Exercises in PDE and Functional Analysis

Exercise Sheet 9

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Exercise 1

(i)

Suppose that $x_k \rightharpoonup x$. Then for any $y' \in Y'$ we have $y' \circ J \in X'$ and hence

$$\langle Jx_k, y' \rangle = \langle x_k, y' \circ J \rangle \to \langle x, y' \circ J \rangle = \langle Jx, y' \rangle .$$

This shows that $Jx_k \rightharpoonup Jx$. If on the other hand $Jx_k \rightharpoonup Jx$ then

$$x_k = J^{-1} J x_k \rightharpoonup J^{-1} J x = x$$

by the above because $J^{-1} \colon Y \to X$ is again an isomorphism of Banach spaces.

(ii)

Suppose that $x^{(k)} \rightharpoonup x$ and let $i \in \{1, \dots, n\}$. Let $x'_i \in X'_i$ for some i. The projection operator $P_i \colon X \to X_i$ is continuous, hence $x' \circ P_i \in X'$. It follows that

$$\langle x_i^{(k)}, x' \rangle = \langle P_i x^{(k)}, x' \rangle = \langle x^{(k)}, x' \circ P_i \rangle \to \langle x, x' \circ P_i \rangle = \langle P_i x, x' \rangle = \langle x_i, x' \rangle$$

This shows that $x_i^{(k)} \rightharpoonup x_i$.

Suppose now that $x_i^{(k)} \rightharpoonup x_i$ for every $i = 1, \dots, n$, and let $x' \in X'$. For every $i = 1, \dots, n$ let $P_i \colon X \to X_i$ be the projection operator, and let $I_i \colon X_i \to X$ be the inclusion operator given by

$$I_i(y) = (0, \dots, y, \dots, 0) .$$

Then both P_i and I_i are linear and bounded, and $\mathrm{Id}_X = \sum_{i=1}^n (I_i \circ P_i)$. Hence

$$x' = x' \circ \mathrm{Id}_X = x' \circ \sum_{i=1}^n (I_i \circ P_i) = \sum_{i=1}^n (x' \circ I_i) \circ P_i = \sum_{i=1}^n x'_i \circ P_i$$

with $x_i' \coloneqq x' \circ I_i \in X_i'$ for every i. It follows for every $i = 1, \dots, n$ that

$$\langle x^{(k)}, x_i' \circ P_i \rangle = \langle P_i x^{(k)}, x_i' \rangle = \langle x_i^{(k)}, x_i' \rangle \to \langle x_i, x_i' \rangle = \langle P_i x, x_i' \rangle = \langle x, x_i' \circ P_i \rangle \,,$$

and hence altogether

$$\langle x^{(k)}, x' \rangle = \sum_{i=1}^{n} \langle x^{(k)}, x_i' \circ P_i \rangle \to \sum_{i=1}^{n} \langle x, x_i' \circ P_i \rangle = \langle x, x' \rangle \,.$$

This shows that $x^{(k)} \rightharpoonup x$.

(iii)

Suppose that $x_k \rightharpoonup x$ in Y. Then for every $x' \in X'$ we have $x' \,|_Y \in Y'$ and hence

$$\langle x_k, x' \rangle = \langle x_k, x' \,|_Y \rangle \to \langle x, x' \,|_Y \rangle = \langle x, x' \rangle \,.$$

This shows that $x_k \rightharpoonup x$ in X.

Suppose now that $x_k \rightharpoonup x$ in X. For $y' \in Y'$ there exists by the Hahn–Banach theorem an extension $x' \in X'$. Then

$$\langle x_k, y' \rangle = \langle x_k, x' \rangle \to \langle x, x' \rangle = \langle x, y' \rangle \,.$$

This shows that $x_k \rightharpoonup x$ in Y.

