Remark and Solutions

Sheet 3

Problem 17

All occuring field are of characteristic zero.

3.

To show that $I \coloneqq [\mathfrak{g}, \mathfrak{g}]^\perp$ is contained in $\operatorname{rad}(\mathfrak{g})$ we show that I is a solvable ideal in $\mathfrak{g}$. The claimed inclusion then follows because $\operatorname{rad}(\mathfrak{g})$ is the unique maximal solvable ideal in $\mathfrak{g}$.

To see that I is an ideal in $\mathfrak{g}$ we observe that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$, and that for every ideal J in $\mathfrak{g}$ the orthogonal $J^\perp$ is again an ideal in $\mathfrak{g}$ because the Killing form $\kappa_{\mathfrak{g}}$ is associative.[1]

To see that I is solvable we observe that $\kappa_{\mathfrak{g}}(x, y) = 0$ for all $x \in I$ and $y \in [\mathfrak{g}, \mathfrak{g}]$ by definition of I and hence especially $\kappa_{\mathfrak{g}}(x, y) = 0$ for all $x \in I$ and $y \in [I, I]$. We know that the restriction of $\kappa_{\mathfrak{g}}$ to I coincides with the Killing form κ_I because I is an ideal in $\mathfrak{g}$. We have thus found that $\kappa_I(x, y) = 0$ for all $x \in I$ and $y \in [I, I]$. It follows from Cartan's criterion (part 2 of this exercise) that the ideal I is indeed solvable.

To show that $\operatorname{rad}(\mathfrak{g})$ is contained in $[\mathfrak{g}, \mathfrak{g}]^\perp$ we need to show that $\kappa_{\mathfrak{g}}(\operatorname{rad}(\mathfrak{g}), [\mathfrak{g}, \mathfrak{g}]) = 0$. It follows from the associativity of $\kappa_{\mathfrak{g}}$ that

$$\kappa_{\mathfrak{g}}(\operatorname{rad}(\mathfrak{g}), [\mathfrak{g}, \mathfrak{g}]) = \kappa_{\mathfrak{g}}([\operatorname{rad}(\mathfrak{g}), \mathfrak{g}], \mathfrak{g})\,.$$

It follows from the upcoming part 4 (that we will prove independent of this part) that the ideal $[\operatorname{rad}(\mathfrak{g}), \mathfrak{g}]$ is nilpotent. That $\kappa_{\mathfrak{g}}([\operatorname{rad}(\mathfrak{g}), \mathfrak{g}], \mathfrak{g}) = 0$ hence follows from the following observation:

Lemma 1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let I be a nilpotent ideal in $\mathfrak{g}$. Then $\kappa_{\mathfrak{g}}(I, \mathfrak{g}) = 0$, i.e. I is contained in the radical of κ.

Proof. Let $x \in I$ and $y \in \mathfrak{g}$. We have for all $n \geq 0$ that $\operatorname{ad}(x)(I^n) \subseteq I^{n+1}$ and $\operatorname{ad}(y)(I^n) \subseteq I^n$ because I^n is an ideal in I. (We are using the convention $I^0 = I$.) It hence follows by induction that the image of $(\operatorname{ad}(x)\operatorname{ad}(y))^{n+1}$ is contained in I^n for every $n \geq 0$.

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.

[1] This was hopefully shown in the lecture for the proof of the equivalence of the characterizations for semisimple Lie algebras.

It therefore follows for n sufficiently large from $I^n = 0$ that also $(\mathrm{ad}(x)\,\mathrm{ad}(y))^n = 0$ whence $\mathrm{ad}(x)\,\mathrm{ad}(y)$ is nilpotent. Therefore $\kappa_{\mathfrak{g}}(x,y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$. □

4.

This solution is longer than necessary but hopefully sheds some light on what's going on. We are taking the technical aspects of our approach from [Bou89, I.§5.3] and [TY05, §19.5].

Recall 2. As a motivation for the upcoming definition of $\mathrm{srad}(\mathfrak{g})$ we recall the equivalent characterizations of the Jacobson radical $\mathrm{J}(A)$ of a finite dimensional k-algebra A:

(1) The intersection of all maximal left ideal of A.

(2) The unique maximal nilpotent ideal of A.

(3) The set of all $x \in A$ that annihilate every simple left A-module.

Moreover:

(4) The algebra A is semisimple if and only if $\mathrm{J}(A) = 0$.

For a finite dimensional Lie algebra $\mathfrak{g}$ one can similarly consider four kinds of "radical":

(1) The sum of any two solvable ideals of $\mathfrak{g}$ is again solvable whence $\mathfrak{g}$ admits a unique maximal solvable ideal. This is the *radical* $\mathrm{rad}(\mathfrak{g})$ as introduced in the lecture. The Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathrm{rad}(\mathfrak{g}) = 0$.

(2) The *radical of the Killing form* of $\mathfrak{g}$. Note that by Cartan's criterion, $\mathrm{rad}(\kappa)$ is solvable. (The restriction of $\kappa_{\mathfrak{g}}$ to $\mathrm{rad}(\kappa_{\mathfrak{g}})$ is the Killing form of $\mathrm{rad}(\kappa_{\mathfrak{g}})$ because $\mathrm{rad}(\kappa_{\mathfrak{g}})$ is an ideal in $\mathfrak{g}$. But this restriction is zero by definition of $\mathrm{rad}(\kappa_{\mathfrak{g}})$ whence $\kappa_{\mathrm{rad}(\kappa_{\mathfrak{g}})} = 0$. Now apply Cartan's criterion to $\mathrm{rad}(\kappa_{\mathfrak{g}})$.) Thus $\mathrm{rad}(\kappa_{\mathfrak{g}})$ is contained in $\mathrm{rad}(\mathfrak{g})$.

(3) The sum of any two nilpotent ideals of $\mathfrak{g}$ is again nilpotent whence $\mathfrak{g}$ admits a unique maximal nilpotent ideal. (This was not shown in the lecture. We will also not show this here, as we won't need this.) This is the *nilradical* $\mathrm{nil}(\mathfrak{g})$. Note that $\mathrm{nil}(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g})$ because every nilpotent ideal is in particular a solvable ideal. Even better, by Lemma 1 we have $\mathrm{nil}(\mathfrak{g}) \subseteq \mathrm{rad}(\kappa_{\mathfrak{g}})$.

(4) The set

$$\mathrm{srad}(\mathfrak{g}) \coloneqq \left\{ x \in \mathfrak{g} \,\middle|\, \begin{array}{c} x \text{ annihilates every finite dimensional} \\ \text{irreducible } \mathfrak{g}\text{-representation} \end{array} \right\}$$

is called the *nilpotent radical* of $\mathfrak{g}$. The author doesn't like this name because it sounds too much like the nilradical, and hence will not use it.[2] Note that $\mathrm{srad}(\mathfrak{g})$ is an ideal because

$$\mathrm{srad}(\mathfrak{g}) = \bigcap \left\{ \ker(\rho) \,\middle|\, \begin{array}{c} (V,\rho) \text{ is a finite dimensional} \\ \text{irreducible } \mathfrak{g}\text{-representation} \end{array} \right\}.$$

[2]The term is taken from [Bou89, I.§5, Definition 3], the notation $\mathrm{srad}(\mathfrak{g})$ is taken from $\mathfrak{s}$ used there.

We show in the following that $\mathrm{srad}(\mathfrak{g})$ is nilpotent (and hence contained in $\mathrm{nil}(\mathfrak{g})$) and that $\mathrm{srad} = \mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. Then altogether

$$\mathrm{rad}(\mathfrak{g}) \supseteq \mathrm{rad}(\kappa_{\mathfrak{g}}) \supseteq \mathrm{nil}(\mathfrak{g}) \supseteq \mathrm{srad} = \mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \,.$$

(We refer to [MO13] for a short comment regarding the history of these radicals.) It then follows from $[\mathrm{rad}(\mathfrak{g}), \mathfrak{g}] \subseteq \mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] = \mathrm{srad}(\mathfrak{g})$ that $[\mathrm{rad}(\mathfrak{g}), \mathfrak{g}]$ is nilpotent, as needed for the exercise.

Showing that $\mathrm{srad}(\mathfrak{g})$ is nilpotent

The nilpotency of $\mathrm{srad}(\mathfrak{g})$ follows from an equivalent characterization of $\mathrm{srad}(\mathfrak{g})$:

Proposition 3. For an ideal I in a Lie algebra $\mathfrak{g}$ the following conditions are equivalent:

(1) I annihilates every finite dimensional irreducible $\mathfrak{g}$-representation V.

(2) I acts nilpotent on every finite dimensional $\mathfrak{g}$-representation (V, ρ) in the sense that $\rho(x)$ is nilpotent for every $x \in I$.

(3) I acts nilpotent on every finite dimensional $\mathfrak{g}$-representation (V, ρ) in the sense that for some $n \geq 1$, $\rho(x)^n = 0$ for every $x \in I$.

Hence $\mathrm{srad}(\mathfrak{g})$ is the unique maximal ideal in $\mathfrak{g}$ that acts nilpotent on every finite dimensional $\mathfrak{g}$-representation.

Lemma 4. Let V be a finite dimensional vector space and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ such that V is irreducible as a $\mathfrak{g}$-representation. If I is an ideal in $\mathfrak{g}$ that consists of nilpotent endomorphisms then $I = 0$.

Proof. It follows from Engel's theorem that I annihilates some nonzero linear subspace U of V. The subspace U is a $\mathfrak{g}$-subrepresentation: For all $x \in I$, $y \in \mathfrak{g}$ and $u \in U$,

$$x.y.u = y.x.u + [x, y].u = 0$$

because $x, [x, y] \in I$. It follows that $U = V$ because V is irreducible. □

Proof of Proposition 3.

(3) $\implies$ (2) Okay.

(2) $\implies$ (1) We may apply Lemma 4 to the ideal $\rho(I)$ of $\rho(\mathfrak{g})$ to find $\rho(I) = 0$.

(1) $\implies$ (3) There exists by the finite dimensionality of V a filtration

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

by $\mathfrak{g}$-subrepresentations of maximal length, i.e. such that the quotients V_i/V_{i-1} are irreducible. Then $I.(V_i/V_{i-1}) = 0$ and thus $I.V_i \subseteq V_{i-1}$ for every $i = 1, \ldots, n$. Whence $\rho(x)^n = 0$ for every $x \in I$. □

Corollary 5. For every finite dimensional Lie algebra $\mathfrak{g}$ the ideal $\mathrm{srad}(\mathfrak{g})$ is nilpotent.

Proof. By applying Proposition 3 to the adjoint action of $\mathrm{srad}(\mathfrak{g})$ on $\mathfrak{g}$ we see that every $x \in \mathrm{srad}(\mathfrak{g})$ is $\mathrm{ad}_{\mathfrak{g}}$-nilpotent nilpotent and hence $\mathrm{ad}_{\mathrm{srad}(\mathfrak{g})}$-nilpotent. It follows from Engel's theorem that $\mathrm{srad}(\mathfrak{g})$ is nilpotent. □

Showing that $\mathrm{srad}(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$

Lemma 6. If $\mathfrak{g}$ is an abelian Lie algebra then elements of $\mathfrak{g}$ can be separated by one-dimensional representations, in the sense that there exists for all $x, y \in \mathfrak{g}$ some one-dimensional representation (V, ρ) of $\mathfrak{g}$ with $\rho(x) \neq \rho(y)$.

