Remark and Solutions

Sheet 5

Problem 25

The relations

$$[h, e] = 2e\,, \qquad [h, f] = -2f\,, \qquad [e, f] = h$$

can be regarded as *rewritting rules*

$$hf \to fh - 2f\,, \qquad eh \to he - 2e\,, \qquad ef \to fe + h\,.$$

To write a monomial $e^l h^m f^n$ as a linear combination of the given basis monomials we repeatedly apply this rewritting rules and expand the resulting expressions if needed. This is best done using computers. We give two solutions for doing so:

The first solution uses `python` together with the multi-purpose `sympy` package to do the necessary work:

```
from sympy import *

e, h, f = symbols('e h f', commutative=False)

def sl2expand(term):
  while True:
    newterm = term
    newterm = newterm.subs(h*f, f*h - 2*f)
    newterm = newterm.subs(e*h, h*e - 2*e)
    newterm = newterm.subs(e*f, f*e + h)
    newterm = expand(newterm)
    if newterm == term:
      break
    else:
      term = newterm
  return term
```

We get from this program the following results:

```
>>> sl2expand(e * h * f)
-4*f*e + f*h*e - 2*h + h**2
>>> sl2expand(e**2 * h**2 * f**2)
-288*f*e + 240*f*h*e - 56*f*h**2*e + 4*f*h**3*e + 64*f**2*e**2 - 16*f**2*h*e
    **2 + f**2*h**2*e**2 - 32*h + 48*h**2 - 18*h**3 + 2*h**4
```

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.

```
>>> sl2expand(h * f**3)
-6*f**3 + f**3*h
>>> sl2expand(h**3 * f)
-8*f + 12*f*h - 6*f*h**2 + f*h**3
```

Hence

$$
\begin{aligned}
ehf &= -4fe + fhe - 2h + h^2\,, \\
e^2h^2f^2 &= -288fe + 240fhe - 56fh^2e + 4fh^3e + 64f^2e^2 \\
&\quad - 16f^2he^2 + f^2h^2e^2 - 32h + 48h^2 - 18h^3 + 2h^4\,, \\
hf^3 &= -6f^3 + f^3h\,, \\
h^3f &= -8f + 12fh - 6fh^2 + fh^3\,.
\end{aligned}
$$

Another software solution is provided by `sage`, a powerful computer algebra system. Lie algebras and their PBW bases are already implemented and an explicit explanation can be found in [Sag13].

```
sage: g = lie_algebras.three_dimensional_by_rank(QQ, 3, names=['E','F','H'])
sage: def sort_key(x):
....:   if x == 'F':
....:     return 0
....:   if x == 'H':
....:     return 1
....:   if x == 'E':
....:     return 2
sage: pbw = g.pbw_basis(basis_key=sort_key)
sage: E,F,H = pbw.algebra_generators()
sage: H * E - E * H
2*PBW['E']
sage: H * F - F * H
-2*PBW['F']
sage: E * F - F * E
PBW['H']
sage: E * H * F
PBW['F']*PBW['H']*PBW['E'] - 4*PBW['F']*PBW['E'] + PBW['H']^2 - 2*PBW['H']
sage: E^2 * H^2 * F^2
PBW['F']^2*PBW['H']^2*PBW['E']^2 - 16*PBW['F']^2*PBW['H']*PBW['E']^2 + 4*PBW[
    'F']*PBW['H']^3*PBW['E'] + 64*PBW['F']^2*PBW['E']^2 - 56*PBW['F']*PBW['H'
    ]^2*PBW['E'] + 2*PBW['H']^4 + 240*PBW['F']*PBW['H']*PBW['E'] - 18*PBW['H'
    ]^3 - 288*PBW['F']*PBW['E'] + 48*PBW['H']^2 - 32*PBW['H']
sage: H * F^3
PBW['F']^3*PBW['H'] - 6*PBW['F']^3
sage: H^3 * F
PBW['F']*PBW['H']^3 - 6*PBW['F']*PBW['H']^2 + 12*PBW['F']*PBW['H'] - 8*PBW['F
    ']
```

(The above output in the lines 12, 14, 16, 18, 20, 22, 24 is in fact autogenerated by `sage` when compiling this document.)

