Remark and Solutions

Sheet 2

Problem 13

3.

We follow the hint: We have for every $v \in V$ and every simple wedge $w_1 \wedge w_2 \in \bigwedge^2(V)$ that

$$v \wedge w_1 \wedge w_2 = -w_1 \wedge v \wedge w_2 = -w_1 \wedge w_2 \wedge v\,.$$

The bilinear map

$$\beta\colon \bigwedge^2(V) \times \bigwedge^2(V) \to \bigwedge^4(V)\,, \quad (x,y) \mapsto x \wedge y$$

is hence symmetric, beause

$$\beta(v_1 \wedge v_2, w_1 \wedge w_2) = v_1 \wedge v_2 \wedge w_1 \wedge w_2 = v_1 \wedge w_1 \wedge w_2 \wedge v_2 = w_1 \wedge w_2 \wedge v_1 \wedge v_1$$

for all simple wedges $v_1 \wedge v_2, w_1 \wedge w_2 \in \bigwedge^2(V)$. The bilinear form β is also non-degenerate: Let e_1, e_2, e_3, e_4 be a basis of V. Then a basis of $\bigwedge^2(V)$ is given by the simple wedges

$$e_1 \wedge e_2\,, \quad e_1 \wedge e_3\,, \quad e_1 \wedge e_4\,, \quad e_2 \wedge e_3\,, \quad e_2 \wedge e_4\,, \quad e_3 \wedge e_4\,.$$

We identify the exterior power $\bigwedge^4(V)$ with the ground field k via the single basis element $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. With respect to this basis of $\bigwedge^2(V)$ the bilinear form β is then given by the following matrix:

$$\begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & \end{pmatrix}$$

This matrics is invertible, which means that β is non-degenerate.

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.

Warning 1. The exterior power $\bigwedge^4(V)$ is *not* the same–and not even isomorphic to—the iterated exterior power $\bigwedge^2(\bigwedge^2(V))$.

We know from the first exercise sheet that

$$\mathfrak{g}_\beta \coloneqq \left\{ \varphi \in \mathfrak{gl}(V) \,\middle|\, \beta(\varphi(x), y) + \beta(x, \varphi(y)) = 0 \text{ for all } x, y \in \bigwedge^2(V) \right\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$ that is isomorphic to $\mathfrak{so}_6(\mathbb{C})$, because β in non-degenerate and symmetric and $\bigwedge^2(V)$ is six-dimensional. We will therefore construct an isomorphism $\mathfrak{sl}_4(V) \cong \mathfrak{g}_\beta$, where $\mathfrak{sl}(V) \cong \mathfrak{sl}_4(\mathbb{C})$ because V is four-dimensional.

Lemma 2. If V is a representation of a Lie algebra $\mathfrak{g}$ then any exterior power $\bigwedge^n(V)$ inherits from V the structure of a $\mathfrak{g}$-representations via

$$x.(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^n v_1 \wedge \cdots \wedge (x.v_i) \wedge \cdots v_n$$

for all $x \in \mathfrak{g}$ and every simple wedge $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n(V)$.

Proof. It follows from the functoriality of the exterior power that for every fixed element $x \in \mathfrak{g}$ the proposed action of x on $\bigwedge^n(V)$ is well-defined. We need to check that this action is compatible with the Lie bracket of $\mathfrak{g}$. We see that

$$\begin{aligned}
&x.y.(v_1 \wedge \cdots \wedge v_n) \\
&= x.\sum_{i=1}^n v_1 \wedge \cdots \wedge (y.v_i) \wedge \cdots v_n \\
&= \sum_{i<j} v_1 \wedge \cdots \wedge (x.v_i) \cdots \wedge (y.v_j) \wedge \cdots v_n \\
&\quad + \sum_{i=1}^n v_1 \wedge \cdots \wedge (y.x.v_i) \cdots \wedge v_j \wedge \cdots v_n \\
&\quad + \sum_{i>j} v_1 \wedge \cdots \wedge (y.v_j) \cdots \wedge (x.v_i) \wedge \cdots v_n
\end{aligned}$$

