Exercises in PDE and Functional Analysis

Exercise Sheet 6

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Exercise 1

For every $n \in \mathbb{N}$ let $e^{(n)} \in \ell^1$ be the sequence with $e^{(n)}_n = 1$ and $e^{(n)}_i = 0$ for all $i \neq n$.

(i)

We may regard A as a map $A\colon \ell^\infty \to \mathbb{R}^\mathbb{N}$, which is then linear. We can then compute

$$\|a\|_\infty = \sup_{n \in \mathbb{N}} |a_n| = \sup_{n \in \mathbb{N}} \|Ae^{(n)}\|_1 \leq \|A\| \,.$$

We also have for every $x \in \ell^1$ that

$$\|Ax\|_1 = \sum_{n \in \mathbb{N}} |a_n x_n| = \sum_{n \in \mathbb{N}} |a_n||x_n| \leq \sum_{n \in \mathbb{N}} \|a\|_\infty |x_n| = \|a\|_\infty \sum_{n \in \mathbb{N}} |x_n| = \|a\|_\infty \|x\|_1 \,.$$

This shows that $\|A\| = \|a\|_\infty$.

Suppose now that $a \in \ell^\infty$, so that $\|A\| = \|a\|_\infty < \infty$. We then find for every $x \in \ell^1$ that $\|Ax\|_1 \leq \|A\|\|x\|_1 < \infty$, and hence that A restricts to a linear map $\ell^1 \to \ell^1$. We then have $A \in \mathcal{L}(\ell^1)$ because $\|A\| < \infty$.

Suppose now that $A \in \mathcal{L}(\ell^1)$. Then in particular $\|a\|_\infty = \|A\| < \infty$ and hence $a \in \ell^1$.

(ii)

We have that

$$\begin{aligned} \mathcal{N}(A) &= \{(x_n)_n \in \ell^1 \mid (a_n x_n)_n = 0\} \\ &= \{(x_n)_n \in \ell^1 \mid x_n = 0 \text{ for every } n \in \mathbb{N} \text{ with } a_n \neq 0\} \,. \end{aligned}$$

Hence $\mathcal{N}(A) = 0$ if and only if $a_n \neq 0$ for every $n \in \mathbb{N}$. We now observe the following:

Claim 1. *Let $y = (y_n)_n \in \ell^1$ be a sequence with support $S \coloneqq \{n \in \mathbb{N} \mid y_n \neq 0\}$. If $\inf_{n \in S} |a_n| > 0$ then the sequence y is contained in the range $A(\ell^1)$.*

Proof. It holds in particular that $a_n \neq 0$ for every $n \in S$. The sequence $x = (x_n)_n$ with

$$x_n \coloneqq \begin{cases} y_n/a_n & \text{if } n \in S\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is therefore well-defined. This sequence satisfies $Ax = y$, and it is again contained in ℓ^1: The constant

$$C \coloneqq \frac{1}{\inf_{n \in S} |a_n|} = \sup_{n \in S} \frac{1}{|a_n|}$$

is by assumption well-defined, and we have that

$$\|x\|_1 = \sum_{n \in \mathbb{N}} |x_n| = \sum_{n \in S} \frac{|y_n|}{|a_n|} \leq C \sum_{n \in S} |y_n| = C \sum_{n \in \mathbb{N}} |y_n| = C \|y\|_1\,.$$

It hence follows from $y \in \ell^1$ that also $x \in \ell^1$. ☐

Suppose now that $y \in \ell^1$ and that $\varepsilon > 0$. Then there exist a sequence $y' \in \ell^1$ with finite support such that $\|y - y'\|_1 < \varepsilon$. (The sequence y' results from y by cutting this sequence off after sufficiently many terms.) It follows from the above claim that y' is contained in the range $A(\ell^1)$, because $a_n \neq 0$ for every $n \in \mathbb{N}$ and y' has only finite support. This shows that $A(\ell^1)$ is dense in ℓ^1.

(iii)

If $\inf_{n \in \mathbb{N}} |a_n| > 0$ then it follows from Claim 1 that $A(\ell^1) = \ell^1$. If on the other hand $\inf_{n \in \mathbb{N}} |a_n| = 0$ then we distinguish between two cases:

- If $a_n = 0$ for some $n \in \mathbb{N}$ then $e^{(n)} \notin A(\ell^1)$ and hence $A(\ell^1) = \ell^1$.
- Suppose otherwise that $a_n \neq 0$ for every $n \in \mathbb{N}$. Then there exist a subsequence $(a_{n(k)})_k$ with $|a_{n(k)}| \leq 1/k$ for every k. Let $y = (y_n)_n \in \ell^1$ be the sequence with $y_{n(k)} = 1/k^2$ for every k and $y_n = 0$ otherwise. Then y is not contained in the range $A(\ell^1)$:

  There would otherwise exist a sequence $x = (x_n)_n \in \ell^1$ with $Ax = y$. It would then follow for every k that

  $$\frac{1}{k^2} = |y_{n(k)}| = |a_{n(k)}||x_{n(k)}| \leq \frac{1}{k}|x_{n(k)}|\,,$$

  and hence $|x_{n(k)}| \geq 1/k$. But then $x \notin \ell^1$, a contradition.

