Remark and Solutions

Sheet 2

Problem 13

3.

We follow the hint: Let V be a four-dimensional vector space with basis e_1, e_2, e_3, e_4. We identify the exterior power $\bigwedge^4(V)$ with the ground field $\mathbb{C}$ via the single basis element $e_1 \wedge e_2 \wedge e_3 \wedge e_4$.

We find that

$$v \wedge w_1 \wedge w_2 = -w_1 \wedge v \wedge w_2 = w_1 \wedge w_2 \wedge v$$

for every vector $v \in V$ and every simple wedge $w_1 \wedge w_2 \in \bigwedge^2(V)$. The bilinear map

$$\beta \colon \bigwedge^2(V) \times \bigwedge^2(V) \to \bigwedge^4(V)\,, \quad (x, y) \mapsto x \wedge y$$

is hence symmetric, beause

$$\begin{aligned} \beta(v_1 \wedge v_2, w_1 \wedge w_2) &= v_1 \wedge v_2 \wedge w_1 \wedge w_2 \\ &= v_1 \wedge w_1 \wedge w_2 \wedge v_2 \\ &= w_1 \wedge w_2 \wedge v_1 \wedge v_2 \end{aligned}$$

for all simple wedges $v_1 \wedge v_2, w_1 \wedge w_2 \in \bigwedge^2(V)$. The bilinear form β is also non-degenerate: A basis of the exterior square $\bigwedge^2(V)$ is given by the simple wedges

$$e_1 \wedge e_2\,, \quad e_1 \wedge e_3\,, \quad e_1 \wedge e_4\,, \quad e_2 \wedge e_3\,, \quad e_2 \wedge e_4\,, \quad e_3 \wedge e_4\,.$$

With respect to this basis of $\bigwedge^2(V)$ the bilinear form β is given by the matrix

$$\begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & \end{pmatrix}.$$

This matrix is invertible, which means that β is non-degenerate.

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.

Warning 1. The fourth exterior power $\bigwedge^4(V)$ is neither the same nor isomorphic to the iterated exterior square $\bigwedge^2(\bigwedge^2(V))$. In particular $\dim \bigwedge^4(V) = \binom{\dim V}{4} = \binom{4}{4} = 1$ whereas $\dim \bigwedge^2(\bigwedge^2(V)) = \binom{\dim \bigwedge^2(V)}{2} = \binom{6}{2} = 15$.

It follows from Problem 8 of the first exercise sheet that

$$\mathfrak{g}_\beta \coloneqq \left\{ \varphi \in \mathfrak{gl}(V) \,\middle|\, \beta(\varphi(x), y) + \beta(x, \varphi(y)) = 0 \text{ for all } x, y \in \bigwedge^2(V) \right\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$ that is isomorphic to $\mathfrak{so}_6(\mathbb{C})$, because β in non-degenerate and symmetric and $\bigwedge^2(V)$ is six-dimensional. We will in the following construct an explicit isomorphism $\mathfrak{sl}_4(V) \cong \mathfrak{g}_\beta$ (where $\mathfrak{sl}(V) \cong \mathfrak{sl}_4(\mathbb{C})$ because V is four-dimensional).

Lemma 2. If V is a representation of a Lie algebra $\mathfrak{g}$ then any exterior power $\bigwedge^n(V)$ inherits from V the structure of a $\mathfrak{g}$-representations via

$$x.(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^n v_1 \wedge \cdots \wedge (x.v_i) \wedge \cdots \wedge v_n$$

for all $x \in \mathfrak{g}$ and every simple wedge $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n(V)$.

Proof. For readability we will denote for all $v_1, \dots, v_n \in V$ the corresponding simple wedge $v_1 \wedge \cdots \wedge v_n$ by $v_1 \cdots v_n$. We first need to show that every $x \in \mathfrak{g}$ the proposed action on x on $\bigwedge^n(V)$ is well-defined and linear. We do so by using the universal property of the exterior power. For this we consider the map

$$\gamma_x \colon V \times \cdots \times V \to \bigwedge^n(V), \quad (v_1, \dots, v_n) \mapsto \sum_{i=1}^n v_1 \cdots (x.v_i) \cdots v_n \,.$$

