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Algebra II

Homogeneous Ideals

Jendrik Stelzner

Let $A = \bigoplus_{d \geq 0} A_d$ be a graded k-algebra and let $I \subseteq A$ be a two-sided ideal.

Lemma 1. The linear subspace $J \coloneqq \bigoplus_{d \geq 0} (I \cap A_d)$ of A is again a two-sided ideal.

Proof. It holds that

$$
\begin{aligned}
AJ &= \left(\sum_{d \geq 0} A_d \right) \left(\sum_{d' \geq 0} (I \cap A_d) \right) \\
&\subseteq \sum_{d,d' \geq 0} A_d (I \cap A_{d'}) \\
&\subseteq \sum_{d,d' \geq 0} \Big((A_d I) \cap (A_d A_{d'}) \Big) \\
&\subseteq \sum_{d,d' \geq 0} (I \cap A_{d+d'}) \\
&= \sum_{d'' \geq 0} (I \cap A_{d''}) \\
&= J
\end{aligned}
$$

and similarly $JA \subseteq J$. □

Proposition 2. The following conditions on I are equivalent:

1) There exist linear subspaces $I_d \subseteq A_d$ with $I = \bigoplus_{d \geq 0} I_d$.
2) It holds that $I = \bigoplus_{d \geq 0} (I \cap A_d)$.
3) For every $x \in I$ with homogeneous decomposition $x = \sum_{d \geq 0} x_d$ all of the homogeneous components x_d are again contained in I.
4) The ideal I is generated by homogeneous elements.
5) It holds for the canonical projection $\pi \colon A \to A/I$ that $A/I = \bigoplus_{d \geq 0} \pi(A_d)$.

Proof.

2) $\implies$ 1) We can choose $I_d = I \cap A_d$ for every $d \geq 0$.

1) $\implies$ 2) We have for every $d \geq 0$ that $I \cap A_d = I_d$.

5) $\implies$ 1) The projection π restrict for every $d \geq 0$ to a linear map $\pi_d \colon A_d \to \pi(A_d)$. With respect to the decompositions $A = \bigoplus_{d \geq 0} A_d$ and $\pi(A) = \bigoplus_{n \geq 0} \pi(A_d)$, the projection π can be written as $\pi = \bigoplus_{d \geq 0} \pi_d$. It follows that

$$\ker(\pi) = \ker\left(\bigoplus_{d \geq 0} \pi_d\right) = \bigoplus_{d \geq 0} \ker(\pi_d)$$

with $\ker(\pi_d) \subseteq A_d$ being a linear subspace for every $d \geq 0$.

1) $\implies$ 5) For every $d \geq 0$ let $\pi_d \colon A_d \to A_d / I_d$ be the canonical projection. The linear map

$$\pi \coloneqq \bigoplus_{d \geq 0} \pi_d \colon \bigoplus_{d \geq 0} A_d \longrightarrow \bigoplus_{d \geq 0} (A_d / I_d)$$

is surjective (because it is surjective in each summand) with kernel

$$\ker(\pi) = \ker\left(\bigoplus_{d \geq 0} \pi_d\right) = \bigoplus_{d \geq 0} \ker(\pi_d) = \bigoplus_{d \geq 0} I_d = I\,.$$

It follows that the map π induces an isomorphism of vector spaces

$$\overline{\pi} \colon A/I \to \bigoplus_{d \geq 0} (A_d / I_d)\,.$$

Under this linear isomorphism $\overline{\pi}$, the summand A_d / I_d of the right hand side corresponds to the linear subspace $\pi(A_d)$ of the left hand side.

1) $\implies$ 3) We may write $x \in I$ as $x = \sum_{d \geq 0} x_d$ with $x_d \in I_d$ for every $d \geq 0$. Then $x_d \in A_d$ for every $d \geq 0$, so $x = \sum_{d \geq 0} x_d$ is the decomposition of x into homogeneous components. It now holds that $x_d \in I_d \subseteq I$.

3) $\implies$ 4) The ideal I is generated by all of its elements, and hence by the homogeneous components of all of its elements.

4) $\implies$ 2) The linear subspace $\bigoplus_{d \geq 0} (I \cap A_d)$ of I is again a two-sided ideal in A, which by assumption contains a generating set of I. Hence $I = \bigoplus_{d \geq 0} (I \cap A_d)$. □

The ideal I is *homogeneous* if it satisfies the equivalent conditions from Proposition 2. It then follows that the quotient A/I inherits a grading from A via

$$(A/I)_d \coloneqq \pi(A_d)$$

for every $d \geq 0$, where $\pi\colon A \to A/I$ denotes the canonical projection. Indeed, it holds that $A/I = \bigoplus_{d \geq 0} (A/I)_d$ by characterization 5), and it holds for all degrees $d, d' \geq 0$ that

$$(A/I)_d (A/I)_{d'} = \pi(A_d)\pi(A_{d'}) = \pi(A_d A_{d'}) \subseteq \pi(A_{d+d'}) = (A/I)_{d+d'} \, .$$

Note that this is the unique grading on A/I which makes π into a homomorphism of graded algebras.