(iv)

Let

$$X \coloneqq \mathrm{L}^p(U) \times \cdots \times \mathrm{L}^p(U)$$

be the $(n+1)$-fold product of copies of $\mathrm{L}^p(U)$, and consider the linear operator

$$J \colon \mathrm{W}^{1,p}(U) \to X \,, \quad f \mapsto (f, \partial_1 f, \dots, \partial_n f) \,.$$

We can endow X with the norm

$$\|(f, f_1, \dots, f_n)\| \coloneqq (\|f\|_p + \| \sqrt{|f_1| + \cdots + |f_n|} \,\|_p) \,.$$

This norm is chosen so that J is an isometric embedding (i.e. we have copied the definition of the norm on $\mathrm{W}^{1,p}(U)$ from the lecture). We can now use the previous parts of the exercise as follows:

- We find from part (i) for a sequence $(x_k)_k \subseteq \mathrm{W}^{1,p}(U)$ and any $x \in \mathrm{W}^{1,p}(U)$ that $x_k \rightharpoonup x$ if and only if $Jx_k \rightharpoonup Jx$, because J is an isometric embedding (and hence an isomorphism of Banach spaces into its image $J(\mathrm{W}^{1,p}(U))$).
- The subspace $J(\mathrm{W}^{1,p}(U)) \subseteq X$ is closed because $J(\mathrm{W}^{1,p}(U)) \cong \mathrm{W}^{1,p}(U)$ is complete. It hence follows from part (iii) for $(x_k)_k \subseteq J(\mathrm{W}^{1,p}(U))$ and $x \in J(\mathrm{W}^{1,p}(U))$ that $x_k \rightharpoonup x$ in $J(\mathrm{W}^{1,p}(U))$ if and only if $x_k \rightharpoonup x$ in X.

- The projections $P_i\colon X \to \mathrm{L}^p(U)$ and inclusions $I_i\colon\ \mathrm{L}^p(U) \to X$ are again continuous. We therefore find as in part (ii) that a sequence in X converges weakly if it converges weakly in each coordinate.

This shows for a sequence $(f_k)_k \subseteq \mathrm{W}^{1,p}(U)$ and any $f \in \mathrm{W}^{1,p}(U)$ that $f_k \rightharpoonup f$ in $\mathrm{W}^{1,p}(U)$ if and only of $f_k \rightharpoonup f$ in $\mathrm{L}^p(U)$ and $\partial_i f_k \rightharpoonup \partial_i f$ in $\mathrm{L}^p(U)$ for every $i = 1, \dots, n$.

Exercise 2

(i)

Suppose that $x_k \rightharpoonup x$. We then know from the lecture (Proposition 8.3, part (iv)) that the sequence $(x_k)_k$ is bounded. The convergence $\langle x_k, x' \rangle \to \langle x, x' \rangle$ holds for every $x' \in X'$, whence for every $x' \in D$.

Suppose now that the sequence $(x_k)_k \subseteq X$ is bounded, say $\|x_k\| \leq C$ for every k for some $C \geq 0$, and that there exists a dense subset $D \subseteq X'$ with $\langle x_k, x' \rangle \to \langle x, x' \rangle$ for every $x' \in D$.

Let $x' \in X'$. There exists some $\tilde{x}' \in D$ with

$$\|x\| \|x' - \tilde{x}'\| \leq \frac{\varepsilon}{2} \quad \text{and} \quad C \|\tilde{x'} - x'\| \leq \frac{\varepsilon}{2}$$

because D is dense in X'. It then holds that

$$\begin{aligned} |\langle x_k, x' \rangle - \langle x, x' \rangle| &\leq |\langle x_k, x' \rangle - \langle x_k, \tilde{x}' \rangle| + |\langle x_k, \tilde{x}' \rangle - \langle x, \tilde{x}' \rangle| + |\langle x, \tilde{x}' \rangle - \langle x, x' \rangle| \\ &= |\langle x_k, x' - \tilde{x}' \rangle| + |\langle x_k, \tilde{x}' \rangle - \langle x, \tilde{x}' \rangle| + |\langle x, \tilde{x}' - x' \rangle| \\ &\leq \|x_k\| \|x' - \tilde{x}'\| + |\langle x_k, \tilde{x}' \rangle - \langle x, \tilde{x}' \rangle| + \|x\| \|\tilde{x'} - x'\| \\ &\leq \frac{\varepsilon}{2} + |\langle x_k, \tilde{x}' \rangle - \langle x, \tilde{x}' \rangle| + \frac{\varepsilon}{2} \\ &= |\langle x_k, \tilde{x}' \rangle - \langle x, \tilde{x}' \rangle| + \varepsilon \,. \end{aligned}$$

When taking the limit $k \to \infty$ we have by assumption that $|\langle x_k, \tilde{x}' \rangle - \langle x, \tilde{x}' \rangle| \to 0$. We hence find that

$$\limsup_{k \to \infty} |\langle x_k, x' \rangle - \langle x, x' \rangle| \leq \varepsilon \,.$$

This holds for every $\varepsilon > 0$, whence $\langle x_k, x' \rangle \to \langle x, x' \rangle$. This shows that $x_k \rightharpoonup x$.

(ii)

Suppose that $x_k \to x$. Then

$$\begin{aligned} \sup_{\|x'\|=1} |\langle x_k, x' \rangle - \langle x, x' \rangle| &= \sup_{\|x'\|=1} |\langle x_k - x, x' \rangle| \\ &\leq \sup_{\|x'\|=1} \|x_k - x\| \|x'\| = \|x_k - x\| \to 0 \,. \end{aligned}$$

Suppose on the other hand that $\sup_{\|x'\|=1} |\langle x_k, x'\rangle - \langle x, x'\rangle| \to 0$. We know from the lecture (Corollary 6.6) that there exists for every k some $x_0' \in X'$ with $\|x_0'\| = 1$ such that

$$\langle x_k - x, x_0'\rangle = \|x_k - x\| \,.$$

Then

$$\|x_k - x\| = |\langle x_k - x, x_0'\rangle| = |\langle x_k, x_0'\rangle - \langle x, x_0'\rangle| \leq \sup_{\|x'\|=1} |\langle x_k, x'\rangle - \langle x, x'\rangle| \to 0 \,,$$

which shows that $x_k \to x$.

Exercise 4

(i)

Boundedness

It holds for every k and $1 \leq p < \infty$ that

$$\|f_k\|_p^p = \int_{-1}^{1} |\sin(k\pi x)|^p \, \mathrm{d}x \leq \int_{-1}^{1} 1 \, \mathrm{d}x = 2 \,.$$

The sequence $(f_k)_k$ is therefore bounded in $\mathrm{L}^p([-1,1])$ for $1 \leq p < \infty$. The sequence is also bounded in $\mathrm{L}^\infty([-\infty,\infty])$ because $\|f_k\|_\infty \leq 1$ for every k.

Weak Convergence

It follows from part (i) of Exercise 3 that $f_k \rightharpoonup 0$ in $\mathrm{L}^p([-1,1])$ for $1 < p < \infty$, and that $f_k \overset{*}{\rightharpoonup} 0$ in $\mathrm{L}^\infty([-1,1])$, where we use that

$$\frac{1}{2} \int_{-1}^{1} \sin(k\pi x) \, \mathrm{d}x = 0 \,.$$

For $p = 1$ we would like to again use Exercise 3, but this case is not covered there. But it was claimed in the lecture (Page 69, (i) at the bottom) that this also works for $p = 1$. So I suspect that also $f_k \rightharpoonup 0$ in $\mathrm{L}^1([-1,1])$, but I don't know how to argue for this.

Strong Convergence

We have for every $1 \leq p < \infty$ and $k \geq 1$ that

$$\begin{aligned}
\|f_{2^{k+2}} - f_{2^{k+1}}\|_p^p &= \int_{-1}^{1} |\sin(2^{k+2}\pi x) - \sin(2^{k+1}\pi x)|^p \, \mathrm{d}x \\
&= 2 \int_{-1/2}^{1/2} |\sin(2^{k+2}\pi x) - \sin(2^{k+1}\pi x)|^p \, \mathrm{d}x \\
&= \int_{-1}^{1} |\sin(2^{k+1}\pi x) - \sin(2^{k}\pi x)|^p \, \mathrm{d}x \\
&= \|f_{2^{k+1}} - f_{2^k}\|_p^p \,,
\end{aligned}$$

where we use for the second equality that the map $|\sin(2^{k+2}\pi x) - \sin(2^{k+1}\pi x)|^p$ is $(1/2)$-periodic (because $k+1 \geq 2$), and for the third equality the change of variables $x \to x/2$. It follows inductively that

$$\|f_{2^{k+1}} - f_{2^k}\|_p = \|f_4 - f_2\|$$

for all $k \geq 1$, and hence that the sequene $(f_k)_k$ is not Cauchy in $\mathrm{L}^p([-1,1])$ because $\|f_4 - f_2\|_p > 0$. This shows that the sequence $(f_k)_k$ is not strongly convergent in $\mathrm{L}^p([-1,1])$.

