

# FOUNDATIONS OF REPRESENTATION THEORY

## 4. EXERCISE SHEET

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### Exercise 13:

A submodule  $S \subseteq N(\lambda)$  is simple if and only if it is of the form

$$S = \left\langle \sum_{i=1}^t \mu_i e_{i1} \right\rangle$$

with  $\mu_1, \dots, \mu_t \in K$  and  $\mu_j \neq 0$  for at least one  $j \in \{1, \dots, t\}$ : Since

$$\phi \left( \sum_{i=1}^t \mu_i e_{i1} \right) = \sum_{i=1}^t \mu_i \phi(e_{i1}) = 0$$

and  $\sum_{i=1}^t \mu_i e_{i1} \neq 0$  these subspaces are 1-dimensional submodules, which are always simple. If  $S \subseteq N(\lambda)$  is simple then  $S$  is nonzero, and because  $\phi$  is nilpotent there is  $v \in S \setminus \{0\}$  with  $\phi(v) = 0$ . So the subspace  $\langle v \rangle \subseteq S$  is a nonzero submodule of  $S$ , so  $S = \langle v \rangle$ . Thus we get

$$\text{soc}(N(\lambda)) = \sum_{\substack{S \subseteq N(\lambda) \\ S \text{ semisimple}}} S = \sum_{\substack{S \subseteq N(\lambda) \\ S \text{ simple}}} S = \langle e_{11}, e_{21}, \dots, e_{t1} \rangle.$$

To determine  $\text{rad}(N(\lambda))$  we notice that for every  $s \in \{1, \dots, t\}$  the subspace

$$M_s := \left\langle \left( \bigcup_{i=1}^t \bigcup_{j=1}^{\lambda_i} \{e_{ij}\} \right) \setminus \{e_{s\lambda_s}\} \right\rangle$$

is a maximal submodule of  $N(\lambda)$ :

It is clear that  $M_s$  is a submodule. To show that  $M_s$  is maximal in  $N(\lambda)$  we notice that for a submodule  $M$  of  $N(\lambda)$  with  $M_s \subsetneq M \subseteq N(\lambda)$  there exists  $v \in M$  and  $\mu_{ij} \in K$  for  $i = 1, \dots, t, j = 1, \dots, \lambda_i$  with  $\mu_{s\lambda_s} = 1$  such that

$$v = \sum_{i=1}^t \sum_{j=1}^{\lambda_i} \mu_{ij} e_{ij},$$

because otherwise  $v \in M_s$  for all  $v \in M$  and thus  $M \subseteq M_s$ . Together with  $M_s \subsetneq M$  this implies that  $e_{ij} \in M$  for  $i = 1, \dots, t$  and  $j = 1, \dots, \lambda_i$ . So  $M = N(\lambda)$ .

Because  $M_s$  is maximal for  $s = 1, \dots, t$  we find that

$$\text{rad}(N(\lambda)) \subseteq \bigcap_{s=1}^t M_s = \left\langle \bigcup_{i=1}^t \bigcup_{j=1}^{\lambda_i-1} \{e_{ij}\} \right\rangle =: r_0.$$

We also find that  $r_0 \supseteq \text{rad}(N(\lambda))$ : If  $W \subsetneq N(\lambda)$  is a maximal submodule with  $r_0 \not\subseteq W$  then there is some  $s \in \{1, \dots, t\}$  with  $e_{s(\lambda_s-2)} \notin W$ , which also means that  $e_{s(\lambda_s-1)} \notin W$ , because

$$\phi(e_{s(\lambda_s-1)}) = e_{s(\lambda_s-2)}.$$

Because  $W$  is maximal in  $N(\lambda)$  the factor module  $N(\lambda)/W$  is simple, and because  $N(\lambda)/W \cong N(\lambda')$  for some  $\lambda' = (\lambda'_1, \dots, \lambda'_t)$  with  $\lambda'_i \leq \lambda_i$  for  $i = 1, \dots, t$  this means that  $N(\lambda)/W$  is one-dimensional. So  $W$  has dimension

