Group 1, Wednesday 8–10, Tutor: Nils Waßmuth November 22, 2018

Exercises in
Foundations in Representation Theory

Exercise Sheet 6

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Exercise 1.

(i)

Let $Z \in \mathrm{Ob}(\mathcal{C})$ be an object and let $\alpha\colon Z \to X$ and $\delta\colon Z \to Y''$ be two morphisms in $\mathcal{C}$ with $f \circ \alpha = g \circ h \circ \delta$, i.e. such the diagram

$$\begin{tikzcd}
Z \arrow[rrrd, dashed, bend left, "\alpha"] \arrow[rdd, dashed, bend right, "\delta"'] & & & \\
 & X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
 & Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. By applying the right-hand pull-back square to the morphisms $\alpha\colon Z \to X$ and $h \circ \delta\colon Z \to Y'$, we find that there exist a unique morphism $\beta\colon Z \to X'$ with $g' \circ \beta = \alpha$ and $f' \circ \beta = h \circ \delta$, i.e. such that the diagram

$$\begin{tikzcd}
Z \arrow[rrrd, bend left, "\alpha"] \arrow[rrd, dashed, bend left, "\beta"] \arrow[rdd, bend right, "\delta"'] & & & \\
 & X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
 & Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. By applying the left-hand pull-back square to the morphisms $\beta\colon Z \to X'$ and and $\delta\colon Z \to Y''$, we find that there exist a unique morphism $\gamma\colon Z \to X''$ with $h' \circ \gamma = \beta$

and $f'' \circ \gamma = \delta$, i.e. such that the diagram

$$\begin{tikzcd}
Z \arrow[drrr, bend left, "\alpha"] \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes.

Suppose that $\gamma' \colon Z \to X''$ is another morphism which makes the diagram

$$\begin{tikzcd}
Z \arrow[drrr, bend left, "\alpha"] \arrow[dr, dashed, "\gamma'"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commute. Then by setting $\beta' \coloneqq h' \circ \gamma'$ we get the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[drrr, bend left, "\alpha"] \arrow[drr, bend left, dashed, "\beta'"] \arrow[dr, "\gamma'"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Then $\beta' = \beta$ by the uniqueness of β and thus $\gamma' = \gamma$ by the uniqueness of γ.

(ii)

Let $Z \in \mathrm{Ob}(\mathcal{C})$ be an object and let $\delta \colon Z \to Y''$ and $\beta \colon Z \to X'$ be two morphisms in $\mathcal{C}$ with $h \circ \delta = f' \circ \beta$, i.e. such that the diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, dashed, "\beta"] \arrow[ddr, bend right, dashed, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. For the morphism $\alpha' := g' \circ \beta \colon Z \to X$ we get following commutative diagram:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "\beta"] \arrow[drrr, bend left, dashed, "\alpha"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

By using the outer pull-back square for the morphisms $\alpha \colon Z \to X$ and $\delta \colon Z \to Y''$, we find that there exists a unique morphism $\gamma \colon Z \to X''$ with $g' \circ h' \circ \gamma = \alpha$ and $f'' \circ \gamma = \delta$, i.e. such that the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\gamma"] \arrow[drrr, bend left, "\alpha"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. We claim that already the whole diagram

$$\begin{tikzcd}
Z \arrow[dr, "\gamma"] \arrow[drr, bend left, "\beta"] \arrow[drrr, bend left, "\alpha"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. For this we still have to check that $h' \circ \gamma = \beta$. The right-hand square is a pushout square, so it follows for any two morphisms $k_1, k_2 \colon Z \to X'$ that $k_1 = k_2$ if and only if both $g' \circ k_1 = g' \circ k_2$ and $f' \circ k_1 = f' \circ k_2$. This holds true for $k_1 = \beta$ and $k_2 = h' \circ \gamma$ because

$$g' \circ \beta = \alpha = g' \circ h' \circ \gamma$$

by construction of γ, and

$$f' \circ \beta = h \circ \delta = h \circ f'' \circ \gamma = f' \circ h' \circ \gamma$$

by choice of β and δ and construction of γ.

Suppose that $\gamma'\colon Z \to X''$ is another morphism which makes the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\gamma'"] \arrow[drr, bend left, "\beta"] \arrow[drrr, bend left, "\alpha"] \arrow[ddr, bend right, "\delta"'] & & & \\
& X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commute. Then $f'' \circ \gamma' = \delta$ and $g' \circ h' \circ \gamma' = \alpha$, and therefore $\gamma' = \gamma$ by the uniqueness of γ.

Exercise 2.

If $h\colon Z \to X$ is any morphism in $\mathcal{C}$, then for $\alpha := g' \circ h$ and $\beta := f' \circ h$ the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, "h"] \arrow[drr, dashed, bend left, "\alpha"] \arrow[ddr, dashed, bend right, "\beta"'] & & \\
& X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

The morphism h is uniquely determined by the compositions α and β because the given diagram is a pull-back square. The morphisms α is uniquely determined by the composition $f \circ \alpha$ because f is a monomorphism, and this composition is given by $f \circ \alpha = g \circ \beta$. Hence h is uniquely determined by $\beta = f' \circ h$, which shows that f' is a monomorphism.

Remark 1. In an abelian category $\mathcal{A}$ the converse also holds, i.e. if the diagram

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

in $\mathcal{A}$ is a pull-back square and f' is a monomorphism then f is also a monomorphism. Indeed, the canonical morphism $\ker(f) \to X$ fits into the following commutative diagram:

$$\begin{tikzcd}
\ker(f) \arrow[drr, dashed, bend left] \arrow[ddr, dashed, bend right, "0"'] & & \\
& X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

It follows that there exist a unique morphism $\lambda\colon \ker(f) \to X'$ which makes the diagram

$$\begin{tikzcd}
\ker(f) \arrow[drr, bend left] \arrow[dr, dashed, "\lambda"] \arrow[ddr, bend right, "0"'] & & \\
& X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

commute. It follows from $0 = f' \circ \lambda$ and f being a monomorphism that also $\lambda = 0$. The canonical morphism $\ker(f) \to X$ is therefore given by

$$g' \circ \lambda = g' \circ 0 = 0\,.$$

This shows that $\ker(f) = 0$ and hence that f is a monomorphism.

Remark 2. It holds dually for a pushout square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"'] \\
Y \arrow[r, "g'"] & Y'
\end{tikzcd}$$

that if f is an epimorphism, then f' is also an epimorphism; in an abelian category, the converse also holds.

Exercise 3.

Proposition 3. Let $\mathcal{A}$ be an additive category. Then a diagram

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

in $\mathcal{A}$ is a pull-back square if and only if in the squence

$$X' \xrightarrow{\begin{bmatrix} g' \\ f' \end{bmatrix}} X \oplus Y' \xrightarrow{[\, f \;\; -g \,]} Y \tag{1}$$

the morphism $X' \to X \oplus Y'$ is a kernel of the morphism $X \oplus Y' \to Y$ (i.e. the sequence is left exact).

Proof. Let $p\colon X \oplus Y' \to X$ and $q\colon X \oplus Y' \to Y'$ be the canonical projections belonging to the biproduct $X \oplus Y$, and let

$$d\colon X \oplus Y' \xrightarrow{[\, f \;\; -g \,]} Y\,.$$

It then holds for every object $Z \in \mathrm{Ob}(\mathcal{C})$ and every two morphisms $\alpha\colon Z \to X$ and $\beta\colon Z \to Y$ that

$$f \circ \alpha = g \circ \beta \iff f \circ \alpha - g \circ \beta = 0 \iff \begin{bmatrix} f & -g \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \iff d \circ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0\,.$$

Let now $k\colon X' \to X \oplus Y'$ be a morphism, which is uniquely of the form

$$k = \begin{bmatrix} g' \\ f' \end{bmatrix}$$

for some morphisms $g'\colon X' \to X$ and $f'\colon X' \to Y'$. We find from the above calculation that the square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{2}$$

commutes if and only if $d \circ k = 0$. We moreover find that

the diagram (2) is a pull-back square

$\iff$ there exist for all morphisms $\alpha\colon Z \to X$ and $\beta\colon Z \to Y'$ with $f \circ \alpha = g \circ \beta$ a unique morphism $\lambda\colon Z \to X'$ with $g' \circ \lambda = \alpha$ and $f' \circ \lambda = \beta$

$\iff$ there exist for all morphisms $\alpha\colon Z \to X$ and $\beta\colon Z \to Y'$ with $d \circ \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] = 0$ a unique morphism $\lambda\colon Z \to X'$ with $k \circ \lambda = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$

$\iff$ there exist for every morphism $\gamma\colon Z \to X \oplus Y'$ with $d \circ \gamma = 0$ a unique morphism $\lambda\colon Z \to X'$ with $k \circ \lambda = \gamma$

$\iff$ k is a kernel for d.

