

FOUNDATIONS OF REPRESENTATION THEORY

6. EXERCISE SHEET

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Exercise 21:

We will assume that V is an artinian module. V is uniform, because $S \neq 0$ is contained in every non-zero submodule of V . This implies that V is indecomposable.

For all $f \in \text{End}(V)$ we have $\text{img } f|_S \subseteq S$: If $f|_S = 0$ this is trivial. Otherwise $\text{img } f|_S \subseteq V$ is a non-zero submodule, so $S \subseteq \text{img } f|_S$. Because S is non-zero, $f|_S^{-1}(S) \subseteq S$ is a non-zero submodule. Because S is simple we get $S = f|_S^{-1}(S)$ and thus $\text{img } f|_S = S$.

This allows us to define $\varphi : \text{End}(V) \rightarrow \text{End}(S)$, $f \mapsto f|_S$. It is obvious that φ is a ring homomorphism. By assumption φ is surjective. We now show that

$$\ker \varphi = \{f \in \text{End}(V) : f \text{ is not invertible}\}.$$

It is clear that

$$\ker \varphi \subseteq \{f \in \text{End}(V) : f \text{ is not invertible}\}.$$

Let $f \in \text{End}(V)$ be not injective. Because $\ker f \neq 0$ is a submodule we have $S \subseteq \ker f$, so $f|_S = 0$. Let $g \in \text{End}(V)$ be injective but not surjective. We get a descending chain

$$V \supseteq \text{img } g \supseteq \text{img } g^2 \supseteq \text{img } g^3 \supseteq \dots$$

of submodules of V . By assumption this chain eventually stabilizes, i.e. there exists some $N \in \mathbb{N}$ with $\text{img } g^n = \text{img } g^{n+1}$ for all $n \geq N$. Because g is injective we also have $\ker g^n = \ker g^{n+1}$ for all $n \in \mathbb{N}$. This implies that

$$V = \ker g^N \oplus \text{img } g^N.$$

Because V is indecomposable this implies that $\ker g^N = 0$ and $\text{img } g^N = V$ or $\ker g^N = V$ and $\text{img } g^N = 0$. So g is either not injective or surjective, which is contradicted either the injectivity or non-surjectivity of g . So g has to be non-injective and thus contained in $\ker \varphi$.

Because

$$\ker \varphi = \{f \in \text{End}(V) : f \text{ is not invertible}\}$$

is an ideal in $\text{End}(V)$, we get that $\text{End}(V)$ is local. so

$$J(\text{End}(V)) = \{f \in \text{End}(V) : f \text{ is not invertible}\} = \ker \varphi$$

and thus

$$\text{End}(V)/J(\text{End}(V)) = \text{End}(V)/\ker \varphi \cong \text{img } \varphi = \text{End}(S).$$

Exercise 22:

(i)

Remark 1. Every endomorphism h of the module $V = (K[T], T \cdot)$ is of the form

$$h : f \mapsto p \cdot f$$

for some polynomial $p \in K[T]$, and each such map is an endomorphism von V .

Proof. It is obvious that every such map is an module endomorphism of V . To show that every endomorphism is of this form we denote $T \cdot$ as $\phi : f \mapsto T \cdot f$ and $p := h(1)$. Because h is a module homomorphism, it follows that for all $\sum_{i=0}^n a_i T^i \in K[T]$

$$\begin{aligned} h \left(\sum_{i=0}^n a_i T^i \right) &= h \left(\sum_{i=0}^n a_i \phi^i(1) \right) = \sum_{i=0}^n a_i \phi^i(h(1)) \\ &= \sum_{i=0}^n a_i \phi^i(p) = \sum_{i=0}^n a_i T^i p = p \sum_{i=0}^n a_i T^i. \end{aligned}$$

□

We assume that the greatest common divisor of some polynomial $p \in K[T]$ and $0 \in K[T]$ is defined as p , because otherwise the statement does not hold true. In this case the direct summands $V \oplus 0$ and $0 \oplus V$ are counterexamples.

Let U be a direct summand of $V \oplus V$ with $U \not\subseteq \{0, V \oplus V\}$. We know that U is the kernel of some idempotent endomorphism of V . Let $h \in \text{End}(V)$ be idempotent with $\ker h = U$. We know that h can be uniquely written as

$$h = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$$

with $h_1, h_2, h_3, h_4 \in \text{End}(V)$. From remark 1 it follows that there exists polynomials $p_1, p_2, p_3, p_4 \in K[T]$ such that

$$h = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}.$$

We get that

$$\ker h = \left\{ \begin{bmatrix} p \\ q \end{bmatrix} \in V \oplus V : \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = 0. \right\}$$

