Foundations of representation theory

4. Exercise sheet

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November 7, 2013

Exercise 13:

Exercise 14:

Exercise 15:

$\neg(ii) \Rightarrow \neg(i)$

Assume that $V = U \oplus C$ is a direct sum decomposition with U simple.

Claim. C is maximal in V.

With this we find that

$$U \subseteq \mathrm{soc}(V) \text{ and } \mathrm{rad}(V) \subseteq C$$

because U is simple and C is maximal in V. Because U nonzero with $U \cap C = 0$, this implies that $\mathrm{soc}(V) \nsubseteq \mathrm{rad}(V)$.

Proof of the claim. Let $C' \subseteq V$ be a submodule with $C \subseteq C' \subseteq V$. Let $C'' := C' \cap U$. Because V is simple we know that $C'' = 0$ or $C'' = U$. If $C'' = 0$ then

$$C = C + C'' = C + (U \cap C') = (C + U) \cap C' = V \cap C' = C'.$$

If $C'' = U$ we get

$$U = C'' = C' \cap U, \text{ so } V = U \cap C \subseteq C', \text{ so } C' = V. \qquad \square$$

$(ii) \Rightarrow (i)$

Assume that V does not have a simple direct summand. If V has no maximal submodule, then $\mathrm{soc}(V) \subseteq \mathrm{rad}(V) = V$ is trivial. If V does have at least one maximal submodule it is easy to see that

$$\mathrm{soc}(V) \subseteq \mathrm{rad}(V) \Leftrightarrow S \subseteq U \text{ for all } S \subseteq V \text{ simple and all } U \subseteq V \text{ maximal.}$$

Assume that $S \subseteq V$ is simple and $U \subseteq W$ is maximal with $S \nsubseteq U$. Then $S \cap U \neq S$ and $S \subsetneq S+U$. Because S is simple this implies $S \cap U = 0$, and because U is maximal it implies $S + U = V$. So $V = S \oplus U$. This is a contradiction to the assumption that V does not have a simple direct summand. So $S \subseteq U$ for all for all $S \subseteq V$ simple and all $U \subseteq V$ maximal.

Exercise 16: