Foundations of representation theory

6. Exercise sheet

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Exercise 21:

We will assume that V is an artinian module. V is uniform, because $S \neq 0$ is contained in every non-zero submodule of V. This implies that V is indecomposable.
For all $f \in \text{End}(V)$ we have $\text{img}\, f_{|S} \subseteq S$: If $f_{|S} = 0$ this is trivial. Otherwise $\text{img}\, f_{|S} \subseteq V$ is a non-zero submodule, so $S \subseteq \text{img}\, f_{|S}$. Because S is non-zero, $f_{|S}^{-1}(S) \subseteq S$ is a non-zero submodule. Because S is simple we get $S = f_{|S}^{-1}(S)$ and thus $\text{img}\, f_{|S} = S$.
This allows us to define $\varphi : \text{End}(V) \to \text{End}(S), f \mapsto f_{|S}$. It is obvious that φ is a ring homomorphism. By assumption φ is surjective. We know show that

$$\ker \varphi = \{f \in \text{End}(V) : f \text{ is not invertible}\}.$$

It is clear that

$$\ker \varphi \subseteq \{f \in \text{End}(V) : f \text{ is not invertible}\}.$$

Let $f \in \text{End}(V)$ be not injective. Because $\ker f \neq 0$ is a submodule we have $S \subseteq \ker f$, so $f_{|S} = 0$. Let $g \in \text{End}(V)$ be injective but not surjective. We get a descending chain

$$V \supseteq \text{img}\, g \supseteq \text{img}\, g^2 \supseteq \text{img}\, g^3 \supseteq \ldots$$

of submodules of V. By assumption this chain eventually stabilizes, i.e. there exists some $N \in \mathbb{N}$ with $\text{img}\, g^n = \text{img}\, g^{n+1}$ for all $n \geq N$. Because g is injective we also have $\ker g^n = \ker g^{n+1}$ for all $n \in \mathbb{N}$. This implies that

$$V = \ker g^N \oplus \text{img}\, g^N.$$

Because V is indecomposable this implies that $\ker g^n = 0$ and $\text{img}\, g^N = V$ or $\ker g^N = V$ and $\text{img}\, g^N = 0$. So g is either not injective or surjective, which is contradicts either the injectivity or non-surjectivity of g. So g has to be non-injective and thus contained in $\ker \varphi$.
Because

$$\ker \varphi = \{f \in \text{End}(V) : f \text{ is not invertible}\}$$

is an ideal in $\text{End}(V)$, we get that $\text{End}(V)$ is local. so

$$J(\text{End}(V)) = \{f \in \text{End}(V) : f \text{ is not invertible}\} = \ker \varphi$$

and thus

$$\text{End}(V)/J(\text{End}(V)) = \text{End}(V)/\ker \varphi \cong \text{img}\, \varphi = \text{End}(S).$$

Exercise 22:

(i)

Remark 1. Every endomorphism h of the module $V = (K[T], T\cdot)$ is of the form

$$h : f \mapsto p \cdot f$$

for some polynomial $p \in K[T]$, and each such map is an endomorphism von V.

Proof. It is obvious that every such map is an module endomorphism of V. To show that every endomorphism is of this form we denote $T\cdot$ as $\phi : f \mapsto T \cdot f$ and $p := h(1)$. Because h is a module homomorphism, it follows that for all $\sum_{i=0}^{n} a_i T^i \in K[T]$

$$\begin{aligned} h\left(\sum_{i=0}^{n} a_i T^i\right) &= h\left(\sum_{i=0}^{n} a_i \phi^i(1)\right) = \sum_{i=0}^{n} a_i \phi^i(h(1)) \\ &= \sum_{i=0}^{n} a_i \phi^i(p) = \sum_{i=0}^{n} a_i T^i p = p \sum_{i=0}^{n} a_i T^i. \end{aligned}$$

