Exercise Group 4, Thursday 8–10, Tutor: Friedrich Klaus

Exercises in
PDE and Functional Analysis

Exercise Sheet 2

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Exercise 1

(i)

If the sequence $(x^{(k)})_k$ would converge in ℓ_2 then it would be a Cauchy sequence. But it holds for all $l > k$ that

$$\|x^{(l)} - x^{(k)}\|_2 = \sqrt{2}\,,$$

which shows that this is not the case.

(ii)

It holds for all $l \geq k$ that

$$\|x^{(l)} - x^{(k)}\|_2 = \left(\sum_{i=k+1}^{l} \frac{1}{i^2} \right)^{1/2} \leq \left(\sum_{i=k+1}^{\infty} \frac{1}{i^2} \right)^{1/2} \longrightarrow 0$$

as $k \to \infty$ because the sum $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges. This shows that the sequence $(x^{(k)})_k$ is a Cauchy sequence. It follows from the completeness of ℓ_2 the sequence converges.

(iii)

It holds for all $l > k$ that

$$\begin{aligned} \|x^{(l)} - x^{(k)}\|_2 = \left(\sum_{i=k}^{l-1} \frac{1}{i} + \sum_{i=2^k+1}^{2^l} \frac{1}{i} \right)^{1/2} \geq \left(\sum_{i=2^k+1}^{2^l} \frac{1}{i} \right)^{1/2} &\geq \left(\sum_{i=2^k+1}^{2^{(k+1)}} \frac{1}{i} \right)^{1/2} \\ &\geq \left(2^k \cdot \frac{1}{2^{(k+1)}} \right)^{1/2} = \frac{1}{2}\,, \end{aligned}$$

which shows that the sequence $(x^{(k)})_k$ is not a Cauchy sequence. It is therefore not convergent.

Exercise 2

(i)

If (X, d) is complete then every Cauchy sequence $(x_n)_n \subseteq X$ converges in X; this holds then in particular for every Cauchy sequence $(x_n)_n \subseteq A$.

Suppose that every Cauchy sequence $(x_n)_n \subseteq A$ converges in X, and let $(y_n)_n \subseteq X$ be a Cauchy sequence. To show that $(y_n)_n$ converges in X we may replace $(y_n)_n$ by any of its subsequences and therefore assume that $d(y_l, y_k) \geq 1/k$ for all $l \geq k$. There exists for every n some $x_n \in A$ with $d(x_n, y_n) < 1/n$. The sequence $(x_n)_n$ is again a Cauchy sequence because it holds for all $l \geq k$ that

$$d(x_l, x_k) = d(x_l, y_l) + d(y_l, y_k) + d(y_k, x_k) < \frac{1}{l} + \frac{1}{l} + \frac{1}{k} < \frac{3}{k}\,.$$

It follows by assumption that the sequence $(x_n)_n$ converges in X to some $y \in X$. It follows that

$$d(y_n, y) = d(y_n, x_n) + d(x_n, y) \leq \frac{1}{n} + d(x_n, y) \longrightarrow 0$$

as $n \to \infty$, which shows that $(y_n)_n$ converges in X.

(ii)

Lemma 1. *Let A and Y be metric spaces and let $f\colon A \to Y$ be a uniformly continuous map. Let $(a_n)_n \subseteq X$ be a sequence.*

1) If $(a_n)_n$ is a Cauchy sequence then the sequence $(f(a_n))_n \subseteq Y$ is again a Cauchy sequence.

2) Suppose that $f(a_n) \to y$ for some $y \in Y$. Let $(a'_n)_n \subseteq A$ be another sequence, for which $d(a'_n, a_n) \to 0$. Then also $f(a'_n) \to y$.

Proof.

1) For every $\varepsilon > 0$ there exists some $\delta > 0$ with $d(f(a), f(a')) < \varepsilon$ for all $a, a' \in X$ with $d(a, a') < \delta$. There exists some k with $d(a_l, a_{l'}) < \delta$ for all $l, l' \geq k$, and hence $d(f(a_l), f(a_{l'})) < \varepsilon$ for all $l, l' \geq k$.

