Exercises in PDE and Functional Analysis

Exercise Sheet 11

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Exercise 3

Suppose first that $T \in \mathcal{K}(X, Y)$, and let $(x_n)_n$ be a sequence in X with $x_n \rightharpoonup x$ for some $x \in X$. We start by showing that the sequence $(x_n)_n$ has a subsequence $(x_{n_k})_k$ with $Tx_{n_k} \to Tx$.

The sequence $(x_n)_n$ is bounded because it is weakly convergent, whence contained in a ball $\mathrm{B}_r(0)$ for some $r > 0$. It follows from the compactness of T that $\overline{T(\mathrm{B}_r(0))}$ is compact. There hence exists a subsequence $(x_{n_k})_k$ such that $(Tx_{n_k})_k \to y$ for some $y \in Y$. It follows from $x_n \rightharpoonup x$ that also $x_{n_k} \rightharpoonup x$, and hence that $Tx_{n_k} \rightharpoonup Tx$ because

$$y' T x_{n_k} = (T' y') x_{n_k} \to (T' y') x = y' T x$$

for every $y' \in Y'$. We find that the sequence $(Tx_{n_k})_k$ converges strongly to y and weakly to Tx. It follows that $Tx = y$ because strong convergence implies weak convergence, and weak limits are unique. This shows altogether that $Tx_{n_k} \to Tx$.

Every subsequence of $(x_n)_n$ is again weakly convergent to x. The above argument hence shows that every subsequence of $(Tx_n)_n$ has a subsequence that converges strongly to Tx. This then means that the sequence $(Tx_n)_n$ already converges strongly to Tx.

Suppose now that $Tx_n \to Tx$ for every sequence $(x_n)_n \subseteq X$ and every $x \in X$ with $x_n \rightharpoonup x$. We need to show that T is continuous and that $\overline{T(\mathrm{B}_1(0))}$ is compact. We have for every sequence $(x_n)_n \subseteq X$ and every $x \in X$ that

$$x_n \to x \implies x_n \rightharpoonup x \implies Tx_n \to Tx\,,$$

which shows that T is continuous. To show that T is compact we need to show that every sequence $(y_n)_n$ in $T(\mathrm{B}_1(0))$ has a subsequence that converges in Y (see equivalence (9.8) in the lecture notes, at the beginning of page 87). There exists a sequence $(x_n)_n$ in X with $(x_n)_n \subseteq \mathrm{B}_1(0)$ such that $y_n = Tx_n$ for every n. The closed unit ball $\overline{\mathrm{B}_1(0)}$ is weakly compact because X is reflexive, hence there exists a subsequence $(x_{n_k})_k$ and some $x \in X$ with $x_{n_k} \rightharpoonup x$. It follows by assumption that

$$y_{n_k} = Tx_{n_k} \to Tx \,,$$

which shows that the subsequence $(y_{n_k})_k$ converges in Y.

Exercise 4

(i)

We fix $\varepsilon > 0$. Suppose that no such constant $C_\varepsilon > 0$ exists. Then there exists for every natural n some $x_n \in X$ with

$$\|Kx_n\| > \varepsilon \|x_n\| + n\|TKx_n\| \,.$$

We see that $x_n \neq 0$ and can therefore assume that $\|x_n\| = 1$. Hence

$$\|Kx_n\| > \varepsilon + n\|TKx_n\| \,. \tag{1}$$

for every n. It follows from $(x_n)_n \subseteq \overline{\mathrm{B}_1(0)}$ that

$$(Kx_n)_n \subseteq K(\overline{\mathrm{B}_1(0)}) \subseteq \overline{K(\mathrm{B}_1(0))} \,.$$

The set $\overline{K(\mathrm{B}_1(0))}$ is compact because K is compact. Hence there exist a subsequence $(x_{n_k})_k$ with $x_{n_k} \to y$ for some $y \in Y$. We may replace the sequence $(x_n)_n$ by this subsequence to assume that $x_n \to y$; the condition (1) remains true because

$$\|Kx_{n_k}\| > \varepsilon \|x_{n_k}\| + n_k \|TKx_{n_k}\| \geq \varepsilon \|x_{n_k}\| + k\|TKx_{n_k}\|$$

for every k. We now find that $\|Kx_n\| \to \|y\|$ and $\|TKx_n\| \to \|Ty\|$. We find from (1) that this is possible only if $\|Ty\| = 0$, which tells us that $Ty = 0$ and hence $y = 0$ because T is injective. But we also find from (1) that

$$\|Kx_n\| > \varepsilon + n\|TKx_n\| \geq \varepsilon$$

for every n, and hence for $n \to \infty$ that $\|y\| \geq \varepsilon > 0$. A contradiction!

(ii)

The inclusion $\mathrm{W}_0^{2,p}(U) \to \mathrm{W}_0^{1,p}(U)$ is continuous and linear, and it is compact by the compact Sobolev embedding theorem (Theorem 9.6 in the lecture) because $1 < 2$. The inclusion $\mathrm{W}_0^{1,p}(U) \to \mathrm{L}^p(U)$ is again continuous and linear, and it is injective. The given statement therefore follows from part (i) of this exercise.

(iii)

We have on the one hand that

$$\|\cdot\| = \|\cdot\|_{\mathrm{L}^p(U)} + \|D^2(\,\cdot\,)\|_{\mathrm{L}^p(U)} \leq 2\|\cdot\|_{\mathrm{W}_0^{2,p}(U)}\,.$$

We have on the other hand that

$$\|\cdot\|_{\mathrm{W}_0^{1,p}(U)} \leq \frac{1}{4}\|\cdot\|_{\mathrm{W}_0^{2,p}(U)} + C\|\cdot\|_{\mathrm{L}^p(U)}$$

for some $C < \infty$ by part (ii) of this exercise. We may assume that $C \geq 1/2$. Then

$$\begin{aligned}
\|\cdot\|_{\mathrm{W}_0^{2,p}(U)} &= \left(\|\cdot\|_{\mathrm{L}^p(U)}^p + \|D(\,\cdot\,)\|_{\mathrm{L}^p(U)}^p + \|D^2(\,\cdot\,)\|_{\mathrm{L}^p(U)}^p\right)^{1/p} \\
&\leq \|\cdot\|_{\mathrm{L}^p(U)} + \|D(\,\cdot\,)\|_{\mathrm{L}^p(U)} + \|D^2(\,\cdot\,)\|_{\mathrm{L}^p(U)} \\
&\leq 2\|\cdot\|_{\mathrm{W}_0^{1,p}(U)} + \|D^2(\,\cdot\,)\|_{\mathrm{L}^p(U)} \\
&\leq \frac{1}{2}\|\cdot\|_{\mathrm{W}_0^{2,p}(U)} + 2C\|\cdot\|_{\mathrm{L}^p(U)} + \|D^2(\,\cdot\,)\|_{\mathrm{L}^p(U)} \\
&\leq \frac{1}{2}\|\cdot\|_{\mathrm{W}_0^{2,p}(U)} + 2C\|\cdot\|_{\mathrm{L}^p(U)} + 2C\|D^2(\,\cdot\,)\|_{\mathrm{L}^p(U)} \\
&= \frac{1}{2}\|\cdot\|_{\mathrm{W}_0^{2,p}(U)} + 2C\|\cdot\|\,,
\end{aligned}$$

which we can rearrange to

$$\|\cdot\|_{\mathrm{W}_0^{2,p}(U)} \leq 4C\|\cdot\|\,.$$