(iv)

Suppose that $A \in \mathcal{K}(\ell^1)$. Then $C \coloneqq \overline{A(\mathrm{B}(0,2))}$ is compact. But suppose that also $|a_n| \nrightarrow 0$. Then there exists for some $\varepsilon > 0$ a subsequence $(|a_{n(k)}|)_k$ with $|a_{n(k)}| > \varepsilon$ for every k. Then

$$\|Ae_{n(k)} - Ae_{n(k')}\| = |a_{n(k)}| + |a_{n(k')}| > 2\varepsilon$$

for all $k' > k$. This shows that the sequence $(Ae_{n(k)})_k$ in C has not subsequence that is Cauchy, and hence no subsequence that is convergent. But this contradicts the compactness of C.

Suppose on the other hand that $a_n \to 0$ let $\varepsilon > 0$. To show that $\overline{A(\mathrm{B}(0,1))}$ is compact it sufficies to show that $A(\mathrm{B}(0,1))$ is precompact because ℓ^1 is complete. So let $\varepsilon > 0$. We need to show for $B \coloneqq \mathrm{B}(1,0)$ that $A(B)$ is covered by finitely many ε-balls.

It follows from $a_n \to 0$ that there exist some N with $|a_n| < \varepsilon/4$ for all $n > N$. Let $C \coloneqq \max(|a_1|, \dots, |a_N|, 1)$. It then holds for all $x, y \in B$ that

$$\begin{aligned}
\|Ax - Ay\|_1 &= \sum_{n=1}^{\infty} |a_n||x_n - y_n| \\
&= \sum_{n=1}^{N} |a_n||x_n - y_n| + \sum_{n=N+1}^{\infty} |a_n||x_n - y_n| \\
&\leq \sum_{n=1}^{N} C|x_n - y_n| + \sum_{n=N+1}^{\infty} \frac{\varepsilon}{4}|x_n - y_n| \\
&\leq C\sum_{n=1}^{N} |x_n - y_n| + \frac{\varepsilon}{4}\|x - y\|_1 \\
&\leq C\sum_{n=1}^{N} |x_n - y_n| + \frac{\varepsilon}{4}(\|x\|_1 + \|y\|_1) \\
&\leq C\sum_{n=1}^{N} |x_n - y_n| + \frac{\varepsilon}{2}\,.
\end{aligned} \tag{1}$$

The normed vector space $(\mathbb{R}^N, \|\cdot\|_1)$ is finite-dimesional and complete, so its unit ball $B' \coloneqq \mathrm{B}(0,1)$ us precompact. Hence there exist finitely many $x'_1, \dots, x'_n \in B'$ such that for every $y' \in B'$ there exists some index i with $\|x'_i - y'\| < \varepsilon/(2C)$. By padding the vectors $x'_1, \dots, x'_r$ with zeroes we get sequences $x_1, \dots, x_r \in \ell^1$ with $\|x_i\|_1 = \|x'_i\|_1 < 1$ and hence $x_i \in B$.

For a sequence $y \in B$ the truncated vector $y' = (y_1, \dots, y_N) \in \mathbb{R}$ is contained in the unit ball B' because $\|y'\|_1 \leq \|y\|_1 \leq 1$. Hence there exist some index i with $\|x'_i - y'\| < \varepsilon/(2C)$. The calculation (1) then shows that

$$\|Ax_i - Ay\|_1 \leq C\|x'_i - y'\|_1 + \frac{\varepsilon}{2} < C\frac{\varepsilon}{2C} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Hence we find that $A(B)$ is covered by the finitely many open balls $\mathrm{B}(Ax_i, \varepsilon)$.

Exercise 2

We note that the closed sets $\overline{U}_1$ and $\overline{U}_2$ are again bounded, and hence compact. Their product $\overline{U}_1 \times \overline{U}_2$ is therefore also compact.

(i)

The integral $(T_K f)(x) = \int_{U_2} K(x,y) f(y) \,\mathrm{d}y$ is well-defined for every $f \in \mathrm{C}(\overline{U}_2)$ and $x \in \overline{U}_1$: The function $K(x,-)f \colon \overline{U}_2 \to \mathbb{R}$ is continuous, and $\overline{U}_2$ is compact. The function $K(x,-)f$ is therefore integrable on $\overline{U}_2$, and hence also on U_2.

For $f \in \mathrm{C}(\overline{U}_2)$ the function $T_K f \colon \overline{U}_1 \to \mathbb{R}$ is again continuous: We have for all $x, x' \in \overline{U}_1$ that

$$\begin{aligned} |(T_K f)(x) - (T_K f)(x'))| &= \left| \int_{U_2} K(x,y) f(y) \,\mathrm{d}y - \int_{U_2} K(x',y) f(y) \,\mathrm{d}y \right| \\ &= \left| \int_{U_2} [K(x,y) - K(x',y)] f(y) \,\mathrm{d}y \right| \\ &\leq \int_{U_2} |K(x,y) - K(x',y)| |f(y)| \,\mathrm{d}y \,. \end{aligned} \tag{2}$$

The function K is uniformly continuous because $\overline{U}_1 \times \overline{U}_2$ is compact. Hence there exist for every $\varepsilon > 0$ some $\delta > 0$ with $|K(x,y) - K(x',y)| < \varepsilon$ whenever $\|x - x'\| < \delta$. It then follows from (2) whenever $\|x - x'\| < \delta$ that

$$|(T_K f)(x) - (T_K f)(x')| \leq \int_{U_2} \varepsilon |f(y)| \,\mathrm{d}y = \varepsilon \int_{U_2} |f(y)| \,\mathrm{d}y = C\varepsilon \,,$$

for the constant $C := \int_{U_2} |f(y)| \,\mathrm{d}y$. It holds that

$$C = \int_{U_2} |f(y)| \,\mathrm{d}y \leq \int_{\overline{U}_2} |f(y)| \,\mathrm{d}y < \infty$$

because $\overline{U}_2$ is compact and f is continuous on $\overline{U}_2$. This shows that $T_K f$ is continuous.

We have thus shown that the map $T_K \colon \mathrm{C}(\overline{U}_2) \to \mathrm{C}(\overline{U}_1)$ is well-defined. The linearity of T_K follows with the linearity of the integral.

It remains to show T_K is continuous. Let

$$C := \sup_{x \in U_1} \int_{U_2} |K(x,y)| \,\mathrm{d}y \,.$$

This constant is finite: We have that $\int_{U_2} |K(x,y)| \,\mathrm{d}y = (T_{|K|} 1)(x)$ where 1 denotes the constant 1-function, and we have seen above that $T_{|K|} 1$ is continuous on $\overline{U}_2$. Hence

$$C = \sup_{x \in U_1} \int_{U_2} |K(x,y)| \,\mathrm{d}y \leq \sup_{x \in U_1} (T_{|K|} 1)(x) \leq \sup_{x \in \overline{U}_1} (T_{|K|} 1)(x) < \infty$$

because $\overline{U}_1$ is compact. We now find that

$$\begin{aligned} |(T_K f)(x)| &= \left| \int_{U_2} K(x,y) f(y) \, \mathrm{d}y \right| = \int_{U_2} |K(x,y)| |f(y)| \, \mathrm{d}y \\ &\leq \int_{U_2} |K(x,y)| \|f\|_\infty \, \mathrm{d}y = \int_{U_2} |K(x,y)| \, \mathrm{d}y \, \|f\|_\infty \leq C \|f\|_\infty \,. \end{aligned}$$

Hence $\|T_K\| \leq C < \infty$.

(ii)

We have shown above that $\|T_K\| \leq \sup_{x \in U_1} \int_{U_2} |K(x,y)| \, \mathrm{d}y$. To show on the other hand that $\sup_{x \in U_1} \int_{U_2} |K(x,y)| \, \mathrm{d}y \leq \|T_K\|$ we need to show that

$$\int_{U_2} |K(x,y)| \, \mathrm{d}y \leq \|T_K\|$$

for every $x \in U_1$. For this we fix $x \in U_1$.

We would like to find a suitable test function $f \in \mathrm{C}(\overline{U}_2)$ with both $\|f\|_\infty \leq 1$ and $\int_{U_2} |K(x,y)| \, \mathrm{d}y \leq |(T_K f)(x)|$ because then $|(T_K f)(x)| \leq \|T_K f\|_\infty \leq \|T_K\|$ and hence $\int_{U_2} |K(x,y)| \, \mathrm{d}y \leq \|T_K\|$. We won't construct such a test function f itself, but instead a sequence $(f_n)_n$ of test functions which play the role of f.

