Jordan Quiver, Part I

Talk 10 on Hall Algebras and Quantum Groups

1. The Jordan Quiver and its Nilpotent Representations

Definition 1.1. The *Jordan quiver* is the quiver that consists of a single vertex and a single edge, which is necessarily a loop.

Throughout this talk the Jordan quiver is denoted by Q. See Figure 1 for a visualization. In the following we write $\Bbbk$ to mean a field or $\mathbb{F}_1$.

A representation of the Jordan quiver over $\Bbbk$ is the same as a pair (V, f) consisting of a $\Bbbk$-vector space V together with an endomorphism f of V. Two such representations (V, f) and (W, g) are isomorphic if and only if V, W are isomorphic as $\Bbbk$-vector spaces and the endomorphisms f, g are similar. We hence find that that the isomorphism classes of Q-representations over $\Bbbk$ are in one-to-one correspondence to conjugacy classes of endomorphisms of $\Bbbk$-vector spaces.

Suppose that V is finite-dimensional, and that $\Bbbk$ is algebraically closed, or that it is $\mathbb{F}_1$, or that we are interested only in nilpotent endomorphisms. Then one can use the usual Jordan normal form to classify these conjugacy classes.

A representation (V, f) of the Jordan quiver is nilpotent if and only if the endomorphism f is nilpotent. We will in the rest of this talk restrict our attention to finite-dimensional, nilpotent representations of the Jordan quiver. As introduced in the previous talks we denote by

$$\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$$

the full subcategory of $\mathbf{Rep}(Q, \Bbbk)$ whose objects are the finite-dimensional, nilpotent representations of Q over $\Bbbk$. We denote the set of isomorphism classes of $\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$ by

$$\mathrm{Iso}(Q, \Bbbk) \coloneqq \mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)/\cong .$$

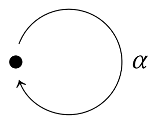

Figure 1: The Jordan quiver Q.

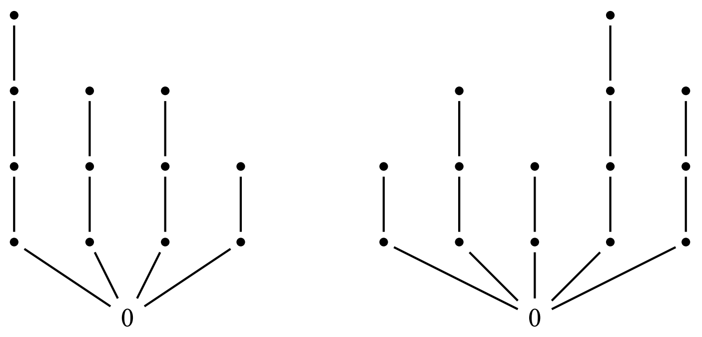

Figure 2: The representations $\mathrm{N}_{(4,3,3,2)}$ and $\mathrm{N}_{(2,3,2,4,3)}$ over $\mathbb{F}_1$.

Definition 1.2. For every dimension $d \geq 0$ let

$$\mathrm{N}_d := \left(\Bbbk^d, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \right)$$

if $\Bbbk$ is a field, and let

$$\mathrm{N}_d := (\{0, 1, \dots, d\}, [d \mapsto (d-1) \mapsto (d-2) \mapsto \cdots \mapsto 1 \mapsto 0 \mapsto 0])$$

if $\Bbbk = \mathbb{F}_1$. For every tupel $(d_1, \dots d_n)$ of dimensions $d_i \geq 0$ let

$$\mathrm{N}_{(d_1,\dots,d_n)} := \mathrm{N}_{d_1} \oplus \cdots \oplus \mathrm{N}_{d_n}\,.$$

Example 1.3. See Figure 2 for visualizations of $\mathrm{N}_{(4,3,3,2)}$ and $\mathrm{N}_{(2,3,2,4,3)}$ over $\mathbb{F}_1$.

Definition 1.4. For every $n \geq 0$ let $\mathrm{Par}(n)$ be the set of partition of the number n, i.e.

$$\mathrm{Par}(n) := \left\{ (\lambda_1, \dots, \lambda_l) \;\middle|\; \begin{array}{l} \lambda_1, \dots, \lambda_l \in \mathbb{N} \\ \lambda_1 \geq \cdots \geq \lambda_l \geq 1 \\ \lambda_1 + \cdots + \lambda_l = n \end{array} \right\}.[1]$$

The set of all partitions is denoted by

$$\mathrm{Par} := \coprod_{n \geq 0} \mathrm{Par}(n)\,.$$

Proposition 1.5. The representations N_λ with $\lambda \in \mathrm{Par}$ form a set of representatives for $\mathrm{Iso}(Q, \Bbbk)$.

Proof. This follows from the existence and uniqueness of the Jordan normal form of nilpotent endomorphisms. (See Appendix A.1 for the Jordan normal form over $\mathbb{F}_1$.) □

[1]We want to point out that in this talk we do not allow a partition to contain zero as an entry. This is done purely for technical reasons.

2. The Hall Algebra of the Jordan Quiver over $\mathbb{F}_q$

We will now consider for $\Bbbk$ the finite field $\mathbb{F}_q$. Then $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is an abelian, finitary, full, exact subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q)$. We want to consider in the following the Hall algebra of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$. For this we need to understand its Euler form.

Proposition 2.1. Let $\mathscr{A} \coloneqq \mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ and let $S \coloneqq \mathrm{N}_1 = (\Bbbk, [0])$.

1. The category $\mathscr{A}$ is hereditary, i.e. $\mathrm{Ext}^n_{\mathscr{A}} = 0$ for every $n \geq 2$.
2. The Grothendieck group $\mathrm{K}(\mathscr{A})$ is freely generated by the class $[S]$. Thus $\mathrm{K}(\mathscr{A}) \cong \mathbb{Z}$ via the map $[M] \mapsto \dim(M)$.
3. The Euler form of $\mathscr{A}$ vanishes.

Proof. See Appendix A.2 and Appendix A.3. □

Since $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is abelian and finitary with vanishing Euler form we can consider its Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$. We find that Green's coproduct makes $\mathbf{H}(Q, \mathbb{F}_q)$ into a graded bialgebra because $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ satisfies the finite subject condition and has vanishing Euler form.[2] It further follows that $\mathbf{H}(Q, \mathbb{F}_q)$ is a graded Hopf algebra since it is connected. (See Appendix A.5 for a more detailed overview.)

The Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ has the classes $[\mathrm{N}_\lambda]$ with $\lambda \in \mathrm{Par}$ as a basis. The multiplication of $\mathbf{H}(Q, \mathbb{F}_q)$ is given by

$$[\mathrm{N}_\lambda] \cdot [\mathrm{N}_\mu] = \sum_{\nu \in \mathrm{Par}} F^\nu_{\lambda,\mu} [\mathrm{N}_\nu]$$

where the structure constants $F^\nu_{\lambda,\mu}$ are given by

$$F^\nu_{\lambda,\mu} = \#\{\text{subrepresentation } L \text{ of } \mathrm{N}_\nu \mid L \cong \mathrm{N}_\mu, \mathrm{N}_\nu / L \cong \mathrm{N}_\lambda\}\,.$$

We will now compute some of the structure constants, following [Sch09, Example 2.2].

Example 2.2 (Structure constants)**.**

1. Let $\lambda = (1^n)$ and $\mu = (1^m)$. We consider the partition $\nu \coloneqq (1^{n+m})$. The action of the edge of the Jordan quiver Q on the representations N_λ, N_μ and N_ν is trivial. We thus find that every m-dimensional linear subspace L of N_ν satisfies the conditions $L \cong \mathrm{N}_\mu$ and $\mathrm{N}_\nu / L \cong \mathrm{N}_\lambda$. The structure constant $F^\nu_{\lambda,\mu}$ is therefore given by

$$\begin{aligned} F^\nu_{\lambda,\mu} &= \text{number of } m\text{-dimensional linear subspaces of } \mathrm{N}_\nu \\ &= \#\mathrm{Gr}(m, n+m, \mathbb{F}_q) \\ &= \begin{bmatrix} n+m \\ m \end{bmatrix}_q . \end{aligned}$$

   (See Lemma A.12 for the last equality.) We see in particular that $F^\nu_{\lambda,\mu}$ depends is a polynomial way on q (with natural coefficients). See Appendix A.7 for some explicit calculations of binomial coefficients.

[2] We say that a category $\mathscr{A}$ satisfies the *finite subobject condition* if every object of $\mathscr{A}$ admits only finitely many subobjects.

2. Let now $\lambda = (n)$ and $\mu = (m)$. We consider the partition $\nu = (n+m)$. The representation N_ν has the standard basis $e_1, \dots, e_{n+m}$, and the subrepresentations of N_ν are given by $\langle e_1, \dots, e_i \rangle$ for $i = 0, \dots, n+m$. The subrepresentations $L \coloneqq \langle e_1, \dots, e_m \rangle$ is the unique one that is isomorphic to N_μ, and its quotient N_ν / L is isomorphic to N_λ. Thus
$$F^{(n+m)}_{(n),(m)} = 1\,.$$
3. One finds for all $n \geq 2$ and $m \geq 1$ that
$$F^{(n,1^m)}_{(n),(1^m)} = q^m = F^{(n,1^m)}_{(1^m),(n)}\,,$$
see Appendix A.8.

We observe that in the above examples the coefficient $F^\nu_{\lambda,\mu}$ are always polynomials in q with integer coefficients. We will see in next week's talk that this is true for any coefficient $F^\nu_{\lambda,\mu}$. This will allow us to define the *generic Hall algebra* of the Jordan quiver.

We also have $F^\nu_{\lambda,\mu} = F^\nu_{\mu,\lambda}$ in each example. We will see in next week's talk that the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ is indeed commutative, which means precisely that $F^\nu_{\lambda,\mu} = F^\nu_{\mu,\lambda}$ for any three partitions $\lambda, \mu, \nu \in \mathrm{Par}$.

3. The Hall Algebra of the Jordan Quiver over $\mathbb{F}_1$

We will now consider the case that $\Bbbk$ is $\mathbb{F}_1$. We have seen in last week's talk how to construct the Hall algebra of Q over $\mathbb{F}_1$, and that it is a cocommutative, graded Hopf algebra. (See Appendix A.9 for a more detailed overview.)

We have also seen that $\mathbf{H}(Q, \mathbb{F}_1)$ is the universal enveloping algebra of its Lie algebra of its primitve elements, which is spanned (as a vector space) by those isomorphism classes $[M]$ for which M is indecomposable. We have the following consequence:

Corollary 3.1. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is generated by $[\mathrm{N}_i]$ for $i \geq 1$.

A more explicit proof of Corollary 3.1 can be found in Remark A.15.

Example 3.2 (Structure constants)**.** We can again compute some structure constants $F^\nu_{\lambda,\mu}$.

