Foundations of representation theory

10. Exercise sheet

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Exercise 38:

(i)

Let $x \in V_3$ with $a_3(x) = 0$. Because the diagram commutes we find that

$$0 = g_3 a_3(x) = a_4 f_3(x).$$

Because a_4 is a monomorphism this means that $f_3(x) = 0$. From the exactness of the upper row we get that $x \in \ker f_3 = \operatorname{im} f_2$, so there exists $y \in V_2$ with $f_2(y) = x$. Using the commutativity of the diagram we get that

$$0 = a_3(x) = a_3 f_2(y) = g_2 a_2(y).$$

Therefor $a_2(y) \in \ker g_2 = \operatorname{im} g_1 = \operatorname{im}(g_1 a_1)$, whereby we used that a_1 is an epimorphism. So we find some $z \in V_1$ with $g_1 a_1(z) = a_2(y)$. Combining all of the above and using the commutativity of the diagram we find that

$$a_2(f_1(z)) = a_2 f_1(z) = g_1 a_1(z) = a_2(y).$$

Because a_2 is a monomorphism it follows that $f_1(z) = y$. Because the upper row is exact we get

$$x = f_2(y) = f_2 f_1(z) = 0.$$

So a_3 is a momomorphism.

(ii)

Let $x \in W_3$. We look at $g_3(x) \in W_4$. Because a_4 is an epimorphism there exists some $y \in V_4$ with $a_4(y) = g_3(x)$. We notice that $f_4(y) = 0$: Using the exactness and commutativity of the diagram we get

$$a_5 f_4(y) = g_4 a_4(y) = g_4 g_3(x) = 0,$$

and because a_5 is a monomorphism it follows that $f_4(y) = 0$. Because the upper row is exact we have

$$y \in \ker f_4 = \operatorname{im} f_3,$$

and therefore there exists some $z \in V_3$ with $f_3(z) = y$.

We now look at $a_3(z)$: From the above and the commutativity of the diagram we get that

$$g_3(x) = a_4(y) = a_4 f_3(z) = g_3 a_3(z) = g_3(a_3(z)),$$

so

$$g_3(x - a_3(z)) = 0.$$

Because the lower row is exact and a_2 is an epimorphism and the diagram commutes it follows that

$$x - a_3(z) \in \ker g_3 = \operatorname{im} g_2 = \operatorname{im}(g_2 a_2) = \operatorname{im}(a_3 f_2) \subseteq \operatorname{im} a_3.$$

Because $a_3(z) \in \operatorname{im} a_3$ this gives us $x \in \operatorname{im} a_3$. So a_3 is an epimorphism.

(iii)

It is enough to assume that a_1 is an epimorphism, a_5 is a monomorphism and a_2 and a_4 are isomorphisms. Combining the two previous statements it then directly follows that a_3 is a mono- and an endomorphism, and therefore an isomorphism.

Exercise 40:

We define the K-vector space

$$K^{\mathbb{N}} := \{(\lambda_i)_{i \in \mathbb{N}} : \lambda_i \in K \text{ for all } i \in \mathbb{N}\}$$

and for all $n \in \mathbb{N}$ the subspaces

$$K_n^{\mathbb{N}} := \{(\lambda_i)_{i \in \mathbb{N}} : \lambda_0 = \ldots = \lambda_{n-1} = 0\}.$$

and

$$K_n := \{(\lambda_i)_{i \in \mathbb{N}} : \lambda_i = 0 \text{ for all } i \geq n\}$$

It is clear that for all $n \in \mathbb{N}$ we have $K^{\mathbb{N}} = K_n \oplus K_n^{\mathbb{N}}$ and $K_n \cong K^n$.
We now look at the short exact sequence sequence

$$0 \longrightarrow K_1 \xrightarrow{f} K^{\mathbb{N}} \xrightarrow{g} K_2^{\mathbb{N}} \longrightarrow 0$$

where f is the canonical inclusion and

$$g : K^{\mathbb{N}} \to K_2^{\mathbb{N}}, (\lambda_0, \lambda_1, \lambda_2, \ldots) \mapsto (0, 0, \lambda_1, \lambda_2, \ldots).$$

Now we just use the direct sum decomposition $K^{\mathbb{N}} = K_2 \oplus K_2^{\mathbb{N}}$, where it is clear that $K_1 \ncong K_2$.