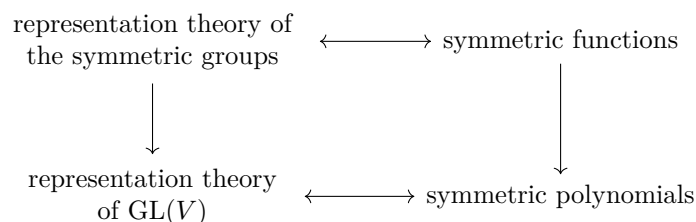


# More on the Representation Theory of $\mathrm{GL}(V)$

We're working over the fixed ground field  $\mathbb{C}$ . We abbreviate  $\otimes_{\mathbb{C}}$  as  $\otimes$ . We denote by  $V$  some  $m$ -dimensional vector space. The set of natural numbers  $\mathbb{N}$  contains 0. For any partition  $\lambda$  we denote by  $\tilde{\lambda}$  the transpose of  $\lambda$ .

In this talk we will continue to explain the following connections:



## 1 Review

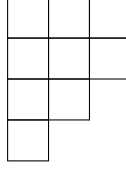
### 1.1 Schur modules

We have previously seen that the isomorphism classes of irreducible finite-dimensional polynomial representations of  $\mathrm{GL}(V)$  are indexed by the set of all partitions: For every partition  $\lambda$  with  $\lambda = (\lambda_1, \dots, \lambda_t)$  let

$$\widetilde{M}^{\lambda}(V) = \bigwedge^{\tilde{\lambda}_1}(V) \otimes \cdots \otimes \bigwedge^{\tilde{\lambda}_n}(V)$$

where  $\tilde{\lambda}$  is the transposed of  $\lambda$ . The tensor factor  $\bigwedge^{\tilde{\lambda}_j}(V)$  hence comes from the  $j$ -th column of the Young diagram associated to  $\lambda$ , which has height  $\tilde{\lambda}_j$ . Note that  $\widetilde{M}^{\lambda}(V) = 0$  if  $\ell(\lambda) > m$ , i.e. if the Young diagram of  $\lambda$  has more than  $m$  rows.

**Example 1.** The Young diagram of  $\lambda = (3, 3, 2, 1)$  is given by



and therefore

$$\widetilde{M}^\lambda(V) = \bigwedge^4(V) \otimes \bigwedge^3(V) \otimes \bigwedge^2(V).$$

The **Schur module** associated to  $\lambda$  is given by

$$S^\lambda(V) = \widetilde{M}^\lambda(V) / Q^\lambda(V)$$

for a certain submodule  $Q^\lambda$  of  $\widetilde{M}^\lambda$ . For  $\ell(\lambda) \leq m$  these are precisely the irreducible finite-dimensional polynomial representations of  $\mathrm{GL}(V)$ .

**Example 2.**

- (1) For  $\lambda = (1, \dots, 1) \vdash n$  we have  $S^\lambda(V) = \bigwedge^n(V)$ .
- (2) For  $\lambda = (n)$  we have  $S^\lambda(V) = \mathrm{Sym}^n(V)$ .

## 1.2 Characters

The polynomial  $\mathrm{GL}(V)$ -representations can also be described by their characters: Using a basis  $e_1, \dots, e_m$  of  $V$  we may identify  $\mathrm{GL}(V)$  with  $\mathrm{GL}_m(\mathbb{C})$ . For any finite-dimensional polynomial representation  $\rho: \mathrm{GL}(V) \rightarrow \mathrm{GL}(W)$  the **weight space**  $W_\beta$  for  $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$  is given by

$$W_\beta = \{w \in W \mid \mathrm{diag}(x_1, \dots, x_m) \cdot w = x_1^{\beta_1} \cdots x_m^{\beta_m} w \text{ for all } x_1, \dots, x_m \in \mathbb{C}^\times\}.$$

The representation  $W$  decomposes into weight spaces in the sense that  $W = \bigoplus_{\beta \in \mathbb{N}^m} W_\beta$  and its **character** is the polynomial  $\mathrm{Char}(W) \in \mathbb{Z}[x_1, \dots, x_m]$  given by

$$\mathrm{Char}(W)(x_1, \dots, x_m) = \mathrm{tr}(\rho(\mathrm{diag}(x_1, \dots, x_m))) = \sum_{\beta \in \mathbb{N}^m} \dim(W_\beta) x^\beta.$$

The character  $\mathrm{Char}(W)$  is actually a symmetric polynomial, and the representation  $W$  is (up to isomorphism) uniquely determined by its character.

**Example 3.**

- (1) The basis  $e_{i_1} \wedge \cdots \wedge e_{i_n}$  of  $\bigwedge^n(V)$  with  $i_1 < \cdots < i_n$  consists of weight vectors, and its character is given by

$$\mathrm{Char}\left(\bigwedge^n(V)\right)(x_1, \dots, x_m) = \sum_{1 \leq i_1 < \cdots < i_n \leq m} x_{i_1} \cdots x_{i_n} = e_n(x_1, \dots, x_m),$$

the  $n$ -th elementary symmetric polynomial in  $m$  variables.

(2) We see similarly that

$$\text{Char}(\text{Sym}^n(V))(x_1, \dots, x_m) = \sum_{1 \leq i_1 \leq \dots \leq i_n \leq m} x_{i_1} \cdots x_{i_n} = h_n(x_1, \dots, x_m)$$

is the  $n$ -th complete homogeneous symmetric polynomial in  $m$  variables.

## 2 From $S_n$ to $\text{GL}(V)$

We denote for any group  $G$  by  $\mathbb{1}_G$  the trivial representation of  $G$ . We denote for  $n \geq 1$  by  $\mathbb{U}_{S_n}$  the sign representation of the symmetric group  $S_n$ .

For every  $n \geq 0$  the tensor power  $V^{\otimes n}$  is again a  $\text{GL}(V)$ -representation, and it is also a right  $S_n$ -representation via

$$(e_1 \otimes \cdots \otimes e_n) \cdot \sigma = e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}.$$

These actions commute. It follows that for every (left)  $S_n$ -representation  $E$  we get a  $\text{GL}(V)$ -representation

$$\mathbb{S}(E) := V^{\otimes n} \otimes_{\mathbb{C}[S_n]} E.$$

This construction results in an exact additive functor

$$\mathbb{S}: S_n\text{-rep} \rightarrow \text{GL}(V)\text{-rep},$$

where  $S_n\text{-rep}$  is the category of finite-dimensional  $S_n$ -representations and  $\text{GL}(V)\text{-rep}$  denotes the category of finite-dimensional polynomial  $\text{GL}(V)$ -representations. The functor  $\mathbb{S}$  is the **Schur functor**. It depends on both  $n$  and  $V$ , but we will omit this in our notation.

**Example 4.**

(1) We have that  $\mathbb{1}_{S_n} \cong \mathbb{C}[S_n] / \langle \sigma - 1 \mid \sigma \in S_n \rangle_{\mathbb{C}}$  and thus

$$\begin{aligned} \mathbb{S}(\mathbb{1}_{S_n}) &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] / \langle \sigma - 1 \mid \sigma \in S_n \rangle_{\mathbb{C}} \\ &\cong V^{\otimes n} / \langle x - x \cdot \sigma \mid t \in V^{\otimes n} \rangle_{\mathbb{C}} \\ &\cong \text{Sym}^n(V). \end{aligned}$$

(2) We find similarly that  $\mathbb{S}(\mathbb{U}_{S_n}) \cong \bigwedge^n(V)$ .

If  $E_1, \dots, E_t$  are representations of the groups  $S_{n_1}, \dots, S_{n_t}$  then for  $n = n_1 + \dots + n_t$ ,

$$E_1 \circ \cdots \circ E_t := \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n_1} \times \cdots \times S_{n_t}]} (E_1 \boxtimes \cdots \boxtimes E_t)$$

is a representation of  $S_n$ .

