

Exercises in Foundations in Representation Theory

Exercise Sheet 6

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Exercise 1.

(i)

Let $Z \in \text{Ob}(\mathcal{C})$ be an object and let $\alpha: Z \rightarrow X$ and $\delta: Z \rightarrow Y''$ be two morphisms in \mathcal{C} with $f \circ \alpha = g \circ h \circ \delta$, i.e. such the diagram

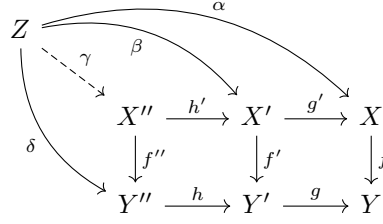
$$\begin{array}{ccccc}
 & & & \alpha & \\
 & & & \curvearrowright & \\
 Z & & & & \\
 & & & \delta & \\
 & & & \curvearrowleft & \\
 & & & & \\
 & X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 & \downarrow f'' & & \downarrow f' & & \downarrow f \\
 & Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}$$

commutes. By applying the right-hand pull-back square to the morphisms $\alpha: Z \rightarrow X$ and $h \circ \delta: Z \rightarrow Y'$, we find that there exist a unique morphism $\beta: Z \rightarrow X'$ with $g' \circ \beta = \alpha$ and $f' \circ \beta = h \circ \delta$, i.e. such that the diagram

$$\begin{array}{ccccc}
 & & & \alpha & \\
 & & & \curvearrowright & \\
 Z & & & & \\
 & & & \beta & \\
 & & & \curvearrowleft & \\
 & & & & \\
 & X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 & \downarrow f'' & & \downarrow f' & & \downarrow f \\
 & Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}$$

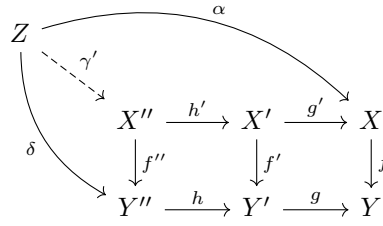
commutes. By applying the left-hand pull-back square to the morphisms $\beta: Z \rightarrow X'$ and $\delta: Z \rightarrow Y''$, we find that there exist a unique morphism $\gamma: Z \rightarrow X''$ with $h' \circ \gamma = \beta$

and $f'' \circ \gamma = \delta$, i.e. such that the diagram

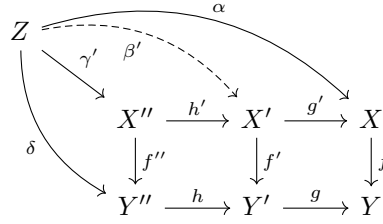


commutes.

Suppose that $\gamma': Z \rightarrow X''$ is another morphism which makes the diagram



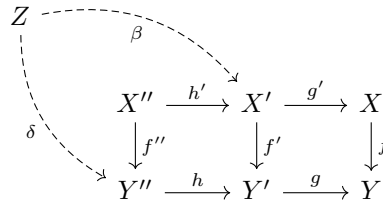
commute. Then by setting $\beta' := h' \circ \gamma'$ we get the following commutative diagram:



Then $\beta' = \beta$ by the uniqueness of β and thus $\gamma' = \gamma$ by the uniqueness of γ .

(ii)

Let $Z \in \text{Ob}(\mathcal{C})$ be an object and let $\delta: Z \rightarrow Y''$ and $\beta: Z \rightarrow X'$ be two morphisms in \mathcal{C} with $h \circ \delta = f' \circ \beta$, i.e. such that the diagram



commutes. For the morphism $\alpha' := g' \circ \beta: Z \rightarrow X$ we get following commutative diagram:

$$\begin{array}{c}
 \begin{array}{ccc}
 & \xrightarrow{\alpha} & \\
 Z & \xrightarrow{\beta} & X \\
 \delta \searrow & & \nearrow \\
 & \xrightarrow{\alpha'} &
 \end{array} \\
 \begin{array}{ccccc}
 X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 \downarrow f'' & & \downarrow f' & & \downarrow f \\
 Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}
 \end{array}$$

