Exercises in PDE and Functional Analysis

Exercise Sheet 10

Jendrik Stelzner

Exercise 1

We suppose that $f = \tilde{f}$ ought to be an equality in $\mathrm{L}^p(U)$, and that f should be to be measurable. We show that $f_k \rightharpoonup f$ for $1 \leq p < \infty$, and that $f_k \overset{*}{\rightharpoonup} f$ for $p = \infty$. It then follows that $f = \tilde{f}$ in $\mathrm{L}^p(U)$ by the uniqueness of weak limits.

We first note that $f \in \mathrm{L}^p(U)$: For $1 \leq p < \infty$ the sequence $(f_k)_k$ is bounded in $\mathrm{L}^p(U)$ because it is weakly convergent. Hence

$$\|f\|_p^p = \int_U |f|^p = \int_U \liminf_{k \to \infty} |f_k|^p \leq \liminf_{k \to \infty} \int_U |f_k|^p = \liminf_{k \to \infty} \|f_k\|_p^p < \infty$$

by Fatou's lemma. For $p = \infty$ we similarly find that the sequence $(f_k)_k$ is bounded in $\mathrm{L}^\infty(U)$ because it is weakly* convergent. It follows that

$$|f(x)| = \lim_{k \to \infty} |f_k(x)| \leq \limsup_{k \to \infty} \|f_k\|_\infty < \infty$$

for almost all $x \in U$, and hence that $\|f\|_\infty < \infty$.

The case $1 \leq p < \infty$

We consider first the case $1 \leq p < \infty$. We fix $T \in \mathrm{L}^p(U)'$ and show that $Tf_k \to Tf$. For this we may replace f_k by $f_k - f$ to assume that $f = 0$. We hence need to show that $Tf_k \to 0$. Let $\varepsilon > 0$.

We know that $\mathrm{L}^p(U)' = \mathrm{L}^q(U)$ with $1/p + 1/q = 1$, in the sense that there exists some $g \in \mathrm{L}^q(U)$ with

$$Th = \int_U gh$$

for every $h \in \mathrm{L}^p(U)$. We therefore have for every k that

$$|Tf_k| = \left| \int_U g f_k \right| \leq \int_U |g f_k| \,.$$

We will split this integral up into three parts:

We first show that there exists a subset $A \subseteq U$ of finite measure with $\|g\|_{q,U\setminus A} < \varepsilon$. Indeed, there exists an increasing sequence

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

of subsets $A_n \subseteq U$ of finite measure with $U = \bigcup_n A_n$ because U is σ-finite. It follows from Lebesgue's monotone convergence theorem that

$$\|g\|_{q,A_n}^q = \int_{A_n} |g|^q = \int_U \chi_{A_n} |g|^q \to \int_U |g|^q = \|g\|_{q,U}^q$$

and hence $\|g\|_{q,U\setminus A_n} \to 0$. For n sufficiently large we hence have $\|g\|_{q,U\setminus A_n} < \varepsilon$, and may choose $A \coloneqq A_n$.

The sequence $(f_k)_k$ is bounded in $\mathrm{L}^p(U)$, hence there exists $C > 0$ with $\|f_k\|_{p,U} < C$ for every k. It follows for every k that

$$\int_U |gf_k| = \int_A |gf_k| + \int_{U\setminus A} |gf_k|$$

with

$$\int_{U\setminus A} |gf_k| \leq \|g\|_{q,U\setminus A} \|f_k\|_{p,U\setminus A} \leq \|g\|_{q,U\setminus A} \|f_k\|_{p,U} \leq C\varepsilon\,.$$