$$\dim N(\lambda) - 1 = \left( \sum_{i=1}^t \lambda_i \right) - 1. \quad (1)$$

Because  $e_{s(\lambda_s-2)} \notin W$  we find that  $\langle e_{s(\lambda_s-2)} \rangle \cap W = 0$  and that  $W$  is a proper subspace of  $W + \langle e_{s(\lambda_s-2)} \rangle$ . Therefore (1) implies that  $W + \langle e_{s(\lambda_s-2)} \rangle = N(\lambda)$ . So  $N(\lambda) = W \oplus \langle e_{s(\lambda_s-2)} \rangle$  (as vector spaces). So there are  $\mu \in K$  and  $v \in W$ ,  $v = \sum_{i=1}^t \sum_{j=1}^{\lambda_i} \mu_{ij} e_{ij}$  with  $\mu_{ij} \in K$  and  $\mu_{s(\lambda_s-2)} = 0$ , such that

$$e_{s(\lambda-1)} = \mu e_{s(\lambda-2)} + v = \mu e_{s(\lambda-2)} + \sum_{i=1}^t \sum_{j=1}^{\lambda_i} \mu_{ij} e_{ij}.$$

Because the  $e_{ij}$  are linear independent we get  $\mu = 0$ , so  $e_{s(\lambda-1)} = v \in W$ , which is a contradiction to  $e_{s(\lambda-1)} \notin W$ . So for every maximal submodule  $W \subsetneq N(\lambda)$  we have  $r_0 \subseteq W$ , so  $r_0 \subseteq \text{rad}(N(\lambda))$ .

## Exercise 14:

**Lemma 1.** Let  $V$  and  $W$  be modules and  $f : V \rightarrow W$  a module homomorphism. Then the following hold:

- i) For every submodule  $U \subseteq W$  the preimage  $f^{-1}(U)$  is a submodule of  $V$ .
- ii) If  $U \subseteq V$  is a submodule,  $\pi : V \rightarrow V/U$  the canonical projection and  $W \subseteq V/U$  a maximal submodule, then the preimage  $\pi^{-1}(W)$  is a maximal submodule of  $V$ .
- iii) If  $W \subseteq V$  is maximal in  $V$  and  $U \subseteq W$  is a submodule then  $W/U$  is maximal in  $V/U$  and  $\pi^{-1}(W/U) = W$ .

*Proof.* i)

We know from algebra that  $f^{-1}(U)$  is an additive subgroup of  $V$ .  $f^{-1}(U)$  is also a subspace of  $V$ , since for all  $\lambda \in K$

$$v \in f^{-1}(U) \Rightarrow f(v) \in U \Rightarrow f(\lambda v) = \lambda f(v) \in U \Rightarrow \lambda v \in f^{-1}(U).$$

$f^{-1}(U)$  is a submodule of  $V$ , since  $f$  is a module homomorphism and  $U$  a submodule and so for all  $j \in J$  and  $v \in f^{-1}(U)$

$$f(\phi_j(v)) = \psi_j(f(v)) \in U,$$

so  $\phi_j(v) \in f^{-1}(U)$ .

ii)

From i) it follows that  $f^{-1}(W)$  is a submodule of  $V$ . Let  $W' \subseteq V$  be a arbitrary submodule with

$$\pi^{-1}(W) \subseteq W' \subseteq V. \quad (2)$$

Then  $W \subseteq W'/U \subseteq V/U$ , so the maximality of  $W$  implies that  $W'/U = W$  or  $W'/U = V/U$ . If  $W'/U = W$  then

$$W' \subseteq \pi^{-1}(W'/U) = \pi^{-1}(W),$$

and with (2) we get  $W' = \pi^{-1}(W)$ . If  $W'/U = V/U$  then the the isomorphism theorems imply that

$$V/W' \cong (V/U)/(W'/U) = 0,$$

so  $W' = V$ .

iii)