This shows the claimed equivalence. □

Remark 4. Instead of the sequence (1) one can also use variations such as

$$X' \xrightarrow{\left[\begin{smallmatrix} g' \\ f' \end{smallmatrix}\right]} X \oplus Y' \xrightarrow{[\,-f\ g\,]} Y\,,$$

$$X' \xrightarrow{\left[\begin{smallmatrix} g' \\ -f' \end{smallmatrix}\right]} X \oplus Y' \xrightarrow{[\,f\ g\,]} Y\,,$$

$$X' \xrightarrow{\left[\begin{smallmatrix} -g' \\ f' \end{smallmatrix}\right]} X \oplus Y' \xrightarrow{[\,f\ g\,]} Y\,.$$

Remark 5. The dual version of Proposition 3 states that in an additive category $\mathcal{A}$, a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow{g'} & Y' \end{array}$$

is a pushout square if and only if in the sequence

$$X \xrightarrow{\begin{bmatrix} g \\ f \end{bmatrix}} X' \oplus Y \xrightarrow{[\,f' \ -g'\,]} Y'$$

the morphism $X' \oplus Y \to Y'$ is a cokernel of the morphism $X \to X' \oplus Y$ (i.e. the sequence is right exact). One can again vary this sequence, just as done in Remark 4 for pullbacks.

Exercise 4.

Lemma 6. Let $X, Y_1, Y_2 \in \mathrm{Ob}(\mathcal{C})$ be objects, let $f_1 \colon X \to Y_1$ be an epimorphism and let $f_2 \colon X \to Y_2$ be a morphism. Then the morphism

$$f \colon X \xrightarrow{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}} Y_1 \oplus Y_2$$

is also an epimorphism.

Proof. Let $g_1, g_2 \colon Y_1 \oplus Y_2 \to Z$ be two parallel morphism with $g_1 \circ f = g_2 \circ f$. If $i \colon Y_1 \to Y_1 \oplus Y_2$ is the canonical morphism into the first summand then

$$g_1 \circ f = g_2 \circ f \implies g_1 \circ f \circ i = g_2 \circ f \circ i \implies g_1 \circ f_1 = g_2 \circ f_1 \implies g_1 = g_2$$

because f_1 is an epimorphism. ☐

Lemma 7. Let $f \colon X \to Y$ be an epimorphism in an abelian category. Then f is a cokernel of its kernel.

Proof. The zero morphism $Y \to \mathrm{coker}(f)$ is a cokernel of f because f is an epimorphism, and the identity morphism $\mathrm{id}_Y \colon Y \to Y$ is therefore an image of f. Together with the canonical factorization $\tilde{f} \colon \mathrm{coim}(f) \to \mathrm{im}(f) = Y$ of the morphism f, which is an isomorphism beause $\mathcal{A}$ is abelian, we get the following commutative triangle:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \\
\mathrm{coim}(f) \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

The coimage $X \to \mathrm{coim}(f)$ is a cokernel of $\ker(f) \to X$. It hence follows from the commutativity of the above diagram and $\tilde{f}$ being an isomorphism that f is also a cokernel of $\ker(f) \to X$. ☐

Let now

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd} \tag{3}$$

be a pullback diagram in $\mathcal{A}$ such that f is an epimorphism. It follows from Proposition 3 that in the sequence

$$X' \xrightarrow{\begin{bmatrix} g' \\ f' \end{bmatrix}} X \oplus Y' \xrightarrow{[\, f \ -g \,]} Y$$

the morphism $X' \to X \oplus Y'$ is a kernel of the morphism $X \oplus Y' \to Y$. It follows from Lemma 6 that the morphism $X \oplus Y' \to Y$ is again an epimorphism, and it hence follows from Lemma 7 that the morphism $X \oplus Y' \to Y$ is a cokernel of the morphism $X' \to X \oplus Y'$. The diagram (3) is therefore a pushout square by Remark 5. It hence follows from Remark 2 that f' is again an epimorphism (beause $\mathcal{A}$ is abelian).