Foundations of representation theory

8. Exercise sheet

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Exercise 29:

We will assume that the vertices of Q are ordered in the most obvious way. We define the subalgebra B of $M_n(K)$ as

$$B := \{M = (m_{ij})_{ij} \in M_n(K) : m_{ij} = 0 \text{ for all } j > i\}.$$

We will show that $KQ \cong B \cong A$.
For all $1 \leq i \leq j \leq n$ let p_{ij} be the unique path in Q from i to j and for all $1 \leq i, j \leq n$ let $E_{ij} \in M_n(K)$ be the matrix with 1 as the (i,j)-entry and 0 otherwise. (E_{ij} maps e_j to e_i.) We know that $(p_{ij})_{1\leq i\leq j\leq n}$ is a basis of KQ, $(E_{ji})_{1\leq i\leq j\leq n}$ is a basis of B and $(E_{ij})_{1\leq i\leq j\leq n}$ is a basis of A.
Let $\phi : KQ \to B$ be the linear map given by $\phi(p_{ij}) = E_{ji}$ for all $1 \leq i \leq j \leq n$. ϕ is a K-algebra homomorphism since for all $1 \leq i \leq j \leq n$ and $1 \leq k \leq l \leq n$

$$\phi(p_{ij}p_{kl}) = \phi(\delta_{il}p_{kj}) = \delta_{il}\phi(p_{kj}) = \delta_{il}E_{jk} = E_{ji}E_{lk} = \phi(p_{ij})\phi(p_{kl}).$$

ϕ is an isomorphism, because for the linear map $\psi : B \to KQ$ given by $\psi(E_{ji}) = p_{ij}$ for all $1 \leq i \leq j \leq n$ we have $\phi\psi = \mathrm{id}_B$ and $\psi\phi = \mathrm{id}_{KQ}$. Thus we have $KQ \cong B$.
To show that $B \cong A$ we notice that for the matrix

$$S := \begin{pmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{pmatrix} \in M_n(K)$$

with $S^2 = 1$ the map

$$\varphi : M_n(K) \to M_n(K), F \mapsto SFS$$

is an vector space automorphism with $\varphi^2 = 1$. φ is an algebra isomorphism because for all $F, G \in M_n(K)$

$$\varphi(FG) = SFGS = SFS^2GS = \varphi(F)\varphi(G).$$

We also notice that φ maps the Basis $(E_{ij})_{1\leq i\leq j\leq n}$ of A to the basis $(E_{ji})_{1\leq i\leq j\leq n}$ of B, thus $\varphi_{|A} \to \varphi_{|B}$ is an algebra isomorphism.

Exercise 30:

We name the vertices of Q as 1 and 2 with $s(\alpha) = t(\alpha) = 1$ and the arrow from 1 to 2 as p. By definition

$$P := \{e_1, e_2, p\} \cup \bigcup_{n\geq 1} \{\alpha^n, p\alpha^n\}$$

is a basis of KQ. It is clear that

$$B = \begin{pmatrix} K[T] & 0 \\ K[T] & K \end{pmatrix}$$

is a K-algebra via the usual matrix multiplication. We define the linear map $\phi : KQ \to B$ by

$$\phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \phi(p) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and}$$
$$\phi(\alpha^n) = \begin{pmatrix} T^n & 0 \\ 0 & 0 \end{pmatrix}, \phi(p\alpha^n) = \begin{pmatrix} 0 & 0 \\ T^n & 0 \end{pmatrix} \text{ for all } n \geq 1.$$

It is clear that ϕ induces a bijection between P and a basis of B, so ϕ is a vector space isomorphism. It is also easy to see that ϕ is an algebra homomorphism, because $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in P$ (this can be directly shown by some boring matrix multiplication which I will not include here). Thus $KQ \cong B$.
The ideal $I = (\alpha^2)$ in KQ generated by the path α^2 corresponds to the ideal $J = (\phi(\alpha)^2)$ in B generated by $\phi(\alpha)^2$. Because

