Lambda^{(k)}$, and similarly for $h_\lambda^{(k)}$ and $p_\lambda^{(k)}$. We can generalize these families of symmetric polynomials to symmetric functions:

Example 21. For every $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we consider the symmetric functions

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_l}, \quad h_\lambda := h_{\lambda_1} \cdots h_{\lambda_l}, \quad p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}.$$

and note that

$$\begin{aligned} e_\lambda(x_1, \dots, x_k) &= e_\lambda^{(k)}(x_1, \dots, x_k), \\ h_\lambda(x_1, \dots, x_k) &= h_\lambda^{(k)}(x_1, \dots, x_k), \\ p_\lambda(x_1, \dots, x_k) &= p_\lambda^{(k)}(x_1, \dots, x_k). \end{aligned}$$

Another important family of symmetric polynomials are the *monomial symmetric polynomials*: For every partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_k)$ the corresponding monomial symmetric polynomial is given by

$$m_\lambda^{(k)}(x_1, \dots, x_k) = \sum \text{distinct permutations of } x_1^{\lambda_1} \cdots x_k^{\lambda_k}.$$

These polynomials also form a basis of $\Lambda^{(k)}$. They too can be generalized to symmetric functions.

Example 22. Let $\lambda \in \mathrm{Par}$ be a partition with $\lambda = (\lambda_1, \dots, \lambda_l)$. For every $k \geq l$ we again define

$$m_\lambda^{(k)}(x_1, \dots, x_k) = \sum \text{distinct permutations of } x_1^{\lambda_1} \cdots x_k^{\lambda_k}.$$

For $k < l$ we set

$$m_\lambda^{(k)} := 0.$$

Then

$$m_\lambda^{(k+1)}(x_1, \dots, x_k, 0) = m_\lambda^{(k)}(x_1, \dots, x_k)$$

for every $k \geq 0$, and each $m_\lambda^{(k)}$ is homogeneous of degree $|\lambda|$. We therefore get a well-defined homogeneous symmetric function

$$m_\lambda \in \Lambda,$$

which we call the *monomial symmetric function* associated to λ.

We now want to generalize the fundamental theorem on symmetric polynomials to symmetric functions. The key observation behind this is the following:

Proposition 23. The map $\Lambda_n^{(k+1)} \to \Lambda_n^{(k)}$ is an isomorphism whenever $k \geq n$.

Proof. A basis of $\Lambda_n^{(k)}$ is given by the symmetric polynomials $e_\lambda^{(k)}$ where λ is of length k and the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfies

$$\lambda_1 + 2\lambda_2 + \cdots + k\lambda_k = n\,.$$

A basis of $\Lambda_n^{(k)}$ is given by the symmetric polynomials $e_\mu^{(k+1)}$ where μ is of length $k+1$ and the partition $\mu = (\mu_1, \dots, \mu_k, \mu_{k+1})$ satisfies

$$\mu_1 + 2\mu_2 + \cdots + k\mu_k + (k+1)\mu_{k+1} = n\,.$$

If $k \geq n$ then $k+1 > n$ and we find that $\mu_{k+1} = 0$. We now find that the map $\Lambda_n^{(k+1)} \to \Lambda_n^{(k)}$ restricts to a bijection between those bases. □

Corollary 24. The map $\Lambda_n \to \Lambda_n^{(k)}$ is an isomorphism whenever $k \geq n$. □

Corollary 25. The following families of symmetric functions form vector space bases of Λ:

1. The elementary symmetric polynomials e_λ with $\lambda \in \mathrm{Par}$.
2. The complete homogeneous symmetric polynomials h_λ with $\lambda \in \mathrm{Par}$.
3. The power symmetric polynomials p_λ with $\lambda \in \mathrm{Par}$.
4. The monomial symmetric polynomials m_λ with $\lambda \in \mathrm{Par}$.

Corollary 26. The elementary symmetric functions e_i with $i \geq 1$ form an algebraically independent algebra generating set for Λ, and similarly for the h_i and the p_i.

Corollary 27. We have $\Lambda \cong \mathbb{C}[X_1, X_2, X_3, \dots]$ as graded algebras, where X_i is of degree i.

Remark 28. It follows from Corollary 24 for any two symmetric functions $f, g \in \Lambda$ that

$$f = g \iff f(x_1, \dots, x_k) = g(x_1, \dots, x_k) \text{ for some } k \geq \deg(f), \deg(g).$$

Remark 29. We have for every number of variables $k \geq 0$ an embedding of graded algebras $\Lambda^{(k)} \to \Lambda$ given by $e_i^{(k)} \to e_i$. (The homomorphism $\Lambda \to \Lambda^{(k)}$ is a retract for this inclusion.) We can thus regard $\Lambda^{(k)}$ as subring of Λ. It then follows that

$$\Lambda \cong \operatorname*{colim}_{k \geq 0} \left(\Lambda^{(k)} \to \Lambda^{(k+1)} \right)$$

where the homomorphism $\Lambda^{(k)} \to \Lambda^{(k+1)}$ are given by the embeddings $e_i^{(k)} \mapsto e_i^{(k+1)}$.

4.3 Hopf Algebra Structure

We can endow the algebra of symmetric functions Λ with the structure of a graded Hopf algebra.

Lemma 30. Let G, H be two groups. Let V be a representation of G and let W be a representation of H. Then

$$(V \otimes W)^{G \times H} = V^G \otimes W^H\,.$$

Proof. The inclusion $V^G \otimes W^H \subseteq (V \otimes W)^{G\times H)}$ can be checked on simple tensors. Let on the other hand $x \in (V \otimes W)^{G\times H}$. We may choose a basis $(v_i)_{i\in I}$ of V and write $x = \sum_{i\in I} v_i \otimes w_i$ for some unique vectors $w_i \in W$. For every element $h \in H$ we then have

$$\sum_{i\in I} v_i \otimes w_i = x = (1,h)x = \sum_{i\in I} v_i \otimes (hw_i)\,.$$

It follows from the uniqueness of the w_i that $hw_i = w_i$ for every $h \in H$ and every $i \in I$, and thus $w_i \in W^H$ for every $i \in I$. This shows that $x \in V \otimes W^H$. We find in the same way that $x \in V^G \otimes W^H$, and thus $x \in (V \otimes W^H) \cap (V^G \otimes W) = V^G \otimes W^H$. $\square$

We have now for any two number of variables $k, l \geq 0$ a homomorphism of graded algebras

$$\begin{aligned} \Delta_{kl} \colon \Lambda^{(k+l)} &= \mathbb{C}[x_1, \dots, x_{k+l}]^{S_{k+l}} \\ &\subseteq \mathbb{C}[x_1, \dots, x_{k+l}]^{S_k\times S_l} \\ &\cong (\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}[x_{k+1}, \dots, x_{k+l}])^{S_k\times S_l} \\ &\cong (\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}[x_1, \dots, x_l])^{S_k\times S_l} \\ &= \mathbb{C}[x_1, \dots, x_k]^{S_k} \otimes \mathbb{C}[x_1, \dots, x_l]^{S_l} \\ &= \Lambda^{(k)} \otimes \Lambda^{(l)}. \end{aligned}$$

We would like to have a homomorphism of graded algebras $\Delta \colon \Lambda \to \Lambda \otimes \Lambda$ such that for all degrees $k, l \geq 0$ the square diagram

$$\begin{tikzcd} \Lambda \arrow[r, dashed, "\Delta"] \arrow[d] & \Lambda \otimes \Lambda \arrow[d] \\ \Lambda^{(k+l)} \arrow[r, "\Delta_{kl}"] & \Lambda^{(k)} \otimes \Lambda^{(l)} \end{tikzcd}$$

commutes. The composition $\Lambda \to \Lambda^{(k)} \otimes \Lambda^{(l)}$ is given on the algebra generators p_i of Λ by

$$p_i \mapsto p_i^{(k)} \otimes 1 + 1 \otimes p_i^{(l)}\,.$$

Such an algebra homomorphism Δ is thus given by

$$\Delta(p_i) = p_i \otimes 1 + 1 \otimes p_i\,.$$

The homomorphism Δ makes the algebra Λ into a cocommutative, graded bialgebra. The counit is given on algebra generators by

$$\varepsilon(p_i) = 0$$

for every $i \geq 0$. Since Λ is graded and connected it follows that it is already a graded Hopf algebra. Its antipode is given on algebra generators by

$$S(p_i) = -p_i$$

for every $i \geq 0$.

We have made Λ into a commutative, cocommutative, graded Hopf algebra.

5 The Isomorphism $\mathbf{H}(Q, \mathbb{F}_1) \cong \Lambda$

Both $\mathbf{H}(Q, \mathbb{F}_1)$ and Λ are commutative, cocommutate, graded Hopf algebras. They are isomorphic as graded Hopf Algebras:

The ring of symmetric functions Λ has is, as a commutative algebra, freely generated by the power symmetric functions $p_1, p_2, \dots$ There hence exists a unique, surjective algebra homomorphism $\Phi \colon \Lambda \to \mathbf{H}(Q, \mathbb{F}_1)$ with $\Phi(p_i) = [\mathrm{N}_i]$ for every $i \geq 1$. We note that Φ is a homomorphism of graded algebras because both p_i and $[\mathrm{N}_i]$ are of degree i. We also have for every degree $n \geq 0$ that

$$\dim \Lambda_n = \#\{(\lambda_1, \dots, \lambda_k) \in \mathrm{Par} \mid \lambda_1 + 2\lambda_2 + \cdots + k\lambda_k = n\} = \dim \mathbf{H}(Q, \mathbb{F}_1)_n \,,$$

with these dimensions being finite. It thus follows from the surjectivity of Φ that it is already an isomorphism of graded algebras.

The algebra isomorphism Φ is already an isomorphism of Hopf algebras: It sufficies to check that Φ is compatible with the comultiplication of the algebra generators $[\mathrm{N}_i]$. This holds since p_i is primitive in Λ and $[\mathrm{N}_i]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$.

We have shown altogether that Φ is an isomorphism of graded Hopf algebras.

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