More on the Representation Theory of $\mathrm{GL}(V)$

We're working over the fixed ground field $\mathbb{C}$. We abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$. We denote by V some m-dimensional vector space. The set of natural numbers $\mathbb{N}$ contains 0. We write $\lambda \vdash n$ to mean that λ is a partition of n.

In this talk we will continue to explain the following connections:

$$
\begin{tikzcd}
\text{representation theory of the symmetric groups} \arrow[r, leftrightarrow] \arrow[d] & \text{symmetric functions} \arrow[d] \\
\text{representation theory of } \mathrm{GL}(V) \arrow[r, leftrightarrow] & \text{symmetric polynomials}
\end{tikzcd}
$$

1 Review

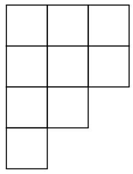
1.1 Schur modules

We have previously seen that the isomorphism classes of irreducible finite-dimensional polynomial representations of $\mathrm{GL}(V)$ are indexed by the set of all partitions: For every partition λ with $\lambda = (\lambda_1, \dots, \lambda_t)$ let

$$\widetilde{M}^{\lambda}(V) = \bigwedge^{\tilde{\lambda}_1}(V) \otimes \cdots \otimes \bigwedge^{\tilde{\lambda}_n}(V)$$

where $\tilde{\lambda}$ is the transposed of λ. The tensor factor $\bigwedge^{\tilde{\lambda}_j}(V)$ hence comes from the j-th column of the Young diagram associated to λ, which has height $\tilde{\lambda}_j$. Note that $\widetilde{M}^{\lambda}(V) = 0$ if $\ell(\lambda) > m$, i.e. if the Young diagram of λ has more than m rows.

Example 1. The Young diagram of $\lambda = (3, 3, 2, 1)$ is given by



and therefore

$$\widetilde{M}^\lambda(V) = \bigwedge^4(V) \otimes \bigwedge^3(V) \otimes \bigwedge^2(V)\,.$$

The **Schur module** associated to λ is given by

$$S^\lambda(V) = \widetilde{M}^\lambda(V)/Q^\lambda(V)$$

for a certain submodule Q_λ of $\widetilde{M}_\lambda$. For $\ell(\lambda) \leq m$ these are precisely the irreducible finite-dimensional polynomial representations of $\mathrm{GL}(V)$.

Example 2.

(1) For $\lambda = (1, \ldots, 1) \vdash n$ we have $S^\lambda(V) = \bigwedge^n(V)$.

(2) For $\lambda = (n)$ we have $S^\lambda(V) = \mathrm{Sym}^n(V)$.

1.2 Characters

The polynomial $\mathrm{GL}(V)$-representations can also be described by their characters: Using a basis $e_1, \ldots, e_m$ of V we may identify $\mathrm{GL}(V)$ with $\mathrm{GL}_m(\mathbb{C})$. For any finite-dimensional polynomial representation $\rho\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$ the **weight space** W_β for $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$ is given by

$$W_\beta = \{w \in W \mid \mathrm{diag}(x_1, \ldots, x_m).w = x_1^{\beta_1} \cdots x_m^{\beta_m} w \text{ for all } x_1, \ldots, x_m \in \mathbb{C}^\times\}\,.$$

The representation W decomposes into weight spaces in the sense that $W = \bigoplus_{\beta \in \mathbb{N}^m} W_\beta$ and its **character** is the polynomial $\mathrm{Char}(W) \in \mathbb{Z}[x_1, \ldots, x_m]$ given by

$$\mathrm{Char}(W)(x_1, \ldots, x_m) = \mathrm{tr}(\rho(\mathrm{diag}(x_1, \ldots, x_m))) = \sum_{\beta \in \mathbb{N}^m} \dim(W_\beta) x^\beta\,.$$

The character $\mathrm{Char}(W)$ is actually a symmetric polynomial, and the representation W is (up to isomorphism) uniquely determined by its character.

Example 3.

(1) The basis $e_{i_1} \wedge \cdots \wedge e_{i_n}$ of $\bigwedge^n(V)$ with $i_1 < \cdots < i_n$ consists of weight vectors, and its character is given by

$$\mathrm{Char}\Big(\bigwedge^n(V)\Big)(x_1, \ldots, x_m) = \sum_{1 \leq i_1 < \cdots < i_n \leq m} x_{i_1} \cdots x_{i_n} = e_n(x_1, \ldots, x_m)\,,$$

the n-th elementary symmetric polynomial in m variables.

(2) We see similarly that

$$\operatorname{Char}(\operatorname{Sym}^n(V))(x_1, \dots, x_m) = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq m} x_{i_1} \cdots x_{i_n} = h_n(x_1, \dots, x_m)$$

is the n-th complete homogeneous symmetric polynomial in m variables.

2 From S_n to $\mathrm{GL}(V)$

We denote for any group G by $\mathbb{1}_G$ the trivial representation of G. We denote for $n \geq 1$ by $\mathbb{U}_{\mathrm{S}_n}$ the sign representation of the symmetric group S_n.

For every $n \geq 0$ the tensor power $V^{\otimes n}$ is again a $\mathrm{GL}(V)$-representation, and it is also a right S_n-representation via

$$(e_1 \otimes \cdots \otimes e_n).\sigma = e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)} \,.$$

It follows that for every (left) S_n-representation E we get a $\mathrm{GL}(V)$-representation

$$\mathbb{S}(E) \coloneqq V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} E \,.$$

This construction results in an exact additive functor

$$\mathbb{S} \colon \mathrm{S}_n\textbf{-rep} \to \mathrm{GL}(V)\textbf{-rep} \,,$$

where S_n **-rep** is the category of finite-dimensional S_n-representations and $\mathrm{GL}(V)$-**rep** denotes the category of finite-dimensional polynomial $\mathrm{GL}(V)$-representations. The functor $\mathbb{S}$ is the **Schur functor**. It depends on both n and V, but we will omit this in our notation.

Example 4.

(1) We have that $\mathbb{1}_{\mathrm{S}_n} \cong \mathbb{C}[\mathrm{S}_n]/\langle \sigma - 1 \mid \sigma \in \mathrm{S}_n \rangle_{\mathbb{C}}$ and thus

$$\begin{aligned} \mathbb{S}(\mathbb{1}_{\mathrm{S}_n}) &\cong V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} \mathbb{C}[\mathrm{S}_n]/\langle \sigma - 1 \mid \sigma \in \mathrm{S}_n \rangle_{\mathbb{C}} \\ &\cong V^{\otimes n}/\langle x - x.\sigma \mid t \in V^{\otimes n} \rangle_{\mathbb{C}} \\ &\cong \operatorname{Sym}^n(V) \,. \end{aligned}$$

(2) We find similarly that $\mathbb{S}(\mathbb{U}_{\mathrm{S}_n}) \cong \bigwedge^n(V)$.

If $E_1, \dots, E_t$ are representations of the groups $\mathrm{S}_{n_1}, \dots, \mathrm{S}_{n_t}$ then for $n = n_1 + \cdots + n_t$,

$$E_1 \circ \cdots \circ E_t \coloneqq \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[\mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_t}]} (E_1 \boxtimes \cdots \boxtimes E_t)$$

is a representation of S_n.

Lemma 5. Let E_i, E and F be representations of symmetric groups. Then

(1) $(E_1 \circ \cdots \circ E_s) \circ \cdots \circ (F_1 \circ \cdots \circ F_t) \cong E_1 \circ \cdots \circ F_t$,

(2) $E \circ F \cong F \circ E$.

Lemma 6. If E and F are representations of symmetric groups S_a and S_b then

$$\mathbb{S}(E \circ F) \cong \mathbb{S}(E) \otimes \mathbb{S}(F) \,.$$

3 Some representation theory of S_n

For better use of the Schur functor $\mathbb{S}$ we need some understand of the representation theory of the symmetric group S_n, where $n \geq 0$. In this section we always assume that $\lambda \vdash n$ unless otherwise specified.

Recall that a **semistandard Young tableaux** of shape λ assigns to each box of the Young diagram of λ a natural number $1, 2, 3, \ldots,$ weakly increasing in each row and strictly increasing in each column. A **numbering** of shape λ fills in the boxes of the Young diagram of λ bijectively with the numbers $1, \ldots, n$. A **standard Young tableaux** is a semistandard Young tableaux that is also a numbering. This means that we assign to the boxes of the Young diagram of λ bijectively the integers $1, 2, \ldots, n$, such that both rows and columns are (necessarily strictly) increasing.

$$\begin{tikzcd}
\text{numbering} \arrow[dr, dash] & & \begin{array}{c}\text{semistandard}\\ \text{Young tableaux}\end{array} \arrow[dl, dash] \\
& \begin{array}{c}\text{standard}\\ \text{Young tableaux}\end{array} &
\end{tikzcd}$$

3.1 The representation M^λ

Two numbering T and T' of a Young diagram of shape λ are **row-equivalent** if they have the same entries in each row. A **row tabloid** is an equivalence class of row-equivalent numbering. A row tabloid can be represented as follows:

$$\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \cline{1-2} \end{array} = \begin{array}{ccc} \hline 2 & 1 & 3 \\ \hline 4 & 6 & 5 \\ \hline 8 & 7 & \\ \cline{1-2} \end{array}$$

Every numbering T defines a row tabloid $[T]$.

The group S_n acts transitive on the set of numberings of shape λ via $(\sigma.T)_{ij} = \sigma(T_{ij})$, which induces a transitive group action on the set of row tabloids of shape λ. The **row group** $R(T)$ of a numbering T is given by all permutations $\sigma \in \mathrm{S}_n$ that act row-wise on T, i.e. the stabilizer of the associated tabloid $[T]$.

It follows that S_n acts linearly on M^λ, the free vector space on the set of row tabloids of shape λ, and that for any numbering T of shape λ,

$$\begin{aligned}
M^\lambda &\cong \mathbb{C}[\mathrm{S}_n]/(p-1 \mid p \in R(T)) \\
&\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}[R(T)]/(p-1 \mid p \in R(T)) \\
&\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{1}_{R(T)} \\
&\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[\mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}]} \mathbb{1}_{\mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}} \\
&\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[\mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}]} (\mathbb{1}_{\mathrm{S}_{\lambda_1}} \boxtimes \cdots \boxtimes \mathbb{1}_{\mathrm{S}_{\lambda_t}}) \\
&= \mathbb{1}_{\mathrm{S}_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{\mathrm{S}_{\lambda_t}} .
\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{S}(M^\lambda) &\cong \mathbb{S}(\mathbb{1}_{\mathrm{S}_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{\mathrm{S}_{\lambda_t}}) \\ &\cong \mathbb{S}(\mathbb{1}_{\mathrm{S}_{\lambda_1}}) \otimes \cdots \otimes \mathbb{S}(\mathbb{1}_{\mathrm{S}_{\lambda_t}}) \\ &\cong \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_t}(V) \\ &=: M^\lambda(V)\,.\end{aligned}$$

3.2 The representation $\widetilde{M}^\lambda$

We can alter the construction of M^λ in two ways: Working with column instead of rows and introducing an alternating sign.

We denote by $C(T)$ the column group of a numbering T, i.e. all permutations that act column-wise on T. We let $\widetilde{M}^\lambda$ be the free vector generated by the numbering T of shape λ subject to the relations $\sigma.T = \mathrm{sgn}(\sigma)T$ for $\sigma \in C(T)$. The resulting vector space generators $[T]$ of $\widetilde{M}^\lambda$ may be visualized as follows:

$$\left|\begin{array}{c|c} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}\right| = - \left|\begin{array}{c|c} 2 & 4 \\ 1 & 5 \\ 3 & \end{array}\right| = \left|\begin{array}{c|c} 2 & 5 \\ 1 & 4 \\ 3 & \end{array}\right|$$

The group S_n acts on $\widetilde{M}^\lambda$ via $\sigma.[T] = [\sigma.T]$ for any numbering T of shape λ, and

$$\begin{aligned}\widetilde{M}^\lambda &\cong \mathbb{C}[\mathrm{S}_n]/(q - \mathrm{sgn}(q)1 \mid q \in C(T)) \\ &\cong \mathbb{U}_{\mathrm{S}_{\tilde{\lambda}_1}} \circ \cdots \circ \mathbb{U}_{\mathrm{S}_{\tilde{\lambda}_t}}\,.\end{aligned}$$

It follows that

$$\mathbb{S}(\widetilde{M}^\lambda) \cong \bigwedge^{\tilde{\lambda}_1}(V) \otimes \cdots \otimes \bigwedge^{\tilde{\lambda}_t}(V) = \widetilde{M}^\lambda(V)\,.$$

3.3 Specht modules

The irreducible representations of the symmetric group S_n can be indexed by the partitions $\lambda \vdash n$ and constructed as follows:

If T is any numbering of shape λ then its **Young centralizer** is the element

$$c_T := \sum_{q \in C(T)} \mathrm{sgn}(q) q \in \mathbb{C}[\mathrm{S}_n]\,.$$

The **Specht module** S^λ is the subspace of M^λ spanned by the elements $v_T := c_T \cdot [T]$ as T ranges through the numbering of shape λ. This is a subrepresentation of M^λ.

Theorem 7 (Classification of irreducible representations of S_n)**.**

(1) The representations S^λ with $\lambda \vdash n$ are pairwise non-isomorphic irreducible representations.

(2) Every irreducible representation of S_n is of the form S^λ for some partition $\lambda \vdash n$.

(3) The elements v_T where T ranges through the standard Young tableaux of shape λ form a basis of S^λ. Hence

$$\dim S^\lambda = \text{number of standard Young tableaux of shape } \lambda\,.$$

One can also construct the Specht module S^λ as a quotient of the representation $\widetilde{M}^\lambda$: The map

$$\widetilde{M}^\lambda \to S^\lambda \quad [T] \mapsto v_T$$

is a well-defined surjective homomorphism of S_n-representations. Its kernel Q^λ can be described similarly to $Q^\lambda(V)$, and it follows that under the isomorphism $\mathbb{S}(\widetilde{M}^\lambda) \cong \widetilde{M}^\lambda(V)$ we have $\mathbb{S}(Q^\lambda) \cong Q^\lambda(V)$.

Theorem 8. For any partition $\lambda \vdash n$,

$$\mathbb{S}(S^\lambda) \cong S^\lambda(V)\,.$$

Proof. By applying the (exact) functor $\mathbb{S}(-)$ to the short exact sequence

$$0 \to Q^\lambda \to \widetilde{M}^\lambda \to S^\lambda \to 0$$

we get the short exact sequence

$$0 \to \mathbb{S}(Q^\lambda) \to \mathbb{S}(\widetilde{M}^\lambda) \to \mathbb{S}(S^\lambda) \to 0e$$

and find that

$$\mathbb{S}(S^\lambda) \cong \mathbb{S}(\widetilde{M}^\lambda)/\mathbb{S}(Q^\lambda) \cong \widetilde{M}^\lambda(V)/Q^\lambda(V) \cong S^\lambda(V)\,,$$

proving the assertion. □

Remark 9. If $\ell(\lambda) > m$, i.e. if the Young diagram of λ has more than $m = \dim(V)$ rows, then $\mathbb{S}(S^\lambda) = S^\lambda(V) = 0$. If $\ell(\lambda) \leq m$ then $\mathbb{S}(S^\lambda) = S^\lambda(V)$ is again an irreducible representation.

Corollary 10. For any $n \geq 0$, $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}$ where f^λ denotes the number of standard Young tableaux of shape λ.

Proof. We find with Theorem 7 and Maschke's theorem that

$$\mathbb{C}[\mathrm{S}_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$$

with

$$\begin{aligned} f^\lambda &= \text{multiplicity of } S_\lambda \text{ in } \mathbb{C}[\mathrm{S}_n] \\ &= \text{dimension of } S_\lambda \\ &= \text{number of standard Young tableaux of shape } \lambda \end{aligned}$$

where the second equality follows (for example) from the Artin–Wedderburn theorem. It follows that

$$\begin{aligned} V^{\otimes n} &= V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} \mathbb{C}[\mathrm{S}_n] \\ &\cong \mathbb{S}(\mathbb{C}[\mathrm{S}_n]) \\ &\cong \mathbb{S}\left(\bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}\right) \\ &\cong \bigoplus_{\lambda \vdash n} \mathbb{S}(S^\lambda)^{\oplus f^\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda} \end{aligned}$$

as claimed. □

4 Translation into rings

4.1 The Grothendieck ring of $\mathrm{GL}(V)$

Recall that the Grothendieck group $\mathrm{K}_0(\mathcal{A})$ of a semisimple abelian category $\mathcal{A}$ is generated by the isomorphism classes of objects of $\mathcal{A}$ subject to the relation $[A]+[B] = [A\oplus B]$ for any two objects $A, B \in \mathcal{A}$. Then in particular

$$[A] + [B] = [A \oplus B] \tag{1}$$

for all $A, B \in \mathcal{A}$, and if $\mathcal{A}$ semisimple then these relations suffice to define $\mathrm{K}_0(\mathrm{GL}(V))$.

Let $\mathcal{A}$ be the semisimple category of finite-dimensional polynomial representations of $\mathrm{GL}(V)$. We abbreviate $\mathrm{K}_0(\mathrm{GL}(V)) \coloneqq \mathrm{K}_0(\mathcal{A})$ This becomes a ring when endowed with the multiplication

$$[V] \cdot [W] = [V \otimes W]\,.$$

It follows from last week's classification of irreducible finite-dimensional polynomial $\mathrm{GL}(V)$-representations that $\mathrm{K}_0(\mathrm{GL}(V))$ admits a $\mathbb{Z}$-basis given by the isomorphism classes $[S^\lambda(V)]$ where λ ranges through all partitions with $\ell(\lambda) \leq m$.

The character of a $\mathrm{GL}(V)$-representation results in a well-defined ring homomorphism

$$\mathrm{Char}\colon\ \mathrm{K}_0(\mathrm{GL}(V)) \to \Lambda(m)$$

where $\Lambda(m)$ denotes the ring of symmetric polynomials in the variables $x_1, \dots, x_m$.

4.2 The representation ring of the symmetric groups

For every $n \geq 0$ let R_n be the Grothendieck group of S_n-**rep**, the category of finite-dimensional S_n-representations. We can define on $R \coloneqq \bigoplus_{n\geq 0} R_n$ a multiplication via

$$[E] \cdot [F] = [E \circ F]\,.$$

This makes R into a commutative graded ring.

The category S_n-**rep** is semisimple by Maschke's theorem and it follows from Theorem 7 that R_n admits a $\mathbb{Z}$-basis given by the isomorphism classes $[S^\lambda]$ with $\lambda \vdash n$. The ring R hence damits a $\mathbb{Z}$-basis given by the isomorphism classes $[S^\lambda]$ where λ ranges through all partitions (of all natural numbers).

The additivity of the Schur functor(s) $\mathbb{S}\colon S_n\text{-}\mathbf{rep} \to \mathcal{A}$ gives for every $n \geq 0$ a group homomorphism $R_n \to \mathrm{K}_0(\mathrm{GL}(V))$, which together give a group homomorphism $R \to \mathrm{K}_0(\mathrm{GL}(V))$. It follows from Lemma 6 that this is a ring homomorphism. By abuse of notation we denote this homomorphism again by $\mathbb{S}$.

We have argued in Remark 9 that the kernel of $\mathbb{S}\colon R \to \mathrm{K}_0(\mathrm{GL}(V))$ is spanned by those $[S^\lambda]$ with $\ell(\lambda) > m$, whereas all other basis vector $[S^\lambda]$ with $\ell(\lambda) \leq m$ are mapped bijectively onto the basis $[S^\lambda(V)]$ of $\mathrm{K}_0(\mathrm{GL}(V))$.

4.3 The ring of symmetric functions

For every $k \geq 0$ let $\Lambda(k)$ denote the ring of symmetric polynomials, a subring of $\mathbb{Z}[x_1, \dots, x_k]$.

When dealing with symmetric polynomials it often happens that the number of variables, k, does not matter: One has a family $(f_k)_{k\geq 0}$ of symmetric polynomials $f_k \in \Lambda(k)$ such that $f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1})$ for every $k \geq 1$, and for every $k \geq 1$ an identity involving f_k which reduces for $x_k \to 0$ to the identity involving f_{k-1}.

To formalize this phenomenon we introduce the **ring of symmetric functions** Λ: For every degree $n \geq 0$ we set

$$\Lambda_n \coloneqq \left\{ (f_k)_{k\geq 0} \;\middle|\; \begin{array}{c} f_k \in \Lambda_n(k) \text{ with} \\ f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1}) \\ \text{for every } k \geq 1 \end{array} \right\}$$

where $\Lambda_n(k) \to \Lambda_n(k-1)$ is the group homomorphism given by setting $x_k \to 0$. We combine these groups into a graded ring $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ with multiplication given by

$$(f_k)_{k\geq 0} \cdot (g_k)_{k\geq 0} \coloneqq (f_k g_k)_{k\geq 0} \,.$$

Remark 11. We have $\Lambda = \lim(\Lambda(0) \leftarrow \Lambda(1) \leftarrow \Lambda(2) \leftarrow \Lambda(3) \leftarrow \cdots)$ in the category of graded rings.

Example 12. For every degree $n \geq 0$ we have an element

$$e_n \coloneqq (e_n(), e_n(x_1), e_n(x_1, x_2), e_n(x_1, x_2, x_3), \dots) \in \Lambda_n$$

and similarly elements $h_n, p_n \in \Lambda_n$. We get for every partition $\lambda = (\lambda_1, \dots, \lambda_t)$ an induced element

$$e_\lambda \coloneqq e_{\lambda_1} \cdots e_{\lambda_t} \in \Lambda_{|\lambda|}$$

and similarly elements $h_\lambda, p_\lambda \in \Lambda_{|\lambda|}$.

Example 13 (Schur polynomials and Schur functions)**.** Let λ be a partition. For every semistandard Young tableaux T of shape λ with entries in $\{1, \dots, k\}$ let

$$x_T \coloneqq \prod_{i \in T} x_i \,.$$

The **Schur polynomial** $s_\lambda(x_1, \dots, x_k)$ is defined as

$$s_\lambda(x_1, \dots, x_k) = \sum_T x_{T'} \in \mathbb{Z}[x_1, \dots, x_n]$$

where T ranges through the semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$.

We observe the following:

(1) The Schur polynomial $s_\lambda(x_1, \dots, x_k)$ is homogeneous of degree $|\lambda|$.

(2) If $\ell(\lambda) > k$, i.e. if the Young diagram of λ has more than k rows, then the Schur polynomial $s_\lambda(x_1, \dots, x_k)$ vanishes since there exist no semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$. (We don't have enough entries to make the first column strictly increasing, which is required for a semistandard Young tableaux.)

(3) The Specht module $S^\lambda(V)$ has a basis e_T where T ranges through the semistandard Young tableaux with entries in $\{1, \dots, m\}$. Each e_T is a weight vector with corresponding weight x_T. Hence

$$s_\lambda(x_1, \dots, x_m) = \mathrm{Char}(S^\lambda(V))(x_1, \dots, x_m) \,.$$

This shows in particular that $s_\lambda(x_1, \dots, x_m)$ is a symmetric polynomial. We also see again that $s_\lambda(x_1, \dots, x_m) = 0$ if $\ell(\lambda) > m$ since then $S^\lambda(V) = 0$.

(4) It holds that $s_\lambda(x_1, \dots, x_{k-1}, 0) = s_\lambda(x_1, \dots, x_{k-1})$.

We find that we get a well-defined element $s_\lambda \in \Lambda_{|\lambda|}$, the **Schur function** associated to λ.

Proposition 14. For every $k \geq 0$ the ring of symmetric polynomials $\Lambda(k)$ has the Schur polynomials $s_\lambda(x_1, \dots, x_k)$ with $\ell(\lambda) \leq k$ as a basis.

Corollary 15. The ring homomorphism $\mathrm{Char}\colon \mathrm{K}_0(\mathrm{GL}(V)) \to \Lambda(m)$ is an isomorphim.

Proof. The basis $[S^\lambda(V)]$ of $\mathrm{K}_0(\mathrm{GL}(V))$ is mapped to the basis $s_\lambda(x_1, \dots, x_m)$ of $\Lambda(m)$, both indexed by the partitions λ with $\ell(\lambda) \leq m$. $\square$

We have for every $k \geq 0$ a homomorphism of graded rings

$$\Lambda \to \Lambda(k) \,, \quad f \mapsto f(x_1, \dots, x_k)$$

that assigns to f the entailed symmetric polynomial in k variables. An equality $f = g$ holds in Λ if and only if for every $k \geq 0$ the equality $f(x_1, \dots, x_k) = g(x_1, \dots, x_k)$ hold.

Proposition 16.

(1) The symmetric functions $e_1, e_2, \ldots$ generate Λ and are algebraically independent.

(2) The symmetric functions $h_1, h_2, \ldots$ generate Λ and are algebraically independent.

(3) The monomials e_λ where λ ranges through all partitions form a $\mathbb{Z}$-basis of Λ.

(4) The monomials h_λ where λ ranges through all partitions form a $\mathbb{Z}$-basis of Λ.

(5) The symmetric functions s_λ where λ ranges through all partitions form a $\mathbb{Z}$-basis of Λ.

Theorem 17. Let $\Phi\colon \Lambda \to R$ be the unique additive group homomorphism that maps the basis element $e_{\tilde{\lambda}}$ to the element $[\widetilde{M}^\lambda]$ where λ ranges through all partitions.

(1) The map Φ is an isomorphism of rings.

(2) It holds that $\Phi(h_\lambda) = [M^\lambda]$.

(3) It holds that $\Phi(s_\lambda) = [S^\lambda]$.

The multiplicity of Φ stems from the identity $\widetilde{M}^\lambda = \mathbb{U}_{\lambda_1} \circ \cdots \circ \mathbb{U}_{\lambda_t}$ for $\lambda = (\lambda_1, \ldots, \lambda_t)$.

5 Conclusion

Corollary 18. The composition

$$\Lambda \xrightarrow{\Phi^{-1}} R \xrightarrow{\mathbb{S}} \mathrm{K}_0(\mathrm{GL}(V)) \xrightarrow{\mathrm{Char}} \Lambda(m)$$

is given by $f \mapsto f(x_1, \ldots, x_m)$.

Proof. The assertion holds for the basis elements s_λ of Λ as

$$s_\lambda \mapsto [S^\lambda] \mapsto [\mathbb{S}(S^\lambda)] = [S^\lambda(V)] \mapsto s_\lambda(x_1, \ldots, x_m)\,.$$

The general assertion follow by additivity of all occuring maps. ☐

We have thus finally arrived at the following commutative diagram of rings:

$$\begin{tikzcd}
R \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathbb{S}"'] & \Lambda \arrow[d, "f \mapsto f(x_1,\ldots,x_m)"] \\
\mathrm{K}_0(\mathrm{GL}(V)) \arrow[r, "\sim", "\mathrm{Char}"'] & \Lambda(m)
\end{tikzcd}$$

We have the following special cases of this diagram:

$$\begin{tikzcd}
{[S^\lambda]} \arrow[r, leftrightarrow] \arrow[d] & s_\lambda \arrow[d] \\
{[S^\lambda(V)]} \arrow[r, leftrightarrow] & s_\lambda(x_1, \dots, x_n)
\end{tikzcd}$$

$$\begin{tikzcd}
{[\widetilde{M}^\lambda]} \arrow[r, leftrightarrow] \arrow[d] & e_{\tilde{\lambda}} \arrow[d] & {[M^\lambda]} \arrow[r, leftrightarrow] \arrow[d] & h_\lambda \arrow[d] \\
{[\widetilde{M}^\lambda(V)]} \arrow[r, leftrightarrow] & e_{\tilde{\lambda}}(x_1, \dots, x_n) & {[M^\lambda(V)]} \arrow[r, leftrightarrow] & h_\lambda(x_1, \dots, x_n)
\end{tikzcd}$$

We can now use these correspondeces to translate between problems about the representation theory of S_n, the representation theory of $\mathrm{GL}(V)$, and the combinatorics of symmetric polynomials.

Example 19.

(1) For every $n \geq 0$ there exists unique natural numbers f^λ for $\lambda \vdash n$ such that one and thus all of the following conditions hold:

a) $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$,

b) $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}$,

c) $(x_1 + \cdots + x_m)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda(x_1, \dots, x_m)$ for every $m \geq 0$,

d) $e_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$ in Λ.

We have already seen that it follows from the first description that f^λ is given by the number of standard Young tableaux of shape λ.

(2) For every partition λ there exist unique natural numbers $K_{\mu,\lambda}$ for $\mu \lhd \lambda$ such that one and thus all of the following conditions hold:

a) $M^\lambda \cong S^\lambda \oplus \bigoplus_{\mu \lhd \lambda} (S^\mu)^{\oplus K_{\mu,\lambda}}$,

b) $M^\lambda(V) \cong S^\lambda(V) \oplus \bigoplus_{\mu \lhd \lambda} S^\mu(V)^{\oplus K_{\mu,\lambda}}$,

c) $h_\lambda(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m) + \sum_{\mu \lhd \lambda} K_{\mu,\lambda} s_\mu(x_1, \dots, x_m)$ for all $m \geq 0$,

d) $h_\lambda = s_\lambda + \sum_{\mu \lhd \lambda} K_{\mu,\lambda} s_\mu$.

The numbers $K_{\mu,\lambda}$ are the **Kostka numbers**.

(3) For any two partitions λ and μ there exist natural numbers $c^\nu_{\lambda,\mu}$ for $\nu \vdash |\lambda| + |\mu|$ such that one and thus all of the following conditions hold:

a) $S^\lambda \circ S^\mu \cong \sum_\nu (S^\nu)^{\oplus c^\nu_{\lambda,\mu}}$,

b) $S^\lambda(V) \otimes S^\mu(V) \cong \sum_\nu S^\nu(V)^{\oplus c^\nu_{\lambda,\mu}}$,

c) $s_\lambda(x_1, \dots, x_m) s_\mu(x_1, \dots, x_m) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x_1, \dots, x_m)$,

d) $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$.

The numbers $c^\nu_{\lambda,\mu}$ are the **Littlewood–Richardson coefficients**.