Foundations of representation theory

6. Exercise sheet

Jendrik Stelzner

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Exercise 21:

We will assume that V is an artinian module.
V is uniform, because $S \neq 0$ is contained in every non-zero submodule of V. Thus V is indecomposable.
For all $f \in \mathrm{End}(V)$ we have $\mathrm{img}\, f_{|S} \subseteq S$: If $f_{|S} = 0$ this is trivial. Otherwise $\mathrm{img}\, f_{|S} \subseteq V$ is a non-zero submodule, so $S \subseteq \mathrm{img}\, f_{|S}$. Because S is non-zero, $f_{|S}^{-1}(S) \subseteq S$ is a non-zero submodule. Because S is simple we get $S = f_{|S}^{-1}(S)$ and therefore $\mathrm{img}\, f_{|S} = S$.
This allows us to define $\varphi : \mathrm{End}(V) \to \mathrm{End}(S), f \mapsto f_{|S}$. It is obvious that φ is a ring homomorphism. By assumption φ is surjective. We now show that

$$\ker \varphi = \{f \in \mathrm{End}(V) : f \text{ is not invertible}\}.$$

It is clear that

$$\ker \varphi \subseteq \{f \in \mathrm{End}(V) : f \text{ is not invertible}\}.$$

Let $f \in \mathrm{End}(V)$ be not injective. Because $\ker f \neq 0$ is a submodule of V we have $S \subseteq \ker f$, so $f_{|S} = 0$. Let $g \in \mathrm{End}(V)$ be not surjective but assume g is injective. We get a descending chain

$$V \supseteq \mathrm{img}\, g \supseteq \mathrm{img}\, g^2 \supseteq \mathrm{img}\, g^3 \supseteq \dots$$

of submodules of V. By assumption this chain eventually stabilizes, i.e. there exists some $N \in \mathbb{N}$ with $\mathrm{img}\, g^n = \mathrm{img}\, g^{n+1}$ for all $n \geq N$. Because g is injective we also have $\ker g^n = \ker g^{n+1}$ for all $n \in \mathbb{N}$. This implies that

$$V = \ker g^N \oplus \mathrm{img}\, g^N.$$

Because V is indecomposable this implies that $\ker g^N = 0$ and $\mathrm{img}\, g^N = V$ or $\ker g^N = V$ and $\mathrm{img}\, g^N = 0$. So g is either not injective or surjective, which is contradicts either the injectivity or non-surjectivity of g. So g has to be non-injective and thus contained in $\ker \varphi$.
Because

$$\ker \varphi = \{f \in \mathrm{End}(V) : f \text{ is not invertible}\}$$

is an ideal in $\mathrm{End}(V)$, we get that $\mathrm{End}(V)$ is local. so

$$J(\mathrm{End}(V)) = \{f \in \mathrm{End}(V) : f \text{ is not invertible}\} = \ker \varphi$$

and therefore

$$\mathrm{End}(V)/J(\mathrm{End}(V)) = \mathrm{End}(V)/\ker \varphi \cong \mathrm{img}\, \varphi = \mathrm{End}(S).$$

Exercise 22:

(i)

Remark 1. Every endomorphism h of the module $V = (K[T], T\cdot)$ is of the form

$$h : f \mapsto p \cdot f$$

for some polynomial $p \in K[T]$, and each such map is an endomorphism of V.

Proof. It is obvious that every such map is an module endomorphism of V. To show that every endomorphism is of this form, we denote $T\cdot$ as $\phi : f \mapsto T \cdot f$ and let $p := h(1)$. Because h is a module homomorphism, it follows that for all $\sum_{i=0}^{n} a_i T^i \in K[T]$

$$\begin{aligned} h\left(\sum_{i=0}^{n} a_i T^i\right) &= h\left(\sum_{i=0}^{n} a_i \phi^i(1)\right) = \sum_{i=0}^{n} a_i \phi^i(h(1)) \\ &= \sum_{i=0}^{n} a_i \phi^i(p) = \sum_{i=0}^{n} a_i T^i p = p \sum_{i=0}^{n} a_i T^i. \end{aligned}$$

□

We assume that the greatest common divisor of some polynomial $p \in K[T]$ and $0 \in K[T]$ is defined as p.
Let U be a direct summand of $V \oplus V$ with $U \notin \{0, V \oplus V\}$. We know that U is the kernel of some idempotent endomorphism of V. Let $h \in \mathrm{End}(V)$ be idempotent with $\ker h = U$. We know that h can be uniquely written as

$$h = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$$

with $h_1, h_2, h_3, h_4 \in \mathrm{End}(V)$. From remark 1 it follows that there exists polynomials $p_1, p_2, p_3, p_4 \in K[T]$ such that

$$h = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}.$$

We get that

$$\ker h = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in V \oplus V : \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = 0. \right\}$$

