Remark and Solutions

Sheet 5

Problem 26

In the following we abbreviate "$\mathfrak{sl}_2(k)$-representation" by "representation".

Lemma 1. If V and W are two representations then for every $\kappa \in k$,

$$(V \otimes W)_\kappa = \bigoplus_{\lambda+\mu=\kappa} V_\lambda \otimes W_\mu \,.$$

Proof. We have $V_\lambda \otimes W_\mu \subseteq (V \otimes W)_{\lambda+\mu}$ because

$$h.(v \otimes w) = (h.v) \otimes w + v \otimes (h.w) = \lambda v \otimes w + \mu v \otimes w = (\lambda+\mu) v \otimes w$$

for every simple tensor $v \otimes w \in V_\lambda \otimes W_\mu$. We also know from Weyl's theorem that

$$V \otimes W = \bigoplus_\kappa (V \otimes W)_\kappa$$

and

$$V \otimes W = \left(\bigoplus_\lambda V_\lambda \right) \otimes \left(\bigoplus_\mu W_\mu \right) = \bigoplus_{\lambda,\mu} V_\lambda \otimes W_\mu = \bigoplus_\kappa \bigoplus_{\lambda+\mu=\kappa} V_\lambda \otimes W_\mu \,.$$

We see that the inclusion $\bigoplus_{\lambda+\mu=\kappa} V_\lambda \otimes W_\mu \subseteq (V \otimes W)_\kappa$ is already an equality. □

For every finite dimensional representation V we denote its formal character by

$$\operatorname{ch}(V) := \sum_{n \in \mathbb{Z}} \dim(V_n) t^n \in k[t, t^{-1}] \,.$$

Lemma 2.

(1) For every $n \geq 0$,

$$\operatorname{ch}(\mathrm{V}(n)) = t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n} \,.$$

(2) If $V^{(1)}, \dots, V^{(n)}$ are finite dimensional representations then

$$\operatorname{ch}\big(V^{(1)} \oplus \cdots \oplus V^{(n)}\big) = \operatorname{ch}\big(V^{(1)}\big) + \cdots + \operatorname{ch}\big(V^{(n)}\big) \,.$$

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.

(3) If V and W are two finite dimensional representations then

$$\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W)\,.$$

Proof.

(1) We see this from the explicit description of $\mathrm{V}(n)$ as done in the lecture.

(2) This follows from $(V^{(1)} \oplus \cdots \oplus V^{(n)})_\lambda = V^{(1)}_\lambda \oplus \cdots \oplus V^{(n)}_\lambda$.

(3) This follows from Lemma 1. □

Proposition 3. Two finite dimensional representations V and W are isomorphic if and only if they have the same formal character.

Proof. If $V \cong W$ then $\dim(V_\lambda) = \dim(W_\lambda)$ for every λ, and hence $\mathrm{ch}(V) = \mathrm{ch}(W)$.

Suppose now that $\mathrm{ch}(V) = \mathrm{ch}(W)$. By Weyl's theorem we can decompose both V and W into irreducibles:

$$V \cong \bigoplus_{n=0}^{N} \mathrm{V}(n)^{\oplus p_n} \quad \text{and} \quad W \cong \bigoplus_{m\geq 0}^{M} \mathrm{V}(m)^{\oplus q_m}$$

with $p_N, q_M \neq 0$. Then $\mathrm{ch}(V)$ has the leading term $p_N t^N$ and $\mathrm{ch}(W)$ has the leading term $q_M t^M$. It follows from $\mathrm{ch}(V) = \mathrm{ch}(W)$ that $N = M$ and $p_N = q_M$. It follows for

$$V' \coloneqq \bigoplus_{n=0}^{N-1} \mathrm{V}(n)^{\oplus p_n} \quad \text{and} \quad W' \coloneqq \bigoplus_{m\geq 0}^{M-1} \mathrm{V}(m)^{\oplus q_m}$$

that

$$\mathrm{ch}(V') = \mathrm{ch}(V) - \mathrm{ch}\big(\mathrm{V}(N)^{\oplus N}\big) = \mathrm{ch}(W) - \mathrm{ch}\big(\mathrm{V}(M)^{\oplus M}\big) = \mathrm{ch}(W')$$

We can now apply induction to find $V' \cong W'$ and hence $p_i = q_i$ for all $i = 0, \dots, N-1$. The induction start is given by $\mathrm{ch}(V) = \mathrm{ch}(W) = 0$, when $V = 0 = W$.

