Jordan Quiver, Part I

Talk 10 on Hall Algebras and Quantum Groups

1. The Jordan Quiver and its Nilpotent Representations

Definition 1.1. The *Jordan quiver* is the quiver that consists of a single vertex and a single edge, which is necessarily a loop.

Throughout this talk the Jordan quiver is denoted by Q. See Figure 1 for a visualization. In the following we write $\Bbbk$ to mean a field or $\mathbb{F}_1$.

A representation of the Jordan quiver over $\Bbbk$ is the same as a pair (V, f) consisting of a $\Bbbk$-vector space V together with an endomorphism f of V. Two such representations (V, f) and (W, g) are isomorphic if and only if V, W are isomorphic as $\Bbbk$-vector spaces and the endomorphisms f, g are similar. We hence find that that the isomorphism classes of Q-representations over $\Bbbk$ are in one-to-one correspondence to conjugacy classes of endomorphisms of $\Bbbk$-vector spaces.

Suppose that V is finite-dimensional, and that $\Bbbk$ is algebraically closed, or that it is $\mathbb{F}_1$, or that we are interested only in nilpotent endomorphisms. Then one can use the usual Jordan normal form to classify these conjugacy classes.

A representation (V, f) of the Jordan quiver is nilpotent if and only if the endomorphism f is nilpotent. We will in the rest of this talk restrict our attention to finite-dimensional, nilpotent representations of the Jordan quiver. As introduced in the previous talks we denote by

$$\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$$

the full subcategory of $\mathbf{Rep}(Q, \Bbbk)$ whose objects are the finite-dimensional, nilpotent representations of Q over $\Bbbk$. We denote the set of isomorphism classes of $\mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)$ by

$$\mathrm{Iso}(Q, \Bbbk) := \mathbf{rep}^{\mathrm{nil}}(Q, \Bbbk)/\cong .$$

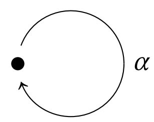

Figure 1: The Jordan quiver Q.

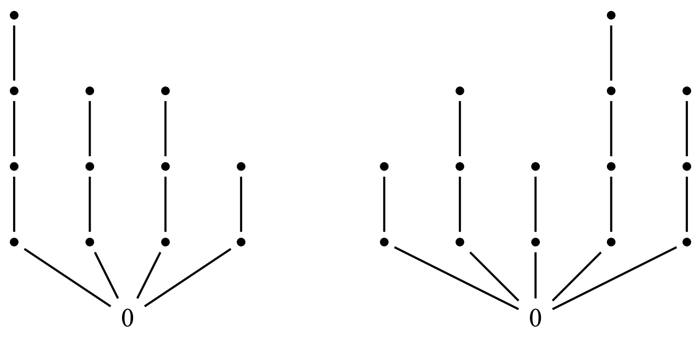

Figure 2: The representations $\mathrm{N}_{(4,3,3,2)}$ and $\mathrm{N}_{(2,3,2,4,3)}$ over $\mathbb{F}_1$.

Definition 1.2. For every dimension $d \geq 0$ let

$$\mathrm{N}_d := \left(\Bbbk^d, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \right)$$

if $\Bbbk$ is a field, and let

$$\mathrm{N}_d := (\{0, 1, \dots, d\}, [d \mapsto (d-1) \mapsto (d-2) \mapsto \cdots \mapsto 1 \mapsto 0 \mapsto 0])$$

if $\Bbbk = \mathbb{F}_1$. For every tupel $(d_1, \dots d_n)$ of dimensions $d_i \geq 0$ let

$$\mathrm{N}_{(d_1,\dots,d_n)} := \mathrm{N}_{d_1} \oplus \cdots \oplus \mathrm{N}_{d_n} .$$

Example 1.3. See Figure 2 for visualizations of $\mathrm{N}_{(4,3,3,2)}$ and $\mathrm{N}_{(2,3,2,4,3)}$ over $\mathbb{F}_1$.

Definition 1.4. For every $n \geq 0$ let $\mathrm{Par}(n)$ be the set of partition of the number n, i.e.

$$\mathrm{Par}(n) := \left\{ (\lambda_1, \dots, \lambda_l) \,\middle|\, \begin{array}{l} \lambda_1, \dots, \lambda_l \in \mathbb{N} \\ \lambda_1 \geq \cdots \geq \lambda_l \geq 1 \\ \lambda_1 + \cdots + \lambda_l = n \end{array} \right\} .^{1}$$

The set of all partitions is denoted by

$$\mathrm{Par} := \coprod_{n \geq 0} \mathrm{Par}(n) .$$

Proposition 1.5. The representations N_λ with $\lambda \in \mathrm{Par}$ form a set of representatives for $\mathrm{Iso}(Q, \Bbbk)$.

Proof. This follows from the existence and uniqueness of the Jordan normal form of nilpotent endomorphisms. (See Appendix A.1 for the Jordan normal form over $\mathbb{F}_1$.) ☐

[1] We want to point out that in this talk we do not allow a partition to contain zero as an entry. This is done purely for technical reasons.

2. The Hall Algebra of the Jordan Quiver over $\mathbb{F}_q$

We will now consider for $\Bbbk$ the finite field $\mathbb{F}_q$. Then $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is an abelian, finitary, full, exact subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q)$. We want to consider in the following the Hall algebra of $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$. For this we need to understand its Euler form.

Proposition 2.1. Let $\mathscr{A} := \mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ and let $S := \mathrm{N}_1 = (\Bbbk, [0])$.

1. The category $\mathscr{A}$ is hereditary, i.e. $\mathrm{Ext}^n_{\mathscr{A}} = 0$ for every $n \geq 2$.
2. The Grothendieck group $\mathrm{K}(\mathscr{A})$ is freely generated by the class $[S]$. Thus $\mathrm{K}(\mathscr{A}) \cong \mathbb{Z}$ via the map $[M] \mapsto \dim(M)$.
3. The Euler form of $\mathscr{A}$ vanishes.

Proof. See Appendix A.2 and Appendix A.3. □

Since $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is abelian and finitary with vanishing Euler form we can consider its Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$. We find that Green's coproduct makes $\mathbf{H}(Q, \mathbb{F}_q)$ into a graded bialgebra because $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ satisfies the finite subject condition and has vanishing Euler form.[2] It further follows that $\mathbf{H}(Q, \mathbb{F}_q)$ is a graded Hopf algebra since it is connected. (See Appendix A.5 for a more detailed overview.)

The Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ has the classes $[\mathrm{N}_\lambda]$ with $\lambda \in \mathrm{Par}$ as a basis. The multiplication of $\mathbf{H}(Q, \mathbb{F}_q)$ is given by

$$[\mathrm{N}_\lambda] \cdot [\mathrm{N}_\mu] = \sum_{\kappa \in \mathrm{Par}} C^\kappa_{\lambda,\mu} [\mathrm{N}_\kappa]$$

where the structure constants $C^\kappa_{\lambda,\mu}$ are given by

$$C^\kappa_{\lambda,\mu} = \#\{\text{subrepresentation } L \text{ of } \mathrm{N}_\kappa \mid L \cong \mathrm{N}_\mu, \mathrm{N}_\kappa / L \cong \mathrm{N}_\lambda\}.$$

We will now compute some of the structure constants, following [Sch09, Example 2.2].

Example 2.2 (Structure constants)**.**

1. Let $\lambda = (1^n)$ and $\mu = (1^m)$. We consider the partition $\kappa := (1^{n+m})$. The action of the edge of the Jordan quiver Q on the representations N_λ, N_μ and N_κ is trivial. We thus find that every m-dimensional linear subspace L of N_κ satisfies the conditions $L \cong \mathrm{N}_\mu$ and $\mathrm{N}_\kappa / L \cong \mathrm{N}_\lambda$. The structure constant $C^\kappa_{\lambda,\mu}$ is therefore given by

$$\begin{aligned} C^\kappa_{\lambda,\mu} &= \text{number of } m\text{-dimensional linear subspaces of } \mathrm{N}_\kappa \\ &= \#\mathrm{Gr}(m, n+m, \mathbb{F}_q) \\ &= \begin{bmatrix} n+m \\ m \end{bmatrix}_q . \end{aligned}$$

   (See Lemma A.12 for the last equality.) We see in particular that $C^\kappa_{\lambda,\mu}$ depends is a polynomial way on q (with natural coefficients). See Appendix A.7 for some explicit calculations of binomial coefficients.

[2]We say that a category $\mathscr{A}$ satisfies the *finite subobject condition* if every object of $\mathscr{A}$ admits only finitely many subobjects.

2. Let now $\lambda = (n)$ and $\mu = (m)$. We consider the partition $\kappa = (n+m)$. The representation N_κ has the standard basis $e_1, \dots, e_{n+m}$, and the subrepresentations of N_κ are given by $\langle e_1, \dots, e_i \rangle$ for $i = 0, \dots, n+m$. The subrepresentations $L \coloneqq \langle e_1, \dots, e_m \rangle$ is the unique one that is isomorphic to N_μ, and its quotient N_κ / L is isomorphic to N_λ. Thus

$$C_{(n),(m)}^{(n+m)} = 1 \,.$$

3. One finds for all $n \geq 2$ and $m \geq 1$ that

$$C_{(n),(1^m)}^{(n,1^m)} = q^m = C_{(1^m),(n)}^{(n,1^m)} \,,$$

   see Appendix A.8.