We have for every k that

$$f_{2^k}\left(\frac{1}{2^{k+1}}\right) - f_{2^{k+1}}\left(\frac{1}{2^{k+1}}\right) = \sin\left(\frac{\pi}{2}\right) - \sin(\pi) = 1 - 0 = 1 \,,$$

and hence

$$\|f_{2^{k+1}} - f_{2^k}\|_\infty \geq 1$$

because $f_{2^{k+1}} - f_{2^k}$ is continuous. This shows that the sequence $(f_k)_k$ is not Cauchy in $\mathrm{L}^\infty([-1,1])$, and hence not strongly convergent in $\mathrm{L}^\infty([-1,1])$.

(ii)

Boundedness

We have for every $1 \leq p < \infty$ that

$$\begin{aligned}
\|f_k\|_p^p = \int_0^1 |f_k(x)|^p \, \mathrm{d}x = \int_0^{1/k} (k^{1/4} - k^{5/4}x)^p \, \mathrm{d}x \xlongequal[x \to x/k]{} \int_0^1 (k^{1/4} - k^{1/4}x)^p \frac{1}{k} \, \mathrm{d}x \\
\xlongequal[x \to x+1]{} \int_{-1}^{0} (-k^{1/4}x)^p \frac{1}{k} \, \mathrm{d}x = \int_0^1 (k^{1/4}x)^p \frac{1}{k} \, \mathrm{d}x = k^{-3/4} \int_0^1 x^p \, \mathrm{d}x = \frac{k^{-3/4}}{p+1} \,.
\end{aligned}$$

We find that $\|f_k\|_p \to 0$ as $k \to \infty$, and hence that the sequence $(f_k)_k$ is bounded in $\mathrm{L}^p([0,1])$.

The functions f_k are continuous with $f_k(0) = k^{1/4}$ and strictly decreasing until they reach 0, whence $\|f_k\|_\infty = k^{1/4}$. The sequence $(f_k)_k$ is therefore not bounded in $\mathrm{L}^\infty([0,1])$.

Weak Convergence

It follows for $1 \leq p < \infty$ from $f_k \to 0$ (which we show below) that $f_k \rightharpoonup 0$. The sequence $(f_k)_k$ is unbounded in $\mathrm{L}^\infty([0,1])$, and is therefore not weakly* convergent.

Strong Convergence

We have seen above that $\|f_k\|_p \to 0$ for $1 \leq p < \infty$, and hence $f_k \to 0$. The sequence $(f_k)_k$ is unbounded in $\mathrm{L}^\infty([0,1])$, and hence not strongly convergent.

(iii)

If $\varphi = 0$ then $f_k = 0$ for every k. Then $f_k \to 0$ and consequently also $f_k \rightharpoonup 0$, and the sequence $(f_k)_k$ is bounded in $\mathrm{L}^p(\mathbb{R})$ for all $1 \leq p \leq \infty$. So let's assume in the following that $\varphi \neq 0$. Then $\|\varphi\|_p > 0$ for every $1 \leq p \leq \infty$ by the continuity of φ.

Boundedness

We have $\|f_k\|_p = \|\varphi\|_p$ for every $1 \leq p \leq \infty$, whence the sequence $(f_k)_k$ is bounded in $\mathrm{L}^p(\mathbb{R})$ for very $1 \leq p \leq \infty$.

Weak Convergence

We note that $\operatorname{supp}(f_k) \subseteq [k-1, k+1]$ for every k.

If $1 < p < \infty$ then there exist for every $f' \in \mathrm{L}^p(\mathbb{R})'$ some $g \in \mathrm{L}^q(\mathbb{R})$ (where p and q are dual exponents) with $\langle \tilde{f}, f' \rangle = \int_{\mathbb{R}} \tilde{f} g$ for every $\tilde{f} \in \mathrm{L}^p(\mathbb{R})$. It follows that

$$|\langle f_k, f' \rangle| = \left| \int_{\mathbb{R}} f_k g \right| = \left| \int_{[k-1,k+1]} f_k g \right| \leq \int_{[k-1,k+1]} |f_k||g| \leq \|\varphi\|_\infty \int_{[k-1,k+1]} |g| \to 0\,.$$

Here we use that $\mathrm{L}^q([k-1,k+1]) \subseteq \mathrm{L}^1([k-1,k+1])$ because the compact interval $[k-1,k+1]$ has finite measure. This shows that $f_k \rightharpoonup 0$ if $1 < p < \infty$.

