Exercises in Foundations in Representation Theory

Exercise Sheet 12

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Exercise 1.

The forgetful fuctor ${}_{\mathbb{Z}}(-)\colon k\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod}$ is exact with

$$ {}_{\mathbb{Z}}\operatorname{Hom}_A(M,-) = {}_{\mathbb{Z}}(-) \circ \operatorname{Hom}_A(M,-) $$

This exercise therefore follows from Exercise 1 of Exercise Sheet 11.

Exercise 2.

(i)

Let A be an abelian group. Then there exists for any sufficiently large index set I a surjective group homomorphism $p_0\colon \mathbb{Z}^{\oplus I} \to A$. The kernel $\ker(p_0) \subseteq \mathbb{Z}^{\oplus I}$ is again free abelian (because subgroups of free abelian groups are again free abelian). Hence $\ker(p_0) \cong \mathbb{Z}^{\oplus J}$ for some suitable index set J. This gives an exact sequence

$$ \cdots \to 0 \to 0 \to \mathbb{Z}^{\oplus J} \to \mathbb{Z}^{\oplus I} \xrightarrow{p_0} A \to 0\,. \tag{1} $$

The abelian groups $P_0 \coloneqq \mathbb{Z}^{\oplus I}$ and $P_1 \coloneqq \mathbb{Z}^{\oplus J}$ are projective since they are free abelian. The exact sequence (1) is therefore a projective resolution for A as desired.

(ii)

For $n = 0$ the abelian group $\mathbb{Z}/n = \mathbb{Z}$ has the projective resolution

$$ \underbrace{\cdots \to 0 \to 0 \to \mathbb{Z}}_{P_\bullet} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0\,. $$

The resulting chain complex

$$ \operatorname{Hom}_{\mathbb{Z}}(P_\bullet, \mathbb{Z}/m) = \big(\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \to 0 \to 0 \to \cdots\big) $$

is isomorphic to the chain complex

$$\mathbb{Z}/m \to 0 \to 0 \to \cdots$$

By taking the cohomology of this cochain complex we find that

$$(\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m))(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m & \text{for } i = 0\,, \\ 0 & \text{for } i > 0\,. \end{cases}$$

(One can also see this by using that $\mathrm{R}^0 \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m) \cong \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m)$, and also that $(\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m))(\mathbb{Z}) = 0$ for $i > 0$ because $\mathbb{Z}$ is free abelian and hence projective in $\mathbf{Ab}$.)

For $n \neq 0$ the abelian group $\mathbb{Z}/n$ has the projective resolution

$$\underbrace{\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}}_{P_\bullet} \xrightarrow{p_0} \mathbb{Z}/n \to 0\,.$$

The resulting chain complex

$$\operatorname{Hom}_{\mathbb{Z}}(P_\bullet, \mathbb{Z}/m) = \bigl(\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \xrightarrow{n} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \to 0 \to 0 \to \cdots\bigr)$$

is isomorphic to the chain complex

$$\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m \to 0 \to 0 \to \cdots$$

By taking the zeroeth cohomology of this cochain complex we find that

$$\begin{aligned} (\mathrm{R}^0 \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m))(\mathbb{Z}/n) &\cong \{x \in \mathbb{Z}/m \mid nx = 0\} \\ &= n\text{-torsion of } \mathbb{Z}/m \\ &= \gcd(n,m)\text{-torsion of } \mathbb{Z}/m \cong \begin{cases} \mathbb{Z}/\gcd(n,m) & \text{if } m \neq 0\,, \\ 0 & \text{if } m = 0\,. \end{cases} \end{aligned}$$

By taking the first cohomology we find that

$$(\mathrm{R}^1 \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m))(\mathbb{Z}/n) \cong n \cdot \mathbb{Z}/m = \gcd(n,m) \cdot \mathbb{Z}/m \cong \mathbb{Z}/\operatorname{lcm}(n,m)\,.$$

We also find that

$$(\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m))(\mathbb{Z}/n) = 0$$

for every $i \geq 2$.

(iii)

For $n = 0$ the abelian group $\mathbb{Z}/n = \mathbb{Z}$ has the projective resolution

$$\underbrace{\cdots \to 0 \to 0 \to \mathbb{Z}}_{P_\bullet} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0\,.$$

The resulting chain complex

$$P_\bullet \otimes_{\mathbb{Z}} (\mathbb{Z}/m) = \bigl(\cdots \to 0 \to 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/m)\bigr)$$

is isomophic to the chain complex

$$\cdots \to 0 \to 0 \to \mathbb{Z}/m\,.$$

By taking the homology of this chain complex we find that

$$(\mathrm{L}^i(- \otimes_{\mathbb{Z}} (\mathbb{Z}/m)))(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m & \text{for } i = 0\,, \\ 0 & \text{for } i > 0\,. \end{cases}$$

(One can also proceed as above, using that $\mathrm{L}^0(- \otimes_{\mathbb{Z}} (\mathbb{Z}/m)) \cong (- \otimes_{\mathbb{Z}} (\mathbb{Z}/m))$ and that $(\mathrm{L}^i(- \otimes_{\mathbb{Z}} (\mathbb{Z}/m)))(\mathbb{Z}) = 0$ for $i > 0$ because $\mathbb{Z}$ is free abelian and hence projective in **Ab**.

