# Remark and Solutions

### Sheet 1

## Problem 14

### 1.

To construct a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ we only need to specify the images $E_i$, $H_i$, $F_i$ of the generators $e_i$, $h_i$, $f_i$ and then check that these proposed images satisfy the given relations (22)–(27). (This is what it means for $\mathfrak{g}$ to be given by the generators $\{e_i, h_i, f_i\}_{i=1}^n$ and relations (22)–(27).)

In the case $n = 1$ we know that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the standard basis $e$, $f$, $h$ constisting of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These three basis vectors satisfy the relations $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$. Motivated by this example we will choose for arbitrary $n \geq 1$ the proposed images $E_i, F_i, H_i \in \mathfrak{sl}_{n+1}(\mathbb{C})$ as

$$E_i \coloneqq E_{i,i+1}, \quad F_i \coloneqq E_{i+1,i}, \quad H_i \coloneqq E_{ii} - E_{i+1,i+1}.$$

We need to check that these elements satisfy the relations (22)–(27):

(22) We find $[H_i, H_j] = 0$ because both $H_i$ and $H_j$ are diagonal matrices and hence commute with each other.

(23) We find

$$\begin{aligned}
[E_i, F_j] &= E_i F_j - F_j E_i \\
&= E_{i,i+1} E_{j+1,j} - E_{j+1,j} E_{i,i+1} \\
&= \delta_{i+1,j+1} E_{ij} - \delta_{ij} E_{j+1,i+1} \\
&= \delta_{ij} (E_{ii} - E_{i+1,i+1}) \\
&= \delta_{ij} H_i
\end{aligned}$$

where we used that $\delta_{i+1,j+1} = \delta_{ij}$ and $\delta_{ij} E_{ij} = \delta_{ij} E_{ii}$.

(24) We find

$$\begin{aligned}
[H_i, E_j] &= [E_{ii} - E_{i+1,i+1}, E_{j,j+1}] \\
&= (E_{ii} - E_{i+1,i+1})E_{j,j+1} - E_{j,j+1}(E_{ii} - E_{i+1,i+1}) \\
&= E_{ii}E_{j,j+1} - E_{i+1,i+1}E_{j,j+1} - E_{j,j+1}E_{ii} + E_{j,j+1}E_{i+1,i+1} \\
&= \delta_{ij}E_{i,j+1} - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} + \delta_{i+1,j+1}E_{j,i+1} \\
&= \delta_{ij}(E_{i,j+1} + E_{j,i+1}) - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} \\
&= 2\delta_{ij}E_{j,j+1} - \delta_{i+1,j}E_{j,j+1} - \delta_{i,j+1}E_{j,j+1} \\
&= (2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1})E_{j,j+1} \\
&= a_{ji}E_j \,.
\end{aligned}$$

(25) This relation can be checked in the same way as (24). But we can also observe that $F_j = E_j^T$ and $H_i = H_i^T$ whence

$$[H_i, F_j] = [H_i^T, E_j^T] = [E_j, H_i]^T = a_{ij}E_j^T = -a_{ij}F_j \,.$$

(26) We find

$$\begin{aligned}
\mathrm{ad}(E_i)(E_j) &= [E_i, E_j] \\
&= [E_{i,i+1}, E_{j,j+1}] \\
&= E_{i,i+1}E_{j,j+1} - E_{j,j+1}E_{i,i+1} \\
&= \delta_{i+1,j}E_{i,j+1} - \delta_{i,j+1}E_{j,i+1} \,.
\end{aligned}$$

We find for $|i-j| \geq 2$ that $\delta_{i+1,j} = \delta_{i,j+1} = 0$ and hence

$$\mathrm{ad}(E_i)^{-a_{ji}+1}(E_j) = \mathrm{ad}(E_i)(E_j) = 0\,.$$

as desired. If $i = j-1$ then $\mathrm{ad}(E_i)(E_j) = E_{i,j+1} = E_{i,i+2}$ and $a_{ji} = -1$ and thus

$$\begin{aligned}
\mathrm{ad}(E_i)^{-a_{ji}+1}(E_j) &= \mathrm{ad}(E_i)^2(E_j) \\
&= [E_i, [E_i, E_j]] \\
&= [E_i, E_{i,i+2}] \\
&= [E_{i,i+1}, E_{i,i+2}] \\
&= E_{i,i+1}E_{i,i+2} - E_{i,i+2}E_{i,i+1} \\
&= \delta_{i,i+1}E_{i,i+2} - \delta_{i,i+2}E_{i,i+1} \\
&= 0
\end{aligned}$$

The case $i = j+1$ works the same.

(27) This can be done by similar calculations as (26) but can also be directly derived from (26) by again using the matrix transpose.

