Group 1, Wednesday 8–10, Tutor: Nils Waßmuth

Exercises in Foundations in Representation Theory

Exercise Sheet 2

Jendrik Stelzner

Exercise 1.

A homomorphism $f\colon X_{(a,b)} \to X_{(c,d)}$ is the same as a pair $f = (\lambda, \mu)$ consisting of scalars $\lambda, \mu \in k$ such that the diagrams

$$\begin{tikzcd} k \arrow[r, "a"] \arrow[d, "\lambda"'] & k \arrow[d, "\mu"] \\ k \arrow[r, "c"] & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \arrow[r, "b"] \arrow[d, "\lambda"'] & k \arrow[d, "\mu"] \\ k \arrow[r, "d"] & k \end{tikzcd}$$

commute, i.e. such that

$$\begin{cases} \mu a = c\lambda\,, \\ \mu b = d\lambda\,, \end{cases} \iff \begin{bmatrix} c & -a \\ d & -b \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \iff \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \in \ker \begin{bmatrix} c & -a \\ d & -b \end{bmatrix}\,.$$

We now distinguish between some cases:

- If (a,b) and (c,d) are linearly independent then the matrix $\left[\begin{smallmatrix} c & -a \\ d & -b \end{smallmatrix}\right]$ is invertible and it follows that $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)}) = 0$.
- If $(a,b) = (0,0)$ but $(c,d) \neq (0,0)$ then $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)}) = \{(0,\mu) \mid \mu \in k\}$.
- If $(a,b) \neq (0,0)$ but $(c,d) = (0,0)$ then $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)}) = \{(\lambda,0) \mid \lambda \in k\}$.
- If $(a,b), (c,d) \neq (0,0)$ are linearly dependent then (c,d) is a nonzero scalar multiple of (a,b) and the space $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)})$ is one-dimensional We further distinguish between two non-exclusive cases:
  - If $a \neq 0$ then also $c \neq 0$ (because (c,d) is a nonzero scalar multiple of (a,b)) and $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)}) = \{\kappa(a,c) \mid \kappa \in k\}$.
  - If $b \neq 0$ then also $d \neq 0$, and $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)}) = \{\kappa(b,d) \mid \kappa \in k\}$.

- If $(a,b) = (c,d) = (0,0)$ then $\operatorname{Hom}(X_{(a,b)}, X_{(c,d)}) = \{(\lambda, \mu) \mid \lambda, \mu \in k\}$.

The two representations $X_{(a,b)}$ and $X_{(c,d)}$ are isomorphic if and only if there exists some $(\lambda, \mu) \in \operatorname{Hom}(X_{(a,b)}, X_{(c,d)})$ with both $\lambda \neq 0$ and $\mu \neq 0$. We see from the discussion above that this happens in only in two cases:

- If (a,b) and (c,d) are both nonzero but linearly dependent.
- If $(a,b) = (c,d) = (0,0)$. In this case it already holds that $X_{(a,b)} = X_{(0,0)} = X_{(c,d)}$.

Exercise 2.

(i)

The path algebra of the quiver Q is given by $kQ \cong k[x]$, and representations of Q over k are therefore "the same" as $k[x]$-modules. To be more precise, a representation

$$V \circlearrowleft \varphi$$

of Q corresponds to the $k[x]$-module whose underlying k-vector space is given by V and for which the action of x on V is given by φ. It follows from the classification of finitely generated $k[x]$-modules and k being algebraically closed that the finite-dimensional indecomposable $k[x]$-modules are up to isomorphism precisely those of the form

$$k[x]/(x-\lambda)^n$$

with $\lambda \in k$ and $n \geq 1$, and that these representations are pairwise nonisomorphic. With respect to the basis $1, (x-\lambda), \dots, (x-\lambda)^{n-1}$ of $k[x]/(x-\lambda)^n$ the action of x is given by the Jordan block matrix

$$\begin{bmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}.$$

This shows altogether that the representations

$$k^n \circlearrowleft \begin{bmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}$$

with $\lambda \in k$ and $n \geq 1$ form a set of representatives for the isomorphism classes of finite-dimensional indecomposable representations of Q over k.