This map is multilinear and we need to check that it is alternating. So let $v_1, \dots, v_n \in V$ with $v_i = v_j$ for some $i < j$. Then $v_1 \cdots (x.v_k) \cdots v_n = 0$ whenever $k \neq i, j$ and thus

$$\begin{aligned}
\gamma_x(v_1, \dots, v_n) &= \sum_{k=1}^n v_1 \cdots (x.v_k) \cdots v_n \\
&= v_1 \cdots (x.v_i) \cdots v_j \cdots v_n + v_1 \cdots v_i \cdots (x.v_j) \cdots v_n \\
&= v_1 \cdots (x.v_i + v_i) \cdots (x.v_j + v_j) \cdots v_n \\
&\quad - v_1 \cdots (x.v_i) \cdots (x.v_j) \cdots v_n \\
&\quad - v_1 \cdots v_i \cdots v_j \cdots v_n \\
&= 0 \,.
\end{aligned}$$

It now follows from the universal property of the exterior power $\bigwedge^n(V)$ that γ_x induces a well-defined linear map

$$\bigwedge^n(V) \to \bigwedge^n(V), \quad v_1 \cdots v_n \mapsto \sum_{i=1}^n v_1 \cdots (x.v_i) \cdots v_n \,.$$

We need to check that this action of $\mathfrak{g}$ on $\bigwedge^n(V)$ is again compatible with the Lie bracket of $\mathfrak{g}$. We see that

$$
\begin{aligned}
&x.y.(v_1 \cdots v_n) \\
&= x.\sum_{i=1}^{n} v_1 \cdots (y.v_i) \cdots v_n \\
&= \sum_{i<j} v_1 \cdots (x.v_i) \cdots (y.v_j) \cdots v_n \\
&\quad + \sum_{i=1}^{n} v_1 \cdots (x.y.v_i) \cdots v_n \\
&\quad + \sum_{i<j} v_1 \cdots (y.v_i) \cdots (x.v_j) \cdots v_n
\end{aligned}
$$

and hence

$$
\begin{aligned}
&x.y.(v_1 \cdots v_n) - y.x.(v_1 \cdots v_n) \\
&= \sum_{i=1}^{n} v_1 \cdots (x.y.v_i) \cdots v_n - \sum_{i=1}^{n} v_1 \cdots (y.x.v_i) \cdots v_n \\
&= \sum_{i=1}^{n} v_1 \cdots (x.y.v_i - y.x.v_i) \cdots v_n \\
&= \sum_{i=1}^{n} v_1 \cdots ([x,y].v_i) \cdots v_n
\end{aligned}
$$

for all $x, y \in \mathfrak{g}$ and every simple wedge $v_1 \cdots v_n \in \bigwedge^n(V)$. $\square$

It follows from Lemma 2 that the natural action of $\mathfrak{sl}(V)$ on V induces on $\bigwedge^2(V)$ the structure of an $\mathfrak{sl}(V)$-representations via

$$X.(v_1 \wedge v_2) = (X.v_1) \wedge v_2 + v_1 \wedge (X.v_2)$$

for all $X \in \mathfrak{sl}(V)$ and every simple wedge $v_1 \wedge v_2 \in \bigwedge^2(V)$. This action of $\mathfrak{sl}(V)$ on $\bigwedge^2(V)$ corresponds to a homomorphism of Lie algebras $\rho\colon \mathfrak{sl}(V) \to \mathfrak{gl}(\bigwedge^2(V))$ that is given by $\rho(X)(x) = X.x$ for all $X \in \mathfrak{sl}(V)$ and $x \in \bigwedge^2(V)$. We show in the following that ρ restricts to an isomorphism $\mathfrak{sl}(V) \to \mathfrak{g}_\beta$.

We first observe that the image of ρ is contained in $\mathfrak{g}_\beta$: We need to show that

$$\beta(\rho(X)(x), y) + \beta(x, \rho(X)(y) = 0$$

for all $X \in \mathfrak{sl}(V)$ and $x, y \in \bigwedge^2(V)$. It sufficies to consider the case that both x and y

