Group 1, Wednesday 8–10, Tutor: Nils Waßmuth December 13, 2018

Exercises in Foundations in Representation Theory

Exercise Sheet 9

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Exercise 1.

We write $C'_\bullet$ instead of $D_\bullet$.

We first note that for any two parallel morphisms $f, g \colon X \to Y$ in $\mathcal{A}$, it holds that $f = g$ if and only if $fx = gx$ for every point $x \in_{\mathcal{A}} X$. Indeed, we may choose $x = \mathrm{id}_X \colon X \to X$ as a test point.

To show that $\mathrm{H}_n(f) = 0$ we hence need to show that $\mathrm{H}_n(f) h_n = 0$ for every point $h_n \in_{\mathcal{A}} \mathrm{H}_n(C_\bullet)$. For this we consider the following commutative diagram:

$$
\begin{tikzcd}
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, two heads, "p"] \arrow[dd, "f_{n+1}"'] & & C_n \arrow[rr, "d_n"] \arrow[dd, "f_n"] & & C_{n-1} \arrow[dd, "f_{n-1}"] & \\
& \mathrm{B}_n(C_\bullet) \arrow[ur, hook, "i"] \arrow[rr, hook, dashed, "\lambda"] \arrow[dd, dashed, "\mathrm{B}_n(f)"] & & \mathrm{Z}_n(C_\bullet) \arrow[ul, hook', "j"'] \arrow[rr, two heads, dashed, "q"] \arrow[dd, dashed, "\mathrm{Z}_n(f)"] & & \mathrm{H}_n(C_\bullet) \arrow[dd, dashed, "\mathrm{H}_n(f)"] \\
C'_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, two heads, "p'"'] & & C'_n \arrow[rr, "d_n"] & & C'_{n-1} & \\
& \mathrm{B}_n(C'_\bullet) \arrow[ur, hook, "i'"] \arrow[rr, hook, dashed, "\lambda'"] & & \mathrm{Z}_n(C'_\bullet) \arrow[ul, hook', "j'"'] \arrow[rr, two heads, dashed, "q'"] & & \mathrm{H}_n(C'_\bullet)
\end{tikzcd}
$$

We also consider a null homotopy $s = (s_n)_{n \in \mathbb{Z}}$ for f, i.e. morphisms $s_n \colon C_n \to C'_{n+1}$ with

$$f_n = d_{n+1} s_n + s_{n-1} d_n$$

for every $n \in \mathbb{Z}$.

There exists a point $z_n \in_{\mathcal{A}} \mathrm{Z}_n(C_\bullet)$ with $h_n \equiv q z_n$ because q is an epimorphism. Let

$$\begin{aligned} c_n &\coloneqq j z_n \in_{\mathcal{A}} C_n \,, \\ z'_n &\coloneqq \mathrm{Z}_n(f) c_n \in_{\mathcal{A}} \mathrm{Z}_n(C'_\bullet) \,, \\ c'_n &\coloneqq f_n c_n = f_n j z_n = j' \mathrm{Z}_n(f) c_n = j' z'_n \in_{\mathcal{A}} C'_n \,. \end{aligned}$$

We have that

$$d_n c_n = d_n j z_n = 0$$

because $d_n j = 0$. It therefore follows from the relation $f_n = d_{n+1} s_n + s_{n-1} d_n$ that

$$c'_n = f_n c_n = (d_{n+1} s_n + s_{n-1} d_n) c_n = d_{n+1} s_n c_n + s_{n-1} d_n c_n = d_{n+1} s_n c_n \,.$$

For the points

$$\begin{aligned} c'_{n+1} &\coloneqq s_n c_n \in_{\mathcal{A}} C'_{n+1} \,, \\ b'_n &\coloneqq p' c'_{n+1} \in_{\mathcal{A}} \mathrm{B}_n(C'_\bullet) \end{aligned}$$

we hence have

$$c'_n = d_{n+1} s_n c_n = d_{n+1} c'_{n+1} = i' p' c'_{n+1} = i' b'_n \,.$$

It holds that

$$\lambda' b'_n = z'_n$$

because

$$j' \lambda' b'_n = i' b'_n = c'_n = j' z'_n$$

and j' is a monomorphism. We find overall that

$$\mathrm{H}_n(f) h_n \equiv \mathrm{H}_n(f) q z_n = q' \mathrm{Z}_n(f) z_n = q' z'_n = q' \lambda' b'_n = 0$$

because $q' \lambda' = 0$, and hence $\mathrm{H}_n(f) h_n = 0$.