Proof. A one dimensional representation of $\mathfrak{g}$ is (up to isomorphism) the same as a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(k) = k$. Both $\mathfrak{g}$ and k are abelian, so every linear map $\mathfrak{g} \to k$ is already a homomorphism of Lie algebras. The assertian is therefore equivalent to $\mathfrak{g}^*$ separating the elements of $\mathfrak{g}$, which is known from linear algebra. $\square$

Corollary 7. If $\mathfrak{g}$ is an abelian Lie algebra then $\mathrm{srad}(\mathfrak{g}) = 0$.

Proof. An element $x \in \mathrm{srad}(\mathfrak{g})$ annihilates every finite dimensional irreducible representation of $\mathfrak{g}$ and thus in particular every one-dimensional representation of $\mathfrak{g}$. This means that x cannot be seperated from 0 by one-dimensional representations of $\mathfrak{g}$. Therefore $x = 0$ by Lemma 6. $\square$

Lemma 8. If $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a surjective homomorphism of Lie algebras then

$$\phi(\mathrm{srad}(\mathfrak{g})) \subseteq \mathrm{srad}(\mathfrak{h})\,.$$

Proof. Every finite dimensional irreducible representation (V, ρ) of $\mathfrak{h}$ can be pulled back to a representation $(V, \rho \circ \phi)$ of $\mathfrak{g}$. It follows from the surjectivity of $\mathfrak{g}$ that $(V, \rho \circ \phi)$ is again irreducible. We find that $(V, \rho \circ \phi)$ is annihilated by $\mathrm{srad}(\mathfrak{g})$, which means that $\phi(\mathrm{srad}(\mathfrak{g}))$ annihilates (V, ρ). This shows that $\phi(\mathrm{srad}(\mathfrak{g}))$ annihilates every finite dimensional irreducible representation of $\mathfrak{h}$, so $\phi(\mathrm{srad}(\mathfrak{g})) \subseteq \mathrm{srad}(\mathfrak{h})$. $\square$

Corollary 9. If $\mathfrak{g}$ is a finite dimensional Lie algebra then $\mathrm{srad}(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$.

Proof. That $\mathrm{srad}(\mathfrak{g})$ is contained in $\mathrm{rad}(\mathfrak{g})$ follows from $\mathrm{srad}(\mathfrak{g})$ being nilpotent and hence solvable. If $\pi\colon \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is the canonical projection then

$$\pi(\mathrm{srad}(\mathfrak{g})) \subseteq \mathrm{srad}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) = 0$$

by Lemma 8 and Corollary 7, and hence $\mathrm{srad}(\mathfrak{g}) \subseteq \ker \pi = [\mathfrak{g}, \mathfrak{g}]$. $\square$

Showing $\mathrm{rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{srad}(\mathfrak{g})$

For this inclusion we will need some preparation. We start with a classical observation from linear algebra:

Lemma 10. Let $\mathrm{char}(k) = 0$. If $A \in \mathrm{M}(n, k)$ is a matrix with $\mathrm{tr}(A^m) = 0$ for all $m \geq 1$ then A is nilpotent.

Proof. We may assume that k is algebraically closed. Let $\lambda_1, \ldots, \lambda_r$ be the pairwise different nonzero (!) eigenvalues of A with corresponding multiplicities $n_1, \ldots, n_r$. The matrix A is triangularizable with diagonal entries $\lambda_1, \ldots, \lambda_r, 0$ so we need to show that $n_1 = \cdots = n_r = 0$.

We have for all $m \geq 0$ that

$$0 = \mathrm{tr}(A^m) = n_1 \lambda_1^m + \cdots + n_r \lambda_r .$$

For $m = 1, \ldots, r$ we can rearrange these equalities in the matrix form

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{pmatrix} = 0 . \tag{1}$$

If we denote the matrix on the left hand side by V then the determinant

$$\det(V) = \lambda_1 \cdots \lambda_r \prod_{j>i} (\lambda_j - \lambda_i)$$

does not vanish. The matrix V is hence invertible. It follows that the column vector in (1) is the zero vector, so that $n_1 = \cdots = n_r = 0$ (here we use that $\mathrm{char}(k) = 0$). $\square$

Lemma 11. Let V be a finite dimensional vector space and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ such that V is irreducible as a $\mathfrak{g}$-representation. If I is any commutative ideal in $\mathfrak{g}$ then $I \cap [\mathfrak{g}, \mathfrak{g}] = 0$.