2) There exists for every $\varepsilon > 0$ some $\delta > 0$ with $d(f(a), f(a')) < \varepsilon/2$ for all $a, a' \in A$ with $d(a, a') < \delta$. There exists some N with $d(a_n, a'_n) < \delta$ and $d(f(a_n), y) < \varepsilon/2$ for all $n \geq N$. It follows that

$$d(f(a'_n), y) = d(f(a'_n), f(a_n)) + d(f(a_n), y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$, and hence $f(a'_n) \to y$. □

For $x \in X$ there exist a sequence $(a_n)_n \subseteq A$ with $a_n \to x$. It follows from Lemma 1 that the sequence $(f(a_n)_n)_n$ is again a Cauchy sequence, hence converges by the completeness of Y. Let $\tilde{f}(x) := \lim_{n\to\infty} f(a_n)$.

If $(a'_n)_n \subseteq A$ is another sequence with $x'_n \to x$ then

$$d(a_n, a'_n) \leq d(a_n, x) + d(x, a'_n) \to 0$$

as $n \to \infty$ and therefore $f(a'_n) \to \tilde{f}(x)$ by Lemma 1. This shows that $\tilde{f}(x)$ is independent of the choice of sequence $(a_n)_n$. (We wont't actually need this independence, but the author thinks that it is nevertheless important enough to warrant a proof.)

To show that $\tilde{f}$ is continuous we show that $\tilde{f}$ is uniformly continuous: So let $\varepsilon > 0$ and let $\delta > 0$ with

$$d(f(a), f(a')) < \frac{\varepsilon}{3}$$

for all $a, a' \in A$ with $d(a, a') < 3\delta$. For $x, x' \in X$ with $d(x, x') < \delta$ we show that $\tilde{f}(x, x') < \varepsilon$: We choose sequences $(a_n)_n, (a'_n)_n \subseteq A$ with $a_n \to x$ and $a'_n \to x'$. It then holds that $f(a_n) \to x$ and $f(a'_n) \to x'$ as seen above. Let n be large enough so that

$$d(a_n, x), d(a'_n, x) < \delta \quad \text{and} \quad d(f(a_n), \tilde{f}(x)), d(f(a'_n), \tilde{f}(x)) < \frac{\varepsilon}{3}\,.$$

It then follows from the inequalities $d(a_n, x), d(x, x'), d(x', a'_n) < \delta$ that $d(a_n, a'_n) < 3\delta$, and hence

$$d(f(a_n), f'(a_n)) < \frac{\varepsilon}{3}$$

by choice of δ. We altogether find that

$$\begin{aligned} d(\tilde{f}(x), \tilde{f}(x')) &\leq d(\tilde{f}(x), f(a_n)) + d(f(a_n), f(a'_n)) + d(f(a'_n), \tilde{f}(x') \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,. \end{aligned}$$

This shows that $\tilde{f}$ is indeed uniformly continuous.

We also note that the continuous extension $\tilde{f}$ is unique because A is dense in X.

(iii)

The isometry f extends by part (ii) (uniquely) to a continuous map $\tilde{f}\colon X \to Y$. For all $x, x' \in X$ there exist sequences $(a_n)_n, (a'_n)_n \subseteq A$ with $a_n \to x$ and $a'_n \to x'$. It then follows from the continuity of $\tilde{f}$ and the continuity of the metrics $d_X\colon X \times X \to \mathbb{R}$ and $d_Y : Y \times Y \to \mathbb{R}$ (with respect to the product topologies on both $X \times X$ and $Y \times Y$)

that

$$\begin{aligned} d_Y(\tilde{f}(x), \tilde{f}(x')) &= d_Y\left(\lim_{n\to\infty} (\tilde{f}(a_n), \tilde{f}(a'_n))\right) \\ &= \lim_{n\to\infty} d_Y(\tilde{f}(a_n), \tilde{f}(a'_n)) \\ &= \lim_{n\to\infty} d_Y(f(a_n), f(a'_n)) \\ &= \lim_{n\to\infty} d_A(a_n, a'_n) \\ &= \lim_{n\to\infty} d_X(a_n, a'_n) \\ &= d_X\left(\lim_{n\to\infty} (a_n, a'_n)\right) \\ &= d_X(x, x') \,. \end{aligned}$$

This shows that $\tilde{f}$ is again an isometry.

