Exercises in Foundations in Representation Theory

Exercise Sheet 4

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Exercise 1.

If the functor $G\colon \mathcal{D} \to \mathcal{C}$ admits a left adjoint functor $F\colon \mathcal{C} \to \mathcal{D}$ then

$$\mathcal{C}(X, G(-)) \cong \mathcal{D}(F(X), -)\,,$$

for every $X \in \mathrm{Ob}(\mathcal{C})$, which shows that the functor $\mathcal{C}(X, G(-))$ is represented by the object $F(X) \in \mathrm{Ob}(\mathcal{D})$.

Suppose on the other hand that the functor $\mathcal{C}(X, G(-))\colon \mathcal{D} \to \mathbf{Set}$ is representable for every object $X \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{E} \subseteq \mathbf{Fun}(\mathcal{D}, \mathbf{Set})$ be the the essential image of the Yoneda embedding

$$E\colon \mathcal{D} \to \mathbf{Fun}(\mathcal{D}, \mathbf{Set})\,,$$

i.e. the full subcategory of $\mathbf{Fun}(\mathcal{D}, \mathbf{Set})$ spanned by all representable functors. It follows from the Yoneda embedding E being fully faithful and essentially surjective (onto $\mathcal{E}$) that there exist a quasi-inverse $E'\colon \mathcal{E} \to \mathcal{D}$ for E. Note that both E and E' are contravariant.

We can now define a functor

$$F'\colon \mathcal{C} \to \mathbf{Fun}(\mathcal{D}, \mathbf{Set})$$

which assigns to every objects $X \in \mathrm{Ob}(\mathcal{C})$ the functor

$$F'(X) \coloneqq \mathcal{C}(X, G(-))\colon \mathcal{D} \to \mathbf{Set}\,,$$

and assigns to every morphism $f\colon X \to X'$ in $\mathcal{C}$ the natural transformation

$$F'(f) \coloneqq f^*\colon \mathcal{C}(X', G(-)) \to \mathcal{C}(X, G(-))\,.$$

The image of F' is by assumption contained in the full subcategory $\mathcal{E}$ of $\mathbf{Fun}(\mathcal{D}, \mathbf{Set})$; we may therefore regard F' instead as a functor $F'\colon \mathcal{C} \to \mathcal{E}$.

We now define the desired left adjoint $F\colon \mathcal{C} \to \mathcal{D}$ as the composition

$$F\colon \mathcal{C} \xrightarrow{F'} \mathcal{E} \xrightarrow{E'} \mathcal{D}\,.$$

That F is indeed left adjoint to G follows from the computation

$$\begin{aligned}\mathcal{D}(F(-_1), -_2) &= (E \circ F)(-_1)(-_2) = (E \circ E' \circ F')(-_1)(-_2) \\ &\cong (\mathrm{Id}_{\mathcal{E}} \circ F')(-_1)(-_2) = F'(-_1)(-_2) = \mathcal{C}(-_1, G(-_2))\,.\end{aligned}$$

Exercise 2.

(i)

Let more generally A and B be two k-algebras and let $f\colon A \to B$ be a homorphism of k-algebras. We show that the induced restrictions of scalars $R\colon B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ has both a left and a right adjoint. For this we show that R can be expressed both as a tensor product and a Hom-functor, and then use the usual $\otimes$-Hom adjunctions.

To construct the left adjoint of R we regard B as an B-A-bimodule and note that $R \cong \mathrm{Hom}_B(B, -)$. It then follows from one of the $\otimes$-Hom adjunctions that

$$(A\text{-}\mathbf{Mod})(-, R) \cong (A\text{-}\mathbf{Mod})(-, \mathrm{Hom}_B(B, -)) \cong (A\text{-}\mathbf{Mod})(B \otimes_A (-), -)\,,$$

which shows that the functor $B \otimes_A (-)\colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ is left adjoint to R.

To construct the right adjoint of R we regard B as an A-B-bimodule and note that $R \cong B \otimes_B (-)$, as shown in the lecture. It then then follows that from another use of the $\otimes$-Hom adjunction

$$(A\text{-}\mathbf{Mod})(R, -) \cong (A\text{-}\mathbf{Mod})(B \otimes_B (-), -) \cong (B\text{-}\mathbf{Mod})(-, \mathrm{Hom}_A(B, -))\,,$$

which shows that the functor $\mathrm{Hom}_A(B, -)\colon A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$ is right adjoint to R.