This shows that $N = M$ and $p_i = q_i$ for all $i = 1, \dots, N$, whence $V \cong W$. □

Let now $n, m \geq 0$. To determine the decomposition of $\mathrm{V}(n) \otimes \mathrm{V}(m)$ into irreducibles we may assume that $n \geq m$ (because $\mathrm{V}(n) \otimes \mathrm{V}(m) \cong \mathrm{V}(m) \otimes \mathrm{V}(n)$). We can now proceed in two ways:

- We have

$$
\begin{aligned}
&\operatorname{ch}(\mathrm{V}(n) \otimes \mathrm{V}(m)) \\
={}& \operatorname{ch}(\mathrm{V}(n)) \operatorname{ch}(\mathrm{V}(m)) \\
={}& (t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n})(t^m + t^{m-2} + \cdots + t^{2-m} + t^{-m}) \\
={}& t^{n+m} + 2t^{n+m-2} + \cdots + mt^{n-m} + \cdots + mt^{m-n} + \cdots + 2t^{2-n-m} + t^{-n-m} . \\
={}& (t^{n+m} + t^{n+m-2} + \cdots + t^{2-n-m} + t^{-n-m}) \\
&+ (t^{n+m-2} + t^{n+m-4} + \cdots + t^{4-n-m} + t^{2-n-m}) \\
&+ \cdots \\
&+ (t^{n-m} + \cdots + t^{m-n}) \\
={}& \operatorname{ch}(\mathrm{V}(n+m)) + \operatorname{ch}(\mathrm{V}(n+m-2)) + \cdots + \operatorname{ch}(\mathrm{V}(n-m)) \\
={}& \operatorname{ch}\big(\mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m)\big)
\end{aligned}
$$

  and therefore

$$
\mathrm{V}(n) \otimes \mathrm{V}(m) \cong \mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m) .
$$

- We note that

$$
\operatorname{ch}(\mathrm{V}(n)) = t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n} = \frac{t^{n+2} - t^{-n}}{t^2 - 1} .
$$

  It follows that

$$
\begin{aligned}
&\operatorname{ch}(\mathrm{V}(n) \otimes \mathrm{V}(m)) \\
={}& \operatorname{ch}(\mathrm{V}(n)) \operatorname{ch}(\mathrm{V}(m)) \\
={}& \frac{t^{n+2} - t^{-n}}{t^2 - 1}(t^m + t^{m-2} + \cdots + t^{2-m} + t^{-m}) \\
={}& \frac{(t^{n+2} - t^{-n})(t^m + t^{m-2} + \cdots + t^{2-m} + t^{-m})}{t^2 - 1} \\
={}& \frac{t^{n+m+2} + t^{n+m} + \cdots + t^{n-m+2} - t^{m-n} - \cdots - t^{-n-m+2} - t^{-n-m}}{t^2 - 1} \\
={}& \frac{t^{n+m+2} - t^{-n-m}}{t^2 - 1} + \frac{t^{n+m} - t^{-n-m+2}}{t^2 - 1} + \cdots + \frac{t^{n-m+2} - t^{m-n}}{t^2 - 1} \\
={}& \operatorname{ch}(\mathrm{V}(n+m)) + \operatorname{ch}(\mathrm{V}(n+m-2)) + \cdots + \operatorname{ch}(n-m) \\
={}& \operatorname{ch}\big(\mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m)\big)
\end{aligned}
$$

  and hence again

$$
\mathrm{V}(n) \otimes \mathrm{V}(m) \cong \mathrm{V}(n+m) \oplus \mathrm{V}(n+m-2) \oplus \cdots \oplus \mathrm{V}(n-m) .
$$

The above decomposition of $\mathrm{V}(n) \otimes \mathrm{V}(m)$ is known as the *Clebsch–Gordan rule*. Under the name of *Glebsch–Gordan coefficients* this plays a role in Quantum mechanics.