(ii)

For the matrix

$$A \coloneqq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{M}_2(\mathbb{R})$$

the only A-invariant subspaces of $\mathbb{R}^2$ are 0 and $\mathbb{R}^2$, because the vectors v and Av are linearly independent for every $v \in V$, $v \neq 0$. The representation

$$\mathbb{R}^2 \circlearrowleft A$$

is therefore irreducible, and hence indecomposable.

But when we change the base field from $\mathbb{R}$ to $\mathbb{C}$ then the matrix A becomes diagonalizable with

$$\mathbb{C}^2 = \langle e_1 + ie_2 \rangle_{\mathbb{C}} \oplus \langle e_1 - ie_2 \rangle_{\mathbb{C}}$$

being a decomposition into nonzero A-invariant subspaces. The representation

$$\mathbb{C}^2 \circlearrowleft A$$

is therefore decomposable.

Exercise 3.

(i)

For $f \in \mathrm{Hom}_A(M, N)$ and $z \in P$ consider the map

$$\tilde{\Phi}(f, z) \colon M \to N \otimes_B P, \quad x \mapsto f(x) \otimes z.$$

The map $\tilde{\Phi}(f, z)$ is a homomorphism of left A-modules because

$$\begin{aligned} \tilde{\Phi}(f, z)(x_1 + x_2) &= f(x_1 + x_2) \otimes z = (f(x_1) + f(x_2)) \otimes z \\ &= f(x_1) \otimes z + f(x_2) \otimes z = \tilde{\Phi}(f, z)(x_1) + \tilde{\Phi}(f, z)(x_2) \end{aligned}$$

and

$$\tilde{\Phi}(f, z)(ax) = f(ax) \otimes z = (af(x)) \otimes z = a(f(x) \otimes z) = a\tilde{\Phi}(f, z)(x)$$

for all $x, x_1, x_2 \in M$, $a \in A$. This shows that $\tilde{\Phi}(f, z)$ is a well-defined element of $\mathrm{Hom}_A(M, N \otimes_B P)$, hence that the map

$$\tilde{\Phi} \colon \mathrm{Hom}_A(M, N) \times P \to \mathrm{Hom}_A(M, N \otimes_B P)$$

is well-defined. The map $\tilde{\Phi}$ is B-balanced because

$$
\begin{aligned}
\tilde{\Phi}(f_1 + f_2, z)(x) &= (f_1 + f_2)(x) \otimes z = (f_1(x) + f_2(x)) \otimes z \\
&= f_1(x) \otimes z + f_2(x) \otimes z = \tilde{\Phi}(f_1, z)(x) + \tilde{\Phi}(f_2, z)(x) \\
&= (\tilde{\Phi}(f_1, z) + \tilde{\Phi}(f_2, z))(x)\,,
\end{aligned}
$$

$$
\begin{gathered}
\tilde{\Phi}(f, z_1 + z_2)(x) = f(x) \otimes (z_1 + z_2) = f(x) \otimes z_1 + f(x) \otimes z_2 = \tilde{\Phi}(f, z_1) + \tilde{\Phi}(f, z_2)\,, \\
\tilde{\Phi}(\lambda f, z)(x) = (\lambda f)(x) \otimes z = (\lambda f(x)) \otimes z = \lambda(f(x) \otimes z) = \lambda\tilde{\Phi}(f, z)(x) = (\lambda\tilde{\Phi}(f, z))(x) \\
\tilde{\Phi}(f, \lambda z)(x) = f(x) \otimes (\lambda z) = \lambda(f(x) \otimes z) = \lambda\tilde{\Phi}(f, z)(x) = (\lambda\tilde{\Phi}(f, z))(x) \\
\tilde{\Phi}(fb, z)(x) = (fb)(x) \otimes z = (f(x)b) \otimes z = f(x) \otimes (bz) = \tilde{\Phi}(f, bz)(x)
\end{gathered}
$$

for all $f, f_1, f_2 \in \operatorname{Hom}_A(M, N)$ $x, x_1, x_2 \in M$, $\lambda \in k$, $z \in P$. It follows that $\tilde{\Phi}$ induces a well-defined k-linear map

$$
\begin{aligned}
\Phi\colon\ \operatorname{Hom}_A(M, N) \otimes_B P &\to \operatorname{Hom}_A(M, N \otimes_B P)\,, \\
f \otimes z &\mapsto \left[\tilde{\Phi}(f, z)\colon x \mapsto f(x) \otimes z\right]\,,
\end{aligned}
$$

as desired.