1. Let again $\lambda = (1^n)$ and $\mu = (1^m)$, and consider $\nu = (1^{n+m})$. We find as before that
$$F^\nu_{\lambda,\mu} = \text{number of } m\text{-dimensional subspaces of } \mathrm{N}_{n+m} = \binom{n+m}{m}.$$
2. Let again $\lambda = (n)$ and $\mu = (m)$ and consider $\nu = (n+m)$. We find as before that
$$F^\nu_{\lambda,\mu} = 1\,.$$
3. For all $n \geq 2$ and $m \geq 1$ we have
$$F^{(n,1^m)}_{(n),(1^m)} = 1 = F^{(n,1^m)}_{(1^m),(n)}\,.$$

4. Let us compute the product $[\mathrm{N}_i] \cdot [\mathrm{N}_j]$.

   We observe that if $[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)$ and L is a subrepresentation of R that is isomorphic to N_j then the quotient R/L results from R by contracting one of the Jordan chains of R by j elements. If $R/L \cong \mathrm{N}_i$ then this means that R consists of a single Jordan chain of length $i + j$, or of two Jordan chains of length i and j respectively. Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i,j)}] + b[\mathrm{N}_{i+j}] \,.$$

   We have seen above that $b = F^{(i+j)}_{(i),(j)} = 1$. The coeffient $a = F^{(i,j)}_{(i),(j)}$ is given by

$$\begin{aligned} a &= \text{how often } j \text{ occurs in } (i,j) \\ &= \begin{cases} 1 & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases} \end{aligned}$$

   Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = \begin{cases} [\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i \neq j, \\ 2[\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i = j. \end{cases} \tag{1}$$

   We see in particular that $[\mathrm{N}_i]$ and $[\mathrm{N}_j]$ commute.

We note that in the first three of the above examples we get the same results as for $\Bbbk = \mathbb{F}_q$ by taking the limit $q \to 1$ (i.e. setting q equal to 1). We will see more on this phenomenon in next week's talk.

Corollary 3.3. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is commutative.

Proof. Its generators $[\mathrm{N}_i]$ with $i \geq 1$ commute. □

We see now that $\mathbf{H}(Q, \mathbb{F}_1)$ is a commutative, cocommutative, graded Hopf algebra. It has a basis indexed by partitions, and its graded parts have the dimension

$$\dim \mathbf{H}(Q, \mathbb{F}_1)_k = \#\{(\lambda_1, \dots, \lambda_l) \in \mathrm{Par} \mid \lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = k\}$$

for every $k \geq 0$.

In the land of combinatorics there is another algebra with these properties, namely the *ring of symmetric functions.*

4. The Ring of Symmetric Functions Λ

4.1. Definition of Λ

For every $n \geq 0$ we denote by

$$\Lambda^{(n)} := \mathbb{C}[x_1, \dots, x_n]^{\mathrm{S}_n}$$

the algebra of symmetric polynomials in n variables. We have for every number of variables $n \geq 0$ a homomorphism of graded algebras

$$\Lambda^{(n+1)} \to \Lambda^{(n)} \,, \quad f^{(n+1)} \mapsto f^{(n+1)}(x_1, \dots, x_n, 0) \,.$$

Definition 4.1. The *ring of symmetric functions* Λ is the limit

$$\Lambda := \lim_{n \geq 0} \Bigl(\Lambda^{(n+1)} \to \Lambda^{(n)} \Bigr)$$

in the category of graded algebras. The elements of Λ are *symmetric functions.*

Warning 4.2. A symmetric function is – contrary to its name – not a function.

Let us make the above definition more explicit: For every degree $k \geq 0$ we have

$$\begin{aligned} \Lambda_k &= \lim_{n \geq 0} \Bigl(\Lambda_k^{(n+1)} \to \Lambda_k^{(n)} \Bigr) \\ &= \left\{ \bigl(f^{(n)} \bigr)_{n \geq 0} \;\middle|\; \begin{array}{l} f^{(n)} \in \Lambda_k^{(n)} \text{ for every } n \geq 0 \text{ such that} \\ f^{(n+1)}(x_1, \dots, x_n, 0) = f^{(n)} \text{ for every } n \geq 0 \end{array} \right\}, \end{aligned}$$

and we have overall

$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$

as vector spaces. A homogeneous symmetric function f, say of degree k, is thus the same as a "consistent choice" of homogeneous symmetric polynomials $f^{(n)}$ of degree k for every number of variables $n \geq 0$.

We have for every number of variables $n \geq 0$ a homomorphism of graded algebras

$$\Lambda \to \Lambda^{(n)}, \quad f \mapsto f(x_1, \dots, x_n)$$

that is given in each degree by projection onto the n-th component.[3]

Example 4.3. We have for every number of variables $n \geq 0$ and every degree $k \geq 0$ the *elementary symmetric polynomial*

$$e_k^{(n)}(x_1, \dots, x_n) := \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k} \in \Lambda_k^{(n)},$$

with $e_k^{(n)} = 0$ whenever $k > n$. These polynomials satisfy the compatibility condition

$$e_k^{(n+1)}(x_1, \dots, x_n, 0) = e_k^{(n)}$$

for all $n \geq 0$. These elementary symmetric polynomials $e_k^{(n)}$ with $n \geq 0$ therefore assemble into a single homogeneous symmetric function

$$e_k \in \Lambda_k .$$

This is the *k-th elementary symmetric function.*

[3] For any two symmetric functions $f, g \in \Lambda$ we have by construction of the algebra Λ that $f = g$ if and only if $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ for every number of variables $n \geq 0$. It also suffices that this equality holds for all sufficiently large n, i.e. if there exists some N such that this equality holds for all $n \geq N$.

We find similarly that the *power symmetric polynomials*

$$p_k^{(n)}(x_1, \dots, x_n) := x_1^k + \cdots + x_n^k ,$$

and the *complete homogenous symmetric polynomials*

$$h_k^{(n)}(x_1, \dots, x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k} = \sum \text{monomials of homogeneous degree } k$$

result in homogeneous symmetric functions

$$p_k, h_k \in \Lambda_k .$$

These are the *power symmetric functions* and *complete homogeneous symmetric functions.*

See Appendix A.11 for some remarks about the definition of Λ.

4.2. The Fundamental Theorem on Symmetric Functions

The *fundamental theorem of symmetric polynomials* asserts that for every number of variables $n \geq 0$ the elementary symmetric polynomials

$$e_1^{(n)}, \dots, e_n^{(n)}$$

form an algebraically independent generating set for the algebra of symmetric polynomials $\Lambda^{(n)}$. It follows from this that both

$$h_1^{(n)}, \dots, h_n^{(n)} \quad \text{and} \quad p_1^{(n)}, \dots, p_n^{(n)}$$

also form algebraically independent algebra generating set for $\Lambda^{(n)}$.[4] We hence have

$$\Lambda^{(n)} \cong \mathbb{C}[x_1, \dots, x_n]^{\mathrm{S}_n}$$

as graded algebras, where x_k is homogeneous of degree k.

F 40 or every partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we can consider the symmetric polynomials

$$e_\lambda^{(n)} := e_{\lambda_1}^{(n)} \cdots e_{\lambda_l}^{(n)} , \qquad h_\lambda^{(n)} := h_{\lambda_1}^{(n)} \cdots h_{\lambda_l}^{(n)} , \qquad p_\lambda^{(n)} := p_{\lambda_1}^{(n)} \cdots p_{\lambda_l}^{(n)} .$$

We have just formulated that the symmetric polynomials

$$e_\lambda^{(n)} \qquad \text{for } \lambda \in \mathrm{Par} \text{ with length } \ell(\lambda) \leq n$$

form a vector space basis for $\Lambda^{(n)}$, and similarly for $h_\lambda^{(n)}$ and $p_\lambda^{(n)}$. We can generalize these families of symmetric polynomials to symmetric functions:

[4] For the elementary symmetric polynomials $e_k^{(n)}$ and homogeneous symmetric polynomials $h_k^{(n)}$ these statements do not only hold over the ground field $\mathbb{C}$, but over every commutative ring. For the power symmetric polynomials $p_k^{(n)}$ we need to work over a field in which the numbers $1, \dots, n$ are invertible.

Example 4.4. For every partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we consider the symmetric functions

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_l}\,, \quad h_\lambda := h_{\lambda_1} \cdots h_{\lambda_l}\,, \quad p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}$$

and note that

$$\begin{aligned} e_\lambda(x_1, \dots, x_n) &= e^{(n)}_\lambda\,, \\ h_\lambda(x_1, \dots, x_n) &= h^{(n)}_\lambda\,, \\ p_\lambda(x_1, \dots, x_n) &= p^{(n)}_\lambda\,. \end{aligned}$$

We now want to generalize the fundamental theorem on symmetric polynomials to symmetric functions. The key observation behind this is the following:

Proposition 4.5. The map $\Lambda^{(n+1)}_k \to \Lambda^{(n)}_k$ is an isomorphism whenever $n \geq k$.

Proof. A vector space basis of $\Lambda^{(n)}_k$ is given by the symmetric polynomials $e^{(n)}_\lambda$ where the partition λ is of length $\ell(\lambda) \leq n$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies

$$\lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = k\,.$$

A vector space basis of $\Lambda^{(n)}_k$ is given by the symmetric polynomials $e^{(n+1)}_\mu$ where the partition μ is of length $\ell(\mu) \leq n+1$ and $\mu = (\mu_1, \dots, \mu_l)$ satisfies

$$\mu_1 + 2\mu_2 + \cdots + l\mu_l = k\,.$$

We find by degree reasons that the case $\ell(\mu) = n+1$ cannot occur. The linear map $\Lambda^{(n+1)}_k \to \Lambda^{(n)}_k$ does therefore restrict to a bijection between the above bases. □

Corollary 4.6. The map $\Lambda_k \to \Lambda^{(n)}_k$ is an isomorphism whenever $n \geq k$. □

Corollary 4.7. The following families of symmetric functions form vector space bases of Λ:

1. The elementary symmetric polynomials e_λ with $\lambda \in \mathrm{Par}$.
2. The complete homogeneous symmetric polynomials h_λ with $\lambda \in \mathrm{Par}$.
3. The power symmetric polynomials p_λ with $\lambda \in \mathrm{Par}$. □

Corollary 4.8. The elementary symmetric functions e_k with $k \geq 1$ form an algebraically independent algebra generating set for Λ, and similarly the h_k and the p_k. □

Corollary 4.9. We have $\Lambda \cong \mathbb{C}[X_1, X_2, X_3, \dots]$ as graded algebras, where each variable X_k is homogeneous of degree k. □

See Appendix A.13 for more consequences.