We observe that in the above examples the coefficient $C_{\lambda,\mu}^{\kappa}$ are always polynomials in q with integer coefficients. We will see in next week's talk that this is true for any coefficient $C_{\lambda,\mu}^{\kappa}$. This will allow us to define the *generic Hall algebra* of the Jordan quiver.

We also have $C_{\lambda,\mu}^{\kappa} = C_{\mu,\lambda}^{\kappa}$ in each example. We will see in next week's talk that the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ is indeed commutative, which means precisely that $C_{\lambda,\mu}^{\kappa} = C_{\mu,\lambda}^{\kappa}$ for any three partitions $\lambda, \mu, \kappa \in \mathrm{Par}$.

3. The Hall Algebra of the Jordan Quiver over $\mathbb{F}_1$

We will now consider the case that $\Bbbk$ is $\mathbb{F}_1$. We have seen in last week's talk how to construct the Hall algebra of Q over $\mathbb{F}_1$, and that it is a cocommutative, graded Hopf algebra. (See Appendix A.9 for a more detailed overview.)

We have also seen that $\mathbf{H}(Q, \mathbb{F}_1)$ is the universal enveloping algebra of its Lie algebra of its primitve elements, which is spanned (as a vector space) by those isomorphism classes $[M]$ for which M is indecomposable. We have the following consequence:

Corollary 3.1. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is generated by $[\mathrm{N}_i]$ for $i \geq 1$.

A more explicit proof of Corollary 3.1 can be found in Remark A.14.

Example 3.2 (Structure constants)**.** We can again compute some structure constants $C_{\lambda,\mu}^{\kappa}$.

1. Let again $\lambda = (1^n)$ and $\mu = (1^m)$, and consider $\kappa = (1^{n+m})$. We find as before that

$$C_{\lambda,\mu}^{\kappa} = \text{number of } m\text{-dimensional subspaces of } \mathrm{N}_{n+m} = \binom{n+m}{m}.$$

2. Let again $\lambda = (n)$ and $\mu = (m)$ and consider $\kappa = (n+m)$. We find as before that

$$C_{\lambda,\mu}^{\kappa} = 1 \,.$$

3. Let us compute the product $[\mathrm{N}_i] \cdot [\mathrm{N}_j]$.

   We observe that if $[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)$ and L is a subrepresentation of R that is isomorphic to N_j then the quotient R/L results from R by contracting one of the Jordan chains of R by j elements. If $R/L \cong \mathrm{N}_i$ then this means that R consists of a single Jordan chain of length $i + j$, or of two Jordan chains of length i and j respectively. Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i,j)}] + b[\mathrm{N}_{i+j}]\,.$$

   We have seen above that $b = C^{(i+j)}_{(i),(j)} = 1$. The coeffient $a = C^{(i,j)}_{(i),(j)}$ is given by

$$\begin{aligned} a &= \text{how often } j \text{ occurs in } (i,j) \\ &= \begin{cases} 1 & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases} \end{aligned}$$

   Thus

$$[\mathrm{N}_i] \cdot [\mathrm{N}_j] = \begin{cases} [\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i \neq j, \\ 2[\mathrm{N}_{(i,j)}] + [\mathrm{N}_{i+j}] & \text{if } i = j. \end{cases}$$

   We see in particular that $[\mathrm{N}_i]$ and $[\mathrm{N}_j]$ commute.

Corollary 3.3. The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is commutative.

Proof. Its generators $[\mathrm{N}_i]$ with $i \geq 1$ commute. □

We see now that $\mathbf{H}(Q, \mathbb{F}_1)$ is a commutative, cocommutative, graded Hopf algebra. It has a basis indexed by partitions, and its graded parts have the dimension

$$\dim \mathbf{H}(Q, \mathbb{F}_1)_k = \#\{(\lambda_1, \dots, \lambda_l) \in \mathrm{Par} \mid \lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = k\}$$

for every $k \geq 0$.

In the land of combinatorics there is another algebra with these properties, namely the *ring of symmetric functions.*

4. The Ring of Symmetric Functions

4.1. Definition

For every $n \geq 0$ we denote by

$$\Lambda^{(n)} := \mathbb{C}[x_1, \dots, x_n]^{\mathrm{S}_n}$$

the algebra of symmetric polynomials in n variables. We have for every number of variables $n \geq 0$ a homomorphism of graded algebras

$$\Lambda^{(n+1)} \to \Lambda^{(n)}\,, \quad f^{(n+1)} \mapsto f^{(n+1)}(x_1, \dots, x_n, 0)\,.$$

Definition 4.1. The *ring of symmetric functions* Λ is the limit

$$\Lambda := \lim_{n \geq 0} \Big(\Lambda^{(n+1)} \to \Lambda^{(n)} \Big)$$

in the category of graded algebras. The elements of Λ are *symmetric functions.*

Warning 4.2. A symmetric function is – contrary to its name – not a function.

Let us make the above definition more explicit: For every degree $k \geq 0$ we have

$$\begin{aligned}\Lambda_k &= \lim_{n \geq 0} \Bigl(\Lambda_k^{(n+1)} \to \Lambda_k^{(n)} \Bigr) \\ &= \left\{ \bigl(f^{(n)} \bigr)_{n \geq 0} \;\middle|\; \begin{array}{l} f^{(n)} \in \Lambda_k^{(n)} \text{ for every } n \geq 0 \text{ such that} \\ f^{(n+1)}(x_1, \dots, x_n, 0) = f^{(n)} \text{ for every } n \geq 0 \end{array} \right\},\end{aligned}$$

and we have overall

$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$

as vector spaces. A homogeneous symmetric function f, say of degree k, is thus the same as a "consistent choice" of homogeneous symmetric polynomials $f^{(n)}$ of degree k for every number of variables $n \geq 0$.

We have for every number of variables $n \geq 0$ a homomorphism of graded algebras

$$\Lambda \to \Lambda^{(n)}, \quad f \mapsto f(x_1, \dots, x_n)$$

that is given in each degree by projection onto the n-th component.[3]

Example 4.3. We have for every number of variables $n \geq 0$ and every degree $k \geq 0$ the *elementary symmetric polynomial*

$$e_k^{(n)}(x_1, \dots, x_n) \coloneqq \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \in \Lambda_k^{(n)},$$

with $e_k^{(n)} = 0$ whenever $k > n$. These polynomials satisfy the compatibility condition

$$e_k^{(n+1)}(x_1, \dots, x_n, 0) = e_k^{(n)}$$

for all $n \geq 0$. These elementary symmetric polynomials $e_k^{(n)}$ with $n \geq 0$ therefore assemble into a single homogeneous symmetric function

$$e_k \in \Lambda_k .$$

This is the *k-th elementary symmetric function.*

We find similarly that the *power symmetric polynomials*

$$p_k^{(n)}(x_1, \dots, x_n) \coloneqq x_1^k + \cdots + x_n^k ,$$

and the *complete homogenous symmetric polynomials*

$$h_k^{(n)}(x_1, \dots, x_n) \coloneqq \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k} = \sum \text{monomials of homogeneous degree } k$$

[3]For any two symmetric functions f, g we have by construction of Λ that

$$f = g \iff f(x_1, \dots, x_n) = g(x_1, \dots, x_n) \text{ for every } n \geq 0.$$

result in homogeneous symmetric functions

$$p_k, h_k \in \Lambda_k .$$

These are the *power symmetric functions* and *complete homogeneous symmetric functions.*

See Appendix A.11 for some remarks about the definition of Λ.

