Foundations of representation theory

5. Exercise sheet

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Exercise 17:

Exercise 18:

Exercise 19:

We can look at the 1-module $V := N(\infty) \times (K, \mathrm{id}_K)$, the endomorphism being $\psi = \phi \times \mathrm{id}_K$.
It is obvious that $W := N(\infty) \times 0$ is a proper submodule of V. W is also maximal in V because $V/W \cong (K, \mathrm{id}_K)$ is simple.
W is the only maximal submodule of V, because every maximal submodule $W' \subseteq V$ has to contain W: Assume $W' \subsetneq V$ is maximal with $W \subsetneq W'$. Then there is some $v = (\sum_{i=1}^n \mu_i e_i, 0) \in W$, $\mu_n \neq 0$, with $v \notin W'$. In particular $(e_n, 0), (e_{n+1}, 0), \ldots \notin W'$, because otherwise $(e_n, 0), (e_{n-1}, 0) = \psi((e_n, 0)), \ldots, e_1 = \psi^{n-1}((e_n, 0)) \in W'$ and thus $v \in W'$. So $W' \subsetneq W' + U((e_n, 0)) \subsetneq V$, which contradicts the maximality of W.
Even though W is the only maximal submodule in V we find that V is not uniform, because the proper submodule $0 \times K$ is not contained in W.

Exercise 20:

Remark 1. Let V be a module of finite length. Then every proper submodule $U \subsetneq V$ is contained in some maximal submodule $W \subsetneq V$.

Proof. Because V is of finite length the same holds for U. Set $U_0 := U$. If U_i is defined and not maximal in V, define U_{i+1} as some submodule of V which contains U_i as a proper submodule. Because $U_i \subsetneq U_{i+1}$ we get $l(U_i) < l(U_{i+1})$. Because V is of finite length the chain

$$U = U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \ldots$$

only contains finitely many submodules. Thus the maximal submodule U_n of this chain has to be a maximal submodule of V containing U. $\square$

Remark 2. A module V is simple if and only if $l(V) = 1$. A semisimple module is local if and only if it is simple. In particular every simple module is local.

Proof. V is simple if and only if $V/0$ is simple, which is equivalent to the filtration $0 = U_0 \subseteq U_1 = V$ being a composition series of V, which is then again equivalent to $l(V) = 1$.
Let V be a semisimple module. Assume V is simple. Because 0 is a maximal submodule of V which contains every proper submodule of V, V is local. Assume that V is not simple and local. Then we can write $V = \bigoplus_{i \in I} V_i$ with V_i being a simple submodule of V, as well as a maximal submodule W of V which contains every proper submodule of V. Because V is not simple the V_i are proper submodules of V, so $V_i \in W$ for all $i \in I$. But this implies that $V = \bigoplus_{i \in I} V_i \subseteq W$, so $V = W$, which contradicts the maximality of W. $\square$

(i)

Let V be a module of finite length. The first statement can be shown by induction over $n := l(V)$.

Base case. Let $n = 0$. Then $V = 0$. Thus V has no local submodules, and $V = 0$ is the corresponding empty sum.

Induction step. Let $n \geq 1$. 0 is a proper submodule of V because V is nonzero. It follows from remark 1 that V contains some maximal submodule W.
If W is the only maximal submodule of V then it follows from remark 1 that every proper submodule of V is contained and W. So V is local.
Otherwise let $(W_i)_{i \in I}$ be the maximal submodules of V. Because the W_i are proper submodules of V we get $l(W_i) < l(V)$. By using the induction hypothesis we can write every W_i as the sum of local submodules $L_j^i \subseteq W_i$. Because there are at least two different W_i we find that $V = \sum_{i \in I} W_i$. It is obvious that the L_j^i are also local submodules of V. Thus V is the sum of the L_j^i.

(ii)

Let V be a module of length $n < \infty$ and assume that V is semisimple. Assume that we get write V as $V = \sum_{i=1}^{n-1} L_i$ where $L_i \subseteq V$ is local. Because

$$n = l(V) = l\left(\sum_{i=1}^{n-1} L_i\right) \leq \sum_{i=1}^{n-1} l(L_i)$$

at least one of the L_i has to be of length greater than 1. W.l.o.g. we can assume that $l(L_1) > 2$. Because V is semisimple it follows that L_1 is semisimple. From remark 2 it follows that L_1 is simple, which contradicts $l(L_1) > 2$ according to remark 2. So V cannot be written as a sum of $n - 1$ local submoduls.
The other implication, namely that if V is not semisimple it can be written as a sum of $n - 1$ local submoduls, is shown by induction; this starts by $n = 2$, because if $n = 1$ then V is simple by remark 2.

Base case. Let $n = 2$. Let $0 = U_0 \subseteq U_1 \subseteq U_2 = V$ be a composition series of V. U_1 is maximal in V because V/U_1 is simple. U_1 is the nonzero proper submodule of V: If W is a different nonzero proper submodule of V, then U_1 and W are both nonzero and of length 1 with $U_1 + W = V$ and $U_1 \cap W = 0$. By remark 2 it follows that $V = U_1 \oplus W$ is the direct sum of simple submodles, which contradicts the assumption that V is not semisimple. Because U_1 is the only nonzero proper submodule of V it follows that every proper submodule is contained in U_1. So V is local.

Induction step. Let $n \geq 3$. We use case differentiation:
If V is not indecomposible we can write $V = \bigoplus_{i=1}^{m} U_i$, $m \geq 2$, as the direct sum of finitely many indecomposible submodules U_i of V, because V is of finite length. Because V is not simple at least one of the U_i has to be not simple. W.l.o.g. we can assume that $U_1, \dots, U_k$ are simple and $U_{k+1}, \dots, U_m$ are not simple. Because $U_{k+1}, \dots, U_m$ are indecomposible and not simple they are also not semisimple. Because the U_i are proper submodules of V we find that $l(U_i) < l(V)$. So by using the induction hypothesis we can write U_i, $k+1 \leq i \leq m$, as the sum $U_i = \sum_{j=1}^{l(U_i)-1} L_j^i$ of $l(U_i) - 1$ local submodules. Thus we can write

$$V = \bigoplus_{i=1}^{m} U_i = \bigoplus_{i=1}^{k} U_i + \bigoplus_{i=k+1}^{m} \sum_{j=1}^{l(U_i)-1} L_j^i$$

as the sum of at most $n-1$ local submodules of V. To write V as the sum of exactly $n-1$ local submoduls we can just use one of the summands more often (note that it is not stated that the local submodules have to be pairwise different on the exercise sheet).
If V is indecomposible then by remark 1 V does contain at least one maximal submodule. If V does contain exactly one maximal submodule V is local itself and can be written as $V = \sum_{i=1}^{n-1} V$.