We notice that the matrix

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

also describes an idempotent vector space endomorphism h' of $K(X)^2$ with respect to the canonical basis, where $K(T)$ is the field of rational functions over K . It is clear that

$$\ker h = \ker h' \cap K[T],$$

and that $\ker h'$ is a subspace of $K(T)^2$. So to show that U is of the form $U_{f,g}$ for some polynomials f, g with greatest common divisor 1, it is enough to show that $\ker h'$ has a basis $\begin{bmatrix} f & g \end{bmatrix}$ with such polynomials f and g . Because $U \not\subseteq \{0, V \oplus V\}$ we know that

h is non-zero and no isomorphism, so $\ker h'$ is one-dimensional. Let $b = {}^t[f \ g]$ be a basis vector of $\ker h'$. We can assume that f and g are polynomials, because otherwise we can multiply b by some non-zero scalar $r \in K[T] \subseteq K(T)$ such that rf and rg are polynomials. We can also assume that the greatest common divisor of f and g is 1, because otherwise we can multiply b with some scalar $r' \in K(T)$ such that the greatest common divisor of $r'f$ and $r'g$ is 1. Because ${}^t[f \ g]$ is a basis vector of $\ker h$ it follows that

$$U = \ker h = \ker h' \cap K[T] = \{(hf, hg) : h \in K[T]\} = U_{f,g}.$$

Now we show the $U_{f,g}$ are direct summands: It is clear that $U_{f,g}$ is a submodule of $V \oplus V$ for alle $f, g \in K[T]$. Assume that the greatest common divisor of f and g is 1. If $f = 0$, then it follows that $g = 1$, so

$$U_{f,g} = 0 \oplus V = \ker \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and if $g = 0$ it follows that $f = 1$, so

$$U_{f,g} = V \oplus 0 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

In both cases the endomorphisms are obviously idempotent. If $f, g \neq 0$ we know from linear algebra that we can find polynomials $r, s \in K[T]$ with $rf + sg = 1$. Then the endomorphism of $V \oplus V$ of the form

$$h = \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix}$$

with

$$\begin{aligned} \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix}^2 &= \begin{bmatrix} rsfg + s^2g^2 & -rsf^2 - s^2fg \\ -r^2fg - rsg^2 & r^2f^2 + rsfg \end{bmatrix} \\ &= \begin{bmatrix} sg(rf + sg) & -sf(rf + sg) \\ -rg(rf + sg) & rf(rf + sg) \end{bmatrix} = \begin{bmatrix} sg & -sf \\ -rg & rf \end{bmatrix} \end{aligned}$$

is idempotent, and thus

$$\ker h = U_{f,g}$$

is a direct summand of $V \oplus V$.

(ii)

We know that $U_{T,T-1}$ is a direct summand of $V \oplus V$, because the greatest common divisor of T and $T - 1$ is 1. Because $U_{T,T-1} \neq 0$ and $U_{T,T-1} \neq V \oplus V$ we know that neither 0 nor $V \oplus V$ is a direct complement of $U_{T,T-1}$ in $V \oplus V$. $U_{1,0}$ and $U_{0,1}$ are also no direct complements of $U_{T,T-1}$ in $V \oplus V$, because $(0, 1) \notin U_{T,T-1} + U_{1,0}$ and $(1, 0) \notin U_{T,T-1} + U_{0,1}$.

Exercise 23:

$\text{soc}(M(w))$

Let b_1, \dots, b_{n+1} be the standard basis of $M(w)$. For all $1 \leq i \leq n+1$ let

$$h_x(b_i) := \min\{k \in \mathbb{N} : \phi_x^k(b_i) = 0\} \text{ and } h_y(b_i) = \min\{k \in \mathbb{N} : \phi_y^k(b_i) = 0\}.$$

Let $c_1, \dots, c_m \in \{b_1, \dots, b_{n+1}\}$ be the pairwise different b_i such that $h_x(b_i) = h_y(b_i) = 1$.

For every $v \in M(w)$ with $h_x(v) = h_y(v) = 1$ the subspace $\langle v \rangle = Kv$ is a simple submodule of $M(w)$, because $\langle x \rangle$ is one-dimensional and $\phi_x(v) = \phi_y(v) = 0$.

Every non-zero submodule $U \subseteq M(w)$ contains such a simple module: Let $v \in U$ with $v \neq 0$. If $h_x(v) = h_y(v) = 1$, $\langle v \rangle$ is such a submodule. Otherwise $h_x(v) > 1$ or $h_y(v) > 1$. In the first case $\langle \phi_x^{h_x(v)-1}(v) \rangle$ is such a submodule, in the second case $\langle \phi_y^{h_y(v)-1}(v) \rangle$ is such a submodule.

It follows that every simple submodule of $M(w)$ is of the form $\langle v \rangle$ for some $v \in M(w)$ with $h_x(v) = h_y(v) = 1$. It is easy to see that each such v is a linear combination of the c_i : For $v = \sum_{i=1}^{n+1} \lambda_i b_i$ we get

$$0 = \phi_x(v) = \sum_{i=1}^{n+1} \lambda_i \phi_x(b_i) \text{ and } 0 = \phi_y(v) = \sum_{i=1}^{n+1} \lambda_i \phi_y(b_i),$$

so for all i we have the implication $\lambda_i \neq 0 \Rightarrow \phi_x(b_i) = \phi_y(b_i) = 1$. So we get

$$\text{soc}(M(w)) = \sum_{\substack{S \subseteq M(w) \\ S \text{ simple}}} S = \bigoplus_{i=1}^m \langle c_i \rangle.$$