Graduate Seminar on Advanced Algebra Jendrik Stelzner December 12, 2019

Jordan Quiver, Part I

Talk 10 on Hall Algebras and Quantum Groups

1. The Jordan Quiver and its Nilpotent Representations

Definition 1.1. The *Jordan quiver* is the quiver that consists of a single vertex and a single edge, which is necessarily a loop.

Throughout this talk the Jordan quiver is denoted by Q. See Figure 1 for a visualization. In the following we write $\Bbbk$ to mean a field or $\mathbb{F}_1$.

A representation of the Jordan quiver over $\Bbbk$ is the same as a pair (V, f) consisting of a $\Bbbk$-vector space V together with an endomorphism f of V. A homomorphism of representations $\varphi\colon (V, f) \to (W, g)$ is then precisely a homomorphism of vector spaces $\varphi\colon V \to W$ that makes the following square diagram commute:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

If $(V, f) \cong (W, g)$ then in particular $V \cong W$ as vector spaces. So to understand the isomorphism classes of Q-representations over $\Bbbk$ we may assume that $V = W$. The commutativity of the above square diagram, together with the requirement that φ is an isomorphism, means precisely that the endomorphisms f, g of V are similar. We hence find that that the isomorphism classes of Q-representations over $\Bbbk$ correspond one-to-one to conjugacy classes of endomorphisms of $\Bbbk$-vector spaces.

Suppose that V is finite-dimensional. If $\Bbbk$ is a field then these conjugacy classes are can be understood via the rational canonical form. In the case that $\Bbbk$ is also algebraically closed, or that it is $\mathbb{F}_1$, or that we are interested only in nilpotent endomorphisms, one can use the usual Jordan normal form. (See Appendix A.1 for the Jordan normal form over $\mathbb{F}_1$.)

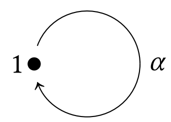

Figure 1: The Jordan quiver Q.

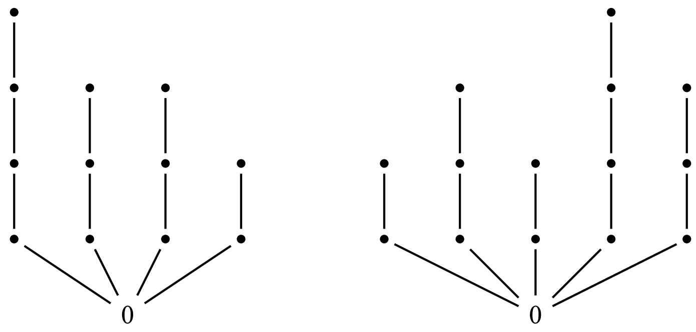

Figure 2: The representations $\mathrm{N}_{(4,3,3,2)}$ and $\mathrm{N}_{(2,3,2,4,3)}$ over $\mathbb{F}_1$.

A representation (V, f) of the Jordan quiver is nilpotent (see Appendix A.2) if and only if the endomorphism f is nilpotent. We will in the rest of this talk restrict our attention to finite-dimensional, nilpotent representations of the Jordan quiver. As introduced in the previous talks we denote by

$$\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$$

the full subcategory of $\mathbf{Rep}(Q, \Bbbk)$ whose objects are the finite-dimensional, nilpotent representations of Q over $\Bbbk$. We denote the set of isomorphism classes of $\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$ by $\mathrm{Iso}(Q, \Bbbk)$. (We will see in Corollary 1.4 that this is indeed a set.)

Every nilpotent endomorphism on a finite-dimensional $\Bbbk$-vector space admits a Jordan normal form. We can therefore classify the isomorphism classes of $\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$: For every dimension $d \geq 0$ let

$$\mathrm{N}_d := \left(\Bbbk^d, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \right)$$

if $\Bbbk$ is a field, and let

$$\mathrm{N}_d := (\{0, 1, \dots, d\}, [d \mapsto (d-1) \mapsto (d-2) \mapsto \cdots \mapsto 1 \mapsto 0 \mapsto 0])$$

if $\Bbbk = \mathbb{F}_1$. For every tupel $(d_1, \dots d_n)$ of dimensions $d_i \geq 0$ let

$$\mathrm{N}_{(d_1,\dots,d_n)} := \mathrm{N}_{d_1} \oplus \cdots \oplus \mathrm{N}_{d_n} .$$

See Figure 2 for a visualization of $\mathrm{N}_{(d_1,\dots,d_n)}$.

Proposition 1.2 (Jordan normal form for nilpotent endomorphisms)**.**

1. Every finite-dimensional, nilpotent representation of Q over $\Bbbk$ is isomorphic to a representation of the form $\mathrm{N}_{(d_1,\dots,d_n)}$ for some $n \geq 0$ and $d_1, \dots, d_n \geq 1$.
2. Two such representations $\mathrm{N}_{(d_1,\dots,d_n)}$ and $\mathrm{N}_{(d'_1,\dots,d'_m)}$ are isomorphic if and only if $n = m$ and the tuples $(d_1, \dots, d_n)$ and $(d'_1, \dots, d'_m)$ are the same up to permutation.

We can reformulate the above proposition in terms of partitions:

Definition 1.3. For every $n \geq 0$ let $\mathrm{Par}(n)$ be the set of partition of the number n, i.e.

$$\mathrm{Par}(n) := \{(\lambda_1, \dots, \lambda_l) \mid \lambda_1 \geq \cdots \geq \lambda_l \geq 1,\ \lambda_1 + \cdots + \lambda_l = n\}.^1$$

The set of all partitions is denoted by

$$\mathrm{Par} := \coprod_{n \geq 0} \mathrm{Par}(n).$$

Corollary 1.4. The representations N_λ with $\lambda \in \mathrm{Par}$ form a set of representatives for $\mathrm{Iso}(Q, \Bbbk)$.

We find in particular that the category $\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$ is essentially small, i.e. that its set of isomorphism classes $\mathrm{Iso}(Q, \Bbbk)$ is indeed a set (and even a countable one).

2. The Hall Algebra of the Jordan Quiver over $\mathbb{F}_q$

We will now consider for $\Bbbk$ the finite field $\mathbb{F}_q$. We want to consider in the following the Hall algebra of the category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$. For this we need to understand its Euler form.

Proposition 2.1. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is hereditary, i.e.

$$\mathrm{Ext}^n_{\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)} = 0$$

for every $n \geq 2$.

Proof. See Appendix A.3. □

Lemma 2.2. Let $S := \mathrm{N}_1$.

1. The representation S is the unique simple object of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ (up to isomorphism).
2. The Groethendieck group $\mathrm{K}_0(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q))$ is freely generated by the class $[S]$. Thus

$$\mathrm{K}_0\big(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)\big) \cong \mathbb{Z}$$

   via the map $[M] \mapsto \dim(M)$.
3. In $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ we have both $\mathrm{Hom}(S, S) = \mathbb{F}_q$ and $\mathrm{Ext}^1(S, S) = \mathbb{F}_q$.

Proof. We have seen the first two assertions in last week's talk. For the first assertion we note that $\mathrm{Hom}(S, S) = \mathbb{F}_q$ because S is one-dimensional. For the computation of $\mathrm{Ext}^1(S, S)$ see Appendix A.4. □

Corollary 2.3. The Euler form of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is trivial.

Proof. See Appendix A.5. □

[1] We want to point out that in this talk we do not allow a partition to contain zero as an entry. This is done purely for technical reasons.

We find from the above that $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F})$ is a abelian, finitary, hereditary category with vanishing Euler form. We are thus well-prepared to consider the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$.