We have that $f_k \to 0$ almost everywhere, and hence there exists by Egorov's theorem for every $\delta > 0$ some measurable subset $A' \subseteq A$ with $\lambda(A \setminus A') < \delta$, such that $f_k \to 0$ uniformly on A'. (Here λ denotes the Lebesgue measure.) It follows that there exists a measurable subset $A' \subseteq A$ with $\|g\|_{q,A\setminus A'} < \varepsilon$ such that $f_k \to 0$ uniformly on A': Indeed, there exists for every $n \geq 1$ some measurable subset $A'_n \subseteq A$ with $\lambda(A \setminus A'_n) < 1/n$ such that $f_k \to 0$ uniformly on A'_n. We may assume that $A'_n \subseteq A_{n+1}$ for every n by replacing the set A'_n with $\bigcup_{m\leq n} A_m$ (as given so far) for every n. Then

$$\lambda\left(U \setminus \bigcup_{n=1}^{\infty} A'_n\right) = \lambda\left(\bigcap_{n=1}^{\infty} (U \setminus A'_n)\right) = \lim_{n\to\infty} \lambda(U \setminus A'_n) = 0$$

and hence $\chi_{A'_n} \to 1$ almost everywhere on A. It follows from another application of Lebesgue's monotone convergence theorem that

$$\|g\|_{q,A'_n}^q = \int_{A'_n} |g|^q = \int_A \chi_{A'_n} |g|^q \to \int_A |g|^q = \|g\|_{q,A}^q\,,$$

and therefore $\|g\|_{q,A\setminus A'_n} \to 0$. For n sufficiently large we hence have $\|g\|_{q,A\setminus A'_n} < \varepsilon$, and may choose $A' \coloneqq A'_n$. We now have with $C'_{A'} \coloneqq \|g\|_{q,U} \lambda(A')^{1/p}$ that

$$\begin{aligned}
\int_A |gf_k| &= \int_{A'} |gf_k| + \int_{A\setminus A'} |gf_k| \\
&\leq \|g\|_{q,A'} \|f_k\|_{p,A'} + \|g\|_{q,A\setminus A'} \|f_k\|_{p,A\setminus A'} \\
&\leq \|g\|_{q,U} \|f_k\|_{\infty,A'} \lambda(A')^{1/p} + \|g\|_{q,A\setminus A'} \|f_k\|_{p,U} \\
&\leq C'_{A'} \|f_k\|_{\infty,A'} + C\varepsilon\,.
\end{aligned}$$

Altogether we have that

$$|Tf_k| \le \int_U |gf_k| = \int_{A'} |gf_k| + \int_{A\setminus A'} |gf_k| + \int_{U\setminus A} |gf_k| \le C'_{A'} \|f_k\|_{\infty,A'} + 2C\varepsilon\,.$$

We have $f_k \to 0$ uniformly on A' and hence $\|f_k\|_{\infty,A'} \to 0$. We find that

$$\limsup_{k\to\infty} |Tf_k| \le 2C\varepsilon\,.$$

By letting $\varepsilon \to 0$ we find that $|Tf_k| \to 0$.

The case $p = \infty$

We fix $g \in \mathrm{L}^1(U)$ and need to show that

$$\int_U f_k g \to \int_U fg\,.$$

This follows from the same argumentation as above.

Exercise 2

(i)

Suppose first that $x^{(k)} \rightharpoonup x$. For every i the projection operator

$$P_i \colon \ell^p \to \mathbb{R}\,, y = (y_n)_n \mapsto y_i$$

is Lipschitz-continuous (with Lipschitz constant 1.) It therefore follows from $x^{(k)} \rightharpoonup x$ that

$$x_i^{(k)} = P_i x^{(k)} \to P_i x = x_i\,.$$

We have seen in the lecture that weakly convergent sequences are bounded.

Suppose now that $x_i^{(k)} \to x_i$ for every i, and that the sequence $(x^{(k)})_k$ is bounded in ℓ^p with $\|x^{(k)}\|_p \le C$ for every k. We fix $T \in (\ell^p)'$ and need to show that $Tx^{(k)} \to Tx$. We may replace $x^{(k)}$ by $x^{(k)} - x$ to assume that $x = 0$. We hence have $x_i^{(k)} \to 0$ for every i and $\|x^{(k)}\|_p \le C$ for every k, and we need to show that $Tx^{(k)} \to 0$.

