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Algebra II, Sheet 8

Remarks and Solutions

Exercise 4.

(a)

It follows from the Artin–Wedderburn theorem that

$$\mathbb{C}[Q] \cong \mathrm{M}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{M}_{n_r}(\mathbb{C})$$

where $n_1, \dots, n_r$ are the dimensions of the irreducible complex representations of Q. We have for the resulting decomposition

$$8 = \dim \mathbb{C}[Q] = n_1^2 + \cdots + n_r^2$$

three possibilities:

$$8 = 4 + 4\,, \quad 8 = 4 + 1 + 1 + 1 + 1\,, \quad 8 = 1 + \cdots + 1\,.$$

The group Q acts trivially on $\mathbb{C}$, and this makes $\mathbb{C}$ into an irreducible complex representation. This shows that $n_i = 1$ for some i, thus we can exclude the possiblity $8 = 4 + 4$. In the case $8 = 1 + \cdots + 1$ we would find that the algebra

$$\mathbb{C}[Q] \cong \mathbb{C} \times \cdots \times \mathbb{C}$$

is commmutative, and hence that the group Q abelian. But this is not the case. We are only left with the possiblity $8 = 4 + 1 + 1 + 1 + 1$, which means that

$$\mathbb{C}[Q] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathrm{M}_2(\mathbb{C})\,.$$

Remark 1. We have only used that the group Q is non-commutative and of order 8. If D is the dihedral group of order 8 then we therefore find that also

$$\mathbb{C}[D] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathrm{M}_2(\mathbb{C})\,.$$

We hence find that $\mathbb{C}[Q] \cong \mathbb{C}[D]$ even though $Q \ncong D$.[1]

[1] This can be fixed by endowing the group algebra with the additional structure of an Hopf algebra.

(b)

There exists a unique linear map $f\colon \mathbb{R}[Q] \to \mathbb{H}$ with $f(q) = a_q$ for every $q \in Q$. This linear map is multiplicative on the basis Q of $\mathbb{R}[Q]$ and is therefore an algebra homomorphism. The basis 1, i, j, k of $\mathbb{H}$ is contained in the image of f, which shows that f is surjective.

(c)

The center of Q is given by $\mathrm{Z}(Q) = \{1, -1\}$. Any two elements of Q commute up to sign, so the quotient group $Q/\mathrm{Z}(Q)$ is abelian. The group $Q/\mathrm{Z}(Q)$ has order 4 and every nontrivial element has order 2. It follows that

$$Q/\mathrm{Z}(Q) \cong \mathbb{Z}/2 \times \mathbb{Z}/2\,.$$

One such isomorphism is given by

$$\pm 1 \mapsto (0,0)\,, \quad \pm i \mapsto (0,1)\,, \quad \pm j \mapsto (1,0)\,, \quad \pm k \mapsto (1,1)\,.$$

There exist four group homomorphisms $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{R}^2$, given by

$$\begin{cases} (1,0) \mapsto 1\,, \\ (0,1) \mapsto 1\,, \end{cases} \qquad \begin{cases} (1,0) \mapsto -1\,, \\ (0,1) \mapsto 1\,, \end{cases} \qquad \begin{cases} (1,0) \mapsto 1\,, \\ (0,1) \mapsto -1\,, \end{cases} \qquad \begin{cases} (1,0) \mapsto -1\,, \\ (0,1) \mapsto -1\,. \end{cases}$$

We hence get four different group homomorphisms $Q \to \mathbb{R}^\times$. These homomorphisms give four nonisomorphic one-dimensional real representations of Q.

(d)

It follows from the Artin–Wedderburn theorem that

$$\mathbb{R}[Q] \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_r}(D_r) \tag{1}$$

where $D_1^{\mathrm{op}}, \dots, D_n^{\mathrm{op}}$ are the the endomorphism ring of the irreducible real representations of Q. We have previously found four nonisomorphic one-dimensional real representations of Q; their endomorphism rings are given by $\mathbb{R}$ (because these representations are one-dimensional).

The quaternions $\mathbb{H}$ become an $\mathbb{R}[Q]$-module, i.e. a representation of Q, via the constructed algebra homomorphism $\mathbb{R}[Q] \to \mathbb{H}$. The quaternions $\mathbb{H}$ are simple as an $\mathbb{H}$-module because $\mathbb{H}$ is a skew field, and hence simple as an $\mathbb{R}[Q]$-module because the homomorphism $\mathbb{R}[Q] \to \mathbb{H}$ is surjective. Thus $\mathbb{H}$ is another irreducible real representation of Q; it is nonisomorphic to the previous four irreducible representations for dimension reasons. We find with the surjectivity of the homomorphism $\mathbb{R}[Q] \to \mathbb{H}$ that

$$\mathrm{End}_Q(\mathbb{H}) = \mathrm{End}_{\mathbb{R}[Q]}(\mathbb{H}) = \mathrm{End}_{\mathbb{H}}(\mathbb{H}) \cong \mathbb{H}^{\mathrm{op}} \cong \mathbb{H}\,,$$

where the last isomorphism is given by $x \mapsto \overline{x}$.

We have found for the decomposition (1) that, up to reordering, $D_1 = \cdots = D_4 = \mathbb{R}$ and $D_5 = \mathbb{H}$. For dimension reasons there can't be any more factors and $n_i = 1$ for all i. Therefore

$$\mathbb{R}[Q] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}\,.$$

Remark 2. We have for the dihedral group D of order 8 that

$$\mathbb{R}[D] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathrm{M}_2(\mathbb{R})\,.$$

Hence $\mathbb{R}[D] \ncong \mathbb{R}[Q]$ even though $\mathbb{C}[D] \cong \mathbb{C}[Q]$.

Indeed, the abelianization of D is given by $\mathbb{Z}/2 \times \mathbb{Z}/2$. We therefore find as for the quaternion group Q that D admits precisely four nonisomorphic real one-dimensional representations. Their endomorphism rings are given by $\mathbb{R}$.

The dihedral group also acts on $\mathbb{R}^2 \cong \mathbb{C}$ in the usual ways. The endomorphisms of this action are those $\mathbb{R}$-linear maps $f\colon \mathbb{C} \to \mathbb{C}$ that are compatible with rotation by $90°$, i.e. multiplication with i, and reflection at the real axis, i.e. complex conjugation. The first conditions ensures that f is already $\mathbb{C}$-linear, and hence given by multiplication with some complex number $z \in \mathbb{C}$. The second conditions tells us that already $z \in \mathbb{R}$. This shows that

$$\mathrm{End}_D(\mathbb{C}) = \mathbb{R}\,.$$

For the Artin–Wedderburn decomposition

$$\mathbb{R}[D] \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_r}(D_r)$$

we now find that, up to reordering, $D_1 = \cdots = D_5 = \mathbb{R}$. We also know that

$$n_i \dim D_i = \dim V_i$$

because $D_i^{n_i}$ is the (up to isomorphism unique) simple module of $\mathrm{M}_{n_i}(D_i)$. We hence find that $n_1 = n_2 = n_3 = n_4 = 1$ and $n_5 = 2$. We find by dimension reasons that there can be no other factors, and hence that

$$\mathbb{R}[D] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathrm{M}_2(\mathbb{R})\,.$$