By using the outer pull-back square for the morphisms $\alpha: Z \rightarrow X$ and $\delta: Z \rightarrow Y''$, we find that there exists a unique morphism $\gamma: Z \rightarrow X''$ with $g' \circ h' \circ \gamma = \alpha$ and $f'' \circ \gamma = \delta$, i.e. such that the diagram

$$\begin{array}{c}
 \begin{array}{ccc}
 & \xrightarrow{\alpha} & \\
 Z & \xrightarrow{\gamma} & X'' \\
 \delta \searrow & & \nearrow \\
 & \xrightarrow{\alpha'} &
 \end{array} \\
 \begin{array}{ccccc}
 X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 \downarrow f'' & & \downarrow f' & & \downarrow f \\
 Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}
 \end{array}$$

commutes. We claim that already the whole diagram

$$\begin{array}{c}
 \begin{array}{ccc}
 & \xrightarrow{\alpha} & \\
 Z & \xrightarrow{\beta} & X \\
 \gamma \searrow & & \nearrow \\
 & \xrightarrow{\alpha'} &
 \end{array} \\
 \begin{array}{ccccc}
 X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 \downarrow f'' & & \downarrow f' & & \downarrow f \\
 Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}
 \end{array}$$

commutes. For this we still have to check that $h' \circ \gamma = \beta$. The right-hand square is a pushout square, so it follows for any two morphisms $k_1, k_2: Z \rightarrow X'$ that $k_1 = k_2$ if and only if both $g' \circ k_1 = g' \circ k_2$ and $f' \circ k_1 = f' \circ k_2$. This holds true for $k_1 = \beta$ and $k_2 = h' \circ \gamma$ because

$$g' \circ \beta = \alpha = g' \circ h' \circ \gamma$$

by construction of γ , and

$$f' \circ \beta = h \circ \delta = h \circ f'' \circ \gamma = f' \circ h' \circ \gamma$$

by choice of β and δ and construction of γ .

Suppose that $\gamma': Z \rightarrow X''$ is another morphism which makes the diagram

$$\begin{array}{ccccc}
 & & & \alpha & \\
 & & & \curvearrowright & \\
 Z & \xrightarrow{\gamma'} & X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 & \searrow \delta & \downarrow f'' & & \downarrow f' & & \downarrow f \\
 & & Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}$$

commute. Then $f'' \circ \gamma' = \delta$ and $g' \circ h' \circ \gamma' = \alpha$, and therefore $\gamma' = \gamma$ by the uniqueness of γ .

Exercise 2.

If $h: Z \rightarrow X$ is any morphism in \mathcal{C} , then for $\alpha := g' \circ h$ and $\beta := f' \circ h$ the following diagram commutes:

$$\begin{array}{ccccc}
 Z & \xrightarrow{h} & X' & \xrightarrow{g'} & X \\
 & \searrow \beta & \downarrow f' & & \downarrow f \\
 & & Y' & \xrightarrow{g} & Y
 \end{array}$$

The morphism h is uniquely determined by the compositions α and β because the given diagram is a pull-back square. The morphisms α is uniquely determined by the composition $f \circ \alpha$ because f is a monomorphism, and this composition is given by $f \circ \alpha = g \circ \beta$. Hence h is uniquely determined by $\beta = f' \circ h$, which shows that f' is a monomorphism.

Remark 1. In an abelian category \mathcal{A} the converse also holds, i.e. if the diagram

$$\begin{array}{ccc}
 X' & \xrightarrow{g'} & X \\
 \downarrow f' & & \downarrow f \\
 Y' & \xrightarrow{g} & Y
 \end{array}$$

in \mathcal{A} is a pull-back square and f' is a monomorphism then f is also a monomorphism. Indeed, the canonical morphism $\ker(f) \rightarrow X$ fits into the following commutative diagram:

$$\begin{array}{ccccc}
 \ker(f) & \xrightarrow{\quad} & X' & \xrightarrow{g'} & X \\
 & \searrow 0 & \downarrow f' & & \downarrow f \\
 & & Y' & \xrightarrow{g} & Y
 \end{array}$$