Exercise 2.

We add another two conditions:

(iv) There exists an isomorphism $X' \oplus X'' \to X$ that makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[d, equal] & X' \oplus X'' \arrow[r, "{[0 \ \mathrm{id}]}"] \arrow[d] & X'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] & X'' \arrow[r] & 0 \end{tikzcd}$$

commute.

(v) There exists an isomorphism $X \to X' \oplus X''$ that makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d] & X'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X' \arrow[r, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] & X' \oplus X'' \arrow[r, "{[0 \ \mathrm{id}]}"] & X'' \arrow[r] & 0 \end{tikzcd}$$

commute.

(i) $\implies$ (iv)

Let $s\colon X'' \to X$ be a morphism with $gs = \mathrm{id}_{X''}$. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[d, equal] & X' \oplus X'' \arrow[r, "\begin{bmatrix} 0 & \mathrm{id} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} f & s \end{bmatrix}"] & X'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] & X'' \arrow[r] & 0
\end{tikzcd}$$

commutes because

$$\begin{bmatrix} f & s \end{bmatrix} \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} = f$$

and

$$g \begin{bmatrix} f & s \end{bmatrix} = \begin{bmatrix} gf & gs \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{id} \end{bmatrix} .$$

It follows from the five lemma that the morphism $\begin{bmatrix} f & s \end{bmatrix}$ is an isomorphism.

(iv) $\implies$ (iii)

We denote the given isomorphism $X' \oplus X'' \to X$ by α. Through α we can also make X into a biproduct $(X, (i, j), (p, q))$ of X' and X'', with

$$i \coloneqq \alpha \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} , \quad j \coloneqq \alpha \begin{bmatrix} 0 \\ \mathrm{id} \end{bmatrix} , \quad p \coloneqq \begin{bmatrix} \mathrm{id} & 0 \end{bmatrix} \alpha^{-1} , \quad q \coloneqq \begin{bmatrix} 0 & \mathrm{id} \end{bmatrix} \alpha^{-1} .$$

It follows from the commutativity of the given diagram that $i = f$ and $q = g$, so we may choose $r = p$ and $s = j$.

(iii) $\implies$ (i)

It holds that $gs = \mathrm{id}$ by definition of a biproduct.

Rest

We have shown that the conditions (i), (iv) and (iii) are equivalent, and we find dually that the conditions (ii), (v) and (iii) are equivalent.

Exercise 3.

We first examine how for any morphism of chain complexes $f\colon C_\bullet \to D_\bullet$ the morphisms of chain complexes $\mathrm{cone}(f) \to E_\bullet$ look like for any other chain complex $E_\bullet$ in $\mathcal{A}$: Such a morphism of chain complexes

$$g = (g_n)_{n \in \mathbb{Z}} \colon \mathrm{cone}(f) \to E_\bullet$$

is given by morphisms

$$g_n = \begin{bmatrix} s_{n-1} & h_n \end{bmatrix} : C_{n-1} \oplus D_n \to E_n$$

for some morphisms $s_{n-1} \colon C_{n-1} \to E_n$ and $h_n \colon D_n \to E_n$, that are subject to the relation

$$\begin{bmatrix} s_{n-2} & h_{n-1} \end{bmatrix} \begin{bmatrix} -d_{n-1} & 0 \\ -f_{n-1} & d_n \end{bmatrix} = d_n \begin{bmatrix} s_{n-1} & h_n \end{bmatrix}$$

for every $n \in \mathbb{Z}$ (where d_n denotes the differentials of $C_\bullet$, $D_\bullet$ and $E_\bullet$). This means that

$$\begin{bmatrix} -s_{n-2}d_{n-1} - h_{n-1}f_{n-1} & h_{n-1}d_n \end{bmatrix} = \begin{bmatrix} d_n s_{n-1} & d_n h_h \end{bmatrix}$$

for every $n \in \mathbb{Z}$, i.e. that

$$\begin{cases} d_n(-s_{n-1}) + (-s_{n-2})d_{n-1} & = & h_{n-1}f_{n-1}\,, \\ h_{n-1}d_n & = & d_n h_n\,. \end{cases}$$