$W/U$  is a proper submodule of  $V/U$ , because  $W$  is maximal in  $V$  and so

$$(V/U)/(W/U) \cong \underbrace{V/W}_{\text{simple}} \neq 0.$$

Let  $W' \subseteq V/U$  be a submodule with

$$W/U \subseteq W' \subsetneq V/U.$$

We know from i) that  $\pi^{-1}(W')$  is a submodule of  $V$ , and obviously

$$W \subseteq \pi^{-1}(W/U) \subseteq \pi^{-1}(W') \subsetneq V.$$

So the maximality of  $W$  implies that  $W = \pi^{-1}(W')$ . So

$$W/U = \pi(W) = \pi(\pi^{-1}(W')) = W'.$$

This shows that  $W/U$  is maximal in  $V/U$ . Now 1 ii) implies that  $\pi^{-1}(W/U)$  is maximal is  $V$ . Because  $W \subseteq \pi^{-1}(W/U)$  it follows that  $W = \pi^{-1}(W/U)$ .  $\square$

(ii)

If  $V/U$  does not have any maximal submodules we get  $\text{rad}(V/U) = V/U$  and

$$(U + \text{rad}(V))/U \subseteq V/U = \text{rad}(V/U)$$

If  $V/U$  does have maximal submodules, then by definition we get

$$\begin{aligned} (U + \text{rad}(V))/U &\subseteq \text{rad}(V/U) \\ \Leftrightarrow (U + \text{rad}(V))/U &\subseteq \bigcap_{\substack{W \subseteq V/U \\ W \text{ maximal}}} W \\ \Leftrightarrow (U + \text{rad}(V))/U &\subseteq W \text{ for all } W \subseteq V/U \text{ maximal.} \end{aligned}$$

Let  $W \subseteq V/U$  be an arbitrary maximal submodule. Using Lemma 1 ii) we find that  $\pi^{-1}(W)$  is a maximal submodule of  $V$ , where  $\pi : V \twoheadrightarrow V/U$  is the canonical projection. Obviously we have  $U \subseteq \pi^{-1}(W)$ , and because  $\pi^{-1}(W)$  is maximal in  $V$  we also get  $\text{rad}(V) \subseteq \pi^{-1}(W)$ , so

$$U + \text{rad}(V) \subseteq \pi^{-1}(W),$$

and thus

$$(U + \text{rad}(V))/U = \pi(U + \text{rad}(V)) \subseteq \pi(\pi^{-1}(W)) = W.$$

(i)

From part (ii) we get

$$\text{rad}(V)/U = (U + \text{rad}(V))/U \subseteq \text{rad}(V/U),$$

so all that's left to show is  $\text{rad}(V/U) \subseteq \text{rad}(V)/U$ . If  $V$  has no maximal submodules, then  $\text{rad}(V) = V$  and

$$\text{rad}(V/U) \subseteq V/U = \text{rad}(V)/U.$$

If  $V$  has maximal submodules, then let  $W$  be an arbitrary maximal submodule of  $V$ . Since  $U \subseteq \text{rad}(V) \subseteq W$  we know from 1 iii) that  $W/U$  is maximal in  $V/U$  and  $\pi^{-1}(W/U) = W$ . So  $\text{rad}(V/U) \subseteq W/U$  and thus

$$\pi^{-1}(\text{rad}(V/U)) \subseteq \pi^{-1}(W/U) = W.$$

Because  $W$  is arbitrary it follows that

$$\pi^{-1}(\text{rad}(V/U)) \subseteq \text{rad}(V)$$

and from this it follows that

$$\text{rad}(V/U) = \pi(\pi^{-1}(\text{rad}(V/U))) \subseteq \pi(\text{rad}(V)) = \text{rad}(V)/U.$$

## Exercise 15:

$$\neg(ii) \Rightarrow \neg(i)$$

Assume that  $V = U \oplus C$  is a direct sum decomposition with  $U$  simple.

