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# Algebra II, Sheet 6

**Remarks and Solutions**

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## Exercise 1.

For every multiindex $\mu = (\mu_1, \dots, \mu_n)$ we denote by $x^\mu \in k[x_1, \dots, x_n]$ the element

$$x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n} .$$

### (a)

For every $f \in k[x_1, \dots, x_n]$ the element

$$f' := \sum_{g \in G} g.f$$

is $G$-invariant because

$$g.f' = g. \sum_{g' \in G} g'.f = \sum_{g' \in G} gg'.f = \sum_{g'' \in G} g''.f = f .$$

For every multiindex $\mu = (\mu_1, \dots, \mu_n)$ we choose $f = x^\mu$, and thus set

$$J_\mu := \sum_{g \in G} g.x^\mu .$$

### (b)

It holds for every $G$-invariant $f \in k[x_1, \dots, x_n]^G$ that

$$\sum_{g \in G} g.f = \sum_{g \in G} f = |G| f .$$

But with $f = \sum_\mu a_\mu x^\mu$ it also holds that

$$\sum_{g \in G} g.f = \sum_{g \in G} g. \sum_\mu a_\mu x^\mu = \sum_\mu a_\mu \sum_{g \in G} g.x^\mu = \sum_\mu a_\mu J_\mu .$$

**Remark 1.** If $\operatorname{char}(k) \nmid |G|$ and $V$ is any representation of $G$ over $k$ then the map

$$R\colon V \to V\,, \quad v \mapsto \frac{1}{|G|}\sum_{g \in G} g.v$$

is a projection of $V$ onto the subspace of invariants $V^G \subseteq V$. This projection is known as the *Reynolds operator*.

## (c)

Let $h \coloneqq |G|$ and let $G = \{g_1, \dots, g_h\}$.

It follows from the previous part of the exercise that $k[x_1, \dots, x_n]$ is generated by the $G$-invariants $J_\mu$ as a vector space, where $\mu \in \mathbb{N}^n$, because the factor $|G|$ is invertible in $k$. It therefore sufficies to show that every $J_\mu$ can be written as polynomial in those $J_\nu$ for which $|\nu| \leq h$.

For every $j \geq 0$ let $p_j = Y_1^j + \cdots + Y_h^j \in k[Y_1, \dots, Y_h]$ be the $j$-th power symmetric polynomial. For the elements

$$y_i \coloneqq (g_i.x_1)Z_1 + \cdots + (g_i.x_n)Z_n \in k[x_1, \dots, x_n][Z_1, \dots, Z_n]$$

with $i = 1, \dots, h$ we then have that

$$\begin{aligned}
p_j(y_1, \dots, y_n) &= y_1^j + \cdots + y_h^j \\
&= \sum_{i=1}^h \big[(g_i.x_1)Z_1 + \cdots + (g_i.x_n)Z_n\big]^j \\
&= \sum_{i=1}^h \sum_{|\mu|=j} \binom{j}{\mu_1, \dots, \mu_n} [(g_i.x_1)Z_1]^{\mu_1} \cdots [(g_i.x_n)Z_n]^{\mu_n} \\
&= \sum_{i=1}^h \sum_{|\mu|=j} \binom{j}{\mu_1, \dots, \mu_n} (g_i.x_1)^{\mu_1} \cdots (g_i.x_n)^{\mu_n} Z^\mu \\
&= \sum_{|\mu|=j} \binom{j}{\mu_1, \dots, \mu_n} \left[\sum_{i=1}^h (g_i.x_1)^{\mu_1} \cdots (g_i.x_n)^{\mu_n}\right] Z^\mu \\
&= \sum_{|\mu|=j} \binom{j}{\mu_1, \dots, \mu_n} \left[\sum_{i=1}^h g_i.x^\mu\right] Z^\mu \\
&= \sum_{|\mu|=j} \binom{j}{\mu_1, \dots, \mu_n} J_\mu Z^\mu\,.
\end{aligned}$$

This shows that $J_\mu$ is, up to the factor

$$C_\mu \coloneqq \binom{|\mu|}{\mu_1, \dots, \mu_n}\,,$$

the coefficient of the monomial $Z^\mu$ in $p_j(y_1, \dots, y_n)$.