For $p = 1$ we can consider a 4-periodic function $g \in \mathrm{L}^\infty(U)$ with

$$g\,|_{(-1,1)} = \operatorname{sign}(\varphi)\,|_{(-1,1)} \quad \text{and} \quad g\,|_{(1,3)} = 0\,.$$

(We do not care about the values $g(n)$ with $n \in \mathbb{Z}$.) We consider the resulting functional $f' \in \mathrm{L}^1(\mathbb{R})'$ given by

$$\langle f, f' \rangle \coloneqq \int_{\mathbb{R}} f g\,.$$

We have that

$$\langle f_{4k}, f \rangle = \int_{\mathbb{R}} f_{4k} g = \int_{4k-1}^{4k+1} f_{4k} g = \int_{-1}^{1} \varphi g = \int_{-1}^{1} \varphi \operatorname{sign}(\varphi) = \int_{-1}^{1} |\varphi| = \|\varphi\|_1 > 0\,,$$

and

$$\langle f_{4k+2}, f \rangle = \int_{\mathbb{R}} f_{4k+2} g = \int_{4k+1}^{4k+3} f_{4k+2} g = \int_1^3 f_2 g = \int_{-1}^1 0 = 0\,.$$

This shows that the sequence $(\langle f_k, f' \rangle)_k$ does not converge, hence that the sequence $(f_k)_k$ does not converge weakly.

Consider now $p = \infty$. We have for every $f \in \mathrm{L}^1(\mathbb{R})$ that

$$\begin{aligned} |\langle f, f_k \rangle| = \left| \int_{\mathbb{R}} f f_k \right| \leq \int_{\mathbb{R}} |f||f_k| = \int_{[k-1,k+1]} |f||f_k| \\ \leq \int_{[k-1,k+1]} \|\varphi\|_\infty |f| \leq \|\varphi\|_\infty \int_{[k-1,k+1]} |f| \to 0\,, \end{aligned}$$

which shows that $f_k \overset{*}{\rightharpoonup} 0$ in $\mathrm{L}^\infty(\mathbb{R})$.

Strong Convergence

We have for all k that f_k and f_{k+3} have disjoint support, because $\operatorname{supp}(f_k) \subseteq \mathrm{B}(k, 1)$. Hence

$$\|f_k - f_{k+3}\|_p^p = \|f_k\|^p + \|f_{k+3}\|_p^p = 2\|\varphi\|_p^p > 0$$

for all k and every $1 \leq p < \infty$. This shows that the sequence $(f_k)_k$ is not Cauchy in $\mathrm{L}^p(\mathbb{R})$ for $1 \leq p < \infty$, and hence not strongly convergent.

We similarly find that $\|f_k - f_{k+3}\|_\infty = \|\varphi\|_\infty$ for all k, and hence that the sequence $(f_k)_k$ is also not strongly convergent in $\mathrm{L}^\infty(\mathbb{R})$.

(For $1 < p \leq \infty$ one can also argue that $f_k \rightharpoonup 0$, and that the sequence $(f_k)_k$ is therefore strongly convergent in $\mathrm{L}^p(\mathbb{R})$ if and only if $f_k \to 0$ in $\mathrm{L}^p(\mathbb{R})$. But $\|f_k\|_p = \|\varphi\| > 0$ for every k, so that $f_k \not\to 0$ in $\mathrm{L}^p(\mathbb{R})$.)

(iv)

If $\varphi = 0$ then $f_k = 0$ for every k. We then find for all $1 \leq p \leq \infty$ that the sequence $(f_k)_k$ isd bounded in $\mathrm{L}^p(\mathbb{R}^n)$, and $f_k \to 0$ and $f_k \rightharpoonup 0$. We will therefore assume in the following that $\varphi \neq 0$, and therefore $\|\varphi\|_p > 0$ for all $1 \leq p \leq \infty$.

