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Algebra II, Sheet 2

Remarks and Solutions

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Exercise 1.

In the following, $\otimes = \otimes_k$ denotes the tensor product over k.

(a)

Let U be a k-vector space. We have seen in the lecture that a $k[G]$-module structure on U (which extends the given vector space structure) is “the same” as the structure of a representation of G on U. We have also seen that for any two representations V and W of G over k their tensor product $V \otimes W$ is again a representation of G, where G acts linearly on $V \otimes W$ so that

$$g.(v \otimes w) = (g.v) \otimes (g.w)$$

for all $g \in G$ and all simple tensors $v \otimes w \in V \otimes W$.

We find by combining these two observations that for any two $k[G]$-modules M and N their tensor product $M \otimes N$ carries again the structure of a $k[G]$-module so that

$$\left(\sum_{g \in G} a_g g \right) \cdot (m \otimes n) = \sum_{g \in G} a_g g.(m \otimes n) = \sum_{g \in G} a_g (gm) \otimes (gn) \tag{1}$$

for all $\sum_{g \in G} a_g g \in k[G]$ and all simple tensors $m \otimes n \in M \otimes N$.

Remark 1.

1) If A is a k-algebra and M and N are two A-modules then their tensor product $M \otimes N$ does in general not have a good A-module structure again. To be more precise:
   - We can define an “action” of A on $M \otimes N$ so that

$$a \cdot (m \otimes n) = (am) \otimes (an)$$

for all $a \in A$ and all simple tensors $m \otimes n \in M \otimes N$. But this does in general not define a module structure, because

$$\begin{aligned}
&(a_1 + a_2) \cdot (m \otimes n) \\
&= ((a_1 + a_2)m) \otimes ((a_1 + a_2)n) \\
&= (a_1 m + a_2 m) \otimes (a_1 n + a_2 n) \\
&= (a_1 m) \otimes (a_1 n) + (a_1 m) \otimes (a_2 n) + (a_2 m) \otimes (a_1 n) + (a_2 m) \otimes (a_2 n) \,,
\end{aligned}$$

while

$$a_1 \cdot (m \otimes n) + a_2 \cdot (m \otimes n) = (a_1 m) \otimes (a_1 n) + (a_2 m) \otimes (a_2 n) \,.$$

- We can define two possible A-module structures on $M \otimes A$ so that

  $$a \cdot (m \otimes n) = (am) \otimes n$$

  or

  $$a \cdot (m \otimes n) = m \otimes (an)$$

  for all $a \in A$ and all simple tensors $m \otimes n \in M \otimes N$, but these do ignore the module structure on N, or that on M.

Note that in the above solution to the exercise, the action of $k[G]$ on $M \otimes N$ in *not* given by $a \cdot (m \otimes n) = (am) \otimes (an)$ for all $a \in k[G]$ and all simple tensors $m \otimes n \in M \otimes N$; this formula only holds for $a \in G$.

2) If A and B are k-algebras and M is an A-modules and N is a B-module then $M \otimes N$ does carry the structure of an $(A \otimes B)$-module so that

$$(a \otimes b) \cdot (m \otimes n) = (am) \otimes (bn)$$

for all simple tensors $a \otimes b \in A \otimes B$ and all simple tensors $m \otimes n \in M \otimes N$.

3) It follows in particular that if A is a k-algebra and $\Delta \colon A \to A \otimes A$ is an algebra homomorphism, then we can for any two A-modules M and N pull back the induced $(A \otimes A)$-module structure of $M \otimes N$ along Δ to an A-module structure on $M \otimes N$. More explicitely, if $a \in A$ with $\Delta(a) = \sum_{i=1}^r a_{i,1} \otimes a_{i,2}$ for some $a_{i,j} \in A$, then

$$\begin{aligned}
a \cdot (m \otimes n) \coloneqq \Delta(a) \cdot (m \otimes n) &= \left(\sum_{i=1}^r a_{1,i} \otimes a_{2,i} \right) \cdot (m \otimes n) \\
&= \sum_{i=1}^r (a_{1,i} m) \otimes (a_{2,i} n) \,.
\end{aligned}$$

There are some remarks to be made about this:

- If we want this tensor product of A-modules to be associative, in the sense that for any three A-modules M, N, P the usual vector space isomorphism

$$(M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad (m \otimes n) \otimes p \to m \otimes (n \otimes p)$$

is already an isomorphism of A-modules, we need the algebra homomorphism Δ to satisfy the following coassociativity diagram:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\mathrm{id} \otimes \Delta"] \\ A \otimes A \arrow[r, "\Delta \otimes \mathrm{id}"] & A \otimes A \otimes A \end{tikzcd}$$

- For the group algebra $k[G]$ we have the following two observations:
  - Every group homomorphism $f \colon G \to H$ induces an algebra homomorphism

$$F \colon k[G] \to k[H], \quad \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g f(g).$$

  - For any two groups G and H we have

$$k[G \times H] \cong k[G] \otimes k[H].$$

  To be more precise, we have a unique k-linear map $k[G \times H] \to k[G] \otimes k[H]$ which maps (g, h) to $g \otimes h$ for all $g \in G$ and $h \in H$, and this map is already an isomorphism of k-algebras.