We know that for every $j > h$ the $j$-th power symmetric polynomial $p_j$ can be expressed as a $k$-polynomial in the power symmetric polynomials $p_1, \dots, p_h$. It follows that the coefficients of $p_j(y_1, \dots, y_n)$ are $k$-polynomials in the coefficients of $p_1(y_1, \dots, y_n), \dots, p_h(y_1, \dots, y_n)$.

This shows that for every multiindex $\mu$ the $G$-invariant $C_\mu J_\mu$ can be expressed as a $k$-polynomial in the terms $C_\nu J_\nu$ with $|\nu| \leq h$. The factor $C_\mu$ is invertible in $k$, hence every $J_\mu$ is a $k$-polynomial in those $J_\nu$ with $|\nu| \leq h$.

**Remark 2.** Let $V$ be a finite-dimensional of a finite group $G$ over $k$. The *Noether number of $V$* is given by

$$\beta(V, G) = \inf\{d \geq 0 \mid \mathcal{P}(V)^G \text{ is generated by homogeneous elements of degree} \leq d\}\,,$$

and the *Noether number of $G$* is given by

$$\beta(G) \coloneqq \sup\{\beta(V, G) \mid V \text{ is a finite-dimensional representation of } G \text{ over } k\}\,.$$

Noether's theorem (1915) shows that $\beta(G) \leq |G|$ if $\operatorname{char}(k) = 0$, which is known as the *Noether bound*. This result can be strengthened in various ways:

- Fogarty (2001) showed that that the Noether bound holds under the weaker assumption that $\operatorname{char}(k) \nmid |G|$.
- Fleischmann (2000) showed the more general result that if $H \subseteq G$ is a normal subgroup with $\operatorname{char}(k) \nmid [G : H]$, then $\beta(V, G) \leq \beta(V, H) \cdot [G : H]$.
- For $\operatorname{char}(k) = 0$ it was proven by Schmid (1991) that $\beta(G) \leq \beta(H)[G : H]$ for every subgroup $H \subseteq G$, and that $\beta(G) \leq \beta(H)\beta(G/H)$ if $H$ is normal in $G$.
- It is an open problem if $\beta(G) \leq \beta(H)[G : H]$ holds for every subgroup $H \subseteq G$ under the weaker condition that $\operatorname{char}(k) \nmid [G : H]$.

# Exercise 2.

## (a)

Let more generally $R$ be any ring and let $M$ be an $R$-module. Recall that a submodule $N \subseteq M$ is *maximal* if it follows for every intermediate submodule $N \subseteq P \subseteq M$ that $P = N$ or $P = M$. This is equivalent to the quotient module $M/N$ having precisely two submodules, i.e. equivalent to $M/N$ being simple. It follows in particular that $R/M$ is simple for every maximal left ideal $M \subseteq R$.

Every simple $R$-module $S$ is already of this form. The module $S$ is cyclic: It holds that $S \neq 0$ and for $x \in S$ with $x \neq 0$ the submodule $\langle x \rangle$ is a nonzero submodule of $S$. Hence $S = \langle x \rangle$ because $S$ is simple. This shows that $S \cong R/M$ for some module $M$. That $S$ is simple is by the above argumentaiton equivalent to $M$ being maximal.

The maximal ideals in $\mathbb{Z}$ are $p\mathbb{Z}$ with $p$ prime. The simple $\mathbb{Z}$-modules are therefore (up to isomorphism) precisely $\mathbb{Z}/p\mathbb{Z}$ with $p$ prime.

