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Algebra II, Sheet 5

Remarks and Solutions

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Exercise 2.

A missing proof

We will give two proof of the following lemma from linear algebra which we used in the tutorial:

Lemma 1. Let M be an R-module and let $P \subseteq N \subseteq M$ be submodules. If a submodule $C \subseteq M$ is a direct complement of P in M, then $C \cap N$ is a direct complement of P in N, i.e. if $M = P \oplus C$ then $N = P \oplus (C \cap N)$.

First Proof

For the first proof we use linear algebra.

Recall 2. Let M be an R-module and let $P \subseteq N \subseteq M$ be submodules. Then

$$P + (C \cap N) = (P + C) \cap N$$

for every submodule $C \subseteq M$

Remark 3. Recall 2 states that the lattice of submodules of M is modular.

First proof. It holds that

$$P \cap (C \cap N) = P \cap C \cap N = 0 \cap N = 0\,,$$

and it holds by Recall 2 that

$$P + (C \cap N) = (P + C) \cap N = M \cap N = N\,.$$

Hence $N = P \oplus (C \cap N)$. □

Second Proof

For the second proof we will also use some linear algebra:

Definition 4. If X is a set then a map $e\colon X \to X$ is *idempotent* if $e^2 = e$.

Definition 5. Let M be an R-module and let $P, C \subseteq M$ be two submodules for which $M = P \oplus C$. Then the map $e\colon M \to M$ given by

$$e(p + c) = p$$

for all $p \in P$ and $c \in C$ is the *projection* onto P along(side) C.

Recall 6. Let M be an R-module.

1) Let $P, C \subseteq M$ be submodules with $M = P \oplus C$. Then the projection $e_{P,C}\colon M \to M$ onto P along C is an idempotent endomorphisms with $\operatorname{im}(e) = P$ and $\ker(e) = C$.

2) If $e\colon M \to M$ is any idempotent endomorphism then

$$M = \operatorname{im}(e) \oplus \ker(e)\,.$$

3) The above constructions yield a one-to-one correspondence

$$\left\{ (P, C) \,\middle|\, \begin{array}{c} \text{submodules} \\ P, C \subseteq M \\ \text{with } M = P \oplus C \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{idempotent} \\ \text{endomorphisms} \\ e\colon M \to M \end{array} \right\},$$
$$(P, C) \longmapsto e_{P,C}\,,$$
$$(\operatorname{im}(e), \ker(e)) \longleftarrow\!\shortmid\, e\,.$$

Second proof. Let $e\colon M \to M$ be the projection onto P along C. The image of e is the submodule P of N, and hence we can restrict the endomorphism e to an endomorphism $e'\colon N \to N$. This restriction e' is again idempotent and therefore leads to a direct sum decomposition

$$N = \operatorname{im}(e') \oplus \ker(e')\,.$$

The first summand is given by $e'(N) = P$. Indeed, the inclusion $e'(N) \subseteq P$ holds by construction of e, and the inclusion $P \subseteq e'(N)$ follows from $P \subseteq N$. The second summand is given by $\ker(e') = \ker(e) \cap N = C \cap N$. We thus find that $N = P \oplus (C \cap N)$, as desired. □

A Remark Regarding Nakayama

We have seen the following lemma in the tutorial.

Lemma 7. Every nonzero fininitely generated R-module M contains a maximal submodule.

We give a short proof Nakayama's lemma which is based on this lemma.

Definition 8.

1) The *radical* of an R-module M is the intersection of all of its maximal submodules; it is denoted by $\mathrm{rad}(M)$.

2) The *Jacobson radical* of M is $\mathrm{J}(R) \coloneqq \mathrm{rad}(R)$.

Remark 9. One may call $\mathrm{J}(R)$ the *left Jacobson radical* of R, as it is the intersection of all maximal left[1] ideals of R. One can then also define the *right Jacobson radical* of R. But it turns out that both notions of Jacobson radical coincide. We will not need this here and will therefore not prove this.[2]

We get from Lemma 7 the following corollary:

Corollary 10. If M is a nonzero finitely generated R-module then its radical $\mathrm{rad}(M)$ is a proper submodule of M.

The radical of a module is functorial in the following sense:

Lemma 11. Let M and N be R-modules and lets $f\colon M \to N$ be a homomorphism of R-modules. Then $f(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$.

Proof. If $P \subseteq N$ is any maximal submodule then its preimage $P' \coloneqq f^{-1}(P)$ is the kernel of the composition $M \xrightarrow{f} N \to N/P$. The homomorphism f therefore induces an injective homomorphism

$$M/P' \hookrightarrow N/P\,.$$

The quotient module N/P is simple because P is maximal. It follows that $M/P' = 0$ or $M/P' \cong N/P$. The submodule P' is therefore either the whole of M or maximal in M. In both cases the radical $\mathrm{rad}(M)$ is contained in P'.

This shows that $\mathrm{rad}(M) \subseteq f^{-1}(P)$ for every maximal submodule $P \subseteq N$, and hence that $f(\mathrm{rad}(M)) \subseteq P$. It follows that $f(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$. ☐

Corollary 12. Let M be an R-module. Then $\mathrm{J}(R)M \subseteq \mathrm{rad}(M)$.