We want to point out that hf^3 can also be computed smartly, and more generally hf^n for every $n \geq 0$: If A is any k-algebra and $a \in A$ then the map $[a, -]\colon A \to A$ is a derivation of A, i.e.

$$[a, xy] = [a, x]y + x[a, y] \,.$$

Hence

$$[h, f^n] = [h, ff \cdots f] = [h, f]f \cdots f + f[h, f] \cdots f + \cdots + ff \cdots [h, f] \,.$$

With $[h, f] = -2f$ we find that

$$[h, f^n] = -2nf^n \,.$$

With $[h, f^n] = hf^n - f^n h$ we find by rearranging that

$$hf^n = f^n h - 2nf^n \,.$$

Problem 26

In the following we abbreviate "$\mathfrak{sl}_2(k)$-representation" by "representation".

Lemma 1. If V and W are two representations then for every $\kappa \in k$,

$$(V \otimes W)_\kappa = \bigoplus_{\lambda+\mu=\kappa} V_\lambda \otimes W_\mu \,.$$

Proof. We have $V_\lambda \otimes W_\mu \subseteq (V \otimes W)_{\lambda+\mu}$ because

$$h.(v \otimes w) = (h.v) \otimes w + v \otimes (h.w) = \lambda v \otimes w + \mu v \otimes w = (\lambda + \mu) v \otimes w$$

for every simple tensor $v \otimes w \in V_\lambda \otimes W_\mu$. We also know from Weyl's theorem that

$$V \otimes W = \bigoplus_\kappa (V \otimes W)_\kappa$$

and

$$V \otimes W = \left(\bigoplus_\lambda V_\lambda \right) \otimes \left(\bigoplus_\mu W_\mu \right) = \bigoplus_{\lambda,\mu} V_\lambda \otimes W_\mu = \bigoplus_\kappa \bigoplus_{\lambda+\mu=\kappa} V_\lambda \otimes W_\mu \,.$$

We see that the inclusion $\bigoplus_{\lambda+\mu=\kappa} V_\lambda \otimes W_\mu \subseteq (V \otimes W)_\kappa$ is already an equality. $\square$

For every finite dimensional representation V we denote its formal character by

$$\operatorname{ch}(V) \coloneqq \sum_{n \in \mathbb{Z}} \dim(V_n) t^n \in k[t, t^{-1}] \,.$$

Lemma 2.

(1) For every $n \geq 0$,

$$\mathrm{ch}(\mathrm{V}(n)) = t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n}\,.$$

(2) If $V^{(1)}, \dots, V^{(n)}$ are finite dimensional representations then

$$\mathrm{ch}\big(V^{(1)} \oplus \cdots \oplus V^{(n)}\big) = \mathrm{ch}\big(V^{(1)}\big) + \cdots + \mathrm{ch}\big(V^{(n)}\big)\,.$$

(3) If V and W are two finite dimensional representations then

$$\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W)\,.$$

Proof.

(1) We see this from the explicit description of $\mathrm{V}(n)$ as done in the lecture.

(2) This follows from $(V^{(1)} \oplus \cdots \oplus V^{(n)})_\lambda = V^{(1)}_\lambda \oplus \cdots \oplus V^{(n)}_\lambda$.

(3) This follows from Lemma 1. □

Proposition 3. Two finite dimensional representations V and W are isomorphic if and only if they have the same formal character.

Proof. If $V \cong W$ then $\dim(V_\lambda) = \dim(W_\lambda)$ for every λ, and hence $\mathrm{ch}(V) = \mathrm{ch}(W)$.