(ii)

We consider the commutative k-algebras $A, B \coloneqq k[t]/(t^2)$ and the A-modules $N \coloneqq A$ and $M, P \coloneqq A/(t) \cong k[t]/(t)$. The A-module $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_A(M, A) = M^\vee$ is the dual module of M, and it holds that $N \otimes_B P \cong P = M$.[1] We thus have to show that the homomorphism

$$
\Phi\colon M^\vee \otimes_A M \to \operatorname{End}_A(M)\,, \quad f \otimes x \mapsto (z \mapsto f(z)x)
$$

is neither surjective nor injective. We do so by showing that $M^\vee \otimes_A M \cong M$ and $\operatorname{End}_A(M) \cong M$, but that the homomorphism $M \to M$ corresponding to Φ is the zero homomorphism.

We first note that

$$
M^\vee = \operatorname{Hom}_A(M, A) = \operatorname{Hom}_A(A/(t), A) = \{t\text{-torsion of } A\} = tA \cong A/(t) = M\,,
$$

where the inverse isomorphism $M \to M^\vee$ is given on representatives given by

$$
M \xrightarrow{\sim} M^\vee\,, \quad [p] \mapsto ([q] \mapsto qtp = tpq)\,.
$$

We also have an isomorphism

$$
M \otimes_A M = A/(t) \otimes_A A/(t) \cong A/((t) + (t)) = A/(t) = M\,,
$$

[1] Here we implicitely use that over commutatives rings the "non-commutative" tensor product coincides with the "commutative" one.

whose inverse is given by

$$M \xrightarrow{\sim} M \otimes_A M\,, \quad [p] \mapsto [p] \otimes [1] = [1] \otimes [p]\,.$$

Lastly, we have an isomorphism

$$\mathrm{End}_A(M) = \mathrm{End}_A(A/(t)) \cong \mathrm{End}_{A/(t)}(A/(t)) \cong A/(t) = M\,,$$

which is given by

$$\mathrm{End}_A(M) \xrightarrow{\sim} M\,, \quad f \mapsto f(1)\,.$$

The overall resulting composition

$$M \xrightarrow{\sim} M \otimes_A M \xrightarrow{\sim} M^\vee \otimes_A M \xrightarrow{\Phi} \mathrm{End}_A(M) \xrightarrow{\sim} M$$

is on elements given by

$$[p] \mapsto [p] \otimes [1] \mapsto ([q] \mapsto tpq) \otimes [1] \mapsto ([q] \mapsto tpq[1] = [tpq]) \mapsto [tp] = t[p]\,,$$

and thus by multiplication with t. It follows that this composition is the zero map because $tM = 0$. We thus have the following commutative diagram in which all vertical arrows are isomorphisms:

$$\begin{tikzcd}
M^\vee \otimes_A M \arrow[r, "\Phi"] & \mathrm{End}_A(M) \arrow[dd, "\sim"] \\
M \otimes M \arrow[u, "\sim"'] & \\
M \arrow[u, "\sim"'] \arrow[r, "0"] & M
\end{tikzcd}$$

Because the zero endomorphism $M \to M$ is neither injective nor surjective, the same follows for Φ.

(iii)

If M is Free

Suppose that M is free of rank n with basis $x_1, \dots, x_n$. Let $\varphi\colon A^n \to M$ be the unique isomorphism of left A-modules with $\varphi(e_i) = x_i$ for all $i = 1, \dots, n$. Then

$$\begin{aligned}
&\mathrm{Hom}_A(M, N) \otimes_B P \\
\cong\ &\mathrm{Hom}_A(A^{\oplus n}, N) \otimes_B P \qquad (1)\\
\cong\ &\mathrm{Hom}_A(A, N)^{\times n} \otimes_B P \\
\cong\ &N^{\times n} \otimes_B P \\
=\ &N^{\oplus n} \otimes_B P \\
\cong\ &(N \otimes_B P)^{\oplus n}
\end{aligned}$$

