# Foundations of representation theory

## 9. Exercise sheet

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## Exercise 33:

Assume that ${}_AA \cong {}_AA \oplus {}_AA$ and let $\phi : {}_AA \to {}_AA \oplus {}_AA$ be an algebra homomorphism. We set

$$(b_0, b_1) := \phi(1)$$

and notice that for all $a \in A$

$$\phi(a) = \phi(a \cdot 1) = a\phi(1) = a(b_0, b_1) = (ab_0, ab_1).$$

Because $\phi$ is surjective we find $a_0, a_1 \in A$ with

$$\begin{aligned} (1,0) &= \phi(a_0) = (a_0b_0, a_0b_1) \text{ and} \\ (0,1) &= \phi(a_1) = (a_1b_0, a_1b_1). \end{aligned}$$

In particular we have $a_0b_0 = a_1b_1 = 1$ and $a_0b_1 = a_1b_0 = 0$. Because

$$\phi(b_0a_0 + b_1a_1) = (b_0a_0b_0 + b_1a_1b_0, b_0a_0b_1 + b_1a_1b_1) = (b_0, b_1) = \phi(1)$$

it follows from the injectivity of $\phi$ that $b_0a_0 + b_1a_1 = 1$.
Now assume that there exist elements $a_0, a_1, b_0, b_1 \in A$ with $a_0b_0 = a_1b_1 = 1$, $a_0b_1 = a_1b_0 = 0$ and $b_0a_0 + b_1a_1 = 1$. We define

$$\psi : {}_AA \to {}_AA \oplus {}_AA, a \mapsto a(b_0, b_1) = (ab_0, ab_1).$$

It is clear that $\psi$ is an $A$-module homomorphism. For all $(c_0, c_1) \in {}_AA \oplus {}_AA$ we have

$$\psi(c_0a_0 + c_1a_1) = (c_0a_0b_0 + c_1a_1b_0, c_0a_0b_1 + c_1a_1b_1) = (c_0, c_1),$$

so $\psi$ is surjective. For $x \in A$ with $\psi(x) = 0$ we have $(xb_0, xb_1) = (0, 0)$, so $(xb_0a_0, xb_1a_1) = (0, 0)$ and thus

$$0 = xb_0a_0 + xb_1a_1 = x(b_0a_0 + b_1a_1) = x \cdot 1 = x.$$

So $\psi$ in injective. This shows that $\psi$ is an $A$-module isomorphism and therefore ${}_AA \cong {}_AA \oplus {}_AA$.
One trivial example of such an algebra is $A = 0$.