are simple wedges $x = v_1 \wedge v_2$ and $y = v_3 \wedge v_4$. Then the term

$$\begin{aligned}
&\gamma(v_1, v_2, v_3, v_4) \\
&\coloneqq \beta(\rho(X)(v_1 \wedge v_2), v_3 \wedge v_4) + \beta(v_1 \wedge v_2, \rho(X)(v_3 \wedge v_4)) \\
&= \rho(X)(v_1 \wedge v_2) \wedge v_3 \wedge v_4 + v_1 \wedge v_2 \wedge \rho(X)(v_3 \wedge v_4) \\
&= (X.v_1) \wedge v_2 \wedge v_3 \wedge v_4 + v_1 \wedge (X.v_2) \wedge v_3 \wedge v_4 \\
&\quad + v_1 \wedge v_2 \wedge (X.v_3) \wedge v_4 + v_1 \wedge v_2 \wedge v_3 \wedge (X.v_4)
\end{aligned}$$

is multilinear and alternating in v_1, v_2, v_3, v_4. To show that $\gamma(v_1, v_2, v_3, v_4) = 0$ for all $v_1, v_2, v_3, v_4 \in V$ it therefore sufficies to show that $\gamma(e_1, e_2, e_3, e_4) = 0$ for the given basis e_1, e_2, e_3, e_4 of V. Let $A \in \mathfrak{sl}_4(\mathbb{C})$ be the matrix that represents $X \in \mathfrak{sl}(V)$ with respect to this basis, which means that $X.e_j = \sum_{i=1}^4 A_{ij} e_i$ for every $j = 1, 2, 3, 4$. Then

$$(X.e_1) \wedge e_2 \wedge e_3 \wedge e_4 = \sum_{i=1}^{4} A_{i1} e_i \wedge e_2 \wedge e_3 \wedge e_4 = A_{11} e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

and similarly

$$\begin{aligned}
e_1 \wedge (X.e_2) \wedge e_3 \wedge e_4 &= A_{22} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,, \\
e_1 \wedge e_2 \wedge (X.e_3) \wedge e_4 &= A_{33} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,, \\
e_1 \wedge e_2 \wedge e_3 \wedge (X.e_4) &= A_{44} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,.
\end{aligned}$$

Hence altogether

$$\begin{aligned}
\gamma(e_1, e_2, e_3, e_4) &= (A_{11} + A_{22} + A_{33} + A_{44}) e_1 \wedge e_2 \wedge e_3 \wedge e_4 \\
&= \operatorname{tr}(A) e_1 \wedge e_2 \wedge e_3 \wedge e_4 = 0 \,.
\end{aligned}$$

We have shown that the image of ρ lies in $\mathfrak{g}_\beta$.

It remains to show that the homomorphism ρ is both surjective onto $\mathfrak{g}_\beta$ and injective. We know that $\dim \mathfrak{sl}(V) = \dim \mathfrak{sl}_4(\mathbb{C}) = 4^2 - 1 = 15$ and $\dim \mathfrak{g}_\beta = \dim \mathfrak{so}_6(\mathbb{C}) = 15$. It therefore sufficies to show that ρ is injective. It moreover sufficies to show that ρ is nonzero because $\mathfrak{sl}(V)$ is simple. If we consider the endomorphism $X \in \mathfrak{sl}(V)$ with $X(e_1) = e_1$, $X(e_2) = -e_2$ and $X(e_3) = X(e_4) = 0$ then

$$\rho(X)(e_1 \wedge e_3) = (X.e_1) \wedge e_3 + e_1 \wedge (X.e_3) = e_1 \wedge e_3 + e_1 \wedge 0 = e_1 \wedge e_3 \neq 0$$

whence $\rho(X) \neq 0$.

Problem 14

1.

To construct a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ we only need to specify the images E_i, H_i, F_i of the generators e_i, h_i, f_i and then check that these proposed images satisfy the given relations (22)–(27). (This is what it means for the Lie algebra $\mathfrak{g}$ to be given by the generators $\{e_i, h_i, f_i\}_{i=1}^n$ and relations (22)–(27).)

In the case $n = 1$ we know that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the standard basis e, f, h constisting of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These three matrices satisfy the relations $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$, which corresponds to the relations (23), (24), (25). Motivated by this example we choose for any $n \geq 1$ the images $E_i, F_i, H_i \in \mathfrak{sl}_{n+1}(\mathbb{C})$ as

$$E_i \coloneqq E_{i,i+1}, \quad F_i \coloneqq E_{i+1,i}, \quad H_i \coloneqq E_{ii} - E_{i+1,i+1}.$$

We need to check that these elements satisfy the relations (22)–(27):

(22) We find $[H_i, H_j] = 0$ because both H_i and H_j are diagonal matrices and hence commute with each other.