It follows that when we restrict $\tilde{f}$ to a map into its image, then $\tilde{f}$ become a bijective isometry.[1] This shows that the subspace $\tilde{f}(X) \subseteq Y$ is complete, and thus closed. But the image $\tilde{f}(X)$ is dense in Y because it contains $\tilde{f}(A) = f(A)$, which is dense in Y. It follows that already $\tilde{f}(X) = Y$. This shows altogether $\tilde{f}$ is a bijective isometry.

Exercise 3

We consider only the case that Ω is nonempty.

(i)

It follows from $\alpha < \beta$ and the boundedness of Ω that

$$C \coloneqq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{1}{\|x-y\|^{\alpha-\beta}} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \|x-y\|^{\beta-\alpha} < \infty$$

with $C > 0$. It therefore holds for every function $f \colon \Omega \to \mathbb{R}^m$ that

$$\begin{aligned} [f]_\alpha = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha} &= \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{1}{\|x-y\|^{\alpha-\beta}} \frac{\|f(x)-f(y)\|}{\|x-y\|^\beta} \\ &\leq \sup_{\substack{x,y\in\Omega \\ x\neq y}} C \frac{\|f(x)-f(y)\|}{\|x-y\|^\beta} = C \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|f(x)-f(y)\|}{\|x-y\|^\beta} = C[f]_\beta \end{aligned}$$

We may replace the constant C by $C+1$ to ensure that $C \geq 1$, while still maintaining that $[f]_\alpha \leq C[f]_\beta$ for every $f \colon \Omega \to \mathbb{R}^m$. It then further follows that

$$\|f\|_{0,\alpha} = \|f\|_\infty + [f]_\alpha \leq C\|f\|_\infty + C[f]_\beta = C\|f\|_{0,\beta} \,.$$

It follows in particular for $f \in \mathrm{C}^{0,\beta}$ that $[f]_\alpha \leq C[f]_\beta < \infty$ and hence $f \in \mathrm{C}^{0,\alpha}$.

[1] We haven't explicitely shown that $\tilde{f}$ is injective, but isometries are always injective.

(ii)

Lemma 2. *If $0 \leq \alpha \leq 1$ then the map $[0,\infty) \to \mathbb{R}$, $x \mapsto x^\alpha$ is subadditive, i.e. it holds that*

$$(x+y)^\alpha \leq x^\alpha + y^\alpha$$

for all $x, y \in [0,\infty)$.

Proof. We fix $x \in [0,\infty)$. If $x = 0$ then the claimed inequality holds, so we may assume that $x > 0$. We then consider the two maps

$$f\colon [0,\infty) \to \mathbb{R}\,, \quad y \mapsto (x+y)^\alpha$$

and

$$g\colon [0,\infty) \to \mathbb{R}\,, \quad y \mapsto x^\alpha + y^\alpha\,.$$

It holds that $f(0) = g(0)$ and and both maps are differentiable. It holds for every $x \in [0,\infty)$ that

$$f'(x) = \alpha(x+y)^{\alpha-1} \leq \alpha y^{\alpha-1} = g'(x)$$

where we use that $0 \leq \alpha \leq 1$. It therefore follows from $f(0) = g(0)$ that $f(x) \leq g(x)$ for all $x \geq 0$. $\square$

Corollary 3. *It holds for all $0 \leq \alpha < 1$ and all $x, y \geq 0$ that*

$$|x^\alpha - y^\alpha| \leq |x-y|^\alpha\,.$$

Proof. We may assume that $x \geq y$. Then

$$x^\alpha = ((x-y)+y)^\alpha \leq (x-y)^\alpha + y^\alpha$$

by Lemma 2 and therefore

$$|x^\alpha - y^\alpha| = x^\alpha - y^\alpha \leq (x-y)^\alpha = |x-y|^\alpha\,,$$

as desired. $\square$

We first consider the case $m = 1$. Fix a base point $z \in \Omega$ and consider the continuous function

$$f\colon \Omega \to \mathbb{R}\,, \quad x \mapsto \|x-z\|^\alpha\,.$$

It holds for all $x, y \in \Omega$ by Corollary 3 and the reverse triangle inequality for $\|\cdot\|$ that

$$\begin{aligned} \|f(x) - f(y)\| &= |\, \|x-z\|^\alpha - \|y-z\|^\alpha \,| \\ &\leq |\, \|x-z\| - \|y-z\| \,|^\alpha \\ &\leq |\, \|(x-z)-(y-z)\| \,|^\alpha \\ &\leq \|x-y\|^\alpha\,. \end{aligned}$$