We know that $(\ell^p)' = \ell^q$ where $1/p + 1/q = 1$, in the sense that there exists a sequence $(a_n)_n \in \ell^q$ with

$$Ty = \sum_i a_i y_i$$

for every sequence $y = (y_n)_n \in \ell^p$. We may consider for any N the sequence

$$a^{(N)} \coloneqq (a_{N+1}, a_{N+2}, \dots) \in \ell^q\,,$$

for which we have by Hölder's inequality

$$|Ty| = \left|\sum_i a_i y_i\right| \leq \left|\sum_{i=1}^{N} a_i y_i\right| + \left|\sum_{i=N+1}^{\infty} a_i y_i\right| \leq \sum_{i=1}^{N} |a_i||y_i| + \|a^{(N)}\|_q \|y\|_p$$

for every $y = (y_n)_n \in \ell^p$. We have in particular that

$$|Tx^{(k)}| \leq \sum_{i=1}^{N} |a_i||x_i^{(k)}| + C\|a^{(N)}\|_q$$

and therefore

$$\limsup_{k\to\infty} |Tx^{(k)}| \leq C\|a^{(N)}\|_q$$

for every N. We have that

$$\|a^{(N)}\|_q^q = \sum_{i=N+1}^{\infty} |a_i|^q \to 0$$

as $N \to \infty$, and hence find that

$$\limsup_{k\to\infty} |Tx^{(k)}| = 0\,.$$

This shows that $Tx^{(k)} \to 0$, as desired.

(ii)

Strong convergence implies weak convergence, so we only need to show that $x^{(k)} \to x$ if $x^{(k)} \rightharpoonup x$. For this we may replace $x^{(k)}$ by $x^{(k)} - x$ to assume that $x = 0$. We hence have that $x^{(k)} \rightharpoonup 0$ and need to show that $x^{(k)} \to 0$. We note that every subsequence of $(x^{(k)})_k$ again weakly converges to 0.

We show that $(x^{(k)})_k$ has a subsequence that converges to 0; i.e. we show that every sequence in ℓ^1 that converges weakly to 0 has a subsequence that converges strongly to 0. It then follows that every subsequence of $(x^{(k)})_k$ has a subsequence that converges strongly to 0 (because every subsequence of $(x^{(k)})_k$ is again weakly convergent to 0). This then means that $x^{(k)} \to 0$.

There exists for every j a subsequence $(x^{(k_i)})_i$ with

$$x_j^{(k_i)} \geq 0 \text{ for every } i \qquad \text{or} \qquad x_j^{(k_i)} \leq 0 \text{ for every } i$$

because there exist inifinitely many k with $x_j^k \geq 0$ or infinitely many k with $x_j^{(k)} \leq 0$. By using the diagonal sequence trick we find that there exists a subsequence $(x^{(k_i)})_i$ such that for every j we have $x_j^{(k_i)} \geq 0$ for all i or $x_j^{(k_i)} \leq 0$ for all i. We may replace the sequence $(x^{(k)})_k$ by this subsequence to assume that for every i, $x_i^{(k)}$ has for all k the same sign (or is 0).

There hence exists a sequence $(a_i)_i \subseteq \{1, -1\}$ with $a_i x_i^{(k)} = |x^{(k)}|_i$ for all i and all k, and therefore

$$\sum_i a_i x_i^{(k)} = \sum_i |x_i^{(k)}| = \|x^{(k)}\|_1$$

for all k. The sequence $(a_i)_i$ defines a linear functional $T \in \ell^\infty = (\ell_1)'$ given by

$$Ty = \sum_i a_i y_i$$

for every $y \in \ell^1$. We find with $x^{(k)} \rightharpoonup 0$ that

$$\|x^{(k)}\|_1 = \sum_i a_i x_i^{(k)} = Tx^{(k)} \to 0\,.$$

Exercise 3

If $x_n \to x$ then also $x_n \rightharpoonup x$, and it follows from the continuity of the norm $\|\cdot\|\colon X \to \mathbb{R}$ that also $\|x_n\| \to \|x\|$.

