Foundations of representation theory

4. Exercise sheet

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Exercise 13:

Exercise 14:

Exercise 15:

$\neg(ii) \Rightarrow \neg(i)$

Assume that $V = U \oplus C$ is a direct sum decomposition with U simple.

Claim. C is maximal in V.

With this we find that

$$U \subseteq \mathrm{soc}(V) \text{ and } \mathrm{rad}(V) \subseteq C$$

because U is simple and C is maximal in V. Because U nonzero with $U \cap C = 0$, this implies that $\mathrm{soc}(V) \nsubseteq \mathrm{rad}(V)$.

Proof of the claim. Let $C' \subseteq V$ be a submodule with $C \subseteq C' \subseteq V$. Let $C'' := C' \cap U$. Because V is simple we know that $C'' = 0$ or $C'' = U$. If $C'' = 0$ then

$$C = C + C'' = C + (U \cap C') = (C + U) \cap C' = V \cap C' = C'.$$

If $C'' = U$ we get

$$U = C'' = C' \cap U, \text{ so } V = U \cap C \subseteq C', \text{ so } C' = V. \qquad \square$$

$(ii) \Rightarrow (i)$

Assume that V does not have a simple direct summand. If V has no maximal submodule, then $\mathrm{soc}(V) \subseteq \mathrm{rad}(V) = V$ is trivial. If V does have at least one maximal submodule it is easy to see that

$$\mathrm{soc}(V) \subseteq \mathrm{rad}(V) \Leftrightarrow S \subseteq U \text{ for all } S \subseteq V \text{ simple and all } U \subseteq V \text{ maximal.}$$

Assume that $S \subseteq V$ is simple and $U \subseteq W$ is maximal with $S \nsubseteq U$. Then $S \cap U \neq S$ and $S \subsetneq S+U$. Because S is simple this implies $S \cap U = 0$, and because U is maximal it implies $S + U = V$. So $V = S \oplus U$. This is a contradiction to the assumption that V does not have a simple direct summand. So $S \subseteq U$ for all for all $S \subseteq V$ simple and all $U \subseteq V$ maximal.

Exercise 16:

(i)

$0 \subseteq K[T]$ is the only small submodule in $(K[T], T\cdot)$: Let $0 \subsetneq U \subseteq K[T]$ be a small submodule. We know that $U = (a)$ for some $a \in K[T]$; because $K[T]$ has to be nonzero and proper we know that $\deg a \geq 1$. We find an irreducible polynomial $p \in K[T]$ with $p \nmid a$. So $(a) + (p) = (1) = K[T]$. Because (p) proper submodule of $(K[T], T\cdot)$ this shows that U is not small.
In $N(\infty)$ all nonzero submodules are small: For all nonzero submodules $V, V' \subseteq N(\infty)$ we have $N(1) \subseteq V, V'$ as a nonzero submodule, so $V \cap V' \supseteq N(1)$ is nonzero.

(ii)

Both modules are uniform:
Let $U, U' \subseteq K[T]$ be nonzero submodules. We know that $U = (a)$ and $U' = (b)$ for $a, b \in K[T] \setminus \{0\}$. Thus (ab) is a nonzero submodule of both U and U' (because $K[T]$ has no zero divisors), so $U \cap U' \supseteq (ab)$ is nonzero.
Let $V, V' \subseteq N(\infty)$ be nonzero submodules. We know that $V = N(i)$ and $V' = N(j)$ for $i, j \in \mathbb{N} \setminus \{0\}$. So $V \cap V' \supseteq N(1)$ is nonzero.

(iii)

Let $V = N(\infty)$ and $U := N(1) \subseteq V$. U is large in V: For every nonzero submodule $U' \subseteq V$ we have $N(1) \subseteq U'$, so $U \cap U' \supseteq N(1)$ is nonzero. U is also small: For every proper submodule $U'' \subseteq V$ we have $U'' = N(i)$ for some $i \in \mathbb{N}$. So $U + U'' = N(1) + N(i) = N(\max 1, i)$ is a proper submodule of V.