**Lemma 5.** Let  $E_i, E$  and  $F$  be representations of symmetric groups. Then

(1)  $(E_1 \circ \cdots \circ E_s) \circ \cdots \circ (F_1 \circ \cdots \circ F_t) \cong E_1 \circ \cdots \circ F_t$ ,

(2)  $E \circ F \cong F \circ E$ .

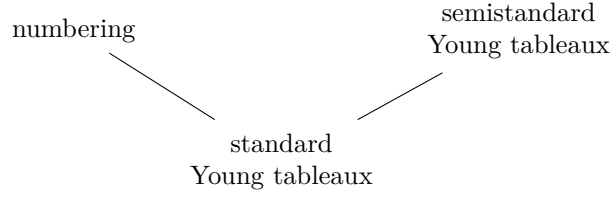
**Lemma 6.** If  $E$  and  $F$  are representations of symmetric groups  $S_a$  and  $S_b$  then

$$\mathbb{S}(E \circ F) \cong \mathbb{S}(E) \otimes \mathbb{S}(F).$$

### 3 Some representation theory of $S_n$

For better use of the Schur functor  $\mathbb{S}$  we need some understanding of the representation theory of the symmetric group  $S_n$ , where  $n \geq 0$ . In this section we always assume that  $\lambda \vdash n$  unless otherwise specified.

Recall that a **semistandard Young tableau** of shape  $\lambda$  assigns to each box of the Young diagram of  $\lambda$  a natural number  $1, 2, 3, \dots$ , weakly increasing in each row and strictly increasing in each column. A **numbering** of shape  $\lambda$  fills in the boxes of the Young diagram of  $\lambda$  bijectively with the numbers  $1, \dots, n$ . A **standard Young tableau** is a semistandard Young tableau that is also a numbering. This means that we assign to the boxes of the Young diagram of  $\lambda$  bijectively the integers  $1, 2, \dots, n$ , such that both rows and columns are (necessarily strictly) increasing.



#### 3.1 The representation $M^\lambda$

Two numberings  $T$  and  $T'$  of a Young diagram of shape  $\lambda$  are **row-equivalent** if they have the same entries in each row. A **row tabloid** is an equivalence class of row-equivalent numberings. A row tabloid can be represented as follows:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline 4 & 6 & 5 \\ \hline 8 & 7 & \\ \hline \end{array}$$

Every numbering  $T$  defines a row tabloid  $[T]$ .

The group  $S_n$  acts transitively on the set of numberings of shape  $\lambda$  via  $(\sigma.T)_{ij} = \sigma(T_{ij})$ , which induces a transitive group action on the set of row tabloids of shape  $\lambda$ . The **row group**  $R(T)$  of a numbering  $T$  is given by all permutations  $\sigma \in S_n$  that act row-wise on  $T$ , i.e. the stabilizer of the associated tabloid  $[T]$ .

It follows that  $S_n$  acts linearly on  $M^\lambda$ , the free vector space on the set of row tabloids of shape  $\lambda$ , and that for any numbering  $T$  of shape  $\lambda$ ,

$$\begin{aligned} M^\lambda &\cong \mathbb{C}[S_n]/(p-1 \mid p \in R(T)) \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}[R(T)]/(p-1 \mid p \in R(T)) \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{1}_{R(T)} \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda_1} \times \dots \times S_{\lambda_t}]} \mathbb{1}_{S_{\lambda_1} \times \dots \times S_{\lambda_t}} \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda_1} \times \dots \times S_{\lambda_t}]} (\mathbb{1}_{S_{\lambda_1}} \boxtimes \dots \boxtimes \mathbb{1}_{S_{\lambda_t}}) \\ &= \mathbb{1}_{S_{\lambda_1}} \circ \dots \circ \mathbb{1}_{S_{\lambda_t}} . \end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{S}(M^\lambda) &\cong \mathbb{S}(\mathbb{1}_{S_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{S_{\lambda_t}}) \\
&\cong \mathbb{S}(\mathbb{1}_{S_{\lambda_1}}) \otimes \cdots \otimes \mathbb{S}(\mathbb{1}_{S_{\lambda_t}}) \\
&\cong \text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_t}(V) \\
&=: M^\lambda(V).
\end{aligned}$$

