November 15, 2018

Algebra II, Sheet 4

Remarks and Solutions

Jendrik Stelzner

Exercise 1.

(a)

The main problem with this exercise is that it can be understood in two ways:

1) We need show that a polynomial map $f\colon W \to V$ is G-equivariant if and only if the induced algebra homomorphism $f^*\colon \mathcal{P}(V) \to \mathcal{P}(W)$ is G-invariant.

2) We need to show that an arbitrary map $f\colon W \to V$ is covariant (i.e. both polynomial and G-equivariant) if and only if $f^*\colon \mathcal{P}(V) \to \mathcal{P}(W)$, $h \mapsto h \circ f$ is a well-defined algebra homomorphism which is also G-invariant.

We start by showing that 1) and 2) are actually equivalent. For this we need to show that a map $f\colon W \to V$ is polynomial if and only if $f^*\colon \mathcal{P}(V) \to \mathcal{P}(W)$, $h \mapsto h \circ f$ is a well-defined algebra homomorphisms. We have already seen in the lecture that f^* is a well-defined algebra homomorphism if f is polynomial.

Suppose on the other hand f^* is well-defined. Let $v_1, \dots, v_n$ be a basis of V. Then there exist unique functions $f_1, \dots, f_n\colon V \to k$ with

$$f(w) = f_1(w)v_1 + \cdots + f_n(w)v_n$$

for every $w \in W$. If $\varphi_1, \dots, \varphi_n \in \mathcal{P}(V)$ denote the coordinate functions with respect to the basis $v_1, \dots, v_n$ then

$$f_i(w) = \varphi_i(f(w)) = f^*(\varphi_i)(w)$$

for every $W \in W$ and hence

$$f_i = f^*(\varphi_i) \in \mathcal{P}(V)$$

for every $i = 1, \dots, n$. This shows that $f_1, \dots, f_n$ are polynomial, and hence that f is polynomial.

We now show 1). For this we calculate for all $g \in G$, $h \in \mathcal{P}(V)$ and $w \in W$ that

$$\begin{aligned}
&(g.f^*)(h)(w) \\
&= (g.f^*(g^{-1}.h))(w) \\
&= f^*(g^{-1}.h)(g^{-1}.w) \\
&= ((g^{-1}.h) \circ f)(g^{-1}.w) \\
&= (g^{-1}.h)(f(g^{-1}.w)) \\
&= h(g.f(g^{-1}.w)) \\
&= h((g.f)(w)) \,.
\end{aligned}$$

We note that for every two elements $v, v' \in V$ with $v \neq v'$ there exist some polynomial function $h \in \mathcal{P}(V)$ with $h(v) \neq h(v')$. Indeed, if we choose again a basis $v_1, \dots, v_n$ of V then it holds for $v = \sum_{i=1}^n a_i v_i$ and $v' = \sum_{i=1}^n a'_i v_i$ with $a_i, a'_i \in k$ that $a_i \neq a'_i$ for some i. For the coordinate functions $\varphi_1, \dots, \varphi_n \in \mathcal{P}(V)$ with respect to the basis $v_1, \dots, v_n$ we therefore have that $\varphi_i(v) = a_i \neq a'_i = \varphi_i(v')$. So we may choose $h = \varphi_i$.

We now find that

$$\begin{aligned}
&f^* \text{ is } G\text{-invariant} \\
\iff{}& g.f^* = f^* \\
\iff{}& (g.f^*)(h)(w) = f^*(h)(w) \text{ for all } h \in \mathcal{P}(V),\ w \in W \\
\iff{}& h((g.f)(w)) = h(f(w)) \text{ for all } h \in \mathcal{P}(V),\ w \in W \\
\iff{}& (g.f)(w) = f(w) \text{ for all } w \in W \\
\iff{}& g.f = f \\
\iff{}& f \text{ is } G\text{-invariant} \\
\iff{}& f \text{ is } G\text{-equivariant}\,,
\end{aligned}$$

where we use again (as seen in the lecture and in the tutorial) that a map $f\colon W \to V$ is G-invariant if and only if it is G-equivariant.

(b)

We know from the lecture that $\mathcal{P}(\mathrm{M}_n(k))^{\mathrm{SL}_n(k)} = k[\det]$, hence that $\mathcal{P}(\mathrm{M}_n(k))^{\mathrm{SL}_n(k)}$ has as a basis the determinant powers $\det^p$ with $p \geq 0$. The grading of $\mathcal{P}(\mathrm{M}_n(k))^{\mathrm{SL}_n(k)}$ is inherited from $\mathcal{P}(\mathrm{M}_n(k))$, and the determinant det is homogeneous of degree n by the Leibniz formula

$$\det = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)} \,.$$

We thus find that the determinant power $\det^p$ has homogeneous degree pn for all $p \geq 0$. We can now pull back the grading of $\mathcal{P}(\mathrm{M}_n(k))^{\mathrm{SL}_n(k)}$ via the isomorphism

$$k[t] \to \mathcal{P}(\mathrm{M}_n(k))^{\mathrm{SL}_n(k)} \,, \quad p(t) \mapsto p(\det)$$

to a grading of $k[t]$; the basis element $\det^p$ of $\mathcal{P}(\mathrm{M}_n(k))^{\mathrm{SL}_n(k)}$ corresponds to the basis element t^p of $k[t]$, which is therefore of homogeneous degree pn. This shows that

$$k[t]_{pn} = \langle t^p \rangle_k$$

for all $p \geq 0$, and $k[t]_d = 0$ otherwise. It follows in particular that

$$\dim k[t]_{pn} = 1$$

for all $p \geq 0$, and $\dim k[t]_d = 0$ otherwise. The Hilbert series of $k[t]$ (with the above grading) is therefore given by the power series (in the variable x) given by

$$\sum_{p=0}^{\infty} x^{pn} = \sum_{p=0}^{\infty} (x^n)^p = \frac{1}{1 - x^n}\,.$$