1. The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_q)$ is free on the set of isomorphism classes, $\mathrm{Iso}(Q, \mathbb{F}_q)$. This basis is indexed by the set of partitions, Par.
2. The multiplication on $\mathbf{H}(Q, \mathbb{F}_q)$ is given by

$$[M] \cdot [N] = \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_q)} C^R_{M,N} [R]$$

where

$$C^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \text{ with } L \cong N \text{ and } R/L \cong M\}\,.$$

The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_q)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_q)} = [0]$.
3. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ satisfies the finite suboject condition and its Euler form vanishes.[2] It follows that Green's coproduct makes the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ into a bialgebra. Its comultiplication is given by

$$\Delta([M]) = \sum_{[M],[N] \in \mathrm{Iso}(Q, \mathbb{F}_q)} \frac{1}{a_R} P^R_{M,N} [M] \otimes [N]$$

where a_R is the size of the automorphism group $\mathrm{Aut}(R)$, and $P^R_{M,N}$ is the number of short exact sequences $0 \to N \to R \to M \to 0$. The counit $\varepsilon \colon \mathbf{H}(Q, \mathbb{F}_q) \to \mathbb{C}$ is given by

$$\varepsilon([M]) = \begin{cases} 1 & \text{if } M = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. We have a grading on $\mathbf{H}(Q, \mathbb{F}_q)$ over the Grothendieck group $\mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)) \cong \mathbb{Z}$, given by

$$\deg([M]) = \dim(M)\,.$$

This grading makes $\mathbf{H}(Q, \mathbb{F}_q)$ into a graded bialgebra.
5. The graded bialgebra $\mathbf{H}(Q, \mathbb{F}_q)$ is connected (i.e. its degree zero part is the ground field). It is therefore already a graded Hopf algebra.

We will in the rest of this talk be mostly concerned with the upcoming Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$, and continue the study of $\mathbf{H}(Q, \mathbb{F}_q)$ in next week's talk. But we will here compute at least some of the structure constants of $\mathbf{H}(Q, \mathbb{F}_q)$. For this we follow [Sch09, Example 2.2].

Example 2.4. For any three partition $\lambda, \mu, \kappa \in \mathrm{Par}$ we abbreviate the structure constant

$$C^{\mathrm{N}_\kappa}_{\mathrm{N}_\lambda, \mathrm{N}_\mu}$$

as $C^\kappa_{\lambda,\mu}$.

[2] We say that a category $\mathscr{A}$ satisfies the *finite suboject condition* if every object of $\mathscr{A}$ admits only finitely many subobjects.

1. Let $\lambda = (1^n)$ and $\mu = (1^m)$. We consider the partition $\kappa := (1^{n+m})$. The action of the edge of the Jordan quiver Q on the representations N_λ, N_μ and N_κ is trivial. We thus find that every m-dimensional linear subspace L of N_κ satisfies the conditions $L \cong \mathrm{N}_\mu$ and $\mathrm{N}_\kappa / L \cong \mathrm{N}_\lambda$. The structure constant $C^\kappa_{\lambda,\mu}$ is therefore given by

$$
\begin{aligned}
C^\kappa_{\lambda,\mu} &= \text{number of } m\text{-dimensional linear subspaces of } \mathrm{N}_\kappa \\
&= \text{number of } m\text{-dimensional linear subspaces of } \mathbb{F}_q^{n+m} \\
&= \#\mathrm{Gr}(m, n+m, \mathbb{F}_q) \\
&= \begin{bmatrix} n+m \\ m \end{bmatrix}_q .
\end{aligned}
$$

(See Appendix A.6 for the last equality.)

We see in particular that $C^\kappa_{\lambda,\mu}$ depends is a polynomial way on q. We have for example

$$
C^{(1^{n+1})}_{(1^n),(1)} = \#\mathrm{Gr}(1, n+1, \mathbb{F}_q) = \#\mathbb{P}^n(\mathbb{F}_q) = \frac{q^{n+1} - 1}{q - 1} = [n+1]_q = 1 + q + \cdots + q^n ,
$$

and also

$$
\begin{aligned}
C^{(1^{n+2})}_{(1^n),(1,1)} = \begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q &= \frac{[n+2]_q [n+1]_q}{[2]_q} \\
&= \frac{(1 + q + \cdots + q^n)(1 + q + \cdots + q^{n+1})}{1+q} \\
&= \begin{cases} (1 + q + \cdots + q^n)(1 + q^2 + \cdots + q^n) & \text{if } n \text{ is even,} \\ (1 + q^2 + \cdots + q^{n-1})(1 + q + \cdots + q^{n+1}) & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

2. Let now $\lambda = (n)$ and $\mu = (m)$. We consider the partition $\kappa = (n+m)$. The representation N_κ has the standard basis $e_1, \dots, e_{n+m}$, and the subrepresentations of N_κ are given by $\langle e_1, \dots, e_i \rangle$ for $i = 0, \dots, n+m$. The subrepresentations $L := \langle e_1, \dots, e_m \rangle$ is the unique one that is isomorphic to N_μ, and its quotient N_κ / L is isomorphic to N_λ. Thus

$$
C^{(n+m)}_{(n),(m)} = 1 .
$$

3. Let us compute the coefficients $C^{(2,1)}_{(1),(2)}$ and $C^{(2,1)}_{(2),(1)}$. We use for $\mathrm{N}_{(2,1)}$ the basis e_1, e_2, e_3 with $\alpha e_1 = e_2$ and $\alpha e_2 = \alpha e_3 = 0$ where α denotes the loop of Q. See Figure 3.

The coefficient $C^{(2,1)}_{(1),(2)}$ is the number of subrepresentations L of $\mathrm{N}_{(2,1)}$ with $L \cong \mathrm{N}_2$ and $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_1$. The condition $L \cong \mathrm{N}_2$ means that L is cyclically generated by a vector $v = a e_1 + b e_2 + c e_3$ with $a \neq 0$. We may assume that $a = 1$. Then

$$
\langle v \rangle_{\mathbb{F}_q Q} = \langle v, \alpha v \rangle_{\mathbb{F}_q} = \langle e_1 + b e_2 + c e_3, e_2 \rangle_{\mathbb{F}_q} = \langle e_1 + c e_3, e_2 \rangle .
$$

For any such subrepresentation L the quotient $\mathrm{N}_{(2,1)}/L$ is one-dimensional and thus isomorphic to N_1. We get for every coefficient $c \in \mathbb{F}_q$ a different subrepresentation of $\mathrm{N}_{(2,1)}$. Hence

$$
C^{(2,1)}_{(1),(2)} = \#\mathbb{F}_q = q .
$$

$$\begin{tikzcd}[column sep=small] e_1 \arrow[d, dash] & & \\ e_2 \arrow[dr, dash] & & e_3 \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

Figure 3: The representations $\mathrm{N}_{(2,1)}$ over $\mathbb{F}_q$.

The coefficient $C^{(2,1)}_{(1),(2)}$ is the number of subrepresentations L of $\mathrm{N}_{(2,1)}$ with $L \cong \mathrm{N}_1$ and $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_2$. The condition $L \cong \mathrm{N}_1$ means that L is cyclically generated by a nonzero vector $v = be_2 + ce_3$ with $a \neq 0$.

If $b \neq 0$ then we may assume that $b = 1$, so that $v = e_2 + ce_3$. Then $\mathrm{N}_{(2,1)}/L$ has the basis vectors $[e_1]$, $[e_3]$ with $\alpha[e_1] = -c[e_3]$ and $\alpha[e_3] = 0$. Thus $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_2$ if $c \neq 0$ and $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_{(1,1)}$ if $c = 0$. In the case $b \neq 0$ we thus have $q - 1$ choices for L. If $b = 0$ then $c \neq 0$ and we may assume that $c = 1$. Then $v = e_3$ and thus $\mathrm{N}_{(2,1)}/L \cong \mathrm{N}_2$.

We find altogether that there are q choices for L, i.e

$$C^{(2,1)}_{(2),(1)} = q\,.$$

4. One finds in the same way as above that more generally

$$C^{(n,1)}_{(n),(1)} = q = C^{(n,1)}_{(1),(n)}$$

for every $n \geq 2$.

We observe that in the above examples the coefficient $C^\kappa\lambda, \mu$ are always polynomials in q with integer coefficients. We will see in next week's talk that this is true for any coefficient $C^\kappa_{\lambda,\mu}$. This will allow us to define the *generaic Hall algebra* of the Jordan quiver.

