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Algebra II, Santa's Extra Sheet

Remarks and Solutions

Exercise 1.

We observe first that instead of $1 \leq i, j \leq n$ the condition $1 \leq i < j \leq n$ is needed. We hence set

$$V \coloneqq \prod_{1 \leq i < j \leq n} (X_i - X_j)$$

and need to show that $V \mid f$.

We consider first the case $i = 1 < 2 = j$: We note that

$$\begin{aligned} k[X_1, \dots, X_n]/(X_1 - X_2) &\cong k[Y_1, \dots, Y_{n-1}]\,, \\ \overline{g(X_1, \dots, X_n)} &\mapsto g(Y_1, Y_1, Y_2, \dots, Y_{n-1})\,. \end{aligned}$$

We find that f is mapped to

$$f(Y_1, Y_1, Y_2, \dots, Y_{n-1}) = -f(Y_1, Y_1, Y_2, \dots, Y_{n-1})\,.$$

It follows from $\mathrm{char}(k) \neq 2$ that f is mapped to 0, and hence that $f \in (X_1 - X_2)$. This shows that $(X_1 - X_2) \mid f$.

We find in the same way that $X_i - X_j \mid f$ for all $1 \leq i < j \leq n$. These elements are pairwise non-equivalent primes of $k[X_1, \dots, X_n]$. that they are prime follows from $k[X_1, \dots, X_n]/(X_i - X_j) \cong k[Y_1, \dots, Y_{n-1}]$ being an integral domain, and they are non-equivalent because they are monic and pairwise different. It follows, because $k[X_1, \dots, X_n]$ is a UFD, that $V \mid f$.

Exercise 2.

Lemma 1. Let X be a topological space, let $U \subseteq X$ be open and let $D \subseteq X$ be dense. Then $D \cap U$ is dense in U.

Proof. Let $x \in U$ and let $V \subseteq U$ be an open neighbourhood of x in U. Then V is also an open neighbourhood of x in X, and hence contains an element $d \in D$. Then $d \in D \cap X$. □

We know from the lecture that $\mathrm{GL}_n(k) \subseteq \mathrm{M}_n(k)$ is open, and that the set

$$D := \{A \in \mathrm{M}_n(k) \mid A \text{ is diagonalizable}\}$$

is dense in $\mathrm{M}_n(k)$. It follows that $G^{\mathrm{ss}} = D \cap \mathrm{GL}_n(k)$ is dense in $\mathrm{GL}_n(k)$. (Here we use that the subspace topology that $\mathrm{GL}_n(k)$ inherits from $\mathrm{M}_n(k)$ coincides with the Zariski topology of $\mathrm{GL}_n(k)$.)

Exercise 3.

The equivalence (a) $\iff$ (b) holds because the $k[G]$-submodules of V are precisely the G-subrepresentations of V. The equivalence (b) $\iff$ (c) holds by Burnside's theorem (which holds because V is finite-dimensional and k is algebraically closed).

Exercise 4.

It follows from the Artin–Wedderburn theorem that

$$\mathbb{C}[G] \cong \mathrm{M}_{m_1}(\mathbb{C}) \times \cdots \times \mathrm{M}_{m_r}(\mathbb{C})$$

where

$$n_i = \dim V_i = m_i \cdot \dim_{\mathbb{C}} \mathbb{C} = m_i \,.$$

We therefore find that

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{r} m_i^2 = \sum_{i=1}^{r} n_i^2 \,.$$

Exercise 5.

(a)

This given set is $V(y^2 - x, z^3 - x)$ where $k[\mathbb{A}^3] = k[x, y, z]$.

(b)

Let $k[\mathbb{A}^1] = k[t]$.

The set $\mathbb{A}^1 = V(0)$ is closed. For every $x \in \mathbb{A}^1$ the singleton $\{x_0\} = V(t - x)$ is closed. Finite unions of closed subsets are again closed, therefore every finite subset is closed.

Let $C \subseteq \mathbb{A}^1$ be any closed subset. Then $C = V(I)$ for some ideal $I \subseteq k[t]$. The algebra $k[t]$ is a PID, therefore the ideal I is of the form $I = (f)$ for some $f \in k[t]$. Then $C = V(I) = V(f)$. If $f = 0$ then $C = V(0) = \mathbb{A}^1$. Otherwise the nonzero polynomial f has only finitey many roots, and then $C = V(f)$ is finite.

(c)

Suppose that the fields k is finite. Let $k[\mathbb{A}^n] = k[t_1, \dots, t_n]$.

For every point $x = (x_1, \dots, x_n) \in \mathbb{A}^n$ the singleton

$$\{x\} = V(t_1 - x_1, \dots, t_n - x_n)$$

is closed. It follows that any subset $C \subseteq \mathbb{A}^n$ is closed because it can be written as the finite union $C = \bigcup_{x \in C} \{x\}$.

Exercise 6.

Let $f \in \mathrm{J}(k[x])$, and recall that the Jacobson radical $\mathrm{J}(k[x])$ is the intersection of all maximal ideals of $k[x]$. The algebra $k[x]$ contains infinitely many monic irreducible polynomials: If $p_1, \dots, p_n$ were all monic irreducible polynomials in $k[x]$ then the polynomial $\prod_{i=1}^n p_i + 1$ would not be divisible by any p_i, which would then contradicts $k[x]$ being a UFD.[1]

It follows for every such monic irreducible polynomial p that $p \mid f$ because $f \in (p)$ (where we use that the algebra $k[x]$ is a PID and the ideal (p) therefore already maximal). This shows that infinitely many non-equivalent primes divides f, which shows that $f = 0$ because $k[x]$ is a UFD.

[1] This is a classical proof technique due to Euclid.