  By combining both of these observations we find that for every group G the diagonal map

$$\tilde{\Delta} \colon G \to G \times G, \quad g \mapsto (g, g),$$

  which is a group homomorphism, induces an algebra homomorphism

$$\Delta \colon k[G] \to k[G] \otimes k[G]$$

  with $\Delta(g) = g \otimes g$ for all $g \in G$ (but not for all $g \in k[G]$!). We can use this algebra homomorphism Δ to form the tensor product of $k[G]$-modules, and this is then the same tensor product as in (1).

(b)

We denote the given vector space isomorphism by

$$\Phi \colon V^\vee \otimes W \to \mathrm{Hom}_k(V, W).$$

It then holds for every $g \in G$ an every simple tensor $\varphi \otimes w \in V^\vee \otimes W$ that

$$\begin{aligned} \Phi(g.(\varphi \otimes w))(v) &= (g.\varphi)(v) \cdot (g.w) = \varphi(g^{-1}.v) \cdot (g.w) \\ &= g.(\varphi(g^{-1}.v) \cdot w) = g.(\Phi(\varphi \otimes w)(g^{-1}.v)) = (g.\Phi(\varphi \otimes w))(v) \end{aligned}$$

for all $v \in V$, and therefore

$$\Phi(g.(\varphi \otimes w)) = g.\Phi(\varphi \otimes w) \,.$$

It follows that

$$\Phi(g.x) = g.\Phi(x)$$

for all $g \in G$ and $x \in V^\vee \otimes W$ because every tensor $x \in V^\vee \otimes W$ is a sum of simple tensors.

(c)

If W is any subspace of V then $\bigoplus_{n=0}^\infty (W \cap V_n)$ is a subrepresentation of V because $c \in k^\times$ acts on $W \cap V_n$ by multiplication with the scalar c^n. If $W = \bigoplus_{n=0}^\infty (W \cap V_n)$ then it follows that W is a subrepresentation.

Suppose on the other hand that W is a subrepresentation. We need to show that for $w \in W$ with $w = \sum_{n=0}^\infty w_n$, where $w_n \in V_n$ for every $n \geq 0$, that the summands w_n are already contained in W. We have that $w_n = 0$ for all but finitely many n, and hence $w = w_0 + \dots + w_m$ for m sufficiently large. We can now proceed in (at least) two ways:

- It holds for every $c \in k^\times$ that

  $$c.w = c.(w_0 + \dots + w_m) = w_0 + cw_1 + c^2 w_2 + \dots + c^m w_m \,.$$

  We therefore have that

  $$\begin{bmatrix} 1 & c_0 & \cdots & c_0^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_m & \cdots & c_m^m \end{bmatrix} \begin{bmatrix} w_0 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} c_0.w \\ \vdots \\ c_m.w \end{bmatrix} \tag{2}$$

  for any choice of scalars $c_0, \dots, c_m \in k^\times$. The determinant of the appearing Vandermonde matrix

  $$V \coloneqq \begin{bmatrix} 1 & c_0 & \cdots & c_0^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_m & \cdots & c_m^m \end{bmatrix}$$

  is given by

  $$\det(V) = \prod_{i > j} (c_i - c_j) \,.$$

  We can choose $c_0, \dots, c_m$ to be pairwise different because k is invertible, which then makes V invertible. We can multipliy reverse (2) with V^{-1} from the left to find that

  $$V^{-1} \begin{bmatrix} c_0.w_0 \\ \vdots \\ c_m.w_m \end{bmatrix} = \begin{bmatrix} w_0 \\ \vdots \\ w_m \end{bmatrix} \,.$$

This shows that the homogeneous components $w_0, \dots, w_m$ can be expressed as linear combinations of the vectors $c_0.w, \dots, c_m.w$, all of which are again contained in the subrepresentation W. This shows that $w_0, \dots, w_m$ are contained in W.