Suppose now that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. If $x = 0$ then $\|x_n\| \to 0$ and hence $x_n \to 0 = x$. In the following we will consider the case $x \neq 0$.

We can replace x by $x/\|x\|$ and x_k by $x_k/\|x\|$ to assume that $\|x\| = 1$. It follows from $\|x_k\| \to \|x\| = 1$ that $x_k \neq 0$ for all but finitely many k, so we may assume that $x_k \neq 0$ for every k. It also follows from $\|x_k\| \to \|x\| = 1$ that

$$\left\| x_k - \frac{x_k}{\|x_k\|} \right\| = \left| 1 - \frac{1}{\|x_k\|} \right| \|x_k\| \to 0\,,$$

and we also have for every $T \in X'$ that

$$T\left(\frac{x_k}{\|x_k\|}\right) = \frac{Tx_k}{\|x_k\|} \to \frac{Tx}{\|x\|} = Tx$$

because $x_k \rightharpoonup x$; this shows that also $x_k/\|x_k\| \rightharpoonup x$. Together this shows that we may also replace x_k by $x_k/\|x_k\|$ to additionally asume that $\|x_k\| = 1$ for every k.

We now have that $\|x\| = 1$ and $\|x_k\| = 1$ for all k, and that $x_k \rightharpoonup x$. We have that

$$\left\| \frac{x_k + x}{2} \right\| \leq \frac{\|x_k\| + \|x\|}{2} = 1$$

for all k. That $x_k \to x$ is therefore equivalent to

$$\left\| \frac{x_k + x}{2} \right\| \to 1$$

because X is uniformly convex. We know from the lecture (Corollary 6.6) that there exists a linear functional $T \in X'$ with $\|T\| = 1$ and $Tx = \|x\| = 1$. It follows that

$$1 \geq \left\| \frac{x_k + x}{2} \right\| \geq \left| T\left(\frac{x_k + x}{2}\right) \right| = \left| \frac{Tx_k + Tx}{2} \right| \to |Tx| = \|x\| = 1$$

and therefore

$$\left\| \frac{x_k + x}{2} \right\| \to 1\,,$$

as desired.

Exercise 6

We assume that the interval I is nonempty, because otherwise $F = 0$ independent of the choice of f. For the length $|I|$ we hence have $|I| > 0$.

The map F is well-defined: A function $u \in \mathrm{L}^\infty(I)$ is essentially bounded, and thus there exists a compact interval $J \subseteq \mathbb{R}$ with $u(t) \subseteq J$ for almost all $t \in I$. The continous map f is bounded on J, and hence $f \circ u$ is almost bounded on I. The composition $f \circ u$ is measurable because both f and c are measurable, and we find that

$$\int_I |f \circ u| \leq \|f \circ u\|_\infty |I| < \infty\,.$$

(i)

Suppose that f is affine. Then there exist $a, b \in \mathbb{R}$ with $f(x) = ax + b$ for all $x \in \mathbb{R}$. Suppose that $(u_k)_k$ is a sequence in $\mathrm{L}^\infty(I)$ and that $u \in \mathrm{L}^\infty(I)$ with $u_k \overset{*}{\rightharpoonup} u$. Then

$$\int_I u_k = \int_I 1 \cdot u_k \to \int_I 1 \cdot u \to \int_I u$$

because $1 \in \mathrm{L}^1(I)$ (since I is bounded), and therefore

$$F(u_k) = a \int_I u_k + b|I| \to a \int_I u + b|I| = F(u)\,.$$

Suppose now on the other hand that F is continuous with respect to weakly* convergence. Let $x, y \in \mathbb{R}$ and let $0 \leq \lambda \leq 1$. We define for every $k \geq 1$ a function $u_k \in \mathrm{L}^\infty(I)$ by subdividing the interval I into $2k$ subintervals of alternating lengths $\lambda|I|/(2k)$ and $(1-\lambda)|I|/(2k)$, and letting u_k assume the alternating values

$$x, y, x, y, \dots, x, y$$

on these subintervals. We know from Exercise 3, part (ii) of sheet 9 that

$$u_k \rightharpoonup \lambda x + (1-\lambda) y =: u\,.$$

We have that $\int_I f \circ u_k = \lambda |I| f(x) + (1-\lambda)|I| f(y)$ for every k, and hence find that