Suppose now that $\mathrm{ch}(V) = \mathrm{ch}(W)$. By Weyl's theorem we can decompose both V and W into irreducibles:

$$V \cong \bigoplus_{n=0}^{N} \mathrm{V}(n)^{\oplus p_n} \quad \text{and} \quad W \cong \bigoplus_{m\geq 0}^{M} \mathrm{V}(m)^{\oplus q_m}$$

with $p_N, q_M \neq 0$. Then $\mathrm{ch}(V)$ has the leading term $p_N t^N$ and $\mathrm{ch}(W)$ has the leading term $q_M t^M$. It follows from $\mathrm{ch}(V) = \mathrm{ch}(W)$ that $N = M$ and $p_N = q_M$. It follows for

$$V' \coloneqq \bigoplus_{n=0}^{N-1} \mathrm{V}(n)^{\oplus p_n} \quad \text{and} \quad W' \coloneqq \bigoplus_{m\geq 0}^{M-1} \mathrm{V}(m)^{\oplus q_m}$$

that

$$\mathrm{ch}(V') = \mathrm{ch}(V) - \mathrm{ch}\big(\mathrm{V}(N)^{\oplus N}\big) = \mathrm{ch}(W) - \mathrm{ch}\big(\mathrm{V}(M)^{\oplus M}\big) = \mathrm{ch}(W')$$

We can now apply induction to find $V' \cong W'$ and hence $p_i = q_i$ for all $i = 0, \dots, N-1$. The induction start is given by $\mathrm{ch}(V) = \mathrm{ch}(W) = 0$, when $V = 0 = W$.

This shows that $N = M$ and $p_i = q_i$ for all $i = 1, \dots, N$, whence $V \cong W$. □

Let now $n, m \geq 0$. To determine the decomposition of $\mathrm{V}(n) \otimes \mathrm{V}(m)$ into irreducibles we may assume that $n \geq m$ (because $\mathrm{V}(n) \otimes \mathrm{V}(m) \cong \mathrm{V}(m) \otimes \mathrm{V}(n)$). We can now proceed in two ways:

- We have

$$
\begin{aligned}
&\operatorname{ch}(\mathrm{V}(n) \otimes \mathrm{V}(m)) \\
=\ &\operatorname{ch}(\mathrm{V}(n)) \operatorname{ch}(\mathrm{V}(m)) \\
=\ &(t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n})(t^m + t^{m-2} + \cdots + t^{2-m} + t^{-m}) \\
=\ &t^{n+m} + 2t^{n+m-2} + \cdots + mt^{n-m} + \cdots + mt^{m-n} + \cdots + 2t^{2-n-m} + t^{-n-m}\,. \\
=\ &(t^{n+m} + t^{n+m-2} + \cdots + t^{2-n-m} + t^{-n-m}) \\
&+ (t^{n+m-2} + t^{n+m-4} + \cdots + t^{4-n-m} + t^{2-n-m}) \\
&+ \cdots \\
&+ (t^{n-m} + \cdots + t^{m-n}) \\
=\ &\operatorname{ch}(\mathrm{V}(n+m)) + \operatorname{ch}(\mathrm{V}(n+m-2)) + \cdots + \operatorname{ch}(\mathrm{V}(n-m)) \\
=\ &\operatorname{ch}\big(\mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m)\big)
\end{aligned}
$$

  and therefore

$$
\mathrm{V}(n) \otimes \mathrm{V}(m) \cong \mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m)\,.
$$

- We note that

$$
\operatorname{ch}(\mathrm{V}(n)) = t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n} = \frac{t^{n+2} - t^{-n}}{t^2 - 1}\,.
$$

  It follows that

$$
\begin{aligned}
&\operatorname{ch}(\mathrm{V}(n) \otimes \mathrm{V}(m)) \\
=\ &\operatorname{ch}(\mathrm{V}(n)) \operatorname{ch}(\mathrm{V}(m)) \\
=\ &\frac{t^{n+2} - t^{-n}}{t^2 - 1}(t^m + t^{m-2} + \cdots + t^{2-m} + t^{-m}) \\
=\ &\frac{(t^{n+2} - t^{-n})(t^m + t^{m-2} + \cdots + t^{2-m} + t^{-m})}{t^2 - 1} \\
=\ &\frac{t^{n+m+2} + t^{n+m} + \cdots + t^{n-m+2} - t^{m-n} - \cdots - t^{-n-m+2} - t^{-n-m}}{t^2 - 1} \\
=\ &\frac{t^{n+m+2} - t^{-n-m}}{t^2 - 1} + \frac{t^{n+m} - t^{-n-m+2}}{t^2 - 1} + \cdots + \frac{t^{n-m+2} - t^{m-n}}{t^2 - 1} \\
=\ &\operatorname{ch}(\mathrm{V}(n+m)) + \operatorname{ch}(\mathrm{V}(n+m-2)) + \cdots + \operatorname{ch}(n-m) \\
=\ &\operatorname{ch}\big(\mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m)\big)
\end{aligned}
$$

  and hence again

$$
\mathrm{V}(n) \otimes \mathrm{V}(m) \cong \mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m)\,.
$$