**Warning 3.** Different maximal ideals $M, M' \subseteq R$ may given isomorphic simple modules $R/M \cong R/M'$. An example for this is $R = \mathrm{M}_n(k)$ with $n \geq 2$ and $k$ a field.

### (b)

If $R$ is an integral domain that is not a field then $R$ not semisimple: It holds for any two nonzero ideals $I, J \subseteq R$ that $I \cap J \supseteq IJ \neq 0$, and hence that the sum $I + J$ is not direct. If $x \in R$ is a nonzero non-unit then this shows that the generated ideal $\langle x \rangle$ has no direct complement.

In particular $2\mathbb{Z} \subseteq \mathbb{Z}$ has no direct complement.

### (c)

Every semisimple $\mathbb{Z}$-module $M$ is by part (a) of the form

$$M \cong \bigoplus_{i \in I} \mathbb{Z}/p_i$$

for some primes $p_i$. The primes $p_i$ are in particular square-free, which proves the statement.

**Remark 4.** One has for every $n \geq 0$ that $\mathbb{Z}/n$ is semisimple if and only if $n$ is square-free. Indeed, if $n = 0$ then $\mathbb{Z}/n = \mathbb{Z}$ is not semisimple. If $n \geq 1$ then we may write $n = p_1^{n_1} \cdots p_r^{n_r}$ for some pairwise different primes $p_i$ and exponents $n_i \geq 0$. Then

$$\mathbb{Z}/n \cong \mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_r^{n_r}$$

by the chinese reminder theorem. This is already a decomposition into indecomposable $\mathbb{Z}$-modules by the classification of finitely generated abelian groups. Hence $\mathbb{Z}/n$ is semisimple if and only if every summand $\mathbb{Z}/p_i^{n_i}$ is already simple. We have seen above that this is the case if and only if every $p_i^{n_i}$ is prime, i.e. if and only if $n_i = 1$ for every $i = 1, \ldots, r$.

It follows that an $\mathbb{Z}$-module $M$ is semisimple if and only if $M \cong \bigoplus_i \mathbb{Z}/n_i$ where the $n_i$ are square-free integers.

## Exercise 3.

Recall the universal properties of the direct sum and the direct product:

- Let $(M_\alpha)_{\alpha \in A}$ be a collection of $R$-modules and let $N$ be another $R$-module. For every $\alpha \in A$ let $i_\alpha \colon M_\alpha \to \bigoplus_{\beta \in A} M_\beta$ be the inclusion into the $\alpha$-th summand.

  Then every choice of homomorphism $f_\alpha \colon M_\alpha \to N$ with $\alpha \in A$ can be uniquely extended to homomorphism $f \colon \bigoplus_{\alpha \in A} M_\alpha \to N$, in the sense that $f \circ i_\alpha = f_\alpha$ for every $\alpha \in A$. The homomorphism $f$ is given on elements by

  $$f\left((m_\alpha)_{\alpha \in A}\right) = \sum_{\alpha \in A} f_\alpha(m_\alpha)$$

for every $(m_\alpha)_{\alpha\in A} \in \bigoplus_{\alpha\in A} M_\alpha$. Note that $f$ is well-defined because $m_\alpha = 0$ for all but finitely many $\alpha \in A$, and hence also $f_\alpha(m_\alpha) = 0$ for all but finitely many $\alpha \in A$.

This construction results in an isomorphism of abelian groups

$$\mathrm{Hom}_R\left(\bigoplus_{\alpha\in A} M_\alpha, N\right) \longleftrightarrow \prod_{\alpha\in A} \mathrm{Hom}_R(M_\alpha, N)\,.$$

- Let $M$ be an $R$-module and let $(N_\alpha)_{\alpha\in A}$ be a collection of $R$-modules. For every $\alpha \in A$ let $p_\alpha\colon \prod_{\beta\in A} N_\beta \to N_\alpha$ be the projection onto the $\alpha$-th factor.