It follows that there exist a unique morphism $\lambda: \ker(f) \rightarrow X'$ which makes the diagram

$$\begin{array}{ccccc}
 \ker(f) & & & & \\
 \swarrow \lambda & & & & \searrow \\
 & X' & \xrightarrow{g'} & X & \\
 & \downarrow f' & & \downarrow f & \\
 & Y' & \xrightarrow{g} & Y & \\
 \searrow 0 & & & &
 \end{array}$$

commute. It follows from $0 = f' \circ \lambda$ and f being a monomorphism that also $\lambda = 0$. The canonical morphism $\ker(f) \rightarrow X$ is therefore given by

$$g' \circ \lambda = g' \circ 0 = 0.$$

This shows that $\ker(f) = 0$ and hence that f is a monomorphism.

Remark 2. It holds dually for a pushout square

$$\begin{array}{ccc}
 X & \xrightarrow{g} & X' \\
 f \downarrow & & \downarrow f' \\
 Y & \xrightarrow{g'} & Y'
 \end{array}$$

that if f is an epimorphism, then f' is also an epimorphism; in an abelian category, the converse also holds.

Exercise 3.

Proposition 3. Let \mathcal{A} be an additive category. Then a diagram

$$\begin{array}{ccc}
 X' & \xrightarrow{g'} & X \\
 \downarrow f' & & \downarrow f \\
 Y' & \xrightarrow{g} & Y
 \end{array}$$

in \mathcal{A} is a pull-back square if and only if in the sequence

$$X' \xrightarrow{\begin{bmatrix} g' \\ f' \end{bmatrix}} X \oplus Y' \xrightarrow{[f \ -g]} Y \quad (1)$$

the morphism $X' \rightarrow X \oplus Y'$ is a kernel of the morphism $X \oplus Y' \rightarrow Y$ (i.e. the sequence is left exact).

Proof. Let $p: X \oplus Y' \rightarrow X$ and $q: X \oplus Y' \rightarrow Y'$ be the canonical projections belonging to the biproduct $X \oplus Y$, and let

$$d: X \oplus Y' \xrightarrow{[f \ -g]} Y.$$

It then holds for every object $Z \in \text{Ob}(\mathcal{C})$ and every two morphisms $\alpha: Z \rightarrow X$ and $\beta: Z \rightarrow Y$ that

$$f \circ \alpha = g \circ \beta \iff f \circ \alpha - g \circ \beta = 0 \iff \begin{bmatrix} f & -g \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \iff d \circ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0.$$

Let now $k: X' \rightarrow X \oplus Y'$ be a morphism, which is uniquely of the form

$$k = \begin{bmatrix} g' \\ f' \end{bmatrix}$$

for some morphisms $g': X' \rightarrow X$ and $f': X' \rightarrow Y'$. We find from the above calculation that the square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array} \quad (2)$$

commutes if and only if $d \circ k = 0$. We moreover find that

$$\begin{aligned} & \text{the diagram (2) is a pull-back square} \\ \iff & \text{there exist for all morphisms } \alpha: Z \rightarrow X \text{ and } \beta: Z \rightarrow Y' \text{ with } f \circ \alpha = g \circ \beta \\ & \text{a unique morphism } \lambda: Z \rightarrow X' \text{ with } g' \circ \lambda = \alpha \text{ and } f' \circ \lambda = \beta \\ \iff & \text{there exist for all morphisms } \alpha: Z \rightarrow X \text{ and } \beta: Z \rightarrow Y' \text{ with } d \circ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \\ & \text{a unique morphism } \lambda: Z \rightarrow X' \text{ with } k \circ \lambda = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ \iff & \text{there exist for every morphism } \gamma: Z \rightarrow X \oplus Y' \text{ with } d \circ \gamma = 0 \\ & \text{a unique morphism } \lambda: Z \rightarrow X' \text{ with } k \circ \lambda = \gamma \\ \iff & k \text{ is a kernel for } d. \end{aligned}$$

