# Exercises in Foundations in Representation Theory

# **Exercise Sheet 3**

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## Exercise 1.

We define a category  $\mathcal{C}$  whose objects are given by  $\mathrm{Ob}(\mathcal{C}) = \mathbb{Z}_{\geq 0}$  and where for any two nonnegative integers  $n, m \in \mathbb{Z}_{\geq 0}$  the morphism set  $\mathcal{C}(n, m)$  is given by the matrix space

$$C(n,m) = M(m \times n, k)$$
.

The compositions of morphisms in  $\mathcal{C}$  is the usual matrix multiplication, which is associative. The identity of an object  $n \in \mathbb{Z}_{\geq 0}$  is given by the  $(n \times n)$ -identity matrix  $I_n$ .

We define a functor  $F: \mathcal{C} \to k$ -mod by mapping any nonnegative integer  $n \in \mathbb{Z}_{\geq 0}$  to the k-vector space  $F(n) = k^n$ , and mapping any matrix  $A \in M(m \times n, k)$  to the k-linear map

$$F(A): k^n \to k^m, \quad x \mapsto Ax.$$

We know from linear algebra that F is both fully faithful and essentially surjective, and hence an equivalence of categories.

#### Exercise 2.

We have for every k-vector space V bijections

- $\{G$ -representations on  $V\}$
- $= \{ \text{group homomorphisms } G \to GL(V) \}$
- = {group homomorphisms  $G \to \operatorname{End}_k(V)^{\times}$ }
- $\cong \{k\text{-algebra homomorphisms } k[G] \to \operatorname{End}_k(V)\}$
- $\cong \{k[G]\text{-module structures on }V\},\$

where we use the adjunction  $k[-] \dashv (-)^{\times}$ . So we can regard every representation  $(V, \rho)$  of G over k as an k[G]-module via

$$\sum_{g \in G} a_g g \cdot v = \sum_{g \in G} a_g \rho(g)(v)$$

for all  $\sum_{g \in G} a_g g \in k[G]$  and all  $v \in V$ ; we can conversely regard every k[G]-modules V as a representation  $(V, \rho)$  of G over k via

$$\rho(g) \colon V \to V$$
,  $v \mapsto g \cdot v$ 

for all  $g \in G \subseteq k[G]$  and all  $v \in V$ . These two constructions are moreover mutually inverse.

Let  $(V, \rho)$  and  $(W, \sigma)$  be two representations of G over k and let  $f \colon V \to W$  be a k-linear map. Then

 $f \text{ is a homomorphisms of representations} \\ \iff f(\rho(g)(v)) = \rho(g)(f(v)) \text{ for all } g \in G, \ v \in V \\ \iff f(g \cdot v) = g \cdot f(v) \text{ for all } g \in G, \ v \in V \\ \iff f(a \cdot v) = a \cdot f(v) \text{ for all } a \in k[G], \ v \in V \\ \iff f \text{ is a homomorphism of } k[G]\text{-modules} \,.$ 

This shows altogether that the categories  $\mathbf{Rep}_k(G)$  and k[G]-Mod are isomorphic, and therefore in particular equivalent.

#### Exercise 3.

We denote the given category by  $\mathcal{G}$ .

(i)

A G-set X consists of a set X and for every  $g \in G$  a map  $X_g \colon X \to X$ , in such a way that  $X_e = \mathrm{id}_X$  and  $X_{g_1g_2} = X_{g_1} \circ X_{g_2}$  for all  $g_1, g_2 \in G$ . A functor  $F \colon \mathcal{G} \to \mathbf{Set}$  consists of a single set F(\*) and for every  $g \in \mathcal{G}(*,*) = G$  a map  $F(g) \colon F(*) \to F(*)$ , in such a way that  $F(e) = \mathrm{id}_X$  and  $F(g_1g_2) = F(g_1)F(g_2)$  for all  $g_1, g_2 \in G$ . We see by setting X = F(\*) and  $X_g = F(g)$  that both constructs consist of precisely the same data. This shows that a G-set X is the same as a functor  $F \colon \mathcal{G} \to \mathbf{Set}$ .

Let X and X' be two G-sets with corresponding functors  $F, F' \colon \mathcal{G} \to \mathbf{Set}$ , and let  $f \colon X \to X'$  be a map. Then

f is a homomorphism of G-sets  $\iff f(gx) = g(f(x))$  for all  $g \in G$ ,  $x \in X$   $\iff f \circ F(g) = F'(g) \circ f$  for every  $g \in G$   $\iff f \colon F(*) \to F'(*)$  defines a natural transformation  $F \to F'$ 

This altogether shows that the category G-**Set** is isomorphic to the functor category  $\operatorname{Fun}(\mathcal{G}, \operatorname{\mathbf{Set}})$ , and therefore in particular equivalent to it.

(ii)

The G-set X which corresponds to the covariant functor  $h^*: \mathcal{G} \to \mathbf{Set}$  is given by the set  $X = h^*(*) = \mathcal{G}(*, *) = G$ , and the action of  $g \in G$  on  $x \in G$  is given by

$$gx = h^*(g)(x) = g_*(x) = g \circ x = gx$$
.

The G-set X is therefore just the regular left G-set, i.e. the group G acting on itself by left multiplication.