  Then every choice of homomorphism $f_\alpha\colon M \to N_\alpha$ with $\alpha \in A$ can be uniquely combined into homomorphism $f\colon M \to \prod_{\alpha\in A} N_\alpha$, in the sense that $p_\alpha \circ f = f_\alpha$ for every $\alpha \in A$. The homomorphism $f$ is given on elements by

  $$f(m) = (f_\alpha(m))_{\alpha\in A}$$

  for every $m \in M$.

  This construction results in an isomorphism of abelian groups

  $$\mathrm{Hom}_R\left(M, \prod_{\alpha\in A} M_\alpha\right) \longleftrightarrow \prod_{\alpha\in A} \mathrm{Hom}_R(M, N_\alpha)\,. \tag{1}$$

**Warning 5.** The isomorphism (1) does in general not restrict to an isomorphism

$$\mathrm{Hom}_R\left(M, \bigoplus_{\alpha\in A} M_\alpha\right) \longleftrightarrow \bigoplus_{\alpha\in A} \mathrm{Hom}_R(M, N_\alpha)\,.$$

This is however true if $M$ is finitely generated.

We now have that

$$\begin{aligned} R' = \mathrm{End}_{\mathbb{Z}}(E) = \mathrm{Hom}_{\mathbb{Z}}(E,E) &= \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_p \mathbb{Z}/p, \bigoplus_q \mathbb{Z}/q\right) \\ &\cong \prod_p \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/p, \bigoplus_q \mathbb{Z}/q\right) \end{aligned}$$

The inclusion $i\colon \bigoplus_q \mathbb{Z}/q \to \prod_q \mathbb{Z}/q$ induces an inclusion of abelian groups

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/p, \bigoplus_q \mathbb{Z}/q\right) \xrightarrow{i_*} \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/p, \prod_q \mathbb{Z}/q\right) \cong \prod_q \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)\,.$$

It follows from Schur's lemma that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) = 0$ if $p \neq q$, and we have for $p = q$ that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathrm{Hom}_{\mathbb{Z}/p}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathrm{End}_{\mathbb{Z}/p}(\mathbb{Z}/p) \cong \mathbb{Z}/p\,.$$

Hence

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/p, \prod_q \mathbb{Z}/q\right) \cong \mathbb{Z}/p\,,$$

where for $\lambda \in \mathbb{Z}/p$ the corresponding homomorphism is given by

$$\mathbb{Z}/p \xrightarrow{\lambda\cdot(-)} \mathbb{Z}/p \hookrightarrow \prod_q \mathbb{Z}/q\,.$$

Every such homomorphism restricts to a homomorphism $\mathbb{Z}/p \to \bigoplus_q \mathbb{Z}/q$, hence the above homomorphism $i_*$ is already an isomorphism. We thus find that

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/p, \bigoplus_q \mathbb{Z}/q\right) \cong \mathbb{Z}/q\,,$$

with the same description as above.

It follows that

$$R' \cong \prod_p \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/p, \bigoplus_q \mathbb{Z}/q\right) \cong \prod_p \mathbb{Z}/p\,,$$

where an element $(\lambda_p)_p \in \prod_p \mathbb{Z}/p$ acts on $(x_p)_p \in E$ via

$$(\lambda_p)_p \cdot (x_p)_p = (\lambda_p x_p)_p\,.$$

To determine $R''$ we use the following observation:

**Lemma 6.** Let $R$ be a commutative ring and let $M$ be an $R$-module. If $R' = \operatorname{End}_R(M)$ is again commutative then $R'' = R'$.

*Proof.* It follows from $R$ being commutative that $R \subseteq R'$, and hence that $R' \supseteq R''$. It holds that $R' \subseteq R''$ because $R'$ is commutative. □

We find that $R'' = R' = \prod_p \mathbb{Z}/p$. The (unique) ring homomorphism

$$\mathbb{Z} = R \to R'' = \prod_p \mathbb{Z}/p$$

is not surjective, so $R'' \neq R$.