Remark and Solutions

Sheet 4

Problem 22

In the following k denotes an algebraically closed field. We start by doing all the preparation to effectively solve the exercise:

Preparation

Definition 1. A *flag* in a vector space V is an increasing sequence of linear subspaces

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = V\,.$$

This flag is *complete* if it has maximal length, i.e. if V is finite dimensional and $\dim F_i = i$ for every $i = 0, \dots, n$.

Recall 2. If A is a k-algebra and $a \in A^\times$ is a unit then the conjugation map $x \mapsto axa^{-1}$ is an algebra automorphism of A and hence also a Lie algebra automorphism.

Recall 3. If V is a representation of a Lie algebra $\mathfrak{h}$ and $F = (F_i)_{i=0}^n$ is a flag of V then $\mathfrak{h}$ *stabilizes* F if $\mathfrak{h}.F_i \subseteq F_i$ for every $i = 0, \dots, n$, i.e. if F consists of subrepresentations.

If V is a finite dimensional vector space then the following conditions on a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(V)$ are equivalent:

(1) $\mathfrak{h}$ is solvable.

(2) $\mathfrak{h}$ stabilizes a complete flag of V.

(3) There exists a basis of $\mathfrak{h}$ with respect to which $\mathfrak{h}$ is represented by upper triangular matrices.

For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_n(k)$ the following conditions are equivalent:

(1) $\mathfrak{h}$ is solvable.

(2) $\mathfrak{h}$ stabilizes a complete flag of k^n.

(3) There exists some $g \in \mathrm{GL}_n(k)$ such that $g\mathfrak{h}g^{-1}$ consists of upper triangular matrices.

*Available online at https://github.com/cionx/representation-theory-1-tutorial-ss-19.

(That k is algebraically closed is used to go from the solvability of $\mathfrak{h}$ to the other two conditions, which are equivalent over any field, and which conversely imply the solvability of $\mathfrak{h}$ over any field.)

We denote by $\mathcal{F}$ the set of all complete flags in k^n and by

$$S \coloneqq (0, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \dots, \langle e_1, \dots, e_n \rangle = k^n)$$

the *standard flag* of k^n.

Recall 4. Let G be a group acting on a set X, i.e. let X be a G-set. The orbit of an element $x \in X$ is the set $G.x = \{g.x \mid g \in G\}$. The action of G on X is *transitive* if the following equivalent conditions hold:

(1) There exists for all $x, y \in X$ some $g \in G$ with $g.x = y$.

(2) There exists some $x \in X$ such that there exists for every $y \in X$ some $g \in G$ with $g.x = y$.

(3) The orbit of every $x \in X$ is all of X.

(4) There exists some $x \in X$ whose orbit is all of X.

The *stabilizer* of a point $x \in X$ is the subgroup $G_x = \{g \in G \mid g.x = x\}$ of G. The map

$$G/G_x \to G.x\,, \quad gG_x \mapsto g.x$$

is a well-defined isomorphism of G-sets, where the action of G on the set of left cosets G/G_x is given by $g.g'G_x = gg'G_x$.

Lemma 5. The group $\mathrm{GL}_n(k)$ acts on the set of complete flags $\mathcal{F}$ via

$$g.(F_0, \dots, F_n) = (g.F_0, \dots, g.F_n)\,.$$

This action is transitive and the stabilizer of the standard flag S is $\mathrm{B}_n(k)$ (the group of upper triangular $(n \times n)$-matrices).

Proof. This is indeed an action of $\mathrm{GL}_n(k)$ on $\mathcal{F}$.

If $F \in \mathcal{F}$ with $F = (F_0, \dots, F_n)$ then let $x_1, \dots, x_n$ be a basis of V so that $x_1, \dots, x_i$ is a basis of F_i for every $i = 0, \dots, n$. Then the matrix $g \in \mathrm{M}(n, k)$ with columns $x_1, \dots, x_n$ is invertible and $g.S = F$ because

$$g.\langle e_1, \dots, e_i \rangle = \langle g.e_1, \dots, g.e_i \rangle = \langle x_1, \dots, x_i \rangle = F_i$$

for every $i = 0, \dots, n$. This shows that the action is transitive.

