Exercises in
Foundations in Representation Theory

Exercise Sheet 11

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Exercise 1.

Lemma 1. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Then F respects exact sequences, i.e. if

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

is an exact sequence in $\mathcal{A}$, then the resulting sequence

$$F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'')$$

in $\mathcal{B}$ is again exact.

Proof. It follows from the exactness of the functor F that it is both left exact and right exact. Hence F respects both kernels and cokernels. Therefore F respect both images and coimages. We have that

$$F(g) \circ F(f) = F(g \circ f) = F(0) = 0$$

because F is additive, and we find from the above discussion that the induced canonical morphism $\operatorname{im}(F(f)) \to \ker(F(g))$ in $\mathcal{B}$ is given by the image of the canonical morphism $\operatorname{im}(f) \to \ker(g)$ in $\mathcal{A}$ under the functor F. Since $\operatorname{im}(f) \to \ker(g)$ is an isomorphism the same follows for $\operatorname{im}(F(f)) \to \ker(F(g))$ (because the functor F respects isomorphisms). □

Corollary 2. If $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, then for every long exact sequence

$$\cdots \to X_{i-1} \to X_i \to X_{i+1} \to \cdots$$

in $\mathcal{A}$ the resulting sequence

$$\cdots \to F(X_{i-1}) \to F(X_i) \to F(X_{i+1}) \to \cdots$$

in $\mathcal{B}$ is again exact. $\square$

We find that if $T^\bullet \colon \mathcal{A} \to \mathcal{B}$ is a cohomological δ-functor given by

$$T^\bullet = ((T^n)_{n \geq 0}, (\delta^n_\xi)_{n \geq 0, \xi})$$

and if $F \colon \mathcal{B} \to \mathcal{C}$ is an exact functor, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are abelian categories, then

$$F \circ T^\bullet \coloneqq ((F \circ T^n)_{n \geq 0}, (F(\delta^n_\xi))_{n \geq 0, \xi})$$

is a δ-functor $\mathcal{A} \to \mathcal{C}$. If the category $\mathcal{A}$ has enough injectives and $T^\bullet$ is universal, then $F \circ T^\bullet$ is again universal. This holds because the δ-functors $T^\bullet$ and $F \circ T^\bullet$ are universal if and only if the functors T^n (resp. $F \circ T^n$) annihilate all injective objects for every $n \geq 1$.

We are now prepared for the exercise: The functor $G \circ F$ is again left exact because both G and F are left exact. We find that $G \circ (\mathrm{R}^\bullet F)$ is a cohomological δ-functor $\mathcal{A} \to \mathcal{C}$ with $G \circ T^0 \cong G \circ F$ because $T^0 \cong F$. The δ-functor $\mathrm{R}^\bullet F$ is universal, and hence $G \circ (\mathrm{R}^\bullet F)$ are universal as seen above.

Exercise 2.

We get from the short exact sequence $0 \to X \to I \to Y \to 0$ a long exact sequence

$$0 \to F(X) \to F(I) \to F(Y) \to (\mathrm{R}^1 F)(X) \to (\mathrm{R}^1 F)(I) \to \cdots$$

We know from the lecture that $(\mathrm{R}^n F)(I) = 0$ for every $n \geq 1$ because I is injective.

(i)

We get for every $n \geq 1$ from the exactness of the sequence

$$0 \to (\mathrm{R}^n F)(Y) \to (\mathrm{R}^{n+1} F)(X) \to 0$$

that the connecting morphism $(\mathrm{R}^n F)(Y) \to (\mathrm{R}^{n+1} F)(X)$ is an isomorphism.

(ii)

We also get from the exactness of the sequence

$$F(I) \to F(Y) \to (\mathrm{R}^1 F)(X) \to 0$$

that the connecting morphism $F(Y) \to (\mathrm{R}^1 F)(X)$ is a cokernel of $F(I) \to F(Y)$.

Exercise 3.