4.2. The Fundamental Theorem on Symmetric Functions

The *fundamental theorem of symmetric polynomials* asserts that for every number of variables $n \geq 0$ the elementary symmetric polynomials

$$e_1^{(n)}, \dots, e_n^{(n)}$$

form an algebraically independent generating set for the algebra of symmetric polynomials $\Lambda^{(n)}$. It follows from this that both

$$h_1^{(n)}, \dots, h_n^{(n)} \quad \text{and} \quad p_1^{(n)}, \dots, p_n^{(n)}$$

also form algebraically independent algebra generating set for $\Lambda^{(n)}$.[4]

For every partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we can consider the symmetric polynomials

$$e_\lambda^{(n)} \coloneqq e_{\lambda_1}^{(n)} \cdots e_{\lambda_l}^{(n)} , \qquad h_\lambda^{(n)} \coloneqq h_{\lambda_1}^{(n)} \cdots h_{\lambda_l}^{(n)} , \qquad p_\lambda^{(n)} \coloneqq p_{\lambda_1}^{(n)} \cdots p_{\lambda_l}^{(n)} .$$

We have just formulated that the symmetric polynomials

$$e_\lambda^{(n)} \qquad \text{for } \lambda \in \mathrm{Par} \text{ with length } \ell(\lambda) \leq n$$

form a vector space basis for $\Lambda^{(n)}$, and similarly for $h_\lambda^{(n)}$ and $p_\lambda^{(n)}$. We can generalize these families of symmetric polynomials to symmetric functions:

Example 4.4. For every partition $\lambda \in \mathrm{Par}$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ we consider the symmetric functions

$$e_\lambda \coloneqq e_{\lambda_1} \cdots e_{\lambda_l} , \quad h_\lambda \coloneqq h_{\lambda_1} \cdots h_{\lambda_l} , \quad p_\lambda \coloneqq p_{\lambda_1} \cdots p_{\lambda_l}$$

and note that

$$\begin{aligned} e_\lambda(x_1, \dots, x_n) &= e_\lambda^{(n)} , \\ h_\lambda(x_1, \dots, x_n) &= h_\lambda^{(n)} , \\ p_\lambda(x_1, \dots, x_n) &= p_\lambda^{(n)} . \end{aligned}$$

We now want to generalize the fundamental theorem on symmetric polynomials to symmetric functions. The key observation behind this is the following:

[4] For the elementary symmetric polynomials $e_k^{(n)}$ and homogeneous symmetric polynomials $h_k^{(n)}$ these statements do not only hold over the ground field $\mathbb{C}$, but over every commutative ring. For the power symmetric polynomials $p_k^{(n)}$ we need to work over a field in which the numbers $1, \dots, n$ are invertible.

Proposition 4.5. The map $\Lambda_k^{(n+1)} \to \Lambda_k^{(n)}$ is an isomorphism whenever $n \geq k$.

Proof. A vector space basis of $\Lambda_k^{(n)}$ is given by the symmetric polynomials $e_\lambda^{(n)}$ where the partition λ is of length $\ell(\lambda) \leq n$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies

$$\lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = k\,.$$

A vector space basis of $\Lambda_k^{(n)}$ is given by the symmetric polynomials $e_\mu^{(n+1)}$ where the partition μ is of length $\ell(\mu) \leq n+1$ and $\mu = (\mu_1, \dots, \mu_l)$ satisfies

$$\mu_1 + 2\mu_2 + \cdots + l\mu_l = k\,.$$

We find by degree reasons that the case $\ell(\mu) = n+1$ cannot occur. The linear map $\Lambda_k^{(n+1)} \to \Lambda_k^{(n)}$ does therefore restrict to a bijection between the above bases. □

Corollary 4.6. The map $\Lambda_k \to \Lambda_k^{(n)}$ is an isomorphism whenever $n \geq k$. □

Corollary 4.7. The following families of symmetric functions form vector space bases of Λ:

1. The elementary symmetric polynomials e_λ with $\lambda \in \mathrm{Par}$.
2. The complete homogeneous symmetric polynomials h_λ with $\lambda \in \mathrm{Par}$.
3. The power symmetric polynomials p_λ with $\lambda \in \mathrm{Par}$. □

Corollary 4.8. The elementary symmetric functions e_k with $k \geq 1$ form an algebraically independent algebra generating set for Λ, and similarly the h_k and the p_k. □

Corollary 4.9. We have $\Lambda \cong \mathbb{C}[X_1, X_2, X_3, \dots]$ as graded algebras, where each variable X_k is homogeneous of degree k. □

See Appendix A.12 for more consequences.

4.3. Hopf Algebra Structure

We can endow the algebra of symmetric functions Λ with the structure of a graded Hopf algebra. We have for any two number of variables $n, m \geq 0$ a homomorphism of graded algebras

$$\begin{aligned}
\Delta_{nm} \colon \Lambda^{(n+m)} &= \mathbb{C}[x_1, \dots, x_{n+m}]^{S_{n+m}} \\
&\subseteq \mathbb{C}[x_1, \dots, x_{n+m}]^{S_n \times S_m} \\
&\cong (\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_{n+1}, \dots, x_{n+m}])^{S_n \times S_m} \\
&\cong (\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_1, \dots, x_m])^{S_n \times S_m} \\
&= \mathbb{C}[x_1, \dots, x_n]^{S_n} \otimes \mathbb{C}[x_1, \dots, x_m]^{S_m} && \text{(Lemma A.17)} \\
&= \Lambda^{(n)} \otimes \Lambda^{(m)}.
\end{aligned}$$

We would like to have a homomorphism of graded algebras $\Delta\colon \Lambda \to \Lambda \otimes \Lambda$ such that for any number of variables $n, m \geq 0$ the square diagram

$$\begin{tikzcd} \Lambda \arrow[r, dashed, "\Delta"] \arrow[d] & \Lambda \otimes \Lambda \arrow[d] \\ \Lambda^{(n+m)} \arrow[r, "\Delta_{nm}"] & \Lambda^{(n)} \otimes \Lambda^{(m)} \end{tikzcd}$$

commutes. The composition $\Lambda \to \Lambda^{(n)} \otimes \Lambda^{(m)}$ is given on the algebra generators p_k of Λ by

$$p_k \mapsto p_k^{(n)} \otimes 1 + 1 \otimes p_k^{(m)} .$$

Such an algebra homomorphism Δ is thus given by

$$\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k .$$

This homomorphism exists because Λ is the free commutative algebra on the generators p_k.

The homomorphism Δ makes the algebra Λ into a cocommutative, graded bialgebra. Since Λ is graded and connected it follows that it is already a graded Hopf algebra.

We see altogether that Λ is a commutative, cocommutative, graded Hopf algebra. It has a basis indexed by partitions, and its graded parts have the dimension

$$\dim \Lambda_k = \#\{(\lambda_1, \dots, \lambda_l) \in \mathrm{Par} \mid \lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = k\} .$$

5. The Isomorphism $\mathbf{H}(Q, \mathbb{F}_1) \cong \Lambda$

Both $\mathbf{H}(Q, \mathbb{F}_1)$ and Λ are commutative, cocommutate, graded Hopf algebras. They are isomorphic as graded Hopf Algebras:

The ring of symmetric functions Λ is, as a commutative algebra, freely generated by the power symmetric functions $p_1, p_2, p_3, \dots$ There hence exists a unique, surjective algebra homomorphism $\Phi\colon \Lambda \to \mathbf{H}(Q, \mathbb{F}_1)$ with

$$\Phi(p_k) = [\mathrm{N}_k]$$

for every degree $k \geq 1$. We note that Φ is a homomorphism of graded algebras because both p_i and $[\mathrm{N}_k]$ are of degree k. We also have for every degree $k \geq 0$ that $\dim \Lambda_k = \dim \mathbf{H}(Q, \mathbb{F}_1)_k$, with this dimension being finite. It thus follows from the surjectivity of Φ that it is already an isomorphism of graded algebras.

The algebra isomorphism Φ is already an isomorphism of Hopf algebras: It sufficies to check that Φ is compatible with the comultiplication of the algebra generators p_k. This holds since p_k is primitive in Λ and $[\mathrm{N}_k]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$.

We have shown altogether that Φ is an isomorphism of graded Hopf algebras.

A. Remarks and Proofs

A.1. Theorem of Krull–Remak–Schmidt and Jordan Normal Form

Let V be an $\mathbb{F}_1$-vector space and let $f\colon V \to V$ be an endomorphism.

Recall A.1. 1. A *subspace* of V is a subset of V that contains the base point 0.

2. If $(U_i)_{i\in I}$ is a collection of subspaces of V then $V = \bigoplus_{i\in I} U_i$ if and only if every nonzero evement of V is contained in precisely one U_i, i.e. if and only if the U_i give a disjoint decomposition of the set $V \setminus \{0\}$.

3. If U is a subspace of V and $V = \bigoplus_{i\in I} W_i$ is a direct sum decomposition then $U = \bigoplus_{i\in I}(U \cap W_i)$.

Definition A.2. A subspace U of V is *f-invariant* if $f(U) \subseteq U$. An f-invariant subspace U of V is *indecomposable* if it is nonzero and there exist no two nonzero f-invariant subspaces W_1, W_2 of U with $U = W_1 \oplus W_2$.

Remark A.3. If U is an indecomposable subspace of V and $U = \bigoplus_{i\in I} W_i$ is any decomposition into f-invariant subspaces W_i then it follows that $U = W_j$ for some $j \in I$ while $W_i = 0$ for every $i \neq j$. Indeed, some W_j must be nonzero because V is nonzero. Then $V = W_j \oplus \bigoplus_{i\in I, i\neq j} W_i$ and thus $\bigoplus_{i\in I, i\neq j} W_i = 0$, and therefore $W_i = 0$ for every $i \in I$.

Proposition A.4 (Krull–Remak–Schmidt)**.** There exists a unique direct sum decomposition of V into indecomposable f-invariant subspaces.

Proof. In the following we mean by a *decomposition* a direct sum decomposition into f-invariant subspaces in which each direct summand is nonzero. We say that a decomposition $V = \bigoplus_{i\in I} U_i$ is *finer* than a decomposition $V = \bigoplus_{j\in J} W_j$ if each U_i is contained in some W_j. This gives a partial order on the set of decompositions of V.

We note that a decomposition $V = \bigoplus_{i\in I} U_i$ consists of indecomposables if and only if it is maximal fine. Indeed, if some U_j is decomposable then there exists a decomposition $U_j = U_j' \oplus U_j''$. Then

$$V = \bigoplus_{i\in I} U_i = \bigoplus_{\substack{i\in I\\ i\neq j}} U_i \oplus U_j = \bigoplus_{\substack{i\in I\\ i\neq j}} U_i \oplus U_j' \oplus U_j'$$

with the last term being a strictly finer that the original decomposition $V = \bigoplus_{i\in I} U_i$. Suppose on the other hand that each U_i is indecomposable and that $V = \bigoplus_{j\in J} W_j$ is a decomposition that is finer than $V = \bigoplus_{i\in I} U_i$. Then $U_i = \bigoplus_{j\in J}(U_i \cap V_j)$ for every $j \in J$. It follows that $U_i = U_i \cap V_j$ for some $j \in J$ and thus $U_i \subseteq V_j$. We also know that V_j is contained in some U_k. Then U_i is contained in U_k whence it follows that $i = k$ and thus $U_i = V_j$. This shows that each U_i equals some V_j, from which it follows that both decompositions must coincide.

We hence need to show that there exists a unique decomposition which is maximal fine. It sufficies to show that any collection of decompositions has a common refinement. Taking a common refinement of all decompositions then gives the desired one.

Let $V = \bigoplus_{j\in J_i} U_j^i$ with $i \in I$ be a collection of decompositions. For every nonzero vector $v \in V$ there exists for every $i \in I$ a unique index $j(i,v) \in J_i$ with $v \in U^i_{j(i,v)}$. We consider

$$W_v := \bigcap_{i\in I} U^i_{j(i,v)}\,.$$

Each W_v is an intersection of f-invariant subspaces and therefore again an f-invariant subspace. Each nonzero vector v of V is contained in some $W_{v'}$, namely for $v' = v$.

Suppose that for two nonzero vectors $v, u \in V$ the subspaces W_v and W_u intersect nonzero. Let w be a nonzero vector contained in both W_v and W_u. Then for every index $i \in I$ the vector w is contained in $U^i_{j(i,v)}$, whence $j(i,v) = j(i,w)$. It follows that $W_v = W_w$, and we find in the same way that $W_u = W_w$. Thus $W_v = W_w$.

This shows that the f-invariant subspaces W_v give a disjoint decomposition of $V \setminus \{0\}$, and hence a decomposition of V. (Once we remove those subspaces which occur multiple times.) This decomposition of V is by construction finer than each decomposition $V = \bigoplus_{j \in J_i} U^i_j$. □

We want to understand how the decomposition from the Krull–Remak–Schmidt theorem looks like. We note that if $v \in V$ is any nonzero vector then there exists at most one preimage of v under f, since f is injective outside of its kernel. Thus we can consider for every nonzero vector $v \in V$ the well-defined two-sided sequence

$$\dots, f^{-2}(v), f^{-1}(v), v, f(v), f^2(v), \dots$$

Here the left part of the sequence consists of as many iterated preimages as exist. The set of all these elements is the *orbit* of v under f. It is denoted by $[v]$.

We note that for any nonzero veector u in $[v]$ we have $[u] = [v]$. If two orbits $[v]$ and $[w]$ intersect in a nonzero vector u then it follows that $[v] = [u] = [v]$. Two distinct orbit do therefore intersect at most in 0. It follows that the orbits give induce a disjoint decomposition of $V \setminus \{0\}$. The vector space V does therefore decompose into the direct sum of the subspaces $[v] \cup \{0\}$. (Once we remove those subspaces which occur multiple times.) Each subspace $[v] \cup \{0\}$ is f-invariant. Any two nonzero f-invariant subspaces of $[v] \cup \{0\}$ intersect nonzero whence the subspaces $[v] \cup \{0\}$ are indecomposable.

This shows that the decomposition of V from the Krull–Remak–Schmidt theorem is given by the orbits with respect to f (together with $\{0\}$).

There exist five kinds of orbits.

Type A. The orbit ends in zero and is finite: It is thus of the form

$$v \to f(v) \to \cdots \to f^n(v) = 0$$

for a unique vector v, that has no preimage under f.

Type B. The orbit ends in zero and is infintite: It is thus of the form

$$\cdots \to f^{-2}(v) \to f^{-1}(v) \to v \to f(v) \to f^2(v) \to \cdots \to f^n(v) = 0\,.$$

Type C. The orbit never reaches zero and has only finitely many preimages. It is thus of the form

$$v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to \cdots$$

for a unique vector v, that has no preimage under f.

Type D. The orbit goes infinite in both directions and is non-circular. It is thus of the form

$$\cdots \to f^{-1}(v) \to v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to \cdots$$

	injective	surjective	bijective	nilpotent	finite-dimensional
Type A				◦	◦
Type B			•	◦	
Type C	•				
Type D	•	•	•		
Type E	•	•	•		◦

Table 1: Possible orbits. Complete characterization via orbits for •. Only locally for ◦.

Type E. The orbit is circular. It is thus of the form

$$v \to f(v) \to f^2(v) \to \cdots \to f^n(v) \to v \to f(v) \to \cdots$$

Depending on the properties of the vector space V and endomorphism f not all kinds of orbits can occur.

- The endomorphism f is injective if and only if no orbits of Type A and Type B occur.
- The endomorphism f is surjective if and only if no orbits of Type A and Type C occur.
- The endomorphism f is bijective if and only if only orbits of Type D and Type E occur.
- The endomorphism f is locally nilpotent if and only if only orbits of Type A and Type B appear.[5]
- The endomorphism f is nilpotent if and only if only orbits of Type A occur, and the lengths of the occuring orbits is bounded.
- If V is finite-dimensional then only orbits of Type A and Type E occur.
- More generally, V is locally finite-dimensional with respect to f if and only if only orbits of Type A and Type E occur.[6]

See Table 1 for an overview.

A.2. Proving that $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is hereditary

Definition A.5. Let $\mathscr{A}$ be an abelian category. A subcategory $\mathscr{B}$ is *closed under extensions* if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ the middle term B is contained in $\mathscr{B}$ provided that both outer terms A, C are contained in $\mathscr{B}$.

Definition A.6. Let $\mathscr{A}$ be an abelian category. A subcategory $\mathscr{B}$ of $\mathscr{A}$ is a *Serre subcategory* if it is abelian, exact, full and closed under extensions.

[5]An endomorphism f is locally nilpotent if there exists for every vector v some power $n \geq 0$ such that $f^n(v) = 0$.

[6]We say that V is locally finite-dimensional if every nonzero vector of V is contained in a finite-dimensional f-invariant subspace.

Example A.7. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ is a Serre subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q) \cong \mathbb{F}_q[x]$-**Mod**. Indeed, it is a full, abelian, exact subcategory of $\mathbf{Rep}(Q, \mathbb{F}_q)$. If in a short exact sequence

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

of Q-representations both A, B are finite-dimensional then the same holds for B. If both A, C are nilpotent then same holds for B: There exists some powers $n, m \geq 0$ with $\alpha^n A = 0$ and $\alpha^m C = 0$. It follows that $\alpha^m B \subseteq \ker(\psi) = \operatorname{im}(\varphi)$ and thus $\alpha^{n+m} B = 0$.

If $\mathscr{A}$ is an abelian category and $\mathscr{B}$ is an abelian, exact subcategory then we have for every $n \geq 0$ and every two objects A, B of $\mathscr{B}$ an induced map

$$\operatorname{Ext}^n_{\mathscr{B}}(A, B) \to \operatorname{Ext}^n_{\mathscr{A}}(A, B) \,.$$

Proposition A.8. Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be an abelian, exact subcategory of $\mathscr{A}$.

1. Suppose that $\mathscr{B}$ is a full subcategory of $\mathscr{A}$. Then the induced map $\operatorname{Ext}^1_{\mathscr{B}}(A, B) \to \operatorname{Ext}^1_{\mathscr{A}}(A, B)$ is injective for any two objects A, B of $\mathscr{B}$. If $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$ then the induced map $\operatorname{Ext}^1_{\mathscr{B}}(A, B) \to \operatorname{Ext}^1_{\mathscr{A}}(A, B)$ is bijective.
2. Suppose that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Suppose furthermore that for some $n \geq 1$ the induced map $\operatorname{Ext}^n_{\mathscr{B}}(A, B) \to \operatorname{Ext}^n_{\mathscr{A}}(A, B)$ is bijective for any two objects A, B of $\mathscr{B}$. Then the induced map $\operatorname{Ext}^{n+1}_{\mathscr{B}}(A, B) \to \operatorname{Ext}^{n+1}_{\mathscr{A}}(A, B)$ is injective for any two objects A, B of $\mathscr{B}$.

Proof.

