Exercises in
Foundations in Representation Theory

Exercise Sheet 5

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Exercise 1.

(i)

The given statement is **false**. We give some counterexamples:

1) Let B be any noncommutative k-algebra and consider the k-algebra
$$A \coloneqq B \times B^{\mathrm{op}} .$$
Then A is again noncommutative, but
$$A^{\mathrm{op}} = (B \times B^{\mathrm{op}}) = B^{\mathrm{op}} \times (B^{\mathrm{op}})^{\mathrm{op}} = B^{\mathrm{op}} \times B \cong B \times B^{\mathrm{op}} = A .$$
We can also do the same construction with the tensor product $\otimes_k$ instead of the product $\times$. (Note that B can be identified with a subalgebra of $B \otimes_k B^{\mathrm{op}}$ via $b \mapsto b \otimes 1$ because k is a field. It hence follows from the noncommutativity of B that A is also noncommutative.)

2) If B is any commutative k-algebra then
$$\mathrm{M}_n(B)^{\mathrm{op}} \cong \mathrm{M}_n(B^{\mathrm{op}}) = \mathrm{M}_n(B)$$
as k-algebras, where the first isomorphism is given by the matrix transpose $M \mapsto M^T$. But the k-algebra $\mathrm{M}_n(B)$ is noncommutative whenever $n \geq 2$ and $B \neq 0$.

3) It holds for any group G that
$$k[G]^{\mathrm{op}} = k[G^{\mathrm{op}}] \cong k[G]$$
as k-algebras, where the second isomorphism is induced by the group isomorphism
$$G^{\mathrm{op}} \to G , \quad g \mapsto g^{-1} .$$
But if G is noncommutative then $k[G]$ is also noncommutative.

4) Consider the quiver Q with a single vertex and and two distinct arrow α and β:

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[loop right] \end{tikzcd}$$

Then $kQ \cong k\langle \alpha, \beta \rangle$ is the free k-algebra on two generators α and β, which in particular not commutative (because α and β don't commute). But

$$(kQ)^{\mathrm{op}} \cong k(Q^{\mathrm{op}}) = kQ$$

as k-algebras because $Q = Q^{\mathrm{op}}$.

5) Let $\mathfrak{g}$ be a Lie algebra over a field k. Then the Lie algebra isomorphism

$$\mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}\,, \quad x \mapsto -x$$

extends to an algebra isomorphism

$$\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}^{\mathrm{op}}) \cong \mathcal{U}(\mathfrak{g})^{\mathrm{op}}\,.$$

(Here we denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.) But if $\mathfrak{g}$ is not abelian (as most Lie algebras tend to be) then $\mathcal{U}(\mathfrak{g})$ is noncommutative (because $\mathfrak{g}$ is a Lie subalgebra of $\mathcal{U}(\mathfrak{g})$).

(ii)

The statement is **true**: One of the $\otimes$-Hom-adjunctions states that the map

$$\mathrm{Hom}_B(N \otimes_A M, P) \to \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))\,, \quad f \mapsto [m \mapsto [n \mapsto f(m \otimes n)]$$

is a well-defined isomorphism of k-vector space.

(iii)

The statement is **true**: The Yoneda embedding $Y \colon \mathcal{C} \to \mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ is fully faithful and thus reflects isomorphisms. It therefore follows for any two ojects $X, X' \in \mathrm{Ob}(\mathcal{C})$ that the map

$$\{\text{isomorphisms } X \to X'\} \to \{\text{natural isomorphisms } h^{X'} \to h^X\}\,, \quad f \mapsto f^*$$

is a well-defined bijection. It therefore follows from $h^X \cong h^{X'}$ that also $X \cong X'$

(iv)