If $g \in \mathrm{GL}_n(k)$ with $g.S = S$ then let $x_1, \dots, x_n$ be the columns of x. That $g.S = S$ means that $g.F_i \subseteq F_i$ for all $i = 0, \dots, n$, which means that

$$\langle x_1, \dots, x_i \rangle \subseteq \langle e_1, \dots, e_i \rangle$$

for all $i = 0, \dots, n$.[1] This happens to be the case precisely when for all $i = 0, \dots, n$ the i-th column of g admits no entries after the i-th row, i.e. if g is upper triangular. This shows that the stabilizer of the standard flag S is precisely $\mathrm{B}_n(k)$. $\square$

We denote by $\mathfrak{b}$ the Lie subalgebra of $\mathfrak{sl}_n(k)$ of traceless upper triangular matrices.

Lemma 6. The only $\mathfrak{b}$-subrepresentations of k^n are $\langle e_1, \dots, e_i \rangle$ for $i = 0, \dots, n$.

Proof. Let U be a $\mathfrak{b}$-subrepresentation and let $x \in U$ be a vector with entries $x_1, \dots, x_n$. Suppose that $x_i \neq 0$ but $x_{i+1} = \cdots = x_n = 0$. Then $x/x_i \in U$ and so we may assume $x_i = 1$. Let $N \in \mathfrak{b}$ be the matrix

$$N := \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

The vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-2} \\ x_{i-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad Nx = \begin{pmatrix} x_2 \\ \vdots \\ x_{i-1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad N^{n-2}x = \begin{pmatrix} x_{i-1} \\ 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad N^{i-1}x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

are again contained in U and they are linearly independent. We also see that

$$\langle x, Nx, N^2x, \dots, N^{i-1}x \rangle = \langle e_1, e_2, \dots, e_i \rangle$$

and hence $\langle e_1, e_2, \dots, e_i \rangle \subseteq U$. Let now i be maximal such that there exists some $x \in U$ with $x_i \neq 0$. Then $U \subseteq \langle e_1, \dots, e_i \rangle$ by choice of i but also $x_{i+1} = \cdots = x_n = 0$ and hence $\langle e_1, \dots, e_i \rangle \subseteq U$ by the above observation. Thus $U = \langle e_1, \dots, e_i \rangle$. $\square$

Corollary 7. The standard flag S is the only complete flag in k^n stabilized by $\mathfrak{b}$.

Proof. The flag S is stabilized by $\mathfrak{b}$. On the other hand every stabilized flag consists of $\mathfrak{b}$-subrepresentations whence the uniqueness follows from Lemma 6. $\square$

Lemma 8. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{gl}_n(k)$ and let $g \in \mathrm{GL}_n(k)$. Then $\mathfrak{h}$ stabilizes a flag F of k^n if and only if $g\mathfrak{h}g^{-1}$ stabilizes the flag $g.F$.

[1]The inclusion $g.F_i \subseteq F_i$ is by equality of dimensions equivalent to $g.F_i = F_i$. We choose to work with the condition $g.F_i \subseteq F_i$ so that we don't need to worry about equality.

Proof. Let $F = (F_0, \dots, F_n)$. Then

$$\begin{aligned} & \mathfrak{h} \text{ stabilizes } F \\ \iff & \mathfrak{h}.F_i \subseteq F_i \text{ for all } i = 0, \dots, n \\ \iff & \mathfrak{h}g^{-1}.gF_i \subseteq F_i \text{ for all } i = 0, \dots, n \\ \iff & g\mathfrak{h}g^{-1}.gF_i \subseteq gF_i \text{ for all } i = 0, \dots, n \\ \iff & g\mathfrak{h}g^{-1} \text{ stabilizes } g.F \end{aligned}$$

as claimed ☐

Corollary 9. If $g \in \mathrm{GL}_n(k)$ then $g.S$ is the only complete flag of k^n stabilized by $g\mathfrak{b}g^{-1}$.

Proof. If $g\mathfrak{b}g^{-1}$ stabilizes a flag F then we may write $F = g'.S$ for some $g' \in \mathrm{GL}_n(k)$ by Lemma 5. Then $\mathfrak{b} = g^{-1}(g\mathfrak{b}g^{-1})g$ stabilizes $g^{-1}g'.S$ by Lemma 8. It follows from Corollary 7 that $g^{-1}g'.S = S$. Therefore $g^{-1}g' \in \mathrm{B}_n(k)$ by Lemma 5. We can therefore write the group element g' as $g' = gh$ with $h \in \mathrm{B}_n(k)$. Then