4.3. Hopf Algebra Structure on Λ

We can endow the algebra of symmetric functions Λ with the structure of a graded Hopf algebra. We have for any two number of variables $n, m \geq 0$ a homomorphism of graded algebras

$$\begin{aligned}\Delta_{nm} : \Lambda^{(n+m)} &= \mathbb{C}[x_1, \dots, x_{n+m}]^{\mathrm{S}_{n+m}} \\ &\subseteq \mathbb{C}[x_1, \dots, x_{n+m}]^{\mathrm{S}_n \times \mathrm{S}_m} \\ &\cong (\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_{n+1}, \dots, x_{n+m}])^{\mathrm{S}_n \times \mathrm{S}_m} \\ &\cong (\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_1, \dots, x_m])^{\mathrm{S}_n \times \mathrm{S}_m} \\ &= \mathbb{C}[x_1, \dots, x_n]^{\mathrm{S}_n} \otimes \mathbb{C}[x_1, \dots, x_m]^{\mathrm{S}_m} && \text{(Lemma A.18)} \\ &= \Lambda^{(n)} \otimes \Lambda^{(m)}.\end{aligned}$$

We would like to have a homomorphism of graded algebras $\Delta : \Lambda \to \Lambda \otimes \Lambda$ such that for any number of variables $n, m \geq 0$ the square diagram

$$\begin{tikzcd} \Lambda \arrow[r, dashed, "\Delta"] \arrow[d] & \Lambda \otimes \Lambda \arrow[d] \\ \Lambda^{(n+m)} \arrow[r, "\Delta_{nm}"] & \Lambda^{(n)} \otimes \Lambda^{(m)} \end{tikzcd}$$

commutes. The composition $\Lambda \to \Lambda^{(n)} \otimes \Lambda^{(m)}$ is given on the algebra generators p_k of Λ by

$$p_k \mapsto p_k^{(n)} \otimes 1 + 1 \otimes p_k^{(m)} .$$

Such an algebra homomorphism Δ is thus given by

$$\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k .$$

This homomorphism exists because Λ is the free commutative algebra on the generators p_k.

The homomorphism Δ makes the algebra Λ into a cocommutative, graded bialgebra. Since Λ is graded and connected it follows that it is already a graded Hopf algebra.

We see altogether that Λ is a commutative, cocommutative, graded Hopf algebra. It has a basis indexed by partitions, and its graded parts have the dimension

$$\dim \Lambda_k = \#\{(\lambda_1, \dots, \lambda_l) \in \mathrm{Par} \mid \lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = k\} .$$

5. The Isomorphism $\mathbf{H}(Q, \mathbb{F}_1) \cong \Lambda$

Both $\mathbf{H}(Q, \mathbb{F}_1)$ and Λ are commutative, cocommutate, graded Hopf algebras. They are actually isomorphic:

The ring of symmetric functions Λ is, as a commutative algebra, freely generated by the power symmetric functions $p_1, p_2, p_3, \dots$ There hence exists a unique, surjective algebra homomorphism $\Phi : \Lambda \to \mathbf{H}(Q, \mathbb{F}_1)$ with

$$\Phi(p_k) = [\mathrm{N}_k]$$

for every degree $k \geq 1$. We note that Φ is a homomorphism of graded algebras because both p_i and $[\mathrm{N}_k]$ are of degree k. We also have for every degree $k \geq 0$ that $\dim \Lambda_k = \dim \mathbf{H}(Q, \mathbb{F}_1)_k$, with this dimension being finite. It thus follows from the surjectivity of Φ that it is already an isomorphism of graded algebras.

The algebra isomorphism Φ is already an isomorphism of Hopf algebras: It sufficies to check that Φ is compatible with the comultiplication of the algebra generators p_k. This holds since p_k is primitive in Λ and $[\mathrm{N}_k]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$.

We have shown altogether that Φ is an isomorphism of graded Hopf algebras.

Remark 5.1. One can show more generall that the isomorphism class $[\mathrm{N}_\lambda]$ for $\lambda \in \mathrm{Par}$ corresponds under the above isomorphism Φ to the monomial symmetric function m_λ. See Appendix A.12 and Appendix A.15 for more details on this.

The stucture constants $F^\nu_{\lambda,\mu}$ of the Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ can thus be understood in the language of symmetric functions as those coeffcients for which

$$m_\lambda m_\mu = \sum_{\nu \in \mathrm{Par}} F^\nu_{\lambda,\mu} m_\nu .$$

A. Further Remarks and Missing Proofs

The following isn't proofread very well and should probably be treated with care.

A.1. Theorem of Krull–Remak–Schmidt and Jordan Normal Form

Let V be an $\mathbb{F}_1$-vector space and let $f\colon V \to V$ be an endomorphism.

Recall A.1.

1. A *subspace* of V is a subset of V that contains the base point 0.
2. If $(U_i)_{i\in I}$ is a collection of subspaces of V then $V = \bigoplus_{i\in I} U_i$ if and only if every nonzero evement of V is contained in precisely one U_i, i.e. if and only if the U_i give a disjoint decomposition of the set $V \setminus \{0\}$.
3. If U is a subspace of V and $V = \bigoplus_{i\in I} W_i$ is a direct sum decomposition then $U = \bigoplus_{i\in I}(U \cap W_i)$.

Definition A.2. A subspace U of V is *f-invariant* if $f(U) \subseteq U$. An f-invariant subspace U of V is *indecomposable* if it is nonzero and there exist no two nonzero f-invariant subspaces W_1, W_2 of U with $U = W_1 \oplus W_2$.

Remark A.3. If U is an indecomposable subspace of V and $U = \bigoplus_{i\in I} W_i$ is any decomposition into f-invariant subspaces W_i then it follows that $U = W_j$ for some $j \in I$ while $W_i = 0$ for every $i \neq j$. Indeed, some W_j must be nonzero because V is nonzero. Then $V = W_j \oplus \bigoplus_{i\in I, i\neq j} W_i$ and thus $\bigoplus_{i\in I, i\neq j} W_i = 0$, and therefore $W_i = 0$ for every $i \in I$.

Proposition A.4 (Krull–Remak–Schmidt)**.** There exists a unique direct sum decomposition of V into indecomposable f-invariant subspaces.

Proof. In the following we mean by a *decomposition* a direct sum decomposition into f-invariant subspaces in which each direct summand is nonzero. We say that a decomposition $V = \bigoplus_{i\in I} U_i$ is *finer* than a decomposition $V = \bigoplus_{j\in J} W_j$ if each U_i is contained in some W_j. This gives a partial order on the set of decompositions of V.

We note that a decomposition $V = \bigoplus_{i\in I} U_i$ consists of indecomposables if and only if it is maximal fine. Indeed, if some U_j is decomposable then there exists a decomposition $U_j = U_j' \oplus U_j''$. Then

$$V = \bigoplus_{i\in I} U_i = \bigoplus_{\substack{i\in I\\ i\neq j}} U_i \oplus U_j = \bigoplus_{\substack{i\in I\\ i\neq j}} U_i \oplus U_j' \oplus U_j'$$

with the last term being a strictly finer that the original decomposition $V = \bigoplus_{i\in I} U_i$. Suppose on the other hand that each U_i is indecomposable and that $V = \bigoplus_{j\in J} W_j$ is a decomposition that is finer than $V = \bigoplus_{i\in I} U_i$. Then $U_i = \bigoplus_{j\in J}(U_i \cap V_j)$ for every $j \in J$. It follows that $U_i = U_i \cap V_j$ for some $j \in J$ and thus $U_i \subseteq V_j$. We also know that V_j is contained in some U_k. Then U_i is contained in U_k whence it follows that $i = k$ and thus $U_i = V_j$. This shows that each U_i equals some V_j, from which it follows that both decompositions must coincide.

We hence need to show that there exists a unique decomposition which is maximal fine. It sufficies to show that any collection of decompositions has a common refinement. Taking a common refinement of all decompositions then gives the desired one.

Let $V = \bigoplus_{j \in J_i} U^i_j$ with $i \in I$ be a collection of decompositions. For every nonzero vector $v \in V$ there exists for every $i \in I$ a unique index $j(i,v) \in J_i$ with $v \in U^i_{j(i,v)}$. We consider

$$W_v := \bigcap_{i \in I} U^i_{j(i,v)} \,.$$

Each W_v is an intersection of f-invariant subspaces and therefore again an f-invariant subspace. Each nonzero vector v of V is contained in some $W_{v'}$, namely for $v' = v$.

Suppose that for two nonzero vectors $v, u \in V$ the subspaces W_v and W_u intersect nonzero. Let w be a nonzero vector contained in both W_v and W_u. Then for every index $i \in I$ the vector w is contained in $U^i_{j(i,v)}$, whence $j(i,v) = j(i,w)$. It follows that $W_v = W_w$, and we find in the same way that $W_u = W_w$. Thus $W_v = W_w$.

This shows that the f-invariant subspaces W_v give a disjoint decomposition of $V \setminus \{0\}$, and hence a decomposition of V. (Once we remove those subspaces which occur multiple times.) This decomposition of V is by construction finer than each decomposition $V = \bigoplus_{j \in J_i} U^i_j$. □

We want to understand how the decomposition from the Krull–Remak–Schmidt theorem looks like. We note that if $v \in V$ is any nonzero vector then there exists at most one preimage of v under f, since f is injective outside of its kernel. Thus we can consider for every nonzero vector $v \in V$ the well-defined two-sided sequence

$$\dots, f^{-2}(v), f^{-1}(v), v, f(v), f^2(v), \dots$$

Here the left part of the sequence consists of as many iterated preimages as exist. The set of all these elements is the *orbit* of v under f. It is denoted by $[v]$.

We note that for any nonzero veector u in $[v]$ we have $[u] = [v]$. If two orbits $[v]$ and $[w]$ intersect in a nonzero vector u then it follows that $[v] = [u] = [v]$. Two distinct orbit do therefore intersect at most in 0. It follows that the orbits give induce a disjoint decomposition of $V \setminus \{0\}$. The vector space V does therefore decompose into the direct sum of the subspaces $[v] \cup \{0\}$. (Once we remove those subspaces which occur multiple times.) Each subspace $[v] \cup \{0\}$ is f-invariant. Any two nonzero f-invariant subspaces of $[v] \cup \{0\}$ intersect nonzero whence the subspaces $[v] \cup \{0\}$ are indecomposable.

This shows that the decomposition of V from the Krull–Remak–Schmidt theorem is given by the orbits with respect to f (together with $\{0\}$).

There exist five kinds of orbits.

Type A. The orbit ends in zero and is finite: It is thus of the form

$$v \to f(v) \to \cdots \to f^n(v) = 0$$

for a unique vector v, that has no preimage under f.

Type B. The orbit ends in zero and is infintite: It is thus of the form

$$\cdots \to f^{-2}(v) \to f^{-1}(v) \to v \to f(v) \to f^2(v) \to \cdots \to f^n(v) = 0 \,.$$

	injective	surjective	bijective	nilpotent	finite-dimensional
Type A				◦	◦
Type B			•	◦	
Type C	•				
Type D	•	•	•		
Type E	•	•	•		◦

Table 1: Possible orbits. Complete characterization via orbits for •. Only locally for ◦.

