Foundations of representation theory

4. Exercise sheet

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November 8, 2013

Exercise 13:

A submodule $S \subseteq N(\lambda)$ is simple if and only if it is of the form

$$S = \left\langle \sum_{i=1}^{t} \mu_i e_{i1} \right\rangle$$

with $\mu_1, \dots, \mu_t \in K$ and $\mu_j \neq 0$ for at least one $j \in \{1, \dots, n\}$: Since

$$\phi\left(\sum_{i=1}^{t} \mu_i e_{i1}\right) = \sum_{i=1}^{t} \mu_i \phi(e_{i1}) = 0$$

and $\sum_{i=1}^{t} \mu_i e_{i1} \neq 0$ these modules are 1-dimensional submodules, which are always simple. If $S \subseteq N(\lambda)$ is simple then S is nonzero and for every $v \in S \setminus \{0\}$ the subspace $\langle v \rangle \subseteq S$ is a nonzero submodule of S, so $S = \langle v \rangle$. Thus we get

$$\operatorname{soc}(N(\lambda)) = \sum_{\substack{S \subseteq N(\lambda) \\ S \text{ semisimple}}} S = \sum_{\substack{S \subseteq N(\lambda) \\ S \text{ simple}}} S = \langle e_{11}, e_{21}, \dots, e_{t1} \rangle .$$

To determine $\operatorname{rad}(N(\lambda))$ we notice that for every $s \in \{1, \dots, t\}$ the subspace

$$M_s := \left\langle \left(\bigcup_{i=1}^{t} \bigcup_{j=1}^{\lambda_i} \{e_{ij}\} \right) \setminus \{e_{s\lambda_s}\} \right\rangle$$

is a maximal submodule of $N(\lambda)$:
It is clear that M_s is a submodule. To show that M_s is maximal in $N(\lambda)$ we notice that for a submodule M of $N(\lambda)$ with $M_s \subsetneq M \subseteq N(\lambda)$ there exists $v \in M$ and $\mu_{ij} \in K$, $i = 1, \dots, t$, $j = 1, \dots, \lambda_i$, with $\mu_{s\lambda_s} = 1$ such that

$$v = \sum_{i=1}^{t} \sum_{j=1}^{\lambda_i} \mu_{ij} e_{ij},$$

because otherwise $v \in M_s$ for all $v \in M$ and thus $M \subseteq M_s$. This implies that $e_{ij} \in M$ for $i = 1, \dots, t$ and $j = 1, \dots, \lambda_i$. So $M = N(\lambda)$.

Because M_s is maximal for $s = 1, \ldots, t$ we find that

$$\operatorname{rad}(N(\lambda)) \subseteq \bigcap_{s=1}^{t} M_s = \left\langle \bigcup_{i=1}^{t} \bigcup_{j=1}^{\lambda_i - 1} \{e_{ij}\} \right\rangle =: r_0.$$

We also find that $r_0 \supseteq \operatorname{rad}(N(\lambda))$: If $W \subsetneq N(\lambda)$ is a maximal submodule with $r_0 \not\subseteq W$ then there is some $s \in \{1, \ldots, t\}$ with $e_{s(\lambda_s - 2)} \notin W$, which also means that $e_{s(\lambda_s-1)} \notin W$, because $\phi(e_{s(\lambda_s-1)}) = e_{s(\lambda_s-2)}$. Because $e_{s(\lambda_s-2)} \notin W$ we find that $\langle e_{s(\lambda_s-2)} \rangle \cap W = 0$ and that W is a proper submodule of $W + \langle e_{s(\lambda_s-2)} \rangle$, so $W + \langle e_{s(\lambda_s-2)} \rangle = N(\lambda)$ because W is maximal. So $N(\lambda) = W \oplus \langle e_{s(\lambda_s-2)} \rangle$. So there are $\mu \in K$ and $v \in W$, $v = \sum_{i=1}^{t} \sum_{j=1}^{\lambda_i} \mu_{ij} e_{ij}$ with $\mu_{ij} \in K$ and $\mu_{s(\lambda-2)} = 0$, such that

$$e_{s(\lambda-1)} = \mu e_{s(\lambda-2)} + v = \mu e_{s(\lambda-2)} + \sum_{i=1}^{t} \sum_{j=1}^{\lambda_i} \mu_{ij} e_{ij}.$$

Because the e_{ij} are linear independent we get $\mu = 0$, so $e_{s(\lambda-1)} = v \in W$, which is a contradiction to $e_{s(\lambda-1)} \notin W$. So for every maximal submodule $W \subsetneq N(\lambda)$ we have $r_0 \subseteq W$, so $r_0 \subseteq \operatorname{rad}(V)$.