and hence

$$\begin{aligned}
&x.y.(v_1 \wedge \cdots \wedge v_n) - y.x.(v_1 \wedge \cdots \wedge v_n) \\
&= \sum_{i=1}^n (v_1 \wedge \cdots \wedge (y.x.v_i) \cdots \wedge v_j \wedge \cdots v_n) - \sum_{i=1}^n v_1 \wedge \cdots \wedge (x.y.v_i) \cdots \wedge v_j \wedge \cdots v_n \\
&= \sum_{i=1}^n v_1 \wedge \cdots \wedge (y.x.v_i - x.y.v_i) \cdots \wedge v_j \wedge \cdots v_n \\
&= \sum_{i=1}^n v_1 \wedge \cdots \wedge ([x, y].v_i) \cdots \wedge v_j \wedge \cdots v_n
\end{aligned}$$

for all $x, y \in \mathfrak{g}$ and every simple wedge $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n(V)$. $\square$

It follows from Lemma 2 that the natural action of $\mathfrak{sl}(V)$ on V induces on $\bigwedge^2(V)$ the structure of a $\mathfrak{sl}(V)$-representations via

$$X.(v_1 \wedge v_2) = (X.v_1) \wedge v_2 + v_1 \wedge (x.V_2)$$

for every $X \in \mathfrak{sl}(V)$ and every simple wedge $v_1 \wedge v_2 \in \bigwedge^2(V)$. This action of $\mathfrak{sl}(V)$ on $\bigwedge^2(V)$ corresponds to a homomorphis of Lie algebras $\rho\colon \mathfrak{sl}(V) \to \bigwedge^2(V)$ given by $\rho(X)(x) = X.x$ for all $X \in \mathfrak{sl}(V)$ and $x \in \bigwedge^2(V)$. We show in the following that ρ restricts to an isomorphism $\mathfrak{sl}(V) \to \mathfrak{g}_\beta$.

We first observe that the image of ρ is contained in $\mathfrak{g}_\beta$: We need to show that

$$\beta(\rho(X)(x), y) + \beta(X, \rho(X)(y) = 0$$

for all $x, y \in \bigwedge^2(V)$. It sufficies to consider the cases that x and y are simple wedges $x = v_1 \wedge v_2$ and $y = v_3 \wedge v_4$. Then

$$\begin{aligned}
&\gamma(v_1, v_2, v_3, v_4) \\
&= \beta(\rho(X)(v_1 \wedge v_2), v_3 \wedge v_4) + \beta(v_1 \wedge v_2, \rho(X)(v_3 \wedge v_4)) \\
&= \beta((X.v_1) \wedge v_2 + v_1 \wedge (X.v_2), v_3 \wedge v_4) + \beta(v_1 \wedge v_2, (X.v_3) \wedge v_n + v_3 \wedge (X.v_4)) \\
&= (X.v_1) \wedge v_2 \wedge v_3 \wedge v_4 + v_1 \wedge (X.v_2) \wedge v_3 \wedge v_4 \\
&\quad + v_1 \wedge v_2 \wedge (X.v_3) \wedge v_4 + v_1 \wedge v_2 \wedge v_3 \wedge (X.v_4)
\end{aligned}$$

is multilinear and alternating in v_1, v_2, v_3, v_4. To show that $\gamma(v_1, v_2, v_3, v_4) = 0$ for all $v_1, v_2, v_3, v_4 \in V$ it therefore sufficies to show that $\gamma(e_1, e_2, e_3, e_4) = 0$ for a basis e_1, e_2, e_3, e_n of V. Let $A \in \mathfrak{sl}_4(\mathbb{C})$ be the matrix that represents $X \in \mathfrak{sl}(V)$ with respect to this basis, which means that $X.e_j = \sum_{i=1}^n A_{ij} e_i$ for every $j = 1, \dots, n$. Then

