Foundations of representation theory

7. Exercise sheet

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Exercise 22:

Let $(e_1, \ldots, e_n)$ be a basis of A as a vector space.

(i) $\Rightarrow$ (ii)

Let $(b_1, \ldots, b_m)$ be a generating set of M as an A-module. We can write $x \in M$ as $x = \sum_{j=1}^m a_j b_j$ with $a_j \in A$ for all j. We can write each a_j as $a_j = \sum_{i=1}^n \lambda_i^j e_i$ with $\lambda_i^j \in K$ for all i. Thus we get

$$x = \sum_{j=1}^m a_j b_j = \sum_{i=1}^n \sum_{j=1}^m \lambda_i^j e_i b_j.$$

Thus x is a linear combination of the $e_i b_j$. Because x is arbitrary it follows that $\{e_i b_j\}_{i=1,\ldots,n, j=1,\ldots,m}$ is a finite generating set of M as a vector space, so M is finite-dimensional.

(ii) $\Rightarrow$ (i)

If $(b_1, \ldots, b_m)$ is a basis of M as a vector space, then $(b_1, \ldots, b_m)$ is also a generating set of M as an A-module, because $\lambda b_i = \lambda 1_A b_i$ for all $\lambda \in K$ and i.

(ii) $\Rightarrow$ (iii)

This follows directly from $l(M) \leq \dim(V) < \infty$.

$\neg$(ii) $\Rightarrow$ $\neg$(iii)

We construct an ascending chain $U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \ldots$ of finite-dimensional submodules of M as follows: We start with $U_0 := 0$. If U_{n-1} is defined we choose some $v \in M \setminus U_{n-1}$ (this is possible because U_{n-1} is finite-dimensional but M is infinite-dimensional). The submodule $W = Av = \langle e_1 v, \ldots, e_n v \rangle$ of M is not contained in U_{n-1}, so $U_{n-1} \subsetneq U_{n-1} + W =: U_n$. U_n is finite-dimensional because U_{n-1} and W are finite-dimensional.
For all $n \in \mathbb{N}$ the filtration

$$0 = U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \ldots \subsetneq U_{n-1} \subsetneq M$$

is of length n, so $l(M) \geq n$ for all $n \in \mathbb{N}$.