November 7, 2018

Algebra II, Sheet 3

Remarks and Solutions

Jendrik Stelzner

Exercise 3.

(a)

The ideal (xy) is homogeneous because it is generated by the homogeneous element $xy \in k[x,y]$. Hence the quotient $k[x,y]/(xy)$ inherits a grading from $k[x,y]$ via the canonical projection $\pi\colon k[x,y] \to k[x,y]/(xy)$. To be more precise, we have for $A \coloneqq k[x,y]/(xy)$ that

$$A_d = \pi(k[x,y]_d)$$

for every $d \geq 0$. The ideal

$$(xy) = \{f \cdot xy \mid f \in k[x,y]\}$$

has as a basis the monomials $x^n y^m$ with $n, m \geq 1$, hence the quotient $A = k[x,y]/(xy)$ has as a basis the residue classes

$$1, \overline{x}^n, \overline{y}^m \qquad \text{with } n, m \geq 1\,.$$

It follows that A_0 is one-dimensional (with single basis element 1) whereas A_d is two-dimensional (with basis elements $\overline{x}^d, \overline{y}^d$) for every $d \geq 0$. The Hilbert series of A is hence given by

$$1 + 2t + 2t^2 + 2t^3 + \cdots = 1 + 2\sum_{n=1}^{\infty} t^n = 1 + 2t \sum_{n=0}^{\infty} t^n = 1 + \frac{2t}{1-t} = \frac{1+t}{1-t}\,.$$

(b)

We have for all $d \geq 0$ that

$$\dim(\mathrm{T}(V)_d) = \dim(V^{\otimes d}) = \dim(V)^d = n^d\,.$$

The Hilbert series of $\mathrm{T}(V)$ is hence given by

$$\sum_{n=0}^{\infty} n^d t^d = \sum_{n=0}^{\infty} (nt)^d = \frac{1}{1-nt}\,.$$

(c)

We first note that the two-sided ideal

$$I \coloneqq (x \otimes y - y \otimes x \mid x, y \in V)$$

is generated by homogeneous elements and is therefore homogeneous. The quotient algebra $\mathrm{T}(V)/I$ therefore inherits a grading from $\mathrm{T}(V)$, which is given by

$$(\mathrm{T}(V)/I)_d = \pi(\mathrm{T}(V)_d) = \pi(V^{\otimes d})$$

for all $d \geq 0$, where $\pi\colon \mathrm{T}(V) \to \mathrm{T}(V)/I$ denotes the canonical projection.

To further determine the quotient algebra $\mathrm{T}(V)/I$ we proceed similarly to Exercise 3, part (b) of the second exercise sheet by showing that $\mathrm{T}(V)/I \cong \mathrm{S}(V)$:

It follows from the universal property of the tensor algebra $\mathrm{T}(V)$ that the inclusion $V \to \mathrm{S}(V)$ induces an algebra homomorphism

$$\tilde{\varphi}\colon \mathrm{T}(V) \to \mathrm{S}(V)\,,$$

which is given on simple tensors by

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_n) = v_1 \cdots v_n\,.$$

It hold that $I \subseteq \ker(\tilde{\varphi})$ because the algebra $\mathrm{S}(V)$ is commutative. The algebra homomorphism $\tilde{\varphi}$ therefore induces a well-defined algebra homomorphism

$$\varphi\colon \mathrm{T}(V)/I \to \mathrm{S}(V)$$

which is on residue classes of simple tensors given by

$$\varphi(\overline{v_1 \otimes \cdots \otimes v_n}) = v_1 \cdots v_n\,.$$

To construct an inverse to φ we note that the algebra $\mathrm{T}(V)/I$ is generated by the residue classes $\overline{v}$ with $v \in V$ (because the tensor algebra $\mathrm{T}(V)$ is generated by the elements $v \in V$) and that these generators commute in the quotient $\mathrm{T}(V)/I$ because