1. Two short exact sequences in $\mathscr{B}$,

   $$0 \to B \to X \to A \to 0 \qquad \text{and} \qquad 0 \to B \to X' \to A \to 0 \,,$$

   are equivalent in $\mathscr{A}$ if there exists an isomorphism $\varphi \colon X \to X'$ that makes the diagram

   $$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

   commute. We find that φ is already an isomorphism in $\mathscr{B}$ because $\mathscr{B}$ is full in $\mathscr{A}$. Thus both sequences are already equivalent in $\mathscr{B}$. This shows the injectivity of $\operatorname{Ext}^1_{\mathscr{B}}(A, B) \to \operatorname{Ext}^1_{\mathscr{A}}(A, B)$.

   Suppose now that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A}$. Let A, B be two objects in $\mathscr{B}$. Every element ξ of $\operatorname{Ext}^1_{\mathscr{A}}(A, B)$ is represented by a short exact sequence $0 \to B \to X \to A \to 0$ in $\mathscr{A}$. The middle term X is already contained in $\mathscr{B}$ because $\mathscr{B}$ is closed under extensions. Thus ξ lies in $\operatorname{Ext}^1_{\mathscr{B}}(A, B)$.
2. We refer to [Oor63, Proposition 3.3]. □

Corollary A.9. Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be a Serre subcategory of $\mathscr{A}$. If $\mathscr{A}$ is hereditary then so is $\mathscr{B}$.

Proof. Let A, B be two objects of $\mathscr{B}$. We show by induction on $n \geq 1$ that the induced map $\operatorname{Ext}^n_{\mathscr{B}}(A,B) \to \operatorname{Ext}^n_{\mathscr{A}}(A,B)$ is bijective. The assertion follows from this.

We know from Proposition A.8 that the induced map $\operatorname{Ext}^1_{\mathscr{B}}(A,B) \to \operatorname{Ext}^1_{\mathscr{A}}(A,B)$ is bijective. If for some $n \geq 1$ the induced map $\operatorname{Ext}^n_{\mathscr{B}}(A,B) \to \operatorname{Ext}^n_{\mathscr{A}}(A,B)$ is bijective then it follows from Proposition A.8 that the induced map $\operatorname{Ext}^{n+1}_{\mathscr{B}}(A,B) \to \operatorname{Ext}^{n+1}_{\mathscr{A}}(A,B)$ is injective. It is also surjective because $\mathscr{A}$ is hereditary. □

Example A.10. Example A.7 shows that $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ is a Serre subcategory of $\mathbf{Rep}(Q,\mathbb{F}_q)$. The category $\mathbf{Rep}(Q,\mathbb{F}_q) \cong \mathbb{F}_q[x]\text{-}\mathbf{Mod}$ is hereditary because it has enough projectives and submodules of projective $\mathbb{F}_q[x]$-modules are again projective, since $\mathbb{F}[x]$ is a principal ideal domain. It thus follows from Corollary A.9 that $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ is again hereditary.

A.3. Understanding $\mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q))$ and the Euler Form of $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$

The indecomposable objects of $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ are precisely N_i with $i \geq 1$. The representation N_i has (up to isomorphism) precisely the subrepresentations N_j with $j = 0,\dots,i$. Thus N_1 is the unique simple object in $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$.

Every objects in $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ admits a composition series, whose composition factors are necessarily the unique simple objects S. It follows that $[S]$ is a free generator of the Grothendieck group $K \coloneqq \mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q))$.

We have $\operatorname{Hom}(S,S) \cong \mathbb{F}_q$ and $\operatorname{Ext}^1(S,S) \cong \mathbb{F}_q$. The first isomorphism holds because S is one-dimensional, and the second is shown in Appendix A.4.

To show that the Euler form of $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ vanishes we regard it as a bilinear form

$$\langle -,- \rangle \colon K \times K \to \mathbb{Q}^\times .$$

Since S is a generator of K is sufficies to show that $\langle S,S \rangle = 1$. This holds true because

$$\langle S,S \rangle = \Big(\#\operatorname{Hom}(S,S)\Big) \cdot \Big(\#\operatorname{Ext}^1(S,S)\Big)^{-1} = q \cdot q^{-1} = 1 .$$

A.4. Computing $\operatorname{Ext}^1(S,S)$

In $\mathbf{rep}^{\mathrm{nil}}(Q,\mathbb{F}_q)$ we can compute $\operatorname{Ext}^1(S,S)$ for $S = \mathrm{N}_1$ in two ways:

A.4.1. Via Homological Algebra

Let $\Bbbk \coloneqq \mathbb{F}_q$. We find with Proposition A.8 that

$$\operatorname{Ext}^1(S,S) = \operatorname{Ext}^1_{\mathbf{rep}^{\mathrm{nil}}(Q,\Bbbk)}(S,S) \cong \operatorname{Ext}^1_{\mathbf{Rep}(Q,\Bbbk)}(S,S) \cong \operatorname{Ext}^1_{\Bbbk[x]\text{-}\mathbf{Mod}}(\Bbbk,\Bbbk) .$$

We can use for $\Bbbk$ (in the first argument) the projective resolution

$$\cdots \to 0 \to \Bbbk[x] \xrightarrow{x} \Bbbk[x] \to \Bbbk \to 0 .$$

Applying the functor $\mathrm{Hom}_{\Bbbk[x]}(-, \Bbbk)$ gives the chain complex

$$0 \to \mathrm{Hom}_{\Bbbk[x]}(\Bbbk[x], \Bbbk) \xrightarrow{x} \mathrm{Hom}_{\Bbbk[x]}(\Bbbk[x], \Bbbk) \to 0 \to \cdots ,$$

which is isomorphic to the chain complex

$$0 \to \Bbbk \xrightarrow{0} \Bbbk \to 0 \to \cdots$$

We find in particular that

$$\mathrm{Hom}_{\Bbbk[x]}(\Bbbk, \Bbbk) \cong \Bbbk\,, \quad \mathrm{Ext}^1_{\Bbbk[x]}(\Bbbk, \Bbbk) \cong \Bbbk\,.$$

A.4.2. Via Counting

We can also count the Yoneda classes of short exact sequences: We have $\mathrm{N}_1 = (\Bbbk, \begin{bmatrix} 0 \end{bmatrix})$, and a short exact sequence

$$0 \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to \,?\, \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to 0$$

can have as its middle term (up to isomorphism) either

$$\mathrm{N}_{(1,1)} = \left(\Bbbk^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{or} \qquad \mathrm{N}_2 = \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

In the first case we get a short exact sequence

$$0 \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to \left(\Bbbk^2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to 0\,.$$

This short exact sequence splits on the level of $\Bbbk$-vector spaces, and any such split is alreday a homomorphism of representations. We hence find that this sequence describes the unique element of $\mathrm{Ext}^1(S, S)$ that is given by the split exact sequences.

We consider now the short exact sequences of the form

$$0 \to (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \xrightarrow{\varphi} \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \xrightarrow{\psi} (\Bbbk, \begin{bmatrix} 0 \end{bmatrix}) \to 0\,. \tag{1}$$

The homomorphism φ must be of the form

$$\varphi = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

for some $a \neq 0$ since the image of φ must be contained in the kernel of $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. It follows from the exactness of the sequence that the homomorphism ψ is of the form

$$\psi = \begin{bmatrix} 0 & b \end{bmatrix}$$

for some $b \neq 0$.

Two such sequences $\xi_{a,b}$ and $\xi_{a',b'}$ for $a, a', b, b' \neq 0$ are Yoneda equivalent if and only if there exists an invertible matrix

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathrm{GL}(2, \Bbbk)$$

such that

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \tag{2}$$

and the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & (k, [0]) \arrow[r, "{\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]}"] \arrow[d, equal] & \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \arrow[r, "{[\,0\ b\,]}"] \arrow[d, "{\left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]}"] & (k, [0]) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (k, [0]) \arrow[r, "{\left[\begin{smallmatrix} a' \\ 0 \end{smallmatrix}\right]}"] & \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \arrow[r, "{[\,0\ b'\,]}"] & (k, [0]) \arrow[r] & 0 \end{tikzcd} \tag{3}$$

The condition (2) means that $w = z$ and $y = 0$, i.e. that the the matrix is of the form

$$\begin{bmatrix} w & x \\ 0 & w \end{bmatrix},$$

The commutativity of the diagram (3) means that

$$w = \frac{a'}{a} \quad \text{and} \quad w = \frac{b}{b'}.$$

We hence find that the extensions $\xi_{a,b}$ and $\xi_{a',b'}$ are Yoneda equivalent if and only if $a'/a = b/b'$.

It follows that the Yoneda equivalence classes of short exact sequences of the form (1) have as a set of representatives the sequences

$$0 \to (\Bbbk, [0]) \xrightarrow{[\,1\ 0\,]} \left(\Bbbk^2, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \xrightarrow{[\,0\ b\,]} (\Bbbk, [0]) \to 0$$

with $b \neq 0$.

We find that overall we have $\#\Bbbk = \#\mathbb{F}_q = q$ many Yoneda equivalence classes of short exact sequences. Thus

$$\#\mathrm{Ext}^1(S, S) = q,$$

from which it follows that $\mathrm{Ext}^1(S, S) \cong \mathbb{F}_q$.

