Foundations of representation theory

6. Exercise sheet

Jendrik Stelzner

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Exercise 21:

We will assume that V is an artinian module. V is uniform, because $S \neq 0$ is contained in every non-zero submodule of V. This implies that V is indecomposable.
For all $f \in \mathrm{End}(V)$ we have $\operatorname{img} f_{|S} \subseteq S$: If $f_{|S} = 0$ this is trivial. Otherwise $\operatorname{img} f_{|S} \subseteq V$ is a non-zero submodule, so $S \subseteq \operatorname{img} f_{|S}$. Because S is non-zero, $f_{|S}^{-1}(S) \subseteq S$ is a non-zero submodule. Because S is simple we get $S = f_{|S}^{-1}(S)$ and thus $\operatorname{img} f_{|S} = S$.
This allows us to define $\varphi : \mathrm{End}(V) \to \mathrm{End}(S), f \mapsto f_{|S}$. It is obvious that φ is a ring homomorphism. By assumption φ is surjective. We know show that

$$\ker \varphi = \{f \in \mathrm{End}(V) : f \text{ is not invertible}\}.$$

It is clear that

$$\ker \varphi \subseteq \{f \in \mathrm{End}(V) : f \text{ is not invertible}\}.$$

Let $f \in \mathrm{End}(V)$ be not injective. Because $\ker f \neq 0$ is a submodule we have $S \subseteq \ker f$, so $f_{|S} = 0$. Let $g \in \mathrm{End}(V)$ be injective but not surjective. We get a descending chain

$$V \supseteq \operatorname{img} g \supseteq \operatorname{img} g^2 \supseteq \operatorname{img} g^3 \supseteq \ldots$$

of submodules of V. By assumption this chain eventually stabilizes, i.e. there exists some $N \in \mathbb{N}$ with $\operatorname{img} g^n = \operatorname{img} g^{n+1}$ for all $n \geq N$. Because g is injective we also have $\ker g^n = \ker g^{n+1}$ for all $n \in \mathbb{N}$. This implies that

$$V = \ker g^N \oplus \operatorname{img} g^N.$$

Because V is indecomposable this implies that $\ker g^n = 0$ and $\operatorname{img} g^N = V$ or $\ker g^N = V$ and $\operatorname{img} g^N = 0$. So g is either not injective or surjective, which is contradicts either the injectivity or non-surjectivity of g. So g has to be non-injective and thus contained in $\ker \varphi$.
Because

$$\ker \varphi = \{f \in \mathrm{End}(V) : f \text{ is not invertible}\}$$

is an ideal in $\mathrm{End}(V)$, we get that $\mathrm{End}(V)$ is local. so

$$J(\mathrm{End}(V)) = \{f \in \mathrm{End}(V) : f \text{ is not invertible}\} = \ker \varphi$$

and thus

$$\mathrm{End}(V)/J(\mathrm{End}(V)) = \mathrm{End}(V)/\ker \varphi \cong \operatorname{img} \varphi = \mathrm{End}(S).$$