The statement is **true**: Suppose more generally that $\mathcal{C}$ is a category which admits an initinal object I. Then the functor $F\colon \mathcal{C} \to \mathbf{Set}$ given by $F(A) = \{*\}$ is represented by I: There exists for every object $X \in \mathrm{Ob}(\mathcal{C})$ a (unique) bijection

$$\eta_X : h^I(X) \to F(X)$$

because both sets are singletons. The square

$$\begin{tikzcd} h^I(X) \arrow[r, "f_*"] \arrow[d, "\eta_X"'] & h^I(X') \arrow[d, "\eta_{X'}"] \\ F(X) \arrow[r, "F(f)"] & F(X') \end{tikzcd}$$

commutes for every morphism $f\colon X \to X'$ in $\mathcal{C}$ because the set $F(X')$ is a singleton. This shows that $\eta\colon h^I \to F$ is a natural isomorphism, hence that I represents the functor F.

In our case $\mathcal{C} = k$-$\mathbf{Alg}$ the initial object—and hence representing object of F—is given by the k-algebra k.

(v)

The statement is **true**: Let $(X^\lambda)_{\lambda\in\Lambda}$ be a family of representations of Q over k. We can then define a new representation $\prod_{\lambda\in\Lambda} X^\lambda$ of Q over k by setting

$$\left(\prod_{\lambda\in\Lambda} X^\lambda\right)_i := \prod_{\lambda\in\Lambda} X_i^\lambda$$

for every $i \in Q_0$, and

$$\left(\prod_{\lambda\in\Lambda} X^\lambda\right)_\alpha := \prod_{\lambda\in\Lambda} X_\alpha^\lambda\colon \prod_{\lambda\in\Lambda} X_{s(\alpha)}^\lambda \to \prod_{\lambda\in\Lambda} X_{t(\alpha)}^\lambda$$

for every $\alpha \in Q_1$.

To construct the needed projections $p^\lambda\colon \prod_{\mu\in\Lambda} X^\mu \to X^\lambda$, let $\lambda \in \Lambda$ and for every $i \in Q_0$ let $p_i^\lambda\colon \prod_{\mu\in\Lambda} X_i^\mu \to X_i^\lambda$ be the canonical projection. Then the family $p^\lambda := (p_i^\lambda)_{i\in Q_0}$ is a morphism of representations $p^\lambda\colon \prod_{\mu\in\Lambda} X^\mu \to X^\lambda$ because the square

$$\begin{tikzcd} \prod_{\mu\in\Lambda} X_{s(\alpha)}^\mu \arrow[r, "\prod_{\mu\in\Lambda} X_\alpha^\mu"] \arrow[d, "p_{s(\alpha)}^\lambda"'] & \prod_{\mu\in\Lambda} X_{t(\alpha)}^\mu \arrow[d, "p_{t(\alpha)}^\lambda"] \\ X_{s(\alpha)}^\lambda \arrow[r, "X_\alpha^\lambda"] & X_{t(\alpha)}^\lambda \end{tikzcd}$$

commutes for every arrow $\alpha \in Q_1$.

Suppose now that Y is a representation of Q over k and that for every $\lambda \in \Lambda$ we are given a morphism of representations $f^\lambda \colon Y \to X^\lambda$.

We then have at every vertex $i \in Q_0$ a family $(f_i^\lambda)_{\lambda \in \Lambda}$ of k-linear maps $f_i^\lambda \colon Y_i \to X_i^\lambda$. It follows for every $i \in Q_0$ by the universal property of the product $\prod_{\lambda \in \Lambda} X_i^\lambda$ that there exists a unique k-linear map $f_i \colon Y_i \to \prod_{\lambda \in \Lambda} X_i^\lambda$ with $p_i^\lambda \circ f_i = f_i^\lambda$ for every $\lambda \in \Lambda$.