(i) $\iff$ (ii): That every object $P \in \mathrm{Ob}(\mathcal{A})$ is projective means that every short exact sequence $0 \to X' \to X \to P \to 0$ in $\mathcal{A}$ ending in any object $P \in \mathrm{Ob}(\mathcal{A})$ splits.

(i) $\iff$ (iii): This equivalence follows from the equivalence (i) $\iff$ (ii) by duality.

(ii) $\iff$ (iv): An object $P \in \mathrm{Ob}(\mathcal{A})$ is projective if and only if the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact (as known from the lecture).

(iii) $\iff$ (v): An object $I \in \mathrm{Ob}(\mathcal{A})$ is injective if and only if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact.

(iii),(v) $\implies$ (vi): For every object $X \in \mathrm{Ob}(\mathcal{A})$ one can use the identity $\mathrm{id}_X \colon X \to X$ to see that the category $\mathcal{A}$ has enough injective, because the object X is injective by (iii). That $\mathrm{R}^1 \mathrm{Hom}_{\mathcal{A}}(-, Y) = 0$ follows from $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ being exact, which holds by (v).

(vi) $\implies$ (v): It follows for every $Y \in \mathrm{Ob}(\mathcal{A})$ from $\mathrm{R}^1 \mathrm{Hom}_{\mathcal{A}}(-, Y) = 0$ via the long exact sequence that $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ is exact.

(ii),(iv) $\iff$ (vii): This follows from the equivalence (iii), (v) $\iff$ (vi) by duality.

Exercise 4.

We assume that the given quiver Q has only finitely many vertices (so that the path algebra kQ is unital) and finitely many arrows (which is needed in part (iii) for $P(i)$ to be finite). We abbreviate $A \coloneqq kQ$. Instead of representations of Q over k we will work with A-modules. Then $X_i = \varepsilon_i X$ for every A-module X.

(ii)

We have for every vertex $i \in Q_0$ that $P(i) = A\varepsilon_i$ It follows that

$$A = \bigoplus_{i \in Q_0} A\varepsilon_i = \bigoplus_{i \in Q_0} P(i)$$

because the basis Q_* of A decomposes as $Q_* = \coprod_{i \in Q_0} Q_* \varepsilon_i$, with $Q_* \varepsilon_i$ being a basis of $P(i)$. This shows that the A-modules $P(i)$ are direct summands of the free A-module A, and hence projective.

(i)

We have for every A-module an isomorphism of k-vector spaces

$$\Phi_2 \colon \mathrm{Hom}_A(A, X) \to X \,, \quad f \mapsto f(1) \,.$$

The A-A-bimodule structure of A leads to a left A-module structure on $\mathrm{Hom}_A(A, X)$ given by

$$(a.f)(a') = f(a'a)$$

for all $a \in A$, $f \in \mathrm{Hom}_A(A, X)$ and $a' \in A$. The above isomorphism of k-vector space is then an isomorphism of (left) A-modules because

$$(a \cdot f)(1) = f(1 \cdot a) = f(a \cdot 1) = a \cdot f(1)$$

for all $a \in A$ and $f \in \mathrm{Hom}_A(A, X)$.

The decomposition $A = \bigoplus_{i \in Q_0} P(i)$ of A-modules results in a decomposition

$$\mathrm{Hom}_A(A, X) = \mathrm{Hom}_A\left(\bigoplus_{i \in Q_0} P(i), X\right) \cong \bigoplus_{i \in Q_0} \mathrm{Hom}_A(P(i), X)$$

of k-vector spaces. The isomorphism

$$\Phi_1 \colon \bigoplus_{i \in Q_0} \mathrm{Hom}_A(P(i), X) \to \mathrm{Hom}_A(A, X)$$

is explicitely given by

$$\Phi_1((f_i)_{i \in Q_0}) = \sum_{i \in Q_0} (f_i \circ \pi_i) = \sum_{i \in Q_0} \pi_i^*(f_i)$$

where $\pi_i \colon A \to P(i)$ denotes the projection along the decomposition $A = \bigoplus_{i \in I}$. This projection π_i is given by right multiplication with the idempotent ε_i. The complete isomorphism of k-vector spaces