We also have $C^\kappa_{\lambda,\mu} = C^\kappa_{\mu,\lambda}$ in each example. We will see in next week's talk that the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ is indeed commutative, which means precisely that $C^\kappa_{\lambda,\mu} = C^\kappa_{\mu,\lambda}$ for any three partitions $\lambda, \mu, \kappa \in \mathrm{Par}$.

3. The Hall Algebra of the Jordan Quiver over $\mathbb{F}_1$

We will now consider the case that $\Bbbk$ is $\mathbb{F}_1$. We have seen in last week's talk how to construct the Hall algebra of Q over $\mathbb{F}_1$:

Recall 3.1. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is a graded, cocommutative Hopf algebra (over the ground field $\mathbb{C}$). Its structure is given as follows:

- The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_1)$ is the free $\mathbb{C}$-vector space on the set $\mathrm{Iso}(Q, \mathbb{F}_1)$. The set $\mathrm{Iso}(Q, \mathbb{F}_1)$ is indexed by the set of partitions Par.
- The grading of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $\deg([M]) = \dim(M)$.

- The multiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

$$[M] \cdot [N] := \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)} C^R_{M,N} [R]$$

where the structure coefficients $C^R_{M,N}$ are given by

$$C^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \mid L \cong N, R/L \cong M\}\,.$$

The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_1)} = [0]$.

- The comultiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

$$\Delta([M]) = \sum_{\substack{[R],[L] \in \mathrm{Iso}(Q,\mathbb{F}_1) \\ M \cong R \oplus L}} [R] \otimes [L]\,.$$

We see in particular that an isomorphism class $[M]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$ if and only if the representation M is indecomposable. We have seen that more generally the Lie algebra of primitive elements of $\mathbf{H}(Q, \mathbb{F}_1)$ has a basis consisting of all such $[M]$.

Example 3.2. We can again compute some structure constants:

1. Let again $\lambda = (1^n)$ and $\mu = (1^m)$, and consider $\kappa = (1^{n+m})$. We find as before that

$$C^\kappa_{\lambda,\mu} = \text{number of } m\text{-dimensional subspaces of } \mathrm{N}_{n+m} = \binom{n+m}{m}.$$

This is the same result as before by taking the limit $q \to 1$.

2. Let again $\lambda = (n)$ and $\mu = (m)$ and consider $\kappa = (n + m)$. We find as before that

$$C^\kappa_{\lambda,\mu} = 1\,.$$

3. Let us compute the product $[\mathrm{N}_i] \cdot [\mathrm{N}_j]$.

We observe that if $[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)$ and L is a subrepresentation of R that is isomorphic to N_j then the quotient R/L results from R by contracting one of the Jordan chains of R by j elements. If $R/L \cong \mathrm{N}_i$ then this means that R consists of a single Jordan chain of length $i + j$, or of two Jordan chains of length i and j respectively. Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i,j)}] + b[\mathrm{N}_{i+j}]\,.$$

We have seen above that $b = C^{(i+j)}_{(i),(j)} = 1$. The coffient a is the number of entries of (i, j) that are of length j. Thus

$$a = \begin{cases} 1 & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = \begin{cases} [\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i \neq j, \\ 2[\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i = j. \end{cases}$$

We see in particular that $[\mathrm{N}_i]$ and $[\mathrm{N}_j]$ commute.

4. We find in the same way that for all $i_1, \dots, i_r \geq 1$ and $j \geq 1$,

$$[\mathrm{N}_{(i_1,\dots,i_r)}] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i_1,\dots,i_r,j)}] + \sum_{\lambda} b_\lambda [\mathrm{N}_\lambda]\,,$$

where λ runs through all distinct tupels of the form $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ with $1 \leq k \leq r$. The coeffiecient a is given by

$$a = \text{how often } j \text{ occurs in } (i_1, \dots, i_r, j)$$

and the coefficient of b_λ for $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ are given by

$$b_\lambda = \text{how often } i_k + j \text{ occurs in } \lambda\,.$$

We have for example

$$\begin{aligned} [\mathrm{N}_{(5,3,3,2,1)}] \cdot [\mathrm{N}_2] = 2[&\mathrm{N}_{(5,3,3,2,1,\overline{2})}] \\ &+ [\mathrm{N}_{(\overline{7},3,3,2,1)}] + 2[\mathrm{N}_{(5,\overline{5},3,2,1)}] + [\mathrm{N}_{(5,3,3,\overline{4},1)}] + 3[\mathrm{N}_{(5,3,3,2,\overline{3})}]\,, \end{aligned}$$

where the entry of interested is overlined.

It follows from the above formula by induction that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated as an algebra by the N_i with $i \geq 1$.[3]

Corollary 3.3. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is commutative.

Remark 3.4. We have seen last week that the Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is the universal enveloping algebra of its Lie algebra of primitive elements, which in turn is spanned (as a vector space) by the N_i. We have thus already seen last week that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated by the N_i as an algebra.

We have hence shown that $\mathbf{H}(Q, \mathbb{F}_1)$ is a commutative, cocommutative, graded Hopf algebra We will in the following show that it is actually the ring of symmetric functions.

4. The Ring of Symmetric Functions

4.1. Definition

For every $k \geq 0$ we denote by

$$\Lambda^{(k)} := \mathbb{C}[x_1, \dots, x_k]^{\mathrm{S}_k}$$

the algebra of symmetric polynomials in k variables. We have for every number of variables $k \geq 0$ a homomorphism of graded algebras

$$\Lambda^{(k+1)} \to \Lambda^{(k)}\,, \quad f^{(k+1)} \mapsto f^{(k+1)}(x_1, \dots, x_k, 0)\,.$$

[3] We have seen in last week's talk that $\mathbf{H}(Q, \mathbb{F}_1)$ is the universal enveloping algebra of its Lie algebra of primitive elements by the Milnor–Moore theorem. We have also seen that this Lie algebra is spanned, as a vector space, by the isomorphism classes N_i with $i \geq 1$, as these are the indecomposable objects in $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_1)$. It therefore also follows from last week's talk that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated by the N_i with $i \geq 0$.

Definition 4.1. The *ring of symmetric functions* Λ is the limit

$$\Lambda := \lim_{k \geq 0} \Big(\Lambda^{(k+1)} \to \Lambda^{(k)} \Big)$$

in the category of graded algebras. The elements of Λ are *symmetric functions.*

Warning 4.2. A symmetric function is – contrary to its name – not a function.

Let us make the above definition more explicit: For every $n \geq 0$ we have

$$\begin{aligned} \Lambda_n &= \lim_{k \geq 0} \Big(\Lambda_n^{(k+1)} \to \Lambda_n^{(k)} \Big) \\ &= \left\{ \big(f^{(k)}\big)_{k \geq 0} \;\middle|\; \begin{array}{l} f^{(k)} \in \Lambda_n^{(k)} \text{ for every } k \geq 0 \text{ such that} \\ f^{(k+1)}(x_1, \dots, x_k, 0) = f^{(k)} \text{ for every } k \geq 0 \end{array} \right\} , \end{aligned}$$

and we have overall

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n$$

as vector spaces. The multiplication on Λ is given by

$$(f^{(k)})_{k \geq 0} \cdot (g^{(k)})_{k \geq 0} = (f^{(k)} \cdot g^{(k)})_{k \geq 0}$$

for all $(f^{(k)})_{k \geq 0} \in \Lambda_n$ and $(g^{(k)})_{k \geq 0} \in \Lambda_m$.