(ii)

We define a functor $D\colon \mathbf{Set} \to \mathbf{Top}$ which assigns to every set X the topological space $D(X)$ which is the set X together with the discrete topology, and assigns to every map $f\colon X \to X'$ between sets X and X' the map f when viewed as a continuous (because $D(X)$ is discrete) map $D(f)\colon D(X) \to D(X')$. It then holds that

$$\mathbf{Top}(D(-), -) = \mathbf{Set}(-, V(-))\,.$$

Indeed, it holds for every set X and every topological space Y that

$$\mathbf{Top}(D(X), Y) = \mathbf{Set}(X, V(Y))$$

because the topology on $D(X)$ is discrete. We have for every map $f\colon X \to X'$ between sets X and X' and every continuous map $g\colon Y \to Y'$ between topological spaces Y

and Y' that the square

$$\begin{array}{ccc}
\mathbf{Top}(D(X'),Y) & \xrightarrow{g_*\circ(-)\circ D(f)^*} & \mathbf{Top}(D(X),Y') \\
\| & & \| \\
\mathbf{Set}(X',V(Y)) & \xrightarrow{V(g)_*\circ(-)\circ f^*} & \mathbf{Set}(X,V(Y'))
\end{array}$$

commutes, which shows that naturality of $\mathbf{Top}(D(-),-) = \mathbf{Set}(-,V(-))$. This shows altogether that the functor D is left adjoint to the functor V.

We also define a functor $T\colon \mathbf{Set} \to \mathbf{Top}$ which assigns to every set Y the topological space $T(Y)$ which is the set Y together with the trivial (i.e. indiscrete) topology, and assigns to every map $g\colon Y \to Y'$ between sets Y and Y' the map g when viewed as a continuous (because $T(Y')$ is trivial) map $T(f)\colon T(Y) \to T(Y')$. It then holds that

$$\mathbf{Top}(-,T(-)) = \mathbf{Set}(V(-),-)\,.$$

Indeed, it holds for every topological space X and every set Y that

$$\mathbf{Top}(X,T(Y)) = \mathbf{Set}(V(X),Y)$$

because the topology on $T(Y)$ is trivial. We have for every continuous map $f\colon X \to X'$ between topological spaces X and X' and every map $g\colon X \to X'$ between sets X and X' that the square

$$\begin{array}{ccc}
\mathbf{Top}(X',T(Y)) & \xrightarrow{T(g)_*\circ(-)\circ f^*} & \mathbf{Top}(X,T(Y')) \\
\| & & \| \\
\mathbf{Set}(X',T(Y)) & \xrightarrow{g_*\circ(-)\circ T(f)^*} & \mathbf{Set}(X,T(Y'))
\end{array}$$

commutes, which shows the naturality of $\mathbf{Top}(-,T(-)) = \mathbf{Set}(V(-),-)$. This shows altogether that the functor T is right adjoint to the functor V.

Exercise 3.

(i)

The adjunction is given by the natural bijections

$$\varphi_{V,M}\colon (A\text{-}\mathbf{Mod})(A \otimes_k V, M) \to (k\text{-}\mathbf{Mod})(V, G(M))$$

which are given by

$$\varphi_{V,M}(f)(v) = f(1 \otimes v) \qquad \text{and} \qquad \varphi^{-1}_{V,M}(g)(a \otimes v) = ag(v)\,.$$

It follows that the unit $\eta\colon \mathrm{Id}_{A\text{-}\mathbf{Mod}} \to G \circ F$ is at an object $V \in \mathrm{Ob}(k\text{-}\mathbf{Mod})$ given by

$$\eta_V = \varphi_{V,A\otimes_k V}(\mathrm{id}_{A\otimes_k V})\colon V \to A \otimes_k V\,, \quad v \mapsto 1 \otimes v\,,$$

and that the counit $\varepsilon\colon F \circ G \to \mathrm{Id}_{k\text{-}\mathbf{Mod}}$ is at an object $M \in \mathrm{Ob}(A\text{-}\mathbf{Mod})$ given by

$$\varepsilon_M = \varphi^{-1}_{G(M),M}(\mathrm{id}_{G(M)})\colon A \otimes_k G(M) \to M\,, \quad a \otimes m \mapsto am\,.$$

(ii)

The adjunction is given by the natural bijections

$$\varphi_{M,P}\colon (\mathbf{Mod}\text{-}B)(M \otimes_A N, P) \to (\mathbf{Mod}\text{-}A)(M, \mathrm{Hom}_B(N,P))$$

which are given by

$$\varphi_{M,P}(f)(m)(n) = f(m \otimes n) \qquad \text{and} \qquad \varphi_{M,P}^{-1}(g)(m \otimes n) = g(m)(n)\,.$$

It follows that the unit $\eta\colon \mathrm{Id}_{\mathbf{Mod}\text{-}A} \to G \circ F$ is at an object M given by

$$\begin{aligned} \eta_M = \varphi_{M, M\otimes_A N}(\mathrm{id}_{M \otimes_A N})\colon M &\to \mathrm{Hom}_B(N, M \otimes_A N)\,, \\ m &\mapsto (n \mapsto m \otimes n)\,, \end{aligned}$$

and that the counit $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathbf{Mod}\text{-}B}$ is at an object P given by

$$\begin{aligned} \varepsilon_P = \varphi^{-1}_{\mathrm{Hom}_B(N,P),P}(\mathrm{id}_{\mathrm{Hom}_B(N,P)})\colon \mathrm{Hom}_B(N,P) \otimes_A N &\to P\,, \\ g \otimes n &\mapsto g(n)\,. \end{aligned}$$