$$(X.e_1) \wedge e_2 \wedge e_3 \wedge e_4 = \sum_{i=1}^n A_{i1} e_i \wedge e_2 \wedge e_3 \wedge e_4 = A_{11} e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

and similarly

$$\begin{aligned}
e_1 \wedge (X.e_2) \wedge e_3 \wedge e_4 &= A_{22} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,, \\
v_1 \wedge v_2 \wedge (X.v_3) \wedge v_4 &= A_{33} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,, \\
v_1 \wedge v_2 \wedge v_3 \wedge (X.v_4) &= A_{44} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,.
\end{aligned}$$

Hence altogether

$$\begin{aligned}
\gamma(e_1, e_2, e_3, e_4) &= (A_{11} + A_{22} + A_{33} + A_{44}) e_1 \wedge e_2 \wedge e_3 \wedge e_4 \\
&= \operatorname{tr}(A) e_1 \wedge e_2 \wedge e_3 \wedge e_4 = 0 \,.
\end{aligned}$$

We have shown that the image of ρ lies in $\mathfrak{g}_\beta$.

We know that $\dim \mathfrak{sl}(V) = \dim \mathfrak{sl}_4(\mathbb{C}) = 4^2 - 1 = 15$ and $\dim \mathfrak{g}_\beta = \dim \mathfrak{so}_6(\mathbb{C}) = 15$. It therefore now sufficies to show that ρ is injective. Since $\mathfrak{sl}(V)$ is simple if sufficies to show that ρ is nonzero. For this we consider the endomorphism $X \in \mathfrak{sl}(V)$

with $X(e_1) = e_1$, $X(e_2) = -e_2$ and $X(e_3) = X_(e_4) = 0$ for some basis e_1, e_2, e_3, e_4 of V. Then

$$\rho(X)(e_1 \wedge e_3) = (X.e_1) \wedge e_3 + e_1 \wedge (X.e_3) = e_1 \wedge e_3 + e_1 \wedge 0 = e_1 \wedge e_3 \neq 0$$

and hence $\rho(X) \neq 0$. This shows that ρ is nonzero.

Problem 14

1.

To construct a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ we only need to specify the images E_i, H_i, F_i of the generators e_i, h_i, f_i and then check that these proposed images satisfy the given relations (22)–(27). (This is what it means for $\mathfrak{g}$ to be given by the generators $\{e_i, h_i, f_i\}_{i=1}^n$ and relations (22)–(27).)

In the case $n = 1$ we know that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the standard basis e, f, h constisting of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These three basis vectors satisfy the relations $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Motivated by this example we will choose for arbitrary $n \geq 1$ the proposed images $E_i, F_i, H_i \in \mathfrak{sl}_{n+1}(\mathbb{C})$ as

$$E_i \coloneqq E_{i,i+1}, \quad F_i \coloneqq E_{i+1,i}, \quad H_i \coloneqq E_{ii} - E_{i+1,i+1}.$$

We need to check that these elements satisfy the relations (22)–(27):

(22) We find $[H_i, H_j] = 0$ because both H_i and H_j are diagonal matrices and hence commute with each other.

(23) We find

$$\begin{aligned} [E_i, F_j] &= E_i F_j - F_j E_i \\ &= E_{i,i+1} E_{j+1,j} - E_{j+1,j} E_{i,i+1} \\ &= \delta_{i+1,j+1} E_{ij} - \delta_{ij} E_{j+1,i+1} \\ &= \delta_{ij}(E_{ii} - E_{i+1,i+1}) \\ &= \delta_{ij} H_i \end{aligned}$$

where we used that $\delta_{i+1,j+1} = \delta_{ij}$ and $\delta_{ij} E_{ij} = \delta_{ij} E_{ii}$.

