Exercises in PDE and Functional Analysis

Exercise Sheet 12

Jendrik Stelzner

Exercise 1

(i)

We show that every operator $T \in \overline{\mathcal{K}(X,Y)}$ is already compact. We do so by showing that every sequence $(y_n)_n$ in $T(\mathrm{B}_1(0))$ has a subsequence which converges in Y. For this it sufficies to show that for every $\varepsilon > 0$ there exists a subsequence $(y_{n_k})_k$ with $\|y_{n_k} - y_{n_l}\| < \varepsilon$ for all k, l. It then follows with the usual diagonal trick that there overall exists a subsequence $(y_{n_k})_k$ with $\|y_{n_k} - y_{n_l}\| < 1/k$ for all $l \geq k$. This subsequence is then a Cauchy sequence and hence convergent by the completeness of Y.

We fix $\varepsilon > 0$. It follows from the assumption $T \in \overline{\mathcal{K}(X,Y)}$ that there exists a compact operator $T' \in \mathcal{K}(X,Y)$ with $\|T - T'\| < \varepsilon/3$. Let $(x_n)_n$ be a sequence in $\mathrm{B}_1(0)$ with $y_n = Tx_n$ and let $(y'_n)_n$ be the sequence in $T'(\mathrm{B}_1(0))$ given by $y'_n = T'x_n$. It follows from the compactness of the operator T' that there exists a subsequence $(y'_{n_k})_k$ that converges in Y, and we may choose this subsequence such that

$$\|y'_{n_k} - y'_{n_l}\| < \frac{\varepsilon}{3}$$

for all k, l. It follows for the corresponding subsequence $(y_{n_k})_k$ that

$$\begin{aligned} \|y_{n_k} - y_{n_l}\| &\leq \|y_{n_k} - y'_{n_k}\| + \|y'_{n_k} - y'_{n_l}\| + \|y'_{n_l} - y_{n_l}\| \\ &= \|Tx_{n_k} - T'x_{n_k}\| + \|y'_{n_k} - y'_{n_l}\| + \|T'x_{n_l} - Tx_{n_l}\| \\ &\leq \|T - T'\|\|x_{n_k}\| + \|y'_{n_k} - y'_{n_l}\| + \|T - T'\|\|x_{n_l}\| \\ &\leq \frac{\varepsilon}{3} \cdot 1 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \cdot 1 = \varepsilon\,. \end{aligned}$$

(ii)

If H is finite-dimensional then

$$\{T \in \mathcal{L}(H) \mid \dim \mathcal{R}(T) < \infty\} = \mathcal{L}(H) = \mathcal{K}(H)\,.$$

We will therefore consider in the following only the case that H is infinite-dimensional.

Lemma 1. *Let X and Y be normed vector spaces, let $(T_n)_n$ be a bounded sequence in $\mathcal{L}(X,Y)$ and let $T \in \mathcal{L}(X,Y)$. If $K \subseteq X$ is a compact subset with $T_n \to T$ pointwise on K then already $T_n \to T$ uniformly on K.*

Proof. There exists by assumption some constant $C > 0$ with $\|T_n\| \leq C$ for all n and also $\|T\| \leq C$. There exists for every $\varepsilon > 0$ finitely many $x_1, \dots, x_k \in X$ with

$$K \subseteq \mathrm{B}_\varepsilon(x_1) \cup \cdots \cup \mathrm{B}_\varepsilon(x_n)\,.$$

There exists some N with $\|T_n x_i - T x_i\| < \varepsilon$ for every $i \geq N$ because $T_n \to T$ pointwise on K. There hence exists for every $x \in K$ some i with $\|x_i - x\| < \varepsilon$, and it follows that

$$\begin{aligned}
\|T_n x - T x\| &= \|T_n x - T_n x_i\| + \|T_n x_i - T x_i\| + \|T x_i - T x\| \\
&\leq \|T_n\| \|x - x_i\| + \|T_n x_i - T x_i\| + \|T\| \|x - x_i\| \\
&\leq (2C + 1)\varepsilon
\end{aligned}$$

for every $i \geq N$. □

If T is an operator of finite rank then $\overline{T(\mathrm{B}_1(0))}$ is a subset of the finite dimensional normed space $\mathcal{R}(T)$ that is both closed and bounded, and hence compact by the Heine–Borel theorem. This shows that

$$\{T \in \mathcal{L}(H) \mid \dim \mathcal{R}(T) < \infty\} \subseteq \mathcal{K}(H)$$

and hence that

$$\overline{\{T \in \mathcal{L}(H) \mid \dim \mathcal{R}(T) < \infty\}} \subseteq \mathcal{K}(H)$$

because $\mathcal{K}(H)$ is closed as a subspace of $\mathcal{L}(H)$.

