Exercises in PDE and Functional Analysis

Exercise Sheet 7

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Exercise 1

We denote as always by $e^{(k)}$ the sequence with $e^{(k)}_k = 1$ and $e^{(k)}_n = 0$ for $n \neq k$.

(i)

The case $1 < p < \infty$

We write $q \coloneqq p'$. It follows with Hölder's inequality that $J(y)$ is for every sequence $y = (y_n)_n \in \ell^q$ a well-defined linear map $J(y)\colon \ell^p \to \mathbb{R}$ with $\|J(y)\| \leq \|y\|_q$. To show that $\|J(y)\| \geq \|y\|_q$ we may assume that $\|y\|_q = 1$ and consider the sequence $x = (y_n)_n \in \ell^p$ given by $x_n = \operatorname{sign}(y_n)|y_n|^{q/p}$. Then $\|x\|_p = 1$ and hence

$$|J(y)(x)| = \sum_{n=1}^{\infty} |y_n|^{1+q/p} = \sum_{n=1}^{\infty} |y_n|^q = \|y\|_q^q = 1 = \|y\|_q \|x\|_p \,,$$

showing that $\|J\| \geq \|y\|_q$. We have shown that $J\colon \ell^q \to (\ell^p)'$ is an isometric linear embedding.

It remains to show that J is surjective, so let $\varphi \in (\ell^p)'$. We note that the linear span $\langle e^{(n)} \mid n \in \mathbb{N} \rangle$ is dense in ℓ^p. The continuous linear map φ is therefore uniquely determined by the values $y_n \coloneqq \varphi(e^{(n)})$ with $n \in \mathbb{N}$. To prove that φ is contained in the range $J(\ell^q)$ we thus need to show that the squence $y \coloneqq (y_n)_n$ is contained in ℓ^q. We show that there exists for every $N \in \mathbb{N}$ some sequence $x_N \in \ell^p$ with $\|x_N\|_p = 1$ such that $|J(y)(x)| = (\sum_{n=1}^N |y_q|^q)^{1/q}$. Then

$$\|y\|_q = \lim_{N \to \infty} \left(\sum_{n=1}^{N} |y_q|^q \right)^{1/q} = \lim_{N \to \infty} |J(y)(x_N)| \leq \lim_{n \to \infty} \|J(y)\| \|x_N\|_p = \|J(y)\|$$

and hence $\|y\|_q < \|J(y)\| < \infty$.

We fix $N \in \mathbb{N}$. We note that for $V \coloneqq (\mathbb{R}^N, \|\cdot\|_q)$ and $W \coloneqq (\mathbb{R}^N, \|\cdot\|_p)$ we find as above that $J\colon V \to W'$ is an isometric embedding. It follows that J is already an isometric isomorphism because $\dim V = N = \dim W'$. It follows for $y' \coloneqq (y_1, \dots, y_N)$ and the compactness of the unit sphere of W (because W is finite-dimensional) that there exist some $x' \in W$ with $\|x'\|_p = 1$ and $|J(y')(x')| = \|y'\|_q$. By padding the vector x' with zeroes we get a sequence $x \in \ell^p$ with

$$|J(y)(x)| = |J(y')(x')| = \|y'\|_q = \left(\sum_{n=1}^{N} |y_n|^q\right)^{1/q},$$

as desired.

The case $p = 1$, $q = \infty$

We find as before with Hölder's inequality that $J\colon \ell^\infty \to (\ell^1)'$ is a well-defined linear map with $\|J(y)\| \leq \|y\|_\infty$. We have on the other hand for $y = (y_n)_n \in \ell^\infty$ that

$$|y_n| = |J(y)(e^{(n)})| \leq \|J(y)\|\|e^{(n)}\|_1 = \|J(y)\|$$

for every $n \in \mathbb{N}$, and hence that $\|y\|_\infty \leq \|J(y)\|$. This shows that J is an isometric embedding.

To show that J is surjective we again pick $\varphi \in (\ell^1)'$ and need to show that the sequence $y = (y_n)_n$ with $y_n \coloneqq \varphi(e^{(n)})_{n\in\mathbb{N}}$ for every $n \in \mathbb{N}$ is contained in ℓ^∞. This holds because

$$|y_n| = |\varphi(e^{(n)})| \leq \|\varphi\|\|e^{(n)}\|_1 = \|\varphi\|$$

for every $n \in \mathbb{N}$, and hence $\|y\|_\infty < \|\varphi\| < \infty$.