- We show that $w_0, \dots, w_m \in W$ by induction on m. If $m = 0$ then $w = w_0$ and hence $w_0 \in W$. For general m we note that the vector

$$\begin{aligned} & c^m w - c.w \\ &= (c^n w_0 + c^n w_1 + \cdots + c^m w_m) - (w_0 + c w_1 + \cdots + c^m w_m) \\ &= (c^m - 1) w_0 + (c^m - c) w_1 + \cdots + (c^m - c^{m-1}) w_{m-1} \end{aligned}$$

is again contained in W for every choice of $c \in k^\times$. It then follows from the induction hypothesis that

$$(c^m - 1) w_0, \dots, (c^m - c^{m-1}) w_{m-1} \in W \,. \tag{3}$$

There exists some scalar $c \in k^\times$ with $c^m \neq 1, c, \dots, c^{m-1}$ (because otherwise every element $c \in k^\times$ would be a j-th root of unity for some $j = 1, \dots, m-1$, but there are only finitely many such roots). It then further follows from (3) that

$$w_0, \dots, w_{m-1} \in W \,,$$

and it then also follows that

$$w_m = w - w_0 - \cdots - w_{m-1} \in W \,.$$

Exercise 2.

(b)

We want to clarify how the described gradings $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$ are supposed to be constructed:

For this let A be a k-algebra and let $a_i \in A$ with $i \in I$ be some elements of A. Then there exist a unique algebra homorphism $f \colon k\langle X_i \mid i \in I \rangle \to A$ with $f(X_i) = a_i$ for every $i \in I$. Suppose in the following that the a_i form a generating set for the algebra A. The algebra homomorphism f is then surjective. If we are given for every $i \in I$ a "degree" d_i, i.e. a natural number $d_i \geq 0$, then we can construct a filtration $\mathcal{F}_\bullet$ on A as follows:

- There exists a unique grading on $k\langle X_i \mid i \in I \rangle$ for which X_i has degree d_i: We need to set

$$k\langle X_i \mid i \in I \rangle_d := \langle X_{i_1} \cdots X_{i_r} \mid r \geq 0, d_{i_1} + \cdots + d_{i_r} = d \rangle$$

for every $d \geq 0$. Then $k\langle X_i \mid i \in I \rangle = \bigoplus_{d \geq 0} k\langle X_i \mid i \in I \rangle_d$, and it holds for all $d, d' \geq 0$ that

$$k\langle X_i \mid i \in I \rangle_d \cdot k\langle X_i \mid i \in I \rangle_{d'} \subseteq k\langle X_i \mid i \in I \rangle_{d+d'} \,.$$

- The above grading results in a filtration $\tilde{\mathcal{F}}_\bullet$ on $k\langle X_i \mid i \in I\rangle$ given by

$$\tilde{\mathcal{F}}_d \coloneqq \bigoplus_{d' \leq d} k\langle X_i \mid i \in I\rangle_{d'} = \langle X_{i_1} \cdots X_{i_r} \mid r \geq 0, d_{i_1} + \cdots + d_{i_r} \leq d\rangle \,.$$

- By using the surjective algebra homomorphism $f\colon k\langle X_i \mid i \in I\rangle \to A$ we can now define a filtration $\mathcal{F}_\bullet$ on A via

$$\mathcal{F}_d \coloneqq f(\tilde{\mathcal{F}}_d) = \langle a_{i_1} \cdots a_{i_r} \mid r \geq 0, d_{i_1} + \cdots + d_{i_r} \leq d\rangle \,.$$

The constructed filtration $\mathcal{F}_\bullet$ constructed in the above way satisfies

$$a_i = f(X_i) \in f(\tilde{\mathcal{F}}_{d_i}) = \mathcal{F}_{d_i}$$

for every $i \in I$.

Remark 2. The filtration $\mathcal{F}_\bullet$ is minimal with this property, i.e. if $\mathcal{F}'_\bullet$ is another filtration on A with $a_i \in \mathcal{F}'_{d_i}$ for every $i \in I$, then $\mathcal{F}_d \subseteq \mathcal{F}'_d$ for every $d \geq 0$. Indeed, the linear subspace $\mathcal{F}_d = f(\tilde{\mathcal{F}}_d)$ of A is generated by the monomials

$$a_{i_1} \cdots a_{i_r} \qquad \text{with } r \geq 0 \text{ and } d_{i_1} + \cdots + d_{i_r} \leq d \,,$$

and these monomials are by assumption contained in $\mathcal{F}'_d$.

Warning 3. It could still happen that a generator a_i is of degree strictly smaller than d_i with respect to the filtration $\mathcal{F}$, i.e. it could happen that $a_i \in \mathcal{F}_d$ for some $d < d_i$.