Suppose now that $T \in \mathcal{K}(H)$ is a compact operator. Let $(e_n)_n$ be an orthonormal basis for H, and for every n let

$$P_n \colon H \to H\,, \quad x \mapsto \sum_{i=1}^{n} (x, e_i) e_i$$

be the orthogonal projection onto the finite dimenisonal subspace $\langle e_1, \dots, e_n \rangle$; this linear operator is continuous because the linear functionals $(-, e_i)$ are continuous. Then $P_n \to \mathrm{Id}$ pointwise on H because $x = \sum_{i=1}^{\infty} (x, e_i) e_i$ for every $x \in H$ (as seen in the lecture). It follows from Lemma 1 that $P_n \to \mathrm{Id}$ uniformly on $\overline{T(\mathrm{B}_1(0))}$. We find for the linear operators $T_n \coloneqq P_n \circ T \in \mathcal{L}(H)$, which have finite dimensional range, that

$$T_n = P_n \circ T \to \mathrm{Id} \circ T = T$$

uniformly on X, and hence that $T_n \to T$ in $\mathcal{L}(H)$. This shows that T can be approximated by operators of finite rank, and hence that

$$\mathcal{K}(H) \subseteq \overline{\{T \in \mathcal{L}(H) \mid \dim \mathcal{R}(H) < \infty\}}\,.$$

Remark 2. We did not need that $P_n \to \mathrm{Id}$ pointwise everywhere on H, but only on the compact subset $\overline{T(\mathrm{B}_1(0))}$. We see from this that we do not need H to be separable: There exists for every $n \geq 1$ finitely many $x_1, \dots, x_k \in \overline{T(\mathrm{B}_1(0))}$ with

$$\overline{T(\mathrm{B}_1(0))} \subseteq \mathrm{B}_{1/n}(x_1) \cup \dots \cup \mathrm{B}_{1/n}(x_k)\,.$$

Let $P_n \in \mathcal{L}(H)$ be the orthogonal projection onto the finite-dimensional linear subspace $\langle x_1, \dots, x_k \rangle$. Then there exists for every $x \in X$ some i with $\|x - x_i\| < \varepsilon$ and it follows that

$$\|Px - x\| \leq \|x_i - x\| < \varepsilon\,.$$

Then $P_n \to \mathrm{Id}$ pointwise on $\overline{T(\mathrm{B}_1(0))}$ and hence again $P_n \circ T \to T$ uniformly on X.

Exercise 4

For $H = 0$ we have

$$\sigma(p(A)) = \emptyset = p(\emptyset) = p(\sigma(A))$$

We therefore consider in the following only the case $H \neq 0$.

Lemma 3. *Let $f_1, \dots, f_n \colon X \to X$ be pairwise commuting maps on a set X. Then the composition $f_1 \circ \dots \circ f_n$ is invertible if and only if all f_i are invertible.*

Proof. If every map f_i is invertible then $f_1 \circ \dots \circ f_n$ is invertible. If on the other hand $f_1 \circ \dots \circ f_n$ is invertible then f_1 is surjective and f_n is injective. We may rearrange the terms $f_1, \dots, f_n$ in the composition $f_1 \circ \dots \circ f_n$ however we want. We then find that every f_i is injective and also that every f_i is surjective. □

If $p = a_0 \in \mathbb{C}$ then

$$\sigma(p(A)) = \sigma(a_0 \,\mathrm{Id}) = \{a_0\}$$

and

$$p(\sigma(A)) = \begin{cases} \{a_0\} & \text{if } \sigma(A) \text{ is nonempty}\,, \\ \emptyset & \text{otherwise}\,. \end{cases}$$

It apparently holds that the spectrum of a bounded operator on a complex Hilbert space is nonempty, so that

$$\sigma(p(A)) = \{a_0\} = p(\sigma(A))$$

as desired. But I don't think that we have proven this requirement in the lecture.

We consider now case $\deg p \geq 1$. We may assume that $a_n \neq 0$ so that $\deg p = n$. For every $\lambda \in \mathbb{C}$ the polynomial $p(z) - \lambda$ is again of degree n with leading coefficient a_n, and may therefore be factorized as

$$p(z) - \lambda = a_n (z - \mu_1) \cdots (z - \mu_n)\,.$$

We find with the above lemma that

$$\begin{aligned}\lambda \in \sigma(p(A)) &\iff p(A) - \lambda \text{ is non-invertible} \\ &\iff a_n(A - \mu_1)\cdots(A - \mu_n) \text{ is non-invertible} \\ &\iff A - \mu_i \text{ is non-invertible for some } i \\ &\iff \mu_i \in \sigma(A) \text{ for some } i\end{aligned}$$

We observe that the complex numbers $\mu_1, \dots, \mu_n$ are precisely the solutions to the polynomial equations $p(z) - \lambda = 0$, and hence the solutions to the equation $p(z) = \lambda$. We therefore find that

$$\begin{aligned}&\lambda \in \sigma(p(A)) \\ \iff &\text{there exists } \mu \text{ with } p(\mu) = \lambda \text{ such that } \mu \in \sigma(A) \\ \iff &\text{there exists } \mu \in \sigma(A) \text{ with } \lambda = p(\mu) \\ \iff &\lambda \in p(\sigma(A))\,.\end{aligned}$$