A.5. Explicit Description of $\mathbf{H}(Q, \mathbb{F}_q)$

1. The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_q)$ is free on the set of isomorphism classes, $\mathrm{Iso}(Q, \mathbb{F}_q)$. This basis is indexed by the set of partitions, Par.

2. The multiplication on $\mathbf{H}(Q, \mathbb{F}_q)$ is given by

$$[M] \cdot [N] = \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_q)} C^R_{M,N} [R]$$

where

$$C^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \mid L \cong N, R/L \cong M\}.$$

The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_q)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_q)} = [0]$.

3. The category $\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)$ satisfies the finite subobject condition and its Euler form vanishes.It follows that Green's coproduct makes the Hall algebra $\mathbf{H}(Q, \mathbb{F}_q)$ into a bialgebra. Its comultiplication is given by

$$\Delta([M]) = \sum_{[R],[L] \in \mathrm{Iso}(Q,\mathbb{F}_q)} \frac{P^M_{R,L}}{a_M} [R] \otimes [L]$$

where a_M is the size of the automorphism group $\mathrm{Aut}(M)$, and $P^M_{R,L}$ is the number of short exact sequences $0 \to L \to M \to R \to 0$. The counit $\varepsilon \colon \mathbf{H}(Q, \mathbb{F}_q) \to \mathbb{C}$ is given by

$$\varepsilon([M]) = \begin{cases} 1 & \text{if } M = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. We have a grading on $\mathbf{H}(Q, \mathbb{F}_q)$ over the Grothendieck group $\mathrm{K}(\mathbf{rep}^{\mathrm{nil}}(Q, \mathbb{F}_q)) \cong \mathbb{Z}$, given by

$$\deg([M]) = \dim(M).$$

This grading makes $\mathbf{H}(Q, \mathbb{F}_q)$ into a graded bialgebra.

5. The graded bialgebra $\mathbf{H}(Q, \mathbb{F}_q)$ is connected (i.e. its degree zero part is the ground field). It is therefore already a graded Hopf algebra.

A.6. Counting $\mathrm{Gr}(d, n, \mathbb{F}_q)$

Recall A.11. For $k \in \mathbb{N}$ the *quantum integer* $[k]_q$ is given by

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1}.$$

We have $[0]_q = 0$ and $[1]_q = 1$. The *quantum factorial* is given by

$$[k]_q! = [k]_q [k-1]_q \cdots [1]_q.$$

For $k, l \in \mathbb{N}$ the *quantum binomial* is given by

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q \cdots [k - l + 1]_q}{[l]_q!}.$$

If $l > k$ then this is zero, and if $l \leq k$ then the quantum binomial can also be expressed as

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q!\,[k-l]_q!}\,.$$

The quantum binomial satsfies the recursive relation

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = q^l \begin{bmatrix} k-1 \\ l \end{bmatrix}_q + \begin{bmatrix} k-1 \\ l-1 \end{bmatrix}_q$$

for all $k, l \in \mathbb{N}$. It hence follows by induction that the quantum binomial $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ is a polynomial in q with natural coefficients, i.e.

$$\begin{bmatrix} k \\ l \end{bmatrix}_q \in \mathbb{N}[q]\,.$$

By taking the limit $q \to 1$ (i.e. by setting q equal to 1) the quantum integer $[k]$ becomes the usual integer k, the quantum factorial $[k]_q!$ becomes the usual factorial $k!$ and the quantum binomial coefficient $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q$ becomes the usual binomial $\binom{k}{l}$.

Lemma A.12. For all dimensions $n, d \geq 0$ we have

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \begin{bmatrix} n \\ d \end{bmatrix}_q\,.$$

Proof. If $d > n$ then both numbers are zero, so suppose that $d \leq n$. Let

$$F_d(n) := (q^n - 1) \cdots (q^n - q^{d-1}) = (q-1)^d\, q^{d(d-1)/2}\, [n]_q \cdots [n-d+1]_q$$

This is the number of linear independent tupels $(v_1, \ldots, v_d)$ of vectors in $\mathbb{F}_q^n$. We find that

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{\#\mathrm{GL}(d, \mathbb{F}_q)}\,.$$

We have $\#\mathrm{GL}(d, \mathbb{F}_q) = F_d(d)$ and thus

$$\#\mathrm{Gr}(d, n, \mathbb{F}_q) = \frac{F_d(n)}{F_d(d)} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q \cdots [1]_q} = \frac{[n]_q \cdots [n-d+1]_q}{[d]_q!} = \begin{bmatrix} n \\ d \end{bmatrix}_q,$$

as claimed. □

A.7. Explicit Computations of Quantum Binomial Coefficients

We have

$$C^{(1^{n+1})}_{(1^n),(1)} = \#\mathrm{Gr}(1, n+1, \mathbb{F}_q) = \#\mathbb{P}^n(\mathbb{F}_q) = \frac{q^{n+1} - 1}{q - 1} = [n+1]_q = 1 + q + \cdots + q^n\,,$$

and also

$$
\begin{aligned}
C^{(1^{n+2})}_{(1^n),(1,1)} = \begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q &= \frac{[n+2]_q\,[n+1]_q}{[2]_q} \\
&= \frac{(1+q+\cdots+q^n)(1+q+\cdots+q^{n+1})}{1+q} \\
&= \begin{cases} (1+q+\cdots+q^n)(1+q^2+\cdots+q^n) & \text{if } n \text{ is even,} \\ (1+q^2+\cdots+q^{n-1})(1+q+\cdots+q^{n+1}) & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

Lastly we compute

$$
\begin{aligned}
\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q = \frac{[n+3]_q\,[n+2]_q\,[n+1]_q}{[3]_q\,[2]_q\,[1]_q} &= \frac{\frac{q^{n+3}-1}{q-1}\,\frac{q^{n+2}-1}{q-1}\,\frac{q^{n+1}-1}{q-1}}{\frac{q^3-1}{q-1}\,\frac{q^2-1}{q-1}\,\frac{q-1}{q-1}} \\
&= \frac{(q^{n+3}-1)(q^{n+2}-1)(q^{n+1}-1)}{(q^3-1)(q^2-1)(q-1)}
\end{aligned}
$$

We recall that the polynomial $x^k - 1$ divides the polynomial $x^l - 1$ (in $\mathbb{Z}[x]$) if and only if the integer k divides the integer l. Then

$$
\frac{x^k-1}{x^l-1} = 1 + x^l + x^{2l} + \cdots + x^{k-l}.
$$

We can therefore compute the above quotient by distinguishing between six cases, depending on how the powers

$$
n+3, \quad n+2, \quad n+1
$$

are divisible by 3 and 2. This in turn is uniquely determined by the residue class of n modulo 6.

Case 1. Suppose that $n \equiv 0$. Then $n+3$ is divisible by 3 and $n+2$ is divisible by 2. In this case

$$
\begin{aligned}
\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+3}-1}{q^3-1} \cdot \frac{q^{n+2}-1}{q^2-1} \cdot \frac{q^{n+1}-1}{q-1} \\
&= (1+q^3+\cdots+q^n)(1+q^2+\cdots+q^n)(1+q+\cdots+q^n).
\end{aligned}
$$

Case 2. Suppose that $n \equiv 1$. Then $n+2$ is divisible by 3 and $n+3$ is divisible by 2. In this case

$$
\begin{aligned}
\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+2}-1}{q^3-1} \cdot \frac{q^{n+3}-1}{q^2-1} \cdot \frac{q^{n+1}-1}{q-1} \\
&= (1+q^3+\cdots+q^{n-1})(1+q^2+\cdots+q^{n+1})(1+q+\cdots+q^n).
\end{aligned}
$$

Case 3. Suppose that $n \equiv 2$. Then $n+1$ is divisible by 3 and $n+2$ is divisible by 2. In this case

$$
\begin{aligned}
\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+1}-1}{q^3-1} \cdot \frac{q^{n+2}-1}{q^2-1} \cdot \frac{q^{n+3}-1}{q-1} \\
&= (1+q^3+\cdots+q^{n-2})(1+q^2+\cdots+q^n)(1+q+\cdots+q^{n+2}).
\end{aligned}
$$

Case 4. Suppose that $n \equiv 3$. Then $n+3$ is divisible by 3 and $n+1$ is divisible by 2. In this case

$$\begin{aligned}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+3}-1}{q^3-1} \cdot \frac{q^{n+1}-1}{q^2-1} \cdot \frac{q^{n+2}-1}{q-1} \\ &= (1+q^3+\cdots+q^n)(1+q^2+\cdots+q^{n-1})(1+q+\cdots+q^{n+1})\,.\end{aligned}$$

Case 5. Suppose that $n \equiv 4$. Then $n+2$ is divisible by both 3 and 2. We observe that

$$\frac{1+q^3}{1+q} = 1-q+q^2$$

and hence for every odd positive integer m that

$$\begin{aligned}\frac{1+q^3+\cdots+q^{3m}}{1+q} &= \frac{(1+q^3)+q^6(1+q^3)+\cdots+q^{3m-3}(1+q^3)}{1+q} \\ &= (1-q+q^2)(1+q^6+\cdots+q^{3m-3})\,.\end{aligned}$$