Consider for example the case that $A = k[x]$ with $a_1 = x$ and $a_2 = x^2$. If we choose $d_1 = 1$ and $d_2 = 3$ then the resulting filtration $\mathcal{F}$ of $k[x]$ is given by

$$\begin{aligned} \mathcal{F}_d = \langle a_1^{n_1} a_2^{n_2} \mid n_1 + 3n_2 \leq d\rangle &= \langle x^{n_1 + 2n_2} \mid n_1 + 3n_2 \leq d\rangle \\ &= \langle 1, x, \dots, x^d\rangle_k = \{f \in k[x] \mid \deg(f) \leq d\} \end{aligned}$$

for every $d \geq 0$. The filtration $\mathcal{F}$ is hence the standard filtration of $k[x]$, and we see that $a_2 = x^2$ does not have degree $d_2 = 3$, but instead degree 2.

Exercise 3.

(a)

We already know that $\mathrm{T}(V)$ is a k-algebra with a decomposition $\mathrm{T}(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ into linear subspaces. It remains to show that this decomposition gives a grading on $\mathrm{T}(V)$, i.e. that for holds for all $x \in V^{\otimes d}$ and $x' \in V^{\otimes d'}$ that $xx' \in V^{\otimes (d+d')}$. For this we may assume that x and x' are simple tensors

$$x = v_1 \otimes \cdots \otimes v_d \quad \text{and} \quad x' = v'_1 \otimes \cdots \otimes v'_{d'} \,.$$

Then

$$\begin{aligned} xx' &= (v_1 \otimes \cdots \otimes v_d)(v'_1 \otimes \cdots \otimes v'_{d'}) \\ &= v_1 \otimes \cdots \otimes v_d \otimes v'_1 \otimes \cdots \otimes v'_{d'} \in V^{\otimes (d+d')} \,. \end{aligned}$$

Remark 4. We find in the same same way as for the tensor algebra $\mathrm{T}(V)$ that both the symmetric algebra $\mathrm{S}(V) = \bigoplus_{d \geq 0} \mathrm{S}^d(V)$ and the exterior algebra $\bigwedge(V) = \bigoplus_{d \geq 0} \bigwedge^d(V)$ are graded k-algebras.

(b)

Let A be a k-algebra. Then for every two elements $x, y \in A$ we denote by

$$[x, y] \coloneqq xy - yx$$

their *commutator*. As a map $[-,-]\colon A \times A \to A$ the commutator is both k-bilinear and alternating.[1] We can now consider the two-sided ideal I in A given by

$$I = ([x, y] \mid x, y \in A)\,.$$

The ideal I is known as the *commutator ideal* of A.

Suppose now that $A = \bigoplus_{d \geq 0} A_d$ is a graded k-algebra. If $x \in A_d$ and $y \in A_{d'}$ are homogenous elements of A then their commutator $[x, y]$ is again homogeneous because

$$[x, y] = xy - yx \in A_{d+d'} + A_{d+d'} = A_{d+d'}\,.$$

It follows that the commutator ideal I of A is a homogeneous ideal: We have for any two elements $x, y \in A$ with homogeneous decompositions $x = \sum_{d \geq 0} x_d$ and $y = \sum_{d \geq 0} y_d$ that

$$[x, y] = \left[\sum_{d \geq 0} x_d, \sum_{d \geq 0} y_d\right] = \sum_{d, d' \geq 0} [x_d, y_{d'}]$$

by the bilinearity of the commutator $[-,-]$. We therefore find that the commutator ideal

$$I = ([x, y] \mid x, y \in A) = ([x', y'] \mid d, d' \geq 0, x' \in A_d, y' \in A_{d'})$$

is generated by homogeneous elements, and is hence homogeneous. It follows that the quotient algebra A/I inherits a grading from A; to be more precise, we have that $A/I = \bigoplus_{d \geq 0} \pi(A_d)$, where $\pi\colon A \to A/I$ denotes the canonical projection.

Remark 5. Let A be a k-algebra and let I be the commutator ideal of A. Then the quotient algebra A/I is commutative, and I is the smallest ideal in A with this property. Indeed, it holds for any two-sided ideal $J \subseteq A$ that

$$\begin{aligned}
& A/J \text{ is commutative} \\
\iff{} & x'y' = y'x' \text{ for all } x', y' \in A/J \\
\iff{} & \overline{x}\,\overline{y} = \overline{y}\,\overline{x} \text{ for all } x, y \in A \\
\iff{} & \overline{xy - yx} = 0 \text{ for all } x, y \in A \\
\iff{} & [x, y] \in J \text{ for all } x, y \in A \\
\iff{} & I \subseteq J\,.
\end{aligned}$$

[1] It is in fact a Lie bracket, and one of the prototypical examples of such.

