Foundations of representation theory
3. exercise sheet

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Exercise 9:

Exercise 10:

Exercise 11:

Let $e_1, \dots, e_n$ be the canonical basis of K^n and $\phi, \psi \in \mathrm{Hom}_K(K^n)$ be defined as

$$\phi(e_i) := \begin{cases} 0 & \text{if } i = 1 \\ e_{i-1} & \text{otherwise} \end{cases} \text{ and } \psi(e_i) := \begin{cases} e_1 & \text{if } i = n \\ e_{i+1} & \text{otherwise} \end{cases}.$$

Let $U \subseteq K^n$ be a submodule. If $U \neq 0$ we find $v \in U$ with $v \neq 0$. Since $e_1, \dots, e_n$ is a basis of K^n we find $\lambda_1, \dots, \lambda_n \in K$ with $v = \sum_{i=1}^n \lambda_i e_i$. Let $m := \max\{i \in \{1, \dots, n\} : \lambda_i \neq 0\}$; this is well-defined because $v \neq 0$, so $\lambda_i \neq 0$ for some $i \in \{1, \dots, n\}$. Because U is a submodule we find that $e_1 = \lambda_m^{-1} \phi^m(v) \in U$. So for all $i \in \{1, \dots, n\}$ we get $e_i = \psi^i(e_1) \in U$. Since $\{e_1, \dots, e_n\} \subseteq U$ it follows that $U = K^n$. So every submodule (K^n, ϕ, ψ) is either 0 or K^n, which means that (K^n, ϕ, ψ) is an n-dimensional simple 2-module.

Exercise 12:

Let K be a field with $\mathrm{char}(K) = 0$ and (V, ϕ_1, ϕ_2) is a 2-module such that $V \neq 0$ and $[\phi_1, \phi_2] = 1$. Assume that V is finite-dimensional. Because $V \neq 0$ we know that $n := \dim V \geq 1$. Let $v_1, \dots, v_n$ be a Basis von V and Φ_1 and Φ_2 the coordinate matrix of ϕ_1, ϕ_2 with respect to the basis $v_1, \dots, v_n$ respectively. Because $[\phi_1, \phi_2] = 1$ we find that $\Phi_1 \Phi_2 - \Phi_2 \Phi_1 = I_n$. It follows, that

$$0 = \mathrm{tr}\,\Phi_1 \,\mathrm{tr}\,\Phi_2 - \mathrm{tr}\,\Phi_2 \,\mathrm{tr}\,\Phi_1 = \mathrm{tr}(\Phi_1 \Phi_2 - \Phi_2 \Phi_1) = \mathrm{tr}\, I_n = n \cdot 1 \neq 0.$$

This shows that (V, ϕ_1, ϕ_2) must be finite dimensional. (We know that such a module exists, because $\left(K[T], T\cdot, \frac{d}{dT}\right)$ is one.)