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &= \frac{1}{|I|} \int_I f \circ u = F(u) = \frac{1}{|I|} \lim_{k \to \infty} F(u_k) = \frac{1}{|I|} \lim_{k \to \infty} \int_I f \circ u_k \\ &= \frac{1}{|I|} \lim_{k \to \infty} \left(\lambda |I| f(x) + (1-\lambda)|I| f(y) \right) = \lambda f(x) + (1-\lambda) f(y)\,. \end{aligned}$$

This shows that f is affine.

(ii)

Suppose now that F is lower semicontinuous with respect to weakly* convergence. To show that f is convex we can proceed as in part (i), and find that

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \frac{1}{|I|}\int_I f \circ u = F(u) \leq \frac{1}{|I|}\liminf_{k\to\infty} F(u_k) = \frac{1}{|I|}\liminf_{k\to\infty}\int_I f \circ u_k \\ &= \frac{1}{|I|}\liminf_{k\to\infty}\Big(\lambda|I|f(x) + (1-\lambda)|I|f(y)\Big) = \lambda f(x) + (1-\lambda)f(y)\,. \end{aligned}$$

I also spent some hours trying to prove the other impliciation, but without success. The best I got is that the map F is lower semicontinuous with respect to the weak convergence in $\mathrm{L}^\infty(I)$:

Let $(u_k)_k$ be a sequence in $\mathrm{L}^\infty(I)$ with $u_k \rightharpoonup u$ for some $u \in \mathrm{L}^\infty(I)$. To show that $F(u) \leq \liminf_{k\to\infty} F(u_k)$ we may replace the sequence $(u_k)_k$ by a suitable subsequence to assume that

$$\liminf_{k\to\infty} F(u_k) = \lim_{k\to\infty} F(u_k)\,. \tag{1}$$

We also note that because the sequence $(u_k)_k$ is bounded in $\mathrm{L}^\infty(I)$, there exists a compact interval $J \subseteq \mathbb{R}$ with $u_k(t) \in J$ and $u(t) \in J$ for almost all $t \in I$. Let $\varepsilon > 0$. There exists some $\delta > 0$ with $|f(x) - f(y)| < \varepsilon$ for all $x, y \in J$ with $|x - y| < \delta$ because f is uniformly continuous on J.

By Mazur's lemma there exists for every j a convex combination $\sum_{k=j}^{n_j} \lambda_k^j u_k$ with

$$\left\| u - \sum_{k=j}^{n_j} \lambda_k^j u_k \right\|_\infty < \delta\,.$$

We observe that F is convex, because f is convex and the integral $\int_I$ is both linear and monotone. It follows that

$$\begin{aligned} F(u) = \int_I f \circ u &\leq \int_I \left(f \circ \left(\sum_{k=j}^{n_j} \lambda_k^j u_k \right) + \varepsilon \right) \\ &= F\left(\sum_{k=j}^{n_j} \lambda_k^j u_k \right) + \varepsilon|I| \leq \sum_{k=j}^{n_j} \lambda_k^j F(u_k) + \varepsilon|I|\,. \end{aligned}$$

It follows from (1) that

$$\sum_{k=j}^{n_j} \lambda_k^j F(u_k) \to \liminf_{k\to\infty} F(u_k)$$

as $j \to \infty$. We therefore find that

$$F(u) \leq \liminf_{k\to\infty} F(u_k) + \varepsilon|I|\,.$$

By taking $\varepsilon \to 0$ we arrive at $F(u) \leq \liminf_{k\to\infty} F(u_k)$.