**Claim.**  $C$  is maximal in  $V$ .

With this we find that

$$U \subseteq \text{soc}(V) \text{ and } \text{rad}(V) \subseteq C$$

because  $U$  is simple and  $C$  is maximal in  $V$ . Because  $U$  nonzero with  $U \cap C = 0$ , this implies that  $\text{soc}(V) \not\subseteq \text{rad}(V)$ .

*Proof of the claim.* Let  $C' \subseteq V$  be a submodule with  $C \subseteq C' \subseteq V$ . Let  $C'' := C' \cap U$ . Because  $U$  is simple we know that  $C'' = 0$  or  $C'' = U$ . If  $C'' = 0$  then by using modularity and  $C \subseteq C'$  we find

$$C = C + C'' = C + (U \cap C') = (C + U) \cap C' = V \cap C' = C'.$$

If  $C'' = U$  we get

$$U = C'' = C' \cap U, \text{ so } V = U \oplus C \subseteq U + C' = C', \text{ so } C' = V. \quad \square$$

(ii)  $\Rightarrow$  (i)

Assume that  $V$  does not have a simple direct summand. If  $V$  has no maximal submodule, then  $\text{soc}(V) \subseteq \text{rad}(V) = V$ .

If  $V$  does have at least one maximal submodule then by definition

$$\text{soc}(V) \subseteq \text{rad}(V) \Leftrightarrow S \subseteq U \text{ for all } S \subseteq V \text{ simple and all } U \subseteq V \text{ maximal.}$$

Assume that  $S \subseteq V$  is simple and  $U \subseteq V$  is maximal with  $S \not\subseteq U$ . Then  $S \cap U \neq S$  and  $S \not\subseteq S + U$ . Because  $S$  is simple this implies  $S \cap U = 0$ , and because  $U$  is maximal it implies  $S + U = V$ . So  $V = S \oplus U$ . This is a contradiction to the assumption that  $V$  does not have a simple direct summand. So  $S \subseteq U$  for all for all  $S \subseteq V$  simple and all  $U \subseteq V$  maximal.

## Exercise 16:

(i)

$0 \subseteq K[T]$  is the only small submodule of  $(K[T], T \cdot)$ : Let  $0 \subsetneq U \subseteq K[T]$  be a small submodule. We know that  $U = (a)$  for some  $a \in K[T]$ ; because  $U$  has to be nonzero and proper we know that  $\deg a \geq 1$ . We find an irreducible polynomial  $p \in K[T]$  with  $p \nmid a$ . So  $(a) + (p) = (1) = K[T]$ . Because  $(p)$  proper submodule of  $(K[T], T \cdot)$  this shows that  $U$  is not small.

In  $N(\infty)$  all nonzero submodules are small: For all nonzero submodules  $V, V' \subseteq N(\infty)$  we have  $N(1) \subseteq V, V'$  as a nonzero submodule, so  $V \cap V' \supseteq N(1)$  is nonzero.

(ii)

Both modules are uniform:

Let  $U, U' \subseteq K[T]$  be nonzero submodules. We know that  $U = (a)$  and  $U' = (b)$  for  $a, b \in K[T] \setminus \{0\}$ . Thus  $(ab)$  is a nonzero submodule of both  $U$  and  $U'$  (because  $K[T]$  has no zero divisors), so  $U \cap U' \supseteq (ab)$  is nonzero.

Let  $V, V' \subseteq N(\infty)$  be nonzero submodules. We know that  $V = N(i)$  and  $V' = N(j)$  for  $i, j \in \mathbb{N} \setminus \{0\}$ . So  $V \cap V' \supseteq N(1)$  is nonzero.

(iii)

Let  $V = N(\infty)$  and  $U := N(1) \subseteq V$ .  $U$  is large in  $V$ : For every nonzero submodule  $U' \subseteq V$  we have  $N(1) \subseteq U'$ , so  $U \cap U' \supseteq N(1)$  is nonzero.  $U$  is also small: For every proper submodule  $U'' \subseteq V$  we have  $U'' = N(i)$  for some  $i \in \mathbb{N}$ . So  $U + U'' = N(1) + N(i) = N(\max\{1, i\})$  is a proper submodule of  $V$ .