Group 1, Wednesday 8–10, Tutor: Nils Waßmuth December 7, 2018

Exercises in
Foundations in Representation Theory

Exercise Sheet 8

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Exercise 1.

(i)

Lemma 1. Let $\mathcal{A}$ be an abelian category.

1) Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\varphi'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

be a commutative square in $\mathcal{A}$.[1] Let $k\colon \ker(f) \to X$ and $k'\colon \ker(f') \to X'$ be kernels of f and f'. Then there exist a unique morphism $\varphi''\colon \ker(f) \to \ker(f'')$ that makes the following diagram commute:

$$\begin{tikzcd} \ker(f) \arrow[r, "k"] \arrow[d, dashed, "\varphi''"'] & X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\varphi'"] \\ \ker(f') \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

[1]We may think about this commutative square a morphism $(\varphi, \varphi')\colon f \to f'$ in the morphism category of $\mathcal{A}$.

2) This induced morphism is functorial in the following sense: Let

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\varphi'"] \\
X' \arrow[r, "f'"] \arrow[d, "\psi"'] & Y' \arrow[d, "\psi''"] \\
X'' \arrow[r, "f''"] & Y''
\end{tikzcd}
$$

be a commutative diagram in $\mathcal{A}$ and let

$$
k\colon \ker(f) \to X\,, \qquad k'\colon \ker(f') \to X'\,, \qquad k''\colon \ker(f'') \to X''
$$

be kernels of f, f' and f''. Let $\varphi''\colon \ker(f) \to \ker(f')$ be the morphism induced by (φ, φ') and let $\psi''\colon \ker(f') \to \ker(f'')$ be the morphism induced by (ψ, ψ'). Then the composition $\psi''\varphi''\colon \ker(f) \to \ker(f'')$ is the morphism induced by $(\psi\varphi, \psi'\varphi')$.

Proof.

1) This follows from the universal property of the kernel $k'\colon \ker(f') \to X'$ of f' because

$$
f'\varphi k = \varphi' f k = \varphi' \circ 0 = 0\,.
$$

2) We have the following commutative diagram:

$$
\begin{tikzcd}
\ker(f) \arrow[r, "k"] \arrow[d, "\varphi'"] \arrow[dd, dashed, bend right, "\psi'\varphi'"'] & X \arrow[r, "f"] \arrow[d, "\varphi"] \arrow[dd, dashed, bend right, "\psi\varphi"'] & Y \arrow[d, "\varphi''"] \arrow[dd, dashed, bend right, "\psi''\varphi''"'] \\
\ker(f') \arrow[r, "k'"] \arrow[d, "\psi'"] & X' \arrow[r, "f'"] \arrow[d, "\psi"] & Y' \arrow[d, "\psi''"] \\
\ker(f'') \arrow[r, "k''"] & X'' \arrow[r, "f''"] & Y''
\end{tikzcd}
$$

The commutativity of the subdiagram

$$
\begin{tikzcd}
\ker(f) \arrow[r, "k"] \arrow[d, dashed, "\psi''\varphi''"'] & X \arrow[r, "f"] \arrow[d, "\psi\varphi"'] & Y \arrow[d, "\psi'\varphi'"] \\
\ker(f'') \arrow[r, "k''"] & X'' \arrow[r, "f''"] & Y'
\end{tikzcd}
$$

shows that $\psi''\varphi''$ satisfies the defining property of the morphisms $\ker(f) \to \ker(f'')$ induced by $(\psi\varphi, \psi'\varphi')$. $\square$

For every $n \in \mathbb{N}$ let $k_n \colon \ker(f_n) \to C_n$ be a kernel of $f_n \colon C_n \to D_n$. It follows by Lemma 1 from the commutativity of the square

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n"] & D_{n-1} \end{tikzcd}$$

that there exists a unique morphism $d'_n \colon \ker(f_n) \to \ker(f_{n-1})$ that makes the following square commute:

$$\begin{tikzcd} \ker(f_n) \arrow[r, dashed, "d'_n"] \arrow[d, "k_n"'] & \ker(f_{n-1}) \arrow[d, "k_{n-1}"] \\ C_n \arrow[r, "d_n"] & C_{n-1} \end{tikzcd} \tag{1}$$