$$F = g'.S = gh.S = g.h.S = g.S$$

because $h.S = S$, which is the desired uniqueness. ☐

The Exercise Itself

The Lie subalgebra $\mathfrak{b}$ of $\mathfrak{sl}_n(k)$ is solvable. If $\mathfrak{b}'$ is any other solvable Lie subalgebra of $\mathfrak{sl}_n(k)$ then by Recall 3 there exists some $g \in \mathrm{GL}_n(k)$ such that $g\mathfrak{b}'g^{-1}$ consists of upper triangular matrices. The conjugation action $g(-)g^{-1}$ gives a Lie algebra automorphism of $\mathfrak{sl}_n(k)$ by Recall 2 and because the condition $\mathrm{tr}(x) = 0$ is invariant under conjugation. Thus $g\mathfrak{b}'g^{-1}$ is a solvable Lie subalgebra of $\mathfrak{b}$.

This shows that $\mathfrak{b}$ is of maximal dimension among all solvable Lie subalgebras of $\mathfrak{sl}_n(k)$ and thus a Borel subalgebra.

If $\mathfrak{b}'$ is itself a Borel subalgebra then $g\mathfrak{b}'g^{-1}$ is again a Borel subalgebra of $\mathfrak{sl}_n(k)$. It then follows from the solvability of $\mathfrak{b}$ and maximality of $g\mathfrak{b}'g^{-1}$ that $g\mathfrak{b}'g^{-1} = \mathfrak{b}$. This shows that all Borel subalgebras of $\mathfrak{sl}_n(k)$ are $\mathrm{GL}_n(k)$-conjugated.

They are also $\mathrm{SL}_n(k)$-conjugated: Suppose that $\mathfrak{b}' = g\mathfrak{b}g^{-1}$ for some $g \in \mathrm{GL}_n(k)$. We may write $g = hd$ where d is the invertible diagonal matrix

$$d = \begin{pmatrix} \det(g) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then $\det h = 1$ and

$$\mathfrak{b}' = g\mathfrak{b}g^{-1} = (hd)\mathfrak{b}(hd)^{-1} = hd\mathfrak{b}d^{-1}h^{-1} = h\mathfrak{b}h^{-1}$$

because $d\mathfrak{b}d^{-1} = \mathfrak{b}$ (since d and d^{-1} are invertible and diagonal). This shows that $\mathfrak{b}'$ and $\mathfrak{b}$ are $\mathrm{SL}_n(k)$-conjugated.[2]

Let $\mathcal{B}$ be the set of all Borel subalgebras of $\mathfrak{sl}_n(k)$. We now construct a bijection

$$\Phi \colon \mathcal{B} \xrightarrow{\sim} \mathcal{F} .$$

The map Φ assigns to each Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{sl}_n(k)$ the unique complete flag of k^n that $\mathfrak{b}'$ stabilizes.

This gives a well-defined map: Every Borel Lie subalgebra $\mathfrak{b}'$ of $\mathfrak{sl}_n(k)$ is of the form $\mathfrak{b}' = g\mathfrak{b}g^{-1}$ for some $g \in \mathrm{SL}_n(k)$, hence stabilizes a unique complete flag by Corollary 9, namely $g.S$ by Lemma 8. Hence $\Phi(g\mathfrak{b}g^{-1}) = g.S$ is well-defined.

The map Φ is surjective: If F is any complete flag in k^n then $F = g.S$ for some $g \in \mathrm{GL}_n(k)$ by Lemma 5 and then $g\mathfrak{b}g^{-1}$ is a Borel subalgebra of $\mathfrak{sl}_n(k)$ that is mapped to F.

The map Φ is also injective: Let $\mathfrak{b}'$ be any Borel subalgebra of $\mathfrak{sl}_n(k)$ and let $F = \Phi(\mathfrak{b}')$ be the complete flag stabilized by $\mathfrak{b}'$. Then

$$\mathfrak{b}_F = \{x \in \mathfrak{sl}_n(k) \mid x \text{ stabilizes } F\}$$

is a Lie subalgebra of $\mathfrak{sl}_n(k)$, and we find by Recall 3 that $\mathfrak{b}_F$ is solvable. The Borel subalgebra $\mathfrak{b}'$ is contained in $\mathfrak{b}_F$ and thus $\mathfrak{b}' = \mathfrak{b}_F$ by the maximality of $\mathfrak{b}'$. This shows that $\mathfrak{b}'$ is uniquely determined by F.

Remark 10.