(iii)

Let X be the left regular G-set, as before. We find that

$$\operatorname{End}_{G\operatorname{-\mathbf{Set}}}(X) = (G\operatorname{-\mathbf{Set}})(X,X) \cong \operatorname{\mathbf{Fun}}(G\operatorname{-\mathbf{Set}},\operatorname{\mathbf{Set}})(h^*,h^*) \cong h^*(*) = \mathcal{G}(*,*) = G.$$

Under this bijection, the group element  $g \in G$  corresponds the the endomorphism of G-sets  $X \to X$  which is given by right multiplication with g. Thus the Yoneda lemma tells us that every endomorphism of X is given by right multiplication with a unique element  $g \in G$ . Moreover, the explicit description of the bijection

$$\operatorname{Fun}(G\operatorname{-Set},\operatorname{Set})(h^*,h^*) \to h^*(*), \quad \eta \mapsto \eta_*(*)$$

tells us that for every endomorphism of G-sets  $\varphi \colon X \to X$ , the corresponding group element  $g \in G$  is given by  $g = \varphi(e)$ .

#### Exercise 4.

We convince ourselves that K is indeed a category:

For any two objects (A, B, h) and (A', B', h') of the proposed category  $\mathcal{K}$ , the collection of morphisms  $(A, B, h) \to (A', B', h')$  is a subset of  $\mathcal{A}(A, A') \times \mathcal{B}(B, B')$ , and is hence again a set (and not a proper class).

The composition of morphisms in  $\mathcal{K}$  is well-defined: If  $(f,g):(A,B,h)\to (A',B',h')$  and  $(f',g'):(A',B',h')\to (A'',B'',h'')$  are two morphisms then in the diagram

$$S(A) \xrightarrow{h} T(B) \xrightarrow{S(f' \circ f)} S(A') \xrightarrow{h'} T(B') \xrightarrow{T(g' \circ g)} T(g' \circ g)$$

$$S(f') \downarrow \qquad \qquad \downarrow T(g') \xrightarrow{T(g')} S(A'') \xrightarrow{h''} T(B'') \leftarrow$$

both squares and both triangles commute; from this it follows that the outer square also commutes. That the composition in  $\mathcal{K}$  is associative follows from the associativity of the compositions in  $\mathcal{A}$  and  $\mathcal{B}$ .

For every object (A, B, h) of  $\mathcal{K}$ , the morphism  $(\mathrm{id}_A, \mathrm{id}_B) : (A, B, h) \to (A, B, h)$  is the identity of (A, B, h) in  $\mathcal{K}$  as it holds for all morphisms  $(f', g') : (A', B', h') \to (A, B, h)$  and all morphisms  $(f'', g'') : (A, B, h) \to (A'', B'', h'')$  that

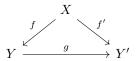
$$(\mathrm{id}_A,\mathrm{id}_B)\circ(f',g')=(\mathrm{id}_A\circ f',\mathrm{id}_B\circ g')=(f',g')$$

and

$$(f'',g'') \circ (\mathrm{id}_A,\mathrm{id}_B) = (f'' \circ \mathrm{id}_A,g'' \circ \mathrm{id}_B) = (f'',g'').$$

(i)

A functor  $S: 1 \to \mathcal{C}$  is the same as an object  $X \in \mathrm{Ob}(\mathcal{C})$  through the assignment X = S(\*). An object of the comma category  $\mathcal{K} = (S, \mathrm{Id}_{\mathcal{C}})$  is therefore the same as a tupel (Y, f) consisting of an object  $Y \in \mathrm{Ob}(\mathcal{C})$  and a morphisms  $f: X \to Y$ . A morphisms  $g: (Y, f) \to (Y', f')$  is then a morphisms  $g: Y \to Y$  with  $g \circ f = f'$ , i.e. such that the triangle



commutes. The given comma category  $\mathcal{K}$  is therefore (isomorphic to) the under-X category.

(ii)

The objects of the given comma category  $\mathcal{K} = (\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$  are tripels (X, X', f) consisting of two objects  $X, X' \in \mathrm{Ob}(\mathcal{C})$  and a morphisms  $f \colon X \to X'$  between them. We can therefore identify the objects of  $\mathcal{K}$  with the morphisms in  $\mathcal{C}$ . For any two morphisms  $(X \xrightarrow{f} X')$  and  $(Y \xrightarrow{g} Y')$ , a morphisms  $f \to f'$  in  $\mathcal{K}$  is then a pair (h, h') of morphisms  $h \colon X \to Y$  and  $h' \colon X' \to Y'$  which make the square

$$X \xrightarrow{f} X'$$

$$\downarrow h \qquad \qquad \downarrow h'$$

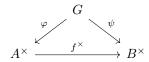
$$Y \xrightarrow{g} Y'$$

commute. The given comma category  $\mathcal K$  is therefore (isomorphic to) the morphism category of  $\mathcal C.$ 

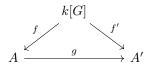
(iii)

A functor  $S: 1 \to \mathbf{Grp}$  is again the same as choosing an object  $S(*) \in \mathbf{Grp}$ , i.e. choosing a group G := S(\*). The objects of the given comma category  $\mathcal{K} = (S, (-)^{\times})$  can therefore be identified with the pairs  $(A, \varphi)$  consisting of a k-algebra A and a

group homomorphism  $\varphi \colon A^{\times} \to G$ . A morphism  $f \colon (A, \varphi) \to (B, \psi)$  is then an algebra homomorphism  $f \colon A \to B$  which makes the triangle



commute. By using the adjunction  $k[-] \vdash (-)^{\times}$  we may further identify the given comma category  $\mathcal K$  with the under-k[G] category, i.e. the category whose objects are pairs (A,f) consisting of a k-algebra A and an algebra homomorphism  $f \colon k[G] \to A$ , and in which a morphism  $g \colon (A,f) \to (A',f')$  is an algebra homomorphism  $g \colon A \to A'$  which makes the triangle



commute.