Group 1, Wednesday 8–10, Tutor: Nils Waßmuth December 20, 2018

Exercises in Foundations in Representation Theory

Exercise Sheet 10

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Exercise 1.

(i)

The statement is **false**. As a counterexample we might consider for $\mathcal{A} = \mathbb{Z}\text{-}\mathbf{Mod}$ the morphism of chain complexes $f \colon C_\bullet \to D_\bullet$ that is given by the following commutative diagram:

$$\begin{tikzcd}
C_\bullet \arrow[d, "f"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 2\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
D_\bullet & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Here $2\mathbb{Z} \to \mathbb{Z}$ is the inclusion and $\mathbb{Z} \to \mathbb{Z}$ is the identity, and the vertical identity $\mathbb{Z} \to \mathbb{Z}$ is in degree 0. This morphism f is a monomorphism because it is a monomorphism in each degree. But the induced morphism $\mathrm{H}_0(f)$ is not a monomorphism because $\mathrm{H}_0(C_\bullet) = \mathbb{Z}/2$ but $\mathrm{H}_0(D_\bullet) = 0$. This shows that $\mathrm{H}_0 \colon \mathbf{Ch}_\bullet(\mathbb{Z}\text{-}\mathbf{Mod}) \to \mathbb{Z}\text{-}\mathbf{Mod}$ does not preserve monomorphisms, and hence is not left exact.

(ii)

The statement is **true**: We get for the identity morphism $\mathrm{id} \colon C_\bullet \to C_\bullet$ the long exact homology sequence of the cone:

$$\cdots \to \mathrm{H}_n(C_\bullet) \xrightarrow{\mathrm{H}_n(\mathrm{id})} \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(\mathrm{cone}(\mathrm{id})) \to \mathrm{H}_{n-1}(C_\bullet) \to \cdots$$

The morphism $\mathrm{H}_n(\mathrm{id}) = \mathrm{id}_{\mathrm{H}_n(C_\bullet)}$ is an isomorphism of every $n \in \mathbb{Z}$, hence it follows that $\mathrm{H}_n(\mathrm{cone}(\mathrm{id})) = 0$ for every $n \in \mathbb{Z}$ by the exactness of the above sequence.

(iii)

The statement is **true**: Every k-vector space V is free, and hence projective by Exercise 4. We can therefore use the identity $\mathrm{id}_V \colon V \to V$ to see that the category k-**Mod** has enough projectives.

(iv)

The statement is **false**: Consider the morphism $\mathrm{id} \colon \mathbb{Z} \to \mathbb{Z}$ and the monomorphism $g \colon \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = 2n$. (This is a monomorphism because it is injective.) Then there does not exist an extension $g' \colon \mathbb{Z} \to \mathbb{Z}$ of g that makes the triangle

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "g"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z} \arrow[dl, dashed, "g'"] \\ \mathbb{Z} & \end{tikzcd}$$

commute, because $g'(1) \in \mathbb{Z}$ would need to be a group element with

$$2g'(1) = g'(2) = g'(g(1)) = \mathrm{id}(1) = 1\,.$$

But such an element does not exist.

(v)

The statement is **false** in its current form, because $\mathbb{Q}$ is not an object in the given category (because $\mathbb{Q}$ is not finitely generated as an abelian group). The original form of the problem, that $\mathbb{Q}$ is injective in the category **Ab**, is **true**:

Let A be an abelian group, let $g \colon A \to \mathbb{Q}$ be a homomorphism and let $f \colon A \to B$ be a monomorphism, i.e. an injective group homomorphism. We need to show that g extends to a group homomorphism $g' \colon B \to \mathbb{Q}$ along f. For this we may assume w.l.o.g. that A is a subgroup of B and that $f \colon A \to B$ is the canonical inclusion.

We find with Zorn's lemma that there exists a maximal extension $\tilde{g} \colon \tilde{B} \to \mathbb{Q}$ of g, i.e. $A \subseteq \tilde{B} \subseteq B$ is an intermediate group, $\tilde{g} \colon \tilde{B} \to \mathbb{Q}$ is an extension of g, and $\tilde{B}$ is maximal with this property.