The family $f = (f_i)_{i \in Q_0}$ is a morphism of representations $f \colon Y \to \prod_{\lambda \in \Lambda} X^\lambda$, i.e. the square

$$\begin{tikzcd}
\prod_{\lambda \in \Lambda} X_{s(\alpha)}^\lambda \arrow[r, "\prod_{\lambda \in \Lambda} X_\alpha^\lambda"] & \prod_{\lambda \in \Lambda} X_{t(\alpha)}^\lambda \\
Y_{s(\alpha)} \arrow[u, "f_{s(\alpha)}"] \arrow[r, "Y_\alpha"] & Y_{t(\alpha)} \arrow[u, "f_{t(\alpha)}"']
\end{tikzcd} \tag{1}$$

commutes for every arrow $\alpha \in Q_1$. Indeed, we have for every $\mu \in \Lambda$ the following diagram:

$$\begin{tikzcd}
X_{s(\alpha)}^\mu \arrow[r, "X_\alpha^\mu"] & X_{t(\alpha)}^\mu \\
\prod_{\lambda \in \Lambda} X_{s(\alpha)}^\lambda \arrow[u, "p_{s(\alpha)}^\mu"] \arrow[r, "\prod_{\lambda \in \Lambda} X_\alpha^\lambda"] & \prod_{\lambda \in \Lambda} X_{t(\alpha)}^\lambda \arrow[u, "p_{t(\alpha)}^\mu"'] \\
Y_{s(\alpha)} \arrow[u, "f_{s(\alpha)}"] \arrow[r, "Y_\alpha"] \arrow[uu, bend left=60, "f_{s(\alpha)}^\mu"] & Y_{t(\alpha)} \arrow[u, "f_{t(\alpha)}"'] \arrow[uu, bend right=60, "f_{t(\alpha)}^\mu"']
\end{tikzcd}$$

The rounded triangles on the left and right commute by construction of f, the upper square commutes by construction of p^μ, and the outer square commutes because $f^\mu \colon Y \to X^\mu$ is a morphism of representations. It follows that

$$p_{t(\alpha)}^\mu \circ \left(\prod_{\lambda \in \Lambda} X_\alpha^\lambda \right) \circ f_{s(\alpha)} = p_{t(\alpha)}^\mu \circ f_{t(\alpha)} \circ Y_\alpha \,.$$

This holds for every $\mu \in \Lambda$, and so it follows that

$$\left(\prod_{\lambda \in \Lambda} X_\alpha^\lambda \right) \circ f_{s(\alpha)} = f_{t(\alpha)} \circ Y_\alpha \,.$$

This shows that the square (1) commutes, and hence that f is indeed a morphism of representations.

It holds that $p^\lambda \circ f = f^\lambda$ because

$$(p^\lambda \circ f)_i = p_i^\lambda \circ f_i = f_i^\lambda$$

at every vertex $i \in Q_0$ by construction of f.

If $\tilde{f} \colon Y \to \prod_{\lambda \in \Lambda} X^\lambda$ is another morphism of representations with $p^\lambda \circ \tilde{f} = f^\lambda$ for every $\lambda \in \Lambda$, then it follows at every vertex $i \in Q_0$ that

$$p_i^\lambda \circ f_i = f_i^\lambda = (p^\lambda \circ \tilde{f})_i = p_i^\lambda \circ \tilde{f}_i$$

for every $\lambda \in \Lambda$, and hence that $f_i = \tilde{f}_i$ by the uniqueness of the f_i. This shows the uniqueness of f.

This shows altogether that the constructed representation $\prod_{\lambda \in \Lambda} X^\lambda$ is together with the projections $p^\lambda \colon \prod_{\mu \in \Lambda} X^\mu \to X^\lambda$ a product of the family $(X^\lambda)_{\lambda \in \Lambda}$.

Remark 1. One can also somewhat avoid this explicit calculations:

1) We have that $\mathbf{Rep}_k(Q) \cong \mathbf{Fun}(\mathbf{Path}(Q), k\text{-}\mathbf{Mod})$ as categories, and one can argue that the functor category $\mathbf{Fun}(\mathbf{Path}(Q), k\text{-}\mathbf{Mod})$ inherits all the nice properties of the module category k-**Mod**. But we haven't (yet) shown in the lecture that a functor category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ between categories $\mathcal{C}$ and $\mathcal{D}$ inherits products (and more generally limits) from $\mathcal{D}$, and checking this by hand amounts to the above calculations.

2) One can argue that $\mathbf{Rep}_k(Q) \cong kQ\text{-}\mathbf{Mod}$, but kQ is for a general quiver Q (no finiteness conditions on Q are given on the exercise sheet) not a unital k-algebra, which may make some people uncomfortable.

(vi)

The statement is **false**: If the category **Field** would have a terminal object K then there would exist (unique) field homomorphisms $\mathbb{F}_2 \to K$ and $\mathbb{F}_3 \to K$. But then K would need to have both characteristic 2 and characteristic 3, which cannot be.

Exercise 2.

(i)

Lemma 2. Let (F, G, φ) be an adjunction from a category $\mathcal{C}$ to a category $\mathcal{D}$. Then F preserves epimorphisms and G preserves monomorphisms, i.e. if f is an epimorphism in $\mathcal{C}$ then $F(f)$ is an epimorphism in $\mathcal{D}$, and if g is a monomorphism in $\mathcal{D}$ then $G(g)$ is a monomorphism in $\mathcal{C}$.

Proof. For every morphism $f \colon X \to X'$ in $\mathcal{C}$ and every object $Y \in \mathrm{Ob}(\mathcal{D})$ the square

$$\begin{tikzcd} \mathcal{D}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] & \mathcal{C}(X, G(Y)) \\ \mathcal{D}(F(X'), Y) \arrow[r, "\varphi_{X',Y}"] \arrow[u, "F(f)^*"] & \mathcal{C}(X', G(Y)) \arrow[u, "f^*"'] \end{tikzcd} \tag{2}$$

commutes by the naturality of φ.

The horizontal arrows in (2) are bijections, so it follows that the map $F(f)^*$ is injective if (and only if) the map f^* is injective. If f is an epimorphism then the map $f^* \colon \mathcal{C}(X, G(Y)) \to \mathcal{C}(X', G(Y))$ is injective for all $Y \in \mathrm{Ob}(\mathcal{D})$, so it then follows that $F(f)^* \colon \mathcal{D}(F(X), Y) \to \mathcal{D}(F(X'), Y)$ is injective for all $Y \in \mathrm{Ob}(\mathcal{D})$. This is precisely what it means for $F(f)$ to be a epimorphism. This shows that the functor F preserves epimorphisms, and it can be shown in the same way that the functor G preserves monomorphisms. $\square$

Lemma 3. If $U\colon \mathcal{C} \to \mathcal{D}$ is a faithful functor then U reflects monomorphisms and epimorphisms, i.e. if $U(f)$ is a monomorphism (resp. epimorphism) then f is a monomorphism (resp. epimorphism).

Proof. Let $f\colon X \to Y$ be a morphism such that $U(f)$ is a monomorphism in $\mathcal{D}$. Let $g_1, g_2\colon W \to X$ be two morphisms in $\mathcal{C}$ with $f \circ g_1 = f \circ g_2$. Then

$$fg_1 = fg_2 \implies U(f)U(g_1) = U(f)U(g_2) \implies U(g_1) = U(g_2) \implies g_1 = g_2$$

because U is faithful. This shows that U reflects monomorphisms. That U also reflects epimorphisms can be shown in the same way. $\square$

Corollary 4. Let $\mathcal{C}$ be a category and let $U\colon \mathcal{C} \to \mathbf{Set}$ be a faithful functor. Let f be a morphism in $\mathcal{C}$. If $U(f)$ is injective then f is a monomorphism, and if $U(f)$ is surjective then f is an epimorphism. $\square$

It follows for the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ from Lemma 2 that it preserves monomorphisms and epimorphisms, because we have seen on the last exercise sheet that U is both a left adjoint and a right adjoint. It also follows from Corollary 4 that U reflects monomorphisms and epimorphisms, because U is faithful. This solves the exercise.