A homogeneous symmetric function f, say of degree n, is thus the same as a "consistent choice" of homogeneous symmetric polynomials $f^{(k)} \in \Lambda^{(k)}$ of degree n for every $k \geq 0$. We have for every number of variables $k \geq 0$ a homomorphism of graded algebras

$$\Lambda \to \Lambda^{(k)} , \quad f \mapsto f(x_1, \dots, x_k)$$

that is given in each degree by projection onto the k-th component. For any two symmetric functions f, g we have by construction of Λ that

$$f = g \iff f(x_1, \dots, x_k) = g(x_1, \dots, x_k) \text{ for every } k \geq 0.$$

Example 4.3. We have for every number of variables $k \geq 0$ and every degree $n \geq 0$ the *elementary symmetric polynomial*

$$e_n^{(k)}(x_1, \dots, x_k) := \sum_{1 \leq i_1 < \dots < i_n \leq k} x_{i_1} \cdots x_{i_n} \in \Lambda_n^{(k)}$$

with $e_n^{(k)} = 0$ whenever $n > k$. These polynomials satisfy the conditions

$$e_n^{(k+1)}(x_1, \dots, x_k, 0) = e_n^{(k)}(x_1, \dots, x_k)$$

for all $k \geq 0$. These elementary symmetric polynomials $e_n^{(k)}$ with $k \geq 0$ therefore assemble into a single homogeneous symmetric function

$$e_n \in \Lambda_n .$$

This is the *n-th elementary symmetric function.*

We find similarly that the *power symmetric polynomials*

$$p_n^{(k)}(x_1, \dots, x_k) := x_1^n + \cdots + x_k^n ,$$

and the *completely homogenous symmetric polynomials*

$$h_n^{(k)}(x_1, \dots, x_k) := \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq k} x_{i_1} \cdots x_{i_n} = \sum \text{monomials of homogeneous degree } n$$

result in homogeneous symmetric functions

$$p_n, h_n \in \Lambda .$$

These are the *power symmetric functions* and *completely homogeneous symmetric functions.*

Warning 4.4.

1. The ring of symmetric functions Λ is *not* the limit of the rings of symmetric polynomials $\Lambda^{(k)}$ in the category of (commutative) rings.
2. The ring of symmetric functions Λ is *not* isomorphic to the rings of symmetric polynomials

   $$\mathbb{C}[x_1, x_2, x_3, \dots]^{\mathrm{S}_{\mathbb{N}}} \quad \text{or} \quad \mathbb{C}[x_1, x_2, x_3, \dots]^{\mathrm{S}_\infty}$$

   where $\mathrm{S}_\infty = \operatorname{colim}_{n \geq 0}(\mathrm{S}_n \hookrightarrow \mathrm{S}_{n+1})$.

(See Appendix A.7 for more details.)

4.2. The Fundamental Theorem on Symmetric Functions

The *fundamental theorem of symmetric polynomials* asserts that for every number of variables $k \geq 0$ the elementary symmetric polynomials

$$e_1^{(k)}, \dots, e_k^{(k)}$$

form an algebraically independent generating set for the algebra of symmetric polynomials $\Lambda^{(k)}$. It follows from this that both

$$h_1^{(k)}, \dots, h_k^{(k)} \quad \text{and} \quad p_1^{(k)}, \dots, p_k^{(k)}$$

also form algebraically independent algebra generating set for $\Lambda^{(k)}$.[4]

For every partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we can consider the symmetric polynomials

$$e_\lambda^{(k)} := e_{\lambda_1}^{(k)} \cdots e_{\lambda_l}^{(k)} , \qquad h_\lambda^{(k)} := h_{\lambda_1}^{(k)} \cdots h_{\lambda_l}^{(k)} , \qquad p_\lambda^{(k)} := p_{\lambda_1}^{(k)} \cdots p_{\lambda_l}^{(k)} .$$

[4] For the elementary symmetric polynomials $e_i^{(k)}$ and homogeneous symmetric polynomials $h_i^{(k)}$ these statements do not only hold over the ground field $\mathbb{C}$, but over every commutative ring. For the power symmetric polynomials $p_i^{(k)}$ we need to work over a field in which the numbers $1, \dots, k$ are invertible.

We have just formulated that the symmetric polynomials

$$e_\lambda^{(k)} \quad \text{for } \lambda \in \mathrm{Par} \text{ with length } \ell(\lambda) \leq k$$

form a vector space basis for $\Lambda^{(k)}$, and similarly for $h_\lambda^{(k)}$ and $p_\lambda^{(k)}$. We can generalize these families of symmetric polynomials to symmetric functions:

Example 4.5. For every $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we consider the symmetric functions

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_l}, \quad h_\lambda := h_{\lambda_1} \cdots h_{\lambda_l}, \quad p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}.$$

and note that

$$\begin{aligned} e_\lambda(x_1, \dots, x_k) &= e_\lambda^{(k)}(x_1, \dots, x_k), \\ h_\lambda(x_1, \dots, x_k) &= h_\lambda^{(k)}(x_1, \dots, x_k), \\ p_\lambda(x_1, \dots, x_k) &= p_\lambda^{(k)}(x_1, \dots, x_k). \end{aligned}$$

Another important family of symmetric polynomials are the *monomial symmetric polynomials*: For every number of variables $k \geq 0$ and partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ of length $l = \ell(\lambda) \leq k$ the corresponding monomial symmetric polynomial is given by

$$m_\lambda^{(k)}(x_1, \dots, x_k) = \sum \text{distinct permutations of } x_1^{\lambda_1} \cdots x_l^{\lambda_l}.$$

These homogeneous polynomials also form a basis of $\Lambda^{(k)}$. They too can be generalized to symmetric functions.

Example 4.6. For $k \geq 0$ any any partition $\lambda \in \mathrm{Par}$ of length $\ell(\lambda) > k$ we set

$$m_\lambda^{(k)} := 0.$$

Then

$$m_\lambda^{(k+1)}(x_1, \dots, x_k, 0) = m_\lambda^{(k)}$$

for every $k \geq 0$, and each $m_\lambda^{(k)}$ is homogeneous of degree $|\lambda|$. We therefore get a well-defined homogeneous symmetric function

$$m_\lambda \in \Lambda_{|\lambda|},$$

which we call the *monomial symmetric function* associated to λ.

We now want to generalize the fundamental theorem on symmetric polynomials to symmetric functions. The key observation behind this is the following:

Proposition 4.7. The map $\Lambda_n^{(k+1)} \to \Lambda_n^{(k)}$ is an isomorphism whenever $k \geq n$.

Proof. A vector space basis of $\Lambda_n^{(k)}$ is given by the symmetric polynomials $e_\lambda^{(k)}$ where the partition λ is of length $\ell(\lambda) \leq k$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies

$$\lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = n.$$

A vector space basis of $\Lambda_n^{(k)}$ is given by the symmetric polynomials $e_\mu^{(k+1)}$ where the partition μ is of length $\ell(\mu) \leq k+1$ and $\mu = (\mu_1, \dots, \mu_l)$ satisfies

$$\mu_1 + 2\mu_2 + \cdots + l\mu_l = n\,.$$

We find by degree reasons that the case $\ell(\mu) = k+1$ cannot occur. The linear map $\Lambda_n^{(k+1)} \to \Lambda_n^{(k)}$ does therefore restrict to a bijection between the above bases. □

Corollary 4.8. The map $\Lambda_n \to \Lambda_n^{(k)}$ is an isomorphism whenever $k \geq n$. □

Corollary 4.9. The following families of symmetric functions form vector space bases of Λ:

1. The elementary symmetric polynomials e_λ with $\lambda \in \mathrm{Par}$.
2. The complete homogeneous symmetric polynomials h_λ with $\lambda \in \mathrm{Par}$.
3. The power symmetric polynomials p_λ with $\lambda \in \mathrm{Par}$.
4. The monomial symmetric polynomials m_λ with $\lambda \in \mathrm{Par}$. □

Corollary 4.10. The elementary symmetric functions e_n with $n \geq 1$ form an algebraically independent algebra generating set for Λ, and similarly for the h_n and the p_n.

Corollary 4.11. We have $\Lambda \cong \mathbb{C}[X_1, X_2, X_3, \dots]$ as graded algebras, where each variable X_n is homogeneous of degree n.

Remark 4.12.