Type C. The orbit never reaches zero and has only finitely many preimages. It is thus of the form

$$v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to \cdots$$

for a unique vector v, that has no preimage under f.

Type D. The orbit goes infinite in both directions and is non-circular. It is thus of the form

$$\cdots \to f^{-1}(v) \to v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to \cdots$$

Type E. The orbit is circular. It is thus of the form

$$v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to v \to f(v) \to \cdots$$

Depending on the properties of the vector space V and endomorphism f not all kinds of orbits can occur.

- The endomorphism f is injective if and only if no orbits of Type A and Type B occur.
- The endomorphism f is surjective if and only if no orbits of Type A and Type C occur.
- The endomorphism f is bijective if and only if only orbits of Type D and Type E occur.
- The endomorphism f is locally nilpotent if and only if only orbits of Type A and Type B appear.[5]
- The endomorphism f is nilpotent if and only if only orbits of Type A occur, and the lengths of the occuring orbits is bounded.
- If V is finite-dimensional then only orbits of Type A and Type E occur.
- More generally, V is locally finite-dimensional with respect to f if and only if only orbits of Type A and Type E occur.[6]

See Table 1 for an overview.

[5] An endomorphism f is locally nilpotent if there exists for every vector v some power $n \geq 0$ such that $f^n(v) = 0$.

[6] We say that V is locally finite-dimensional if every nonzero vector of V is contained in a finite-dimensional f-invariant subspace.

A.2. Proving That $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is Hereditary

Definition A.5. Let $\mathscr{A}$ be an abelian category. A subcategory $\mathscr{B}$ is *closed under extensions* if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ the middle term B is contained in $\mathscr{B}$ provided that both outer terms A, C are contained in $\mathscr{B}$.

Definition A.6. Let $\mathscr{A}$ be an abelian category. A subcategory $\mathscr{B}$ of $\mathscr{A}$ is a *Serre subcategory* if it is abelian, exact, full and closed under extensions.

Example A.7. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is a Serre subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q) \cong \mathbb{F}_q[x]$-**Mod**. Indeed, it is a full, abelian, exact subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q)$. If in a short exact sequence

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

of Q-representations both A, B are finite-dimensional then the same holds for B. If both A, C are nilpotent then same holds for B: There exists some powers $n, m \geq 0$ with $\alpha^n A = 0$ and $\alpha^m C = 0$. It follows that $\alpha^m B \subseteq \ker(\psi) = \mathrm{im}(\varphi)$ and thus $\alpha^{n+m} B = 0$.

If $\mathscr{A}$ is an abelian category and $\mathscr{B}$ is an abelian, exact subcategory then we have for every $n \geq 0$ and every two objects A, B of $\mathscr{B}$ an induced map

$$\mathrm{Ext}^n_{\mathscr{B}}(A, B) \to \mathrm{Ext}^n_{\mathscr{A}}(A, B)\,.$$

Proposition A.8. Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be an abelian, exact subcategory of $\mathscr{A}$.

1. Suppose that $\mathscr{B}$ is a full subcategory of $\mathscr{A}$. Then the induced map $\mathrm{Ext}^1_{\mathscr{B}}(A, B) \to \mathrm{Ext}^1_{\mathscr{A}}(A, B)$ is injective for any two objects A, B of $\mathscr{B}$. If $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$ then the induced map $\mathrm{Ext}^1_{\mathscr{B}}(A, B) \to \mathrm{Ext}^1_{\mathscr{A}}(A, B)$ is bijective.
2. Suppose that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Suppose furthermore that for some $n \geq 1$ the induced map $\mathrm{Ext}^n_{\mathscr{B}}(A, B) \to \mathrm{Ext}^n_{\mathscr{A}}(A, B)$ is bijective for any two objects A, B of $\mathscr{B}$. Then the induced map $\mathrm{Ext}^{n+1}_{\mathscr{B}}(A, B) \to \mathrm{Ext}^{n+1}_{\mathscr{A}}(A, B)$ is injective for any two objects A, B of $\mathscr{B}$.

Proof.

1. Two short exact sequences in $\mathscr{B}$,

$$0 \to B \to X \to A \to 0 \qquad \text{and} \qquad 0 \to B \to X' \to A \to 0\,,$$

are equivalent in $\mathscr{A}$ if there exists an isomorphism $\varphi\colon X \to X'$ that makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commute. We find that φ is already an isomorphism in $\mathscr{B}$ because $\mathscr{B}$ is full in $\mathscr{A}$. Thus both sequences are already equivalent in $\mathscr{B}$. This shows the injectivity of $\mathrm{Ext}^1_{\mathscr{B}}(A, B) \to \mathrm{Ext}^1_{\mathscr{A}}(A, B)$.

Suppose now that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Let A, B be two objects in $\mathscr{B}$. Every element ξ of $\mathrm{Ext}^1_{\mathscr{A}}(A, B)$ is represented by a short exact sequence $0 \to B \to X \to A \to 0$ in $\mathscr{A}$. The middle term X is already contained in $\mathscr{B}$ because $\mathscr{B}$ is closed under extensions. Thus ξ lies in $\mathrm{Ext}^1_{\mathscr{B}}(A, B)$.

2. We refer to [Oor63, Proposition 3.3]. □

Corollary A.9. Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be a Serre subcategory of $\mathscr{A}$. If $\mathscr{A}$ is hereditary then so is $\mathscr{B}$.

Proof. Let A, B be two objects of $\mathscr{B}$. We show by induction on $n \geq 1$ that the induced map $\mathrm{Ext}^n_{\mathscr{B}}(A, B) \to \mathrm{Ext}^n_{\mathscr{A}}(A, B)$ is bijective. The assertion follows from this.

We know from Proposition A.8 that the induced map $\mathrm{Ext}^1_{\mathscr{B}}(A, B) \to \mathrm{Ext}^1_{\mathscr{A}}(A, B)$ is bijective. If for some $n \geq 1$ the induced map $\mathrm{Ext}^n_{\mathscr{B}}(A, B) \to \mathrm{Ext}^n_{\mathscr{A}}(A, B)$ is bijective then it follows from Proposition A.8 that the induced map $\mathrm{Ext}^{n+1}_{\mathscr{B}}(A, B) \to \mathrm{Ext}^{n+1}_{\mathscr{A}}(A, B)$ is injective. It is also surjective because $\mathscr{A}$ is hereditary. □

Example A.10. Example A.7 shows that $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is a Serre subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q)$. The category $\mathbf{Rep}(Q, \mathbb{F}_q) \cong \mathbb{F}_q[x]\text{-}\mathbf{Mod}$ is hereditary because it has enough projectives and submodules of projective $\mathbb{F}_q[x]$-modules are again projective, since $\mathbb{F}[x]$ is a principal ideal domain. It thus follows from Corollary A.9 that $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is again hereditary.

A.3. Understanding $\mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q))$ and the Euler Form of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$

The indecomposable objects of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ are precisely N_i with $i \geq 1$. The representation N_i has (up to isomorphism) precisely the subrepresentations N_j with $j = 0, \dots, i$. Thus N_1 is the unique simple object in $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$.

Every objects in $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ admits a composition series, whose composition factors are necessarily the unique simple objects S. It follows that $[S]$ is a free generator of the Grothendieck group $K := \mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q))$.

We have $\mathrm{Hom}(S, S) \cong \mathbb{F}_q$ and $\mathrm{Ext}^1(S, S) \cong \mathbb{F}_q$. The first isomorphism holds because S is one-dimensional, and the second is shown in Appendix A.4.

To show that the Euler form of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ vanishes we regard it as a bilinear form

$$\langle -, - \rangle \colon K \times K \to \mathbb{Q}^\times .$$

Since S is a generator of K is sufficies to show that $\langle S, S \rangle = 1$. This holds true because

$$\langle S, S \rangle = \Big(\#\mathrm{Hom}(S, S) \Big) \cdot \Big(\#\,\mathrm{Ext}^1(S, S) \Big)^{-1} = q \cdot q^{-1} = 1 .$$

A.4. Computing $\mathrm{Ext}^1(S, S)$

In $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ we can compute $\mathrm{Ext}^1(S, S)$ for $S = \mathrm{N}_1$ in two ways:

A.4.1. Via Homological Algebra

Let $\Bbbk := \mathbb{F}_q$. We find with Proposition A.8 that

$$\mathrm{Ext}^1(S, S) = \mathrm{Ext}^1_{\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)}(S, S) \cong \mathrm{Ext}^1_{\mathbf{Rep}(Q, \Bbbk)}(S, S) \cong \mathrm{Ext}^1_{\Bbbk[x]\text{-}\mathbf{Mod}}(\Bbbk, \Bbbk) .$$

We can use for $\Bbbk$ (in the first argument) the projective resolution

$$\cdots \to 0 \to \Bbbk[x] \xrightarrow{x} \Bbbk[x] \to \Bbbk \to 0 .$$

Applying the functor $\operatorname{Hom}_{\Bbbk[x]}(-,\Bbbk)$ gives the chain complex

$$0 \to \operatorname{Hom}_{\Bbbk[x]}(\Bbbk[x],\Bbbk) \xrightarrow{x} \operatorname{Hom}_{\Bbbk[x]}(\Bbbk[x],\Bbbk) \to 0 \to \cdots ,$$

which is isomorphic to the chain complex

$$0 \to \Bbbk \xrightarrow{0} \Bbbk \to 0 \to \cdots$$

We find in particular that

$$\operatorname{Hom}_{\Bbbk[x]}(\Bbbk,\Bbbk) \cong \Bbbk , \quad \operatorname{Ext}^1_{\Bbbk[x]}(\Bbbk,\Bbbk) \cong \Bbbk .$$

A.4.2. Via Counting

We can also count the Yoneda classes of short exact sequences: We have $\mathrm{N}_1 = (\Bbbk, \begin{bmatrix} 0 \end{bmatrix})$, and a short exact sequence

$$0 \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to ? \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to 0$$

can have as its middle term (up to isomorphism) either

$$\mathrm{N}_{(1,1)} = \left(\Bbbk^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{or} \qquad \mathrm{N}_2 = \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

In the first case we get a short exact sequence

$$0 \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to \left(\Bbbk^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to 0 .$$

This short exact sequence splits on the level of $\Bbbk$-vector spaces, and any such split is alreday a homomorphism of representations. We hence find that this sequence describes the unique element of $\operatorname{Ext}^1(S,S)$ that is given by the split exact sequences.

We consider now the short exact sequences of the form

$$0 \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \xrightarrow{\varphi} \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \xrightarrow{\psi} (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to 0 . \tag{2}$$

The homomorphism φ must be of the form

$$\varphi = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

for some $a \neq 0$ since the image of φ must be contained in the kernel of $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. It follows from the exactness of the sequence that the homomorphism ψ is of the form

$$\psi = \begin{bmatrix} 0 & b \end{bmatrix}$$

for some $b \neq 0$.