$$\Phi \coloneqq \Phi_2 \circ \Phi_1 \colon \bigoplus_{i \in Q_0} \mathrm{Hom}_A(P(i), X) \to X$$

is therefore given by

$$\Phi((f_i)_{i \in I}) = \Phi_2(\Phi_1((f_i)_{i \in I})) = \left(\sum_{i \in Q_0} \pi_i^*(f_i)\right)(1) = \sum_{i \in Q_0} f(\pi_i(1)) = \sum_{i \in Q_0} f_i(\varepsilon_i)$$

for every $(f_i)_{i \in I} \in \bigoplus_{i \in I} \mathrm{Hom}_A(P(i), X)$. The restriction

$$\mathrm{Hom}_A(P(j), X) \hookrightarrow \bigoplus_{i \in Q_0} \mathrm{Hom}_A(P(i), X) \xrightarrow{\Phi} X$$

is therefore given by

$$f \mapsto f(\varepsilon_i)\,.$$

We note that

$$f(\varepsilon_i) = f(\varepsilon_i^2) = \varepsilon_i f(\varepsilon_i) \in \varepsilon_i X = X_i$$

for every $f \in \mathrm{Hom}_A(P(i), X)$. This means that the isomorphism Φ maps the direct summand $\mathrm{Hom}_A(P(i), X)$ into the direct summand X_i of X. It follows that the resulting restriction

$$\mathrm{Hom}_A(P(i), X) \to X_i\,, \quad f \mapsto f(\varepsilon_i)$$

is an isomorphism.

(iii)

The quiver Q admits only finitely many paths because Q contains only finitely many arrows and no oriented cycles. It follows that the path algebra A is finite-dimensional, and hence that every module $P(i)$ (which is a direct summand of A) is finite-dimensional. We also find that

$$\mathrm{End}_A(P(i)) = \mathrm{Hom}_A(P(i), P(i)) \cong P(i)_i$$

has as a basis the set

$$Q_*(i,i) = \{p \in Q_* \mid s(p) = i = t(p)\} = \{\text{oriented cycles at } i\} \cup \{\varepsilon_i\} = \{\varepsilon_i\}\,.$$

This shows that the k-algebra $\mathrm{End}_A(P(i))$ is one-dimensional, which reveals to us that $\mathrm{End}_A(P(i)) = k$ as k-algebras.[1]

(iv)

If M is a decomposable A-module then there exists a decomposition $M = N \oplus P$ into two nonzero submodules N and P of M. The projection $e\colon M \to M$ onto N along the decomposition $M = N \oplus P$ is then an idempotent in the k-algebra $\mathrm{End}_A(M)$ with

$$\mathrm{im}(e) = N \quad \text{and} \quad \ker(e) = P\,.$$

It follows from $N \neq 0$ that $e \neq 0$, and from $P \neq M$ that $e \neq \mathrm{id}_M$. We hence see that the endomorphism algebra $\mathrm{End}_A(M)$ contains a non-trivial idempotent if the module M is decomposable.[2]

We have seen in the previous part of the exercise that $\mathrm{End}_A(P(i)) = k$ as k-algebras. The field k contains no non-trivial idemponents, and hence $P(i)$ is indecomposable.

Exercise 5.

(i)

The short exact sequence $0 \to A' \to A \to A'' \to 0$ results in a long exact sequence

$$0 \to F(A') \to F(A) \to F(A'') \to (\mathrm{R}^1\, F)(A') \to (\mathrm{R}^1\, F)(A) \to \cdots$$

[1] Here we use that for every one-dimensional k-algebra B the unique morphism of k-algebras $k \to B$, that is given by $\lambda \to \lambda \cdot 1_B$, is an isomorphism of k-algebras.

[2] This construction does in fact give a bijection between the direct sum decompositions $M = N \oplus P$ of M and the idempotents in the endomorphism algebra $\mathrm{End}_A(M)$.

We have for every $n \geq 1$ that $(\mathrm{R}^n F)(A') = 0$ and $(\mathrm{R}^n F)(A) = 0$ because the objects A' and A are F-acyclic. We find for every $n \geq 1$ from the exactness of the sequence

$$0 \to \mathrm{R}^n(A'') \to 0$$

that $\mathrm{R}^n(A'') = 0$, which shows that A'' is also F-acyclic. We also find that the sequence

$$0 \to F(A') \to F(A) \to F(A'') \to 0$$

is again exact.

(ii)

We set $A^n := 0$ for every $n < 0$. It follows from the exactness of the sequence

$$\cdots \to A^{n-1} \to A^n \to A^{n+1} \to \cdots$$

that we have for every $n \in \mathbb{Z}$ a short exact sequence

$$0 \to Z^n \to A^n \to Z^{n+1} \to 0\,.$$

We have for every $n < 0$ that $Z^n = 0$, and hence that Z^n is F-acyclic. If Z^n is acyclic then we get from the above short exact sequence and part (i) of this exercise that Z^{n+1} is also F-acyclic, since then both Z^n and A^n are F-acyclic. Whence we find inductively that Z^n is F-acyclic for every $n \in \mathbb{Z}$.

(iii)

Instead of doing this exercise I spent my holidays being sick and watching UK policits going crazy.

Exercise 6.

(i)

I didn't do this exercise, as it is nontrivial and I rather watched Trump's attempt at a record length government shutdown.

(ii)

We have the following commutative diagram with (short) exact columns, in which the first two rows are (long) exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "a^0"] \arrow[d, "\mathrm{id}_X"'] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "i^0"] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & B^0 \arrow[r] \arrow[d] & B^1 \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

We regard the rows of this diagram as chain complexes and the overall diagram as a short exact sequence of these chain complexes. The upper two rows are acyclic, hence the third row is acyclic by part (i) of Exercise 2 of Exercise sheet 8.

(iii)

We have for every $n \geq 0$ a short exact sequence

$$0 \to A^n \to I^n \to B^n \to 0\,.$$

The object A^n is F-acyclic and the object I^n is injective, and hence also F-acyclic. It follows from part (i) of Exercise 5 that B^n is also F-acyclic.

(iv)

We have for every $n \geq 0$ the short exact sequence

$$0 \to A^n \to I^n \to B^n \to 0$$

of F-acyclic objects. It follows from part (i) of Exercise 5 that the sequence

$$0 \to F(A^n) \to F(I^n) \to F(B^n) \to 0$$

is again (short) exact for every $n \geq 0$. This shows the exactness of the sequence

$$0 \to F(A^\bullet) \to F(I^\bullet) \to F(B^\bullet) \to 0\,,$$

because this exactness is computed degreewise.

(v)

We get from the short exact sequence $0 \to F(A^\bullet) \to F(I^\bullet) \to F(B^\bullet) \to 0$ the long exact cohomology sequence

$$\cdots \to \mathrm{H}^{n-1}(F(B^\bullet)) \to \mathrm{H}^n(F(A^\bullet)) \to \mathrm{H}^n(F(I^\bullet)) \to \mathrm{H}^n(F(B^\bullet)) \to \cdots$$

The chain complex $B^\bullet$ is exact and consists of F-acyclic objects, hence the chain complex $F(B^\bullet)$ is again exact by part (iii) of Exercise 5. We therefore have $\mathrm{H}^n(F(B^\bullet)) = 0$ for every n. We hence have for every $n \geq 0$ an exact sequence

$$0 \to \mathrm{H}^n(F(A^\bullet)) \to \mathrm{H}^n(F(I^\bullet)) \to 0\,,$$

which tells us that the morphism $\mathrm{H}^n(F(A^\bullet)) \to \mathrm{H}^n(F(I^\bullet))$ is an isomorphism. We have that $\mathrm{H}^n(F(I^\bullet)) \cong (\mathrm{R}^n F)(X)$ for every $n \geq 0$ by the explicit computation of $(\mathrm{R}^n F)(X)$ via the injective resolution $(I^\bullet, i^0)$ of X.