Exercise Group 4, Thursday 8–10, Tutor: Friedrich Klaus

Exercises in PDE and Functional Analysis

Exercise Sheet 1

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Exercise 1

(i)

It holds for all $x, y \in X$ that

$$d^*(x,y) = 0 \iff d(x,y) = 0 \iff x = y\,,$$

which shows that d^* is reflexive. It holds for all $x, y \in X$ that

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(d,x)}{1+d(y,x)} = d^*(y,x)\,,$$

which shows that d^* symmetric. It holds for all $x, y, z \in X$ that

$$\begin{aligned} d^*(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\ &= 1 - \frac{1}{1+d(x,z)} \\ &\leq 1 - \frac{1}{1+d(x,y)+d(y,z)} \\ &= \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \\ &= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \\ &\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z))} \\ &= d^*(x,y) + d^*(y,z)\,, \end{aligned}$$

which shows that d^* satisfies the triangle inequality. This shows altogether that d^* is a metric.

(ii) and (iii)

That d is a metric on $\mathbb{R}^n$ follows from part (i) and the following lemma. Part (iii) also follows from the following lemma.

Lemma 1. *Let (X_n, d_n) with $n \geq 1$ be a family of metric spaces such that the metrics d_n are uniformly bounded, i.e. such that there exists a constant $C > 0$ with $d_n \leq C$ for every n. Let $X \coloneqq \prod_{n=1}^{\infty} X_n = X_1 \times X_2 \times \cdots$*

1) The function $d \colon X \times X \to \mathbb{R}$ given by

$$d(x, y) \coloneqq \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$$

is a well-defined metric on X, which is again bounded by C.

2) A sequence $(x^{(k)})_k$ in X converges to $x \in X$ if and only if it does so in each coordinate, i.e. if and only if $x_n^{(k)} \to x_n$ for every n.

Proof. We may replace the metrics d_n by d_n/C to assume that $C = 1$.

1) It holds for all $x, y \in X$ that

$$d(x, y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1\,,$$

which shows that d is well-defined and again bounded by 1. It holds for all $x, y \in X$ that

$$\begin{aligned} d(x, y) = 0 \iff \sum_{n=1}^{\infty} \underbrace{\frac{d_n(x_n, y_n)}{2^n}}_{\geq 0} = 0 &\iff \forall n : d_n(x_n, y_n) = 0 \\ &\iff \forall n : x_n = y_n \iff x = y\,, \end{aligned}$$

which shows that d is reflexive. That d is symmetric follows from all d_n being symmetric. It holds for all $x, y, z \in X$ that

$$\begin{aligned} d(x, z) &= \sum_{n=1}^{\infty} \frac{d_n(x_n, z_n)}{2^n} \\ &\leq \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n) + d_n(y_n, z_n)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n} + \sum_{n=1}^{\infty} \frac{d_n(y_n, z_n)}{2^n} \\ &= d(x, y) + d(y, z)\,, \end{aligned}$$

which shows that d satisfies the triangle inequality.

2) The projection $\pi_n \colon X \to X_n$ onto the n-th factor is for every n Lipschitz-continuous with Lipschitz-constant 2^n. It follows from this continuity that if $x^{(k)} \to x$ then also $x_n^{(k)} \to x_n$ for every n.

   Suppose on the other hand that $x_n^{(k)} \to x_n$ for every n and let $\varepsilon > 0$. Let $N \geq 1$ with $\sum_{n=N+1}^{\infty} 1/2^n = \varepsilon/2$ and let $K \geq 1$ with $\sum_{n=1}^{N} d_n(x_n^{(k)}, x_n)/2^n < \varepsilon/2$ for all $k \geq K$. It follows for all $k \geq K$ that

$$
\begin{aligned}
d(x^{(k)}, x) &= \sum_{n=1}^{\infty} \frac{d_n(x_n^{(k)}, x_n)}{2^n} \\
&= \sum_{n=1}^{N} \frac{d_n(x_n^{(k)}, x_n)}{2^n} + \sum_{n=N+1}^{\infty} \frac{d_n(x_n^{(k)}, x_n)}{2^n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,.
\end{aligned}
$$

   This shows that $x^{(k)} \to x$ with respect to d. □

(iv)

Suppose that such a norm $\|\cdot\|$ and constant $C_1 > 0$ exist. We have seen in Lemma 1 that the metric d is bounded, hence it follows from the inequality $\|\cdot\| \leq d(\,\cdot\,, 0)/C_1$ that the norm $\|\cdot\|$ is also bounded. But a norm on a nonzero vector space is never bounded (because it is reflexive and homogeneous).

Exercise 2

We assume that A is nonempty because otherwise $d(-, A) = \infty$.

(i)

It holds that $d(x, A) = 0$ if and only if there exists for every $n \geq 1$ some $a_n \in A$ with $d(x, a_n) < 1/n$, which means precisely that there exists a sequence $(a_n)_n$ in A which converges in X to x. Such a sequence exists if and only if x is contained in the sequential closure of A, which conincides with the closure of A because (X, d) is a metric (and not just topological) space.