(23) We find

$$\begin{aligned}
[E_i, F_j] &= [E_{i,i+1}, E_{j+1,j}] \\
&= E_{i,i+1}E_{j+1,j} - E_{j+1,j}E_{i,i+1} \\
&= \delta_{i+1,j+1}E_{ij} - \delta_{ij}E_{j+1,i+1} \\
&= \delta_{ij}E_{ij} - \delta_{ij}E_{j+1,i+1} \\
&= \delta_{ij}E_{ii} - \delta_{ij}E_{i+1,i+1} \\
&= \delta_{ij}(E_{ii} - E_{i+1,i+1}) \\
&= \delta_{ij}H_i.
\end{aligned}$$

(24) We find

$$\begin{aligned}
[H_i, E_j] &= [E_{ii} - E_{i+1,i+1}, E_{j,j+1}] \\
&= (E_{ii} - E_{i+1,i+1})E_{j,j+1} - E_{j,j+1}(E_{ii} - E_{i+1,i+1}) \\
&= E_{ii}E_{j,j+1} - E_{i+1,i+1}E_{j,j+1} - E_{j,j+1}E_{ii} + E_{j,j+1}E_{i+1,i+1} \\
&= \delta_{ij}E_{i,j+1} - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} + \delta_{i+1,j+1}E_{j,i+1} \\
&= \delta_{ij}E_{i,j+1} - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} + \delta_{ij}E_{j,i+1} \\
&= \delta_{ij}(E_{i,j+1} + E_{j,i+1}) - \delta_{i+1,j}E_{i+1,j+1} - \delta_{i,j+1}E_{ji} \\
&= 2\delta_{ij}E_{j,j+1} - \delta_{i+1,j}E_{j,j+1} - \delta_{i,j+1}E_{j,j+1} \\
&= (2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1})E_{j,j+1} \\
&= a_{ji}E_j.
\end{aligned}$$

(25) This relation can be checked in the same way as (24). But we can also observe that $F_j = E_j^T$ and $H_i = H_i^T$ and hence

$$[H_i, F_j] = [H_i^T, E_j^T] = [E_j, H_i]^T = a_{ji}E_j^T = -a_{ji}F_j.$$

(26) We find

$$\begin{aligned}\operatorname{ad}(E_i)(E_j) &= [E_i, E_j] \\ &= [E_{i,i+1}, E_{j,j+1}] \\ &= E_{i,i+1}E_{j,j+1} - E_{j,j+1}E_{i,i+1} \\ &= \delta_{i+1,j}E_{i,j+1} - \delta_{i,j+1}E_{j,i+1}\,.\end{aligned}$$

If $|i-j| \geq 2$ then $\delta_{i+1,j} = \delta_{i,j+1} = 0$ and $a_{ji} = 0$ and hence

$$\operatorname{ad}(E_i)^{-a_{ji}+1}(E_j) = \operatorname{ad}(E_i)(E_j) = 0$$

as needed. If $i = j-1$ then $[E_i, E_j] = \operatorname{ad}(E_i)(E_j) = E_{i,j+1} = E_{i,i+2}$ and $a_{ji} = -1$ and thus

$$\begin{aligned}\operatorname{ad}(E_i)^{-a_{ji}+1}(E_j) &= \operatorname{ad}(E_i)^2(E_j) \\ &= [E_i, [E_i, E_j]] \\ &= [E_{i,i+1}, E_{i,i+2}] \\ &= E_{i,i+1}E_{i,i+2} - E_{i,i+2}E_{i,i+1} \\ &= \delta_{i,i+1}E_{i,i+2} - \delta_{i,i+2}E_{i,i+1} \\ &= 0\,.\end{aligned}$$

The case $i = j+1$ works the same.

(27) This can be done by similar calculations as in (26) but can also be directly derived from (26) by again using the matrix transpose.

Remark 3.