Two such sequences $\xi_{a,b}$ and $\xi_{a',b'}$ for $a, a', b, b' \neq 0$ are Yoneda equivalent if and only if there exists an invertible matrix

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathrm{GL}(2, \Bbbk)$$

such that

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} \tag{3}$$

and the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & (k, [0]) \arrow[r, "{\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]}"] \arrow[d, equal] & \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \arrow[r, "{[\,0\ b\,]}"] \arrow[d, "{\left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]}"] & (k, [0]) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (k, [0]) \arrow[r, "{\left[\begin{smallmatrix} a' \\ 0 \end{smallmatrix}\right]}"] & \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \arrow[r, "{[\,0\ b'\,]}"] & (k, [0]) \arrow[r] & 0
\end{tikzcd} \tag{4}$$

The condition (3) means that $w = z$ and $y = 0$, i.e. that the the matrix is of the form

$$\begin{bmatrix} w & x \\ 0 & w \end{bmatrix},$$

The commutativity of the diagram (4) means that

$$w = \frac{a'}{a} \quad \text{and} \quad w = \frac{b}{b'}\,.$$

We hence find that the extensions $\xi_{a,b}$ and $\xi_{a',b'}$ are Yoneda equivalent if and only if $a'/a = b/b'$.

It follows that the Yoneda equivalence classes of short exact sequences of the form (2) have as a set of representatives the sequences

$$0 \to (\Bbbk, [0]) \xrightarrow{[\,1\ 0\,]} \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \xrightarrow{[\,0\ b\,]} (\Bbbk, [0]) \to 0$$

with $b \neq 0$.

We find that overall we have $\#\Bbbk = \#\mathbb{F}_q = q$ many Yoneda equivalence classes of short exact sequences. Thus

$$\#\mathrm{Ext}^1(S, S) = q\,,$$

from which it follows that $\mathrm{Ext}^1(S, S) \cong \mathbb{F}_q$.

A.5. Explicit Description of $\mathbf{H}(Q, \mathbb{F}_q)$

1. The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_q)$ is free on the set of isomorphism classes, $\mathrm{Iso}(Q, \mathbb{F}_q)$. This basis is indexed by the set of partitions, Par.

2. The multiplication on $\mathbf{H}(Q, \mathbb{F}_q)$ is given by

$$[M] \cdot [N] = \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_q)} F^R_{M,N}[R]$$

where

$$F^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \mid L \cong N, R/L \cong M\}.$$

The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_q)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_q)} = [0]$.

3. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ satisfies the finite subobject condition and its Euler form vanishes.It follows that Green's coproduct makes the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ into a bialgebra. Its comultiplication is given by

$$\Delta([M]) = \sum_{[R],[L] \in \mathrm{Iso}(Q, \mathbb{F}_q)} \frac{P^M_{R,L}}{a_M} [R] \otimes [L]$$

where a_M is the size of the automorphism group $\mathrm{Aut}(M)$, and $P^M_{R,L}$ is the number of short exact sequences $0 \to L \to M \to R \to 0$. The counit $\varepsilon \colon \mathbf{H}(Q, \mathbb{F}_q) \to \mathbb{C}$ is given by

$$\varepsilon([M]) = \begin{cases} 1 & \text{if } M = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. We have a grading on $\mathbf{H}(Q, \mathbb{F}_q)$ over the Grothendieck group $\mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)) \cong \mathbb{Z}$, given by

$$\deg([M]) = \dim(M).$$

This grading makes $\mathbf{H}(Q, \mathbb{F}_q)$ into a graded bialgebra.

5. The graded bialgebra $\mathbf{H}(Q, \mathbb{F}_q)$ is connected (i.e. its degree zero part is the ground field). It is therefore already a graded Hopf algebra.

A.6. Counting $\mathrm{Gr}(d, n, \mathbb{F}_q)$

Recall A.11. For $k \in \mathbb{N}$ the *quantum integer* $[k]_q$ is given by

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1}.$$

We have $[0]_q = 0$ and $[1]_q = 1$. The *quantum factorial* is given by

$$[k]_q! = [k]_q [k-1]_q \cdots [1]_q.$$

For $k, l \in \mathbb{N}$ the *quantum binomial* is given by

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q \cdots [k-l+1]_q}{[l]_q!}.$$

If $l > k$ then this is zero, and if $l \leq k$ then the quantum binomial can also be expressed as

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q!\,[k-l]_q!} .$$

The quantum binomial satsfies the recursive relation

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = q^l \begin{bmatrix} k-1 \\ l \end{bmatrix}_q + \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q$$

for all $k, l \in \mathbb{N}$. It hence follows by induction that the quantum binomial $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ is a polynomial in q with natural coefficients, i.e.

$$\begin{bmatrix} k \\ l \end{bmatrix}_q \in \mathbb{N}[q] .$$

By taking the limit $q \to 1$ (i.e. by setting q equal to 1) the quantum integer $[k]$ becomes the usual integer k, the quantum factorial $[k]_q!$ becomes the usual factorial $k!$ and the quantum binomial coefficient $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ becomes the usual binomial $\binom{k}{l}$.

Lemma A.12. For all dimensions $n, d \geq 0$ we have

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \begin{bmatrix} n \\ d \end{bmatrix}_q .$$

Proof. If $d > n$ then both numbers are zero, so suppose that $d \leq n$. Let

$$F_d(n) := (q^n - 1) \cdots (q^n - q^{d-1}) = (q-1)^d \, q^{d(d-1)/2} \, [n]_q \cdots [n-d+1]_q$$

This is the number of linear independent tupels $(v_1, \dots, v_d)$ of vectors in $\mathbb{F}_q^n$. We find that

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{\#\mathrm{GL}(d, \mathbb{F}_q)} .$$

We have $\#\mathrm{GL}(d, \mathbb{F}_q) = F_d(d)$ and thus

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{F_d(d)} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q \cdots [1]_q} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q!} = \begin{bmatrix} n \\ d \end{bmatrix}_q ,$$

as claimed. □

Remark A.13. To see that $\#\mathrm{Gr}(d, n, \mathbb{F}_q)$ is a polynomial in q with natural coefficients one can also use the disjoint decomposition of $\mathrm{Gr}(d, n, \mathbb{F}_q)$ into Schubert cells. These cells are affine spaces, in the sense that they are isomorphic to $\mathbb{F}_q^n$ for suitable n. It follows that $\#\mathrm{Gr}(d, n, \mathbb{F}_q)$ is a sum of $q^{n_1} + \cdots + q^{n_t}$ for suitable powers $n_i \geq 1$. These powers do not depend on the choice of q.

A.7. Explicit Computations of Quantum Binomial Coefficients

We have

$$F^{(1^{n+1})}_{(1^n),(1)} = \#\mathrm{Gr}(1, n+1, \mathbb{F}_q) = \#\mathbb{P}^n(\mathbb{F}_q) = \frac{q^{n+1}-1}{q-1} = [n+1]_q = 1 + q + \dots + q^n\,,$$

and also

$$\begin{aligned} F^{(1^{n+2})}_{(1^n),(1,1)} = \begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q &= \frac{[n+2]_q\,[n+1]_q}{[2]_q} \\ &= \frac{(1+q+\dots+q^n)(1+q+\dots+q^{n+1})}{1+q} \\ &= \begin{cases} (1+q+\dots+q^n)(1+q^2+\dots+q^n) & \text{if } n \text{ is even,} \\ (1+q^2+\dots+q^{n-1})(1+q+\dots+q^{n+1}) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Lastly we compute

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q = \frac{[n+3]_q\,[n+2]_q\,[n+1]_q}{[3]_q\,[2]_q\,[1]_q} &= \frac{\frac{q^{n+3}-1}{q-1}\,\frac{q^{n+2}-1}{q-1}\,\frac{q^{n+1}-1}{q-1}}{\frac{q^3-1}{q-1}\,\frac{q^2-1}{q-1}\,\frac{q-1}{q-1}} \\ &= \frac{(q^{n+3}-1)(q^{n+2}-1)(q^{n+1}-1)}{(q^3-1)(q^2-1)(q-1)} \end{aligned}$$

We recall that the polynomial $x^k - 1$ divides the polynomial $x^l - 1$ (in $\mathbb{Z}[x]$) if and only if the integer k divides the integer l. Then

$$\frac{x^k-1}{x^l-1} = 1 + x^l + x^{2l} + \dots + x^{k-l}\,.$$

We can therefore compute the above quotient by distinguishing between six cases, depending on how the powers

$$n+3\,,\quad n+2\,,\quad n+1$$

are divisible by 3 and 2. This in turn is uniquely determined by the residue class of n modulo 6.

Case 1. Suppose that $n \equiv 0$. Then $n+3$ is divisible by 3 and $n+2$ is divisible by 2. In this case

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+3}-1}{q^3-1} \cdot \frac{q^{n+2}-1}{q^2-1} \cdot \frac{q^{n+1}-1}{q-1} \\ &= (1+q^3+\dots+q^n)(1+q^2+\dots+q^n)(1+q+\dots+q^n)\,. \end{aligned}$$

Case 2. Suppose that $n \equiv 1$. Then $n+2$ is divisible by 3 and $n+3$ is divisible by 2. In this case

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+2}-1}{q^3-1} \cdot \frac{q^{n+3}-1}{q^2-1} \cdot \frac{q^{n+1}-1}{q-1} \\ &= (1+q^3+\dots+q^{n-1})(1+q^2+\dots+q^{n+1})(1+q+\dots+q^n)\,. \end{aligned}$$

Case 3. Suppose that $n \equiv 2$. Then $n+1$ is divisible by 3 and $n+2$ is divisible by 2. In this case

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+1}-1}{q^3-1} \cdot \frac{q^{n+2}-1}{q^2-1} \cdot \frac{q^{n+3}-1}{q-1} \\ &= (1+q^3+\cdots+q^{n-2})(1+q^2+\cdots+q^n)(1+q+\cdots+q^{n+2}). \end{aligned}$$

Case 4. Suppose that $n \equiv 3$. Then $n+3$ is divisible by 3 and $n+1$ is divisible by 2. In this case

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+3}-1}{q^3-1} \cdot \frac{q^{n+1}-1}{q^2-1} \cdot \frac{q^{n+2}-1}{q-1} \\ &= (1+q^3+\cdots+q^n)(1+q^2+\cdots+q^{n-1})(1+q+\cdots+q^{n+1}). \end{aligned}$$

Case 5. Suppose that $n \equiv 4$. Then $n+2$ is divisible by both 3 and 2. We observe that

$$\frac{1+q^3}{1+q} = 1-q+q^2$$

and hence for every odd positive integer m that

$$\begin{aligned} \frac{1+q^3+\cdots+q^{3m}}{1+q} &= \frac{(1+q^3)+q^6(1+q^3)+\cdots+q^{3m-3}(1+q^3)}{1+q} \\ &= (1-q+q^2)(1+q^6+\cdots+q^{3m-3}). \end{aligned}$$

