Foundations of representation theory
3. exercise sheet

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Exercise 9:

$\mathrm{End}(V)$

$\mathrm{End}(V)$ (as a set of 3×3-matrices) consists of all $A = (a_{ij}) \in M(3 \times 3, K)$ with

$$\begin{pmatrix} 0 & a_{11} & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{pmatrix} = A \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is the case if and only if $a_{11} = a_{22}$ and $a_{21} = a_{23} = a_{31} = 0$. So

$$\mathrm{End}(V) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} : a, b, c, d, e \in K \right\}.$$

idempotent endomorphisms

An endomorphism $f = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} \in \mathrm{End}(V)$ is idempotent if and only f

$$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix}^2 = \begin{pmatrix} a^2 & 2ab + cd & ac + ce \\ 0 & a^2 & 0 \\ 0 & ad + de & e^2 \end{pmatrix},$$

which is equivalent to the following conditions holding all at once: (i) $a^2 = a$, (ii) $e^2 = e$, (iii) $b = 2ab + cd$, (iv) $c = ac+ce = c(a+e)$ and (v) $d = ad+de = d(a+e)$. From (i) it follows that $a = 0$ or $a = 1$ and from (ii) it follows that $e = 0$ or $e = 1$. From (iv) it follows that $e = 1 - a$ if $c \neq 0$ and from (v) it follows that $e = 1 - a$ if $d \neq 0$. From (iii) it follows that if $c = 0$ or $d = 0$ then $b = 0$ (because $a = 0$ or $a = 1$). The idempotent endomorphisms of V can now be easily found by case differentiation.

$c = d = 0$

If $c = d = 0$ then $b = 0$. This gives us the idempotent endomorphisms

$$f_1 = 0, f_2 = 1, f_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \text{ and } f_4 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

$c \neq 0$ and $d = 0$

If $c \neq 0$ and $d = 0$ then $b = 0$ and $e = 1 - a$. This gives us the idempotent endomorphisms

$$f_5^c = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & \end{pmatrix} \text{ and } f_6^c = \begin{pmatrix} 0 & & c \\ & 0 & \\ & & 1 \end{pmatrix}.$$

$c = 0$ and $d \neq 0$

If $c = 0$ and $d \neq 0$ then $b = 0$ and $e = 1 - a$. Thus we get the idempotent endomorphisms

$$f_7^d = \begin{pmatrix} 1 & & \\ & 1 & \\ & d & 0 \end{pmatrix} \text{ and } f_8^d = \begin{pmatrix} 0 & & \\ & 0 & \\ & d & 1 \end{pmatrix}$$

$c \neq 0$ and $d \neq 0$

If $c \neq 0$ and $d \neq 0$ then $e = 1 - a$. If $a = 0$ then $b = cd$, if $a = 1$ then $b = 2b + cd$, so $b = -cd$. Thus we get the idempotent endomorphisms

$$f_9^{c,d} = \begin{pmatrix} 0 & cd & c \\ & 0 & \\ & d & 1 \end{pmatrix} \text{ and } f_{10}^{c,d} = \begin{pmatrix} 1 & -cd & c \\ & 1 & \\ & d & 0 \end{pmatrix}$$

direct sum decompositions

We know that submodules $V_1, V_2 \subseteq V$ are a direct sum decomposition $V = V_1 \oplus V_2$ if and only if there exists an idempotent endomorphism $f \in \mathrm{End}(V)$ with $V_1 = \ker f$ and $V_2 = \operatorname{im} f$. So the direct sum decompositions of V are

$$\begin{aligned}
V &= \operatorname{im}(f_1) \oplus \ker(f_1) = \ker(f_2) \oplus \operatorname{im}(f_2) = 0 \oplus V && (1)\\
&= \operatorname{im}(f_3) \oplus \ker(f_3) = \ker(f_4) \oplus \operatorname{im}(f_4) = \langle e_3 \rangle \oplus \langle e_1, e_2 \rangle && (2)\\
&= \operatorname{im}(f_5^c) \oplus \ker(f_5^c) = \ker(f_6^{-c}) \oplus \operatorname{im}(f_6^{-c}) = \langle e_1, e_2 \rangle \oplus \left\langle (-c, 0, 1)^T \right\rangle && (3)\\
&= \operatorname{im}(f_7^d) \oplus \ker(f_7^d) = \ker(f_8^{-d}) \oplus \operatorname{im}(f_8^{-d}) = \left\langle e_1, (0, 1, d)^T \right\rangle \oplus \langle e_3 \rangle && (4)\\
&= \operatorname{im}(f_9^{c,d}) \oplus \ker(f_9^{c,d}) = \ker(f_{10}^{-c,-d}) \oplus \operatorname{im}(f_{10}^{-c,-d})\\
&= \left\langle (c, 0, 1)^T \right\rangle \oplus \left\langle e_1, (0, 1, -d)^T \right\rangle && (5)
\end{aligned}$$

with $c, d \in K^*$ arbitrary and $e_1, \ldots, e_n$ denoting the canonical basis of K^n. ($V_1 \oplus V_2$ and $V_2 \oplus V_1$ are seen as the same decomposition.)