The above decomposition of $\mathrm{V}(n) \otimes \mathrm{V}(m)$ is known as the *Clebsch–Gordan rule*. Under the name of *Glebsch–Gordan coefficients* this plays a role in Quantum mechanics.

Problem 27

Suppose that A is not a domain. There are two possible situation for this to happen:

If $A = 0$ then $\mathrm{gr}_i(A) = A_{(i)}/A_{(i-1)} = 0$ for every i and thus $\mathrm{gr}(A) = 0$. In this case $\mathrm{gr}(A)$ is again not a domain.

If $A \neq 0$ then there exist nonzero $a, b \in A$ with $ab = 0$. Then there exist $i, j \geq 0$ with $a \in A_{(i)}$ and $b \in A_{(j)}$ but $a \notin A_{(i-1)}$ and $b \notin A_{(j-1)}$ (where $A_{(-1)} = 0$). (This means that a is of degree i and b is of degree j.) Then the resulting elements $[a]_i \in \mathrm{gr}_i(A)$ and $[b]_j \in \mathrm{gr}_j(A)$ are nonzero with

$$[a]_i \cdot [b]_j = [ab]_{i+j} = [0]_{i+j} = 0\,.$$

In this case $\mathrm{gr}(A)$ is again not a domain.

If $\mathfrak{g}$ is a Lie algebra then $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = \mathrm{S}(\mathfrak{g})$ is a domain because $\mathrm{S}(\mathfrak{g})$ is a polynomial algebra. (More specifically, if $(x_i)_{i \in I}$ is a basis of $\mathfrak{g}$ then $\mathrm{S}(\mathfrak{g}) \cong k[t_i \mid i \in I]$.) Thus $\mathrm{U}(\mathfrak{g})$ is a domain.

Problem 28

The main idea of our approach is taken from [MRS87, 6.7].

Let $A = \bigcup_{i \geq -1} A_{(i)}$ be a filtered algebra (where $A_{(-1)} = 0$). We denote for $x \in A_{(i)}$ the corresponding equivalence class in $\mathrm{gr}_i(A) = A_{(i)}/A_{(i-1)}$ by $[x]_i$. We denote for every $a \in A$ by $\deg(a)$ the degree of a, i.e.

$$\deg(a) = \min\{i \geq 0 \mid a \in A_{(i)}\}\,.$$

For every element $a \in A$ the equivalence class $[a]_i$ is defined for all $i \geq \deg(a)$, but $[a]_i = 0$ if $i > \deg(a)$. To every element $a \in A$ we can hence associate a canonical element $\gamma(a) \in \mathrm{gr}(A)$, namely $\gamma(a) = [a]_{\deg(a)}$.

To every left (resp. right) ideal I in A we can associate a homogeneous left (resp. right) ideal $\mathrm{gr}(I)$ in A, given by the homogeneous parts

$$\mathrm{gr}_i(I) \coloneqq (A_{(i-1)} + I \cap A_{(i)})/A_{(i-1)} = \{[x]_i \mid x \in I \cap A_{(i)}\} \subseteq \mathrm{gr}_i(A)$$

The space $\mathrm{gr}(I) = \bigoplus_{i \geq 0} \mathrm{gr}_i(I)$ is indeed a left ideal in $\mathrm{gr}(A)$ because

$$\begin{aligned}
\mathrm{gr}_i(A)\mathrm{gr}_j(I) &= \{[a]_i[x]_j \mid [a]_i \in \mathrm{gr}_i(A), [x]_j \in \mathrm{gr}_j(I)\} \\
&= \{[a]_i[x]_j \mid a \in A_{(i)}, x \in I \cap A_{(j)}\} \\
&= \{[ax]_{i+j} \mid a \in A_{(i)}, x \in I \cap A_{(j)}\} \\
&\subseteq \{[y]_{i+j} \mid y \in I \cap A_{i+j}\} \\
&= \mathrm{gr}_{i+j}(I)\,.
\end{aligned}$$

If I is a right ideal then $\mathrm{gr}(I)$ is again a right ideal.