It then follows that $f \in \mathrm{C}^{0,\alpha}(\Omega)$ because

$$[f]_\alpha = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha} \leq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|x-y\|^\alpha}{\|x-y\|^\alpha} = 1\,.$$

But $f \notin \mathrm{C}^{0,\beta}(\Omega)$ because

$$\begin{aligned} [f]_\beta = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|f(x)-f(y)\|}{\|x-y\|^\beta} \underset{y=z}{\geq} \sup_{\substack{x\in\Omega \\ x\neq z}} \frac{\|f(x)-f(z)\|}{\|x-z\|^\beta} &= \sup_{\substack{x\in\Omega \\ x\neq z}} \frac{\|x-z\|^\alpha}{\|x-z\|^\beta} \\ &= \sup_{\substack{x\in\Omega \\ x\neq z}} \frac{1}{\|x-z\|^{\beta-\alpha}} = \infty\,, \end{aligned}$$

where we have used that Ω is open and thus contains points arbitrarily close to z (but distinct to z).

For $m \geq 1$ we can replace f by the function

$$\Omega \to \mathbb{R}^m\,, \quad x \mapsto \begin{pmatrix} \|x-z\|^\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and all of the above calculations stay true.

(iii)

Every Hölder-continuous function is uniformly continuous, so it suffices to construct a continuous function $f\colon \Omega \to \mathbb{R}^n$ which is not uniformly continuous. If $z \in \partial\Omega$ is a boundary point of Ω then the function

$$f\colon \Omega \to \mathbb{R}\,, \quad z \mapsto \frac{1}{\|x-z\|}$$

should do the trick, but I currently don't have the time to work out the details.

Exercise 4

(i)

Suppose there exists a countable dense subset $D \subseteq X$. Then there exists for every $a \in A$ some $x_a \in D$ with $d(x_a, a) < \delta/2$. It follows from D being countable but A being uncountable that there exist $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $x_{a_1} = x_{a_2} =: x$. It follows that

$$d(a_1,a_2) \leq d(a_1,x) + d(x,a_2) < \frac{\delta}{2} + \frac{\delta}{2} = \delta\,,$$

which contradicts the choice of A.

(ii)

For every subset $S \subseteq \mathbb{Z}$ there exists a continuous bounded function $f_S \colon \mathbb{R} \to \mathbb{R}$ with

$$f_S(n) = \begin{cases} 1 & \text{if } n \in S\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(One can take for every $n \in S$ a small hat of height 1 with peak at the point n and support in $[n - 1/2, n + 1/2]$, and then connects these hats by the zero function.) It then holds for any two distinct subsets $S, T \subseteq \mathbb{Z}$ that some $n \in \mathbb{Z}$ is contained in precisely one of the sets S and T; it then follows that $|f_S(n) - f_T(n)| = 1$. This shows that $\|f_S - f_T\| \geq 1$ for all $S, T \subseteq \mathcal{P}(\mathbb{Z})$ with $S \neq T$ (where $\mathcal{P}(\mathbb{Z})$ denotes the power set of $\mathbb{Z}$). It follows froem the uncountability of the power set $\mathcal{P}(\mathbb{Z})$ and part (i) of the exercise that $\mathrm{C}^0_b(\mathbb{R})$ is not seperable.

(iii)

If $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ are functions for which the limits $\lim_{x\to\infty} f_1(x)$ and $\lim_{x\to\infty} f_2(x)$ exist, then it holds for every $\alpha \in \mathbb{R}$ that the limit $\lim_{x\to\infty}(\alpha f_1 + f_2)(x)$ also exists, and is given by

$$\lim_{x\to\infty}(\alpha f_1 + f_2)(x) = \alpha \left(\lim_{x\to\infty} f_1(x)\right) + \left(\lim_{x\to\infty} f_2(x)\right)\,.$$

The analogous statement for $x \to -\infty$ also holds. It follows that for all functions $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ with $\lim_{|x|\to\infty} f(x) = 0$ and all $\alpha \in \mathbb{R}$, the function $\alpha f_1 + f_2$ again satisfies $\lim_{|x|\to\infty}(\alpha f_1 + f_2)(x) = 0$. It also holds that $0 \in \mathrm{C}^0_0(\mathbb{R})$. This shows altogether that $\mathrm{C}^0_0(\mathbb{R})$ is a linear subspace of $\mathrm{C}^0_b(\mathbb{R})$.

