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Algebra II, Sheet 10

Remarks and Solutions

Exercise 2.

Recall 1. A map $f\colon X \to Y$ between topological spaces X and Y is continuous if and only if

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

(a)

It holds that $e \in U \subseteq \overline{U}$. The inversion map $i\colon G \to G$ is a morphism and hence continuous, and therefore satisfies

$$i(\overline{U}) \subseteq \overline{i(U)} = \overline{U}\,.$$

This shows that $\overline{U}^{-1} \subseteq \overline{U}$. For every $g \in U$ the left multiplication

$$\ell_g\colon G \to G\,,\quad h \mapsto gh$$

is a morphism and hence continuous, and therefore satisfies

$$\ell_g(\overline{U}) \subseteq \overline{\ell_g(U)} = \overline{U}\,.$$

This shows that $U\overline{U} \subseteq \overline{U}$. We now use that for every $h \in \overline{U}$ the right multiplication

$$r_h\colon G \to G\,,\quad g \mapsto gh$$

is a morphism and hence continuous with $r_g(U) \subseteq \overline{U}$, and therefore satisfies

$$r_h(\overline{U}) \subseteq \overline{r_h(U)} \subseteq \overline{\overline{U}} = \overline{U}\,.$$

This shows that $\overline{U}\,\overline{U} \subseteq \overline{U}$. We have altogether shown that $\overline{U}$ is again a subgroup of G.

(b)

For every $g \in U$ the commutator map

$$c_g \colon G \to G\,, \quad h \mapsto ghg^{-1}h^{-1}$$

in a morphisms and hence continuous, and the singleton $\{e\} \subseteq G$ is closed. Therefore

$$c_g(\overline{U}) \subseteq \overline{c_g(U)} = \overline{\{e\}} = \{e\}\,,$$

where we use that $c_g(U) = \{e\}$ since U is abelian. This shows that every $g \in U$ commutes with every $h \in \overline{U}$. By repeating this argument we find that every $h \in \overline{U}$ commutes with every $g \in \overline{U}$. This then shows that $\overline{U}$ is abelian.