1. We used in (25) that that $[A,B]^T = (AB-BA)^T = B^TA^T - A^TB^T = [B^T, A^T]$.
2. For (24) and (25) it is useful to understand how a commutator $[D,A]$ looks like if D is a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n \in \mathbb{C}$:

   The matrix DA results from A by multiplying for every $i = 1, \dots, n$ the i-th row of A by the corresponding diagonal entry λ_i. Similarly the matrix AD results from A by multiplying for every $j = 1, \dots, n$ the j-th column of A by the corresponding diagonal entry λ_j. This means in formulae that

   $$(DA)_{ij} = \lambda_i A_{ij} \quad \text{and} \quad (AD)_{ij} = \lambda_j A_{ij}$$

   for all $i, j = 1, \dots, n$, and thus

   $$[D,A]_{ij} = (DA-AD)_{ij} = (DA)_{ij} - (AD)_{ij} = \lambda_i A_{ij} - \lambda_j A_{ij} = (\lambda_i - \lambda_j)A_{ij}$$

   for all $i, j = 1, \dots, n$.

   It follows that

   $$[H_i, E_j] = [H_i, E_{j,j+1}] = ((H_i)_{jj} - (H_i)_{j+1,j+1})E_{j,j+1}$$

where

$$(H_i)_{jj} - (H_i)_{j+1,j+1} = \begin{cases} 0 & \text{if } j+1 < i\,, \\ -1 & \text{if } j+1 = i\,, \\ 2 & \text{if } j = i\,, \\ -1 & \text{if } j = i+1\,, \\ 0 & \text{if } j > i+1\,. \end{cases}$$
$$= a_{ji}$$

Hence $[H_i, E_j] = a_{j,i} E_j$ as desired.

We have shown that the matrices E_i, F_i, H_i satisfy the given relations. There hence exists a unique homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ with $\varphi(e_i) = E_i$, $\varphi(f_i) = F_i$ and $\varphi(h_i) = H_i$. It remains to show that φ is surjective.

The image of φ is a Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$. It therefore sufficies to show that $\mathfrak{sl}_{n+1}(\mathbb{C})$ is generated by the elements $\{E_i, F_i, H_i\}_{i=1}^n$ as a Lie algebra. Let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$ generated by these elements. We know that $\mathfrak{sl}_{n+1}(\mathbb{C})$ has as a basis the diagonal matrices $H_1, \dots, H_n$ together with the off-diagonal matrices E_{ij} where $1 \leq i \neq j \leq n+1$. It sufficies to show that these matrices are contained in $\mathfrak{s}$. This holds for $H_1, \dots, H_n$ by construction of $\mathfrak{s}$.

Let us consider the off-diagonal matrices E_{ij} with $j > i$. We fix the index i and show that $E_{i,i+1}, E_{i,i+2}, \dots, E_{i,n+1} \in \mathfrak{s}$. This holds for $E_{i,i+1} = E_i$ by construction of $\mathfrak{s}$. If $E_{ij} \in \mathfrak{s}$ for some $i+1 \leq j < n+1$ then we find inductively that the matrix

$$[E_{ij}, E_j] = [E_{ij}, E_{j,j+1}] = E_{ij}E_{j,j+1} - E_{j,j+1}E_{ij} = E_{i,j+1}$$

is again contained $\mathfrak{s}$. This shows that all off-diagonal matrices E_{ij} with $j > i$ are contained in $\mathfrak{s}$.

For the off-diagonal matrices E_{ij} with $i < j$ we can argue in the same way by using the matrices F_i instead of E_i. But we could also observe that the Lie algebra generating set $\{E_i, F_i, H_i\}_{i=1}^n$ of $\mathfrak{s}$ is closed under matrix transposition whence $\mathfrak{s}$ is closed under matrix transposition (because matrix transposition is a Lie algebra anti-automorphism). It thus follows for all $i < j$ from $E_{ji} \in \mathfrak{s}$ that also $E_{ij} = E_{ji}^T \in \mathfrak{s}$.

2.

We construct an inverse $\psi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ to φ. We define ψ to be the unique linear map with $\psi(e) = e_1$, $\psi(f) = f_1$ and $\psi(h) = h_1$. Recall that $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$. The relations (23), (24), (25) therefore ensure that ψ is a homomorphism of Lie algebras. Then $\psi\varphi = \mathrm{id}_{\mathfrak{sl}_2(\mathbb{C})}$ because this holds on the basis e, h, f of $\mathfrak{sl}_2(\mathbb{C})$ and $\varphi\psi = \mathrm{id}_{\mathfrak{g}}$ because this holds on the Lie algebra generators e_1, h_1, f_1 of $\mathfrak{g}$.