Exercises in Foundations in Representation Theory

Exercise Sheet 7

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Exercise 1.

We write $S(F')$ instead of sF'. We show that $S(F')$—together with suitable restriction maps—is a sheaf, and that $S(F')$—together with a suitable homomorphism of presheaves $i\colon F' \to S(F')$—satisfies the universal property of the sheafification of F'.

$S(F)(U)$ is a Subgroup of $F(U)$

We start by showing that the subset $S(F')(U)$ of $F(U)$ is for any open subset $U \subseteq X$ an abelian subgroup, and that $S(F')(U)$ contains $F'(U)$.

By taking for U the open cover $\{U\}$ we see that $F'(U) \subseteq S(F')(U)$. This shows in particular that $0 \in S(F')(U)$.

Suppose that $s, t \in S(F')(U)$ are any two elements. Then there exist an open cover $\{U_i\}_{i \in I}$ of U and an open cover $\{V_j\}_{j \in J}$ of V such that $s|_{U_i} \in F'(U_i)$ for every $i \in I$ and $t|_{V_j} \in F'(V_j)$ for every $j \in I$. Then

$$U = U \cap U = \left(\bigcup_{i \in I} U_i \right) \cap \left(\bigcup_{j \in J} V_j \right) = \bigcup_{(i,j) \in I \times J} U_i \cap V_j \,,$$

hence $\{W_{ij}\}_{(i,j) \in I \times J}$ with $W_{ij} = U_i \cap V_j$ for every $(i,j) \in I \times J$ is again an open cover of U. It follows for all $(i,j) \in I \times J$ that

$$s|_{W_{ij}} = (s|_{U_i})|_{W_{ij}} \in F'(W_{ij})$$

because $s|_{U_i} \in F'(U_i)$. We similarly find that $t|_{W_{ij}} \in F'(W_{ij})$ for every $(i,j) \in I \times J$. It follows that

$$(s - t)|_{W_{ij}} = s|_{W_{ij}} - t|_{W_{ij}} \in F'(W_{ij})$$

for every $(i,j) \in I \times J$. Hence $s - t \in S(F')(U)$ by definition of $S(F')$.

This shows that $S(F')(U)$ is indeed a subgroup of $F(U)$, and that $S(F')(U)$ contains $F'(U)$.

$S(F')$ is a Subpresheaf of F

We claim that $S(F')$ is a subpresheaf of F, i.e. that

$$\rho_{V,U}(S(F')(V)) \subseteq S(F')(U)$$

for all open subsets $U \subseteq V \subseteq X$.

If $s \in S(F')(V)$ is a section then there exist an open cover $\{V_i\}_{i\in I}$ with $s\,|_{V_i} \in F'(U)$ for every $i \in I$. Then

$$U = U \cap V = U \cap \bigcup_{i\in I} V_i = \bigcup_{i\in I} (U \cap V_i)\,,$$

hence $\{U_i\}_{i\in I}$ with $U_i \coloneqq U \cap V_i$ for every $i \in I$ is an open cover of U. It holds for every $i \in I$ that

$$s\,|_{U_i} = (s\,|_{V_i})\,|_{U_i} \in F'(U_i)$$

because $s\,|_{V_i} \in F'(V_i)$. Hence $s\,|_U \in S(F')(U)$ by definition of $S(F')$.

$S(F')$ is Already a Sheaf

If a presheaf satisfies the separability axiom then the same holds for any of its subpresheafs. It holds in particular that subpresheafs of sheafs do satisfy the separability axiom. Hence $S(F')$, which is a subpresheaf of the sheaf F, satisfies the separability axiom.