(ii)

It holds that for all $x, y \in X$ that

$$
\begin{aligned}
d(x, A) = \inf_{a \in A} d(x, a) &\leq \inf_{a \in A} \left(d(x, y) + d(y, a) \right) \\
&= d(x, y) + \inf_{a \in A} d(y, a) = d(x, y) + d(y, A)
\end{aligned}
$$

and therefore $d(x, A) - d(y, A) \leq d(x, y)$. It then also follows that

$$
d(y, A) - d(x, A) \leq d(y, x) = d(x, y) \,.
$$

Together this shows that $|d(x, A) - d(y, A)| \leq d(x, y)$.

(iii)

It follows from part (i) that the function $d(-, A) + d(-, B)$ is positive because

$$\{x \in X \mid d(x, A) = 0\} \cap \{x \in X \mid d(x, B) = 0\} = \overline{A} \cap \overline{B} = A \cap B = \emptyset\,.$$

The function

$$f \coloneqq \frac{d(-, A)}{d(-, A) + d(-, B)} : X \to [0, 1]\,, \quad x \mapsto \frac{d(x, A)}{d(x, A) + d(x, B)}$$

is therefore well-defined. It follows from part (ii) that f is continous. It holds that

$$f^{-1}(0) = \{x \in X \mid d(x, A) = 0\} = \overline{A} = A$$

and

$$\begin{aligned} f^{-1}(1) &= \{x \in X \mid d(x, A) = d(x, A) + d(x, B)\} \\ &= \{x \in X \mid d(x, B) = 0\} = \overline{B} = B\,, \end{aligned}$$

as desired.

Exercise 3

(i)

For $p = \infty$ we have that $(\ell_\infty, \|\cdot\|_\infty) = \mathrm{B}(\mathbb{N}, \mathbb{R})$, which is a Banach space by Proposition 2.2 from the lecture (because $\mathbb{R}$ is complete). So in the following we only consider the case $1 \leq p < \infty$.

We first show that $\|\cdot\|_p$ defines a norm on ℓ_p, and then show that ℓ_p is complete with respect to $\|\cdot\|_p$.

It holds for every $x \in \ell_p$ that

$$\|x\|_p = 0 \iff \|x\|_p^p = 0 \iff \sum_{n=1}^{\infty} x_n^p = 0 \iff x_n = 0 \text{ for all } n \iff x = 0\,,$$

which shows that $\|\cdot\|_p$ is reflexive. It holds for all $\lambda \in \mathbb{R}$, $x \in \ell_p$ that

$$\|\lambda x\| = \sqrt[p]{\sum_{n=1}^{\infty} |\lambda x_n|^p} = \sqrt[p]{\sum_{n=1}^{\infty} |\lambda|^p |x_n|^p} = |\lambda| \sqrt[p]{\sum_{n=1}^{\infty} |x_n|^p} = |\lambda|\, \|x\|_p\,,$$

which shows that $\|\cdot\|$ is homogeneous. The triangle inequality for $\|\cdot\|$ is precisely Minkowski's inequality, which we are allowed to assume without proof according to the exercise.

To show that ℓ_p is complete with respect to $\|\cdot\|_p$ we consider a Cauchy sequence $(x^{(k)})_k$ in $(\ell_p, \|\cdot\|)$. It then holds for every $n \geq 1$ that the projection $\pi_n \colon \ell_p \to \mathbb{R}$, $x \mapsto x_n$ is

Lipschitz-continuous with Lipschitz-constant 1, from which it follows that $(x_n^{(k)})_k$ is again a Cauchy sequence, and thus convergent. For every $n \geq 1$ let $x_n \coloneqq \lim_{k\to\infty} x_n^{(k)}$.

The sequence x is again contained in ℓ_p since it follows from Fatou's lemma that

$$\|x\|_p^p = \sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} \lim_{k\to\infty} |x_n^{(k)}|^p \leq \liminf_{k\to\infty} \sum_{n=1}^{\infty} |x_n^{(k)}|^p = \liminf_{k\to\infty} \|x^{(k)}\|_p^p < \infty\,,$$

because the sequence $(x^{(k)})_k$ is bounded.

To show that the Cauchy sequence $(x^{(k)})_k$ converges to x with respect to the norm $\|\cdot\|_p$ we may replace the sequence $(x^{(k)})_k$ by any of its subsequences. We may therefore assume that $\|x^{(l)} - x^{(k)}\| \leq 1/2^k$ for all $l \geq k$. It then follows from Fatou's lemma that

$$\begin{aligned}\|x - x^{(k)}\|_p^p &= \sum_{n=1}^{\infty} \left|x_n - x_n^{(k)}\right|^p = \sum_{n=1}^{\infty} \lim_{l\to\infty} \left|x_n^{(l)} - x_n^{(k)}\right|^p \\ &\leq \liminf_{l\to\infty} \sum_{n=1}^{\infty} \left|x_n^{(l)} - x_n^{(k)}\right|^p = \liminf_{l\to\infty} \left\|x^{(l)} - x^{(k)}\right\| \leq \liminf_{l\to\infty} \frac{1}{2^k} = \frac{1}{2^k}\,,\end{aligned}$$

for every k, which shows that $x^{(k)} \to x$ with respect to $\|\cdot\|_p$. Altogether this shows that the sequence $(x^{(k)})_k$ converges in $(\ell_p, \|\cdot\|_p)$.

