Group 1, Wednesday 8–10, Tutor: Nils Waßmuth

Exercises in
Foundations in Representation Theory

Exercise Sheet 1

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Exercise 1.

We convince ourselves that $A^\times$ is a group with respect to the multiplication of A.

- If $a_1, a_2 \in A^\times$ are units in A then there exist $b_1, b_2 \in A$ with $a_i b_i = 1 = b_i a_i$ for both $i = 1, 2$. It then follows that

  $$(a_1 a_2)(b_2 b_1) = a_1 a_2 b_2 b_1 = a_1 \cdot 1 \cdot b_1 = a_1 b_1 = 1$$

  and similarly $(b_2 b_1)(a_1 a_2) = 1$. This shows that a_1, a_2 is again a unit, which shows that $A^\times$ is closed under the multiplication of A.

- It follows from the equation $1 \cdot 1 = 1$ that $1 \in A$ is a unit with inverse given by 1, and that 1 is a neutral element for $A^\times$.

- If $a \in A^\times$ is a unit with $ab = 1 = ba$ for some $b \in A$ then b is again a unit and an inverse for a in $A^\times$.

If $f \colon A \to B$ is a k-algebra homomorphism then it holds for $a \in A^\times$ that

$$f(a)f(a^{-1}) = f(aa^{-1}) = f(1) = 1$$

and similarly $f(a^{-1})f(a) = 1$, which shows that $f(a)$ is a unit in B. This shows that f restricts to a map $f \colon A^\times \to B^\times$, which is then still multiplicative and therefore a group homomorphism.

(i)

It follows from G being a k-basis of $k[G]$ that there exists a unique k-linear extension $f \colon k[G] \to A$ of ψ, which is given by $f(g) = \psi(g)$ for every $g \in G$. It remains to show that f is already a homomorphism of k-algebra: The map f is multiplicative on the k-basis G of $k[G]$ and therefore multiplicative on the whole of $k[G]$, because both

$$k[G] \times k[G] \to k[G]\,, \quad (x,y) \mapsto f(xy)$$

and

$$k[G] \times k[G] \to k[G]\,, \quad (x,y) \mapsto f(x)f(y)$$

are k-linear. It also holds that $f(1) = \psi(1) = 1$.

(ii)

The group algebra $k[\mathbb{Z}^n]$

We denote the basis element of $k[\mathbb{Z}^n]$ associated to the group element $(a_1, \dots, a_n) \in \mathbb{Z}^n$ by $t_1^{a_1} \cdots t_n^{a_n}$. The group algebra $k[\mathbb{Z}^n]$ has then a basis $t_1^{a_1} \cdots t_n^{a_n}$ with $a_1, \dots, a_n \in \mathbb{Z}$ on which the k-bilinear multiplication of $k[\mathbb{Z}^n]$ is given by

$$t_1^{a_1} \cdots t_n^{a_n} \cdot t_1^{b_1} \cdots t_n^{b_n} = t_1^{a_1+b_1} \cdots t_n^{a_n+b_n}\,.$$

This shows that $k[\mathbb{Z}^n] \cong k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ is the k-algebra of Lauraunt polynomials in n variables $t_1, \dots, t_n$ over k.

The group algebra $k[\mathbb{Z}/2]$

The group algebra $k[\mathbb{Z}/2]$ has a k-basis consisting of two elements $1, s$, with 1 being the unit of A and s satisfying the equation $s^2 = 1$. This shows that $k[\mathbb{Z}/2] \cong k[x]/(x^2-1)$. If $\mathrm{char}(k) = 2$ then it follows that

$$k[\mathbb{Z}/2] \cong k[x]/(x^2-1) = k[x]/((x-1)^2) \cong k[x]/(x^2)$$

is the algebra of dual numbers. If $\mathrm{char}(k) = 2$ then it follows from the chinese remainder theorem that

$$\begin{aligned} k[\mathbb{Z}/2] \cong k[x]/(x^2-1) &= k[x]/((x+1)(x-1)) \\ &\cong k[x]/(x+1) \times k[x]/(x-1) \cong k \times k\,. \end{aligned}$$

Exercise 2.

In the given quiver Q there exists for every pair (i,j) of vertices $i, j \in \{1, \dots, n\} = Q_0$ a unique path e_{ij} from i to j in Q if $i \leq j$, and no path otherwise. The paths e_{ij} with $1 \leq i \leq j \leq n$ are by definition a k-basis of kQ.