It follows (both from this result itself, and from a similiar calculation) that if C is a commutative k-algebra then there exists for every algebra homomorphism $f\colon A \to C$ a unique algebra homomorphism $\overline{f}\colon A/I \to C$ which makes the triangle

$$\begin{tikzcd}
A/I \arrow[r, dashed, "\overline{f}"] & C \\
A \arrow[u] \arrow[ur, "f"'] &
\end{tikzcd}$$

commute. This gives rise to a one-to-one correspondence

$$\begin{aligned}
&\{\text{algebra homomorphisms } f\colon A \to C\} \\
\longleftrightarrow\, &\{\text{algebra homomorphisms } \overline{f}\colon A/I \to C\}\,.
\end{aligned}$$

(We have thus constructed a left adjoint to the forgetful functor k-**CommAlg** $\to$ k-**Alg**. The author encourages the reader to compare this construction to the abelianization $G/[G,G]$ of a group G.) Hence the quotient algebra A/I is the "most general way" to make the algebra A commutative.

We now return to the original algebra $\mathrm{T}(V)$ and its commutator ideal I. Thinking about the algebra $\mathrm{T}(V)$ as the "most general way of making the vector space V into a k-algebra" and about the quotient $\mathrm{T}(V)/I$ as the "most general way to make the algebra $\mathrm{T}(V)$ commutative", we may guess that the quotient algebra $\mathrm{T}(V)/I$ should be the "most general way to make the vector space V into a commutative k-algebra". In this way we see that the quotient algebra $\mathrm{T}(V)/I$ ought to be the symmetric algebra $\mathrm{S}(V)$.

Indeed, the inclusion $i\colon V \to \mathrm{S}(V)$ is a k-linear map and hence induces by the universal property of the tensor algebra $\mathrm{T}(V)$ an algebra homomorphism

$$\tilde{\varphi}\colon\; \mathrm{T}(V) \to \mathrm{S}(V)\,,$$

which is given on simple tensors by

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_n) = v_1 \cdots v_n\,.$$

It follows from $\mathrm{S}(V)$ being commutative that φ extends to an algebra homomorphism

$$\varphi\colon\; \mathrm{T}(V)/I \to \mathrm{S}(V)\,,$$

which is given by

$$\varphi(\overline{v_1 \otimes \cdots \otimes v_n}) = v_1 \cdots v_n$$

for all $v_1, \dots, v_n$. It follows on the other hand from $\mathrm{T}(V)/I$ being commutative that the k-linear map

$$j\colon V \to \mathrm{T}(V) \to \mathrm{T}(V)/I\,, \quad v \mapsto \overline{v}$$

induces an algebra homomorphism

$$\psi\colon\; \mathrm{S}(V) \to \mathrm{T}(V)/I\,,$$

which is on monomials given by

$$\psi(v_1 \cdots v_n) = \overline{v_1} \cdots \overline{v_n} = \overline{v_1 \otimes \cdots \otimes v_n}$$

for all $v_1, \dots, v_n \in V$. The algebra homomorphisms φ and ψ are mutually inverse on vector space generators (the residue classes $\overline{v_1 \otimes \cdots \otimes v_n}$ for $\mathrm{T}(V)/I$ and the monomials $v_1 \cdots v_n$ for $\mathrm{S}(V)$), and are hence mutually inverse. In other words, φ is an isomorphism with $\varphi^{-1} = \psi$.

Remark 6. One could also reformulate the above discussion by noting that for every commutative k-algebra C we have bijections

$$\begin{aligned} &\{\text{algebra homomorphisms } \mathrm{T}(V)/I \to C\} \\ \cong &\{\text{algebra homomorphisms } \mathrm{T}(V) \to C\} \\ \cong &\{\text{linear maps } V \to C\} \\ \cong &\{\text{algebra homomorphisms } \mathrm{S}(V) \to C\} \end{aligned}$$

by the universal property of the quotient algebra $\mathrm{T}(V)/I$, the tensor algebra $\mathrm{T}(V)$, and the symmetric algebra $\mathrm{S}(V)$. These bijections are actually natural in C, and hence give a natural isomorphism between convariant Hom-functors

$$\mathrm{Hom}_{k\text{-}\mathbf{CommAlg}}(\mathrm{T}(V)/I, -) \cong \mathrm{Hom}_{k\text{-}\mathbf{CommAlg}}(\mathrm{S}(V), -)\,.$$

It now follows from Yoneda's lemma that $\mathrm{T}(V)/I \cong \mathrm{S}(V)$.

