Group 1, Wednesday 8–10, Tutor: Nils Waßmuth

**Exercises in**
**Foundations in Representation Theory**

# Exercise Sheet 3

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## Exercise 1.

We define a category $\mathcal{C}$ whose objects are given by $\mathrm{Ob}(\mathcal{C}) = \mathbb{Z}_{\geq 0}$ and where for any two nonnegative integers $n, m \in \mathbb{Z}_{\geq 0}$ the morphism set $\mathcal{C}(n, m)$ is given by the matrix space

$$\mathcal{C}(n,m) = \mathrm{M}(m \times n, k)\,.$$

The compositions of morphisms in $\mathcal{C}$ is the usual matrix multiplication, which is associative. The identity of an object $n \in \mathbb{Z}_{\geq 0}$ is given by the $(n \times n)$-identity matrix $I_n$.

We define a functor $F \colon \mathcal{C} \to k\textbf{-mod}$ by mapping any nonnegative integer $n \in \mathbb{Z}_{\geq 0}$ to the $k$-vector space $F(n) = k^n$, and mapping any matrix $A \in \mathrm{M}(m \times n, k)$ to the $k$-linear map

$$F(A) \colon k^n \to k^m\,, \quad x \mapsto Ax\,.$$

We know from linear algebra that $F$ is both fully faithful and essentially surjective, and hence an equivalence of categories.

## Exercise 2.

We have for every $k$-vector space $V$ bijections

$$\begin{aligned} &\{G\text{-representations on } V\} \\ ={}& \{\text{group homomorphisms } G \to \mathrm{GL}(V)\} \\ ={}& \{\text{group homomorphisms } G \to \mathrm{End}_k(V)^\times\} \\ \cong{}& \{k\text{-algebra homomorphisms } k[G] \to \mathrm{End}_k(V)\} \\ \cong{}& \{k[G]\text{-module structures on } V\}\,, \end{aligned}$$

where we use the adjunction $k[-] \dashv (-)^{\times}$. So we can regard every representation $(V, \rho)$ of $G$ over $k$ as an $k[G]$-module via

$$\sum_{g \in G} a_g g \cdot v = \sum_{g \in G} a_g \rho(g)(v)$$

for all $\sum_{g \in G} a_g g \in k[G]$ and all $v \in V$; we can conversely regard every $k[G]$-modules $V$ as a representation $(V, \rho)$ of $G$ over $k$ via

$$\rho(g) \colon V \to V\,, \quad v \mapsto g \cdot v$$

for all $g \in G \subseteq k[G]$ and all $v \in V$. These two constructions are moreover mutually inverse.

Let $(V, \rho)$ and $(W, \sigma)$ be two representations of $G$ over $k$ and let $f \colon V \to W$ be a $k$-linear map. Then

$$\begin{aligned}
& f \text{ is a homomorphisms of representations} \\
\iff & f(\rho(g)(v)) = \rho(g)(f(v)) \text{ for all } g \in G,\, v \in V \\
\iff & f(g \cdot v) = g \cdot f(v) \text{ for all } g \in G,\, v \in V \\
\iff & f(a \cdot v) = a \cdot f(v) \text{ for all } a \in k[G],\, v \in V \\
\iff & f \text{ is a homomorphism of } k[G]\text{-modules}\,.
\end{aligned}$$

This shows altogether that the categories $\mathbf{Rep}_k(G)$ and $k[G]$-$\mathbf{Mod}$ are isomorphic, and therefore in particular equivalent.

# Exercise 3.

We denote the given category by $\mathcal{G}$.