Let $f \in \mathrm{C}^0_b(\mathbb{R})$ be in the closure of $\mathrm{C}^0_0(\mathbb{R})$. Then there exist for every $\varepsilon > 0$ some $g \in \mathrm{C}^0_0(\mathbb{R})$ with $\|f - g\| < \varepsilon$, and hence

$$|f(x)| \leq |g(x)| + |f(x) - g(x)| \leq |g(x)| + \varepsilon$$

for every $x \in \mathbb{R}$. It follows that

$$\limsup_{x\to\infty} |f(x)| \leq \varepsilon\,.$$

This shows that $\limsup_{x\to\infty}|f(x)| = 0$, which in turn shows that $\lim_{x\to\infty} f(x) = 0$. It can be shown in the same way that $\lim_{x\to-\infty} f(x) = 0$. This shows together that $\lim_{|x|\to\infty} f(x) = 0$, and hence that $\mathrm{C}^0_0(\mathbb{R})$ is closed in $\mathrm{C}^0_b(\mathbb{R})$.

It remains to show that $\mathrm{C}^0_0(\mathbb{R})$ is separable. We know from the lecture that $\mathrm{C}^0([-n,n])$ is separable for every $n \in \mathbb{N}$ (see Theorem 2.5 in the lecture notes). Let $B_n \subseteq \mathrm{C}^0([-n,n])$ be a countable dense subset. We extend every $g \in B_n$ to a function $\hat{g} \in \mathrm{C}^0_0(\mathbb{R})$ by letting $\hat{g}$ tend linearly to 0 on the intervals $[-n-1, -n]$ and $[n, n+1]$ and setting $\hat{g} \equiv 0$

outside of $[-n-1, n+1]$. This means explicitely that

$$\hat{g}(x) = \begin{cases} 0 & \text{if } x \leq -n-1\,, \\ g(-n)\cdot(x+n+1) & \text{if } -n-1 \leq x \leq -n\,, \\ g(x) & \text{if } -n \leq x \leq n\,, \\ g(n)\cdot(n+1-x) & \text{if } n \leq x \leq n+1\,, \\ 0 & \text{if } x \geq n+1\,. \end{cases}\,.$$

Let $\hat{B}_n \coloneqq \{\hat{g} \mid g \in B_n\}$ and set $\hat{B} \coloneqq \bigcup_{n\geq 0} \hat{B}_n$. Then $\hat{B}$ is a countable subset of $\mathrm{C}^0_0(\mathbb{R})$, and we claim that it is dense:

Let $f \in \mathrm{C}^0_0(\mathbb{R})$ and let $\varepsilon > 0$. There exist some $n \in \mathbb{N}$ with $|f(x)| \leq \varepsilon$ whenever $|x| \geq n$. It holds in particular that

$$|f(n)|, |f(-n)| \leq \varepsilon \tag{1}$$

There exist by choice of B_n some $g \in B_n$ with $\|f-g\|_{\mathrm{C}^0([-n,n])} \leq \varepsilon$. It follows in particular that $|f(x)-g(x)| \leq \varepsilon$ for $x = n, -n$, and hence

$$|g(n)|, |g(-n)| \leq 2\varepsilon$$

by (1). It then follows from the construction of the extension $\hat{g}$ that $|g(x)| \leq 2\varepsilon$ for all $|x| \geq n$. It follow from the triangle inequality that

$$|f(x)-g(x)| \leq |f(x)| + |g(x)| \leq 2\varepsilon + \varepsilon = 3\varepsilon$$

for all $|x| \geq n$. Together with $\|f-g\|_{\mathrm{C}^0([-n,n])} \leq \varepsilon$ this shows that $\|f-g\|_{\mathrm{C}^0_0(\mathbb{R})} \leq 3\varepsilon$.