Note that the constructed homomorphism $\tilde{\varphi}\colon \mathrm{T}(V) \to \mathrm{S}(V)$ maps the homogeneous component $\mathrm{T}(V)_d = V^{\otimes d}$ onto the homogeneous component $\mathrm{S}^d(V)$, and is therefore a homomorphism of graded k-algebras. It follows that the induced isomorphism $\varphi\colon \mathrm{T}(V)/I \to \mathrm{S}(V)$ is also a homomorphism of k-algebras because

$$\varphi((\mathrm{T}(V)/I)_d) = \varphi(\pi(\mathrm{T}(V)_d)) = \tilde{\varphi}(\mathrm{T}(V)_d) = \mathrm{S}^d(V)\,,$$

where $\pi\colon \mathrm{T}(V) \to \mathrm{T}(V)/I$ denotes the canonical projection. The algebra isomorphism φ is therefore already an isomorphism of graded k-algebras.

Note that if $(v_i)_{i \in I}$ is a basis of V then we may further identify the symmetric algebra $\mathrm{S}(V)$ with the poylnomial ring $k[X_i \mid i \in I]$. Indeed, there exist a unique linear map

$$f\colon V \to k[X_i \mid i \in I]$$

with $f(v_i) = X_i$ for every $i \in I$, and this linear maps induces by the universal property of the symmetric algebra $\mathrm{S}(V)$ an algebra homomorphism

$$\alpha\colon \mathrm{S}(V) \to k[X_i \mid i \in I]$$

with $\alpha(v_i) = X_i$ for every $i \in I$. It follows on the other hand from the universal property of the polynomial ring $k[X_i \mid i \in I]$ that there exist a unique algebra homomorphism

$$\beta\colon k[X_i \mid i \in I] \to \mathrm{S}(V)$$

with $\beta(X_i) = v_i$ for every $i \in I$. As the symmetric algebra $\mathrm{S}(V)$ is generated by the basis elements v_i, and the polynomial ring $k[X_i \mid i \in I]$ is generated by the variables X_i, we find that φ and ψ are mutually inverse on a set of algebra generators, and are hence mutually inverse.

The isomorphism α maps for every $d \geq 0$ the homogeneous component $\mathrm{S}^d(V)$ onto the homogeneous component $k[X_i \mid i \in I]_d$, and is hence already an isomorphism of graded k-algebras. We therefore find that

$$\mathrm{T}(V)/I \cong \mathrm{S}(V) \cong k[X_i \mid i \in I]$$

as graded k-algebras.

(c)

This is just the filtration associated to the grading of the tensor algebra from part (a) of the exercise.

(d)

We won't give a complete solution to this problem, but instead state the relevant results and sketch how one can proceed to prove them:

Let A be an associative algebra and for all $x, y \in A$ let

$$[x, y] \coloneqq xy - yx$$

be the their commutator. We can then consider the algebra

$$U(A) \coloneqq \mathrm{T}(A)/I$$

where $I \subseteq \mathrm{T}(A)$ is the two-sided ideal given by

$$I \coloneqq (x \otimes y - y \otimes x - [x, y] \mid x, y \in A)\,.$$

(The ideal I is built in precisely such a way that

$$[\overline{x}, \overline{y}]_{(\mathrm{T}(V)/I)} = \overline{x}\,\overline{y} - \overline{y}\,\overline{x} = \overline{x \otimes y - y \otimes x} = \overline{[x, y]_A}$$

for all $x, y \in A$, i.e. such that the commutator in A coincides with the one in $\mathrm{T}(V)/I$.) We then have the following classical result:

Theorem 7 (Poincaré–Birkhoff–Witt)**.** Let A be a k-algebra and let $(x_i)_{i \in I}$ be a basis of A where $(I, \leq)$ is a linearly ordered set. Then the ordered monomials

$$x_{i_1}^{n_1} \cdots x_{i_r}^{n_r}$$

with $r \geq 0$, $n_1, \dots, n_r \geq 1$ and $i_1 < \cdots < i_r$ form a basis of $U(A)$. (Here we write for $x \in A$ the resulting element $\overline{x} \in U(A)$ by abuse of notation again as x.)

Remark 8. The Poincaré–Birkhoff–Witt theorem actually holds for every Lie algebra $\mathfrak{g}$ over the field k, where one defines the *universal enveloping algebra* of $\mathfrak{g}$ as

$$\mathcal{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g})\,.$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the "most general" k-algebra which contains $\mathfrak{g}$ as a Lie subalgebra. As a functor, $\mathcal{U}$ it is left adjoint to the forgetful functor k-**Alg** $\to$ k-**Lie**.