Suppose that $\tilde{B} \neq B$. Then there exists some $x \in B$ with $x \notin \tilde{B}$. We show for $B' \coloneqq \tilde{B} + \mathbb{Z}x$ that the homomorphism $\tilde{g} \colon \tilde{B} \to \mathbb{Q}$ can be extended to a homomorphism $B' \to \mathbb{Q}$. We distinguish between two cases:

- If $\tilde{B} \cap \mathbb{Z}x = 0$ then $B' = \tilde{B} \oplus \mathbb{Z}x$. We can then choose the extension g' as

  $$g'(\tilde{b} + nx) = \tilde{g}(\tilde{b})$$

  for all $\tilde{b} \in \tilde{B}$ and $n \in \mathbb{Z}$.

- Otherwise let $n > 1$ be minimal such that $nx \in \tilde{B}$. (We do not have to consider the case $n = 1$ because $x \notin \tilde{B}$.) Then $\tilde{B} \cap \mathbb{Z}x = \mathbb{Z}nx$, and we hence have a (right) exact sequence

  $$\mathbb{Z} \xrightarrow{\psi} \tilde{B} \oplus \mathbb{Z} \xrightarrow{\varphi} B' \to 0\,, \tag{1}$$

where $\varphi(\tilde{b}, k) = \tilde{b} + kx$ and $\varphi(k) = k(nx, -n)$. It follows for the homomorphism

$$\tilde{g}' : \tilde{B} \oplus \mathbb{Z} \to \mathbb{Q}\,, \quad (\tilde{b}, k) \mapsto \tilde{g}(\tilde{b}) + k\frac{\tilde{g}(nx)}{n}$$

that $\tilde{g}' \circ \psi = 0$ because

$$\tilde{g}'(nx, -n) = \tilde{g}(nx) - n\frac{\tilde{g}(nx)}{n} = \tilde{g}(nx) - \tilde{g}(nx) = 0\,.$$

It follows from the right exactness of (1) hat $\tilde{g}'$ factors through a homomorphism $g' : B' \to \mathbb{Q}$ with

$$g'(\tilde{b} + kx) = \tilde{g}(\tilde{b}) + k\frac{\tilde{g}(nx)}{n}$$

for all $\tilde{b} \in \tilde{B}$ and $k \in \mathbb{Z}$. This is the desired extension of $\tilde{g}$ to B'.

It follows from the maximality of the extension $(\tilde{B}, \tilde{g})$ that already $\tilde{B} = B'$, which contradicts $x \notin \tilde{B}$ (but $x \in B'$). We hence find that already $\tilde{B} = B$, and hence that $\tilde{g}$ is the desired extension of g onto B.

(vi)

The statement ist **false**. We see in Exercise 4 that an abelian group (i.e. $\mathbb{Z}$-module) is projective if and only if it is a direct summand of a free abelian group. Free abelian groups are torsion-free, and hence projective abelian groups are also torsion-free. But $\mathbb{Z}/2$ is a nontrivial torsion group, and hence not projective.

Exercise 2.

We denote by $i : P' \to P$ and $j : P'' \to P$ the canonical morphisms belonging to the coproduct structure of $P' \oplus P'' = P$. We then have for every morphism $f : X \to Y$ in $\mathcal{A}$ the following commutative square:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(P, X) \arrow[r, "f_*"] \arrow[d, "\left[\begin{smallmatrix} i^* \\ j^* \end{smallmatrix}\right]"'] & \operatorname{Hom}_{\mathcal{A}}(P, Y) \arrow[d, "\left[\begin{smallmatrix} i^* \\ j^* \end{smallmatrix}\right]"] \\
\operatorname{Hom}_{\mathcal{A}}(P', X) \oplus \operatorname{Hom}_{\mathcal{A}}(P'', X) \arrow[r, "\left[\begin{smallmatrix} f_* & 0 \\ 0 & f_* \end{smallmatrix}\right]"'] & \operatorname{Hom}_{\mathcal{A}}(P', Y) \oplus \operatorname{Hom}_{\mathcal{A}}(P'', Y)
\end{tikzcd}$$

The vertical arrows are isomorphisms by the universal property of the coproduct.

We have seen in the lecture that the object P is projective if and only if the group homomorphism $f_* : \operatorname{Hom}_{\mathcal{A}}(P, X) \to \operatorname{Hom}_{\mathcal{A}}(P, Y)$ is surjective whenever the morphism f is an epimorphism. It follows from the above commutative square with isomorphisms for vertical arrows that this holds if and only if both group homomorphisms $f_*\colon \operatorname{Hom}_{\mathcal{A}}(P', X) \to \operatorname{Hom}_{\mathcal{A}}(P', Y)$ and $f_*\colon \operatorname{Hom}_{\mathcal{A}}(P'', X) \to \operatorname{Hom}_{\mathcal{A}}(P'', Y)$ are surjective whenever f is an epimorphism. But this is what it means for both P' and P'' to be projective.