## (i)

A $G$-set $X$ consists of a set $X$ and for every $g \in G$ a map $X_g \colon X \to X$, in such a way that $X_e = \mathrm{id}_X$ and $X_{g_1 g_2} = X_{g_1} \circ X_{g_2}$ for all $g_1, g_2 \in G$. A functor $F \colon \mathcal{G} \to \mathbf{Set}$ consists of a single set $F(*)$ and for every $g \in \mathcal{G}(*, *) = G$ a map $F(g) \colon F(*) \to F(*)$, in such a way that $F(e) = \mathrm{id}_X$ and $F(g_1 g_2) = F(g_1) F(g_2)$ for all $g_1, g_2 \in G$. We see by setting $X = F(*)$ and $X_g = F(g)$ that both constructs consist of precisely the same data. This shows that a $G$-set $X$ is the same as a functor $F \colon \mathcal{G} \to \mathbf{Set}$.

Let $X$ and $X'$ be two $G$-sets with corresponding functors $F, F' \colon \mathcal{G} \to \mathbf{Set}$, and let $f \colon X \to X'$ be a map. Then

$$\begin{aligned}
& f \text{ is a homomorphism of } G\text{-sets} \\
\iff & f(gx) = g(f(x)) \text{ for all } g \in G,\, x \in X \\
\iff & f \circ F(g) = F'(g) \circ f \text{ for every } g \in G \\
\iff & f \colon F(*) \to F'(*) \text{ defines a natural transformation } F \to F'
\end{aligned}$$

This altogether shows that the category $G$-$\mathbf{Set}$ is isomorphic to the functor category $\mathbf{Fun}(\mathcal{G}, \mathbf{Set})$, and therefore in particular equivalent to it.

## (ii)

The $G$-set $X$ which corresponds to the covariant functor $h^* \colon \mathcal{G} \to \mathbf{Set}$ is given by the set $X = h^*(*) = \mathcal{G}(*, *) = G$, and the action of $g \in G$ on $x \in G$ is given by

$$gx = h^*(g)(x) = g_*(x) = g \circ x = gx\,.$$

The $G$-set $X$ is therefore just the regular left $G$-set, i.e. the group $G$ acting on itself by left multiplication.

## (iii)

Let $X$ be the left regular $G$-set, as before. We find that

$$\mathrm{End}_{G\text{-}\mathbf{Set}}(X) = (G\text{-}\mathbf{Set})(X, X) \cong \mathbf{Fun}(G\text{-}\mathbf{Set}, \mathbf{Set})(h^*, h^*) \cong h^*(*) = \mathcal{G}(*, *) = G\,.$$

Under this bijection, the group element $g \in G$ corresponds the the endomorphism of $G$-sets $X \to X$ which is given by right multiplication with $g$. Thus the Yoneda lemma tells us that every endomorphism of $X$ is given by right multiplication with a unique element $g \in G$. Moreover, the explicit description of the bijection

$$\mathbf{Fun}(G\text{-}\mathbf{Set}, \mathbf{Set})(h^*, h^*) \to h^*(*)\,, \quad \eta \mapsto \eta_*(*)$$

tells us that for every endomorphism of $G$-sets $\varphi \colon X \to X$, the corresponding group element $g \in G$ is given by $g = \varphi(e)$.

# Exercise 4.

We convince ourselves that $\mathcal{K}$ is indeed a category:

For any two objects $(A, B, h)$ and $(A', B', h')$ of the proposed category $\mathcal{K}$, the collection of morphisms $(A, B, h) \to (A', B', h')$ is a subset of $\mathcal{A}(A, A') \times \mathcal{B}(B, B')$, and is hence again a set (and not a proper class).

The composition of morphisms in $\mathcal{K}$ is well-defined: If $(f, g) \colon (A, B, h) \to (A', B', h')$ and $(f', g') \colon (A', B', h') \to (A'', B'', h'')$ are two morphisms then in the diagram

$$\begin{tikzcd}
S(A) \arrow[r, "h"] \arrow[d, "S(f)"'] \arrow[dd, bend right=60, "S(f' \circ f)"'] & T(B) \arrow[d, "T(g)"] \arrow[dd, bend left=60, "T(g' \circ g)"] \\
S(A') \arrow[r, "h'"] \arrow[d, "S(f')"'] & T(B') \arrow[d, "T(g')"] \\
S(A'') \arrow[r, "h''"] & T(B'')
\end{tikzcd}$$

both squares and both triangles commute; from this it follows that the outer square also commutes. That the composition in $\mathcal{K}$ is associative follows from the associativity of the compositions in $\mathcal{A}$ and $\mathcal{B}$.