The (standard) proof of the Poincaré–Birkhoff–Witt theorem proceeds similar to the proof that the monomials $x^n y^m$ with $n, m \geq 0$ form a basis of the first Weyl algebra $A_1 \cong k\langle x, y\rangle/(yx - xy - 1)$:[2]

- One first notes that the algebra $U(A) = \mathrm{T}(A)/I$ is spanned by the residue classes

  $$x_{i_1} \cdots x_{i_n} \qquad \text{where } n \geq 0 \text{ and } i_1, \dots, i_n \in I \tag{4}$$

  because the tensor algebra $\mathrm{T}(V)$ has the simple tensors $x_{i_1} \otimes \cdots \otimes x_{i_n}$ with $n \geq 0$ and $i_1, \dots, i_n \in I$ as a basis. One can then use the rearrangement rule

  $$yx = \overline{y \otimes x} = \overline{x \otimes y} + \overline{[y, x]} = xy + [y, x]$$

  for $x, y \in A$ to show that every monomial of the form (4) can be written as a linear combination of the monomials

  $$x_{i_1} \cdots x_{i_n} \qquad \text{where } n \geq 0 \text{ and } i_1, \dots, i_n \in I \text{ with } i_1 \leq \cdots \leq i_n\,. \tag{5}$$

  (Note that this is precisely the proposed basis from the Poincaré–Birkhoff–Witt theorem.)

- The hard part is to show that the monomials (4) are linearly independent. For this one uses the trick introduced in the tutorial: One constructs an $U(A)$-module M with k-basis

  $$X_{i_1} \cdots X_{i_r} \qquad \text{where } n \geq 0 \text{ and } i_1, \dots, i_n \in I \text{ with } i_1 \leq \cdots \leq i_n \tag{6}$$

  such that the action of every x_i on the basis (6) behaves the same as the action of x_i on (5). By using the linear independence of (6) it can then be conluded that (5) is also linearly independent. (The construction of M is rather complicated, and we will not attempt to do this here.)

The above version of the Poincaré–Birkhoff–Witt theorem is also known as the "concrete version", and it is equivalent to the following "abstract version":

Theorem 9 (Poincaré–Birkhoff–Witt)**.** Let A be a k-algebra and let $\pi\colon \mathrm{T}(A) \to U(A)$ be the canonical projection (which is a homomorphism of filtered k-algebra). Then the homomorphisms of graded k-algebras

$$\mathrm{gr}(\pi)\colon \mathrm{T}(A) \to \mathrm{gr}(U(A)) \qquad \text{and} \qquad \mathrm{T}(A) \to \mathrm{S}(A)$$

have the same kernel, and hence induce an isomorphism of k-algebras

$$\mathrm{gr}(U(A)) \cong \mathrm{S}(A)\,.$$

[2]One can in fact derive the result about the Weyl algebra from the Poincaré–Birkhoff–Witt theorem.

One can then also further identify the symmetric algebra $\mathrm{S}(A)$ with a polynomial ring $k[X_i \mid i \in I]$ (as graded k-algebras) by choosing a basis $(a_i)_{i\in I}$ of A. It follows in particular for $A = \mathrm{M}_n(k)$ that

$$\mathrm{gr}(U(\mathrm{M}_n(k))) \cong \mathrm{S}(\mathrm{M}_n(k)) \cong k[X_1, \dots, X_{n^2}] .$$

Exercise 4.

(a)

It follows from the Newton identities of part (b) inductively that

$$\begin{aligned}
p_1 &= e_1 , \\
p_2 &= e_1 p_1 - 2e_2 \\
&= e_1^2 - 2e_2 , \\
p_3 &= e_1 p_2 - e_2 p_1 + 3e_3 \\
&= e_1(e_1^2 - 2e_2) - e_2 e_1 + 3e_3 \\
&= e_1^3 - 3e_1 e_2 + 3e_3 , \\
p_4 &= e_1 p_3 - e_2 p_2 + e_3 p_1 - 4e_4 \\
&= e_1(e_1^3 - 3e_1 e_2 + 3e_3) - e_2(e_1^2 - 2e_2) + e_3 e_1 - 4e_4 \\
&= e_1^4 - 3e_1^2 e_2 + 3e_1 e_3 - e_1^2 e_2 + 2e_2^2 + e_1 e_3 - 4e_4 \\
&= e_1^4 - 4e_1^2 e_2 + 4e_1 e_3 + 2e_2^2 - 4e_4 .
\end{aligned}$$

(b)

We consider the generating series

$$E(t) = \sum_{r \geq 0} e_r t^r \qquad \text{and} \qquad P(t) = \sum_{r \geq 0} p_{r+1} t^r ,$$

for which we know from the lecture that

$$E(t) = \prod_{i=1}^n (1 + X_i t) \qquad \text{and} \qquad P(t) = \sum_{i=1}^n \frac{X_i}{1 - X_i t} .$$