Problem 27

Suppose that A is not a domain. There are two possible situation for this to happen:

If $A = 0$ then $\mathrm{gr}_i(A) = A_{(i)}/A_{(i-1)} = 0$ for every i and thus $\mathrm{gr}(A) = 0$. In this case $\mathrm{gr}(A)$ is again not a domain.

If $A \neq 0$ then there exist nonzero $a, b \in A$ with $ab = 0$. Then there exist $i, j \geq 0$ with $a \in A_{(i)}$ and $b \in A_{(j)}$ but $a \notin A_{(i-1)}$ and $b \notin A_{(j-1)}$ (where $A_{(-1)} = 0$). (This means that a is of degree i and b is of degree j.) Then the resulting elements $[a]_i \in \mathrm{gr}_i(A)$ and $[b]_j \in \mathrm{gr}_j(A)$ are nonzero with

$$[a]_i \cdot [b]_j = [ab]_{i+j} = [0]_{i+j} = 0\,.$$

In this case $\mathrm{gr}(A)$ is again not a domain.

If $\mathfrak{g}$ is a Lie algebra then $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = \mathrm{S}(\mathfrak{g})$ is a domain because $\mathrm{S}(\mathfrak{g})$ is a polynomial algebra. (More specifically, if $(x_i)_{i \in I}$ is a basis of $\mathfrak{g}$ then $\mathrm{S}(\mathfrak{g}) \cong k[t_i \mid i \in I]$.) Thus $\mathrm{U}(\mathfrak{g})$ is a domain.

Problem 28

The main idea of our approach is taken from [MRS87, 6.7].

Let $A = \bigcup_{i \geq -1} A_{(i)}$ be a filtered algebra (where $A_{(-1)} = 0$). We denote for $x \in A_{(i)}$ the corresponding equivalence class in $\mathrm{gr}_i(A) = A_{(i)}/A_{(i-1)}$ by $[x]_i$. We denote for every $a \in A$ by $\deg(a)$ the degree of a, i.e.

$$\deg(a) = \min\{i \geq 0 \mid a \in A_{(i)}\}\,.$$

For every element $a \in A$ the equivalence class $[a]_i$ is defined for all $i \geq \deg(a)$, but $[a]_i = 0$ if $i > \deg(a)$. To every element $a \in A$ we can hence associate a canonical element $\gamma(a) \in \mathrm{gr}(A)$, namely $\gamma(a) = [a]_{\deg(a)}$.

To every left (resp. right) ideal I in A we can associate a homogeneous left (resp. right) ideal $\mathrm{gr}(I)$ in A, given by the homogeneous parts

$$\mathrm{gr}_i(I) \coloneqq (A_{(i-1)} + I \cap A_{(i)})/A_{(i-1)} = \{[x]_i \mid x \in I \cap A_{(i)}\} \subseteq \mathrm{gr}_i(A)$$

The space $\mathrm{gr}(I) = \bigoplus_{i \geq 0} \mathrm{gr}_i(I)$ is indeed a left ideal in $\mathrm{gr}(A)$ because

$$\begin{aligned}
\mathrm{gr}_i(A)\mathrm{gr}_j(I) &= \{[a]_i[x]_j \mid [a]_i \in \mathrm{gr}_i(A), [x]_j \in \mathrm{gr}_j(I)\} \\
&= \{[a]_i[x]_j \mid a \in A_{(i)}, x \in I \cap A_{(j)}\} \\
&= \{[ax]_{i+j} \mid a \in A_{(i)}, x \in I \cap A_{(j)}\} \\
&\subseteq \{[y]_{i+j} \mid y \in I \cap A_{i+j}\} \\
&= \mathrm{gr}_{i+j}(I)\,.
\end{aligned}$$

If I is a right ideal then $\mathrm{gr}(I)$ is again a right ideal.

Recall 4. If M is an R-module (where R is some ring) and A, B, C are submodules of M with $A \subseteq C$ then

$$A + (B \cap C) = (A + B) \cap C\,.$$

We can therefore just write $A + B \cap C$.

Remark 5. The ideal I inherits from A a filtration given by $I_{(i)} \coloneqq I \cap A_{(i)}$. Then

$$\begin{aligned}
\operatorname{gr}_i(I) &= (A_{(i-1)} + I \cap A_{(i)})/A_{(i-1)} \\
&\cong (I \cap A_{(i)})/(A_{(i-1)} \cap I \cap A_{(i)}) \\
&= (I \cap A_{(i)})/(I \cap A_{(i-1)}) \\
&= I_{(i)}/I_{(i-1)}\,.
\end{aligned}$$

This justifies the notation $\operatorname{gr}_i(I)$.

If I and J are two ideals in A with $I \subseteq J$ then also $\operatorname{gr}_i(I) \subseteq \operatorname{gr}_i(J)$ for all i and thus $\operatorname{gr}(I) \subsetneq \operatorname{gr}(J)$. We will see that if I is strictly contained in J, then $\operatorname{gr}(I)$ is also strictly contained in $\operatorname{gr}(J)$. This will be a consequence of the following observation, that is interesting in its own right.

Proposition 6. Let I be a left (resp. right) ideal in A and let S be a subset of I. If $\{[s]_{\deg(s)} \mid s \in S\}$ is a generating set for the left (resp. right) ideal $\operatorname{gr}(I)$ then S is a generating set for I.

Proof. Let $a \in A$. The associated element $[a]_{\deg(a)} \in A$ can be written as a linear combination

$$[a]_{\deg(a)} = \sum_{s \in S} b_s [s]_{\deg(s)}$$

for some $b_s \in \operatorname{gr}(A)$. By decomposing the coefficients b_s into homogeneous components we see that we can replace each b_s by its homogeneous component of degree $\deg(a) - \deg(s)$. We may hence assume that each b_s is homogeneous of degree $\deg(a) - \deg(s)$; in particular $b_s = 0$ whenever $\deg(s) > \deg(a)$. For

$$S' \coloneqq \{s \in S \mid \deg(s) \leq \deg(a)\}$$

we thus have

$$[a]_{\deg(a)} = \sum_{s \in S'} b_s [s]_{\deg(s)}\,.$$

We can write every b_s as $b_s = [a_s]_{\deg(a)-\deg(s)}$ for some $a_s \in A_{\deg(a)-\deg(s)}$ because b_s

is homogeneous of degree $\deg(a) - \deg(s)$. Then

$$\begin{aligned}
[a]_{\deg(a)} &= \sum_{s \in S'} b_s [s]_{\deg(s)} \\
&= \sum_{s \in S'} [a_s]_{\deg(a)-\deg(s)} [s]_{\deg(s)} \\
&= \sum_{s \in S'} [a_s s]_{\deg(a)} \\
&= \left[\sum_{s \in S'} a_s s \right]_{\deg(a)}
\end{aligned}$$

and hence $a - \sum_{s \in S'} a_s s \in A_{\deg(a)-1}$. Now we can proceed inductively to express this difference as a linear combination of S. □

Remark 7. One may think for $a \in A$ about the associated element $[a]_{\deg(a)} \in \mathrm{gr}(A)$ as the "leading term" of a, and as $\mathrm{gr}(I)$ as the "ideal of leading terms" of I. (Note that if A is a graded algebra then $\mathrm{gr}(A) = A$ and $[a]_{\deg(a)}$ is precisely the leading term of a.) Then the above statement and its proof may be compared to Hilbert's basis theorem[1], or the concept of a Gröbner basis.