Boundedness

We have for every $1 \leq p < \infty$ that

$$\begin{aligned} \|f_k\|_p^p = \int_{\mathbb{R}^n} \left| k^{-n/2} \varphi\left(\frac{x}{k}\right) \right|^p \mathrm{d}x \underset{x \to kx}{=\!=\!=} \int_{\mathbb{R}^n} |k^{-n/2} \varphi(x)|^p k^n \,\mathrm{d}x \\ = k^{n - np/2} \int_{\mathbb{R}^n} |\varphi(x)|^p \,\mathrm{d}x = k^{n(1-p/2)} \|\varphi\|_p^p\,. \end{aligned}$$

This shows that the sequence $(f_k)_k$ is bounded in $\mathrm{L}^p(\mathbb{R}^n)$ if and only if $1 - p/2 \leq 0$, i.e. if and only if $p \geq 2$.

We have for $p = \infty$ that $\|f_k\|_\infty = k^{-n/2} \|\varphi\|_\infty$, where $k^{-n/2} \to 0$ as $k \to \infty$. The sequence $(f_k)_k$ is therefore bounded in $\mathrm{L}^\infty(\mathbb{R}^n)$.

Weak Convergence

If $p < 2$ then the sequence $(f_k)_k$ is unbounded in $\mathrm{L}^p(\mathbb{R}^n)$ because

$$\|f_k\|_p^p = k^{n(1-p/2)} \|\varphi\|_p^p \to \infty\,.$$

The sequence is then not weakly convergent in $\mathrm{L}^p(\mathbb{R}^n)$.

If $p > 2$ then $f_k \to 0$ (see below) and hence $f_k \rightharpoonup 0$ (resp. $f_k \overset{*}{\rightharpoonup} 0$). It similarly follows from $f_k \to 0$ in $\mathrm{L}^\infty(\mathbb{R}^n)$ (see again below) that $f_k \overset{*}{\rightharpoonup} 0$.

For $p = 2$ we show that $f_k \rightharpoonup 0$. For this we use that $\mathrm{L}^2(\mathbb{R}^n)' = \mathrm{L}^2(\mathbb{R}^n)$. We therefore need to show that

$$\int_{\mathbb{R}^n} f_k g \to 0$$

for every $g \in \mathrm{L}^2(\mathbb{R}^n)$. By part (i) of Exercise 2 is sufficies to show this for $g \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ because $\mathrm{C}_c^\infty(\mathbb{R}^n)$ is dense in $\mathrm{L}^2(\mathbb{R}^n)$. We have for every $g \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ that

$$\begin{aligned}\int_{\mathbb{R}^n} f_k(x) g(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} k^{-n/2} \varphi\left(\frac{x}{k}\right) g(x)\,\mathrm{d}x \\ &\underset{k \to kx}{=\!=\!=} \int_{\mathbb{R}^n} k^{-n/2} \varphi(x) g(kx) k^n\,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x) g_k(x)\,\mathrm{d}x\end{aligned}$$

where $g_k(x) \coloneqq k^{n/2} g(kx)$. We have seen in the lecture (page 70, part (ii) near the beginning of the page) that

$$\int_{\mathbb{R}_n} \varphi(x) g_k(x) \to 0\,,$$

as desired.

Strong Convergence

We have seen above that the sequence $(f_k)_k$ is for $p < 2$ unbounded, and hence not strongly convergent.

We find for $2 < p < \infty$ that $\|f_k\|_p^p = k^{n(1-p/2)} \|\varphi\|_p^p \to 0$ and hence $f_k \to 0$. For $p = \infty$ we have seen above that $\|f_k\|_\infty = k^{-n/2} \|\varphi\|_\infty \to 0$, whence $f_k \to 0$ in $\mathrm{L}^\infty(\mathbb{R})$.

If the sequence $(f_k)_k$ would strongly converge in $\mathrm{L}^2(\mathbb{R}^2)$, say $f_k \to f$, then also $f_k \rightharpoonup f$. We have seen above that $f_k \rightharpoonup 0$, hence we would have $f = 0$. But

$$\|f_k\|_2^2 = k^{n(1-2/2)} \|\varphi\|_p^p = k^0 \|\varphi\|_p^p = \|\varphi\|_2^2$$

for every k, so $f \nrightarrow 0$. This shows that the sequence $(f_k)_k$ does not strongly converge in $\mathrm{L}^2(\mathbb{R}^n)$.