Foundations of representation theory
2. exercise sheet

Jendik Stelzner

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Exercise 5 and 6:

Let V be a 2-dimensional module. The following are equivalent:

(i) V has at least 5 submodules.

(ii) Every subspace of V is a submodule.

(iii) V is not cyclic.

Proof. $(i) \Rightarrow (ii)$

Let $j \in J$ be fixed but arbitrary. Since V has at least 5 submodules, at least 3 of these submodules have to be nontrivial. Since these three have to be 1-dimensional, we find pairwise linear independent $v_1, v_2, v_3 \in V$ with $v_i \neq 0$ for all i, such that $\langle v_1 \rangle$, $\langle v_2 \rangle$ and $\langle v_3 \rangle$ are submodules. Since $\phi_j(v_i) \in \langle v_i \rangle$ for all i, it follows that every v_i is an eigenvector of ϕ_j. Let λ_i be the eigenvalue of v_i. If the λ_i were pairwise different, $\{v_1, v_2, v_3\}$ would be linear independent, which is not possible because V is only 2-dimensional. So there must be $i_1, i_2 \in \{1,2,3\}$ with $i_1 \neq i_2$ but $\lambda_{i_1} = \lambda_{i_2}$. Since v_{i_1} and v_{i_2} are linear independent, it follows that $\{v_{i_1}, v_{i_2}\}$ is a basis of V. From this it follows that every every $v \in V, v \neq 0$ is an eigenvector of ϕ_j with eigenvalue λ_{i_1}. This directly implies that every subspace of V is ϕ_j invariant. Because j is arbitrary it follows that every subspace is ϕ_j invariant for all $j \in J$, so every subspace is a submodule.

$(ii) \Rightarrow (i)$

Let $\{v_1, v_2\}$ be a basis of V. Since v_1, v_2 and $v_1 + v_2$ are pairwise linear independent, $\{0\}$, $\langle v_1 \rangle$, $\langle v_2 \rangle$, $\langle v_1 + v_2 \rangle$ and V are five pairwise different submodules of V.

$(ii) \Rightarrow (iii)$

Since for every $v \in V, U(v) = \langle v \rangle$ is an at most 1-dimensional submodule of V, there is no $v \in V$ such that $U(v) = V$. So V is not cyclic.

$(iii) \Rightarrow (ii)$

Assume that not every subspace of V is a submodule. Than we find a subspace $U \subseteq V$ such that U is not a submodule of V. This implies that U is nontrivial, so U must be 1-dimensional. Let $v \in U$ be a basis vector of U. Because U is a subspace but no submodule of V there exist $j \in J$ such that $\phi_j(v) \notin U = \langle v \rangle$. This means that v and $\phi_j(v)$ are linear independent and thus $\{v, \phi_j(v)\}$ is a basis of V and $V = U(v)$. So V is cyclic. $\square$

$(i) \Rightarrow (ii)$ shows Exercise 5, and $\neg(i) \Rightarrow \neg(iii)$ shows Exercise 6.

Exercise 7:

A matrice $A = (a_{ij}) \in M_3(K)$ describes a vector space endomorphism of K^3, which is a module endomorphism of V if and only if

$$A \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A \text{ and } A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} A. \tag{1}$$

By matrix multiplication we get

$$\begin{pmatrix} a_{12} & 0 & 0 \\ a_{22} & 0 & 0 \\ a_{32} & 0 & 0 \end{pmatrix} = A \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a_{13} & 0 & 0 \\ a_{23} & 0 & 0 \\ a_{33} & 0 & 0 \end{pmatrix} = A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \end{pmatrix}.$$

So A does satisfy (1) if and only if $a_{12} = a_{13} = a_{23} = a_{32} = 0$ and $a_{11} = a_{22} = a_{33}$. The set of all matrices satisfying this conditions is

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix} : a, b, c \in K \right\}.$$

V is not simple, because

$$U = \langle e_2, e_3 \rangle = \ker \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is a nonzero proper submodule of V.

V is indecomposable: For submodules $U_1, U_2 \subseteq V$ with $U_1 \oplus U_2 = V$ there exist $\lambda, \mu \in K$ such that $e_1 + \lambda e_2 + \mu e_3 \in U_1$ or $e_1 + \lambda e_2 + \mu e_3 \in U_2$, because otherwise v and e_1 would be linear independent for all $v \in U_1$ and $v \in U_2$, so v and e_1 would be linear independent for all $v \in U_1 \oplus U_2 = V$, which is obviously not the case. W.l.o.g. we can assume that $e_1 + \lambda e_2 + \mu e_3 \in U_1$. Because U_1 is a submodule of V it follows that

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (e_1 + \lambda e_2 + \mu e_3) = e_2 \in U_1 \text{ and}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} (e_1 + \lambda e_2 + \mu e_3) = e_3 \in U_1.$$

This implies that

$$V = \langle e_1, e_2, e_3 \rangle = \langle e_1 + \lambda e_2 + \mu e_3, e_2, e_3 \rangle \subseteq U_1 \subseteq V,$$

so $U_1 = V$. Because $U_1 \cap U_2 = \{0\}$ this means that $U_2 = \{0\}$ and that V is indecomposable.