Let now $U \subseteq X$ be an open subset and let $\{U_i\}_{i\in I}$ be an open cover of U. For every $i \in I$ let $s_i \in S(F')(U_i)$ be a section, such that $s_i\,|_{U_i\cap U_j} = s_j\,|_{U_i\cap U_j}$ for all $i, j \in I$. It follows from F being a sheaf (and hence satisfying the glueing axiom) that there exist a section $s \in F(U)$ with $s\,|_{U_i} = s_i$ for every $i \in I$. The section s is already containd in $S(F')(U)$ by definiton of $s(F')$, and the relation $s\,|_{U_i} = s_i$ still holds in $S(F')(U_i)$. This shows that the presheaf $S(F')$ also satisfies the glueing axiom.

We have thus shown that $S(F')$ is a subsheaf of F.

Construction of the Structure Morphism $i\colon F' \to S(F')$

We have seen above that $F'(U)$ is for every open subset $U \subseteq X$ a subgroup of $S(F')(U)$. It follows that F' is a subpresheaf of $S(F')$, as both F' and $S(F')$ are subpresheafs of the sheaf F. Hence the inclusions $i_U\colon F'(U) \to S(F')(U)$, where $U \subseteq X$ ranges trough the open subsets, defines a homomorphism of presheaves $i\colon F' \to S(F')$.

Checking the Universal Property

It remains to show that the sheaf $S(F')$ together with the homomorphism $i\colon F' \to S(F')$ satisfies the universal property of the sheafification of F'. So let G be another sheaf and let $f'\colon F' \to G$ be a homomorphism of presheaves. We need to show that there exist a unique homomorphism of sheaves $f\colon S(F') \to G$ that makes the following triangle commute:

$$\begin{tikzcd} F' \arrow[r, "i"] \arrow[dr, "f'"'] & S(F') \arrow[d, dashed, "f"] \\ & G \end{tikzcd}$$

The commutativity of this triangle means that for every open subset $U \subseteq X$ the following triangle commutes:

$$\begin{tikzcd} F'(U) \arrow[r, "i_U"] \arrow[dr, "f'_U"'] & S(F')(U) \arrow[d, dashed, "f_U"] \\ & G(U) \end{tikzcd}$$

The commutativity of this triangle means that $f_U(s) = f'_U(s)$ for every open subset $U \subseteq X$ and every section $s \in F'(U)$ because i_U is just the set-theoretic inclusion.

Uniqueness of f

Suppose that such a homomorphism f exists. Let $U \subseteq X$ be an open set and let $s \in S(F')(U)$ be a section. Then there exists an open cover $\{U_i\}_{i \in I}$ of U with $s\,|_{U_i} \in F'(U_i)$ for every $i \in I$. It then holds for the image section $f_U(s) \in G(U)$ that

$$f_U(s)\,|_{U_i} = f_{U_i}(s\,|_{U_i}) = f'_{U_i}(s\,|_{U_i})$$

because $s\,|_{U_i} \in F'(U_i)$. This shows the uniqueness of f.

Construction of f_U

We have seen above how we need to construct f_U for an open subset $U \subseteq X$: For every section $s \in S(F')(U)$ there exist an open cover $\{U_i\}_{i \in I}$ of U with $s\,|_{U_i} \in F'(U_i)$ for every $i \in I$. We set $s_i \coloneqq s\,|_{U_i}$ for every $i \in I$ and consider the sections $\overline{s}_i \in G(U_i)$ given by $\overline{s}_i \coloneqq f'_U(s_i)$ for every $i \in I$. It holds for all $i, j \in I$ that

$$\begin{aligned} \overline{s}_i\,|_{U_i \cap U_j} = f'_{U_i}(s_i)\,|_{U_i \cap U_j} = f'_{U_i \cap U_j}(s_i\,|_{U_i \cap U_j}) &= f'_{U_i \cap U_j}((s\,|_{U_i})\,|_{U_i \cap U_j}) \\ &= f'_{U_i \cap U_j}(s\,|_{U_i \cap U_j}), \end{aligned}$$

and hence

$$\overline{s}_i\,|_{U_i \cap U_j} = f'_{U_i \cap U_j}(s\,|_{U_i \cap U_j}) = \overline{s}_j\,|_{U_i \cap U_j}.$$

It follows from G being a sheaf that there exist a unique section $\overline{s} \in G(U)$ with $\overline{s}\,|_{U_i} = \overline{s}_i$ for every $i \in I$. This means that the section $\overline{s}$ is the unique section $\overline{s} \in G(U)$ with

$$\overline{s}\,|_{U_i} = f'_{U_i}(s\,|_{U_i})$$

for every $i \in I$. We will refer to this property as the *defining property* of $\overline{s}$.

