Remark and Solutions

Sheet 5

Problem 27

Suppose that A is not a domain. There are two possible situation for this to happen:

If $A = 0$ then $\mathrm{gr}_i(A) = A_{(i)}/A_{(i-1)} = 0$ for every i and thus $\mathrm{gr}(A) = 0$. In this case $\mathrm{gr}(A)$ is again not a domain.

If $A \neq 0$ then there exist nonzero $a, b \in A$ with $ab = 0$. Then there exist $i, j \geq 0$ with $a \in A_{(i)}$ and $b \in A_{(j)}$ but $a \notin A_{(i-1)}$ and $b \notin A_{(j-1)}$ (where $A_{(-1)} = 0$). (This means that a is of degree i and b is of degree j.) Then the resulting elements $[a]_i \in \mathrm{gr}_i(A)$ and $[b]_j \in \mathrm{gr}_j(A)$ are nonzero with

$$[a]_i \cdot [b]_j = [ab]_{i+j} = [0]_{i+j} = 0\,.$$

In this case $\mathrm{gr}(A)$ is again not a domain.

If $\mathfrak{g}$ is a Lie algebra then $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = \mathrm{S}(\mathfrak{g})$ is a domain because $\mathrm{S}(\mathfrak{g})$ is a polynomial algebra. (More specifically, if $(x_i)_{i \in I}$ is a basis of $\mathfrak{g}$ then $\mathrm{S}(\mathfrak{g}) \cong k[t_i \mid i \in I]$.) Thus $\mathrm{U}(\mathfrak{g})$ is a domain.

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.