and also

$$
\begin{aligned}
&\operatorname{Hom}_A(M, N \otimes_B P) \\
\cong\ &\operatorname{Hom}_A(A^{\oplus n}, N \otimes_B P) \\
\cong\ &\operatorname{Hom}_A(A, N \otimes_B P)^{\times n} \\
\cong\ &(N \otimes_B P)^{\times n} \\
=\ &(N \otimes_B P)^{\oplus n}\,,
\end{aligned} \tag{2}
$$

where the isomorphisms (1) and (2) are induced by φ. The first of the above two isomorphisms is altogether given by

$$
\begin{aligned}
\operatorname{Hom}_A(M, N) \otimes P &\to (N \otimes_B P)^{\oplus n}\,, \\
f \otimes z &\mapsto (f(x_1) \otimes z, \dots, f(x_n) \otimes z)\,,
\end{aligned}
$$

and the second isomorphism is altogether given by

$$
\begin{aligned}
\operatorname{Hom}_A(M, N \otimes_B P) &\to (N \otimes P)^{\oplus n}\,, \\
g &\mapsto (g(x_1), \dots, g(x_n))\,.
\end{aligned}
$$

The diagram

$$
\begin{tikzcd}
\operatorname{Hom}_A(M, N) \otimes_B P \arrow[rr, "\Phi"] \arrow[dr, "\sim"'] & & \operatorname{Hom}_A(M, N \otimes_B P) \arrow[dl, "\sim"] \\
& (N \otimes_B P)^{\oplus n} &
\end{tikzcd}
$$

commutes, and so it follows that Φ is an isomorphism.

If P is Free

Suppose that P is free of rank n and with basis $z_1, \dots, z_n$. Let $\varphi\colon B^n \to P$ be the unique isomorphism of left B-modules with $\varphi(z_i) = e_i$ for every $i = 1 \dots, n$. Then

$$
\begin{aligned}
&\operatorname{Hom}_A(M, N) \otimes_B P \\
\cong\ &\operatorname{Hom}_A(M, N) \otimes_B B^{\oplus n} \\
\cong\ &(\operatorname{Hom}_A(M, N) \otimes_B B)^{\oplus n} \\
\cong\ &\operatorname{Hom}_A(M, N)^{\oplus n}
\end{aligned} \tag{3}
$$

and also

$$
\begin{aligned}
&\operatorname{Hom}_A(M, N \otimes_B P) \\
\cong\ &\operatorname{Hom}_A(M, N \otimes_B B^{\oplus n}) \\
\cong\ &\operatorname{Hom}_A(M, N^{\oplus n}) \\
\cong\ &\operatorname{Hom}_A(M, N^{\times n}) \\
\cong\ &\operatorname{Hom}_A(M, N)^{\times n} \\
\cong\ &\operatorname{Hom}_A(M, N)^{\oplus n}\,,
\end{aligned} \tag{4}
$$

where the isomorphisms (3) and (4) are induced by φ. The inverse of the first isomorphism is given by

$$\begin{aligned}\operatorname{Hom}_A(M,N)^{\oplus n} &\to \operatorname{Hom}_A(M,N) \otimes_B P\,,\\ (f_1,\dots,f_n) &\mapsto f_1 \otimes z_1 + \cdots + f_n \otimes z_n\,,\end{aligned}$$

and the inverse of the second isomorphism is given by

$$\begin{aligned}\operatorname{Hom}_A(M,N)^{\oplus n} &\to \operatorname{Hom}_A(M, N \otimes_B P)\,,\\ (f_1,\dots,f_n) &\mapsto [x \mapsto f_1(x) \otimes z_1 + \cdots + f_n(x) \otimes z_n]\\ &= \Phi(f_1 \otimes z_1 + \cdots + f_n \otimes z_n)\,.\end{aligned}$$

This shows that the diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(M,N) \otimes_B P \arrow[rr, "\Phi"] & & \operatorname{Hom}_A(M, N \otimes_B P) \\
& \operatorname{Hom}_A(M,N)^{\oplus n} \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

commutes, hence that Φ is an isomorphism.

