Group 1, Wednesday 8–10, Tutor: Nils Waßmuth January 25, 2019

Exercises in Foundations in Representation Theory

Exercise Sheet 13

Jendrik Stelzner

Exercise 1.

Let $f\colon X \to I$ be a morphism and let $m\colon X \to Y$ be a monomorphism. For every $\beta \in B$ the composition $f_\beta \coloneqq p_\beta \circ f\colon X \to I_\beta$ has an extension $g_\beta\colon Y \to I_\beta$, i.e. there exists a morphism g_β with $g_\beta \circ m = f_\beta$, because I_β is injective. There exists a unique morphism $g\colon Y \to I$ with $g_\beta = p_\beta \circ g$ for every $\beta \in B$ by the universal property of the product $(I, (p_\beta)_{\beta \in B})$. It holds that

$$p_\beta \circ g \circ m = g_\beta \circ m = f_\beta = p_\beta \circ f$$

for every $\beta \in B$, and hence

$$g \circ m = f$$

by the universal property of the product of the product $(I, (p_\beta)_{\beta \in B})$. This shows that I is indeed injective.

$$\begin{tikzcd}
X \arrow[r, hook, "m"] \arrow[d, "f"'] \arrow[dd, bend right=60, "f_\beta"'] & Y \arrow[dl, dashed, "g"] \arrow[ddl, dashed, bend left, "g_\beta"] \\
I \arrow[d, "p_\beta"'] & \\
I_\beta &
\end{tikzcd}$$

Exercise 2.

(i)

The abelian groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible and hence injective (by Baer's criterion). We find that

$$0 \to \mathbb{Z} \to \underbrace{\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \to 0 \to \cdots}_{I^\bullet}$$

is an injective resolution of $\mathbb{Z}$ in $\mathbf{Ab}$.

(ii)

For $n = 0$ the functor

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -) \cong \mathrm{Id}_{\mathbf{Ab}}$$

is exact, whence

$$\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -) \cong \begin{cases} \mathrm{Id}_{\mathbf{Ab}} & \text{for } i = 0\,, \\ 0 & \text{for } i > 0\,. \end{cases}$$

We have in particular that

$$(\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -))(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0\,, \\ 0 & \text{for } i > 0\,. \end{cases}$$

Let $n > 0$. The cochain complex

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, I^\bullet) = \big(\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \to 0 \to 0 \to \cdots\big)$$

is isomorphic to the cochain complex

$$0 \to \mathbb{Z}/n \to 0 \to 0 \to \cdots \tag{1}$$

Here we used on the one hand that

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}) \cong n\text{-torsion of } \mathbb{Q} = 0\,.$$

We think on the other hand about $\mathbb{Q}/\mathbb{Z}$ as a subgroup of the circle group $\mathbb{S}^1$ via the exponential map, and find for the subgroup $\mu_n \subseteq \mathbb{S}^1$ of n-th root of unity that

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) = n\text{-torsion of } \mathbb{Q}/\mathbb{Z} \cong \mu_n \cong \mathbb{Z}/n\,. \tag{2}$$

By taking the cohomology of the cochain complex (1) we find that

$$(\mathrm{R}^i \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, -))(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n & \text{if } n = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Exercise 3.

We know from the lecture that the A-module $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is injective because $\mathbb{Q}/\mathbb{Z}$ is injective (see Corollary 6.26). We have seen in (2) that

$$\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A$$

as abelian groups, and hence also A-modules. Thus A is injective as an A-module.

Exercise 4.

(i)

The sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, Y') \xrightarrow{h^X(f)} \mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{h^X(g)} \mathrm{Hom}_{\mathcal{A}}(X, Y'')$$

is precisely the sequence

$$Y'(X) \xrightarrow{f} Y(X) \xrightarrow{g} Y''(X)$$

of X-valued points. The statement follows from this observation because exactness can be computed on (generalized) points (see Theorem 3.42).

(ii)

We show that the functor F respects finite coproducts; that F is additive then follows from Theorem 3.24. It can then also be shown dually that the functor G respects finite products, which then also shows that G is additive.