We also observe that the integer $n-1$ is divisible by 3, and that it is an odd multiple of 3 because $n+2$ is divisible by 6 and therefore

$$\frac{n-1}{3}+1 = \frac{n+2}{3}$$

is even. We now find that

$$\begin{aligned}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+3}-1}{q-1} \cdot \frac{q^{n+2}-1}{(q^3-1)(q+1)} \cdot \frac{q^{n+1}-1}{q-1} \\ &= \frac{q^{n+3}-1}{q-1} \cdot \frac{1+q^3+\cdots+q^{n-1}}{q+1} \cdot \frac{q^{n+1}-1}{q-1} \\ &= (1+q+\cdots+q^{n+2})(1-q+q^2)(1+q^6+\cdots+q^{n-4})(1+q+\cdots+q^n)\,.\end{aligned}$$

Case 6. Suppose that $n \equiv 5$. Then $n+1$ is divisible by 3 and $n+3$ is divisible by 2. In this case

$$\begin{aligned}\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q &= \frac{q^{n+1}-1}{q^3-1} \cdot \frac{q^{n+3}-1}{q^2-1} \cdot \frac{q^{n+2}-1}{q-1} \\ &= (1+q^3+\cdots+q^{n-2})(1+q^2+\cdots+q^{n+1})(1+q+\cdots+q^{n+1})\,.\end{aligned}$$

A.8. Computing more Structure Constants for $\mathrm{H}(Q, \mathbb{F}_q)$

We use for $\mathrm{N}_{(n,1)}$ the basis $e_1, \dots, e_n, f_1, \dots, f_m$ with

$$\alpha e_i = e_{i+1} \quad \text{and} \quad \alpha e_n = \alpha f_1 = \cdots = \alpha f_m = 0$$

for $i = 1, \dots, n-1$, where α denotes the loop of Q. See Figure 3.

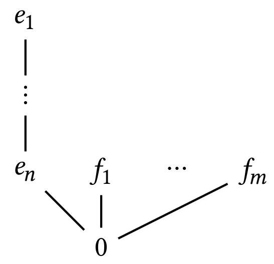

Figure 3: The representations $\mathrm{N}_{(n,1^m)}$.

The coefficient $C^{(n,1^m)}_{(1^m),(n)}$ is the number of subrepresentations L of $\mathrm{N}_{(n,1^m)}$ with $L \cong \mathrm{N}_n$ and $\mathrm{N}_{(n,1)}/L \cong \mathrm{N}_{(1^m)}$. The condition $L \cong \mathrm{N}_n$ means that L is cyclically generated by a vector

$$v = a_1 e_1 + \cdots + a_n e_n + b_1 f_1 + \cdots + b_m f_m$$

with $a_1 \neq 0$. We may assume that $a_1 = 1$. Then

$$\begin{aligned} L &= \langle \alpha^k v \mid k \geq 0 \rangle_{\mathbb{F}_q} \\ &= \langle e_1 + a_2 e_2 + \cdots + a_n e_n + b_1 f_1 + \cdots + b_m f_m,\ e_2 + a_2 e_3 + \cdots + a_n e_{n-1},\ \ldots,\ e_n \rangle_{\mathbb{F}_q} \\ &= \langle e_1 + b_1 f_1 + \cdots + b_m f_m, e_2, \ldots, e_n \rangle\,. \end{aligned}$$

For any such subrepresentation L the quotient $\mathrm{N}_{(n,1^m)}/L$ is m-dimensional and spanned by the residue classes $[f_1], \ldots, [f_m]$. It is thus isomorphic to $\mathrm{N}_{(1^m)}$. We get for every choice of coefficient $b_1, \ldots, b_m \in \mathbb{F}_q$ a different subrepresentation of $\mathrm{N}_{(n,1^m)}$. Hence

$$C^{(n,1^m)}_{(1^m),(n)} = \#\mathbb{F}_q^m = q^m\,.$$

The coefficient $C^{(n,1^m)}_{(n),(1^m)}$ is the number of subrepresentations L of $\mathrm{N}_{(n,1^m)}$ with $L \cong \mathrm{N}_{(1^m)}$ and $\mathrm{N}_{(n,1^m)}/L \cong \mathrm{N}_n$. The condition $L \cong \mathrm{N}_{1^m}$ means precisely that L is an m-dimensional linear subspace of $\langle e_n, f_1, \ldots, f_m \rangle_{\mathbb{F}_q}$. We note that if g is a vector of $\langle e_n, f_1, \ldots, f_m \rangle_{\mathbb{F}_q}$ that is not contained in L then the residue classes

$$[e_1], \ldots, [e_n], [g]$$

will be a basis of the quotient $\mathrm{N}_{(n,1^m)}/L$. We claim that $\mathrm{N}_{(n,1^m)}/L$ is isomorphic to N_n if and only if the subspace L does not contain the vector e_n.

Suppose first that e_1 is contained in L. Then we take for g any vector of $\langle e_n, f_1, \ldots, f_n \rangle$ not contained in L. Then $\alpha[e_{n-1}] = [e_n] = 0$ and we find that

$$[e_1], \ldots, [e_{n-1}] \quad \text{and} \quad [g]$$

are two Jordan chains for $\mathrm{N}_{(n,1^m)}/L$. Thus $\mathrm{N}_{(n,1^m)}/L$ is not isomorphic to N_n.

Suppose on the other hand that e_1 not contained in L. Then we can choose g as e_1 and find that the residue classes

$$[e_n], \ldots, [e_1]$$

form a basis of $\mathrm{N}_{(n,1^m)}/L$. This basis forms a single Jordan chain, whence $\mathrm{N}_{(n,1^m)}/L$ is isomorphic to N_n.

We thus find that the structure constant $C^{(n,1^m)}_{(n),(1^m)}$ is the number of m-dimensional subspaces of $\langle e_n, f_1, \dots, f_m\rangle$ that does not contain e_n. The number of m-dimensional subspaces of $\langle e_n, f_1, \dots, f_m\rangle$ is according to Lemma A.12 given by the quantum binomial

$$\begin{bmatrix} m+1 \\ m \end{bmatrix}_q .$$

The number of m-dimensional subspaces of $\langle e_n, f_1, \dots, f_m\rangle$ that contains e_1 is equals the number of $(m-1)$-dimensional subspaces of the quotient vector spaces $\langle e_n, f_1, \dots, f_n\rangle / \langle e_n\rangle \cong \langle f_1, \dots, f_m\rangle$. We see again from Lemma A.12 that there are

$$\begin{bmatrix} m \\ m-1 \end{bmatrix}_q$$

such subspaces. We hence find that

$$C^{(n,1^m)}_{(n),(1^m)} = \begin{bmatrix} m+1 \\ m \end{bmatrix}_q - \begin{bmatrix} m \\ m-1 \end{bmatrix}_q = \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} m \\ 1 \end{bmatrix}_q = [m+1]_q - [m]_q = q^m .$$

A.9. Explicit Description of $\mathbf{H}(Q, \mathbb{F}_1)$

The Hall algebra $\mathbf{H}(Q, \mathbb{F}_1)$ is a graded, cocommutative Hopf algebra (over the ground field $\mathbb{C}$). It is defined as follows:

- The underlying vector space of $\mathbf{H}(Q, \mathbb{F}_1)$ is the free $\mathbb{C}$-vector space on the set $\mathrm{Iso}(Q, \mathbb{F}_1)$. The set $\mathrm{Iso}(Q, \mathbb{F}_1)$ is indexed by the set of partitions Par.
- The multiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

  $$[M] \cdot [N] \coloneqq \sum_{[R] \in \mathrm{Iso}(Q, \mathbb{F}_1)} C^R_{M,N} [R]$$

  where the structure coefficients $C^R_{M,N}$ are given by

  $$C^R_{M,N} = \#\{\text{subrepresentations } L \text{ of } R \mid L \cong N, R/L \cong M\} .$$

  The multiplicative neutral element of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $1_{\mathbf{H}(Q,\mathbb{F}_1)} = [0]$.
- The grading of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by $\deg([M]) = \dim(M)$.
- The comultiplication of $\mathbf{H}(Q, \mathbb{F}_1)$ is given by

  $$\Delta([M]) = \sum_{\substack{[R],[L] \in \mathrm{Iso}(Q, \mathbb{F}_1) \\ M \cong R \oplus L}} [R] \otimes [L] .$$

We see in particular that an isomorphism class $[M]$ is primitive in $\mathbf{H}(Q, \mathbb{F}_1)$ if and only if the representation M is indecomposable. We have seen that more generally the Lie algebra of primitive elements of $\mathbf{H}(Q, \mathbb{F}_1)$ has a basis consisting of all such $[M]$. We have concluded from the theorem of Milnor–Moore that $\mathbf{H}(Q, \mathbb{F}_q)$ is the universal enveloping algebra of its Lie algebra of primitive elements.

