Foundations of representation theory

7. Exercise sheet

Jendrik Stelzner

November 29, 2013

Exercise 26:

Let $(e_1, \dots, e_n)$ be a basis of A as a vector space.

(i) $\Rightarrow$ (ii)

Let $(b_1, \dots, b_m)$ be a generating set of M as an A-module. We can write $x \in M$ as $x = \sum_{j=1}^m a_j b_j$ with $a_j \in A$ for all j. We can write each a_j as $a_j = \sum_{i=1}^n \lambda_i^j e_i$ with $\lambda_i^j \in K$ for all i. Thus we get

$$x = \sum_{j=1}^m a_j b_j = \sum_{i=1}^n \sum_{j=1}^m \lambda_i^j e_i b_j.$$

Thus x is a linear combination of the $e_i b_j$. Because x is arbitrary it follows that $\{e_i b_j\}_{i=1,\dots,n, j=1,\dots,m}$ is a finite generating set of M as a vector space, so M is finite-dimensional.

(ii) $\Rightarrow$ (i)

If $(b_1, \dots, b_m)$ is a basis of M as a vector space, then $(b_1, \dots, b_m)$ is also a generating set of M as an A-module, because $\lambda b_i = \lambda 1_A b_i$ for all $\lambda \in K$ and i.

(ii) $\Rightarrow$ (iii)

This follows directly from $l(M) \leq \dim(V) < \infty$.

$\neg$(ii) $\Rightarrow$ $\neg$(iii)

We construct an ascending chain $U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \dots$ of finite-dimensional submodules of M as follows: We start with $U_0 := 0$. If U_{n-1} is defined we choose some $v \in M \setminus U_{n-1}$ (this is possible because U_{n-1} is finite-dimensional but M is infinite-dimensional). The submodule $W = Av = \langle e_1 v, \dots, e_n v \rangle$ of M is not contained in U_{n-1}, so $U_{n-1} \subsetneq U_{n-1} + W =: U_n$. U_n is finite-dimensional because U_{n-1} and W are finite-dimensional.
For all $n \in \mathbb{N}$ the filtration

$$0 = U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \dots \subsetneq U_{n-1} \subsetneq M$$

is of length n, so $l(M) \geq n$ for all $n \in \mathbb{N}$.

Exercise 27:

We can assume that Q is weakly connected: Otherwise, if $Q_1, \ldots, Q_n$ are the weakly connected component of Q, it is easy to see that $KQ = \oplus_{i=1}^n KQ_i$ and $C(KQ) = \oplus_{i=1}^n C(KQ_i)$. Also notice that for every quiver Q the center $C(KQ)$ is a subalgebra. I didn't manage to determine $C(KQ)$ for every weakly connected quiver, but at least for one kind: We look at a quiver with vertices $Q_0 = (1, \ldots, n)$ and arrows $(a_1, \ldots, a_n)$ where a_i goes from i to $i+1$ for $1 \leq i \leq n-1$ and a_n goes from n to 1. One can see this quiver as a cycle with n vertices.
Let $x \in C(A)$ with $x = \sum_{k=1}^n \lambda_k p_k$ where $\lambda_k \in K$ and p_k is a path in Q for all k. We notice that $\lambda_k \neq 0$ implies that $s(p_k) = t(p_k)$: For all $i \in Q_0$ let

$$I_i^s := \{1 \leq k \leq n : s(p_k) = i\} \text{ and } I_i^t := \{1 \leq k \leq n : t(p_k) = i\}.$$

Because $x \in C(A)$ we find that

$$\sum_{k \in I_i^t} \lambda_k p_k = e_i x = x e_i = \sum_{k \in I_i^s} \lambda_k p_k. \tag{1}$$

Because the paths p_k are linear independent we find that $I_i^s = I_i^t$. (This observation holds true for every quiver.)
Because of the special form of Q we can classify all paths p in Q with $s(p) = t(p)$ by C_i^n, where $C_i := (a_i, \ldots, a_n, a_1, a_2, \ldots, a_{i-1})$ and $n \in \mathbb{N}$.
So we know that $x = \sum_{k=1}^n \lambda_k C_{i_k}^{\nu_k}$ with $\lambda_k \neq 0$ for all k. We find that for every k we have $C_j^{\nu_i}$ as a non-zero linear factor of x for all i: This follows directly from the equality given by (1). So every $x \in C(A)$ is of the form

$$x = \sum_{k=1}^d \lambda_k \sum_{i=1}^n C_i^k = \sum_{k=1}^d \lambda_k \left(\sum_{i=1}^n C_i \right)^k.$$