(ii)

Let $g_1, g_2\colon Y \to Z$ be morphisms in **Haus** with $g_1 \circ f = g_2 \circ f$. We can then consider the equivalizer

$$E = \{y \in Y \mid g_1(y) = g_2(y)\}\,.$$

The set E is closed in Y: It can be written as the preimage $E = (g_1, g_2)^{-1}(\Delta_Z)$ of the diagonal $\Delta_Z \subseteq Z \times Z$ under the continuous map $(g_1, g_2)\colon Y \to Z \times Z$, where the diagonal Δ_Z is closed in $Z \times Z$ because Z is Hausdorff. The equalizer E contains the image $f(X)$ because $g \circ f = g_2 \circ f$. But this image is dense in Y, and so it follows that

$$E = \overline{E} \supseteq \overline{f(X)} = Y\,.$$

This shows that $E = Y$ and hence that $g_1 = g_2$.

(iii)

Let more generally $(X, x_0), (Y, y_0)$ be pointed, connected topological spaces, let X be Hausdorff and let $f\colon (X, x_0) \to (Y, y_0)$ be a morphism in $\mathbf{Conn}_*$ for which the underlying continuous map $f\colon X \to Y$ is a local homeomorphism. Then f is a monomorphism in $\mathbf{Conn}_*$:

Let $g_1, g_2 \colon (Z, z_0) \to (X, x_0)$ be morphisms in $\mathbf{Conn}_*$ with $f \circ g_1 = f \circ g_2 =: h$. We then consider the equalizer

$$E := \{z \in Z \mid g_1(z) = g_2(z)\} .$$

We find as in the previous part of the exercise that the set E is closed in Z because X is Hausdorff.

The set E is also open: Let $z \in E$ and set $x := g_1(z) = g_2(z)$ and $y := f(x) = h(z)$. Let $V \subseteq Y$ be an open neighbourhood of y and let $U \subseteq X$ be an open neighbourhood of x such that f restricts to a homeomorphism $f' \colon U \to V$. The set $W := g_1^{-1}(U) \cap g_2^{-1}(U)$ is then an open neighbourhood of z in Z for which we get the following diagram:

$$\begin{tikzcd}
(W, z) \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[dr, "h"'] & (X, x) \arrow[d, "f'"] \\
& (V, y)
\end{tikzcd}$$

It follows from f' being a homeomorphism that already $g_1 \equiv g_2$ on W. This shows that E contais an open neighbourhood (namely W) around z.

The set E is also nonempty because it contains the base point z_0. This shows altogether that E is a nonempty subset of Z which is both closed and open. It follows from Z being connected that $E = Z$ and hence that $g_1 = g_2$.

Exercise 3.

(i)

(a) $\implies$ (d) There exists a unique morphism $Z \to Y$ for every object $Y \in \mathrm{Ob}(\mathcal{A})$; this holds in particular for $Y = Z$.

(d) $\implies$ (c) The morphisms id_Z and $0_{\mathcal{A}(Z,Z)}$ are both elements of the one-point set $\mathcal{A}(Z, Z)$, hence they are equal.

(c) $\implies$ (a) It holds for every morphism $f \colon Z \to Y$ in $\mathcal{A}$ that

$$f = f \circ \mathrm{id}_Z = f \circ 0_{\mathcal{A}(Z,Z)} = 0_{\mathcal{A}(X,Y)} .$$

This shows that the morphism $0_{\mathcal{A}(X,Y)}$ is the unique morphism $X \to Y$ in $\mathcal{A}$.

The implication circle (b) $\implies$ (d) $\implies$ (c) $\implies$ (b) can be shown in the same way.