Proof. Let A be the (associative and unital) k-subalgebra of $\mathrm{End}_k(V)$ generated by I. Note that A is commutative because I is abelian.

Claim. If J is a Lie ideal in $\mathfrak{g}$ with $J \subseteq I$ and $\mathrm{tr}(xa) = 0$ for all $x \in J$, $a \in A$ then $J = 0$.

Proof of the Claim. It follows from $x \in J \subseteq I \subseteq A$ that $x^n \in A$ for all $n \geq 0$ and thus by assumption $\mathrm{tr}(x^n) = \mathrm{tr}(x \cdot x^{n-1}) = 0$ for all $n \geq 1$. It follows from Lemma 10 that x is nilpotent and it therefore further follows from Lemma 4 that $J = 0$. $\square$

We find that $\mathfrak{g}$ and I commute: Letting $x \in \mathfrak{g}$ and $y \in I$ we find for all $a \in A$ that $ay = ya$ because A is commutative. As a consequence,

$$\begin{aligned} \mathrm{tr}([x,y]a) &= \mathrm{tr}(xya - yxa) \\ &= \mathrm{tr}(xya) - \mathrm{tr}(yxa) \\ &= \mathrm{tr}(xya) - \mathrm{tr}(xay) \\ &= \mathrm{tr}(xya) - \mathrm{tr}(xya) \\ &= 0 . \end{aligned}$$

It follows with the claim that $[\mathfrak{g}, I] = 0$.

It follows that $\mathfrak{g}$ and A commute: The centralizer of $\mathfrak{g}$ in $\mathrm{End}_k(V)$ is an (associative and unital) subalgebra of $\mathrm{End}_k(V)$ that contains I, and hence also contains A.

We see that $\mathrm{tr}(za) = 0$ for all $z \in [\mathfrak{g}, \mathfrak{g}]$ and $a \in A$: We have for all $x, y \in \mathfrak{g}$ that $ay = ya$ because $\mathfrak{g}$ and A commute, and hence $\mathrm{tr}([x,y]a) = 0$ by the same calculation as above.

It follows for $J = [\mathfrak{g},\mathfrak{g}] \cap I$, which is a Lie ideal in $\mathfrak{g}$ that is contained in I, that $\operatorname{tr}(za) = 0$ for all $z \in J$ and $a \in A$ and thus $J = 0$ by the claim. This is what we want to prove. $\square$

Corollary 12. If $\mathfrak{g}$ is a Lie algebra then $\operatorname{rad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}] \subseteq \operatorname{srad}(\mathfrak{g})$.

Proof. Let (V,ρ) be a finite dimensional irreducible representation of $\mathfrak{g}$. Then $\rho(\operatorname{rad}(\mathfrak{g}))$ is solvable and hence there exists some minimal $n \geq 0$ with $\rho(\operatorname{rad}(\mathfrak{g}))^{(n+1)} = 0$; here we use for every ideal I the convention $I^{(0)} = I$, so that $I^{(m+1)} = [I^{(m)}, I^{(m)}]$ for all $m \geq 0$. Let $\mathfrak{g}' \coloneqq \rho(\mathfrak{g})$ and $I' \coloneqq \rho(\operatorname{rad}(\mathfrak{g}))^{(n)}$. Then I' is an ideal in $\mathfrak{g}'$ that is abelian because

$$[I', I'] = [\rho(\operatorname{rad}(\mathfrak{g}))^{(n)}, \rho(\operatorname{rad}(\mathfrak{g}))^{(n)}] = \rho(\operatorname{rad}(\mathfrak{g}))^{(n+1)} = 0\,.$$