The quiver Q has a representation over k with vertices k and edges $k \xrightarrow{1} k$:

$$k \xrightarrow{\ 1\ } k \xrightarrow{\ 1\ } k \xrightarrow{\ 1\ } \cdots \xrightarrow{\ 1\ } k \xrightarrow{\ 1\ } k$$

This representation of Q over k corresponds to a kQ-module structure on $k^{\oplus n} = k^n$ as discussed in the lecture (i.e. as in the standard way). The action of e_{ij} on k^n is with respect to the standard basis given by the lower triangular matrix E_{ij}. The k-algebra homomorphism $\varphi\colon kQ \to \mathrm{End}_k(k^n) = \mathrm{M}_n(k)$ which corresponds to this kQ-module structure therefore maps e_{ij} onto E_{ij}. The homomorphisms φ therefore maps the path k-basis of kQ bijectively onto the standard k-basis of L_n, and it is hence an isomorphism of k-algebras.

Exercise 3.

(i) + (ii)

Recall that we have for all sets X, Y and Z the usual adjuction

$$\mathrm{Maps}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Maps}(X, \mathrm{Maps}(Y, Z))\,, \quad \mu \mapsto [y \mapsto \mu(y, -)]$$

is a bijection. It follows in particular that for a k-vector space V we get a bijection

$$\mathrm{Maps}(A \times V, V) \to \mathrm{Maps}(A, \mathrm{Maps}(V, V))\,, \quad \mu \mapsto [a \mapsto \mu(a, -)]\,.$$

It sufficies to show that $\mu \in \mathrm{Maps}(A \times V, V)$ is an A-module structure if and only if the corresponding element $\varphi \in \mathrm{Maps}(A, \mathrm{Maps}(V, V))$ is a homomorphism of k-algebra into $\mathrm{End}_k(V)$.

The map $\mu \in \mathrm{Maps}(A \times V, V)$ is an A-module structure if and only if the following axioms are satisfied for all $a, a_1, a_2 \in A$, all $\lambda \in k$ and all $m, m_1, m_2 \in M$:

(MS1) $\mu(a, m_1 + m_2) = \mu(a, m_1) + \mu(a, m_2)$,

(MS2) $\mu(a_1 + a_2, m) = \mu(a_1, m) + \mu(a_2, m)$,

(MS3) $\mu(a_1, \mu(a_2, m)) = \mu(a_1 a_2, m)$,

(MS4) $\mu(1, m) = m$,

(MS5) $\mu(\lambda a, m) = \lambda \mu(a, m) = \mu(a, \lambda m)$.

The map φ is a homomorphism of k-algebra into $\mathrm{End}_k(V)$ if and only if the following conditions are satisfied for all $a, a_1, a_2 \in A$ and all $\lambda \in k$:

(AH1) $\varphi(a) \in \mathrm{End}_k(M)$,

(AH2) $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$,

(AH3) $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$,

(AH4) $\varphi(\lambda a) = \lambda \varphi(a)$,

(AH5) $\varphi(1) = \mathrm{id}_V$.

Spelling out these conditions in more detail, we see that the following conditions need to be satisfied for all $a, a_1, a_2 \in A$, all $\lambda \in k$ and all $m, m_1, m_2 \in M$

(AH1*) $\varphi(a)(m_1 + m_2) = \varphi(a)(m_1) + \varphi(m_2)$,

(AH2*) $\varphi(a)(\lambda m) = \lambda\varphi(a)(m)$,

(AH3*) $\varphi(a_1 + a_2)(m) = \varphi(a_1)(m) + \varphi(a_2)(m)$,

(AH4*) $\varphi(a_1 a_2)(m) = \varphi(a_1)(\varphi(a_2)(m))$,

(AH5*) $\varphi(\lambda a)(m) = \lambda\varphi(a)(m)$,

(AH6*) $\varphi(1) = \mathrm{id}_V$.

The relation between μ and φ is so that $\mu(a, m) = \varphi(a)(m)$ for all $a \in A$ and all $m \in M$. We therefore find that condition (MS1) is equivalent to (AH1*), (MS2) is equivalent to (AH3*), (MS3) is equivalent to (AH4*), (MS4) is equivalent to (AH6*), and (MS5) is equivalent to the combination of (AH2*) and (AH5*).

(iii)

Since the map f is already k-linear, we find that

$$\begin{aligned} & f \text{ is a homomorphism of } A\text{-module} \\ \iff & f(av) = af(v) \text{ for all } a \in A,\, v \in V \\ \iff & f(\varphi(a)) = \psi(a)(f(v) \text{ for all } a \in A,\, v \in V \\ \iff & f \circ \varphi(a) = \psi(a) \circ f \text{ for all } a \in A\,. \end{aligned}$$

Exercise 4.

(i)

We give two solutions.

First Solution

Lemma 1. *Let R be a ring and let $n \geq 1$. Then the map*

$$\{\textit{two-sided ideals } I \subseteq R\} \longrightarrow \{\textit{two-sided ideals } J \subseteq \mathrm{M}_n(R)\}\,, \quad I \mapsto \mathrm{M}_n(I)$$

is a well-defined bijection.