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \underbrace{\begin{pmatrix} T^2 & 0 \\ 0 & 0 \end{pmatrix}}_{=\phi(\alpha)^2} \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} adT^2 & 0 \\ bdT^2 & 0 \end{pmatrix}$$

we find that

$$J = (\phi(\alpha)^2) = \begin{pmatrix} (T^2) & 0 \\ (T^2) & 0 \end{pmatrix}.$$

Thus ϕ induces an algebra isomorphism $\bar{\phi}$ between the algebras KQ/I and $B/J = A$.

Exercise 31:

¬(ii) ⇒ ¬(i)

Let $Q_V^1, \dots, Q_V^n$ be the (weakly) connected components of Q_V. By assumption $n \geq 2$. For $j = 1, \dots, n$ we define the non-zero representation $V^j = (V_i^j, V_a^j)_{i\in Q_0, a\in Q_1}$ of Q as

$$V_i^j = \begin{cases} V_i & \text{if } i \in (Q_V^j)_0, \\ 0 & \text{if } i \notin (Q_V^j)_0, \end{cases} \text{ and } V_a^j = \begin{cases} V_a & \text{if } a \in (Q_V^j)_1, \\ 0 & \text{if } a \notin (Q_V^j)_1. \end{cases}$$

From the definition of Q_V and because V is thin it directly follows that $V = \bigoplus_{j=1}^n V^j$. Thus V is decomposable.

$\neg$(i) $\Rightarrow$ $\neg$(ii)

Let $V = V^1 \oplus V^2$ with $V^1, V^2 \neq 0$. Because V is thin it follows that for all $i, j \in Q_0$ with $V_i^1 \neq 0$, $V_i^2 = 0$ and $V_j^2 \neq 0$, $V_j^1 = 0$ the corresponding vertices i and j in Q_V have no arrows between them: We find that for any $a \in Q_1$ from i to j or from j to i we have $V_a^1 = 0$ and $V_a^2 = 0$, thus $V_a = V_a^1 \oplus V_a^2 = 0$. So Q_V has at least two connected components, one containing i and one containing j.

(ii) $\Rightarrow$ (iii)

Let $f \in \mathrm{End}_Q(V)$. It is clear that $f_i = 0$ for all $i \in Q_0 \setminus (Q_V)_0$. For all $i \in (Q_V)_0$ we have $\dim(V_i) = 1$ because V is thin, and thus $f_i = \lambda_i 1_{V_i}$. Now let $i \in (Q_V)_0$ be fixed. Because Q_V is connected we find $j \in (Q_V)_0$ s.t. an arrow a from i to j or from j to i exists in $(Q_V)_1$. Thus we get one of the following commutative diagramms:

$$\begin{tikzcd} V_i \arrow[r, "\lambda_i"] \arrow[d, "V_a"] & V_i \arrow[d, "V_a"] \\ V_j \arrow[r, "\lambda_j"] & V_j \end{tikzcd} \qquad \begin{tikzcd} V_i \arrow[r, "\lambda_i"] & V_i \\ V_j \arrow[r, "\lambda_j"] \arrow[u, "V_a"] & V_j \arrow[u, "V_a"'] \end{tikzcd}$$

Because V_i and V_j are one-dimensional and $V_a \neq 0$ we find that in both cases $\lambda_i = \lambda_j$. Because Q_V is connected we find inductively that $\lambda_i = \lambda_j$ for all $j \in (Q_V)_0$. Thus we get $f_j = \lambda_i \,\mathrm{id}_{V_j}$ for all $j \in (Q_V)_0$ and therefore $f = \lambda_i 1_V$. It follows that $\mathrm{End}_K(Q) \cong K$.

(iii) $\Rightarrow$ (i)

From $\mathrm{End}_Q(V) \cong K$ it directly follows that V is indecomposable.