That V is irreducible as a representation of $\mathfrak{g}$ means that it is irreducible as a representation of $\mathfrak{g}'$. We can therefore apply Lemma 11 to find that $I' \cap [\mathfrak{g}', \mathfrak{g}'] = 0$. Hence

$$0 = I' \cap [\mathfrak{g}', \mathfrak{g}'] = \rho(\operatorname{rad}(\mathfrak{g}))^{(n)} \cap [\rho(\mathfrak{g}), \rho(\mathfrak{g})] = \rho(\operatorname{rad}(\mathfrak{g})^{(n)} \cap [\mathfrak{g},\mathfrak{g}])\,. \tag{2}$$

If now $n \geq 1$ then $\operatorname{rad}(\mathfrak{g})^{(n)} \subseteq [\mathfrak{g},\mathfrak{g}]$ and then (2) simplifies to $\rho(\operatorname{rad}(\mathfrak{g})^{(n)}) = 0$. But this would contradict the minimality of n, hence $n = 0$. We hence find

$$0 = \rho(\operatorname{rad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}])\,.$$

This shows that $\operatorname{rad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$ annihilates every finite dimensional irreducible representation of $\mathfrak{g}$, which means that $\operatorname{rad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}] \subseteq \operatorname{srad}(\mathfrak{g})$. $\square$

With this we finally arrive at the desired equality:

Theorem 13. If $\mathfrak{g}$ is a finite dimensional Lie algebra then $\operatorname{srad}(\mathfrak{g}) = \operatorname{rad}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$.

Proof. We combine Corollary 9 and Corollary 12. $\square$

A Generalization

The nilpotence of $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$ can also be generalized by using Problem 20 from sheet 4, as explained in [Bou89, I.§5.5] and [TY05, §19.6]:

Theorem 14. If D is a derivation of a finite dimensional Lie algeba $\mathfrak{g}$ then

$$D(\operatorname{rad}(\mathfrak{g})) \subseteq \operatorname{nil}(\mathfrak{g})\,.$$

We are now gonna prove this theorem. We have split up the proof in multiple small steps each of which is interesting on its own.

Definition 15. An ideal I is a Lie algebra $\mathfrak{g}$ is *characteristic* if $D(I) \subseteq I$ for every ideal I.

Example 16. The derived ideal $[\mathfrak{g},\mathfrak{g}]$ is characteristic because

$$D([\mathfrak{g},\mathfrak{g}]) \subseteq [D(\mathfrak{g}), \mathfrak{g}] + [\mathfrak{g}, D(\mathfrak{g})] \subseteq [\mathfrak{g},\mathfrak{g}]\,.$$

Lemma 17. If I is a characteristic ideal in a Lie algebra $\mathfrak{g}$ and J is an ideal in I then J is also an ideal in $\mathfrak{g}$.

Proof. For every $x \in \mathfrak{g}$ the adjoint action $\operatorname{ad}(x)$ restricts to a derivation of I, which then leaves J invariant. □

Lemma 18. If $\mathfrak{g}$ is a finite dimensional Lie algebra with Killing form κ then every derivation D of $\mathfrak{g}$ is skew-selfadjoint with respect to κ, i.e. for all $x, y \in \mathfrak{g}$,

$$\kappa(D(x), y) = -\kappa(x, D(y)) \,.$$

Proof. We find with $[D, \operatorname{ad}(x)] = \operatorname{ad}(D(x))$[3] that

$$\begin{aligned}
\kappa(D(x), y) &= \operatorname{tr}(\operatorname{ad}(D(x)) \operatorname{ad}(y)) \\
&= \operatorname{tr}([D, \operatorname{ad}(x)], \operatorname{ad}(y)) \\
&= \operatorname{tr}(D \operatorname{ad}(x) \operatorname{ad}(y) - \operatorname{ad}(x) D \operatorname{ad}(y)) \\
&= \operatorname{tr}(D \operatorname{ad}(x) \operatorname{ad}(y)) - \operatorname{tr}(\operatorname{ad}(x) D \operatorname{ad}(y)) \\
&= \operatorname{tr}(\operatorname{ad}(x) \operatorname{ad}(y) D) - \operatorname{tr}(\operatorname{ad}(x) D \operatorname{ad}(y)) \\
&= \operatorname{tr}(\operatorname{ad}(x)[\operatorname{ad}(y), D]) \\
&= -\operatorname{tr}(\operatorname{ad}(x)[D, \operatorname{ad}(y)]) \\
&= -\operatorname{tr}(\operatorname{ad}(x) \operatorname{ad}(D(y))) \\
&= -\kappa(x, D(y))
\end{aligned}$$