(ii)

We assume that c_0 is to be endowed with the norm $\|\cdot\|_\infty$, i.e. that c_0 is a subspace of ℓ^∞. We can then proceed as before:

It follows with Hölder's inequality that $J\colon \ell^1 \to c_0'$ is a well-defined linear map with $\|J(y)\| \leq \|y\|_1$ for every $y \in \ell^1$. To see that also $\|J(y)\| \geq \|y\|_1$ we can consider for every $N \in \mathbb{N}$ the sequence $x^{(N)} = (x_n^{(N)})_n \in c_0$ with

$$x_n^{(N)} = \begin{cases} \operatorname{sign}(y_n) & \text{if } n \leq N\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This sequence satisfies $\|x^{(N)}\|_\infty \leq 1$ and

$$\sum_{n=1}^{N} |y_n| = |J(y)(x^{(N)})| \leq \|J(y)\|\|x^{(N)}\| \leq \|J(y)\|$$

for every $N \in \mathbb{N}$. Hence $\|y\|_1 = \sum_{n=1}^{\infty} |y_n| \leq \|J(y)\|$. This shows that J is an isometric embedding.

To show that J is surjective we note that the linear span $\langle e^{(n)} \mid n \in \mathbb{N} \rangle$ is dense in c_0.[1] (Every sequence $x \in c_0$ can be approximated by a finite sequence by truncation.) To show that $\varphi \in c_0'$ is contained in the range $J(\ell^1)$ is therefore sufficies (by the same reasoning as before) to show that the sequence $y = (y_n)_n$ with $y_n = \varphi(e^{(n)})$ is contained in ℓ^1. This is the case because we have for every $N \in \mathbb{N}$ for the sequences $x^{(N)} \in c_0$ as above that

$$\sum_{n=1}^{N} |y_n| = \left| \sum_{n=1}^{\infty} \varphi(e^{(n)}) x_n^{(N)} \right| = |\varphi(x^{(N)})| \leq \|\varphi\| \|x^{(N)}\|_\infty \leq \|\varphi\| ,$$

and hence $\|y\|_1 = \sum_{n=1}^{\infty} |y_n| \leq \|\varphi\| < \infty$.

Exercise 4

(i)

For every $x \in X$ let

$$I_x \coloneqq \{ \alpha > 0 \mid a^{-1} x \in C \} .$$

It follows from C being convex and containing 0 that for every $\alpha \in I_x$ also $[\alpha, \infty) \subseteq I_x$. This shows that I_x is an interval, namely a subinterval of $(0, \infty)$ that is unbounded from above.

It follows from 0 being contained in the interior of C that there exist some $\varepsilon > 0$ with $\overline{\mathrm{B}}(0, \varepsilon) \subseteq C$. It then holds for every nonzero $x \in X$ that $(\varepsilon / \|x\|) x \in C$ and hence $\|x\| / \varepsilon \in X$. It follows for $M \coloneqq 1/\varepsilon$ that

$$p(x) = \inf I_x \leq \frac{1}{\varepsilon} \|x\| = M \|x\| .$$

For on the other hand $x = 0$ we have that $I_x = (0, \infty)$ and hence $p(0) = 0$. This shows that p is well-defined with $0 \leq p \leq M\|x\| < \infty$ for every $x \in X$.

(ii)

It holds for every $x \in X$ and every nonzero $\lambda > 0$ that $I_{\lambda x} = \lambda I_x$. It follows that

$$p(\lambda x) = \inf I_{\lambda x} = \inf (\lambda I_x) = \lambda \inf I_x = \lambda p(x) ,$$

where we again use that $\lambda > 0$.

To show that p is subaddive let $x, y \in X$. It sufficies to show that $I_x + I_y \subseteq I_{x+y}$ because then

$$p(x+y) = \inf I_{x+y} \leq \inf (I_x + I_y) = (\inf I_x) + (\inf I_y) = p(x) + p(y) .$$

[1] This is where the argument breaks down for ℓ^∞ instead of c_0.

So let $\alpha \in I_x$ and $\beta \in I_y$. Then both x/α and y/β are contained in C and we want to show that also $(x+y)/(\alpha+\beta) \in C$. This holds true because

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} \cdot \frac{x}{\alpha} + \frac{\beta}{\alpha+\beta} \cdot \frac{y}{\beta}$$

is a convex combination of x/α and y/β and hence again contained in C.

(iii)

For every $x \in X$ the map $h_x\colon (0,\infty) \to X$ given by $h(\alpha) = x/\alpha$ is continuous, hence $I_x = h^{-1}(C)$ is closed in $(0,\infty)$. Since I_x is also a subinterval of $(0,\infty)$ we find that either $I_x = (0,\infty)$ or $I_x = [p(x),\infty)$. (Because otherwise $I_x = (p(x),\infty)$ with $p(x) > 0$, but this is not a closed subset of $(0,\infty)$.) We find in both cases that

$$p(x) \leq 1 \iff \inf I_x \leq 1 \iff 1 \in I_x \iff x \in C\,.$$