We find as the dual result that two objects I' and I'' in $\mathcal{A}$ are injective if and only if their biproduct $I' \oplus I''$ is injective.

Exercise 3.

(i)

For an object $X \in \operatorname{Ob}(\mathcal{A})$ and any bounded chain complex $C_\bullet \in \operatorname{Ob}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$, a morphism of chain complexes $f\colon C_\bullet \to I_0(X)$ corresponds to the choice of a morphism $f_0\colon C_0 \to X$ that makes the diagram

$$\begin{tikzcd}
{} \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d, dashed] & C_1 \arrow[r, "d_1"] \arrow[d, dashed] & C_0 \arrow[d, dashed, "f_0"] \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X
\end{tikzcd}$$

commute. The commutativity of this diagram means that $f_0 d_1 = 0$. Such morphisms are in one-to-one correspondence to morphism $\overline{f_0}\colon \operatorname{coker}(d_1) \to X$ through the universal property of the cokernel. The morphisms f_0 and $\overline{f_0}$ are more specifically related by the commutativity of the following triangle:

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[d, "p_{C_\bullet}"'] & X \\
\operatorname{coker}(d_1) \arrow[ur, dashed, "\overline{f_0}"'] &
\end{tikzcd}$$

We hence define the functor $L\colon \mathbf{Ch}_{\geq 0}(\mathcal{A})$ on objects by

$$L(C_\bullet) = \operatorname{coker}(d_1^C)\,.$$

For every morphism of chain complexes $f\colon C_\bullet \to D_\bullet$ we let

$$L(f)\colon \operatorname{coker}(d_1^C) \to \operatorname{coker}(d_1^C)$$

be the unique morphism in $\mathcal{A}$ that makes the square

$$\begin{tikzcd}
C_0 \arrow[r, "p_{C_\bullet}"] \arrow[d, "f_0"'] & L(C_\bullet) \arrow[d, dashed, "L(f)"] \\
D_0 \arrow[r, "p_{D_\bullet}"'] & L(D_\bullet)
\end{tikzcd}$$

commute. (That this defines a functor follows from the functoriality of the cokernel.)

We have argued above as to why the map

$$\begin{aligned} \varphi_{C_\bullet, X} \colon \operatorname{Hom}_{\mathbf{Ch}_{\geq 0}(\mathcal{A})}(C_\bullet, I_0(X)) &\to \operatorname{Hom}_{\mathcal{A}}(L(C_\bullet), X)\,, \\ f &\mapsto \overline{f_0} \end{aligned}$$

is a bijection. This bijection in natural in both $C_\bullet$ and X: Let $g \colon D_\bullet \to C_\bullet$ be a morphism of $\mathcal{A}$-valued bounded chain complexes and let $h \colon X \to Y$ be a morphism in $\mathcal{A}$. Then the diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathbf{Ch}_{\geq 0}(\mathcal{A})}(C_\bullet, I_0(X)) \arrow[r, "I_0(h)\circ(-)\circ g"] \arrow[d, "\varphi_{C_\bullet, X}"'] & \operatorname{Hom}_{\mathbf{Ch}_{\geq 0}(\mathcal{A})}(D_\bullet, I_0(Y)) \arrow[d, "\varphi_{D_\bullet, Y}"] \\ \operatorname{Hom}_{\mathcal{A}}(L(C_\bullet), X) \arrow[r, "h\circ(-)\circ L(g)"'] & \operatorname{Hom}_{\mathcal{A}}(L(D_\bullet), Y) \end{tikzcd} \tag{2}$$

commutes. Indeed, it follows for every morphism $f \in \operatorname{Hom}_{\mathbf{Ch}_{\geq 0}(\mathcal{A})}(C_\bullet, I_0(X))$ from the commutativity of the square

$$\begin{tikzcd} C_\bullet \arrow[r, "f"] & I_0(X) \arrow[d, "I_0(h)"] \\ D_\bullet \arrow[u, "g"] \arrow[r, "I_0(h)\circ f\circ g"'] & I_0(Y) \end{tikzcd}$$

that in degree 0 the following square commutes:

$$\begin{tikzcd} C_0 \arrow[r, "f_0"] & X \arrow[d, "h"] \\ D_0 \arrow[u, "g_0"] \arrow[r, "(I(h)\circ f\circ g)_0"'] & Y \end{tikzcd}$$