□

We assume that the greatest common divisor of some polynomial $p \in K[T]$ and $0 \in K[T]$ is defined as p, because otherwise the statement does not hold true. In this case the direct summands $V \oplus 0$ and $0 \oplus V$ are counterexamples.
Let U be a direct summand of $V \oplus V$ with $U \notin \{0, V \oplus V\}$. We know that U is the kernel of some idempotent endomorphism of V. Let $h \in \mathrm{End}(V)$ be idempotent with $\ker h = U$. We know that h can be uniquely written as

$$h = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$$

with $h_1, h_2, h_3, h_4 \in \mathrm{End}(V)$. From remark 1 it follows that there exists polynomials $p_1, p_2, p_3, p_4 \in K[T]$ such that

$$h = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}.$$

We get that

$$\ker h = \left\{ \begin{bmatrix} p \\ q \end{bmatrix} \in V \oplus V : \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = 0. \right\}$$

We notice that the matrix

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

also describes an idempotent vector space endomorphism h' of $K(X)^2$ with respect to the canonical basis, where $K(T)$ is the field of rational functions over K. It is clear that

$$\ker h = \ker h' \cap K[T],$$

and that $\ker h'$ is a subspace of $K(T)^2$. So to show that U is of the form $U_{f,g}$ for some polynomials f, g with greatest common divisor 1, it is enough to show that $\ker h'$ has a basis ${}^{\mathrm{t}}\begin{bmatrix} f & g \end{bmatrix}$ with such polynomials f and g. Because $U \notin \{0, V \oplus V\}$ we know that

h is non-zero and no isomorphism, so $\ker h'$ is one-dimensional. Let $b = {}^{\mathrm{t}}\begin{bmatrix} f & g \end{bmatrix}$ be a basis vector of $\ker h'$. We can assume that f and g are polynomials, because otherwise we can multiply b by some non-zero scalar $r \in K[T] \subseteq K(T)$ such that rf and rg are polynomials. We can also assume that the greatest common divisor of f and g is 1, because otherwise we can multiply b with some scalar $r' \in K(T)$ such that the greatest common divisor of $r'f$ and $r'g$ is 1. Because ${}^{\mathrm{t}}\begin{bmatrix} f & g \end{bmatrix}$ is a basis vector of $\ker h$ it follows that

$$U = \ker h = \ker h' \cap K[T] = \{(hf, hg) : h \in K[T]\} = U_{f,g}.$$

Now we show the $U_{f,g}$ are direct summands: It is clear that $U_{f,g}$ is a submodule of $V \oplus V$ for alle $f, g \in K[T]$. Assume that the greatest common divisor of f and g is 1. If $f = 0$, then it follows that $g = 1$, so

$$U_{f,g} = 0 \oplus V = \ker \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and if $g = 0$ it follows that $f = 1$, so

$$U_{f,g} = V \oplus 0 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

In both cases the endomorphisms are obvioulsy idempotent. If $f, g \neq 0$ we know from linear algebra that we can find polynomials $r, s \in K[T]$ with $rf + sg = 1$. Then the endomorphism of $V \oplus V$ of the form

$$h = \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix}$$

with

$$\begin{aligned} \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix}^2 &= \begin{bmatrix} rsfg + s^2g^2 & -rsf^2 - s^2fg \\ -r^2fg - rsg^2 & r^2f^2 + rsfg \end{bmatrix} \\ &= \begin{bmatrix} sg(rf + sg) & -sf(rf + sg) \\ -rg(rf + sg) & rf(rf + sg) \end{bmatrix} = \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix} \end{aligned}$$

is idempotent, and thus

$$\ker h = U_{f,g}$$

is a direct summand of $V \oplus V$.

(ii)

We know that $U_{T,T-1}$ is a direct summand of $V \oplus V$, because the greatest common divisor of T and $T - 1$ is 1. Because $U_{T,T-1} \neq 0$ and $U_{T,T-1} \neq V \oplus V$ we know that neither 0 nor $V \oplus V$ is a direct complement of $U_{T,T-1}$ in $V \oplus V$. $U_{1,0}$ and $U_{0,1}$ are also no direct complements of $U_{T,T-1}$ in $V \oplus V$, because $(0, 1) \notin U_{T,T-1} + U_{1,0}$ and $(1, 0) \notin U_{T,T-1} + U_{0,1}$.