The composition $d'_{n-1} d'_n$ is by Lemma 1 the unique morphism $\ker(f_n) \to \ker(f_{n-2})$ that makes the following diagram commute:

$$\begin{tikzcd} \ker(f_n) \arrow[r, dashed, "d'_n"] \arrow[d, "k_n"'] \arrow[rr, dashed, bend left, "d'_n d'_{n-1}"] & \ker(f_{n-1}) \arrow[r, dashed, "d'_{n-1}"] \arrow[d, "k_{n-1}"] & \ker(f_{n-2}) \arrow[d, "k_{n-2}"] \\ C_n \arrow[r, "d_n"] \arrow[rr, bend right, "0"'] & C_{n-1} \arrow[r, "d_{n-1}"] & C_{n-2} \end{tikzcd}$$

The zero morphism also makes this diagram commute, hence $d'_{n-1} d'_n = 0$. This shows that $\ker(f) = ((\ker(f_n))_{n \in \mathbb{Z}}, (d'_n)_{n \in \mathbb{Z}})$ is a chain complex. The commutativity of the square (1) tells us furtheromer that $k \coloneqq (k_n)_{n \in \mathbb{Z}}$ is a morphism of chain complexes $k \colon \ker(f) \to C_\bullet$.

The composition fk vanishes because $(fk)_n = f_n k_n = 0$ for every $n \in \mathbb{Z}$. Suppose that $g \colon B_\bullet \to C_\bullet$ is another morphism of chain complexs for which $fg = 0$. Then $0 = (fg)_n = f_n g_n$ for every $n \in \mathbb{Z}$, and it follows from the universal property of the kernel $k_n \colon \ker(f_n) \to C_n$ that there exist a unique morphism $h_n \colon B_n \to \ker(f_n)$ that makes the following diagram commute:

$$\begin{tikzcd} \ker(f_n) \arrow[r, "k_n"] & C_n \\ B_n \arrow[u, dashed, "h_n"] \arrow[ur, "g_n"'] & \end{tikzcd} \tag{2}$$

Then $h \coloneqq (h_n)_{n \in \mathbb{Z}}$ is a morphism of chain complexes: We have for every $n \in \mathbb{Z}$ the following diagram:

$$\begin{tikzcd}
B_n \arrow[r, dashed, "h_n"] \arrow[d, "d_n"] \arrow[rr, bend left, "g_n"] & \ker(f_n) \arrow[r, "k_n"] \arrow[d, "d'_n"] & C_n \arrow[d, "d_n"] \\
B_{n-1} \arrow[r, dashed, "h_{n-1}"] \arrow[rr, bend right, "g_{n-1}"'] & \ker(f_{n-1}) \arrow[r, "k_{n-1}"] & C_{n-1}
\end{tikzcd} \tag{3}$$

The right square commutes because k is a morphism of chain complexes, the outer square commutes because g is a morphism of chain complexes and the upper and lower triangles commute by choice of h_n and h_{n-1}. It follows that the left square commutes, because

$$k_{n-1}d'_n h_n = d_n k_n h_n = d_n g_n = g_{n-1}d_n = k_{n-1}h_{n-1}d_n$$

and hence $d'_n h_n = h_{n-1}d_n$ since k_{n-1} is a monomorphism.

That the morphism h_n makes for every $n \in \mathbb{Z}$ the triangle (2) commute gives altogether that the morphism of chain complexes $h\colon B_\bullet \to \ker(f)$ makes the following diagram commute:

$$\begin{tikzcd}
\ker(f) \arrow[r, "k"] & C_\bullet \\
B_\bullet \arrow[u, dashed, "h"] \arrow[ur, "g"'] &
\end{tikzcd}$$

The morphism h is unique with this property: If $h'\colon B_\bullet \to \ker(f)$ is another morphim of chain complexes with $kh' = g$ then $k_n h'_n = g_n$ for every $n \in \mathbb{Z}$, and hence the triangle

$$\begin{tikzcd}
\ker(f_n) \arrow[r, "k_n"] & C_n \\
B_n \arrow[u, dashed, "h'_n"] \arrow[ur, "g_n"'] &
\end{tikzcd}$$

commutes for every $n \in \mathbb{Z}$. It then follows for every $n \in \mathbb{Z}$ from the uniqueness of h_n that $h'_n = h_n$, and hence overall $h' = h$.