(iii)

The adjunction is given by the natural bijections

$$\varphi_{(A,S),B}\colon \mathbf{CRing}(S^{-1}A, B) \to \mathcal{C}((A,S),(B,B^\times))\,,$$

which are given by

$$\varphi_{(A,S),B}(f) = f \circ i_{A,S} \qquad \text{and} \qquad \varphi^{-1}_{(A,S),B}(g)\left(\frac{a}{s}\right) = g(a)g(s)^{-1}\,,$$

where $i_{A,S}\colon A \to S^{-1}A$, $a \mapsto a/1$ denotes the canonical homomorphism. It follows that the unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$ is at an object (A,S) given by

$$\begin{aligned} \eta_{(A,S)} = \varphi_{(A,S),S^{-1}A}(\mathrm{id}_{S^{-1}A})\colon (A,S) &\to (S^{-1}A, (S^{-1}A)^\times)\,, \\ \eta_{(A,S)} &= i_{A,S}\,, \end{aligned}$$

and that the counit $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathbf{CRing}}$ is at an object B given by

$$\varepsilon_{(B^\times)^{-1}B} = \varphi_{(B^\times)^{-1}B,B}(\mathrm{id}_{(B,B^\times)})\colon (B^\times)^{-1}B \to B\,, \quad \frac{b}{u} \mapsto bu^{-1}\,.$$

Exercise 4.

(i)

The given functor F is represented by the polynomial ring $k[X_1,\dots,X_n]$. Indeed, there exist for every commutative k-algebra B a bijection

$$\eta_B\colon \mathrm{Hom}_{k\text{-}\mathbf{CAlg}}(k[X_1,\dots,X_n], B) \to B^n\,, \quad f \mapsto (f(X_1),\dots,f(X_n))$$

by the universal property of the polynomial ring. These bijections are natural and hence yield an isomorphism of functors $\eta\colon \mathrm{Hom}_{k\text{-}\mathbf{CAlg}}(k[X_1,\dots,X_n], -) \to F$.

(iii)

An A-valued $(n \times n)$-matrix M is invertible if and only if its determinant $\det(M) \in A$ is a unit, i.e. if there exists an element $a \in A$ with $\det(M)a = 1$; the value a is then uniquely determined as $a = \det(M)^{-1}$. It follows that we have bijections

$$\begin{aligned}
&\mathrm{GL}_n(A) \\
&= \{M \in \mathrm{M}_n(A) \mid \exists a \in A : \det(M)a = 1\} \\
&\cong \{(M, a) \in \mathrm{M}_n(A) \times A \mid \det(M)a = 1\} \\
&\cong \{(a_{11}, \dots, a_{nn}, a) \in A^{n^2+1} \mid \det(a_{11}, \dots, a_{nn})a = 1\} \\
&= \{(a_{11}, \dots, a_{nn}, a) \in A^{n^2+1} \mid \det(a_{11}, \dots, a_{nn})a - 1 = 0\} \\
&\cong \mathrm{Hom}_{k\text{-}\mathbf{CAlg}}(k[X_{11}, \dots, X_{nn}, Y]/(\det \cdot Y - 1), A) \\
&\cong \mathrm{Hom}_{k\text{-}\mathbf{CAlg}}(k[X_{11}, \dots, X_{nn}]_{\det}, A)
\end{aligned}$$

where we denote by abuse of notation with $\det \in k[X_{11}, \dots, X_{nn}]$ the polynomial

$$\det = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) X_{1\sigma(1)} \cdots X_{n\sigma(n)} \,.$$

The above bijection is (in reverse order) overall given by

$$\begin{aligned}
\eta_A \colon \mathrm{Hom}_{k\text{-}\mathbf{CAlg}}(k[X_{11}, \dots, X_{nn}]_{\det}, A) &\to \mathrm{GL}_n(A) \,, \\
f &\mapsto \begin{bmatrix} f(X_{11}) & \cdots & f(X_{1n}) \\ \vdots & \ddots & \vdots \\ f(X_{n1}) & \cdots & f(X_{nn}) \end{bmatrix} ,
\end{aligned}$$

where we identify $k[X_{11}, \dots, X_{nn}]$ with a subalgebra of $k[X_{11}, \dots, X_{nn}]_{\det}$ via the canonical homomorphism $f \mapsto f/1$ (which is possible because $k[X_{11}, \dots, X_{nn}]$ is an integral domain). These bijections are natural and hence give rise to a natural isomorphism $\eta \colon \mathrm{Hom}_{k\text{-}\mathbf{CAlg}}(k[X_{11}, \dots, X_{nn}]_{\det}, -) \to F$. The functor F is therefore representable by $k[X_{11}, \dots, X_{nn}]_{\det}$.