We define the desired map $f_U\colon S(F')(U) \to G(U)$ by $f_U(s) = \overline{s}$.

Independence of $\overline{s}$ from the Choosen Cover

The section $\overline{s}$ constructed above does not depend on the choice of open cover of U, and hence the map f_U does not depend on this choice of cover either, i.e. the map f_U is well-defined.

To show this, let $\{V_j\}_{j\in J}$ be another open cover of U such that $s|_{V_j} \in F'(V_j)$ for every $j \in J$. Let $\overline{s}' \in G(U)$ be the section constructed as above, but via the cover $\{V_j\}_{j\in J}$ instead of the cover $\{U_i\}_{i\in I}$. Hence the section $\overline{s}'$ is uniquely determined by the property that

$$\overline{s}'|_{V_j} = f'_{V_j}(s|_{V_j})$$

for every $j \in J$.

Suppose first that this alternative open cover $\{V_j\}_{j\in J}$ is a refinement of the previous open cover $\{U_i\}_{i\in I}$, i.e. that for every $j \in J$ there exist some $i \in I$ with $V_j \subseteq U_i$. It then follows that

$$\overline{s}|_{V_j} = (\overline{s}|_{U_i})|_{V_j} = f'_{U_i}(s|_{U_i})|_{V_j} = f'_{V_j}((s|_{U_i})|_{V_j}) = f'_{V_j}(s|_{V_j}).$$

This shows that the section $\overline{s}$ already satisfies the defining property of the section $\overline{s}'$, and hence that $\overline{s} = \overline{s}'$.

For the general case we note that the two open covers $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ of U have a common refinement $\{W_{ij}\}_{(i,j)\in I\times J}$ given by $W_{ij} = U_i \cap U_j$ for every $(i,j) \in I \times J$. It then follows from the above discussion that all three covers lead to the same section $\overline{s} \in G(U)$. This shows the claimed independence, and hence that f_U is well-defined.

f_U is a Group Homomorphism

We need to show that the map $f_U\colon S(F')(U) \to G(U)$ is for every open subset $U \subseteq X$ a group homomorphism.

For this let $s, t \in S(F')(U)$ be any two sections. We have previously seen (in the proof that $S(F')(U)$ is a subgroup of $F(U)$) that there exist an open cover $\{U_i\}_{i\in I}$ of U such that $s|_{U_i}, t|_{U_i} \in F'(U_i)$ for every $i \in I$. It then holds for the section $\overline{s} + \overline{t} \in G(U)$ that

$$(\overline{s} + \overline{t})|_{U_i} = \overline{s}|_{U_i} + \overline{t}|_{U_i} = f'_{U_i}(s|_{U_i}) + f'_{U_i}(t|_{U_i}) = f'_{U_i}(s|_{U_i} + t|_{U_i}) = f'_{U_i}((s+t)|_{U_i})$$

for every $i \in I$. This shows that the section $\overline{s} + \overline{t}$ satisfies the defining property of the section $\overline{s+t}$, and hence that $\overline{s} + \overline{t} = \overline{s+t}$. In other words, $f_U(s+t) = f_U(s) + f_U(t)$.

f is a Homomorphism of Sheaves

The group homomorphisms $f_U\colon S(F')(U) \to G(U)$ with $U \subseteq X$ open define a homomorphism of sheaves $f\colon S(F') \to G$:

Let $U \subseteq V \subseteq X$ be open subsets and let $s \in S(F')(V)$ be a section. Let $\{V_i\}_{i\in I}$ be an open cover of V with $s|_{V_i} \in F'(V_i)$ for every $i \in I$. Then $\{U_i\}_{i\in I}$ with $U_i = U \cap V_i$ for every $i \in I$ is an open cover of U, and it holds for every $i \in I$ that

$$s|_{U_i} = (s|_{V_i})|_{U_i} \in F'(V_i)$$

because $s|_{V_i} \in F'(V_i)$. It follows that

$$\begin{aligned}(\overline{s}|_U)|_{U_i} = \overline{s}|_{U_i} = (\overline{s}|_{V_i})|_{U_i} = f'_{V_i}(s|_{V_i})|_{U_i} = f'_{U_i}((s|_{V_i})|_{U_i}) &= f'_{U_i}(s|_{U_i}) \\ &= f'_{U_i}((s|_U)|_{U_i}) \,.\end{aligned}$$

This shows that the restriction $\overline{s}|_U$ satisfies the defining property of the section $\overline{(s|_U)}$, and hence that $= \overline{s}|_U = \overline{(s|_U)}$. In other words, $f_U(s|_U) = f_V(s)|_U$.

f Makes the Needed Triangle Commute

It remains to show that the homomorphism $f\colon S(F') \to G$ makes for every open subset $U \subseteq X$ the triangle

$$\begin{tikzcd} F'(U) \arrow[r, "i_U"] \arrow[dr, "f'_U"'] & S(F')(U) \arrow[d, dashed, "f_U"] \\ & G(U) \end{tikzcd}$$

commute, i.e. that $f_U(s) = f'_U(s)$ for every section $s \in F'(U)$.

For this we can use the open cover $\{U\}$ of U and that

$$f'_U(s)|_U = f'_U(s|_U)$$

to see that $f'_U(s)$ satisfies the defining property of $\overline{s}$, and hence that $f_U(s) = f'_U(s)$.

Exercise 2.

We define the first homomorphism $\Phi\colon F(U) \to \prod_{i\in I} F(U_i)$ in the i-th component as the restriction ρ_{U,U_i}, hence overall by

$$\Phi(s) = (s|_{U_i})_{i\in I}$$

for every section $s \in F(U)$. The second homomorphism

$$\Psi\colon \prod_{i\in I} F(U_i) \to \prod_{(j,k)\in I\times I} F(U_j \cap U_k)$$

is in components given by

$$\begin{aligned}\Psi_{j,k}\colon \prod_{i\in I} F(U_i) &\xrightarrow{\text{projection}} F(U_j) \times F(U_k) \\ &\xrightarrow{\text{restriction}\times\text{restriction}} F(U_j\cap U_k) \times F(U_j \cap U_k) \\ &\xrightarrow{\text{difference}} F(U_j \cap U_k)\,,\end{aligned}$$

and hence overall by

$$\Psi((s_i)_{i\in I}) = (s_j|_{U_j\cap U_k} - s_k|_{U_j\cap U_k})_{(j,k)\in I\times I}$$

for every collection of sections $(s_i)_{i \in I} \in \prod_{i \in I} F(U_i)$. We consider the resulting sequence

$$0 \to F(U) \xrightarrow{\Phi} \prod_{i \in I} F(U_i) \xrightarrow{\Psi} \prod_{(j,k) \in I \in} F(U_j \cap U_k)\,. \tag{1}$$

The exactness of this sequence (1) at $F(U)$ is equivalent to Φ being a monmorphisms, and hence equivalent to $\ker(\Phi) = 0$. This is equivalent to the presheaf F satisfying the separability axiom.

For the exactness of the sequence (1) at $\prod_{i \in I} F(U_i)$ we first note that $\Psi \circ \Phi = 0$, because F is a presheaf and therefore

$$\begin{aligned}
\Psi_{j,k}(\Phi(s)) &= \Psi_{j,k}((s\,|_{U_i})_{i \in I}) \\
&= (s\,|_{U_j})\,|_{U_j \cap U_k} - (s\,|_{U_k})\,|_{U_j \cap U_k} \\
&= s\,|_{U_j \cap U_k} - s\,|_{U_j \cap U_k} \\
&= 0
\end{aligned}$$

for every section $s \in F(U)$ and all $(j,k) \in I \times I$. Hence $\operatorname{im}(\Phi) \subseteq \ker(\Psi)$.