the map $\Gamma : f \mapsto (\operatorname{im} e, \ker e)$

Since I don't understand what is meant by describing the map Γ, I will just write down some properties and observations:
Γ is injective: If f and g are idempotent endomorphisms then

$$\Gamma(f) = \Gamma(g) \Rightarrow \operatorname{im}(f) = \operatorname{im}(g) \text{ and } \ker(f) = \ker(g)$$

so $f(x) = 0 = g(x)$ for all $x \in \ker(f) = \ker(g)$ and $f(x) = x = g(x)$ for all $x \in \operatorname{im}(f) = \operatorname{im}(g)$ (because f and g are idempotent). Since $V = \operatorname{im}(f) \oplus \ker(f) = \operatorname{im}(g) \oplus \ker(g)$ it follows that $f(x) = g(x)$ for all $x \in V$, so $f = g$.
It is easy to see that for any idempotent endomorphism $f \in \operatorname{End}(V)$:

$$\Gamma(f) = (\operatorname{im} f, \ker f) = (\ker(1-f), \operatorname{im}(1-f)) \in \operatorname{im}\Gamma, .$$

The following observation is an interesting one: All direct sum decompositiots of V except for $V = 0 \oplus V = V \oplus U$ are formed using only four kinds of different submodules: $\langle e_3 \rangle$ and $\langle c, 0, 1)^T \rangle$ (both 1-dimensional) and $\langle e_1, (0,1,d)^T \rangle$ and $\langle e_1, e_2 \rangle$ (both 2-dimensonial). For an idempotent endomorphism $f \in \operatorname{Hom}(V)$ there is a connection between the matrix of f and the corresponding direct sum decomposition $V = U_1 \oplus U_2$ with U_1 is 1-dimensional and U_2 is 2-dimensional: If f is given by the matrix $\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix}$ then $U_1 = \langle e_3 \rangle$ if $c = 0$ and $U_1 = \langle (c,0,1)^T \rangle$ if $c \neq 0$, as well as $U_2 = \langle e_1, e_2 \rangle$ if $d = 0$ and $U_2 = \langle e_1, (0,1,d)^T \rangle$ if $d \neq 0$. So the 1-dimensional component is solely determined by c, and the 2-dimensional component is solely determined by d, and both in a very simple way.

Exercise 11:

Let $e_1, \dots, e_n$ be the canonical basis of K^n and $\phi, \psi \in \operatorname{Hom}_K(K^n)$ be defined as

$$\phi(e_i) := \begin{cases} 0 & \text{if } i = 1 \\ e_{i-1} & \text{otherwise} \end{cases} \text{ and } \psi(e_i) := \begin{cases} 0 & \text{if } i = n \\ e_{i+1} & \text{otherwise} \end{cases}.$$

Let $U \subseteq K^n$ be a submodule. If $U \neq 0$ we find $v \in U$ with $v \neq 0$. Since $e_1, \dots, e_n$ is a basis of K^n we find $\lambda_1, \dots, \lambda_n \in K$ with $v = \sum_{i=1}^n \lambda_i e_i$. Let $m := \max\{i \in \{1, \dots, n\} : \lambda_i \neq 0\}$; this is well-defined because $v \neq 0$, so $\lambda_i \neq 0$ for some $i \in \{1, \dots, n\}$. Because U is a submodule we find that $e_1 = \lambda_m^{-1} \phi^{m-1}(v) \in U$. So for all $i \in \{1, \dots, n\}$ we get $e_i = \psi^i(e_1) \in U$. Since $\{e_1, \dots, e_n\} \subseteq U$ it follows that $U = K^n$. So every submodule (K^n, ϕ, ψ) is either 0 or K^n, which means that (K^n, ϕ, ψ) is an n-dimensional simple 2-module.

Exercise 12:

Let K be a field with $\operatorname{char}(K) = 0$ and (V, ϕ_1, ϕ_2) is a 2-module such that $V \neq 0$ and $[\phi_1, \phi_2] = 1$. Assume that V is finite-dimensional. Because $V \neq 0$ we know that $n := \dim V \geq 1$. Let $v_1, \dots, v_n$ be a Basis von V and Φ_1 and Φ_2 the coordinate matrix of ϕ_1, ϕ_2 with respect to the basis $v_1, \dots, v_n$ respectively. Because $[\phi_1, \phi_2] = 1$ we find that $\Phi_1 \Phi_2 - \Phi_2 \Phi_1 = I_n$. It follows, that

$$0 = \operatorname{tr}\Phi_1 \operatorname{tr}\Phi_2 - \operatorname{tr}\Phi_2 \operatorname{tr}\Phi_1 = \operatorname{tr}(\Phi_1 \Phi_2 - \Phi_2 \Phi_1) = \operatorname{tr} I_n = n \cdot 1 \neq 0.$$

This shows that (V, ϕ_1, ϕ_2) must be infinite dimensional. (We know that such a module exists, because $\left(K[T], T\cdot, \frac{d}{dT}\right)$ is one.)