1. It follows from Corollary 4.8 for any two symmetric functions $f, g \in \Lambda$ that

$$f = g \iff f(x_1, \dots, x_k) = g(x_1, \dots, x_k) \text{ for some } k \geq \deg(f), \deg(g).$$

2. We can regard Λ as a colimit of $\Lambda^{(k)} \to \Lambda^{(k+1)}$ for $k \geq 0$, see Appendix A.8.

4.3. Hopf Algebra Structure

We can endow the algebra of symmetric functions Λ with the structure of a graded Hopf algebra. We have now for any two number of variables $k, l \geq 0$ a homomorphism of graded algebras

$$\begin{aligned}
\Delta_{kl} \colon \Lambda^{(k+l)} &= \mathbb{C}[x_1, \dots, x_{k+l}]^{\mathrm{S}_{k+l}} \\
&\subseteq \mathbb{C}[x_1, \dots, x_{k+l}]^{\mathrm{S}_k \times \mathrm{S}_l} \\
&\cong (\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}[x_{k+1}, \dots, x_{k+l}])^{\mathrm{S}_k \times \mathrm{S}_l} \\
&\cong (\mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}[x_1, \dots, x_l])^{\mathrm{S}_k \times \mathrm{S}_l} \\
&= \mathbb{C}[x_1, \dots, x_k]^{\mathrm{S}_k} \otimes \mathbb{C}[x_1, \dots, x_l]^{\mathrm{S}_l} && \text{(Appendix A.9)} \\
&= \Lambda^{(k)} \otimes \Lambda^{(l)}.
\end{aligned}$$

We would like to have a homomorphism of graded algebras $\Delta \colon \Lambda \to \Lambda \otimes \Lambda$ such that for all degrees $k, l \geq 0$ the square diagram

$$\begin{tikzcd} \Lambda \arrow[r, dashed, "\Delta"] \arrow[d] & \Lambda \otimes \Lambda \arrow[d] \\ \Lambda^{(k+l)} \arrow[r, "\Delta_{kl}"] & \Lambda^{(k)} \otimes \Lambda^{(l)} \end{tikzcd}$$

commutes. The composition $\Lambda \to \Lambda^{(k)} \otimes \Lambda^{(l)}$ is given on the algebra generators p_n of Λ by

$$p_n \mapsto p_n^{(k)} \otimes 1 + 1 \otimes p_n^{(l)} .$$

Such an algebra homomorphism Δ is thus given by

$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n .$$

This homomorphism exists because Λ is the free commutative algebra on the generators p_n.

The homomorphism Δ makes the algebra Λ into a cocommutative, graded bialgebra. The counit is given on algebra generators by

$$\varepsilon(p_n) = 0$$

for every $n \geq 0$. Since Λ is graded and connected it follows that it is already a graded Hopf algebra. Its antipode is given on algebra generators by

$$S(p_n) = -p_n$$

for every $n \geq 0$.

We see altogether that Λ is a commutative, cocommutative, graded Hopf algebra.

5. The Isomorphism $\mathbf{H}(Q, \mathbb{F}_1) \cong \Lambda$

Both $\mathbf{H}(Q, \mathbb{F}_1)$ and Λ are commutative, cocommutate, graded Hopf algebras. They are isomorphic as graded Hopf Algebras:

The ring of symmetric functions Λ has is, as a commutative algebra, freely generated by the power symmetric functions $p_1, p_2, \dots$ There hence exists a unique, surjective algebra homomorphism $\Phi \colon \Lambda \to \mathbf{H}(Q, \mathbb{F}_1)$ with

$$\Phi(p_i) = [\mathrm{N}_i]$$

for every $i \geq 1$. We note that Φ is a homomorphism of graded algebras because both p_i and $[\mathrm{N}_i]$ are of degree i. We also have for every degree $n \geq 0$ that

$$\dim \Lambda_n = \#\{(\lambda_1, \dots, \lambda_k) \in \mathrm{Par} \mid \lambda_1 + 2\lambda_2 + \cdots + k\lambda_k = n\} = \dim \mathbf{H}(Q, \mathbb{F}_1)_n ,$$

with these dimensions being finite. It thus follows from the surjectivity of Φ that it is already an isomorphism of graded algebras.

The algebra isomorphism Φ is already an isomorphism of Hopf algebras: It sufficies to check that Φ is compatible with the comultiplication of the algebra generators p_i. This holds since p_i is primitive in Λ and $[\mathrm{N}_i]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$.

We have shown altogether that Φ is an isomorphism of graded Hopf algebras.

A. Appendices

A.1. Theorem of Krull–Remak–Schmidt and Jordan Normal Form

Let V be an $\mathbb{F}_1$-vector space and let $f\colon V \to V$ be an endomorphism.

Recall A.1. 1. A *subspace* of V is a subset of V that contains the base point 0.

2. If $(U_i)_{i\in I}$ is a collection of subspaces of V then $V = \bigoplus_{i\in I} U_i$ if and only if every nonzero evement of V is contained in precisely one U_i, i.e. if and only if the U_i give a disjoint decomposition of the set $V \setminus \{0\}$.
3. If U is a subspace of V and $V = \bigoplus_{i\in I} W_i$ is a direct sum decomposition then $U = \bigoplus_{i\in I}(U \cap W_i)$.

Definition A.2. A subspace U of V is *f-invariant* if $f(U) \subseteq U$. An f-invariant subspace U of V is *indecomposable* if it is nonzero and there exist no two nonzero f-invariant subspaces W_1, W_2 of U with $U = W_1 \oplus W_2$.

Remark A.3. If U is an indecomposable subspace of V and $U = \bigoplus_{i\in I} W_i$ is any decomposition into f-invariant subspaces W_i then it follows that $U = W_j$ for some $j \in I$ while $W_i = 0$ for every $i \neq j$. Indeed, some W_j must be nonzero because V is nonzero. Then $V = W_j \oplus \bigoplus_{i\in I, i\neq j} W_i$ and thus $\bigoplus_{i\in I, i\neq j} W_i = 0$, and therefore $W_i = 0$ for every $i \in I$.

Proposition A.4 (Krull–Remak–Schmidt)**.** There exists a unique direct sum decomposition of V into indecomposable f-invariant subspaces.

Proof. In the following we mean by a *decomposition* a direct sum decomposition into f-invariant subspaces in which each direct summand is nonzero. We say that a decomposition $V = \bigoplus_{i\in I} U_i$ is *finer* than a decomposition $V = \bigoplus_{j\in J} W_j$ if each U_i is contained in some W_j. This gives a partial order on the set of decompositions of V.

We note that a decomposition $V = \bigoplus_{i\in I} U_i$ consists of indecomposables if and only if it is maximal fine. Indeed, if some U_j is decomposable then there exists a decomposition $U_j = U_j' \oplus U_j''$. Then

$$V = \bigoplus_{i\in I} U_i = \bigoplus_{\substack{i\in I\\ i\neq j}} U_i \oplus U_j = \bigoplus_{\substack{i\in I\\ i\neq j}} U_i \oplus U_j' \oplus U_j'$$

with the last term being a strictly finer that the original decomposition $V = \bigoplus_{i\in I} U_i$. Suppose on the other hand that each U_i is indecomposable and that $V = \bigoplus_{j\in J} W_j$ is a decomposition that is finer than $V = \bigoplus_{i\in I} U_i$. Then $U_i = \bigoplus_{j\in J}(U_i \cap V_j)$ for every $j \in J$. It follows that $U_i = U_i \cap V_j$ for some $j \in J$ and thus $U_i \subseteq V_j$. We also know that V_j is contained in some U_k. Then U_i is contained in U_k whence it follows that $i = k$ and thus $U_i = V_j$. This shows that each U_i equals some V_j, from which it follows that both decompositions must coincide.

We hence need to show that there exists a unique decomposition which is maximal fine. It sufficies to show that any collection of decompositions has a common refinement. Taking a common refinement of all decompositions then gives the desired one.

Let $V = \bigoplus_{j\in J_i} U_j^i$ with $i \in I$ be a collection of decompositions. For every nonzero vector $v \in V$ there exists for every $i \in I$ a unique index $j(i,v) \in J_i$ with $v \in U_{j(i,v)}^i$. We consider

$$W_v := \bigcap_{i\in I} U_{j(i,v)}^i \,.$$

Each W_v is an intersection of f-invariant subspaces and therefore again an f-invariant subspace. Each nonzero vector v of V is contained in some $W_{v'}$, namely for $v' = v$.

Suppose that for two nonzero vectors $v, u \in V$ the subspaces W_v and W_u intersect nonzero. Let w be a nonzero vector contained in both W_v and W_u. Then for every index $i \in I$ the vector w is contained in $U^i_{j(i,v)}$, whence $j(i, v) = j(i, w)$. It follows that $W_v = W_w$, and we find in the same way that $W_u = W_w$. Thus $W_v = W_w$.