Exercise 4.

(i)

The k-module of R comes from identifying R with $A \times X \times B$ via the bijection

$$R \to A \times X \times B\,, \quad \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \mapsto (a, x, b)\,.$$

The proposed multiplication is well-defined because

$$\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & a_1 x_2 + x_1 b_2 \\ 0 & b_1 b_2 \end{bmatrix} \in R\,.$$

The multiplication is associative because

$$\begin{aligned}
&\left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \right) \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 a_2 & a_1 x_2 + x_1 b_2 \\ 0 & b_1 b_2 \end{bmatrix} \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 a_2 a_3 & a_1 a_2 x_3 + a_1 x_2 b_3 + x_1 b_2 b_3 \\ 0 & b_1 b_2 b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 a_3 & a_2 x_3 + x_2 b_3 \\ 0 & b_2 b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \left(\begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} \right).
\end{aligned}$$

The multiplication is distributive on the left because

$$\begin{aligned}
&\left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix}\right) \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 + a_2 & x_1 + x_2 \\ 0 & b_1 + b_2 \end{bmatrix} \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} \\
&= \begin{bmatrix} (a_1 + a_2)a_3 & (a_1 + a_2)x_3 + (x_1 + x_2)b_3 \\ 0 & (b_1 + b_2)b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 a_3 + a_2 a_3 & a_1 x_3 + a_2 x_3 + x_1 b_3 + x_2 b_3 \\ 0 & b_1 b_3 + b_2 b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 a_3 & a_1 x_3 + x_1 b_3 \\ 0 & b_1 b_3 \end{bmatrix} + \begin{bmatrix} a_2 a_3 & a_2 x_3 + x_2 b_3 \\ 0 & b_2 b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} + \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_3 & x_3 \\ 0 & b_3 \end{bmatrix} .
\end{aligned}$$

The distributivity on the right can be shown in the same way. The multiplication is already k-bilinear because

$$\begin{aligned}
&\left(\lambda \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix}\right) \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda a_1 & \lambda x_1 \\ 0 & \lambda b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda a_1 a_2 & \lambda a_1 x_2 + \lambda x_1 b_2 \\ 0 & \lambda b_1 b_2 \end{bmatrix} \\
&= \lambda \begin{bmatrix} a_1 a_2 & a_1 x_2 + x_1 b_2 \\ 0 & b_1 b_2 \end{bmatrix} \\
&= \lambda \left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix}\right) ,
\end{aligned}$$

und similarly

$$\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \left(\lambda \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix}\right) = \cdots = \lambda \left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix}\right) .$$

The multiplicative unit of R is given by the identity matrix because

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

This shows altogether that R together with the given k-module structure and multiplication is a k-algebra.

(ii)

The k-linear map

$$X \otimes_k N \xrightarrow{\lambda \otimes \mathrm{id}_N} \mathrm{Hom}_k(N, M) \otimes_k N \xrightarrow[f \otimes n \mapsto f(n)]{\text{eval}} M$$

corresponds to a k-bilinear map

$$X \times M \to N\,, \quad (x,n) \mapsto \lambda(x)(n)\,, \tag{5}$$

which we might think about as an action of X on N to M. We hence write

$$x \cdot n \coloneqq \lambda(x)(n)$$

for all $x \in X$, $n \in N$. It follows from the k-bilinearity of (5) that

$$(x_1 + x_2)n = x_1 n + x_2 n\,, \quad x(n_1 + n_2) = x n_1 + x n_2\,, \quad (\lambda x)n = \lambda(xn) = x(\lambda n)$$

for all $x, x_1, x_2 \in X$, $n, n_1, n_2 \in N$, $\lambda \in k$. It further follows from λ being a homomorphism of A-B-bimodules that

$$(ax)n = \lambda(ax)(n) = (a\lambda(x))(n) = a\lambda(x)(n) = a(xn) \tag{6}$$

and

$$(xb)n = \lambda(xb)(n) = (\lambda(x)b)(n) = \lambda(x)(bn) = x(bn)\,. \tag{7}$$