For $n \neq 0$ the abelian group $\mathbb{Z}/n$ has the projective resolution

$$\underbrace{\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}}_{P_\bullet} \xrightarrow{p_0} \mathbb{Z}/n \to 0\,.$$

The resulting chain complex

$$P_\bullet \otimes_{\mathbb{Z}} (\mathbb{Z}/m) = \bigl(\cdots \to 0 \to 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/m) \xrightarrow{n} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/m)\bigr)$$

is isomorphic to the chain complex

$$\cdots \to 0 \to 0 \to \mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m\,.$$

We find as before that

$$(\mathrm{L}^0(- \otimes_{\mathbb{Z}} (\mathbb{Z}/m)))(\mathbb{Z}/n) \cong n \cdot \mathbb{Z}/m = \gcd(n,m) \cdot \mathbb{Z}/m \cong \mathbb{Z}/\operatorname{lcm}(n,m)$$

and

$$\begin{aligned} (\mathrm{L}^1(- \otimes_{\mathbb{Z}} (\mathbb{Z}/m)))(\mathbb{Z}/n) &\cong \{x \in \mathbb{Z}/m \mid nx = 0\} \\ &= n\text{-torsion of } \mathbb{Z}/m \\ &\cong \begin{cases} \mathbb{Z}/\gcd(n,m) & \text{if } m \neq 0\,, \\ 0 & \text{if } m = 0\,, \end{cases} \end{aligned}$$

as well as

$$(\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m))(\mathbb{Z}/n) = 0$$

for every $i \geq 2$.

Exercise 3.

We observe that $A \cong \mathbb{Z}[\mathbb{Z}/n]$ is a free $\mathbb{Z}$-module with basis $1, t, \dots, t^{n-1}$. We will denote the powers of t in A as both t^i with $i \in \mathbb{N}$ and t^i with $i \in \mathbb{Z}/n$, where $t^{[i]} \coloneqq t^i$ for every $[i] \in \mathbb{Z}/n$. For such a linear combination $\sum_{i=0}^{n-1} a_i t^i$ with $a_i \in \mathbb{Z}$ we also set $a_{[i]} \coloneqq a_i$ for every $[i] \in \mathbb{Z}/n$, so that

$$\sum_{i=0}^{n-1} a_i t^i = \sum_{i \in \mathbb{Z}/n} a_i t^i \,.$$

(i)

We have

$$q(1-t) = (1 + t + \dots + t^{n-1})(1-t) = 1 - t^n = 0$$

which shows that

$$\dots \to A \xrightarrow{q} A \xrightarrow{1-t} A \xrightarrow{q} A \xrightarrow{1-t} A$$

is a complex. The A-module A is free and hence projective.

(ii)

We have for every $x \in A$ with $x = \sum_{i=0}^{n-1} a_i t^i$ that

$$\begin{aligned} qx = \sum_{i \in \mathbb{Z}/n} t^i \sum_{j \in \mathbb{Z}/n} a_j t^j &= \sum_{i,j \in \mathbb{Z}/n} a_j t^{i+j} = \sum_{j \in \mathbb{Z}/n} a_j \sum_{i \in \mathbb{Z}/n} t^{i+j} \\ &= \sum_{j \in \mathbb{Z}/n} a_j \sum_{k \in \mathbb{Z}/n} t^k = \left(\sum_{j \in \mathbb{Z}/n} a_j \right) q \,. \end{aligned}$$

It follows that

$$qx = 0 \iff \sum_{i=0}^{n-1} a_i = 0 \,.$$

This means that for $qx = 0$ we may write the tupel $(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n$ as a linear combination of $e_0 - e_1, \dots, e_{n-2} - e_{n-1}$, say

$$(a_0, \dots, a_{n-1}) = b_0(e_0 - e_1) + \dots + b_{n-2}(e_{n-2} - e_{n-1}) \,.$$

It follows that

$$x = \sum_{i=0}^{n-1} a_i t^i = \sum_{j=0}^{n-2} b_j (t^j - t^{j+1}) = \sum_{j=0}^{n-2} b_j t^j (1-t) \,.$$

This shows the exactness at P_{2n} for $n > 0$.