Exercise 8:

well-defined

It first needs to be shown that the function is well-defined, i.e. that for all $f \in \mathrm{Hom}_J(V, W)$ Γ_f is a submodule of $V \times W$ with $\Gamma_f \oplus (0 \times W) = V \times W$. (It is clear that $0 \times W$ is a submodule of $V \times W$, since 0 is a submodule of V and W is a submodule of itself.)

Let $f \in \mathrm{Hom}_J(V, W)$. $\Gamma_f \subseteq V \times W$ is a subspace: It is $(0, 0) = (0, f(0)) \in \Gamma_f$, for $(v, f(v)) \in \Gamma_f$ is $-(v, f(v)) = (-v, f(-v)) \in \Gamma_f$, for $(v, f(v)), (v', f(v')) \in \Gamma_f$ is $(v, f(v)) + (v', f(v')) = (v + v', f(v + v')) \in \Gamma_f$, and for $(v, f(v)) \in \Gamma_f$ and $\lambda \in K$ we find $\lambda(v, f(v)) = (\lambda v, f(\lambda v)) \in \Gamma_f$. To show that Γ_f is a submodule we also need to show that $(\phi_j \oplus \psi_j)(v, f(v)) \in \Gamma_f$ for all $j \in J$ and $v \in V$. This holds because f is an module homomorphism and so

$$(\phi_j \oplus \psi_j)(v, f(v)) = (\phi_j(v), \psi_j(f(v))) = (\phi_j(v), f(\phi_j(i))) \in \Gamma_f.$$

To show that $\Gamma_f \oplus (0 \times W) = V \times W$ we first notice that $\Gamma_f \cap (0 \times W) = \{(0, 0)\}$: For $(x, y) \in \Gamma_f \cap (0 \times W)$ we find $v \in V$ und $w \in W$ such that $(v, f(v)) = (x, y) = (0, w)$. So $v = 0$ and $w = f(v) = f(0) = 0$ and thus $(x, y) = (0, 0)$. To show that $\Gamma_f + (0 \times W) = V \times W$ we notice that for all $(v, w) \in V \times W$

$$(v, w) = (v, f(v)) + (0, w - f(v)) \in \Gamma_f + (0 \times W).$$

injective

To show that the function is injective we notice that for $f, g \in \mathrm{Hom}_J(V, W)$

$$\begin{aligned} \Gamma_f = \Gamma_g &\Rightarrow \{(v, f(v)) : v \in V\} = \{(v, g(v)) : v \in V\} \\ &\Rightarrow (v, f(v)) = (v, g(v)) \text{ for all } v \in V \\ &\Rightarrow f(v) = g(v) \text{ for all } v \in V \\ &\Rightarrow f = g. \end{aligned}$$

surjective

To show that the function is surjective we construct for each submodule $U \subseteq V \times W$ with $U \oplus (0 \times W) = V \times W$ a homomorphism $f_U \in \operatorname{Hom}_J(V, W)$ with $\Gamma_{f_U} = U$. Let $U \subseteq V \times W$ be a submodule with $U \oplus (0 \times W) = V \times W$. For every $v \in V$ there exist a unique $w_v \in W$ with $(v, w_v) \in U$: To show the uniqueness we notice that for $(v, w), (v, w') \in U$ with $(v, w) = (v, w')$

$$U \ni (v, w) - (v, w') = (0, w - w') \in W \Rightarrow w - w' = 0 \Rightarrow w = w',$$

since $U \subseteq V \times W$ is a subspace and $U \cap (0 \times W) = \{(0, 0)\}$. To show that such a w_v exists we write $(v, w) \in V \times W$ uniquely as $(v, w) = (v, w') + (0, w'')$ with $(v, w') \in U$ and $(0, w'') \in 0 \times W$; this is possible because $V = U \oplus (0 \times W)$.
This now allows us to define a function $f : V \to W, v \mapsto w_v$. Note that from the definition of f it directly follows that $f(v) = w \Leftrightarrow (v, w) \in U$ and $\Gamma_f = U$. It turns out that f is a module homomorphism: Because $U \subseteq V \times W$ is a subspace we find that for $v, v' \in V$

$$(v + v', f(v) + f(v')) = (v, f(v)) + (v', f(v')) \in U,$$

so $f(v + v') = f(v) + f(v')$. We also find that for all $v \in V$ and $\lambda \in K$

$$(\lambda v, \lambda f(v)) = \lambda(v, f(v)) \in U,$$

so $f(\lambda v) = \lambda f(v)$. This shows that f is K-linear. To show that f is a module homomorphism we notice that for all $v \in V$ and $j \in J$

$$(v, f(v)) \in U \Rightarrow (\phi_j(v), \psi_j(f(v))) \in U,$$

because U is a submodule of $V \times W$. This implies $f(\phi_j(v)) = \psi_j(f(v))$ for all $v \in V$ and $j \in J$, so $f\phi_j = \psi_j f$ for all $j \in J$. This shows that f is a module homomorphism.