This shows altogether that the morphism of chain complexes $k\colon \ker(f) \to C_\bullet$ is a kernel of f. This explicit construction of the kernel also shows that kernels in $\mathbf{Ch}_\bullet(\mathcal{A})$ can be computed degree-wise.

Remark 2. We similary find that the cokernel of f is given by a chain complex

$$\operatorname{coker}(f) = ((\operatorname{coker}(f_n))_{n\in\mathbb{Z}}, (d''_n)_{n\in\mathbb{Z}})$$

together with a morphism of chain complexes $c\colon D_\bullet \to \operatorname{coker}(f)$ such that

- the morphism $c_n\colon D_n \to \operatorname{coker}(f_n)$ is for every $n \in \mathbb{Z}$ a cokernel of the morphism $f_n\colon C_n \to D_n$, and

- the differential $d''_n\colon \operatorname{coker}(f_n) \to \operatorname{coker}(f_{n-1})$ is for every $n \in \mathbb{Z}$ the unique morphism $\operatorname{coker}(f_n) \to \operatorname{coker}(f_{n-1})$ that makes the following square commute:

$$\begin{tikzcd}
D_n \arrow[r, "d_n"] \arrow[d, "c_n"'] & D_{n-1} \arrow[d, "c_{n-1}"] \\
\operatorname{coker}(f_n) \arrow[r, dashed, "d''_n"'] & \operatorname{coker}(f_{n-1})
\end{tikzcd}$$

- If $g\colon D_\bullet \to E_\bullet$ is another morphism of chain complexes with $gf = 0$, then the unique morphism of chain complexes $h\colon \operatorname{coker}(f) \to E_\bullet$ that makes the triangle

$$\begin{tikzcd}
D_\bullet \arrow[r, "c"] \arrow[dr, "g"'] & \operatorname{coker}(f) \arrow[d, dashed, "h"] \\
& E_\bullet
\end{tikzcd}$$

commute can be computed degree-wise, i.e. the component $h_n\colon \operatorname{coker}(f_n) \to E_n$ is for every $n \in \mathbb{Z}$ the unique morphism $\operatorname{coker}(f_n) \to E_n$ that makes the following triangle commute:

$$\begin{tikzcd}
D_n \arrow[r, "c"] \arrow[dr, "g_n"'] & \operatorname{coker}(f_n) \arrow[d, dashed, "h_n"] \\
& E_n
\end{tikzcd}$$

We can similarly calculate the image and coimage of f degree-wise

(ii)

Consider the case $\mathcal{A} = \mathbb{Z}$-**Mod** and the acylic chain complexes

$$\begin{aligned}
C_\bullet &= (\cdots \to 0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2 \to 0 \to \cdots)\,, \\
D_\bullet &= (\cdots \to 0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0 \to 0 \cdots)\,,
\end{aligned}$$

where i denotes the inclusion and p the canonical projection. We have0a morphism of chain complexes $f\colon C_\bullet \to D_\bullet$ that is given by the following commutative ladder:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, dashed] & 2\mathbb{Z} \arrow[r, "i"] \arrow[d, dashed, "i"] & \mathbb{Z} \arrow[r, "p"] \arrow[d, dashed, "\mathrm{id}_\mathbb{Z}"] & \mathbb{Z}/2 \arrow[r] \arrow[d, dashed] & 0 \arrow[r] \arrow[d, dashed] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}_\mathbb{Z}"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

We can compute the kernel, cokernel and image of f degree-wise, and hence find that

$$\begin{aligned}
\ker(f) &= (\cdots \to 0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \cdots)\,, \\
\operatorname{coker}(f) &= (\cdots \to 0 \to \mathbb{Z}/2 \to 0 \to 0 \to 0 \to \cdots)\,, \\
\operatorname{im}(f) &= (\cdots \to 0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \to 0 \to 0 \to \cdots)\,.
\end{aligned}$$

None of those three chain complexes is acyclic.

Exercise 2.

Lemma 3. Let $0 \to C'_\bullet \xrightarrow{f} C_\bullet \xrightarrow{g} C''_\bullet \to$ be a short exact sequence of chain complexes in $\mathcal{A}$.