(ii)

Lemma 2. *Let (X, d) be a complete metric space. Then a subspace A of X is closed if and only if it again complete.*

Proof. If A is closed then every Cauchy sequence in A has a limit in X, which is then again contained in A. If A is complete and $x \in X$ is the limit of a sequence $(a_n)_n \subseteq A$, then the sequence $(a_n)_n$ is in particular Cauchy and therefore has a limit a in A, from which it follows that $x = a \in A$ (because limits in Hausdorff spaces are unique). □

It sufficies by Lemma 2 to show that ℓ_1 is not closed in ℓ_∞ with respect to the norm $\|\cdot\|_\infty$. For this we consider the sequence $x \in \ell_\infty$ with $x_n = 1/n$. The sequence x is not contained in ℓ_1. But it is with respect to $\|\cdot\|_\infty$ the limit of the sequence $(x^{(k)})_k \subseteq \ell_1$ given by

$$x_n^{(k)} \coloneqq \begin{cases} 1/n & \text{if } n \leq k\,, \\ 0 & \text{else}\,, \end{cases}$$

because

$$\left\|x - x^{(k)}\right\|_\infty = \sup_{n>k} \frac{1}{n} = \frac{1}{k+1} \longrightarrow 0$$

as $k \to \infty$. This shows that ℓ_1 is with respect to the norm $\|\cdot\|_\infty$ not closed in ℓ_∞, and therefore not complete with respect to $\|\cdot\|_\infty$.

Exercise 4

Interior of A_1

The interior of A_1 is empty: Let $x \in A_1$ and let $\varepsilon > 0$. It follows from x being a positive null sequence that there exists some $N \in \mathbb{N}$ with $0 < x_N < \varepsilon/2$. The sequence $(y_n)_n \subseteq \mathbb{R}$ with

$$y_n \coloneqq \begin{cases} x_n - \varepsilon & \text{if } n = N\,, \\ x_n & \text{else}\,, \end{cases}$$

differs with x only by a single value, and is therefore again in ℓ_1. It holds that $y_N < -\varepsilon/2$ and therefore that $y_N \notin A_1$, but

$$\|x - y\|_1 = \varepsilon\,.$$

This shows that A_1 does not contain an open ball around x.

Closure of A_1

The closure of A_1 is given by the set

$$C \coloneqq \{x \in \ell_1 \mid x_n \geq 0 \text{ for all } n\}\,.$$

To show that $\overline{A_1} \subseteq C$ we note that the projection map

$$\pi_n \colon \ell_1 \to \mathbb{R}\,, \quad x \mapsto x_n$$

is for every $n \geq 1$ Lipschitz-continuous with Lipschitz-constant 1, hence continuous, and that therefore

$$\pi_n(\overline{A_1}) \subseteq \overline{\pi_n(A_1)} = \overline{(0,\infty)} = [0,\infty)\,.$$

To show the inclusion $C \subseteq \overline{A_1}$ we define for $x \in C$ and $\varepsilon > 0$ the sequence $(y_n)_n \subseteq \mathbb{R}$ with

$$y_n \coloneqq \begin{cases} x_n & \text{if } x_n > 0\,, \\ \varepsilon/2^n & \text{otherwise}\,. \end{cases}$$

The sequence y is again in ℓ_1 because both x and $(1/2^n)_{n\geq 1}$ are contained in ℓ_1, and it is even contained in A_1. It holds that

$$\|x - y\|_1 = \sum_{\substack{n \geq 1 \\ x_n = 0}} \frac{\varepsilon}{2^n} \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon\,,$$

as desired.

Interior of A_2

The interior of A_2 is empty: If $x \in A_2$ and $\varepsilon > 0$ then the sequence $(y_n)_n \subseteq \mathbb{R}$ with

$$y_n := \begin{cases} x_n & \text{if } x_n \neq 0\,, \\ \varepsilon/2^n & \text{otherwise}\,, \end{cases}$$

is again in ℓ_1 with $\|x - y\|_1 \leq \varepsilon$, as seen above. But the sequence y is not contained in A_2, since $y_n \neq 0$ for every n. This shows that A_2 does not contain an open ball around x.

Closure of A_2

The set A_2 is dense in ℓ_1: If $x \in \ell_1$ then we can define the sequence $(x^{(k)})_k \subseteq A_1$ by cutting off the sequence $(x_n)_n$ after k terms:

$$x_n^{(k)} := \begin{cases} x_n & \text{if } n \leq k\,, \\ 0 & \text{else}\,. \end{cases}$$

It then holds that

$$\left\| x - x^{(k)} \right\|_1 = \sum_{n=k+1}^{\infty} |x_n| \longrightarrow 0$$

as $k \to \infty$ because the series $\sum_{n=1}^{\infty} \|x_n\|$ converges.

This shows that every sequences $x \in \ell_1$ is the limit of a sequence $(x^{(k)})_k \subseteq A_2$, which shows that $\overline{A_2} = \ell_1$.