For every object $(A, B, h)$ of $\mathcal{K}$, the morphism $(\mathrm{id}_A, \mathrm{id}_B)\colon (A, B, h) \to (A, B, h)$ is the identity of $(A, B, h)$ in $\mathcal{K}$ as it holds for all morphisms $(f', g')\colon (A', B', h') \to (A, B, h)$ and all morphisms $(f'', g'')\colon (A, B, h) \to (A'', B'', h'')$ that

$$(\mathrm{id}_A, \mathrm{id}_B) \circ (f', g') = (\mathrm{id}_A \circ f', \mathrm{id}_B \circ g') = (f', g')$$

and

$$(f'', g'') \circ (\mathrm{id}_A, \mathrm{id}_B) = (f'' \circ \mathrm{id}_A, g'' \circ \mathrm{id}_B) = (f'', g'') \,.$$

## (i)

A functor $S\colon 1 \to \mathcal{C}$ is the same as an object $X \in \mathrm{Ob}(\mathcal{C})$ through the assignment $X = S(*)$. An object of the comma category $\mathcal{K} = (S, \mathrm{Id}_{\mathcal{C}})$ is therefore the same as a tupel $(Y, f)$ consisting of an object $Y \in \mathrm{Ob}(\mathcal{C})$ and a morphisms $f\colon X \to Y$. A morphisms $g\colon (Y, f) \to (Y', f')$ is then a morphisms $g\colon Y \to Y$ with $g \circ f = f'$, i.e. such that the triangle

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y \arrow[rr, "g"] & & Y' \end{tikzcd}$$

commutes. The givencomma category $\mathcal{K}$ is therefore (isomorphic to) the under-$X$ category.

## (ii)

The objects of the given comma category $\mathcal{K} = (\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$ are tripels $(X, X', f)$ consisting of two objects $X, X' \in \mathrm{Ob}(\mathcal{C})$ and a morphisms $f\colon X \to X'$ between them. We can therefore identify the objects of $\mathcal{K}$ with the morphisms in $\mathcal{C}$. For any two morphisms $(X \xrightarrow{f} X')$ and $(Y \xrightarrow{g} Y')$, a morphisms $f \to f'$ in $\mathcal{K}$ is then a pair $(h, h')$ of morphisms $h\colon X \to Y$ and $h'\colon X' \to Y'$ which make the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

commute. The given comma category $\mathcal{K}$ is therefore (isomorphic to) the morphism category of $\mathcal{C}$.

## (iii)

A functor $S\colon 1 \to \mathbf{Grp}$ is again the same as choosing an object $S(*) \in \mathbf{Grp}$, i.e. choosing a group $G \coloneqq S(*)$. The objects of the given comma category $\mathcal{K} = (S, (-)^{\times})$ can therefore be identified with the pairs $(A, \varphi)$ consisting of a $k$-algebra $A$ and a group homomorphism $\varphi\colon A^\times \to G$. A morphism $f\colon (A,\varphi) \to (B,\psi)$ is then an algebra homomorphism $f\colon A \to B$ which makes the triangle

$$\begin{tikzcd}
 & G \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
A^\times \arrow[rr, "f^\times"] & & B^\times
\end{tikzcd}$$

commute. By using the adjunction $k[-] \vdash (-)^\times$ we may further identify the given comma category $\mathcal{K}$ with the under-$k[G]$ category, i.e. the category whose objects are pairs $(A, f)$ consisting of a $k$-algebra $A$ and an algebra homomorphism $f\colon k[G] \to A$, and in which a morphism $g\colon (A,f) \to (A',f')$ is an algebra homomorphism $g\colon A \to A'$ which makes the triangle

$$\begin{tikzcd}
 & k[G] \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
A \arrow[rr, "g"] & & A'
\end{tikzcd}$$

commute.