Proof. If $I \subseteq R$ is a two-sided ideal then the canonical projection $\pi\colon R \to R/I$ is a ring homomorphism with kernel I. The kernel of the induced ring homomorphism $\pi_*\colon \mathrm{M}_n(R) \to \mathrm{M}_n(R/I)$ is given by $\ker(\pi_*) = \mathrm{M}_n(I)$, which shows that $\mathrm{M}_n(I)$ is a two-sided ideal in $\mathrm{M}_n(R)$. This shows that the proposed map is well-defined. The map is injective because one can retrieve I from $\mathrm{M}_n(I)$ as the set of all possible entries of matrices in $\mathrm{M}_n(I)$.

To show the surjectivity let $J \subseteq \mathrm{M}_n(R)$ be a two-sided ideal. For all entry positions $i, j = 1, \dots, n$ let

$$I_{ij} = \{a \in R \mid \text{there exists a matrix } A \in J \text{ whose } (i,j)\text{-th entry is } a\}\,.$$

If $\pi_{ij} \colon \mathrm{M}_n(R) \to R$ denotes the projection onto the (i,j)-th component then π_{ij} is a homomorphism of both left R-modules and of right R-modules. It follows that $I_{ij} = \pi_{ij}(I)$ is both a left R-submodule of R and a right R-submodule of R, and therefore a two-sided ideal in R.

For $A \in J$ and $i, j = 1, \dots, n$ we can multiply A from the left and from the right with permutation matrices to move the (i,j)-th entry of A to any other position. We don't leave J when doing so because J is a two-sided ideal in $\mathrm{M}_n()$. This shows that the two-sided ideal I_{ij} does not depend on the choice of position (i,j), i.e. that there exists a unique two-sided ideal $I \subseteq R$ with $\pi_{ij}(J) = I$ for all i, j. It follows from the construction of I that $J = \mathrm{M}_n(I)$. ☐

We now get the statement from the exercise for free: If $n = 0$ then $\mathrm{M}_n(k) = 0$ and there exist no nonzero two-sided ideals. If $n \geq 1$ then the two-sided ideals in $\mathrm{M}_n(k)$ correspond bijectively to the two-sided ideals in k, of which there are only two. It then follows from $\mathrm{M}_n(k) \neq 0$ that $\mathrm{M}_n(k)$ contains no proper nonzero ideal.

Second Solution

Let $I \subseteq \mathrm{M}_n(k)$ be a nonzero two-sided ideal. There then exists some matrix $A \in I$ whose (i,j)-th entry is nonzero for suitable i, j. It follows with $c \coloneqq A_{ij} \neq 0$ that

$$E_{ij} = \frac{1}{c} E_{ii} A E_{jj} \in I\,.$$

By multiplying E_{ij} from both sides with permutation matrices it then follows that I contains the standard k-basis of $\mathrm{M}_n(k)$. This shows that $I = \mathrm{M}_n(k)$ because I is a k-submodule of $\mathrm{M}_n(k)$.

(ii)

We give three solutions.

First Solution

Let M be a nonzero A-modules and let $\varphi \colon A \to \mathrm{End}_k(M)$ be the associated k-algebra homomorphism. The homomorphism φ is nonzero since its image contains the identity id_M (which is nonzero because M is nonzero). This shows that the kernel $\ker(\varphi)$, which is a two-sided ideal in A, is a proper ideal. It follows from part (i) of the exercise that $\ker(\varphi) = 0$ and therefore that φ is injective. It follows that

$$\dim_k \mathrm{End}_k(V) \geq \dim_k A = n^2$$

and therefore that $\dim_k M \geq n$.

Second Solutions

Recall that the k-algebra A is semisimple, and that k^n is up to isomorpism the only simple A-modules. It follows that every A-modules M is of the form $M = (k^n)^{\oplus I}$ for some index set I, and thus in particular that $\dim_k M \geq \dim_k k^n = n$ if M is nonzero.

Third Solution

We can also give a more ad hoc version of the above argument: For every $i = 1, \dots, n$ let $C_i = AE_{ii}$ be the left ideal of A which consist of all matrices for which all columns except for the i-th one vanish. It then holds that $A = C_1 \oplus \cdots \oplus C_n$. It holds for every i that $C_i \cong k^n$ as A-modules, and k^n is simple as an A-modules (because every nonzero element $x \in k^n$ generates k^n as an A-modules).

Let M be a nonzero A-modules and let $m \in M$ be a nonzero element. The map

$$\varphi \colon A \to M\,, \quad a \mapsto am$$

is then a nonzero homomorphism of A-modules. It follows from $A = C_1 + \cdots + C_n$ that the kernel $\ker(\varphi)$ does not contain C_i for some i. It then follows that $C_i \cap \ker(\varphi)$ is a proper A-submodule of C_i, and therefore that $C_i \cap \ker(\varphi) = 0$. This shows that the restriction $\varphi|_{C_i} \colon C_i \to M$ is injective, from which it follows that

$$\dim_k M \geq \dim_k C_i = \dim_k k^n = n\,.$$