Exercises in PDE and Functional Analysis

Exercise Sheet 8

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Exercise 2

(i)

We may assume that $c \geq 0$, because otherwise we can replace c by $-c$ and x by $-x$; we may also assume that $c \neq 0$ and hence $c > 0$. Suppose that not $\|x - c\mathbf{1}\|_\infty \geq c$. Then

$$\sup_{n \in \mathbb{N}} |x_n - c| = \|x - c\mathbf{1}\|_\infty < c$$

and hence there exist $0 < c' < c$ with $|x_n - c| < c'$ for every $n \in \mathbb{N}$. It follows from $x \in W$ that there exists some $y \in \ell^\infty$ with $x = Sy - y$, and hence $x_n = y_{n+1} - y_n$ for every $n \in \mathbb{N}$. We now have for every $n \in \mathbb{N}$ that

$$|y_{n+1} - y_n - c| < c'$$

and hence

$$c - c' < y_{n+1} - y_n < c + c' \,.$$

It follows from $c' < c$ that $c - c' > 0$ for every $n \in \mathbb{N}$, hence we find that the sequence y is strictly increasing. But this contradicts y being bounded.

(ii)

It follows from part (i) that $\mathbf{1} \notin W$ because for $x = \mathbf{1}$ and $c = -1$ we get

$$\|x - c\mathbf{1}\|_\infty = \|0\|_\infty = 0 < 1 = |c| \,.$$

The sum $Y \coloneqq W + \mathbb{R}\mathbf{1}$ is therefore direct, i.e. it holds that $Y = W \oplus \mathbb{R}\mathbf{1}$. We can hence define a map

$$\varphi \colon Y \to \mathbb{R} \,, \quad x + c\mathbf{1} \mapsto c$$

where $x \in W$ and $c \in \mathbb{R}$. The map φ is linear, and it follows from part (i) that

$$\varphi(y) = \varphi(x + c\mathbf{1}) = c \leq |c| \leq \|x + c\mathbf{1}\|_\infty = \|y\|_\infty$$

for every $y \in Y$ with $y = x + c\mathbf{1}$, where $x \in W$ and $c \in \mathbb{R}$. Hence $\varphi \in Y'$ with $\|\varphi\| \leq 1$. By considering $y = \mathbf{1}$ we see that actually $\|\varphi\| = 1$ because $\varphi(\mathbf{1}) = 1$ and $\|\mathbf{1}\|_\infty = 1$.

It follows from a version of the Hahn–Banch theorem (Theorem 6.4 from the lecture) that there exist an extension $\Phi \in (\ell^\infty)'$ of φ with $\|\Phi\| = \|\varphi\| = 1$. It again holds that $\Phi(\mathbf{1}) = \varphi(\mathbf{1}) = 1$. The functional Φ is shift invariant: We have for every $x \in \ell^\infty$ that

$$\Phi(Sx) = \Phi(x) \iff \Phi(Sx) - \Phi(x) = 0 \iff \Phi(Sx - x) = 0\,,$$

and the last condition holds because $Sx - x \in W$ and hence

$$\Phi(Sx - x) = \varphi(Sx - x) = 0$$

by construction of Φ via φ, and the definition of φ.

That Φ is positive, i.e. that $\Phi(x) \geq 0$ for $x \geq 0$, follows from the upcoming part (iii) because $\liminf_{n\to\infty} x_n \geq 0$.

(iii)

Suppose first that $x \geq 0$. We then have for every $N \in \mathbb{N}$ that

$$\Phi(x) = \Phi(S^N x) \leq |\Phi(S^N x)| \leq \|S^N x\|_\infty = \sup_{n\in\mathbb{N}} |x_{n+N}| = \sup_{n\geq N} |x_n| = \sup_{n\geq N} x_n\,.$$

By taking the limit $n \to \infty$ we find that

$$\Phi(x) \leq \limsup_{n\to\infty} x_n\,.$$

If now $x \in \ell^\infty$ then we get for $c \coloneqq \inf_{n\in\mathbb{N}} x_n \in \mathbb{R}$ that $x - c\mathbf{1} \geq 0$ and hence

$$\Phi(x) - c = \Phi(x - c\mathbf{1}) \leq \limsup_{n\to\infty}(x_n - c) = \left(\limsup_{n\to\infty} x_n\right) - c\,,$$

which again shows that

$$\Phi(x) \leq \limsup_{n\to\infty} x_n\,.$$

We now also find for every $x \in \ell^\infty$ that

$$-\Phi(x) = \Phi(-x) \leq \limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n\,,$$

and hence

$$\Phi(x) \geq \liminf_{n\to\infty} x_n\,.$$

Exercise 3

We only consider the case $X \neq 0$.