We may extend this commutative square to the following diagram:

$$\begin{tikzcd} L(C_\bullet) \arrow[rrrr, "\overline{f_0}"] & & & & X \arrow[dddd, "h"] \\ & C_0 \arrow[ul, "p_{C_\bullet}"'] \arrow[rr, "f_0"] & & X \arrow[ur, equal] \arrow[dd, "h"] & \\ & & & & \\ & D_0 \arrow[uu, "g_0"] \arrow[dl, "p_{D_\bullet}"] \arrow[rr, "(I(h)\circ f\circ g)_0"'] & & Y \arrow[dr, equal] & \\ L(D_\bullet) \arrow[uuuu, "L(g)"] \arrow[rrrr, "\overline{(I(h)\circ f\circ g)_0}"'] & & & & Y \end{tikzcd}$$

The four added trapezoids commute by the constructions of both L and $\overline{(-)}$. It follows that the outer square commutes, because the morphism $p_{D_\bullet}$ is an epimorphsm and

$$\begin{aligned} \overline{(I(h)\circ f\circ g)_0} \circ p_{D_\bullet} &= (I(h)\circ f\circ g)_0 = h \circ f_0 \circ g_0 \\ &= h \circ \overline{f_0} \circ p_{C_\bullet} \circ g_0 = h \circ \overline{f_0} \circ L(g) \circ p_{D_\bullet}. \end{aligned}$$

This shows that
$$\overline{(I(h) \circ f \circ g)_0} = h \circ \overline{f_0} \circ L(g)\,.$$
This proves the desired commutativity of the square (2), and whence that the functor L is left adjoint to the functor I_0.

Remark 1. We have for every bounded chain complex $C_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$ that
$$\operatorname{coker}(d_1) = \mathrm{H}_0(C_\bullet)\,.$$
The desired left adjoint can therefore also be described as the zeroeth homology H_0. But the above explicit description of $L(C_\bullet)$ as $\operatorname{coker}(d_1)$ works also for unbouded chain complexes, and hence also gives a left adjoint $\mathbf{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ of $I_0 \colon \mathcal{A} \to \mathbf{Ch}_\bullet(\mathcal{A})$.

(ii)

This follows from part (i) by duality. The right adjoint $R\colon \mathbf{Ch}^{\geq 0}(\mathcal{A})$ of the functor I^0 is given on objects by
$$R(C^\bullet) = \ker(d_C^0)\,,$$
and assigns to every morphism $f\colon C^\bullet \to D^\bullet$ of (bounded) cochain complexes the unique morphism $\ker(d_C^0) \to \ker(d_D^0)$ that makes the square
$$\begin{tikzcd}
\ker(d_C^0) \arrow[r] \arrow[d, dashed, "R(f)"'] & C^0 \arrow[d, "f^0"] \\
\ker(d_D^0) \arrow[r] & D^0
\end{tikzcd}$$
commute.

Exercise 4.

(i)

This follows from part (ii) because A is free as an A-module.

(ii)

We start by showing that every free A-module F is projective: Let $f\colon M \to N$ be an epimorphism of A-modules and let $g\colon F \to N$ be any homomorphism of A-modules. There exists by assumption a basis $(b_i)_{i\in I}$ of F, and for every $i \in I$ an element $m_i \in M$ with $f(m_i) = g(b_i)$. It follows for the unique homomorphism of A-modules $g'\colon F \to M$ with $g'(b_i) = m_i$ for every $i \in I$ that
$$f(g'(b_i)) = f(m_i) = g(b_i)$$

for every $i \in I$, and hence that $f \circ g' = g$. This means that the triangle

$$\begin{tikzcd} & P \arrow[dl, dashed, "g'"'] \arrow[d, "g"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commutes. This shows that F is projective.

It follows that direct summands of free modules are again projective, because direct summands of projective modules are again projective. (As known from Exercise 2.)

Suppose on the other hand that P is a projective A-module. Then there exists for the free A-module F with basis $(b_p)_{p \in P}$ a unique homomorphism of A-modules $f \colon F \to P$ with $f(b_p) = p$ for every $p \in P$. The homomorphism f is an epimorphism and the resulting short exact sequence

$$0 \to \ker(f) \to F \to P \to 0$$

splits because P is projective. Hence $P \oplus \ker(f) \cong F$ is free, which shows that P is a direct summand of a free module.