(ii)

We denote the zero object of $\mathcal{A}$ by Z (because 0 is already overloaded enough). Then $\mathrm{id}_Z = 0_{\mathcal{A}(Z,Z)}$ by the previous part of the exercise, and therefore

$$0_{X,Y} = 0_{Z,Y} \circ 0_{X,Z} = 0_{Z,Y} \circ \mathrm{id}_Z \circ 0_{X,Z} = 0_{Z,Y} \circ 0_{\mathcal{A}(Z,Z)} \circ 0_{X,Z} = 0_{\mathcal{A}(X,Y)}$$

for all $X, Y \in \mathcal{A}(X, Y)$.

Exercise 4.

(i)

Let M be a field and let $f\colon M \to K = \mathbb{Q}(i)$ and $g\colon M \to L = \mathbb{Q}(\sqrt{2})$ be field homomorphisms. Then $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are field extensions of M. The subfield of $\mathbb{Q}(i)$ are $\mathbb{Q}(i)$, $\mathbb{Q}$, and the subfield of $\mathbb{Q}(\sqrt{2})$ are $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}$; the only isomorphic subfields of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are $\mathbb{Q} \subseteq \mathbb{Q}(i)$ and $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})$. Hence we find that $M = \mathbb{Q}$. There exist only a single field homorphism $\mathbb{Q} \to \mathbb{Q}$, namely the identity $\mathrm{id}_{\mathbb{Q}}$. Hence we find that $M = \mathbb{Q}$ and $f = g = \mathrm{id}_{\mathbb{Q}}$.

It follows that $P = \mathbb{Q}$ together with the inclusions $p\colon P \to \mathbb{Q}(i)$ and $q\colon P \to \mathbb{Q}(\sqrt{2})$ is a product of K and L in **Field**. (That there exist a *unique* morphism $M \to P$ follows from $\mathrm{id}_{\mathbb{Q}}$ follows from $\mathrm{id}_{\mathbb{Q}}$ being the only such homomorphism.)

(ii)

Suppose that a product P of K and L exists in the category **Field**. We find as above that P is isomorphic to a subfield of $\mathbb{Q}(i)$, and hence that either $P = \mathbb{Q}$ or $P = \mathbb{Q}(i)$. It follows from the existence of the (usual) diagonal homomorphism $\Delta_{\mathbb{Q}(i)}\colon \mathbb{Q}(i) \to P$ that P is a field extension of $\mathbb{Q}(i)$. We thus find that $P = \mathbb{Q}(i)$.

There exists precisely two field homomorphisms $\mathbb{Q}(i) \to \mathbb{Q}(i)$, the identity $\mathrm{id}_{\mathbb{Q}(i)}$ and the conjugation $c\colon \mathbb{Q}(i) \to \mathbb{Q}(i)$, $a + ib \mapsto a - ib$. If $p\colon P \to \mathbb{Q}(i)$ and $q\colon P \to \mathbb{Q}(i)$ are the structure morphisms belonging to the product P then it follows that p and q are isomorphisms, and that for every choice of field homomorphisms $f_1, f_2\colon \mathbb{Q}(i) \to \mathbb{Q}(i)$ there exists a unique field homomorphism $g\colon \mathbb{Q}(i) \to \mathbb{Q}(i)$ such that the diagram

$$\begin{tikzcd}
 & P \arrow[dl, bend right, "p"'] \arrow[dr, bend left, "q"] & \\
\mathbb{Q}(i) & \mathbb{Q}(i) \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[u, dashed, "g"'] & \mathbb{Q}(i)
\end{tikzcd}$$

commutes. But then

$$f_1 = pg = pq^{-1} f_2 \,,$$

which cannot holds for both the choice $f_1 = \mathrm{id}_{\mathbb{Q}(i)}$, $f_2 = \mathrm{id}_{\mathbb{Q}(i)}$ (for which we get $f_1 = f_2$) and also the choice $f_1 = c$, $f_2 = \mathrm{id}_{\mathbb{Q}(i)}$ (for which we get $f_1 \neq f_2$).

This shows that no product of K and L exist in **Field**.