Exercise 14:

Lemma 1. Let V and W be modules and $f : V \to W$ a module homomorphism. Then the following hold:

i) For every submodule $U \subseteq W$ the preimage $f^{-1}(U)$ is a submodule of V.

ii) If $U \subseteq V$ is a submodule, $\pi : V \twoheadrightarrow V/U$ the canonical projection and $W \subseteq V/U$ a maximal submodule, then $\pi^{-1}(W)$ is a maximal submodule of V.

iii) If $W \subseteq V$ is maximal in V and $U \subseteq W$ is a submodule then W/U is maximal in V/U and $\pi^{-1}(W/U) = W$.

Proof. **i)**

We know from algebra that $f^{-1}(U)$ is an additive subgroup of V. $f^{-1}(U)$ is also a subspace of V, since for all $\lambda \in K$

$$v \in f^{-1}(U) \Rightarrow f(v) \in U \Rightarrow f(\lambda v) = \lambda f(v) \in U \Rightarrow \lambda v \in f^{-1}(U).$$

$f^{-1}(U)$ is a submodule of V, since f is a module homomorphism and U a submodule and so for all $j \in J$ and $v \in f^{-1}(U)$

$$f(\phi_j(v)) = \psi_j(f(v)) \in U,$$

so $\phi_j(v) \in f^{-1}(U)$.

ii)

From i) it follows that $f^{-1}(W)$ is a submodule of V. Let $W' \subseteq V$ be a arbitrary submodule with

$$\pi^{-1}(W) \subseteq W' \subseteq V. \tag{1}$$

Then $W \subseteq W'/U \subseteq V/U$, so the maximality of W implies that $W'/U = W$ or $W'/U = V/U$. If $W'/U = W$ then

$$W' \subseteq \pi^{-1}(W'/U) = \pi^{-1}(W),$$

and with (1) we get $W' = \pi^{-1}(W)$. If $W'/U = V/U$ then the the isomorphism theorems imply that

$$V/W' \cong (V/U)/(W'/U) = 0,$$

so $W' = V$.

iii)

W/U is a proper submodule of V/U, because W is maximal in V and so

$$(V/U)/(W/U) \cong \underbrace{V/W}_{\text{simple}} \neq 0.$$

Let $W' \subseteq V/U$ be a submodule with

$$W/U \subseteq W' \subsetneq V/U.$$

We know from i) that $\pi^{-1}(W')$ is a submodule of V, and obviously

$$W \subseteq \pi^{-1}(W/U) \subseteq \pi^{-1}(W') \subsetneq V.$$

So the maximality of W implies that $W = \pi^{-1}(W')$. So

$$W/U = \pi(W) = \pi(\pi^{-1}(W')) = W'.$$

This shows that W/U is maximal in V/U. Now **1 ii)** implies that $\pi^{-1}(W/U)$ is maximal is V. Because $W \subseteq \pi^{-1}(W/U)$ it follows that $W = \pi^{-1}(W/U)$. $\square$

(ii)

If V/U does not have any maximal submodules we get $\text{rad}(V/U) = V/U$ and

$$(U + \text{rad}(V))/U \subseteq V/U = \text{rad}(V/U)$$

If V/U does have maximal submodules, then by definition we get

$$\begin{aligned} & (U + \text{rad}(V))/U \subseteq \text{rad}(V/U) \\ \Leftrightarrow\ & (U + \text{rad}(V))/U \subseteq \bigcap_{\substack{W \subseteq V/U \\ W \text{ maximal}}} W \\ \Leftrightarrow\ & (U + \text{rad}(V))/U \subseteq W \text{ for all } W \subseteq V/U \text{ maximal.} \end{aligned}$$

Let $W \subseteq V/U$ be an arbitrary maximal submodule. Using Lemma **1 ii)** we find that $\pi^{-1}(W)$ is a maximal submodule of V, where $\pi : V \twoheadrightarrow V/U$ is the canonical projection. Obviously we have $U \subseteq \pi^{-1}(W)$, and because $\pi^{-1}(W)$ is maximal in V we also get $\text{rad}(V) \subseteq \pi^{-1}(W)$, so

$$U + \text{rad}(V) \subseteq \pi^{-1}(W),$$

and thus

$$(U + \text{rad}(V))/U = \pi(U + \text{rad}(V)) \subseteq \pi(\pi^{-1}(W)) = W.$$

(i)