Proof. Consider for $x \in M$ the map

$$\rho_x\colon R \to M\,, \quad r \mapsto rx\,.$$

The map ρ_x is a homomorphism of R-modules, and so it follows that

$$\mathrm{J}(R)x = \rho_x(\mathrm{J}(R)) = \rho_x(\mathrm{rad}(R)) \subseteq \mathrm{rad}(M)$$

by Lemma 11. ☐

We are now ready to state and prove Nakayama's lemma.

Lemma 13 (Nakayama)**.** If M is a finitely generated R-module with $\mathrm{J}(R)M = M$ then $M = 0$.

Proof. If M were nonzero then it would follow from Corollary 10 and Corollary 12 that $\mathrm{J}(R)M \subseteq \mathrm{rad}(M) \subsetneq M$. ☐

[1]Because for us, "module" means "left module".
[2]The author also doesn't know a good proof. of this

Exercise 3.

We instead show the following more general lemma, from which we then deduce the equivalence from the exercise:

Lemma 14. Let $A = \bigoplus_{d \geq 0} A_d$ be a graded, commutative k-algebra. Then for a family $(x_i)_{i \in I}$ of homogenenous elements x_i of degree $\deg(x_i) \geq 1$ the following conditions are equivalent:

1) The elements $(x_i)_{i \in I}$ generate the ideal $A^+ \coloneqq \bigoplus_{d \geq 1} A_d$ of A.

2) The elements $(x_i)_{i \in I}$ generate A as an A_0-algebra.

Proof. Suppose that the ideal A^+ is generated by $(x_i)_{i \in I}$ and let $A' \coloneqq A_0[x_i \mid i \in I]$ be the A_0-subalgebra of A generated by $(x_i)_{i \in I}$. We show that $A_d \subseteq A'$ for all $d \geq 0$ by induction.

For $d = 0$ we have $A_0 \subseteq A'$ by definition of A'. Suppose that $d \geq 1$ and that $A_0, \dots, A_{d-1} \subseteq A'$. It then holds that $A_d \subseteq A^+$ and hence for $x \in A_d$ that $x \in A^+$. We may therefore write

$$x = \sum_{i \in I} a_i x_i$$

for some coefficients $a_i \in A$ (where $a_i = 0$ for all but finitely many i). We can then decompose every coefficient a_i as $a_i = \sum_{d' \geq 0} a'_{i,d'}$ with $a'_{i,d'}$ homogeneous of degree d'. We get that

$$x = \sum_{i \in I} a_i x_i = \sum_{d' \geq 0} \sum_{i \in I} a'_{i,d'} x_i \,.$$

The element x on the left-hand side is homogeneous of degree d, and the summand $a_{i,d'} x_i$ on the right-hand side is homogeneous of degree $d' + \deg(x_i)$. We therefore find by comparing homogeneous components that

$$x = \sum_{i \in I} a'_{i,d-\deg(x_i)} x_i$$

where we set $a'_{i,d'} = 0$ for $d' < 0$. Hence we may assume that the coefficients a_i are homogeneous of degree $d - \deg(x_i) < d - 1$. It then follows by induction hypothesis that the coefficients a_i are contained in A'. Hence

$$x = \sum_{i \in I} a_i x_i \in \sum_{i \in I} A' x_i \subseteq \sum_{i \in I} A' = A' \,.$$

This shows that also $A_d \subseteq A'$.

Suppose on the other hand that A_0 is generated by $(x_i)_{i \in I}$ as an A_0-algebra. Then the monomials $x_{i_1} \cdots x_{i_r}$ with $i_1, \dots, i_r \in I$ generate A as an A_0-module. These generators are homogeneous, and hence the A_0-submodule A^+ is generated by those monomials of degree ≥ 1, which are those monomials with $r \geq 1$. We have for these monomials that

$$x_{i_1} \cdots x_{i_r} = x_{i_1} \cdots x_{i_{r-1}} x_{i_r} \in A x_{i_r} \,.$$

This shows that A^+ is generated by $(x_i)_{i \in I}$ as an ideal. $\square$

We now show how the exercise follows from the above lemma:

If A^+ is generated by finitely many elements $x_1, \dots, x_n$, then we may replace each generator x_i by all of its homogeneous components to assume that $x_1, \dots, x_n$ are homogeneous; these generators are necessarily of homogeneous degree $\deg(x_i) \geq 1$ because $x_i \in A^+$. It then follows from the above lemma that A is generated by $x_1, \dots, x_n$ as an A_0-algebra.

Suppose on the other hand A is generated by finitely many elements $x_1, \dots, x_n$ as an A_0-algebra We may replace each generator x_i by all of its homogeneous components to assume that $x_1, \dots, x_n$ are homogeneous. We may then also remove the generators of degree 0, as they are not needed. It then follows from the above lemma that $x_1, \dots, x_n$ generate A^+ as an ideal.

Remark 15. If it not needed that A_0 is noetherian. But if A_0 is notherian, then one can given a shorter proof of one of the impliciation:

If A_0 is noetherian and A is finitely generated as an A_0-algebra then A is also noetherian by Hilbert's basis theorem. It then follows that the ideal A^+ of A is finitely generated.