This shows that the f-invariant subspaces W_v give a disjoint decomposition of $V \setminus \{0\}$, and hence a decomposition of V. (Once we remove those subspaces which occur multiple times.) This decomposition of V is by construction finer than each decomposition $V = \bigoplus_{j \in J_i} U^i_j$. $\square$

We want to understand how the decomposition from the Krull–Remak–Schmidt theorem looks like. We note that if $v \in V$ is any nonzero vector then there exists at most one preimage of v under f, since f is injective outside of its kernel. Thus we can consider for every nonzero vector $v \in V$ the well-defined two-sided sequence

$$\dots, f^{-2}(v), f^{-1}(v), v, f(v), f^2(v), \dots$$

Here the left part of the sequence consists of as many iterated preimages as exist. The set of all these elements is the *orbit* of v under f. It is denoted by $[v]$.

We note that for any nonzero veector u in $[v]$ we have $[u] = [v]$. If two orbits $[v]$ and $[w]$ intersect in a nonzero vector u then it follows that $[v] = [u] = [v]$. Two distinct orbit do therefore intersect at most in 0. It follows that the orbits give induce a disjoint decomposition of $V \setminus \{0\}$. The vector space V does therefore decompose into the direct sum of the subspaces $[v] \cup \{0\}$. (Once we remove those subspaces which occur multiple times.) Each subspace $[v] \cup \{0\}$ is f-invariant. Any two nonzero f-invariant subspaces of $[v] \cup \{0\}$ intersect nonzero whence the subspaces $[v] \cup \{0\}$ are indecomposable.

This shows that the decomposition of V from the Krull–Remak–Schmidt theorem is given by the orbits with respect to f (together with $\{0\}$).

There exist five kinds of orbits.

Type A. The orbit ends in zero and is finite: It is thus of the form

$$v \to f(v) \to \cdots \to f^n(v) = 0$$

for a unique vector v, that has no preimage under f.

Type B. The orbit ends in zero and is infintite: It is thus of the form

$$\cdots \to f^{-2}(v) \to f^{-1}(v) \to v \to f(v) \to f^2(v) \to \cdots \to f^n(v) = 0\,.$$

Type C. The orbit never reaches zero and has only finitely many preimages. It is thus of the form

$$v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to \cdots$$

for a unique vector v, that has no preimage under f.

Type D. The orbit goes infinite in both directions and is non-circular. It is thus of the form

$$\cdots \to f^{-1}(v) \to v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to \cdots$$

	injective	surjective	bijective	nilpotent	finite-dimensional
Type A				∘	∘
Type B			•	∘	
Type C	•				
Type D	•	•	•		
Type E	•	•	•		∘

Table 1: Possible orbits. Complete characterization via orbits for •. Only locally for ∘.

Type E. The orbit is circular. It is thus of the form

$$v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to v \to f(v) \to \cdots$$

Depending on the properties of the vector space V and endomorphism f not all kinds of orbits can occur.

- The endomorphism f is injective if and only if no orbits of Type A and Type B occur.
- The endomorphism f is surjective if and only if no orbits of Type A and Type C occur.
- The endomorphism f is bijective if and only if only orbits of Type D and Type E occur.
- The endomorphism f is locally nilpotent if and only if only orbits of Type A and Type B appear.[5]
- The endomorphism f is nilpotent if and only if only orbits of Type A occur, and the lengths of the occuring orbits is bounded.
- If V is finite-dimensional then only orbits of Type A and Type E occur.
- More generally, V is locally finite-dimensional with respect to f if and only if only orbits of Type A and Type E occur.[6]

See Table 1 for an overview.

A.2. Nilpotent Representations

A representation $V = ((V_i)_{i\in\Gamma_0}, (f_\alpha)_{\alpha\in\Gamma_1})$ of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is *nilpotent* if there exists some $N \geq 0$ such that for every path $\alpha_n, \dots, \alpha_1$ in Γ of length $n \geq N$ we have $f_{\alpha_n} \circ \cdots \circ f_{\alpha_1} = 0$. (See [Szc11, Definition 4.4].)

If Γ is finite and has no oriented cycles then every representation of Γ is nilpotent.

A.3. Proof of Proposition 2.1

Definition A.5. Let $\mathscr{A}$ be an abelian category. A subcategory $\mathscr{B}$ is *closed under extensions* if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ the middle term B is contained in $\mathscr{B}$ provided that both outer terms A, C are contained in $\mathscr{B}$.

[5]An endomorphism f is locally nilpotent if there exists for every vector v some power $n \geq 0$ such that $f^n(v) = 0$.

[6]We say that V is locally finite-dimensional if every nonzero vector of V is contained in a finite-dimensional f-invariant subspace.

Definition A.6. Let $\mathscr{A}$ be an abelian category. A subcategory $\mathscr{B}$ of $\mathscr{A}$ is a *Serre subcategory* if it is abelian, exact, full and closed under extensions.

Example A.7. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is a Serre subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q) \cong \mathbb{F}_q[x]$-$\mathbf{Mod}$. Indeed, it is a full, abelian, exact subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q)$. If in a short exact sequence

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

of Q-representations both A, B are finite-dimensional then the same holds for B. If both A, C are nilpotent then same holds for B: There exists some powers $n, m \geq 0$ with $\alpha^n A = 0$ and $\alpha^m C = 0$. It follows that $\alpha^m B \subseteq \ker(\psi) = \operatorname{im}(\varphi)$ and thus $\alpha^{n+m} B = 0$.

If $\mathscr{A}$ is an abelian category and $\mathscr{B}$ is an abelian, exact subcategory then we have for every $n \geq 0$ and every two objects A, B of $\mathscr{B}$ an induced map

$$\operatorname{Ext}^n_{\mathscr{B}}(A, B) \to \operatorname{Ext}^n_{\mathscr{A}}(A, B) .$$

Proposition A.8. Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be an abelian, exact subcategory of $\mathscr{A}$.

1. Suppose that $\mathscr{B}$ is a full subcategory of $\mathscr{A}$. Then the induced map $\operatorname{Ext}^1_{\mathscr{B}}(A, B) \to \operatorname{Ext}^1_{\mathscr{A}}(A, B)$ is injective for any two objects A, B of $\mathscr{B}$. If $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$ then the induced map $\operatorname{Ext}^1_{\mathscr{B}}(A, B) \to \operatorname{Ext}^1_{\mathscr{A}}(A, B)$ is bijective.
2. Suppose that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Suppose furthermore that for some $n \geq 1$ the induced map $\operatorname{Ext}^n_{\mathscr{B}}(A, B) \to \operatorname{Ext}^n_{\mathscr{A}}(A, B)$ is bijective for any two objects A, B of $\mathscr{B}$. Then the induced map $\operatorname{Ext}^{n+1}_{\mathscr{B}}(A, B) \to \operatorname{Ext}^{n+1}_{\mathscr{A}}(A, B)$ is injective for any two objects A, B of $\mathscr{B}$.

Proof.

1. Two short exact sequences in $\mathscr{B}$,

   $$0 \to B \to X \to A \to 0 \qquad \text{and} \qquad 0 \to B \to X' \to A \to 0 ,$$

   are equivalent in $\mathscr{A}$ if there exists an isomorphism $\varphi \colon X \to X'$ that makes the diagram

   $$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

   commute. We find that φ is already an isomorphism in $\mathscr{B}$ because $\mathscr{B}$ is full in $\mathscr{A}$. Thus both sequences are already equivalent in $\mathscr{B}$. This shows the injectivity of $\operatorname{Ext}^1_{\mathscr{B}}(A, B) \to \operatorname{Ext}^1_{\mathscr{A}}(A, B)$.
   Suppose now that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Let A, B be two objects in $\mathscr{B}$. Every element ξ of $\operatorname{Ext}^1_{\mathscr{A}}(A, B)$ is represented by a short exact sequence $0 \to B \to X \to A \to 0$ in $\mathscr{A}$. The middle term X is already contained in $\mathscr{B}$ because $\mathscr{B}$ is closed under extensions. Thus ξ lies in $\operatorname{Ext}^1_{\mathscr{B}}(A, B)$.
2. We refer to [Oor63, Proposition 3.3]. □

Corollary A.9. Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be a Serre subcategory of $\mathscr{A}$. If $\mathscr{A}$ is hereditary then so is $\mathscr{B}$.