It follows that

$$E'(t) = \sum_{i=1}^n X_i \prod_{j \neq i} (1 + X_j t) = \sum_{i=1}^n \frac{X_i}{1 + X_i t} \prod_{j=1}^n (1 + X_j t) = P(-t) E(t) .$$

By comparing the $(r-1)$-th coefficients of both $E'(t)$ and $P(-t)E(t)$ we find that

$$\begin{aligned}
& r e_r \\
&= p_1 e_{r-1} - p_2 e_{r-2} + \cdots + (-1)^r p_{r-1} e_1 + (-1)^{r+1} p_r e_0 \\
&= p_1 e_{r-1} - p_2 e_{r-2} + \cdots + (-1)^r p_{r-1} e_1 + (-1)^{r+1} p_r ,
\end{aligned}$$

which can now be rearranged to the desired equality.

(c)

We denote the given matrix on the right hand side by M_r. We give two possible solutions:

First Solution

We will show that $\det(M_r) = p_r$ by induction on $r \geq 1$. For $r = 1$ we have that

$$\det(M_1) = \det \begin{bmatrix} e_1 \end{bmatrix} = e_1 = p_1 \, .$$

For general m we expand the determinant of M_r with respect to the last row, i.e. the r-th row. The $(r,1)$-th minor of M_r is given by

$$\det(M_r^{(r,1)}) = \det \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ e_{r-2} & \cdots & 1 \end{bmatrix} = 1 \, ,$$

and for every $j = 2, \dots, r$ the (r,j)-th minor of M_r is by induction hypotheses given by

$$\det(M_r^{(r,j)}) = \det \begin{bmatrix} M_{j-1} & & & \\ * & 1 & & \\ * & \vdots & \ddots & \\ * & * & \cdots & 1 \end{bmatrix} = \det(M_{j-1}) = p_{j-1} \, .$$

It follows that

$$\begin{aligned} &\det(M_r) \\ &= (-1)^{r+1} r e_r \det(M_r^{(r,1)}) + (-1)^{r+2} e_{r-1} \det(M_r^{(r,2)}) + \cdots + (-1)^{2r} e_1 \det(M_r^{(r,r)}) \\ &= (-1)^{r+1} r e_r + (-1)^{r+2} e_{r-1} p_1 + \cdots + (-1)^{2r} e_1 p_{r-1} \\ &= e_1 p_{r-1} - e_2 p_{r-2} + \cdots + (-1)^{r+1} r e_r \\ &= p_r \end{aligned}$$

by the Newton identities from part (b) of the exercise.

Second Solution

We may rewrite the Newton identities from part (b) as

$$r e_r = (-1)^{r-1} p_r + (-1)^{r-2} p_{r-1} e_1 + \cdots + p_1 e_{r-1} \, ,$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & & & & & \\ e_1 & 1 & & & & \\ e_2 & e_1 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ e_{r-2} & e_{r-3} & e_{r-4} & \cdots & 1 & \\ e_{r-1} & e_{r-2} & e_{r-3} & \cdots & e_1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ -p_2 \\ p_3 \\ \vdots \\ (-1)^{r-2}p_{r-1} \\ (-1)^{r-1}p_r \end{bmatrix} = \begin{bmatrix} e_1 \\ 2e_2 \\ 3e_3 \\ \vdots \\ (r-1)e_{r-1} \\ re_r \end{bmatrix}.$$

The matrix A on the left hand side of this equation is invertible (because it has determinant 1). It follows from Cramer's rule and $\det(A) = 1$ that

$$\begin{aligned} (-1)^{r-1}p_r &= \det \begin{bmatrix} 1 & & & & & e_1 \\ e_1 & 1 & & & & 2e_2 \\ e_2 & e_1 & 1 & & & 3e_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ e_{r-2} & e_{r-3} & e_{r-4} & \cdots & 1 & (r-1)e_{r-1} \\ e_{r-1} & e_{r-2} & e_{r-3} & \cdots & e_1 & re_r \end{bmatrix} \\ &= (-1)^{r-1} \det \begin{bmatrix} e_1 & 1 & & & & \\ 2e_2 & e_1 & 1 & & & \\ 3e_3 & e_2 & e_1 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ (r-1)e_{r-1} & e_{r-2} & e_{r-3} & e_{r-4} & \cdots & 1 \\ re_r & e_{r-1} & e_{r-2} & e_{r-3} & \cdots & e_1 \end{bmatrix}. \end{aligned}$$