This shows the claimed equivalence. \square

Remark 4. Instead of the sequence (1) one can also use variations such as

$$\begin{aligned} X' & \xrightarrow{\begin{bmatrix} g' \\ f' \end{bmatrix}} X \oplus Y' \xrightarrow{[-f \ g]} Y, \\ X' & \xrightarrow{\begin{bmatrix} g' \\ -f' \end{bmatrix}} X \oplus Y' \xrightarrow{[f \ g]} Y, \\ X' & \xrightarrow{\begin{bmatrix} -g' \\ f' \end{bmatrix}} X \oplus Y' \xrightarrow{[f \ g]} Y. \end{aligned}$$

Remark 5. The dual version of Proposition 3 states that in an additive category \mathcal{A} , a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{g'} & Y' \end{array}$$

is a pushout square if and only if in the sequence

$$X \xrightarrow{\begin{bmatrix} g \\ f \end{bmatrix}} X' \oplus Y \xrightarrow{[f' - g']} Y'$$

the morphism $X' \oplus Y \rightarrow Y'$ is a cokernel of the morphism $X \rightarrow X' \oplus Y$ (i.e. the sequence is right exact). One can again vary this sequence, just as done in Remark 4 for pullbacks.

Exercise 4.

Lemma 6. Let $X, Y_1, Y_2 \in \text{Ob}(\mathcal{C})$ be objects, let $f_1: X \rightarrow Y_1$ be an epimorphism and let $f_2: X \rightarrow Y_2$ be a morphism. Then the morphism

$$f: X \xrightarrow{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}} Y_1 \oplus Y_2$$

is also an epimorphism.

Proof. Let $g_1, g_2: Y_1 \oplus Y_2 \rightarrow Z$ be two parallel morphism with $g_1 \circ f = g_2 \circ f$. If $i: Y_1 \rightarrow Y_1 \oplus Y_2$ is the canonical morphism into the first summand then

$$g_1 \circ f = g_2 \circ f \implies g_1 \circ f \circ i = g_2 \circ f \circ i \implies g_1 \circ f_1 = g_2 \circ f_1 \implies g_1 = g_2$$

because f_1 is an epimorphism. \square

Lemma 7. Let $f: X \rightarrow Y$ be an epimorphism in an abelian category. Then f is a cokernel of its kernel.

Proof. The zero morphism $Y \rightarrow \text{coker}(f)$ is a cokernel of f because f is an epimorphism, and the identity morphism $\text{id}_Y: Y \rightarrow Y$ is therefore an image of f . Together with the canonical factorization $\tilde{f}: \text{coim}(f) \rightarrow \text{im}(f) = Y$ of the morphism f , which is an isomorphism because \mathcal{A} is abelian, we get the following commutative triangle:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \nearrow \tilde{f} & \\ \text{coim}(f) & & \end{array}$$

The coimage $X \rightarrow \text{coim}(f)$ is a cokernel of $\ker(f) \rightarrow X$. It hence follows from the commutativity of the above diagram and \tilde{f} being an isomorphism that f is also a cokernel of $\ker(f) \rightarrow X$. \square

Let now

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array} \quad (3)$$

be a pullback diagram in \mathcal{A} such that f is an epimorphism. It follows from Proposition 3 that in the sequence

$$X' \xrightarrow{\begin{bmatrix} g' \\ f' \end{bmatrix}} X \oplus Y' \xrightarrow{[f \ -g]} Y$$

the morphism $X' \rightarrow X \oplus Y'$ is a kernel of the morphism $X \oplus Y' \rightarrow Y$. It follows from Lemma 6 that the morphism $X \oplus Y' \rightarrow Y$ is again an epimorphism, and it hence follows from Lemma 7 that the morphism $X \oplus Y' \rightarrow Y$ is a cokernel of the morphism $X' \rightarrow X \oplus Y'$. The diagram (3) is therefore a pushout square by Remark 5. It hence follows from Remark 2 that f' is again an epimorphism (because \mathcal{A} is abelian).