That on the other hand $\operatorname{im}(\Phi) \supseteq \ker(\Psi)$ means that for every collection $(s_i)_{i \in I}$ of sections $s_i \in F(U_i)$ with $s_j\,|_{U_j \cap U_k} = s_k\,|_{U_j \cap U_k}$ for all $k, j \in I$, there exist a section $s \in F(U)$ with $s\,|_{U_i} = s_i$ for every $i \in I$. Hence $\operatorname{im}(\Phi) \supseteq \ker(\Psi)$ if and only if F satisfies the glueing axiom.

This shows together that the sequence (1) is exact if and only if the presheaf F satisfies with respect to the given open subset U and cover $\{U_i\}_{i \in I}$ of U both the separability axiom and the glueing axiom. (We have also seen that the composition $\Psi \circ \Phi$ vanishes if and only if the restrictions of F are compatible with each other.)

That the sequence (1) is exact for every open subset $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U is therefore equivalent to F being a sheaf.

Exercise 3.

We denote the given morphisms as follows:

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & X_4 & \xrightarrow{\alpha_4} & X_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & Y_4 & \xrightarrow{\beta_4} & Y_5
\end{array}$$

(i)

This follows from part (ii) by duality.

(ii)

Let $x_3 \in_{\mathcal{A}} X_3$ with $f_3 x_3 \equiv 0$. Then

$$0 \equiv \beta_3 f_3 x_3 = f_4 \alpha_3 x_3\,.$$

It follows that $\alpha_3 x_3 \equiv 0$ because f_4 is a monomorphism. It follows from the exactness of the sequence $X_2 \to X_3 \to X_4$ that there exist some $x_2 \in X_2$ with $\alpha_2 x_2 \equiv x_3$. Then

$$0 \equiv f_3 x_3 \equiv f_3 \alpha_2 x_2 = \beta_2 f_2 x_2 \,.$$

Hence there exists for $y_2 \coloneqq f_2 x_2 \in_{\mathcal{A}} Y_2$ by the exactness of the sequence $Y_1 \to Y_2 \to Y_3$ some $y_1 \in Y_1$ with $\beta_1 y_1 \equiv y_2$. It follows that there exist some $x_1 \in X_1$ with $f_1 x_1 \equiv y_1$ because f_1 is an epimorphism. It now holds that

$$f_2 \alpha_1 x_1 = \beta_1 f_1 x_1 \equiv \beta_1 y_1 \equiv y_2 = f_2 x_2$$

and hence $\alpha_1 x_1 \equiv x_2$ because f_2 is a monomorphism. It follows overall that

$$x_3 \equiv \alpha_2 x_2 \equiv \alpha_2 \alpha_1 x_1 = 0$$

because $\alpha_2 \alpha_1 = 0$. This shows that f_3 is a monomorphism.

(iii)

It follows from part (i) that f_3 is an epimorphism (because the isomorphisms f_2 and f_4 isomorphisms are in particular epimorphisms), and it follows from part (ii) that f_3 is a monomorphism (because the isomorphisms f_2 and f_4 are in particular monomorphisms). This shows that the morphism f_3 is already an isomorphim because the category $\mathcal{A}$ is abelian.

Exercise 4.

We denote the given morphims as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r, "\alpha'"] \arrow[d, "f'"] & X \arrow[r, "\beta'"] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Y' \arrow[r, "\alpha"] \arrow[d, "g'"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Y'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & Z' \arrow[r, "\alpha''"] \arrow[d] & Z \arrow[r, "\beta''"] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2}$$

We consider the following three cases:

(i) The upper two rows are exact.

(ii) The lower two rows are exact.

(iii) The upper row and the lower row are exact.