The second condition states that the family $h \coloneqq (h_n)_{n \in \mathbb{Z}}$ is a morphism of chain complexes $h \colon D_\bullet \to E_\bullet$, whereas the first condition states that $s \coloneqq (-s_n)_{n \in \mathbb{Z}}$ defines a null homotopy for the morphism $hf \colon C_\bullet \to E_\bullet$.

A morphism of chain complexes $\operatorname{cone}(f) \to E_\bullet$ is therefore the same as a morphism of chain complexes $h \colon D_\bullet \to E_\bullet$ such that hf is null homotopic, together with the choice of a null homotopy for hf. This entails that a morphism of chain complexes $h \colon D_\bullet \to E_\bullet$ can be extended to a morphism of chain complexes $\operatorname{cone}(f) \to E_\bullet$ if and only if the composition hf is nilpotent.

It follows that a morphism of chain complexes $f \colon C_\bullet \to D_\bullet$ can be extended to a morphism of chain complexes $\operatorname{cone}(\mathrm{id}_{C_\bullet}) \to D_\bullet$ if and only if the composition $f\,\mathrm{id}_{C_\bullet} = f$ is null homotopic.

Exercise 4.

(i)

The components of the morphism $fh \colon \operatorname{cone}(f)[1] \to D_\bullet$ are given by

$$(fh)_n = f_n h_n = f_n \begin{bmatrix} -\mathrm{id} & 0 \end{bmatrix} = \begin{bmatrix} -f_n & 0 \end{bmatrix}$$

for every $n \in \mathbb{Z}$. The collection $s = (s_n)_{n \in \mathbb{Z}}$ of morphisms

$$s_n \coloneqq \begin{bmatrix} 0 & -\mathrm{id} \end{bmatrix} : C_n \oplus D_{n+1} \to D_{n+1}$$

are a null homotopy for fh because

$$\begin{aligned} d_{n+1}s_n + s_{n-1}d_n &= d_{n+1} \begin{bmatrix} 0 & -\mathrm{id} \end{bmatrix} + \begin{bmatrix} 0 & -\mathrm{id} \end{bmatrix} \begin{bmatrix} d_n & 0 \\ f_n & -d_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -d_{n+1} \end{bmatrix} \begin{bmatrix} -f_n & d_{n+1} \end{bmatrix} = \begin{bmatrix} -f_n & 0 \end{bmatrix} = (fh)_n \end{aligned}$$

for every $n \in \mathbb{Z}$.

(ii)

We know from the previous part of the exercise that hf is null homotpic in $\mathbf{Ch}_\bullet(\mathcal{A})$, hence that $hf = 0$ in $\mathbf{K}_\bullet(\mathcal{A})$. It follows that $h = 0$ in $\mathbf{K}_\bullet(\mathcal{A})$ because f is a monomorphism in $\mathbf{K}_\bullet(\mathcal{A})$, hence that h in null homotopic in $\mathbf{Ch}_\bullet(\mathcal{A})$.