(24) We find

$$
\begin{aligned}
[H_i, E_j] &= [E_{ii} - E_{i+1,i+1}, E_{j,j+1}] \\
&= (E_{ii} - E_{i+1,i+1})E_{j,j+1} - E_{j,j+1}(E_{ii} - E_{i+1,i+1}) \\
&= E_{ii}E_{j,j+1} - E_{i+1,i+1}E_{j,j+1} - E_{j,j+1}E_{ii} + E_{j,j+1}E_{i+1,i+1} \\
&= \delta_{ij}E_{i,j+1} - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} + \delta_{i+1,j+1}E_{j,i+1} \\
&= \delta_{ij}(E_{i,j+1} + E_{j,i+1}) - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} \\
&= 2\delta_{ij}E_{j,j+1} - \delta_{i+1,j}E_{j,j+1} - \delta_{i,j+1}E_{j,j+1} \\
&= (2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1})E_{j,j+1} \\
&= a_{ji}E_j \,.
\end{aligned}
$$

(25) This relation can be checked in the same way as (24). But we can also observe that $F_j = E_j^T$ and $H_i = H_i^T$ whence

$$[H_i, F_j] = [H_i^T, E_j^T] = [E_j, H_i]^T = a_{ij}E_j^T = -a_{ij}F_j \,.$$

(26) We find

$$
\begin{aligned}
\mathrm{ad}(E_i)(E_j) &= [E_i, E_j] \\
&= [E_{i,i+1}, E_{j,j+1}] \\
&= E_{i,i+1}E_{j,j+1} - E_{j,j+1}E_{i,i+1} \\
&= \delta_{i+1,j}E_{i,j+1} - \delta_{i,j+1}E_{j,i+1} \,.
\end{aligned}
$$

We find for $|i-j| \geq 2$ that $\delta_{i+1,j} = \delta_{i,j+1} = 0$ and hence

$$\mathrm{ad}(E_i)^{-a_{ji}+1}(E_j) = \mathrm{ad}(E_i)(E_j) = 0\,.$$

as desired. If $i = j-1$ then $\mathrm{ad}(E_i)(E_j) = E_{i,j+1} = E_{i,i+2}$ and $a_{ji} = -1$ and thus

$$
\begin{aligned}
\mathrm{ad}(E_i)^{-a_{ji}+1}(E_j) &= \mathrm{ad}(E_i)^2(E_j) \\
&= [E_i, [E_i, E_j]] \\
&= [E_i, E_{i,i+2}] \\
&= [E_{i,i+1}, E_{i,i+2}] \\
&= E_{i,i+1}E_{i,i+2} - E_{i,i+2}E_{i,i+1} \\
&= \delta_{i,i+1}E_{i,i+2} - \delta_{i,i+2}E_{i,i+1} \\
&= 0
\end{aligned}
$$

The case $i = j+1$ works the same.

(27) This can be done by similar calculations as (26) but can also be directly derived from (26) by again using the matrix transpose.

Remark 3.

1. We used for (25) that that $[A,B]^T = (AB - BA)^T = B^T A^T - A^T B^T = [B^T, A^T]$.

2. For (24) and (25) it is useful to understand how a commutator $[D,A]$ looks like if D is a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n \in k$:

   The matrix DA results from A by multiplying for every $i = 1, \dots, n$ the i-th row of A by the corresponding diagonal entry λ_i. Similarly the matrix AD results from A by multiplying for every $j = 1, \dots, n$ the jth column of A by the corresponding diagonal entry λ_j. This means in formulae that

   $$(DA)_{ij} = \lambda_i A_{ij} \quad \text{and} \quad (AD)_{ij} = \lambda_j A_{ij}$$

   for all $i, j = 1, \dots, n$, and thus

   $$[D,A]_{ij} = (DA - AD)_{ij} = (DA)_{ij} - (AD)_{ij} = \lambda_i A_{ij} - \lambda_j A_{ij} = (\lambda_i - \lambda_j) A_{ij}$$

   for all $i, j = 1, \dots, n$.