A.10. More on the Multiplication of $\mathbf{H}(Q, \mathbb{F}_1)$

The computation of the product $[\mathrm{N}_i] \cdot [\mathrm{N}_j]$ in $\mathbf{H}(Q, \mathbb{F}_1)$ can be further generalized:
We find for all $i_1, \dots, i_r \geq 1$ and $j \geq 1$ that

$$[\mathrm{N}_{(i_1,\dots,i_r)}] \cdot [\mathrm{N}_j] = a[\mathrm{N}_{(i_1,\dots,i_r,j)}] + \sum_{\lambda} b_\lambda [\mathrm{N}_\lambda]\,,$$

where λ runs through all distinct tupels of the form $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ with $1 \leq k \leq r$. The coeffiecient a is given by

$$a = \text{how often } j \text{ occurs in } (i_1, \dots, i_r, j)$$

and the coefficient of b_λ for $\lambda = (i_1, \dots, i_k + j, \dots, i_r)$ are given by

$$b_\lambda = \text{how often } i_k + j \text{ occurs in } \lambda\,.$$

Example A.13. We find that

$$\begin{aligned}[\mathrm{N}_{(5,3,3,2,1)}] \cdot [\mathrm{N}_2] = {} & 2[\mathrm{N}_{(5,3,3,2,1,\overline{2})}] \\ & + [\mathrm{N}_{(\overline{7},3,3,2,1)}] + 2[\mathrm{N}_{(5,\overline{5},3,2,1)}] + [\mathrm{N}_{(5,3,3,\overline{4},1)}] + 3[\mathrm{N}_{(5,3,3,2,\overline{3})}]\,,\end{aligned}$$

where the entry of interest is overlined.

Remark A.14. It follows from the above formula by induction that $\mathbf{H}(Q, \mathbb{F}_1)$ is generated as an algebra by the N_i with $i \geq 1$.

A.11. More on Symmetric Functions

Warning A.15.

1. The ring of symmetric functions Λ is *not* the limit of $\Lambda^{(n+1)} \to \Lambda^{(n)}$ for $n \geq 0$ in the category of (commutative) algebras.
   We consider for this the symmetric polyonmials
   $$f^{(n)}(x_1, \dots, x_n) := x_1 + x_1 x_2 + \cdots + x_1 \cdots x_n\,.$$
   The polynomials satisfy the compatibility condition
   $$f^{(n+1)}(x_1, \dots, x_n, 0) = f^{(n)}$$
   for every number of variables $n \geq 0$. But there exists no symmetric function $f \in \Lambda$ with
   $$f(x_1, \dots, x_n) = f^{(n)}(x_1, \dots, x_n)$$
   for every $n \geq 0$ since otherwise
   $$n = \deg(f^{(n)}) \leq \deg(f)$$
   for every $n \geq 0$, which is not possible. This shows that Λ together with the homomorphisms $\Lambda \to \Lambda^{(n)}$ is not the limit of the homomorphisms $\Lambda^{(n+1)} \to \Lambda^{(n)}$ for $n \geq 0$ in the category of rings.

2. The ring of symmetric functions Λ is *not* isomorphic to the algebras of symmetric polynomials

$$\mathbb{C}[x_1, x_2, x_3, \dots]^{S_{\mathbb{N}}} \quad \text{or} \quad \mathbb{C}[x_1, x_2, x_3, \dots]^{S_\infty}$$

where $S_\infty = \operatorname{colim}_{n \geq 0}(S_n \hookrightarrow S_{n+1})$.

Indeed, we observe that both $\mathbb{C}[x_1, x_2, x_3, \dots]^{S_{\mathbb{N}}}$ and $\mathbb{C}[x_1, x_2, x_3, \dots]^{S_\infty}$ are just the ground field $\mathbb{C}$: If a symmetric polynomial $f \in \mathbb{C}[x_1, x_2, \dots]$ were to contain a nontrivial monomial then it must also contain all permutations of this monomial. But there are infinitely many such permutations, while f contains only finitely many polynomials.

Suppose that more generally $(f^{(n)})_{n \geq 0}$ is any sequence of symmetric polynomials $f^{(n)} \in \Lambda^{(n)}$ that are compatible in the sense that $f(x_1, \dots, x_n) = f^{(n)}(x_1, \dots, x_n)$ for every $n \geq 0$. Then the $f^{(n)}$ define a symmetric function $f \in \Lambda$ with $f^{(n)}(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ for every $n \geq 0$ if and only if the degrees $\deg(f^{(n)})$ are bounded, i.e. if and only if there exists some $K \geq 0$ with $\deg(f^{(N)}) \leq K$ for every $n \geq 0$.

Indeed, if such a symmetric function f exists then $\deg(f^{(n)}) \leq \deg(f)$ for every $n \geq 0$. If on the other hand such a bound K exists then we consider for every $k = 0, \dots, K$ the sequence $(f_k^{(n)})_{n \geq 0}$ of degree k parts. That the symmetric polynomials $f^{(n)}$ are compatible means that for every degree $k = 0, \dots, K$ the homogeneous symmetric polynomials $f_k^{(n)}$ are compatible. Thus there exists for every degree $k = 0, \dots, K$ a homogeneous symmetric function $f_k \in \Lambda_k$ with

$$f_k(x_1, \dots, x_n) = f_k^{(n)}(x_1, \dots, x_n)$$

for every number of variables $n \geq 0$. It then follows for the symmetric function

$$f \coloneqq f_0 + f_1 + \cdots + f_K$$

in each degree $l = 0, \dots, K$ that

$$f(x_1, \dots, x_n)_l = f_0(x_1, \dots, x_n)_k + \cdots + f_k(x_1, \dots, x_n)_l = f_l(x_1, \dots, x_n) = f_l^{(n)}(x_1, \dots, x_n)$$

for every $n \geq 0$ since each $f_l(x_1, \dots, x_n)$ is homogeneous of degree l. This shows that

$$f(x_1, \dots, x_n) = f^{(n)}(x_1, \dots, x_n)$$

for every number of variables $n \geq 0$.

A.12. Regarding Λ as a Colimit

Remark A.16.

1. It follows from Corollary 4.6 for any two symmetric functions $f, g \in \Lambda$ that

$$f = g \iff f(x_1, \dots, x_n) = g(x_1, \dots, x_n) \text{ for some } n \geq \deg(f), \deg(g).$$

2. We can regard Λ as a colimit of suitable inclusions $\Lambda^{(n)} \to \Lambda^{(n+1)}$ for $n \geq 0$:

We have for every number of variables $n \geq 0$ an injective homomorphism of graded algebras $\Lambda^{(n)} \to \Lambda^{(n+1)}$ that is given on algebra generators by $e_k^{(n)} \to e_k^{(n+1)}$ for every $k = 0, \dots, n$. This is a right sided inverse for the homomorphism $\Lambda^{(n+1)} \to \Lambda^{(n)}$. In this way a symmetric polynomial in n variables can be extended to a symmetric polynomial in $n+1$ variables.

We similarly have for every number of variables $n \geq 0$ a homomorphism of graded algebras $\Lambda^{(n)} \to \Lambda$ that is given on algebra generators by $e_k^{(n)} \to e_k$ for every $k = 0, \dots, k$.

We now find that Λ together with the homomorphisms $\Lambda^{(n)} \to \Lambda$ is a colimit of the homomorphisms $\Lambda^{(n)} \to \Lambda^{(n+1)}$ for $n \geq 0$. In this way we can regard Λ as a sort of increasing union of the algebras of symmetric polynomials.

A.13. Invariants of a Tensor Product

Lemma A.17. Let G, H be two groups. Let V be a representation of G and let W be a representation of H. Then

$$(V \otimes W)^{G \times H} = V^G \otimes W^H .$$

Proof. The inclusion $V^G \otimes W^H \subseteq (V \otimes W)^{G \times H}$ can be checked on simple tensors. Let on the other hand $x \in (V \otimes W)^{G \times H}$. We may choose a basis $(v_i)_{i \in I}$ of V and write $x = \sum_{i \in I} v_i \otimes w_i$ for some unique vectors $w_i \in W$. For every element $h \in H$ we then have

$$\sum_{i \in I} v_i \otimes w_i = x = (1, h)x = \sum_{i \in I} v_i \otimes (h w_i) .$$

It follows from the uniqueness of the w_i that $h w_i = w_i$ for every $h \in H$ and every $i \in I$, and thus $w_i \in W^H$ for every $i \in I$. This shows that $x \in V \otimes W^H$. We find in the same way that $x \in V^G \otimes W^H$, and thus $x \in (V \otimes W^H) \cap (V^G \otimes W) = V^G \otimes W^H$. □

References

[Oor63] Frans Oort. "Yoneda Extensions in Abelian Categories". In: *Mathematische Annalen* 153.3 (February 11, 1963), pp. 227–235. DOI: BF01360318.

[Sch09] Olivier Schiffmann. *Lectures on Hall Algebras.* October 23, 2009. arXiv: 0611617v2 [math.RT].