Foundations of representation theory

8. Exercise sheet

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Exercise 29:

We will assume that the vertices of Q are ordered in the most obvious way. We define the subalgebra B of $M_n(K)$ as

$$B := \{M = (m_{ij})_{ij} \in M_n(K) : m_{ij} = 0 \text{ for all } j > i\}.$$

We will show that $KQ \cong B \cong A$.
For all $1 \leq i \leq j \leq n$ let p_{ij} be the unique path in Q from i to j and for all $1 \leq i, j \leq n$ let $E_{ij} \in M_n(K)$ be the matrix with 1 as the (i, j)-entry and 0 otherwise. (E_{ij} maps e_j to e_i.) We know that $(p_{ij})_{1 \leq i \leq j \leq n}$ is a basis of KQ, $(E_{ij})_{1 \leq j \leq i \leq n}$ is a basis of B and $(E_{ij})_{1 \leq i \leq j \leq n}$ is a basis of A.
Let $\phi : KQ \to B$ be the linear map given by $\phi(p_{ij}) = E_{ji}$ for all $1 \leq i \leq j \leq n$. ϕ is a K-algebra homomorphism since for all $1 \leq i \leq j \leq n$ and $1 \leq l \leq k \leq n$

$$\phi(p_{ij}p_{lk}) = \phi(\delta_{ik}p_{lj}) = \delta_{ik}\phi(p_{lj}) = \delta_{ik}E_{jl} = E_{ji}E_{kl} = \phi(p_{ij})\phi(p_{lk}).$$

ϕ is an isomorphism, because for the linear map $\psi : B \to KQ$ given by $\psi(E_{ij}) = p_{ji}$ for all $1 \leq i \leq j \leq n$ we have $\phi\psi = \mathrm{id}_B$ and $\psi\phi = \mathrm{id}_{KQ}$. Thus we have $KQ \cong B$.
To show that $B \cong A$ we notice that for the matrix

$$S := \begin{pmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{pmatrix} \in M_n(K)$$

with $S^2 = 1$ the map

$$f : M_n(K) \to M_n(K), F \mapsto SFS$$

is an vector space automorphism with $f^2 = \mathrm{id}$. f is an algebra isomorphism because for all $F, G \in M_n(K)$

$$f(FG) = SFGS = SFS^2GS = f(F)f(G).$$

We also notice that f maps the Basis $(E_{ij})_{1 \leq i \leq j \leq n}$ of A to the basis $(E_{ij})_{1 \leq j \leq i \leq n}$ of B, thus $f_{|A} \to f_{|B}$ is an algebra isomorphism.

Exercise 30:

We name the vertices of Q as 1 and 2 with $s(\alpha) = t(\alpha) = 1$ and the arrow from 1 to 2 as p. By definition

$$P := \{e_1, e_2, p\} \cup \bigcup_{n \geq 1} \{\alpha^n, p\alpha^n\}$$

is a basis of KQ. It is obvious that

$$B = \begin{pmatrix} K[T] & 0 \\ K[T] & K \end{pmatrix}$$

is a K-algebra via the usual matrix multiplication. We define the linear map $\phi : KQ \to B$ by

$$\phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \phi(p) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and}$$
$$\phi(\alpha^n) = \begin{pmatrix} T^n & 0 \\ 0 & 0 \end{pmatrix}, \phi(p\alpha^n) = \begin{pmatrix} 0 & 0 \\ T^n & 0 \end{pmatrix} \text{ for all } n \geq 1.$$

It is clear that ϕ induces a bijection between P and a basis of B, so ϕ is a vector space isomorphism. It is also easy to see that ϕ is an algebra homomorphism, because $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in P$ (this can be directly shown by some boring matrix multiplication which I will not include here). Thus $KQ \cong B$.
The ideal $I = (\alpha^2)$ in KQ generated by the path α^2 corresponds to the ideal $J = (\phi(\alpha)^2)$ in B generated by $\phi(\alpha)^2$. Because

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \underbrace{\begin{pmatrix} T^2 & 0 \\ 0 & 0 \end{pmatrix}}_{=\phi(\alpha)^2} \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} adT^2 & 0 \\ bdT^2 & 0 \end{pmatrix}$$

we find that

$$J = (\phi(\alpha)^2) = \begin{pmatrix} (T^2) & 0 \\ (T^2) & 0 \end{pmatrix}.$$

Thus ϕ induces an algebra isomorphism $\bar{\phi}$ between the algebras KQ/I and $B/J = A$.