We notice that the matrix

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

also describes an idempotent vector space endomorphism h' of $K(T)^2$ with respect to the canonical basis, where $K(T)$ is the field of rational functions over K. It is clear that

$$\ker h' = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in K(T)^2 : \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = 0. \right\}$$

and

$$\ker h = \ker h' \cap (K[T] \oplus K[T]),$$

So to show that U is of the form $U_{f,g}$ for some polynomials f, g with $\gcd(f,g) = 1$, it is enough to show that $\ker h'$ has a basis $\left\{ {}^{\mathrm{t}}\begin{bmatrix} f & g \end{bmatrix} \right\}$ with such polynomials f and g.
From $U \notin \{0, V \oplus V\}$ we know that h is non-zero and no isomorphism, so $\ker h'$ is a one-dimensional subspace of $K(T)^2$. Let $b = {}^{\mathrm{t}}\begin{bmatrix} f & g \end{bmatrix}$ be a basis vector of $\ker h'$. We can assume that f and g are polynomials, because otherwise we can multiply b by some non-zero scalar $r \in K[T] \subseteq K(T)$ such that rf and rg are polynomials. We can also assume that the $\gcd(f,g) = 1$, because otherwise we can multiply b with some scalar $r' \in K(T)$ such that the greatest common divisor of $\gcd(r'f, r'g) = 1$. Because ${}^{\mathrm{t}}\begin{bmatrix} f & g \end{bmatrix}$ is a basis vector of $\ker h'$ it follows that

$$U = \ker h = \ker h' \cap (K[T] \oplus K[T]) = \{(hf, hg) : h \in K[T]\} = U_{f,g}.$$

We now show the modules of the form $U_{f,g}$ with $\gcd(f,g) = 1$ are direct summands: It is clear that $U_{f,g}$ is a submodule of $V \oplus V$ for alle $f, g \in K[T]$. Assume that $\gcd(f,g) = 1$. If $f = 0$, then it follows that $g = 1$, so

$$U_{f,g} = 0 \oplus V = \ker \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and if $g = 0$ it follows that $f = 1$, so

$$U_{f,g} = V \oplus 0 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

In both cases the endomorphisms, expressed as matrices, are obvioulsy idempotent. If $f, g \neq 0$ we know from linear algebra that we can find polynomials $r, s \in K[T]$ with $rf + sg = 1$. Then the endomorphism of $V \oplus V$ of the form

$$h = \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix}$$

with

$$\begin{aligned} \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix}^2 &= \begin{bmatrix} rsfg + s^2g^2 & -rsf^2 - s^2fg \\ -r^2fg - rsg^2 & r^2f^2 + rsfg \end{bmatrix} \\ &= \begin{bmatrix} sg(rf+sg) & -sf(rf+sg) \\ -rg(rf+sg) & rf(rf+sg) \end{bmatrix} = \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix} \end{aligned}$$

is idempotent, and thus

$$\ker h = U_{f,g}$$

is a direct summand of $V \oplus V$.

(ii)

We know that $U_{T,T-1}$ is a direct summand of $V \oplus V$, because $\gcd(T, T-1) = 1$. Because $U_{T,T-1} \neq 0$ and $U_{T,T-1} \neq V \oplus V$ we know that neither 0 nor $V \oplus V$ is a direct complement of $U_{T,T-1}$ in $V \oplus V$. $U_{1,0}$ and $U_{0,1}$ are also no direct complements of $U_{T,T-1}$ in $V \oplus V$, because $(0,1) \notin U_{T,T-1} + U_{1,0}$ and $(1,0) \notin U_{T,T-1} + U_{0,1}$.

Exercise 23:

soc$(M(w))$

Let $b_1, \ldots, b_{n+1}$ be the standard basis of $M(w)$. For all $v \in M(w)$ let

$$h_x(v) := \min\{k \in \mathbb{N} : \phi_x^k(v) = 0\} \text{ and } h_y(v) := \min\{k \in \mathbb{N} : \phi_y^k(v) = 0\}.$$

Notice that

$$h_x(v) = 0 \Leftrightarrow v = 0 \Leftrightarrow h_y(v) = 0$$

and

$$h_x(v) = 1 \Leftrightarrow h_y(v) = 1$$

for all $v \in M(w)$. Let $c_1, \ldots, c_m \in \{b_1, \ldots, b_{n+1}\}$, $1 \leq m \leq n+1$, be the pairwise different b_i with $h_x(b_i) = h_y(b_i) = 1$.

For every $v \in M(w)$ with $h_x(v) = h_y(v) = 1$ the subspace $\langle v \rangle = Kv$ is a simple submodule of $M(w)$, because $\langle x \rangle$ is one-dimensional and $\phi_x(v) = \phi_y(v) = 0$.

Every non-zero submodule $U \subseteq M(w)$ contains such a simple module: Let $v \in U$ with $v \neq 0$. If $h_x(v) = h_y(v) = 1$, $\langle v \rangle$ is such a submodule. Otherwise $h_x(v) > 1$ or $h_x(v) > 1$. In the first case $\left\langle \phi_x^{h_x(v)-1}(v) \right\rangle$ is such a submodule, in the second case $\left\langle \phi_y^{h_y(v)-1}(v) \right\rangle$ is such a submodule.

It follows that every simple submodule of $M(w)$ is of the form $\langle v \rangle$ for some $v \in M(w)$ with $h_x(v) = h_y(v) = 1$. It is easy so see that each such v is a linear combination of the c_i: For $v = \sum_{i=1}^{n+1} \lambda_i b_i$ we get

$$0 = \phi_x(v) = \sum_{i=1}^{n+1} \lambda_i \phi_x(b_i) \text{ and } 0 = \phi_y(v) = \sum_{i=1}^{n+1} \lambda_i \phi_y(b_i),$$

so for all i the implication $\lambda_i \neq 0 \Rightarrow \phi_x(b_i) = \phi_y(b_i) = 1$ holds. So we get

$$\operatorname{soc}(M(w)) = \sum_{\substack{S \subseteq M(w) \\ S \text{ simple}}} S = \bigoplus_{i=1}^{m} \langle c_i \rangle .$$