Recall 4. If M is an R-module (where R is some ring) and A, B, C are submodules of M with $A \subseteq C$ then

$$A + (B \cap C) = (A + B) \cap C\,.$$

We can therefore just write $A + B \cap C$.

Remark 5. The ideal I inherits from A a filtration given by $I_{(i)} := I \cap A_{(i)}$. Then

$$\begin{aligned}\operatorname{gr}_i(I) &= (A_{(i-1)} + I \cap A_{(i)})/A_{(i-1)} \\ &\cong (I \cap A_{(i)})/(A_{(i-1)} \cap I \cap A_{(i)}) \\ &= (I \cap A_{(i)})/(I \cap A_{(i-1)}) \\ &= I_{(i)}/I_{(i-1)}\,.\end{aligned}$$

This justifies the notation $\operatorname{gr}_i(I)$.

If I and J are two ideals in A with $I \subseteq J$ then also $\operatorname{gr}_i(I) \subseteq \operatorname{gr}_i(J)$ for all i and thus $\operatorname{gr}(I) \subsetneq \operatorname{gr}(J)$. We will see that if I is strictly contained in J, then $\operatorname{gr}(I)$ is also strictly contained in $\operatorname{gr}(J)$. This will be a consequence of the following observation, that is interesting in its own right.

Proposition 6. Let I be a left (resp. right) ideal in A and let S be a subset of I. If $\{[s]_{\deg(s)} \mid s \in S\}$ is a generating set for the left (resp. right) ideal $\operatorname{gr}(I)$ then S is a generating set for I.

Proof. Let $a \in A$. The associated element $[a]_{\deg(a)} \in A$ can be written as a linear combination

$$[a]_{\deg(a)} = \sum_{s \in S} b_s [s]_{\deg(s)}$$

for some $b_s \in \operatorname{gr}(A)$. By decomposing the coefficients b_s into homogeneous components we see that we can replace each b_s by its homogeneous component of degree $\deg(a) - \deg(s)$. We may hence assume that each b_s is homogeneous of degree $\deg(a) - \deg(s)$; in particular $b_s = 0$ whenever $\deg(s) > \deg(a)$. For

$$S' := \{s \in S \mid \deg(s) \le \deg(a)\}$$

we thus have

$$[a]_{\deg(a)} = \sum_{s \in S'} b_s [s]_{\deg(s)}\,.$$

We can write every b_s as $b_s = [a_s]_{\deg(a)-\deg(s)}$ for some $a_s \in A_{\deg(a)-\deg(s)}$ because b_s

is homogeneous of degree $\deg(a) - \deg(s)$. Then

$$\begin{aligned}
[a]_{\deg(a)} &= \sum_{s \in S'} b_s [s]_{\deg(s)} \\
&= \sum_{s \in S'} [a_s]_{\deg(a)-\deg(s)} [s]_{\deg(s)} \\
&= \sum_{s \in S'} [a_s s]_{\deg(a)} \\
&= \left[\sum_{s \in S'} a_s s \right]_{\deg(a)}
\end{aligned}$$

and hence $a - \sum_{s \in S'} a_s s \in A_{\deg(a)-1}$. Now we can proceed inductively to express this difference as a linear combination of S. □

Remark 7. One may think for $a \in A$ about the associated element $[a]_{\deg(a)} \in \mathrm{gr}(A)$ as the "leading term" of a, and as $\mathrm{gr}(I)$ as the "ideal of leading terms" of I. (Note that if A is a graded algebra then $\mathrm{gr}(A) = A$ and $[a]_{\deg(a)}$ is precisely the leading term of a.) Then the above statement and its proof may be compared to Hilbert's basis theorem[1], or the concept of a Gröbner basis.

Corollary 8. If I and J are left (resp. right) ideals in A with $I \subsetneq J$ then $\mathrm{gr}(I) \subsetneq \mathrm{gr}(J)$.