As the underlying k-module of $F(M,N,\lambda)$ we choose $M \oplus N$, and write the elements of $F(M,N,\lambda)$ as column vectors

$$\begin{bmatrix} m \\ n \end{bmatrix}$$

with $m \in M$, $n \in N$. The action of R on $F(M,N,\lambda)$ is given by

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} am + xn \\ bn \end{bmatrix}\,.$$

This action is distributive on the left because

$$\begin{aligned}
&\left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} \\
&= \begin{bmatrix} a_1 + a_2 & x_1 + x_2 \\ 0 & b_1 + b_2 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \\
&= \begin{bmatrix} (a_1 + a_2)m + (x_1 + x_2)n \\ (b_1 + b_2)n \end{bmatrix} \\
&= \begin{bmatrix} a_1 m + a_2 m + x_1 n + x_2 n \\ b_1 n + b_2 n \end{bmatrix} \\
&= \begin{bmatrix} a_1 m + x_1 n \\ b_1 n \end{bmatrix} + \begin{bmatrix} a_2 m + x_2 n \\ b_2 n \end{bmatrix} \\
&= \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} + \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}\,,
\end{aligned}$$

and distributive on the right because

$$
\begin{aligned}
&\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \left(\begin{bmatrix} m_1 \\ n_1 \end{bmatrix} + \begin{bmatrix} m_2 \\ n_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} m_1 + m_2 \\ n_1 + n_2 \end{bmatrix} \\
&= \begin{bmatrix} a(m_1 + m_2) + x(n_1 + n_2) \\ b(n_1 + n_2) \end{bmatrix} \\
&= \begin{bmatrix} am_1 + am_2 + xn_1 + xn_2 \\ bn_1 + bn_2 \end{bmatrix} \\
&= \begin{bmatrix} am_1 + xn_1 \\ bn_1 \end{bmatrix} + \begin{bmatrix} am_2 + xn_2 \\ bn_2 \end{bmatrix} \\
&= \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} m_1 \\ n_1 \end{bmatrix} + \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} m_2 \\ n_2 \end{bmatrix} .
\end{aligned}
$$

The multiplicaiton is already k-bilinear because

$$
\begin{aligned}
\left(\mu \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} \mu a & \mu x \\ 0 & \mu b \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} \mu am + \mu xn \\ \mu bn \end{bmatrix} &= \mu \begin{bmatrix} am + xn \\ bn \end{bmatrix} \\
&= \mu \left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \right)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \left(\mu \begin{bmatrix} m \\ n \end{bmatrix} \right) = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} \mu m \\ \mu n \end{bmatrix} = \begin{bmatrix} \mu am + \mu xn \\ \mu bn \end{bmatrix} &= \mu \begin{bmatrix} am + xn \\ bn \end{bmatrix} \\
&= \mu \left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \right)
\end{aligned}
$$

The multiplication is associative because

$$
\begin{aligned}
&\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \left(\begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \right) \\
&= \begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 m + x_2 n \\ b_2 n \end{bmatrix} \\
&= \begin{bmatrix} a_1 a_2 m + a_1 x_2 n + x_1 b_2 n \\ b_1 b_2 n \end{bmatrix} \\
&= \begin{bmatrix} (a_1 a_2) m + (a_1 x_2 + x_1 b_2) n \\ (b_1 b_2) n \end{bmatrix} \\
&= \begin{bmatrix} a_1 + a_2 & a_1 x_2 + x_1 b_2 \\ 0 & b_1 + b_2 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \\
&= \left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \right) \begin{bmatrix} m \\ n \end{bmatrix} ,
\end{aligned}
$$

where we use the associativity of (6) and (7). It also holds that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 1 \cdot m + 0 \cdot n \\ 1 \cdot n \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} .$$

Altogether we have defined on $F(M, N, \lambda)$ the structure of a left R-module.