We also observe that the integer $n-1$ is divisible by 3, and that it is an odd multiple of 3 because $n+2$ is divisible by 6 and therefore

$$\frac{n-1}{3} + 1 = \frac{n+2}{3}$$

is even. We now find that

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+3}-1}{q-1} \cdot \frac{q^{n+2}-1}{(q^3-1)(q+1)} \cdot \frac{q^{n+1}-1}{q-1} \\ &= \frac{q^{n+3}-1}{q-1} \cdot \frac{1+q^3+\cdots+q^{n-1}}{q+1} \cdot \frac{q^{n+1}-1}{q-1} \\ &= (1+q+\cdots+q^{n+2})(1-q+q^2)(1+q^6+\cdots+q^{n-4})(1+q+\cdots+q^n). \end{aligned}$$

Case 6. Suppose that $n \equiv 5$. Then $n+1$ is divisible by 3 and $n+3$ is divisible by 2. In this case

$$\begin{aligned} \begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+1}-1}{q^3-1} \cdot \frac{q^{n+3}-1}{q^2-1} \cdot \frac{q^{n+2}-1}{q-1} \\ &= (1+q^3+\cdots+q^{n-2})(1+q^2+\cdots+q^{n+1})(1+q+\cdots+q^{n+1}). \end{aligned}$$

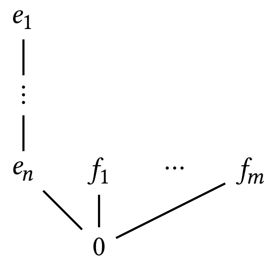

Figure 3: The representations $\mathrm{N}_{(n,1^m)}$.

A.8. Computing More Structure Constants for $\mathbf{H}(Q,\mathbb{F}_q)$

We use for $\mathrm{N}_{(n,1)}$ the basis $e_1,\dots,e_n,f_1,\dots,f_m$ with

$$\alpha e_i = e_{i+1} \quad \text{and} \quad \alpha e_n = \alpha f_1 = \cdots = \alpha f_m = 0$$

for $i = 1,\dots,n-1$, where α denotes the loop of Q. See Figure 3.

The coefficient $F^{(n,1^m)}_{(1^m),(n)}$ is the number of subrepresentations L of $\mathrm{N}_{(n,1^m)}$ with $L \cong \mathrm{N}_n$ and $\mathrm{N}_{(n,1)}/L \cong \mathrm{N}_{(1^m)}$. The condition $L \cong \mathrm{N}_n$ means that L is cyclically generated by a vector

$$v = a_1e_1 + \cdots + a_ne_n + b_1f_1 + \cdots + b_mf_m$$

with $a_1 \neq 0$. We may assume that $a_1 = 1$. Then

$$\begin{aligned} L &= \langle \alpha^k v \mid k \geq 0\rangle_{\mathbb{F}_q} \\ &= \langle e_1 + a_2e_2 + \cdots + a_ne_n + b_1f_1 + \cdots + b_mf_m,\ e_2 + a_2e_3 + \cdots + a_ne_{n-1},\ \dots,\ e_n\rangle_{\mathbb{F}_q} \\ &= \langle e_1 + b_1f_1 + \cdots + b_mf_m, e_2,\dots,e_n\rangle\,. \end{aligned}$$

For any such subrepresentation L the quotient $\mathrm{N}_{(n,1^m)}/L$ is m-dimensional and spanned by the residue classes $[f_1],\dots,[f_m]$. It is thus isomorphic to $\mathrm{N}_{(1^m)}$. We get for every choice of coefficient $b_1,\dots,b_m \in \mathbb{F}_q$ a different subrepresentation of $\mathrm{N}_{(n,1^m)}$. Hence

$$F^{(n,1^m)}_{(1^m),(n)} = \#\mathbb{F}_q^m = q^m\,.$$

The coefficient $F^{(n,1^m)}_{(n),(1^m)}$ is the number of subrepresentations L of $\mathrm{N}_{(n,1^m)}$ with $L \cong \mathrm{N}_{(1^m)}$ and $\mathrm{N}_{(n,1^m)}/L \cong \mathrm{N}_n$. The condition $L \cong \mathrm{N}_{1^m}$ means precisely that L is an m-dimensional linear subspace of $\langle e_n, f_1,\dots,f_m\rangle_{\mathbb{F}_q}$. We note that if g is a vector of $\langle e_n, f_1,\dots,f_m\rangle_{\mathbb{F}_q}$ that is not contained in L then the residue classes

$$[e_1],\dots,[e_n],[g]$$

will be a basis of the quotient $\mathrm{N}_{(n,1^m)}/L$. We claim that $\mathrm{N}_{(n,1^m)}/L$ is isomorphic to N_n if and only if the subspace L does not contain the vector e_n.

Suppose first that e_1 is contained in L. Then we take for g any vector of $\langle e_n, f_1,\dots,f_n\rangle$ not contained in L. Then $\alpha[e_{n-1}] = [e_n] = 0$ and we find that

$$[e_1],\dots,[e_{n-1}] \quad \text{and} \quad [g]$$

are two Jordan chains for $\mathrm{N}_{(n,1^m)}/L$. Thus $\mathrm{N}_{(n,1^m)}/L$ is not isomorphic to N_n.

Suppose on the other hand that e_1 not contained in L. Then we can choose g as e_1 and find that the residue classes

$$[e_n], \dots, [e_1]$$

form a basis of $\mathrm{N}_{(n,1^m)}/L$. This basis forms a single Jordan chain, whence $\mathrm{N}_{(n,1^m)}/L$ is isomorphic to N_n.

We thus find that the structure constant $F^{(n,1^m)}_{(n),(1^m)}$ is the number of m-dimensional subspaces of $\langle e_n, f_1, \dots, f_m \rangle$ that does not contain e_n. The number of m-dimensional subspaces of $\langle e_n, f_1, \dots, f_m \rangle$ is according to Lemma A.12 given by the quantum binomial

$$\begin{bmatrix} m+1 \\ m \end{bmatrix}_q .$$

The number of m-dimensional subspaces of $\langle e_n, f_1, \dots, f_m \rangle$ that contains e_1 is equals the number of $(m-1)$-dimensional subspaces of the quotient vector spaces $\langle e_n, f_1, \dots, f_n \rangle / \langle e_n \rangle \cong \langle f_1, \dots, f_m \rangle$. We see again from Lemma A.12 that there are

$$\begin{bmatrix} m \\ m-1 \end{bmatrix}_q$$

such subspaces. We hence find that

$$F^{(n,1^m)}_{(n),(1^m)} = \begin{bmatrix} m+1 \\ m \end{bmatrix}_q - \begin{bmatrix} m \\ m-1 \end{bmatrix}_q = \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} m \\ 1 \end{bmatrix}_q = [m+1]_q - [m]_q = q^m .$$

A.9. Explicit Description of $\mathbf{H}(Q, \mathbb{F}_1)$

The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is a graded, cocommutative Hopf algebra (over the ground field $\mathbb{C}$). It is defined as follows:

- The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_1)$ is the free $\mathbb{C}$-vector space on the set $\mathrm{Iso}(Q, \mathbb{F}_1)$. The set $\mathrm{Iso}(Q, \mathbb{F}_1)$ is indexed by the set of partitions Par.
- The multiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

  $$[M] \cdot [N] := \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)} F^R_{M,N} [R]$$

  where the structure coefficients $F^R_{M,N}$ are given by

  $$F^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \mid L \cong N, R/L \cong M\} .$$

  The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_1)} = [0]$.
- The grading of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $\deg([M]) = \dim(M)$.

- The comultiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

$$\Delta([M]) = \sum_{\substack{[R],[L] \in \mathrm{Iso}(Q, \mathbb{F}_1) \\ M \cong R \oplus L}} [R] \otimes [L] .$$

We see in particular that an isomorphism class $[M]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$ if and only if the representation M is indecomposable. We have seen that more generally the Lie algebra of primitive elements of $\mathbf{H}(Q, \mathbb{F}_1)$ has a basis consisting of all such $[M]$. We have concluded from the theorem of Milnor–Moore that $\mathbf{H}(Q, \mathbb{F}_q)$ is the universal enveloping algebra of its Lie algebra of primitive elements.

A.10. More on the Multiplication of $\mathbf{H}(Q, \mathbb{F}_1)$

The computation of the product $[\mathrm{N}_i] \cdot [\mathrm{N}_j]$ in $\mathbf{H}(Q, \mathbb{F}_1)$ can be further generalized:

We find for all $i_1, \dots, i_r \geq 1$ and $j \geq 1$ that

$$[\mathrm{N}_{(i_1,\dots,i_r)}] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i_1,\dots,i_r,j)}] + \sum_\lambda b_\lambda [\mathrm{N}_\lambda] ,$$

where λ runs through all distinct tupels of the form $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ with $1 \leq k \leq r$. The coeffiecient a is given by

$$a = \text{how often } j \text{ occurs in } (i_1, \dots, i_r, j)$$

and the coefficient of b_λ for $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ are given by

$$b_\lambda = \text{how often } i_k + j \text{ occurs in } \lambda .$$

Example A.14. We find that

$$\begin{aligned} [\mathrm{N}_{(5,3,3,2,1)}] \cdot [\mathrm{N}_2] = {} & 2[\mathrm{N}_{(5,3,3,2,1,\overline{2})}] \\ & + [\mathrm{N}_{(\overline{7},3,3,2,1)}] + 2[\mathrm{N}_{(5,\overline{5},3,2,1)}] + [\mathrm{N}_{(5,3,3,\overline{4},1)}] + 3[\mathrm{N}_{(5,3,3,2,\overline{3})}] , \end{aligned}$$

where the entry of interest is overlined.

We can rewrite the above identity in a more systematic way: For every tupel $\lambda = (\lambda_1, \dots, \lambda_l)$ consisting of nonnegative integers $\lambda_i \geq 0$ and every $j \geq 0$ let

$$a(\lambda, j) := (\text{how often } i \text{ occurs in } \lambda) = \#\{1 \leq i \leq l \mid \lambda_i = j\} .$$

We denote for every $i = 1, \dots, l$ by ε_i the tupel that has 1 as its i-th entry and 0 otherwise. Then

$$\begin{aligned} [\mathrm{N}_\lambda] \cdot [\mathrm{N}_j] &= a((\lambda, j), j)[\mathrm{N}_{(\lambda,j)}] + \sum_{i=1}^{l} \frac{a(\lambda + j\varepsilon_i, \lambda_i + j)}{a(\lambda, \lambda_i)} [\mathrm{N}_{\lambda + j\varepsilon_i}] \\ &= (a(\lambda, j) + 1)[\mathrm{N}_{(\lambda,j)}] + \sum_{i=1}^{l} \frac{a(\lambda, \lambda_i + j) + 1}{a(\lambda, \lambda_i)} [\mathrm{N}_{\lambda + j\varepsilon_i}] \end{aligned} \tag{5}$$

The coefficients $a(\lambda + j\varepsilon_i, \lambda_i + j)$ are the same as the $b_{\lambda + j\varepsilon_i}$ above. The normalization $a(\lambda, \lambda_i)$ ensures that those summands which occur multiple times are effectively only counted once.