1) If $C'_\bullet$ is acyclic then g is a quasi-isomorphism.

2) If $C''_\bullet$ is acyclic then f is a quasi-isomorphism.

Proof. This follows from the long exact sequence in homology. $\square$

(i)

If $C''_\bullet$ is acyclic then $\mathrm{H}_n(C'_\bullet) \cong \mathrm{H}_n(C_\bullet)$ for every $n \in \mathbb{Z}$ by Lemma 3, and hence $C'_\bullet$ is acyclic if and only if $C_\bullet$ is acylic. Similarly, if $C'_\bullet$ is acyclic then $C_\bullet$ is acyclic if and only if $C''_\bullet$ is acyclic. This shows together that if any two of the chain complexes $C'_\bullet$, $C_\bullet$, $C''_\bullet$ are acyclic, then so is the third one.

(ii)

The canonical morphism $p\colon C_\bullet \to \operatorname{coim}(f)$ fits into the short exact sequence

$$0 \to \ker(f) \to C_\bullet \xrightarrow{p} \operatorname{im}(f) \to 0 \,.$$

It follows by Lemma 3 from $\ker(f)$ being acylic that the canonical morphism p is a quasi-isomorphism. We similarly find that the canonical morphism $i\colon \operatorname{im}(f) \to D_\bullet$ is a quasi-isomorphism because $\operatorname{coker}(f)$ is acyclic. The canonical induced morphism $\tilde{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism because $\mathbf{Ch}_\bullet(\mathcal{A})$ is abelian, and hence also a quasi-isomorphism. We find that

$$f = i\tilde{f}p$$

is a composition of three quasi-isomorphism and hence a quasi-isomorphism itself.

Exercise 3.

We assume that all vector spaces V_i are finite-dimensional to ensure that the expression $\sum_{n\in\mathbb{Z}}(-1)^n \dim V_n$ is well-defined.

We abbreviate $H_n \coloneqq \mathrm{H}_n(V_\bullet)$, $Z_n \coloneqq \mathrm{Z}_n(V_\bullet)$ and $B_n \coloneqq \mathrm{B}_n(V_\bullet)$ for every $n \in \mathbb{Z}$. We have for every $n \in \mathbb{Z}$ that

$$\dim H_n = \dim Z_n/B_n = \dim Z_n - \dim B_n \,,$$

and also that

$$\dim V_n = \dim Z_n + \dim B_{n-1} \,.$$

We hence find that

$$\begin{aligned}
\sum_n (-1)^n \dim V_n &= \sum_n (-1)^n (\dim Z_n + \dim B_{n-1}) \\
&= \sum_n (-1)^n \dim Z_n + \sum_n (-1)^n \dim B_{n-1} \\
&= \sum_n (-1)^n \dim Z_n - \sum_n (-1)^n \dim B_n \\
&= \sum_n (-1)^n (\dim Z_n - \dim B_n) = \sum_n (-1)^n \dim H_n \,.
\end{aligned}$$

Exercise 4.

We have for all $i = 0, \dots, n$ and $j = 0, \dots, n+1$ that

- $f_j^{n+1} f_i^n = f_i^{n+1} f_{j-1}^n$ if $j > i$ as both compositions are the unique order-preserving map $\{0, \dots, n-1\} \to \{0, \dots, n+1\}$ whose image doesn't contain i and j, and
- $f_j^{n+1} f_i^n = f_{i+1}^{n+1} f_j^n$ if $j \leq i$ as both compositions are the unique order-presvering map $\{0, \dots, n-1\} \to \{0, \dots, n+1\}$ whose image doesn't contain $i+1$ and j.