From part **(ii)** we get

$$\mathrm{rad}(V)/U = (U + \mathrm{rad}(V))/U \subseteq \mathrm{rad}(V/U),$$

so all that's left to show is $\mathrm{rad}(V/U) \subseteq \mathrm{rad}(V)/U$. If V has no maximal submodules, then $\mathrm{rad}(V) = V$ and

$$\mathrm{rad}(V/U) \subseteq V/U = \mathrm{rad}(V)/U.$$

If V has maximal submodules, then let W be an arbitrary maximal submodule of V. Since $U \subseteq \mathrm{rad}(V) \subseteq W$ we know from **1 iii)** that W/U is maximal in V/U and $\pi^{-1}(W/U) = W$. So $\mathrm{rad}(V/U) \subseteq W/U$ and thus

$$\pi^{-1}(\mathrm{rad}(V/U)) \subseteq \pi^{-1}(W/U) = W.$$

Because W is arbitrary it follows that

$$\pi^{-1}(\mathrm{rad}(V/U)) \subseteq \mathrm{rad}(V)$$

and from this it follows that

$$\mathrm{rad}(V/U) = \pi(\pi^{-1}(\mathrm{rad}(V/U))) \subseteq \pi(\mathrm{rad}(V)) = \mathrm{rad}(V)/U.$$

Exercise 15:

$\neg(ii) \Rightarrow \neg(i)$

Assume that $V = U \oplus C$ is a direct sum decomposition with U simple.

Claim. C is maximal in V.

With this we find that

$$U \subseteq \mathrm{soc}(V) \text{ and } \mathrm{rad}(V) \subseteq C$$

because U is simple and C is maximal in V. Because U nonzero with $U \cap C = 0$, this implies that $\mathrm{soc}(V) \nsubseteq \mathrm{rad}(V)$.

Proof of the claim. Let $C' \subseteq V$ be a submodule with $C \subseteq C' \subseteq V$. Let $C'' := C' \cap U$. Because V is simple we know that $C'' = 0$ or $C'' = U$. If $C'' = 0$ then

$$C = C + C'' = C + (U \cap C') = (C + U) \cap C' = V \cap C' = C'.$$

If $C'' = U$ we get

$$U = C'' = C' \cap U, \text{ so } V = U \cap C \subseteq C', \text{ so } C' = V. \qquad \square$$

$(ii) \Rightarrow (i)$

Assume that V does not have a simple direct summand. If V has no maximal submodule, then $\mathrm{soc}(V) \subseteq \mathrm{rad}(V) = V$ is trivial. If V does have at least one maximal submodule it is easy to see that

$$\mathrm{soc}(V) \subseteq \mathrm{rad}(V) \Leftrightarrow S \subseteq U \text{ for all } S \subseteq V \text{ simple and all } U \subseteq V \text{ maximal.}$$

Assume that $S \subseteq V$ is simple and $U \subseteq W$ is maximal with $S \nsubseteq U$. Then $S \cap U \neq S$ and $S \subsetneq S+U$. Because S is simple this implies $S \cap U = 0$, and because U is maximal it implies $S + U = V$. So $V = S \oplus U$. This is a contradiction to the assumption that V does not have a simple direct summand. So $S \subseteq U$ for all for all $S \subseteq V$ simple and all $U \subseteq V$ maximal.

Exercise 16:

(i)

$0 \subseteq K[T]$ is the only small submodule of $(K[T], T\cdot)$: Let $0 \subsetneq U \subseteq K[T]$ be a small submodule. We know that $U = (a)$ for some $a \in K[T]$; because $K[T]$ has to be nonzero and proper we know that $\deg a \geq 1$. We find an irreducible polynomial $p \in K[T]$ with $p \nmid a$. So $(a) + (p) = (1) = K[T]$. Because (p) proper submodule of $(K[T], T\cdot)$ this shows that U is not small.
In $N(\infty)$ all nonzero submodules are small: For all nonzero submodules $V, V' \subseteq N(\infty)$ we have $N(1) \subseteq V, V'$ as a nonzero submodule, so $V \cap V' \supseteq N(1)$ is nonzero.

(ii)

Both modules are uniform:
Let $U, U' \subseteq K[T]$ be nonzero submodules. We know that $U = (a)$ and $U' = (b)$ for $a, b \in K[T] \setminus \{0\}$. Thus (ab) is a nonzero submodule of both U and U' (because $K[T]$ has no zero divisors), so $U \cap U' \supseteq (ab)$ is nonzero.
Let $V, V' \subseteq N(\infty)$ be nonzero submodules. We know that $V = N(i)$ and $V' = N(j)$ for $i, j \in \mathbb{N} \setminus \{0\}$. So $V \cap V' \supseteq N(1)$ is nonzero.

(iii)

Let $V = N(\infty)$ and $U := N(1) \subseteq V$. U is large in V: For every nonzero submodule $U' \subseteq V$ we have $N(1) \subseteq U'$, so $U \cap U' \supseteq N(1)$ is nonzero. U is also small: For every proper submodule $U'' \subseteq V$ we have $U'' = N(i)$ for some $i \in \mathbb{N}$. So $U + U'' = N(1) + N(i) = N(\max 1, i)$ is a proper submodule of V.