We also have that

$$
\begin{aligned}
(1-t)x = (1-t)\sum_{i\in\mathbb{Z}/n} a_i t^i &= \sum_{i\in\mathbb{Z}/n} a_i t^i - \sum_{i\in\mathbb{Z}/n} a_i t^{i+1} \\
&= \sum_{i\in\mathbb{Z}/n} a_i t^i - \sum_{i\in\mathbb{Z}/n} a_{i-1} t^i = \sum_{i\in\mathbb{Z}/n} (a_i - a_{i-1}) t^i
\end{aligned}
$$

and therefore that

$$
(1-t)x = 0 \iff a_0 = \cdots = a_{n-1}\,.
$$

This shows that for $(1-t)x = 0$ there exists some $a \in \mathbb{Z}$ with $x = aq$, which gives the exactness at P_{2n+1} for $n \geq 0$.

(iii)

The map p_0 is surjective because $p_0(n) = n$ for every $n \in \mathbb{Z}$. It is also A-linear because it is additive (i.e. $\mathbb{Z}$-linear) with

$$
p_0(tx) = p_0(x) = tp_0(x)
$$

for every $x \in A$.

We have for $x = \sum_{i\in\mathbb{Z}/n} a_i t^i \in A$ that $x \in \ker(p_0)$ if and only if $\sum_{i\in\mathbb{Z}/n} a_i = 0$. We have seen above that this is equivalent to $qx = 0$. The exactness at P_0 does therefore follow from the exactness of $A \xrightarrow{1-t} A \xrightarrow{q} A$.

(iv)

We use the constructed projective resolution

$$
\underbrace{\cdots \to A \xrightarrow{q} A \xrightarrow{1-t} A \xrightarrow{q} A \xrightarrow{1-t} A}_{P_\bullet} \xrightarrow{p_0} \mathbb{Z} \to 0\,.
$$

By applying the functor $- \otimes_A \mathbb{Z}$ we get the chain complex

$$
P_\bullet \otimes_A \mathbb{Z} = \left(\cdots \to A \otimes_A \mathbb{Z} \xrightarrow{q} A \otimes_A \mathbb{Z} \xrightarrow{1-t} A \otimes_A \mathbb{Z} \xrightarrow{q} A \otimes_A \mathbb{Z} \xrightarrow{1-t} A \otimes_A \mathbb{Z}\right),
$$

which is isomorphic to the chain complex

$$
\cdots \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}\,.
$$

By taking the homology of this chain complex we find that

$$
(\mathrm{L}^i(- \otimes_A \mathbb{Z}))(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0\,, \\ \mathbb{Z}/n & \text{for } i \geq 1 \text{ odd}\,, \\ 0 & \text{for } i \geq 2 \text{ even}\,. \end{cases}
$$

Exercise 4.

(i)

We can use the (very explicit) standard projective resolution (from Theorem 6.11).

(ii)

The short exact sequence

$$0 \to P(2)^{\oplus 2} \to P(1) \oplus P(2)^{\oplus 2} \to M \to 0$$

results in a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{End}(M) \arrow[r] & \operatorname{Hom}(P(1) \oplus P(2)^{\oplus 2}, M) \arrow[r] & \operatorname{Hom}(P(2)^{\oplus 2}, M) \arrow[dlll, rounded corners] \\
(\mathrm{R}^1 \operatorname{Hom}(-, M))(M) \arrow[r] & (\mathrm{R}^1 \operatorname{Hom}(-, M))(P(1) \oplus P(2)^{\oplus 2}) \arrow[r] & \cdots &
\end{tikzcd}$$

The representation $P(1) \oplus P(2)^{\oplus 2}$ is again projective, whence

$$(\mathrm{R}^1 \operatorname{Hom}(-, M))(P(1) \oplus P(2)^{\oplus 2}) = 0\,.$$

We therefore have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{End}(M) \arrow[r] & \operatorname{Hom}(P(1) \oplus P(2)^{\oplus 2}, M) \arrow[r] & \operatorname{Hom}(P(2)^{\oplus 2}, M) \arrow[dlll, rounded corners] \\
(\mathrm{R}^1 \operatorname{Hom}(-, M))(M) \arrow[r] & 0 & &
\end{tikzcd}$$

It follows that

$$\begin{aligned}
&\dim\,(\mathrm{R}^1 \operatorname{Hom}(-, M))(M) \\
=\ &\dim \operatorname{Hom}(P(2)^{\oplus 2}, M) - \dim \operatorname{Hom}(P(1) \oplus P(2)^{\oplus 2}, M) + \dim \operatorname{End}(M) \\
=\ &2 \dim M_2 - (\dim M_1 + 2 \dim M_2) + \dim \operatorname{End}(M) \\
=\ &\dim \operatorname{End}(M) - 1\,,
\end{aligned}$$

where we use that

$$\operatorname{Hom}(N_1 \oplus N_2, M) \cong \operatorname{Hom}(N_1, M) \oplus \operatorname{Hom}(N_2, M)$$

for any two representations N_1 and N_2, and

$$\operatorname{Hom}(P(i), M) \cong M_i$$

for every $i \in Q_0$.