Remark A.15. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition and let $\lambda' = (\lambda_1, \dots, \lambda_{l-1})$. Then

$$\lambda = (\lambda', \lambda_l)\,.$$

The above formulas do therefore yield the identity

$$\begin{aligned}
\mathrm{N}_\lambda = \mathrm{N}_{(\lambda',\lambda_l)} &= \frac{1}{a((\lambda',\lambda_l),\lambda_l)}\left([\mathrm{N}_{\lambda'}]\cdot[\mathrm{N}_{\lambda_l}] - \sum_{i=1}^{l-1}\frac{a(\lambda',\lambda'_i+\lambda_l)+1}{a(\lambda',\lambda'_i)}[\mathrm{N}_{\lambda'+\lambda_l\varepsilon_i}]\right)\\
&= \frac{1}{a(\lambda,\lambda_l)}\left([\mathrm{N}_{\lambda'}]\cdot[\mathrm{N}_{\lambda_l}] - \sum_{i=1}^{l-1}\frac{a(\lambda',\lambda'_i+\lambda_l)+1}{a(\lambda',\lambda'_i)}[\mathrm{N}_{\lambda'+\lambda_l\varepsilon_i}]\right)
\end{aligned} \tag{6}$$

The partitions λ' and $\lambda' + j\varepsilon_i$ that appear on the right hand side are of strictly smaller length than λ. It hence follows from this formula by induction that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated as an algebra by the classes $[\mathrm{N}_i]$ with $i \geq 1$.

A.11. More on Symmetric Functions

Warning A.16.

1. The ring of symmetric functions Λ is *not* the limit of $\Lambda^{(n+1)} \to \Lambda^{(n)}$ for $n \geq 0$ in the category of (commutative) algebras.

   We consider for this the symmetric polyonmials

   $$f^{(n)}(x_1, \dots, x_n) := x_1 + x_1x_2 + \cdots + x_1 \cdots x_n\,.$$

   The polynomials satisfy the compatibility condition

   $$f^{(n+1)}(x_1, \dots, x_n, 0) = f^{(n)}$$

   for every number of variables $n \geq 0$. But there exists no symmetric function $f \in \Lambda$ with

   $$f(x_1, \dots, x_n) = f^{(n)}(x_1, \dots, x_n)$$

   for every $n \geq 0$ since otherwise

   $$n = \deg(f^{(n)}) \leq \deg(f)$$

   for every $n \geq 0$, which is not possible. This shows that Λ together with the homomorphisms $\Lambda \to \Lambda^{(n)}$ is not the limit of the homomorphisms $\Lambda^{(n+1)} \to \Lambda^{(n)}$ for $n \geq 0$ in the category of rings.

2. The ring of symmetric functions Λ is *not* isomorphic to the algebras of symmetric polynomials

   $$\mathbb{C}[x_1, x_2, x_3, \dots]^{\mathrm{S}_{\mathbb{N}}} \quad \text{or} \quad \mathbb{C}[x_1, x_2, x_3, \dots]^{\mathrm{S}_\infty}$$

   where $\mathrm{S}_\infty = \operatorname{colim}_{n\geq 0}(\mathrm{S}_n \hookrightarrow \mathrm{S}_{n+1})$.

   Indeed, we observe that both $\mathbb{C}[x_1, x_2, x_3, \dots]^{\mathrm{S}_{\mathbb{N}}}$ and $\mathbb{C}[x_1, x_2, x_3, \dots]^{\mathrm{S}_\infty}$ are just the ground field $\mathbb{C}$: If a symmetric polynomial $f \in \mathbb{C}[x_1, x_2, \dots]$ were to contain a nontrivial monomial then it must also contain all permutations of this monomial. But there are infinitely many such permutations, while f contains only finitely many polynomials.

Suppose that more generally $(f^{(n)})_{n\geq 0}$ is any sequence of symmetric polynomials $f^{(n)} \in \Lambda^{(n)}$ that are compatible in the sense that $f(x_1, \dots, x_n) = f^{(n)}(x_1, \dots, x_n)$ for every $n \geq 0$. Then the $f^{(n)}$ define a symmetric function $f \in \Lambda$ with $f^{(n)}(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ for every $n \geq 0$ if and only if the degrees $\deg(f^{(n)})$ are bounded, i.e. if and only if there exists some $K \geq 0$ with $\deg(f^{(N)}) \leq K$ for every $n \geq 0$.

Indeed, if such a symmetric function f exists then $\deg(f^{(n)}) \leq \deg(f)$ for every $n \geq 0$. If on the other hand such a bound K exists then we consider for every $k = 0, \dots, K$ the sequence $(f_k^{(n)})_{n\geq 0}$ of degree k parts. That the symmetric polynomials $f^{(n)}$ are compatible means that for every degree $k = 0, \dots, K$ the homogeneous symmetric polynomials $f_k^{(n)}$ are compatible. Thus there exists for every degree $k = 0, \dots, K$ a homogeneous symmetric function $f_k \in \Lambda_k$ with

$$f_k(x_1, \dots, x_n) = f_k^{(n)}(x_1, \dots, x_n)$$

for every number of variables $n \geq 0$. It then follows for the symmetric function

$$f := f_0 + f_1 + \cdots + f_K$$

in each degree $l = 0, \dots, K$ that

$$f(x_1, \dots, x_n)_l = f_0(x_1, \dots, x_n)_k + \cdots + f_k(x_1, \dots, x_n)_l = f_l(x_1, \dots, x_n) = f_l^{(n)}(x_1, \dots, x_n)$$

for every $n \geq 0$ since each $f_l(x_1, \dots, x_n)$ is homogeneous of degree l. This shows that

$$f(x_1, \dots, x_n) = f^{(n)}(x_1, \dots, x_n)$$

for every number of variables $n \geq 0$.

A.12. Monomial Symmetric Functions

For every number of variables $n \geq 0$ and partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ of length $l \leq n$ the corresponding *monomial symmetric polynomial* is given by

$$m_\lambda^{(n)}(x_1, \dots, x_n) := \sum \text{distinct permutations of } x_1^{\lambda_1} \cdots x_l^{\lambda_l}\,.$$

if $n \geq l$, and by

$$m_\lambda^{(n)} := 0$$

if $n < l$. We have

$$m_\lambda^{(n+1)}(x_1, \dots, x_n, 0) = m_\lambda^{(n)}$$

for every number of variables $n \geq 0$, and each $m_\lambda^{(n)}$ is homogeneous of degree $|\lambda|$. We therefore get a well-defined homogeneous symmetric function

$$m_\lambda \in \Lambda_{|\lambda|}\,,$$

which we call the *monomial symmetric function* associated to λ.

The symmetric polynomials $m_\lambda^{(n)}$ where λ is of length $\ell(\lambda) \leq n$ form a vector space basis of $\Lambda^{(n)}$, for every number of variables $n \geq 0$. It hence follows from Corollary 4.6 (similarly to Corollary 4.7) that the monomial symmetric functions m_λ with $\lambda \in \mathrm{Par}$ form a vector space basis for Λ.

A.13. Regarding Λ as a Colimit

Remark A.17.

1. It follows from Corollary 4.6 for any two symmetric functions $f, g \in \Lambda$ that

$$f = g \iff f(x_1, \dots, x_n) = g(x_1, \dots, x_n) \text{ for some } n \geq \deg(f), \deg(g).$$

2. We can regard Λ as a colimit of suitable inclusions $\Lambda^{(n)} \to \Lambda^{(n+1)}$ for $n \geq 0$:

   We have for every number of variables $n \geq 0$ an injective homomorphism of graded algebras $\Lambda^{(n)} \to \Lambda^{(n+1)}$ that is given on algebra generators by $e_k^{(n)} \to e_k^{(n+1)}$ for every $k = 0, \dots, n$. This is a right sided inverse for the homomorphism $\Lambda^{(n+1)} \to \Lambda^{(n)}$. In this way a symmetric polynomial in n variables can be extended to a symmetric polynomial in $n + 1$ variables.

   We similarly have for every number of variables $n \geq 0$ a homomorphism of graded algebras $\Lambda^{(n)} \to \Lambda$ that is given on algebra generators by $e_k^{(n)} \to e_k$ for every $k = 0, \dots, k$.

   We now find that Λ together with the homomorphisms $\Lambda^{(n)} \to \Lambda$ is a colimit of the homomorphisms $\Lambda^{(n)} \to \Lambda^{(n+1)}$ for $n \geq 0$. In this way we can regard Λ as a sort of increasing union of the algebras of symmetric polynomials.

A.14. Invariants of a Tensor Product

Lemma A.18. Let G, H be two groups. Let V be a representation of G and let W be a representation of H. Then

$$(V \otimes W)^{G \times H} = V^G \otimes W^H.$$

Proof. The inclusion $V^G \otimes W^H \subseteq (V \otimes W)^{G \times H}$ can be checked on simple tensors. Let on the other hand $x \in (V \otimes W)^{G \times H}$. We may choose a basis $(v_i)_{i \in I}$ of V and write $x = \sum_{i \in I} v_i \otimes w_i$ for some unique vectors $w_i \in W$. For every element $h \in H$ we then have

$$\sum_{i \in I} v_i \otimes w_i = x = (1, h)x = \sum_{i \in I} v_i \otimes (hw_i).$$

It follows from the uniqueness of the w_i that $hw_i = w_i$ for every $h \in H$ and every $i \in I$, and thus $w_i \in W^H$ for every $i \in I$. This shows that $x \in V \otimes W^H$. We find in the same way that $x \in V^G \otimes W^H$, and thus $x \in (V \otimes W^H) \cap (V^G \otimes W) = V^G \otimes W^H$. ☐

A.15. Understanding the Isomorphism $\mathbf{H}(Q, \mathbb{F}_1) \cong \Lambda$ on the Basis $[\mathrm{N}_\lambda]$

A.15.1. Fixing Notation

Fer every tupel $\lambda = (\lambda_1, \dots, \lambda_l)$ of nonnegative integers $\lambda_i \geq 0$ and every $j \geq 0$ let

$$a(\lambda, k) := (\text{how often } k \text{ occurs in } \lambda) = \#\{1 \leq i \leq l \mid \lambda_i = k\},$$

and let

$$A(\lambda) := \prod_{k=0}^{\infty} a(\lambda, k)!\,.$$

This product is well-defined because $a(\lambda, j) = 0$ for all but finitely many $j \geq 0$. The number $A(\lambda)$ is the cardinality of the stabilizer of the tupel λ under the permutation action of the symmetric group S_l.