$$\overline{v_1}\,\overline{v_2} - \overline{v_2}\,\overline{v_1} = \overline{v_1 \otimes v_2 - v_2 \otimes v_1} = 0$$

for all $v_1, v_2 \in V$. The quotient algebra $\mathrm{T}(V)$ is therefore commutative. It follows from the universal property of the symmetric algebra $\mathrm{S}(V)$ that the linear map

$$V \to \mathrm{T}(V) \to \mathrm{T}(V)/I\,, \quad v \mapsto \overline{v}$$

induces a well-defined algebra homomorphism

$$\psi\colon \mathrm{S}(V) \to \mathrm{T}(V)\,,$$

which is given on monomials by

$$\psi(v_1 \cdots v_n) = \overline{v_1} \cdots \overline{v_n} = \overline{v_1 \otimes \cdots \otimes v_n}$$

for all $v_1, \dots, v_n \in V$.

The two algebra homomorphisms φ and ψ are mutually inverse on the algebra generators $\overline{v}$ of $\mathrm{T}(V)$ and v of $\mathrm{S}(V)$, where $v \in V$, and are hence mutually inverse. In other words, the algebra homomorphism φ is an isomorphism with $\varphi^{-1} = \psi$.

The algebra homomorphism $\tilde{\varphi}\colon \mathrm{T}(V) \to \mathrm{S}(V)$ maps for all $d \geq 0$ the homogeneous component $\mathrm{T}(V)_d = V^{\otimes d}$ onto the homogeneous component $\mathrm{S}^d(V)$. It follows that the induced homomorphism $\varphi\colon \mathrm{T}(V)/I \to \mathrm{S}(V)$ also maps the homogeneous component $(\mathrm{T}(V)/I)_d$ onto the homogeneous component $\mathrm{S}^d(V)$. The algebra isomorphism φ is therefore aready an isomorphism of graded k-algebras.

Remark 1. Let $J = (xy - yx \mid x, y \in \mathrm{T}(V))$ be the commutator ideal of $\mathrm{T}(V)$ from Exercise 3, part (b) of the second exercise sheet. It holds that $I \subseteq J$, and because the quotient $\mathrm{T}(V)/I$ is commutative it also holds that $J \subseteq I$. (Recall that the commutator ideal I is the smallest two-sided ideal in A whose quotient is commutative.) Together this shows that $I = J$. We could have therefore also referred to Exercise 3, part (b) of the second exercise sheet to conclude that $\mathrm{T}(V) \cong \mathrm{S}(V)$ as graded k-algebras.

By choosing a basis $v_1, \dots, v_n$ of the vector space V we can further identify the symmetric algebra $\mathrm{S}(V)$ with the polynomial ring $k[x_1, \dots, x_n]$ as graded k-algebras. (See the solutions to Exercise 3, part (b) of the second exercise sheet for the detailed calculations.) Altogether we have that

$$\mathrm{T}(V)/I \cong \mathrm{S}(V) \cong k[x_1, \dots, x_n]$$

as graded k-algebras. The Hilbert series of $\mathrm{T}(V)/I$ is therefore the same as the one of $k[x_1, \dots, x_n]$, which we can compute and express in two possible ways:

- By using the counting method “stars and bars” we can compute for every $d \geq 0$ that

  $$\dim k[x_1, \dots, x_n]_d = \binom{d+n-1}{n-1}.$$

  The searched Hilbert series is therefore given by

  $$\sum_{d=0}^{\infty} \binom{d+n-1}{n-1} t^n .$$

- We may use that

  $$k[x_1, \dots, x_n] \cong k[x_1] \otimes \dots \otimes k[x_n]$$

  as graded k-algebras, and that the Hibert series of $k[x]$ is given by

  $$1 + t + t^2 + t^3 + \dots = \frac{1}{1-t} .$$

  It then follows that the Hilbert series of $k[x_1, \dots, x_n]$ is given by

  $$\underbrace{\frac{1}{1-t} \cdots \frac{1}{1-t}}_{n \text{ times}} = \frac{1}{(1-t)^n} .$$