It is also easy to see, that $C := \sum_{i=1}^n C_i \in C(A)$, because every path p of Q commutes with $C := \sum_{i=1}^n C_i$: If $s(p) = i$ and $t(p) = j$ we get that

$$pC = pC_i^1 = C_j^1 p = Cp.$$

Because the paths are a K-Basis of KQ it follows that C commutes with every element in KQ. Because $C(A)$ as a subalgebra is closed under scalar multiplication, addition and multiplication, we find that

$$C(A) = \left\{ \sum_{k=1}^d \lambda_k C^k : \lambda_k \in K \text{ for all } k \right\}.$$

In particular we find that $C(A) \cong K[T]$.

Exercise 28:

Let

$$P := \{p : p \text{ is a path from } i \text{ to } j \text{ such that there is no path from } j \text{ to } i\}.$$

First we show that $P \subseteq J(KQ)$: Let $p \in P$ with $s(p) = i$ and $t(p) = j$, i.e. with $p = pe_i = e_j p$ and $e_i q e_j = 0$ for all paths q in Q. Let $x \in A$ with $x = \sum_{k=1}^{n} \lambda_k q_k$ for paths q_k and $\lambda_k \in K$. We find that

$$(px)^2 = \left(\sum_{k=1}^{n} \lambda_k p q_k \right)^2 = \sum_{k,l=1}^{n} \lambda_k \lambda_l p q_k p q_l = 0,$$

because $pq_k p = pe_i q_k e_j p = 0$ for all paths k. Because px is nilpotent we find that $1 - px$ is invertible. Because x is arbitrary is follows that $p \in J(KQ)$.
Now we show that P is a basis of $J(KQ)$. We already know that P is linear independent, because all paths are linear independent, so all that's left to show is that every $x \in J(KQ)$ can be written as a linear combination of elements in P.
Assume that there is some $x \in J(KQ)$ which cannot be written as a linear combination of elements in P. Let $x = \sum_{k=1}^{n} \lambda_k x_k$ be a linear combination of x with $\lambda_k \neq 0$ for all k and the x_k being pairwise different paths. We can assume w.l.o.g. that $x_k \notin P$ for all k, because $P \subseteq J(KQ)$. We can also assume that $s(x_k) = s(x_l)$ and $t(x_k) = t(x_l)$ for all k, l, because otherwise we can replace x by $e_{t(x_1)} x e_{s(x_1)} \in e_{t(x_1)} J(KQ) e_{s(x_1)} \subseteq J(KQ)$.
Because $x_1 \notin P$ we find some path $\bar{x}$ with $s(\bar{x}) = t(x_1)$ and $t(\bar{x}) = s(x_1)$. Because $x \in J(KQ)$ we know that $1 - \bar{x}x$ is invertible. Notice that

$$y := \bar{x}x = \sum_{k=1}^{n} \lambda_k \underbrace{\bar{x} x_k}_{:= y_k}$$

is non-zero with $s(y_k) = t(y_k) = s(y_l) = t(y_l) =: i$ for all k, l, i.e. the y_k are closed paths from i to i.
Because $1 - y$ is invertible we find som $y' = \sum_{k=1}^{m} \mu_k w_k$ with $\mu_k \neq 0$ and the w_k being pairwise different paths, such that $yy' = 1$. So we get

$$\sum_{j \in Q_0} e_j = 1 = (1 - y)y' = y' - yy' = \sum_{k=1}^{m} \mu_k w_k - \sum_{k=1}^{n} \sum_{l=1}^{m} \lambda_k \mu_l y_k w_l. \tag{2}$$

For $j \in Q_0$ let

$$J_j := \{1 \leq k \leq m : t(w_k) = j\}$$

and $I := J_i$. By multiplying (2) from the left with e_i we get

$$e_i = \sum_{k \in I} \mu_k w_k - \sum_{k=1}^{n} \sum_{l \in I} \lambda_k \mu_l y_k w_l.$$

which contradicts the linear independence of the paths in Q.