(1) The transitive action of $\mathrm{GL}_n(k)$ on $\mathcal{F}$ from Lemma 5 restricts to an action of $\mathrm{SL}_n(k)$ on $\mathcal{F}$, which is again transitive: We have seen above that any $g \in \mathrm{GL}_n(k)$ can be written as $g = hd$ for some $h \in \mathrm{SL}_n(k)$ and $d \in \mathrm{B}_n(k)$. Then

$$g.S = hd.S = h.d.S = h.S$$

since $d.S = S$ (because d is upper triangular). The $\mathrm{SL}_n(k)$-orbit of S thus coincides with the $\mathrm{GL}_n(k)$-orbit of S, which is all of $\mathcal{F}$. The stabilizer of S in $\mathrm{SL}_n(k)$ is $\mathrm{SB}_n(k) \coloneqq \mathrm{B}_n(k) \cap \mathrm{SL}_n(k)$.[3] It follows from Recall 4 that we get an induced bijection

$$\mathrm{SL}_n(k)/\mathrm{SB}_n(k) \to \mathcal{F} , \quad g\,\mathrm{SB}_n(k) \mapsto g.S .$$

The group $\mathrm{SL}_n(k)$ acts (transitively) on $\mathcal{B}$ via conjugation and the subgroup $\mathrm{SB}_n(k)$ stabilizes the standard flag $\mathfrak{b}$. We therefore get an induced (surjective) map

$$\mathrm{SL}_n(k)/\mathrm{SB}_n(k) \to \mathcal{B} , \quad g\,\mathrm{SB}_n(k) \mapsto g\mathfrak{b}g^{-1} .$$

[2] Geometrically speaking, suppose that $F = (F_0, \dots, F_n)$ is a complete flag in k^n and $x_1, \dots, x_n$ is a basis of k^n such that $x_1, \dots, x_i$ is a basis of F_i for every $i = 0, \dots, n$. If $g \in \mathrm{GL}_n(k)$ is the matrix with columns $x_1, \dots, x_n$ then it may happen that $\det(g) \neq 1$. But we can normalize the first basis vector x_1 by replacing it with $x_1/\deg(g)$. The resulting basis $y_1, \dots, y_n$ of k^n with $y_1 = x_1/\det(g)$ and $y_i = x_i$ for $i = 2, \dots, n$ then again satisfies $\langle y_1, \dots, y_i \rangle = F_i$ for all $i = 1, \dots, n$, but the resulting matrix h with columns $y_1, \dots, y_n$ now satisfies $\det(h) = \deg(g)/\det(g) = 1$. We can therefore adjust the determinant of the occuring matrix g without changing the underlying flag.

[3] The notation $\mathrm{SB}_n(k)$ is (probably) non-standard and the author's fault.

We get overall the following commutative diagram:

$$\begin{tikzcd} & \mathrm{SL}_n(k)/\mathrm{SB}_n(k) \arrow[dl, "g \mapsto g\mathfrak{b}g^{-1}"'] \arrow[dr, "g \mapsto g.S"] & \\ \mathcal{B} \arrow[rr, "\Phi", "g\mathfrak{b}g^{-1} \mapsto g.S"'] & & \mathcal{F} \end{tikzcd}$$

The bottom map and right map are bijections, so the same holds for the left map. We have thus constructed a commutative triangle of one-to-one correspondences.

(2) Instead of $\mathfrak{sl}_n(k)$ and $\mathrm{SL}_n(k)$ we could have used $\mathfrak{gl}_n(k)$ and $\mathrm{GL}_n(k)$. We then get another commutative triangle of one-to-one correspondences:

$$\begin{tikzcd} & \mathrm{GL}_n(k)/\mathrm{B}_n(k) \arrow[dl, "g \mapsto g\mathfrak{b}g^{-1}"'] \arrow[dr, "g \mapsto g.S"] & \\ \left\{\begin{matrix}\text{Borel subalgebras} \\ \text{of } \mathfrak{gl}_n(k)\end{matrix}\right\} \arrow[rr, "g\mathfrak{b}g^{-1} \mapsto g.S"'] & & \left\{\begin{matrix}\text{complete flags} \\ \text{in } k^n\end{matrix}\right\} \end{tikzcd}$$

The set $\mathcal{F}$ of complete flags can be regarded as a geometric object, a so called *flag variety*. The above diagrams then give us connections between linear algebra groups, Lie theory and (algebraic) geometry.