(i)

We assume that $C_n(A) = 0$ for all $n < 0$, and accordingly $d_n = 0$ for all $n \leq 0$. We then have that $d_n d_{n+1} = 0$ for all $n \leq 0$. For $n > 0$ we have that

$$\begin{aligned}
d_n d_{n+1} &= \left(\sum_{i=0}^{n} (-1)^i A(f_i^n) \right) \left(\sum_{j=0}^{n+1} (-1)^j A(f_j^{n+1}) \right) = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} A(f_j^{n+1} f_i^n) \\
&= \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} A(f_j^{n+1} f_i^n) + \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} A(f_j^{n+1} f_i^n) \,.
\end{aligned}$$

By using that $f_j^{n+1} f_i^n = f_{i+1}^{n+1} f_j^n$ for $j \leq i$, we can rearrange the second sum as

$$\begin{aligned}
&\sum_{0 \leq j \leq i \leq n} (-1)^{i+j} A(f_j^{n+1} f_i^n) \\
&= \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} A(f_{i+1}^{n+1} f_j^n) \\
&= \sum_{0 \leq j < i \leq n+1} (-1)^{i-1+j} A(f_i^{n+1} f_j^n) \\
&= - \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} A(f_i^{n+1} f_j^n) \\
&= - \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} A(f_j^{n+1} f_i^n) \,.
\end{aligned}$$

We hence find that $d_n d_{n+1} = 0$.

(ii)

We start by defining a cosimplical object $\Sigma\colon \Delta \to \mathbf{Top}$ and then define the desired simplicial set as $\mathbf{Top}(-, Y) \circ \Sigma\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$.

For every $n \geq 0$ let $e_0, \dots, e_n$ be the standard basis of $\mathbb{R}^{n+1}$ and let

$$\Sigma(n) \coloneqq \operatorname{conv}(e_0, \dots, e_n) \subseteq \mathbb{R}^{n+1}$$

be the standard n-simplex, where conv denotes the convex hull operator.[2] For all $0 \leq m \leq n$ and every $f \in \Delta(m, n)$ let $\Sigma(f)\colon \Sigma(m) \to \Sigma(n)$ be the unique affine map with

$$\Sigma(f)(e_i) = e_{f(i)}$$

for every $i = 0, \dots, m$. It then holds that $\Sigma(\mathrm{id}_n) = \mathrm{id}_{\Sigma(n)}$ for every $n \geq 0$, and that $\Sigma(gf) = \Sigma(g)\Sigma(f)$ for all composable morphisms $f\colon k \to m$ and $g\colon m \to n$ in Δ. We have hence constructed a covariant functor $\Sigma\colon \Delta \to \mathbf{Top}$, i.e. a cosimplical object in the category $\mathbf{Top}$.

We set $S \coloneqq \mathbf{Top}(-, Y) \circ \Sigma\colon \Delta \to \mathbf{Set}$. The elements of the set $S(n)$ are then precisely the continuous map $s\colon \Delta^n \to Y$, i.e. the n-simplices in Y.

Let $\mathrm{C}^{\mathrm{sing}}_\bullet(Y)$ be the singular chain complex of Y. We have just seen for $n \geq 0$ that

$$\mathrm{C}^{\mathrm{sing}}_n(Y) = FS(n) = C_n(FS)\,.$$

Note that for every $n \geq 0$ and $i = 0, \dots, n$, the map $\Sigma(f^n_i)\colon \Sigma(n-1) \to \Sigma(n)$ is precisely the inclusion of $\Sigma(n-1)$ into $\Sigma(n)$ as the i-th face. The differential of the singular chain complex $\mathrm{C}^{\mathrm{sing}}_n(Y)$ is therefore given by

$$d^{\mathrm{sing}}_n(s) = \sum_{i=0}^{n} (-1)^i s \circ \Sigma(f^n_i)$$

for every simplex $s \in S(n)$. The differential of the chain complex $C_\bullet(FS)$ is given by

$$\begin{aligned} d_n(s) &= \sum_{i=0}^{n} (-1)^i (FS)(f^n_i)(s) = \sum_{i=0}^{n} (-1)^i (F \circ \mathbf{Top}(-, Y) \circ \Sigma)(f^n_i)(s) \\ &= \sum_{i=0}^{n} (-1)^i F(\Sigma(f^n_i)^*)(s) = \sum_{i=0}^{n} (-1)^i \Sigma(f^n_i)^*(s) = \sum_{i=0}^{n} (-1)^i s \circ \Sigma(f^n_i)\,. \end{aligned}$$

This shows that the two chain complexes $\mathrm{C}^{\mathrm{sing}}_\bullet$ and $C_\bullet(FS)$ do have not only the same components but also the same differential, hence that they are the same.

[2] We avoid the usual notation Δ^n for the standard n-simplex because enough things are named Δ already.