Foundations of representation theory

9. Exercise sheet

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Exercise 33:

Assume that ${}_AA \cong {}_AA \oplus {}_AA$ and let $\phi : {}_AA \to {}_AA \oplus {}_AA$ be an algebra homomorphism. We set

$$(b_0, b_1) := \phi(1)$$

and notice that for all $a \in A$

$$\phi(a) = \phi(a \cdot 1) = a\phi(1) = a(b_0, b_1) = (ab_0, ab_1).$$

Because ϕ is surjective we find $a_0, a_1 \in A$ with

$$\begin{aligned}(1, 0) &= \phi(a_0) = (a_0b_0, a_0b_1) \text{ and}\\ (0, 1) &= \phi(a_1) = (a_1b_0, a_1b_1).\end{aligned}$$

In particular we have $a_0b_0 = a_1b_1 = 1$ and $a_0b_1 = a_1b_0 = 0$. Because

$$\phi(b_0a_0 + b_1a_1) = (b_0a_0b_0 + b_1a_1b_0, b_0a_0b_1 + b_1a_1b_1) = (b_0, b_1) = \phi(1)$$

it follows from the injectivity of ϕ that $b_0a_0 + b_1a_1 = 1$.
Now assume that there exist elements $a_0, a_1, b_0, b_1 \in A$ with $a_0b_0 = a_1b_1 = 1$, $a_0b_1 = a_1b_0 = 0$ and $b_0a_0 + b_1a_1 = 1$. We define

$$\psi : {}_AA \to {}_AA \oplus {}_AA, a \mapsto a(b_0, b_1) = (ab_0, ab_1).$$

It is clear that ψ is an A-module homomorphism. For all $(c_0, c_1) \in {}_AA \oplus {}_AA$ we have

$$\psi(c_0a_0 + c_1a_1) = (c_0a_0b_0 + c_1a_1b_0, c_0a_0b_1 + c_1a_1b_1) = (c_0, c_1),$$

so ψ is surjective. For $x \in A$ with $\psi(x) = 0$ we have $(xb_0, xb_1) = (0, 0)$, so $(xb_0a_0, xb_1a_1) = (0, 0)$ and thus

$$0 = xb_0a_0 + xb_1a_1 = x(b_0a_0 + b_1a_1) = x \cdot 1 = x.$$

So ψ in injective. This shows that ψ is an A-module isomorphism and therefore ${}_AA \cong {}_AA \oplus {}_AA$.
One trivial example of such an algebra is $A = 0$.

Exercise 34:

We notice that $\operatorname{im}(h_2 f_1) \subseteq \operatorname{im} g_1$: Because the upper row is exact we have $f_2 f_1 = 0$ and from the commutativity of the diagram it follows that

$$0 = h_3 f_2 f_1 = g_2 h_2 f_1.$$

Because the lower row is exact this gives us

$$\operatorname{im}(h_2 f_1) \subseteq \ker g_2 = \operatorname{im} g_1.$$

Because g_1 is injective it induces an isomorphism $\bar{g}_1 : Y_1 \to \operatorname{im} g_1$. In particular $1_{Y_1} = \bar{g}_1^{-1} g_1$ and $1_{\operatorname{im} g_1} = g_1 \bar{g}_1^{-1}$. Therefore $h_1 := \bar{g}_1^{-1} h_2 f_1$ is an homomorphism with

$$g_1 h_1 = g_1 \bar{g}_1^{-1} h_2 f_1 = 1_{\operatorname{im} g_1} h_2 f_1 = h_2 f_1.$$

This homomorphism is unique because for $h_2' : X_1 \to Y_1$ with $g_1 h_1' = h_2 f_1$ we have

$$h_1' = 1_{Y_1} h_1' = \bar{g}_1^{-1} g_1 h_1' = \bar{g}_1^{-1} h_2 f_1 = h_1.$$