We denote for every i by ε_i the tupel whose i-th entry is 1 and whose other entries are 0. We note that for every $j \geq 1$ we have

$$a(\lambda + j\varepsilon_i, k) = \begin{cases} a(\lambda, k) & \text{if } k \neq \lambda_i, \lambda_i + j, \\ a(\lambda, k) - 1 & \text{if } k = \lambda_i, \\ a(\lambda, k) + 1 & \text{if } k = \lambda_i + j. \end{cases}$$

It follows that

$$A(\lambda + j\varepsilon_i) = \prod_{k=0}^{\infty} a(\lambda + j\varepsilon_i, k)! = \prod_{k=0}^{\infty} a(\lambda, k)! \cdot \frac{a(\lambda, \lambda_i + j) + 1}{a(\lambda, \lambda_i)} = A(\lambda) \cdot \frac{a(\lambda + j\varepsilon_i, \lambda_i + j)}{a(\lambda, \lambda_i)} .$$

For every polynomial $f^{(n)} \in \mathbb{C}[x_1, \dots, x_n]^{\mathrm{S}_n}$ we denote its *symmetrization* by

$$R^{(n)}(f^{(n)}) := \sum_{w \in \mathrm{S}_n} w.f^{(n)} .$$

We note that if $\mathrm{g}^{(n)} \in \Lambda^{(n)}$ is a symmetric polynomial then

$$\begin{aligned} R^{(n)}(f^{(n)}\mathrm{g}^{(n)}) &= \sum_{w \in \mathrm{S}_n} w.(f^{(n)}\mathrm{g}^{(n)}) \\ &= \sum_{w \in \mathrm{S}_n} (w.f^{(n)})(w.\mathrm{g}^{(n)}) \\ &= \sum_{w \in \mathrm{S}_n} (w.f^{(n)})\mathrm{g}^{(n)} \\ &= R^{(n)}(f^{(n)})\mathrm{g}^{(n)} . \end{aligned}$$

A.15.2. A Calculation

Suppose now that $\lambda = (\lambda_1, \dots, \lambda_l)$ is any of positive integers $\lambda_i \geq 1$ and that $j \geq 0$ is any nonnegative integer. We compute the product

$$m_\lambda^{(n)} \cdot p_j^{(n)}$$

for any number of variables n with $n > l$. For this we fill the tupel λ on the right with zeroes to reach a tupel μ of length n, i.e.

$$\mu = (\mu_1, \dots, \mu_n, \mu_{n+1}, \dots, \mu_n) = (\lambda_1 \dots, \lambda_l, 0, \dots, 0) .$$

We have

$$m_\lambda^{(n)}(x_1, \dots, x_n) = m_\mu^{(n)}(x_1, \dots, x_n) = \frac{1}{A(\mu)} R^{(n)}(x_1^{\mu_1} \cdots x_n^{\mu_n}) .$$

We can hence compute the above product as

$$\begin{aligned}
m_\lambda^{(n)}(x_1, \dots, x_n) \cdot p_j^{(n)}(x_1, \dots, x_n) &= \frac{1}{A(\mu)} R^{(n)}\Big(x_1^{\mu_1} \cdots x_n^{\mu_n}\Big) \cdot p_j^{(n)}(x_1, \dots, x_n) \\
&= \frac{1}{A(\mu)} R^{(n)}\Big(x_1^{\mu_1} \cdots x_n^{\mu_n} \cdot p_j^{(n)}(x_1, \dots, x_n)\Big) \\
&= \frac{1}{A(\mu)} R^{(n)}\Big(x_1^{\mu_1} \cdots x_n^{\mu_n} \cdot \sum_{i=1}^n x_i^j\Big) \\
&= \sum_{i=1}^n \frac{1}{A(\mu)} R^{(n)}\Big(x_1^{\mu_1} \cdots x_i^{\mu_i + j} \cdots x_n^{\mu_n}\Big) \\
&= \sum_{i=1}^n \frac{A(\mu + j\varepsilon_i)}{A(\mu)} m_{\mu + j\varepsilon_i}^{(n)}(x_1, \dots, x_n) \\
&= \sum_{i=1}^n \frac{a(\mu + j\varepsilon_i, \mu_i + j)}{a(\mu, \mu_i)} m_{\mu + j\varepsilon_i}^{(n)}(x_1, \dots, x_n)
\end{aligned}$$

For the summands $i = 1, \dots, l$ we can replace the role of μ by that of λ. For the summands $i = l + 1, \dots, n$ we have $\mu_i = 0$ and the first l entries of $\mu + j\lambda\varepsilon_i$ equal that of λ. We therefore have for every $i = l + 1, \dots, n$ that

$$a(\mu + j\varepsilon_i, \mu_i + j) = a(\mu + j\varepsilon_i, j) = a(\lambda, j) + 1 = a((\lambda, j), j)$$

as well as

$$a(\mu, \mu_i) = a(\mu, 0) = n - l.$$

We also have for every $i = l + 1, \dots, n$ that

$$\begin{aligned}
m_{\mu + j\varepsilon_i}^{(n)}(x_1, \dots, x_n) &= \frac{1}{A(\mu + j\varepsilon_i)} R^{(n)}(x_1^{\mu_1} \cdots x_l^{\mu_l} x_i^j) \\
&= \frac{1}{A(\mu + j\varepsilon_{l+1})} R^{(n)}(x_1^{\mu_1} \cdots x_l^{\mu_l} x_{l+1}^j) \\
&= m_{\mu + j\varepsilon_{l+1}}^{(n)} \\
&= m_{(\lambda, j)}^{(n)}
\end{aligned}$$

We hence find that the summands for $i = l + 1, \dots, n$ give

$$\sum_{i=l+1}^n \frac{a((\lambda, j), j)}{n - l} m_{(\lambda, j)}^{(n)} = m_{(\lambda, j)}^{(n)}.$$

We find altogether that

$$m_\lambda^{(n)} \cdot p_j^{(n)} = a((\lambda, j), j) m_{(\lambda, j)}^{(n)} + \sum_{i=1}^l \frac{a(\lambda + j\varepsilon_i, \lambda_i + j)}{a(\lambda, \lambda_i)} m_{\lambda + j\varepsilon_i}^{(n)}.$$

This is a relation for symmetric polynonomials that holds for all $n > l$. It thus gives the equality of symmetric functions

$$m_\lambda \cdot p_j = a((\lambda, j), j) m_{(\lambda,j)} + \sum_{i=1}^{l} \frac{a(\lambda + j\varepsilon_i, \lambda_i + j)}{a(\lambda, \lambda_i)} m_{\lambda + j\varepsilon_i} .$$

This can also be expressed as

$$m_{(\lambda,j)} = \frac{1}{a((\lambda, j), j)} \left(m_\lambda \cdot p_j - \sum_{i=1}^{l} \frac{a(\lambda + j\varepsilon_i, \lambda_i + j)}{a(\lambda, \lambda_i)} m_{\lambda + j\varepsilon_i} \right) \tag{7}$$

$$= \frac{1}{a((\lambda, j), j)} \left(m_\lambda \cdot p_j - \sum_{i=1}^{l} \frac{a(\lambda, \lambda_i + j) + 1}{a(\lambda, \lambda_i)} m_{\lambda + j\varepsilon_i} \right) . \tag{8}$$

Suppose that $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition of positive length $l \geq 1$. Then it follows for the partition $\lambda' := (\lambda_1, \dots, \lambda_{l-1})$ that

$$\lambda = (\lambda', \lambda_l)$$

and therefore

$$m_\lambda = m_{(\lambda',\lambda_l)} = \frac{1}{a((\lambda', \lambda_l), \lambda_l)} \left(m_{\lambda'} \cdot p_{\lambda_l} - \sum_{i=1}^{l-1} \frac{a(\lambda', \lambda'_i + \lambda_l) + 1}{a(\lambda', \lambda'_i)} m_{\lambda' + \lambda_l \varepsilon_i} \right)$$

$$= \frac{1}{a(\lambda, \lambda_l)} \left(m_{\lambda'} \cdot p_{\lambda_l} - \sum_{i=1}^{l-1} \frac{a(\lambda', \lambda'_i + \lambda_l) + 1}{a(\lambda', \lambda'_i)} m_{\lambda' + \lambda_l \varepsilon_i} \right) . \tag{9}$$

A.15.3. Consequences

Proposition A.19. Under the isomorphism $\mathbf{H}(Q, \mathbb{F}_1)$ from Section 5 the class $[\mathrm{N}_\lambda]$ corresponds to the monomial symmetric function m_λ, for every partition $\lambda \in \mathrm{Par}$.

Proof. We denote the isomorphism as in Section 5 by $\Phi : \Lambda \to \mathbf{H}(Q, \mathbb{F}_1)$. We show the assertion by induction over the length of λ. If $\ell(\lambda) = 0$ then $[\mathrm{N}_\lambda] = [0] = 1$ and also $[m_\lambda] = 1$. If $\ell(\lambda) = 1$ then $\lambda = (k)$ for some positive integer $k \geq 1$ and thus

$$\Phi([\mathrm{N}_\lambda]) = \Phi([\mathrm{N}_k]) = p_k = m_{(k)} = m_\lambda .$$

The second to last equality (which is an equality of symmetric functions) holds because

$$m_{(k)}(x_1, \dots, x_1) = \sum \{\text{distinct permutations of } x_1^k\} = x_1^k + \cdots + x_n^k = p_k(x_1, \dots, x_n)$$

for every number of variables $n \geq 0$.

For the induction step we consider the two identities (6) and (9). If $\lambda = (\lambda_1, \dots, \lambda_l)$ then for $\lambda' := (\lambda_1, \dots, \lambda_{l-1})$ these identities tell us that

$$[\mathrm{N}_\lambda] = \frac{1}{a(\lambda, \lambda_l)} \left([\mathrm{N}_{\lambda'}] \cdot [\mathrm{N}_{\lambda_l}] - \sum_{i=1}^{l-1} \frac{a(\lambda', \lambda'_i + \lambda_l) + 1}{a(\lambda', \lambda'_i)} [\mathrm{N}_{\lambda' + \lambda_l \varepsilon_i}] \right)$$

and similarly

$$m_\lambda = \frac{1}{a(\lambda,\lambda_l)}\left(m_{\lambda'}\cdot p_{\lambda_l} - \sum_{i=1}^{l-1}\frac{a(\lambda',\lambda'_i+\lambda_l)+1}{a(\lambda',\lambda'_i)} m_{\lambda'+\lambda_l\varepsilon_i}\right).$$

We note that the partitions that appear on the right hand sides of these equations are all of strictly smaller length than λ. It follows from the induction hypothesis that the right hands sides correspond to each other under Φ. The same does therefore hold for the left hand sides. □

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