Let $(C, (c_1, \dots, c_n))$ be a coproduct of finitely many objects $X_1, \dots, X_n$ of $\mathcal{A}$. Then

$$\mathrm{Hom}_{\mathcal{B}}(F(C), -) \cong \mathrm{Hom}_{\mathcal{A}}(C, G(-)) \cong \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{A}}(X_i, G(-)) \cong \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{B}}(F(X_i), -)\,.$$

This shows that the object $F(C)$ represents the functor $\prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{B}}(F(X_i), -)$. This means that the object $F(C)$ becomes a coproduct of the objects $F(X_1), \dots, F(X_n)$ with respect to suitable morphisms $c'_i \colon F(X_i) \to F(C)$. Such morphisms $c'_1, \dots, c'_n$ can be determined via the above isomorphism of functors: The identity

$$\mathrm{id}_{F(C)} \in \mathrm{Hom}_{\mathcal{B}}(F(C), F(C))\,.$$

corresponds under the above isomorphism to the one such tupel

$$(c'_1, \dots, c'_n) \in \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{B}}(F(X_i), F(X))\,.$$

Under the above isomorphisms we have

$$\mathrm{id}_{F(C)} \mapsto \varphi(\mathrm{id}_{F(C)}) \mapsto (\varphi(\mathrm{id}_{F(C)}) \circ c_i)_{i=1}^{n} \mapsto (\varphi^{-1}(\varphi(\mathrm{id}_{F(C)}) \circ c_i))_{i=1}^{n}\,,$$

and hence

$$c'_i = \varphi^{-1}(\varphi(\mathrm{id}_{F(C)}) \circ c_i)$$

for every i. To show that F respects the given coproduct we need to show that $c'_i = F(c_i)$ for every i, and hence that

$$F(c_i) = \varphi^{-1}(\varphi(\mathrm{id}_{F(C)}) \circ c_i)$$

for every i.

We recall that for all composable morphisms

$$F(X) \xrightarrow{f} Y \xrightarrow{g} Y'$$

in $\mathcal{B}$ we have the identity

$$\varphi(g \circ f) = G(g) \circ \varphi(f)\,,$$

and that for all composable morphisms

$$X \xrightarrow{f} X' \xrightarrow{g} G(Y)$$

in $\mathcal{A}$ we have the identity

$$\varphi^{-1}(g \circ f) = \varphi^{-1}(g) \circ F(f)\,.$$

We find with this second identity the desired equality

$$\varphi^{-1}(\varphi(\mathrm{id}_{F(C)}) \circ c_i) = \varphi^{-1}(\varphi(\mathrm{id}_{F(C)})) \circ F(c_i) = \mathrm{id}_{F(C)} \circ F(c_i) = F(c_i)\,.$$

(iii)

We show that the right adjoint functor G is left exact. The right exactness of F then follows by duality.

We need to show that for every short exact sequence

$$0 \to Y' \to Y \to Y'' \to 0$$

in $\mathcal{B}$ the resulting sequence

$$0 \to G(Y') \to G(Y) \to G(Y'')$$

in $\mathcal{A}$ is again (left) exact. We use part (i) of this exercise, and show that for every object X of $\mathcal{A}$ the resulting sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, 0) \to \mathrm{Hom}_{\mathcal{A}}(X, G(Y')) \to \mathrm{Hom}_{\mathcal{A}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{A}}(X, G(Y''))$$

is exact. We have that $0 = G(0)$ because the functor G is additive, and hence need to show that the sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, G(0)) \to \mathrm{Hom}_{\mathcal{A}}(X, G(Y') \to \mathrm{Hom}_{\mathcal{A}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{A}}(X, G(Y''))$$

is exact. This sequence is via the natural isomorphism φ isomorphic to the sequence

$$\mathrm{Hom}_{\mathcal{B}}(F(X), 0) \to \mathrm{Hom}_{\mathcal{B}}(F(X), Y') \to \mathrm{Hom}_{\mathcal{B}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), Y'')\,,$$

which is the same as the sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(F(X), Y') \to \mathrm{Hom}_{\mathcal{B}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), Y'')\,.$$

This sequence is indeed (left) exact—as desired—by the left exactness of the functor $\mathrm{Hom}_{\mathcal{B}}(F(X), -)$.