Corollary 8. If I and J are left (resp. right) ideals in A with $I \subsetneq J$ then $\mathrm{gr}(I) \subsetneq \mathrm{gr}(J)$.

Proof. Suppose that $\mathrm{gr}(I) = \mathrm{gr}(J)$. The ideal $\mathrm{gr}(I)$ is homogeneous and thus generated by homogeneous elements. There hence exists some subset S of I such that $\mathrm{gr}(I)$ is generated by the set $\{[s]_{\deg(s)} \mid s \in S\}$. It follows from Proposition 6 and $\mathrm{gr}(I) = \mathrm{gr}(J)$ that S is a generating set of both I and J. Thus $I = J$. □

Corollary 9. If $\mathrm{gr}(A)$ is left (resp. right) noetherian then A is left (resp. right) noetherian. The same holds for left (resp. right) artinian.

Proof. Every strictly increasing sequence of left (resp. right) ideals

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

in A results by Corollary 8 in a strictly increasing seqenuece of left (resp. right) ideals

$$\mathrm{gr}(I_0) \subsetneq \mathrm{gr}(I_1) \subsetneq \mathrm{gr}(I_2) \subsetneq \mathrm{gr}(I_3) \subsetneq \cdots$$

in $\mathrm{gr}(A)$. So if A is not noetherian then neither is $\mathrm{gr}(A)$. Similarly for artinian. □

Corollary 10. If $\mathfrak{g}$ is a finite dimensional Lie algebra then $\mathrm{U}(\mathfrak{g})$ is both left and right noetherian.

Proof. If $\dim \mathfrak{g} = n$ then $\mathrm{gr}(\mathrm{U}(\mathfrak{g})) = k[x_1, \ldots, x_n]$ is both left and right noetherian whence the assertion follows from Corollary 9. □

[1] In [MRS87, Proposition 6.7] the above proposition is implicitely used, and for a missing calculations the authors refer to an earlier proof — namely that of Hilbert's basis theorem.

Remark 11. Recall that for any $n \geq 1$ the polynomial algebra $k[x_1, \dots, x_n]$ is not artinian because

$$(x_n) \supsetneq (x_n^2) \supsetneq (x_n^3) \supsetneq \cdots$$

is a strictly decreasing sequence of ideals of infinite length. We can see similarly that if $\mathfrak{g}$ is any nonzero Lie algebra then $\mathrm{U}(\mathfrak{g})$ is neither left artianian nor right artinian:

We show that that $\mathrm{U}(\mathfrak{g})$ is not left artinian: Let $(x_i)_{i \in I}$ be basis of $\mathfrak{g}$ such that the index set $(I, \leq)$ is linearly ordered and admits a maximal element $j \in I$. (Here we use the well ordering theorem.) Then $\mathrm{U}(\mathfrak{g})$ admits the associated PBW basis consisting of all monomials $x_{i_1}^{n_1} \cdots x_{i_r}^{n_r}$ with $r \geq 0$, $n_i \geq 1$, $i_1 < \cdots < i_r$. It follows that for all $m \geq 0$ the left ideal $\mathrm{U}(\mathfrak{g}) x_j^m$ has as a basis all monomials $x_{i_1}^{n_1} \cdots x_{i_r}^{n_r} x_j^{m'}$ with $r \geq 0$, $n_i \geq 1$, $i_1 < \cdots < i_r < j$ and $m' \geq m$. Thus

$$\mathrm{U}(\mathfrak{g}) x_j \supsetneq \mathrm{U}(\mathfrak{g}) x_j^2 \supsetneq \mathrm{U}(\mathfrak{g}) x_j^3 \supsetneq \cdots$$

is a strictly increasing sequence of left ideals of infinite length.

That $\mathrm{U}(\mathfrak{g})$ is not right artinian can be seen in the same way.

References

[MRS87] J. C. McConnel, J. C. Robson, and L. W. Small. *Noncommutative Noetherian Rings*. Graduate Studies in Mathematics 30. American Mathematical Society, 1987, pp. xx+636. ISBN: 978-0-8218-2169-5.