(iii)

Let s be a null homotopy for the morphism h, i.e. let $s = (s_n)_{n\in\mathbb{Z}}$ be a family of morphisms $s_n \colon C_n \oplus D_{n+1} \to C_{n+1}$ with

$$h_n = d_{n+1}s_n + s_{n-1}d_n$$

for every $n \in \mathbb{Z}$. We may write every morphism s_n as

$$s_n = \begin{bmatrix} t_n & -u_{n+1} \end{bmatrix}$$

for some morphisms $t_n \colon C_n \to C_{n+1}$ and $u_n \colon D_{n+1} \to C_{n+1}$. Then

$$\begin{aligned} \begin{bmatrix} -\mathrm{id} & 0 \end{bmatrix} &= h_n \\ &= d_{n+1}s_n + s_{n-1}d_n \\ &= d_{n+1}\begin{bmatrix} t_n & -u_{n+1} \end{bmatrix} + \begin{bmatrix} t_{n-1} & -u_n \end{bmatrix}\begin{bmatrix} d_n & \\ f_n & -d_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} d_{n+1}t_n & -d_{n+1}u_{n+1} \end{bmatrix}\begin{bmatrix} t_{n-1}d_n - u_nf_n & u_nd_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} d_{n+1}t_n + t_{n-1}d_n - u_nf_n & u_nd_{n+1} - d_{n+1}u_{n+1} \end{bmatrix}. \end{aligned}$$

We hence have for every $n \in \mathbb{Z}$ that

$$\begin{cases} u_nf_n - \mathrm{id} &= d_{n+1}t_n + t_{n-1}d_n\,, \\ u_nd_{n+1} - d_{n+1}u_{n+1} &= 0\,. \end{cases}$$

The second conditions tells us that $u \coloneqq (u_n)_{n\in\mathbb{Z}}$ is a morphism of chain complexes $u \colon D_\bullet \to C_\bullet$, and the first condition states that $uf - \mathrm{id}$ is null homotopic in $\mathbf{Ch}_\bullet(\mathcal{A})$, i.e. that $uf = \mathrm{id}$ in the category $\mathbf{K}_\bullet(\mathcal{A})$.

(iv)

Parts (ii) and (iii) show that every monomorphism in $\mathbf{K}_\bullet(\mathcal{A})$ splits, hence that every kernel in $\mathbf{K}_\bullet(\mathcal{A})$ splits. Suppose that a kernel $k \colon K \to \mathbb{Z}/4$ of g in $\mathbf{K}_\bullet(\mathcal{A})$ exists.

The induced group homomorphism $\mathrm{H}_0(k) \colon \mathrm{H}_0(K) \to \mathbb{Z}/4$ is a section because k is a section. It follows from $\mathbb{Z}/4$ being indecomposable that either $\mathrm{H}_0(k) = 0$ or that $\mathrm{H}_0(k)$ is an isomorphism. We now show that neither can happen:

We find that $\mathrm{H}_n(k) \neq 0$: The inclusion $i \colon \mathbb{Z}/2 \to \mathbb{Z}/4$ satisfies $gi = 0$, hence factors through k in $\mathbf{K}_\bullet(\mathcal{A})$.

$$\begin{tikzcd} K \arrow[r, "k"] & \mathbb{Z}/4 \arrow[r, "g"] & \mathbb{Z}/2 \\ & \mathbb{Z}/2 \arrow[u, "i"'] \arrow[ul, dashed] & \end{tikzcd}$$

We get in the zeroeth homology the following induced commutative diagram:

$$\begin{tikzcd}
\mathrm{H}_0(K) \arrow[r, "\mathrm{H}_0(k)"] & \mathbb{Z}/4 \arrow[r, "g"] & \mathbb{Z}/2 \\
& \mathbb{Z}/2 \arrow[u, "i"'] \arrow[ul, dashed] &
\end{tikzcd}$$

Hence $\mathrm{H}_0(K) \neq 0$ because $i \neq 0$.

But we also find that $\mathrm{H}_0(k)$ is not an isomorphism: The zeroeth homology of the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[rr, bend right, "0"'] & \mathbb{Z}/4 \arrow[r, "g"] & \mathbb{Z}/2
\end{tikzcd}$$

is the following commutive diagram:

$$\begin{tikzcd}
\mathrm{H}_0(K) \arrow[r, "\mathrm{H}_0(k)"] \arrow[rr, bend right, "0"'] & \mathbb{Z}/4 \arrow[r, "g"] & \mathbb{Z}/2
\end{tikzcd}$$

It follows from $g \neq 0$ that $\mathrm{H}_0(k)$ is not an isomorphism.