as claimed. □

Corollary 19. If I is a characteristic ideal in a finite dimensional Lie algebra $\mathfrak{g}$ then its orthogonal $I^\perp$ with respect to the Killing form κ of $\mathfrak{g}$ is again a characterestic ideal.

Proof. The orthogonal of I is again an ideal by the associativity of κ. If D is any derivation of $\mathfrak{g}$ and $x \in I^\perp$ then for every $y \in I$ again $D(y) \in I$ and hence

$$\kappa(D(x), y) = -\kappa(x, D(y)) = 0 \,.$$

Thus $D(I^\perp) \subseteq I^\perp$. □

Example 20. If $\mathfrak{g}$ is a finite dimensional Lie algebra then the radical $\operatorname{rad}(\mathfrak{g})$ is the orthogonal of the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ (by part 3) and hence again a characteristic ideal.

Corollary 21. If I is an ideal in a finite dimensional Lie algebra $\mathfrak{g}$ then

$$\operatorname{rad}(I) = \operatorname{rad}(\mathfrak{g}) \cap I \,.$$

Proof. The intersection $\operatorname{rad}(\mathfrak{g}) \cap I$ is a solvable ideal in I and hence contained in $\operatorname{rad}(I)$. The radical $\operatorname{rad}(I)$ is a characteristic ideal in I by Example 20 and hence an ideal in $\mathfrak{g}$ by Lemma 17. It is solvable and hence contained in $\operatorname{rad}(\mathfrak{g})$, and thus contained in $\operatorname{rad}(\mathfrak{g}) \cap I$. □

[3] This is the formula that show that the inner derivations form an ideal in $\operatorname{Der}(\mathfrak{g})$.

Proof of Theorem 14. We apply the construction of Problem 20 from sheet 4: We regard the element $D \in \mathrm{Der}(\mathfrak{g})$ as a Lie algebra homomorphism $\theta\colon k \to \mathrm{Der}(\mathfrak{g})$ and form the semidirect product $\mathfrak{g}' := \mathfrak{g} \rtimes_\theta k$. Then the derivation D of $\mathfrak{g}$ extends to the inner derivation $\mathrm{ad}_{(0,1)}$ of $\mathfrak{g}'$.

It follows from Corollary 21 that $\mathrm{rad}(\mathfrak{g}) = \mathrm{rad}(\mathfrak{g}') \cap \mathfrak{g}$ is contained in $\mathrm{rad}(\mathfrak{g}')$ because $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$. Now $[\mathfrak{g}', \mathrm{rad}(\mathfrak{g}')]$ is contained in the nilradical $\mathrm{nil}(\mathfrak{g}')$ as seen in Corollary 5. Hence

$$D(\mathrm{rad}(\mathfrak{g})) = \mathrm{ad}_{(0,1)}(\mathrm{rad}(\mathfrak{g})) = [(0,1), \mathrm{rad}(\mathfrak{g})]$$

is contained in $\mathrm{nil}(\mathfrak{g}')$, and thus also in $\mathfrak{g} \cap \mathrm{nil}(\mathfrak{g}')$. This intersection is a nilpotent ideal in $\mathfrak{g}$ and hence contained in $\mathrm{nil}(\mathfrak{g})$. Thus $D(\mathrm{rad}(\mathfrak{g})) \subseteq \mathfrak{g} \cap \mathrm{nil}(\mathfrak{g}') \subseteq \mathrm{nil}(\mathfrak{g})$. $\square$

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