   It follows that

   $$[H_i, E_j] = [H_i, E_{j,j+1}] = ((H_i)_{jj} - (H_i)_{j+1,j+1}) E_{j,j+1}$$

   where

   $$(H_i)_{jj} - (H_i)_{j+1,j+1} = \begin{cases} 0 & \text{if } j+1 < i\,, \\ -1 & \text{if } j+1 = i\,, \\ 2 & \text{if } j = i\,, \\ -1 & \text{if } j = i+1\,, \\ 0 & \text{if } j > i+1\,. \end{cases}$$

   Hence $[H_i, E_j] = a_{j,i} E_j$ as desired.

We have shown that E_i, F_i, H_i with $i = 1, \dots, n$ satisfy the given relations, and all these matrices are contained in $\mathfrak{sl}_{n+1}(\mathbb{C})$. There hence exists a unique homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ with $\varphi(e_i) = E_i$, $\varphi(f_i) = F_i$ and $\varphi(h_i) = H_i$. It remains to show that φ is surjective.

The image of φ is a Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$. It therefore suffieces to show that $\mathfrak{sl}_{n+1}(\mathbb{C})$ is generated by the elements $\{E_i, F_i, H_i\}_{i=1}^n$ as a Lie algebra. Let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$ generated by these elements. We know that $\mathfrak{sl}_{n+1}(\mathbb{C})$ has a basis the diagonal matrices $H_1, \dots, H_n$ together with the off-diagonal matrices E_{ij} where $1 \leq i \neq j \leq n+1$. It sufficies to show that these matrices are contained in $\mathfrak{s}$. This holds for $H_1, \dots, H_n$ by construction of $\mathfrak{s}$.

Let us consider the off-diagonal matrices E_{ij} with $j > i$. We fix the index i and show that $E_{i,i+1}, E_{i,i+2}, \dots, E_{i,n+1} \in \mathfrak{s}$. This holds for $E_{i,i+1} = E_i$ by construction of $\mathfrak{s}$. If $E_{ij} \in \mathfrak{s}$ for some $i+1 \leq j < n+1$ then we find inductively that the matrix

$$[E_{ij}, E_j] = [E_{ij}, E_{j,j+1}] = E_{ij} E_{j,j+1} - E_{j,j+1} E_{ij} = E_{i,j+1}$$

is again contained $\mathfrak{s}$. This shows that all off-diagonal matrices E_{ij} with $j > i$ are contained in $\mathfrak{s}$.

For the off-diagonal matrices E_{ij} with $i < j$ we can argue in the same way by using the matrices F_i instead of E_i. But we could also observe that the Lie algebra generating set $\{E_i, F_i, H_i\}_{i=1}^n$ of $\mathfrak{s}$ is closed under matrix transpose whence $\mathfrak{s}$ is closed under matrix transpose (because matrix transpose is a Lie algebra anti-isomorphism). It thus follows for all $i < j$ from $E_{ji} \in \mathfrak{s}$ that also $E_{ij} \in \mathfrak{s}$.

2.

We construct an inverse $\psi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ to φ. We define ψ to be the unique linear map with $\psi(e) = e_1$, $\psi(f) = f_1$ and $\psi(h) = h_1$. Recall that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. The relations (23), (24), (25) therefore ensure that ψ is a homomorphism of Lie algebras. Then $\psi\varphi = \mathrm{id}_{\mathfrak{sl}_2(\mathbb{C})}$ because this holds on the basis e, h, f of $\mathfrak{sl}_2(\mathbb{C})$ and $\varphi\psi = \mathrm{id}_{\mathfrak{g}}$ because this holds on the Lie algebra generators e_1, h_1, f_1 of $\mathfrak{g}$.