We start with the function $g \colon \overline{U}_2 \to \mathbb{R}$ given by

$$g(y) \coloneqq \operatorname{sign}(K(x,y)) = \begin{cases} -1 & \text{if } K(x,y) < 0 \,, \\ 0 & \text{if } K(x,y) = 0 \,, \\ 1 & \text{if } K(x,y) > 0 \,. \end{cases}$$

This function satisfies both $\|g\|_\infty \leq 1$ and

$$(T_K g)(x) = \int_{U_2} |K(x,y)| \, \mathrm{d}y \,, \tag{3}$$

but g will in general not be continuous. We will therefore approximate g by continuous functions:

The function g is measurable and bounded, and $\overline{U}_1$ is bounded. Hence $g \in \mathrm{L}^2(\overline{U}_1)$. We similarly have that $K(x,-) \in \mathrm{L}^2(\overline{U}_2)$ because $K(x,-)$ is continuous and hence bounded on the compact set $\overline{U}_2$. We note that (3) can be expressed as

$$\langle K(x,-), g \rangle_{\mathrm{L}^2(U_2)} = \int_{U_2} |K(x,y)| \, \mathrm{d}y \,.$$

We can now approximate the function g in $\mathrm{L}^2(\overline{U}_2)$ by continuous (even smooth) functions, in the sense that there exists a sequence $(g_n)_n$ of continuous maps $g_n \colon \overline{U}_1 \to \mathbb{R}$ such that $g_n \to g$ in $\mathrm{L}^2(\overline{U}_2)$, i.e.

$$\|g - g_n\|_2 \to 0 \,.$$

But the functions g_n may not necessarily satisfy $\|g_n\|_\infty \leq 1$ anymore, so we have adjust them a bit: We replace g_n by the function $f_n \colon \overline{U}_2 \to \mathbb{R}$ given by

$$f_n(y) \coloneqq \begin{cases} 1 & \text{if } g_n(y) \geq 1\,, \\ g_n(y) & \text{if } g_n(y) \in [-1,1]\,, \\ -1 & \text{if } g_n(y) \leq -1\,. \end{cases}$$

We then have the following:

- It again holds that $f_n \in \mathrm{L}^2(\overline{U}_2)$ because $|f_n(y)| \leq |g_n(y)|$ for every $y \in \overline{U}_2$.
- The functions f_n are again continuous because $f_n = h \circ g_n$ for the continuous map $h \colon \mathbb{R} \to \mathbb{R}$ given by

$$h(z) \coloneqq \begin{cases} 1 & \text{if } z \geq 1\,, \\ z & \text{if } z \in [-1,1]\,, \\ -1 & \text{if } z \leq -1\,. \end{cases}$$

- It again holds that $f_n \to g$ in $\mathrm{L}^2(\overline{U}_2)$, because $|g(y) - f_n(y)| \leq |g(y) - g_n(y)|$ for every $y \in \overline{U}_2$, and hence $\|g - f_n\|_2 \leq \|g - g_n\|_2$: If $g_n(y) \in [-1,1]$ then then this holds true because $f_n(y) = g_n(y)$, and if $g_n(y) \geq 1$ or $g_n(y) \leq -1$ then this holds because $g(y) \in [-1,1]$.

We have more specifically $\|g - f_n\|_{\mathrm{L}^2(\overline{U}_2)} \to 0$ and hence also $\|g - f_n\|_{\mathrm{L}^2(U_2)} \to 0$ because $\|g - f_n\|_{\mathrm{L}^2(U_2)} \leq \|g - f_n\|_{\mathrm{L}^2(\overline{U}_2)}$. It follows that

$$\begin{aligned} |(T_K f_n)(x)| &= \left| \int_{U_2} K(x,y) f_n(y) \,\mathrm{d}y \right| = \left| \langle K(x,-), f_n \rangle_{\mathrm{L}^2(U_2)} \right| \to \left| \langle K(x,-), g \rangle_{\mathrm{L}^2(U_2)} \right| \\ &= \left| \int_{U_2} K(x,y) g(y) \,\mathrm{d}y \right| = \left| \int_{U_2} |K(x,y)| \,\mathrm{d}y \right| = \int_{U_2} |K(x,y)| \,\mathrm{d}y\,. \end{aligned}$$

But we also have that

$$|(T_K f_n)(x)| \leq \|T_K f_n\|_\infty \leq \|T_K\| \|f_n\|_\infty \leq \|T_K\|$$

because $\|f_n\|_\infty \leq 1$. It follows that $\int_{U_2} |K(x,y)| \,\mathrm{d}y \leq \|T_K\|$.