Proof. Suppose that $\mathrm{gr}(I) = \mathrm{gr}(J)$. The ideal $\mathrm{gr}(I)$ is homogeneous and thus generated by homogeneous elements. There hence exists some subset S of I such that $\mathrm{gr}(I)$ is generated by the set $\{[s]_{\deg(s)} \mid s \in S\}$. It follows from Proposition 6 and $\mathrm{gr}(I) = \mathrm{gr}(J)$ that S is a generating set of both I and J. Thus $I = J$. □

Corollary 9. If $\mathrm{gr}(A)$ is left (resp. right) noetherian then A is left (resp. right) noetherian. The same holds for left (resp. right) artinian.

Proof. Every strictly increasing sequence of left (resp. right) ideals

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

in A results by Corollary 8 in a strictly increasing seqenuece of left (resp. right) ideals

$$\mathrm{gr}(I_0) \subsetneq \mathrm{gr}(I_1) \subsetneq \mathrm{gr}(I_2) \subsetneq \mathrm{gr}(I_3) \subsetneq \cdots$$

in $\mathrm{gr}(A)$. So if A is not noetherian then neither is $\mathrm{gr}(A)$. Similarly for artinian. □

Corollary 10. If $\mathfrak{g}$ is a finite dimensional Lie algebra then $\mathrm{U}(\mathfrak{g})$ is both left and right noetherian.

Proof. If $\dim \mathfrak{g} = n$ then $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = k[x_1, \ldots, x_n]$ is both left and right noetherian whence the assertion follows from Corollary 9. □

[1] In [MRS87, Proposition 6.7] the above proposition is implicitely used, and for a missing calculations the authors refer to an earlier proof — namely that of Hilbert's basis theorem.

Remark 11. Recall that for any $n \geq 1$ the polynomial algebra $k[x_1, \dots, x_n]$ is not artinian because

$$(x_n) \supsetneq (x_n^2) \supsetneq (x_n^3) \supsetneq \cdots$$

is a strictly decreasing sequence of ideals of infinite length. We can see similarly that if $\mathfrak{g}$ is any nonzero Lie algebra then $\mathrm{U}(\mathfrak{g})$ is neither left artianian nor right artinian:

We show that that $\mathrm{U}(\mathfrak{g})$ is not left artinian: Let $(x_i)_{i \in I}$ be basis of $\mathfrak{g}$ such that the index set $(I, \leq)$ is linearly ordered and admits a maximal element $j \in I$. (Here we use the well ordering theorem.) Then $\mathrm{U}(\mathfrak{g})$ admits the associated PBW basis consisting of all monomials $x_{i_1}^{n_1} \cdots x_{i_r}^{n_r}$ with $r \geq 0$, $n_i \geq 1$, $i_1 < \cdots < i_r$. It follows that for all $m \geq 0$ the left ideal $\mathrm{U}(\mathfrak{g})x_j^m$ has as a basis all monomials $x_{i_1}^{n_1} \cdots x_{i_r}^{n_r} x_j^{m'}$ with $r \geq 0$, $n_i \geq 1$, $i_1 < \cdots < i_r < j$ and $m' \geq m$. Thus

$$\mathrm{U}(\mathfrak{g})x_j \supsetneq \mathrm{U}(\mathfrak{g})x_j^2 \supsetneq \mathrm{U}(\mathfrak{g})x_j^3 \supsetneq \cdots$$

is a strictly increasing sequence of left ideals of infinite length.

That $\mathrm{U}(\mathfrak{g})$ is not right artinian can be seen in the same way.

References

[MRS87] J. C. McConnel, J. C. Robson, and L. W. Small. *Noncommutative Noetherian Rings*. Graduate Studies in Mathematics 30. American Mathematical Society, 1987, pp. xx+636. ISBN: 978-0-8218-2169-5.

[Sag13] Travis Scrimshaw. *The Poincaré–Birkhoff–Witt Basis For A Universal Enveloping Algebra*. November 3, 2013. URL: http://doc.sagemath.org/html/en/reference/algebras/sage/algebras/lie_algebras/poincare_birkhoff_witt.html (visited on May 23, 2019).