(iii)

Let (M, N, λ) and (M', N', λ') be two triples consisting of left A-modules M and M', left B-modules N and N', and homomorphism of A-B-bimodules $\lambda\colon X \to \operatorname{Hom}_k(N, M)$ and $\lambda'\colon X \to \operatorname{Hom}_k(N', M')$. We define a *morphism* $f\colon (M, N, \lambda) \to (M', N', \lambda')$ to be a pair $f = (f_1, f_2)$ consisting of a homomorphism of left A-modules $f_1\colon M \to M'$ and a homomorphism of left B-modules $f_2\colon N \to N'$, subject to the relation that

$$f_1 \circ \lambda(x) = \lambda'(x) \circ f_2$$

for all $x \in X$; this means for the actions of X on N to M and on N' to M' that

$$f_1(xn) = x f_2(n)$$

for all $x \in X$, $n \in N$. This may be represented pictorially by the following “commutative diagram”:

$$\begin{tikzcd} M \arrow[d, "f_1"'] & N \arrow[l, "x\cdot(-)"'] \arrow[d, "f_2"] \\ M' & N' \arrow[l, "x\cdot(-)"'] \end{tikzcd}$$

The *composition* of two such morphisms

$$f\colon (M, N, \lambda) \to (M', N', \lambda') ,$$
$$f'\colon (M', N', \lambda') \to (M'', N'', \lambda'')$$

is given by

$$f' \circ f = (f_1', f_2') \circ (f_1, f_2) = (f_1' \circ f_1, f_2' \circ f_2) .$$

This is again a morphism because

$$f_1' \circ f_1 \circ \lambda(x) = f_1' \circ \lambda'(x) \circ f_2 = \lambda''(x) \circ f_2' \circ f_2$$

for every $x \in X$. The identity morphism of (M, N, λ) is given by $\mathrm{id}_{(M,N,\lambda)} = (\mathrm{id}_M, \mathrm{id}_N)$; this is a morphisms because

$$\mathrm{id}_M \circ \lambda(x) = \lambda(x) = \lambda(x) \circ \mathrm{id}_N$$

for every $x \in X$; it holds for every morphism $f\colon (M, N, \lambda) \to (M', N', \lambda')$ that $f \circ \mathrm{id} = f$, and for every morphisms $f\colon (M', N', \lambda') \to (M, N, \lambda)$ that $\mathrm{id} \circ f = f$.

The construction F is functorial: Every morphism $f\colon (M,N,\lambda) \to (M',N',\lambda')$ induces a map

$$F(f)\colon F(M,N,\lambda) \to F(M',N',\lambda')\,, \quad \begin{bmatrix} m \\ n \end{bmatrix} \mapsto \begin{bmatrix} f_1(m) \\ f_2(n) \end{bmatrix}\,;$$

we hence have that $F(f) = f_1 \oplus f_2$. The additive map $F(f)$ is a homomorphism of left R-modules because

$$\begin{aligned} F(f)\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}\right) &= F(f)\left(\begin{bmatrix} am+xn \\ bn \end{bmatrix}\right) = \begin{bmatrix} f_1(am+xn) \\ f_2(bn) \end{bmatrix} \\ &= \begin{bmatrix} f_1(am)+f_1(xn) \\ f_2(bn) \end{bmatrix} = \begin{bmatrix} af_1(m)+xf_2(n) \\ bf_2(n) \end{bmatrix} \\ &= \begin{bmatrix} a & x \\ 0 & b \end{bmatrix}\begin{bmatrix} f_1(m) \\ f_2(n) \end{bmatrix} = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} F(f)\left(\begin{bmatrix} m \\ n \end{bmatrix}\right)\,. \end{aligned}$$

It furthermore holds for any two composable morphisms $f\colon (M,N,\lambda) \to (M',N',\lambda')$ and $f'\colon (M',N',\lambda') \to (M'',N'',\lambda'')$ that

$$\begin{aligned} F(f') \circ F(f) &= (f'_1 \oplus f'_2) \circ (f_1 \oplus f_2) = (f'_1 \circ f_1) \oplus (f'_2 \circ f_2) \\ &= (f' \circ f)_1 \oplus (f' \circ f)_2 = F(f' \circ f)\,, \end{aligned}$$

and it holds that

$$F(\mathrm{id}_{(M,N,\lambda)}) = \mathrm{id}_M \oplus \mathrm{id}_N = \mathrm{id}_{M \oplus N} = \mathrm{id}_{F(M,N,\lambda)}\,.$$

This shows altogether the claimed functoriality of F.