Proof. Let A, B be two objects of $\mathscr{B}$. We show by induction on $n \geq 1$ that the induced map $\operatorname{Ext}^n_{\mathscr{B}}(A,B) \to \operatorname{Ext}^n_{\mathscr{A}}(A,B)$ is bijective. The assertion follows from this.

We know from Proposition A.8 that the induced map $\operatorname{Ext}^1_{\mathscr{B}}(A,B) \to \operatorname{Ext}^1_{\mathscr{A}}(A,B)$ is bijective. If for some $n \geq 1$ the induced map $\operatorname{Ext}^n_{\mathscr{B}}(A,B) \to \operatorname{Ext}^n_{\mathscr{A}}(A,B)$ is bijective then it follows from Proposition A.8 that the induced map $\operatorname{Ext}^{n+1}_{\mathscr{B}}(A,B) \to \operatorname{Ext}^{n+1}_{\mathscr{A}}(A,B)$ is injective. It is also surjective because $\mathscr{A}$ is hereditary. □

Proof of Proposition 2.1. We have seen in Example A.7 that $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is a Serre subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q) \cong \mathbb{F}_q[x]$-$\mathbf{Mod}$. The module category $\mathbb{F}_q[x]$-$\mathbf{Mod}$ is hereditary because it has enough projectives and submodules of projective $\mathbb{F}_q[x]$-modules are again projective, since $\mathbb{F}[x]$ is a principal ideal domain. It thus follows from Corollary A.9 that $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is again hereditary. □

A.4. Computing $\operatorname{Ext}^1(S,S)$

In $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ we can compute $\operatorname{Ext}^1(S,S)$ for $S = \mathrm{N}_1$ in two ways:

A.4.1. Via Homological Algebra

Let $\Bbbk \coloneqq \mathbb{F}_q$. We find with Proposition A.8 that

$$\operatorname{Ext}^1(S,S) = \operatorname{Ext}^1_{\mathbf{rep}^{\mathrm{nil}}(Q,\Bbbk)}(S,S) \cong \operatorname{Ext}^1_{\mathbf{Rep}(Q,\Bbbk)}(S,S) \cong \operatorname{Ext}^1_{\Bbbk[x]\text{-}\mathbf{Mod}}(\Bbbk,\Bbbk)\,.$$

We can use for $\Bbbk$ (in the first argument) the projective resolution

$$\cdots \to 0 \to \Bbbk[x] \xrightarrow{x} \Bbbk[x] \to \Bbbk \to 0\,.$$

Applying the functor $\operatorname{Hom}_{\Bbbk[x]}(-,\Bbbk)$ gives the chain complex

$$0 \to \operatorname{Hom}_{\Bbbk[x]}(\Bbbk[x],\Bbbk) \xrightarrow{x} \operatorname{Hom}_{\Bbbk[x]}(\Bbbk[x],\Bbbk) \to 0 \to \cdots\,,$$

which is isomorphic to the chain complex

$$0 \to \Bbbk \xrightarrow{0} \Bbbk \to 0 \to \cdots$$

We find in particular that

$$\operatorname{Hom}_{\Bbbk[x]}(\Bbbk,\Bbbk) \cong \Bbbk\,, \quad \operatorname{Ext}^1_{\Bbbk[x]}(\Bbbk,\Bbbk) \cong \Bbbk\,.$$

A.4.2. Via Counting

We can also count the Yoneda classes of short exact sequences: We have $\mathrm{N}_1 = (\Bbbk, [0])$, and a short exact sequence

$$0 \to (\Bbbk, [0]) \to ? \to (\Bbbk, [0]) \to 0$$

can have as its middle term (up to isomorphism) either

$$\mathrm{N}_{(1,1)} = \left(\Bbbk^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{or} \qquad \mathrm{N}_2 = \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

In the first case we get a short exact sequence

$$0 \to (\Bbbk, [0]) \to \left(\Bbbk^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \to (\Bbbk, [0]) \to 0\,.$$

This short exact sequence splits on the level of $\Bbbk$-vector spaces, and any such split is alreday a homomorphism of representations. We hence find that this sequence describes the unique element of $\mathrm{Ext}^1(S, S)$ that is given by the split exact sequences.

We consider now the short exact sequences of the form

$$0 \to (\Bbbk, [0]) \xrightarrow{\varphi} \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \xrightarrow{\psi} (\Bbbk, [0]) \to 0\,. \tag{1}$$

The homomorphism φ must be of the form

$$\varphi = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

for some $a \neq 0$ since the image of φ must be contained in the kernel of $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. It follows from the exactness of the sequence that the homomorphism ψ is of the form

$$\psi = \begin{bmatrix} 0 & b \end{bmatrix}$$

for some $b \neq 0$.

Two such sequences $\xi_{a,b}$ and $\xi_{a',b'}$ for $a, a', b, b' \neq 0$ are Yoneda equivalent if and only if there exists an invertible matrix

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathrm{GL}(2, \Bbbk)$$

such that

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} \tag{2}$$

and the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & (k, [0]) \arrow[r, "{\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]}"] \arrow[d, equal] & \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \arrow[r, "{\left[\begin{smallmatrix} 0 & b \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]}"] & (k, [0]) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (k, [0]) \arrow[r, "{\left[\begin{smallmatrix} a' \\ 0 \end{smallmatrix}\right]}"] & \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \arrow[r, "{\left[\begin{smallmatrix} 0 & b' \end{smallmatrix}\right]}"] & (k, [0]) \arrow[r] & 0
\end{tikzcd} \tag{3}$$

The condition (2) means that $w = z$ and $y = 0$, i.e. that the the matrix is of the form

$$\begin{bmatrix} w & x \\ 0 & w \end{bmatrix},$$

The commutativity of the diagram (3) means that

$$w = \frac{a'}{a} \quad \text{and} \quad w = \frac{b}{b'} .$$

We hence find that the extensions $\xi_{a,b}$ and $\xi_{a',b'}$ are Yoneda equivalent if and only if $a'/a = b/b'$.

It follows that the Yoneda equivalence classes of short exact sequences of the form (1) have as a set of representatives the sequences

$$0 \to (\Bbbk, [0]) \xrightarrow{[1\ 0]} \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \xrightarrow{[0\ b]} (\Bbbk, [0]) \to 0$$

with $b \neq 0$.

We find that overall we have $\#\Bbbk = \#\mathbb{F}_q = q$ many Yoneda equivalence classes of short exact sequences. Thus

$$\#\mathrm{Ext}^1(S, S) = q ,$$

from which it follows that $\mathrm{Ext}^1(S, S) \cong \mathbb{F}_q$.

A.5. Proof of Corollary 2.3

Let $K := \mathrm{K}_0(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q))$. We can regard the Euler form of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ as a bilinear form

$$\langle -, - \rangle \colon K \times K \to \mathbb{Q}^\times .$$

Since S is a generator of K is sufficies to show that $\langle S, S \rangle = 1$. This holds true because

$$\langle S, S \rangle = \Big(\#\mathrm{Hom}(S, S)\Big) \cdot \Big(\#\,\mathrm{Ext}^1(S, S)\Big)^{-1} = q \cdot q^{-1} = 1 .$$

This proves the assertion.