**Remark 1.**

1. We used for (25) that that $[A,B]^T = (AB - BA)^T = B^T A^T - A^T B^T = [B^T, A^T]$.
2. For (24) and (25) it is useful to understand how a commutator $[D, A]$ looks like if $D$ is a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n \in k$:

   The matrix $DA$ results from $A$ by multiplying for every $i = 1, \dots, n$ the $i$-th row of $A$ by the corresponding diagonal entry $\lambda_i$. Similarly the matrix $AD$ results from $A$ by multiplying for every $j = 1, \dots, n$ the $j$th column of $A$ by the corresponding diagonal entry $\lambda_j$. This means in formulae that

   $$(DA)_{ij} = \lambda_i A_{ij} \quad \text{and} \quad (AD)_{ij} = \lambda_j A_{ij}$$

   for all $i, j = 1, \dots, n$, and thus

   $$[D, A]_{ij} = (DA - AD)_{ij} = (DA)_{ij} - (AD)_{ij} = \lambda_i A_{ij} - \lambda_j A_{ij} = (\lambda_i - \lambda_j) A_{ij}$$

   for all $i, j = 1, \dots, n$.

   It follows that

   $$[H_i, E_j] = [H_i, E_{j,j+1}] = ((H_i)_{jj} - (H_i)_{j+1,j+1}) E_{j,j+1}$$

   where

   $$(H_i)_{jj} - (H_i)_{j+1,j+1} = \begin{cases} 0 & \text{if } j+1 < i\,, \\ -1 & \text{if } j+1 = i\,, \\ 2 & \text{if } j = i\,, \\ -1 & \text{if } j = i+1\,, \\ 0 & \text{if } j > i+1\,. \end{cases}$$

   Hence $[H_i, E_j] = a_{j,i} E_j$ as desired.

We have shown that $E_i, F_i, H_i$ with $i = 1, \dots, n$ satisfy the given relations, and all these matrices are contained in $\mathfrak{sl}_{n+1}(\mathbb{C})$. There hence exists a unique homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ with $\varphi(e_i) = E_i$, $\varphi(f_i) = F_i$ and $\varphi(h_i) = H_i$. It remains to show that $\varphi$ is surjective.

The image of $\varphi$ is a Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$. It therefore suffieces to show that $\mathfrak{sl}_{n+1}(\mathbb{C})$ is generated by the elements $\{E_i, F_i, H_i\}_{i=1}^n$ as a Lie algebra. Let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$ generated by these elements. We know that $\mathfrak{sl}_{n+1}(\mathbb{C})$ has a basis the diagonal matrices $H_1, \dots, H_n$ together with the off-diagonal matrices $E_{ij}$ where $1 \leq i \neq j \leq n+1$. It sufficies to show that these matrices are contained in $\mathfrak{s}$. This holds for $H_1, \dots, H_n$ by construction of $\mathfrak{s}$.

Let us consider the off-diagonal matrices $E_{ij}$ with $j > i$. We fix the index $i$ and show that $E_{i,i+1}, E_{i,i+2}, \dots, E_{i,n+1} \in \mathfrak{s}$. This holds for $E_{i,i+1} = E_i$ by construction of $\mathfrak{s}$. If $E_{ij} \in \mathfrak{s}$ for some $i+1 \leq j < n+1$ then we find inductively that the matrix

$$[E_{ij}, E_j] = [E_{ij}, E_{j,j+1}] = E_{ij} E_{j,j+1} - E_{j,j+1} E_{ij} = E_{i,j+1}$$

is again contained $\mathfrak{s}$. This shows that all off-diagonal matrices $E_{ij}$ with $j > i$ are contained in $\mathfrak{s}$.

For the off-diagonal matrices $E_{ij}$ with $i < j$ we can argue in the same way by using the matrices $F_i$ instead of $E_i$. But we could also observe that the Lie algebra generating set $\{E_i, F_i, H_i\}_{i=1}^n$ of $\mathfrak{s}$ is closed under matrix transpose whence $\mathfrak{s}$ is closed under matrix transpose (because matrix transpose is a Lie algebra anti-isomorphism). It thus follows for all $i < j$ from $E_{ji} \in \mathfrak{s}$ that also $E_{ij} \in \mathfrak{s}$.

## 2.

We construct an inverse $\psi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ to $\varphi$. We define $\psi$ to be the unique linear map with $\psi(e) = e_1$, $\psi(f) = f_1$ and $\psi(h) = h_1$. Recall that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. The relations (23), (24), (25) therefore ensure that $\psi$ is a homomorphism of Lie algebras. Then $\psi\varphi = \mathrm{id}_{\mathfrak{sl}_2(\mathbb{C})}$ because this holds on the basis $e$, $h$, $f$ of $\mathfrak{sl}_2(\mathbb{C})$ and $\varphi\psi = \mathrm{id}_{\mathfrak{g}}$ because this holds on the Lie algebra generators $e_1$, $h_1$, $f_1$ of $\mathfrak{g}$.