(i)

By applying the snake lemma to the upper two rows we get an exact sequence

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'') \to 0\,. \tag{3}$$

The morphisms between the cokernels are the unique ones which make the diagram

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[d] \\
\operatorname{coker}(f') \arrow[r, dashed] & \operatorname{coker}(f) \arrow[r, dashed] & \operatorname{coker}(f'')
\end{tikzcd}$$

commute. The morphisms f', f, f'' are monomorphisms, hence their kernels are given by $\ker(f') = \ker(f) = \ker(f'') = 0$. It follows from the (right) exactness of the columns of (2) that g' is a cokernel of f', that g is a cokernel of f and that g'' is a cokernel of f''. The induced morphisms $Z' \to Z$ and $Z \to Z''$ that make the diagram

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[d] \\
Z' \arrow[r, dashed] & Z \arrow[r, dashed] & Z''
\end{tikzcd}$$

commute and precisely α'' and β'' from the last row of (2) (by the uniqueness of these induced morphisms).

We have shows that the exact sequence (3) is (up to isomorphism) given by

$$0 \to 0 \to 0 \to 0 \to Z' \xrightarrow{\alpha''} Z \xrightarrow{\beta''} Z'' \to 0\,.$$

This shows that the last row of (2) is exact.

(ii)

This case follows by duality from case (i)

(iii)

We need to show the exactness of the sequence

$$0 \to Y' \xrightarrow{\alpha} Y \xrightarrow{\beta} Y'' \to 0\,,$$

which amounts to showing the exactness of the three sequences

$$0 \to Y' \xrightarrow{\alpha} Y\,, \qquad Y' \xrightarrow{\alpha} Y \xrightarrow{\beta} Y''\,, \qquad Y \xrightarrow{\beta} Y'' \to 0\,.$$

The exactness of the first sequence

$$0 \to Y' \xrightarrow{\alpha} Y$$

is equivalent to α being a monomorphisms. That α is indeed a monomorphism follows by applying one of the four lemmata (namely part (ii) of Exercise 3) to the first two columns of the diagram (2); because both α' and α'' are monomorphisms it follows that α is also one.

We find dually that the third sequence

$$Y \xrightarrow{\beta} Y'' \to 0$$

is exact.

To show the exactness of the second sequence

$$Y' \xrightarrow{\alpha} Y \xrightarrow{\beta} Y''$$

it remains to show that the induced morphism $\lambda\colon \operatorname{im}(\alpha) \to \ker(\beta)$ is an isomorphism. By λ we mean the unique morphism $\operatorname{im}(\alpha) \to \ker(\beta)$ that makes the following diagram commute:

$$\begin{tikzcd}
Y' \arrow[rr, "\alpha"] \arrow[dr] & & Y \arrow[rr, "\beta"] & & Y'' \\
& \operatorname{im}(\alpha) \arrow[ur] \arrow[rr, dashed, "\lambda"] & & \ker(\beta) \arrow[ul] &
\end{tikzcd}$$

For the existence of this morphism we use the assumption $\beta\alpha = 0$ (the uniqueness follows from the universal property of the kernel $\ker(\beta) \to Y$). To show that λ is indeed an isomorphism, we further expand the commutative diagram (2) to the following commutative beauty:

$$\begin{tikzcd}
& 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & \\
& & 0 \arrow[dd, dashed] & & 0 \arrow[dd, dashed] & & \\
0 \arrow[r] & X' \arrow[rr, "\alpha'"] \arrow[dr] \arrow[dd, "f'"] & & X \arrow[rr, "\beta'"] \arrow[dd, "f"] & & X'' \arrow[r] \arrow[dd, "f''"] & 0 \\
& & \operatorname{im}(\alpha') \arrow[ur] \arrow[rr, dashed, "\lambda'"] \arrow[dd, dashed] & & \ker(\beta') \arrow[ul] \arrow[dd, dashed] & & \\
0 \arrow[r] & Y' \arrow[rr, "\alpha"] \arrow[dr] \arrow[dd, "g'"] & & Y \arrow[rr, "\beta"] \arrow[dd, "g"] & & Y'' \arrow[r] \arrow[dd, "g''"] & 0 \\
& & \operatorname{im}(\alpha) \arrow[ur] \arrow[rr, dashed, "\lambda"] \arrow[dd, dashed] & & \ker(\beta) \arrow[ul] \arrow[dd, dashed] & & \\
0 \arrow[r] & Z' \arrow[rr, "\alpha''"] \arrow[dr] \arrow[dd] & & Z \arrow[rr, "\beta''"] \arrow[dd] & & Z'' \arrow[r] \arrow[dd] & 0 \\
& & \operatorname{im}(\alpha'') \arrow[ur] \arrow[rr, dashed, "\lambda''"] \arrow[dd, dashed] & & \ker(\alpha'') \arrow[ul] \arrow[dd, dashed] & & \\
& 0 & & 0 & & 0 & \\
& & 0 & & 0 & &
\end{tikzcd} \tag{4}$$

We have in particular the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{im}(\alpha') \arrow[r] \arrow[d, "\lambda'"] & \mathrm{im}(\alpha) \arrow[r] \arrow[d, "\lambda"] & \mathrm{im}(\alpha'') \arrow[r] \arrow[d, "\lambda''"] & 0 \\
0 \arrow[r] & \ker(\beta') \arrow[r] & \ker(\beta) \arrow[r] & \ker(\beta'') \arrow[r] & 0
\end{tikzcd}
\tag{5}
$$

The rows of this diagram are again exact:

To see the exactness of the lower row we apply the snake lemma to the diagram

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\beta'"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\beta''"] & 0 \\
0 \arrow[r] & X'' \arrow[r, "f''"] & Y'' \arrow[r, "g''"] & Z'' \arrow[r] & 0
\end{tikzcd}
$$

to get the exact sequence

$$
0 \to \ker(\beta') \to \ker(\beta) \to \ker(\beta'') \xrightarrow{\delta} 0 \to 0 \to 0 \to 0\,,
$$

where we use that $\operatorname{coker}(\beta') = \operatorname{coker}(\beta) = \operatorname{coker}(\beta'') = 0$ because β', β, β'' are epimorphisms.

We argue in a similar way for the exactness of the lower row of (5): We first apply the snake lemma to the diagram

$$
\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f'"] \arrow[d, "\alpha'"] & Y' \arrow[r, "g'"] \arrow[d, "\alpha"] & Z' \arrow[r] \arrow[d, "\alpha''"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}
$$

to get the exact sequence

$$
0 \to 0 \to 0 \to 0 \xrightarrow{\delta} \operatorname{coker}(\alpha') \to \operatorname{coker}(\alpha) \to \operatorname{coker}(\alpha'') \to 0\,,
$$

where we use that $\ker(\alpha') = \ker(\alpha) = \ker(\alpha'') = 0$ because α', α, α'' are monomorphisms. This shows that in the intermediate diagram

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker}(\alpha') \arrow[r] & \operatorname{coker}(\alpha) \arrow[r] & \operatorname{coker}(\alpha'') \arrow[r] & 0
\end{tikzcd}
$$

the rows are again exact. We can therefore apply the snake lemma again. This time we get the exact sequence

$$
0 \to \mathrm{im}(\alpha') \to \mathrm{im}(\alpha) \to \mathrm{im}(\alpha'') \xrightarrow{\delta} 0 \to 0 \to 0 \to 0\,,
$$

where we use that the morphisms $X \to \operatorname{coker}(\alpha')$, $Y \to \operatorname{coker}(\alpha)$ and $Z \to \operatorname{coker}(\alpha'')$ are epimorphisms.

We have now shown that both rows of (5) are exact. We know that both λ' and λ'' are isomorphisms because the upper and lower rows of (2) are exact. It follows from the five lemma that λ is again an isomorphism, which shows the desired exactness.