A.6. Counting $\mathrm{Gr}(d, n, \mathbb{F}_q)$

Recall A.10. For $k \in \mathbb{N}$ the *quantum integer* $[k]_q$ is given by

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1} .$$

We have $[0]_q = 0$ and $[1]_q = 1$. The *quantum factorial* is given by

$$[k]_q! = [k]_q [k-1]_q \cdots [1]_q .$$

For $k, l \in \mathbb{N}$ the *quantum binomial* is given by

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q \cdots [k-l+1]_q}{[l]_q!} .$$

If $l > k$ then this is zero, and if $l \leq k$ then the quantum binomial can also be expressed as

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q! \, [k-l]_q!} .$$

The quantum binomial satsfies the recursive relation

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = q^l \begin{bmatrix} k-1 \\ l \end{bmatrix}_q + \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q$$

for all $k, l \in \mathbb{N}$. It hence follows by induction that the quantum binomial $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ is a polynomial in q with natural coefficients, i.e.

$$\begin{bmatrix} k \\ l \end{bmatrix}_q \in \mathbb{N}[q] .$$

By taking the limit $q \to 1$ (i.e. by setting q equal to 1) the quantum integer $[k]$ becomes the usual integer k, the quantum factorial $[k]_q!$ becomes the usual factorial $k!$ and the quantum binomial coefficient $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ becomes the usual binomial $\binom{k}{l}$.

Lemma A.11. For all dimensions $n, d \geq 0$ we have

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \begin{bmatrix} n \\ d \end{bmatrix}_q .$$

Proof. If $d > n$ then both numbers are zero, so suppose that $d \leq n$. Let

$$F_d(n) := (q^n - 1) \cdots (q^n - q^{d-1}) = (q-1)^d \, q^{d(d-1)/2} \, [n]_q \cdots [n-d+1]_q$$

This is the number of linear independent tupels $(v_1, \ldots, v_d)$ of vectors in $\mathbb{F}_q^n$. We find that

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{\#\mathrm{GL}(d, \mathbb{F}_q)} .$$

We have $\#\mathrm{GL}(d, \mathbb{F}_q) = F_d(d)$ and thus

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{F_d(d)} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q \cdots [1]_q} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q!} = \begin{bmatrix} n \\ d \end{bmatrix}_q ,$$

as claimed. □

A.7. More on Symmetric Functions

For the first claim consider the symmetric polyonmials

$$f^{(k)}(x_1, \dots, x_k) := x_1 + x_1x_2 + \cdots + x_1 \cdots x_k .$$

The polynomials satisfy the compatibility condition

$$f^{(k+1)}(x_1, \dots, x_k, 0) = f^{(k)}(x_1, \dots, x_k)$$

for every $k \geq 0$. But there exists no symmetric function $f \in \Lambda$ with $f(x_1, \dots, x_k) = f^{(k)}(x_1, \dots, x_k)$ for every $k \geq 0$ since otherwise

$$k = \deg(f^{(k)}) \leq \deg(f)$$

for every $k \geq 0$, which is not possible. This shows that Λ together with the homomorphisms $\Lambda \to \Lambda^{(k)}$ is not the limit of the homomorphisms $\Lambda^{(k+1)} \to \Lambda^{(k)}$ for $k \geq 0$ in the category of rings.

Suppose that more generally $(f^{(k)})_{k\geq 0}$ is any sequence of symmetric polynomials $f^{(k)} \in \Lambda^{(k)}$ that are compatible in the sense that $f(x_1, \dots, x_k) = f^{(k)}(x_1, \dots, x_k)$ for every $k \geq 0$. Then the $f^{(k)}$ define a symmetric function $f \in \Lambda$ with $f^{(k)}(x_1, \dots, x_k) = f(x_1, \dots, x_k)$ for every $k \geq 0$ if and only if the degrees $\deg(f^{(k)})$ are bounded, i.e. if and only if there exists some $N \geq 0$ with $\deg(f^{(k)}) \leq N$ for every $k \geq 0$.

Indeed, if such a symmetric function f exists then $\deg(f^{(k)}) \leq \deg(f)$ for every $k \geq 0$. If on the other hand such a bound N exists then we consider for every $n = 0, \dots, N$ the sequence $(f_n^{(k)})_{k\geq 0}$ of degree n parts. That the symmetric polynomials $f^{(k)}$ are compatible means that for every degree $n = 0, \dots, N$ the homogeneous symmetric polynomials $f_n^{(k)}$ are compatible. Thus there exists for every degree $n = 0, \dots, N$ a homogeneous symmetric function $f_n \in \Lambda_n$ with

$$f_n(x_1, \dots, x_k) = f_n^{(k)}(x_1, \dots, x_k)$$

for every $k \geq 0$. It then follows for the symmetric function

$$f := f_0 + f_1 + \cdots + f_N$$

in each degree $m = 0, \dots, N$ that

$$f(x_1, \dots, x_k)_m = f_0(x_1, \dots, x_k)_n + \cdots + f_n(x_1, \dots, x_k)_m = f_m(x_1, \dots, x_k) = f_m^{(k)}(x_1, \dots, x_k)$$

for every $k \geq 0$ since each $f_l(x_1, \dots, x_k)$ is homogeneous of degree m. This shows that

$$f(x_1, \dots, x_k) = f^{(k)}(x_1, \dots, x_k)$$

for every $k \geq 0$.

To see the second claim we note that both $\mathbb{C}[x_1, x_2, \dots]^{\mathrm{S}_{\mathbb{N}}}$ and $\mathbb{C}[x_1, \dots, x_n]^{\mathrm{S}_\infty}$ are just $\mathbb{C}$. Indeed, if a symmetric polynomial $f \in \mathbb{C}[x_1, x_2, \dots]$ were to contain a nontrivial monomial then it must also contain all permutations of this monomial. But there are infinitely many such permutations, while f contains only finitely many polynomials.

A.8. Regarding Λ as a Colimit

We have for every number of variables $k \geq 0$ an injective homomorphism of graded algebras $\Lambda^{(k)} \to \Lambda^{(k+1)}$ that is given on algebra generators by $e_n^{(k)} \to e_n^{(k+1)}$ for every $n = 0, \dots, k$. This is a right sided inverse for the homomorphism $\Lambda^{(k+1)} \to \Lambda^{(k)}$. In this way a symmetric polynomial in k variables can be extended to a symmetric polynomial in $k+1$ variables.

We similarly have for every number of variables $k \geq 0$ a homomorphism of graded algebras $\Lambda^{(k)} \to \Lambda$ that is given on algebra generators by $e_n^{(k)} \to e_n$ for every $n = 0, \dots, k$.

We now find that Λ together with the homomorphisms $\Lambda^{(k)} \to \Lambda$ is a colimit of the homomorphisms $\Lambda^{(k)} \to \Lambda^{(k+1)}$ for $k \geq 0$. In this way we can regard Λ as a sort of increasing union of the algebras of symmetric polynomials.

A.9. Invariants of a Tensor Product

Lemma A.12. Let G, H be two groups. Let V be a representation of G and let W be a representation of H. Then

$$(V \otimes W)^{G \times H} = V^G \otimes W^H .$$

Proof. The inclusion $V^G \otimes W^H \subseteq (V \otimes W)^{G \times H)}$ can be checked on simple tensors. Let on the other hand $x \in (V \otimes W)^{G \times H}$. We may choose a basis $(v_i)_{i \in I}$ of V and write $x = \sum_{i \in I} v_i \otimes w_i$ for some unique vectors $w_i \in W$. For every element $h \in H$ we then have

$$\sum_{i \in I} v_i \otimes w_i = x = (1, h)x = \sum_{i \in I} v_i \otimes (hw_i) .$$

It follows from the uniqueness of the w_i that $hw_i = w_i$ for every $h \in H$ and every $i \in I$, and thus $w_i \in W^H$ for every $i \in I$. This shows that $x \in V \otimes W^H$. We find in the same way that $x \in V^G \otimes W^H$, and thus $x \in (V \otimes W^H) \cap (V^G \otimes W) = V^G \otimes W^H$. □

References

[Oor63] Frans Oort. "Yoneda Extensions in Abelian Categories". In: *Mathematische Annalen* 153.3 (February 11, 1963), pp. 227–235. DOI: `BF01360318`.

[Sch09] Olivier Schiffmann. *Lectures on Hall Algebras.* October 23, 2009. arXiv: `0611617v2 [math.RT]`.

[Szc11] Matthew Szczesny. *Representations of Quivers over $\mathbb{F}_1$ and Hall Algebras.* July 25, 2011. arXiv: `1006.0912v3 [math.QA]`.