(i)

It holds for all $x, y \in X$ that

$$(Tx) + (Ty) = \left(\lim_{n\to\infty} T_n x\right) + \left(\lim_{n\to\infty} T_n y\right)$$
$$= \lim_{n\to\infty} [(T_n x) + (T_n y)] = \lim_{n\to\infty} T_n(x+y) = T(x+y)\,,$$

and for all $x \in X$ and $\lambda \in \mathbb{K}$ that

$$T(\lambda x) = \lim_{n\to\infty} T_n(\lambda x) = \lim_{n\to\infty} \lambda T_n x = \lambda \lim_{n\to\infty} T_n x = \lambda T x\,.$$

This shows that T is linear.

We have for every $x \in X$ with $\|x\| = 1$ by the continuity of the norm (of Y) that

$$\|Tx\| = \left\|\lim_{n\to\infty} T_n x\right\| = \lim_{n\to\infty} \|T_n x\| = \liminf_{n\to\infty} \|T_n x\| \leq \liminf_{n\to\infty} \|T_n\|\,.$$

That $\liminf_{n\to\infty} \|T_n\| < \infty$ follows from the Banach–Steinhaus theorem (Theorem 7.3 in the lecture). (That the theorem can be applied, i.e. that $\sup_{n\in\mathbb{N}} \|T_n x\| < \infty$ for every $x \in X$, follows from the convergence of the sequence $(T_n x)_n$.) It follows in particular that T is bounded, and hence that $T \in \mathcal{L}(X, Y)$.

(ii)

For every $n \in \mathbb{N}$ let $T_n \colon \ell^1 \to \ell^1$ be the linear map that multiplies the n-th position with 2, i.e. that is given by

$$(T_n x)_m = \begin{cases} 2x_m & \text{if } m = n\,, \\ x_m & \text{if } m \neq n \end{cases}$$

for every $x \in \ell^1$. Then

$$\|T_n x\|_1 \leq \sum_{m\in\mathbb{N}} 2|x_m| = 2\|x\|_1$$

for every $x \in \ell^1$ and therefore $\|T_n\| \leq 2$ for every $n \in \mathbb{N}$. Hence $T_n \in \mathcal{L}(X, Y)$ for every $n \in \mathbb{N}$.

We have for every $x \in \ell^1$ that

$$\|T_n x - x\|_1 = |x_n| \to 0$$

as $n \to \infty$, and hence that $T_n x \to x$. But the sequence $(T_n)_n$ does not converge in the operator norm: We have for all $n > m$ that

$$\|(T_n - T_m)e^{(n)}\|_1 = \|2e^{(n)} - e^{(n)}\|_1 = \|e^{(n)}\|_1 = 1\,,$$

and hence $\|T_n - T_m\| \geq 1$. The sequence $(T_n)_n$ is therefore not a Cauchy sequence, and hence not convergent.

Exercise 4

(i)

We need to show that the family of linear functionals

$$\mathcal{T} \coloneqq \{T(y) \mid y \in X, \|y\| \leq 1\}$$

is bounded. We have for every $x \in X$ that $T(y)x = T(x)y$ by assumption, and therefore

$$\sup_{S \in \mathcal{T}} |Sx| = \sup_{\substack{y \in X \\ \|y\| \leq 1}} |(Ty)x| = \sup_{\substack{y \in X \\ \|y\| \leq 1}} |(Tx)y| \leq \|Tx\| < \infty\,.$$

It hence follows from the Banach–Steinhaus theorem that T is bounded.

(ii)

By the closed graph theorem we need to show for every sequence $(x_n)_n$ in X and all $x \in X$ and $S \in X'$ with both $x_n \to x$ and $Tx_n \to S$ we already have $S = Tx$. For this we may replace x_n by $x_n - x$ and S by $S - Tx$ to additionally assume that $x_n \to 0$. We hence need to show that $S = 0$, whence that $Sy = 0$ for every $y \in X$.

We have for every $\lambda \in \mathbb{R}$ that

$$\begin{aligned} 0 &\leq \langle x_n + \lambda y, T(x_n + \lambda y) \rangle \\ &= \langle x_n, Tx_n \rangle + \lambda(\langle x_n, Ty \rangle + \langle y, Tx_n \rangle) + \lambda^2 \langle y, Ty \rangle\,. \end{aligned}$$

It follows from $x_n \to 0$ that $\langle x_n, Tx_n \rangle \to 0$ and $\langle x_n, Ty \rangle \to 0$, and we also have that $\langle y, Tx_n \rangle \to \langle y, S \rangle$. By taking the limit $n \to \infty$ we therefore find that

$$0 \leq \lambda \langle y, S \rangle + \lambda^2 \langle y, Ty \rangle = \lambda \cdot (\langle y, S \rangle + \lambda \langle y, Ty \rangle)$$

for every $\lambda \in \mathbb{R}$. This shows that

$$\begin{cases} \langle y, S \rangle + \lambda \langle y, Ty \rangle \geq 0 & \text{if } \lambda > 0\,, \\ \langle y, S \rangle + \lambda \langle y, Ty \rangle \leq 0 & \text{if } \lambda < 0\,. \end{cases}$$

It follows for $\lambda = 0$ by continuity that $\langle y, S \rangle \geq 0$ and $\langle y, S \rangle \leq 0$, and hence $\langle y, S \rangle = 0$.