### 3.2 The representation $\widetilde{M}^\lambda$

We can alter the construction of  $M^\lambda$  in two ways: Working with column instead of rows and introducing an alternating sign.

We denote by  $C(T)$  the column group of a numbering  $T$ , i.e. all permutations that act column-wise on  $T$ . We let  $\widetilde{M}^\lambda$  be the free vector generated by the numbering  $T$  of shape  $\lambda$  subject to the relations  $\sigma.T = \text{sgn}(\sigma)T$  for  $\sigma \in C(T)$ . The resulting vector space generators  $[T]$  of  $\widetilde{M}^\lambda$  may be visualized as follows:

$$\begin{vmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{vmatrix} = - \begin{vmatrix} 2 & 4 \\ 1 & 5 \\ 3 & \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \\ 3 & \end{vmatrix}$$

The group  $S_n$  acts on  $\widetilde{M}^\lambda$  via  $\sigma.[T] = [\sigma.T]$  for any numbering  $T$  of shape  $\lambda$ , and

$$\begin{aligned}
\widetilde{M}^\lambda &\cong \mathbb{C}[S_n]/(q - \text{sgn}(q)1 \mid q \in C(T)) \\
&\cong \mathbb{U}_{S_{\lambda_1}} \circ \cdots \circ \mathbb{U}_{S_{\lambda_t}}.
\end{aligned}$$

It follows that

$$\mathbb{S}(\widetilde{M}^\lambda) \cong \bigwedge^{\lambda_1}(V) \otimes \cdots \otimes \bigwedge^{\lambda_t}(V) = \widetilde{M}^\lambda(V).$$

### 3.3 Specht modules

The irreducible representations of the symmetric group  $S_n$  can be indexed by the partitions  $\lambda \vdash n$  and constructed as follows:

If  $T$  is any numbering of shape  $\lambda$  then its **Young centralizer** is the element

$$c_T := \sum_{q \in C(T)} \text{sgn}(q)q \in \mathbb{C}[S_n].$$

The **Specht module**  $S^\lambda$  is the subspace of  $M^\lambda$  spanned by the elements  $v_T := c_T \cdot [T]$  as  $T$  ranges through the numbering of shape  $\lambda$ . This is a subrepresentation of  $M^\lambda$ .

**Theorem 7** (Classification of irreducible representations of  $S_n$ ).

- (1) The representations  $S^\lambda$  with  $\lambda \vdash n$  are pairwise non-isomorphic irreducible representations.

- (2) Every irreducible representation of  $S_n$  is of the form  $S^\lambda$  for some partition  $\lambda \vdash n$ .
- (3) The elements  $v_T$  where  $T$  ranges through the standard Young tableaux of shape  $\lambda$  form a basis of  $S^\lambda$ . Hence

$$\dim S^\lambda = \text{number of standard Young tableaux of shape } \lambda.$$

One can also construct the Specht module  $S^\lambda$  as a quotient of the representation  $\widetilde{M}^\lambda$ : The map

$$\widetilde{M}^\lambda \rightarrow S^\lambda \quad [T] \mapsto v_T$$

is a well-defined surjective homomorphism of  $S_n$ -representations. Its kernel  $Q^\lambda$  can be described similarly to  $Q^\lambda(V)$ , and it follows that under the isomorphism  $\mathbb{S}(\widetilde{M}^\lambda) \cong \widetilde{M}^\lambda(V)$  we have  $\mathbb{S}(Q^\lambda) \cong Q^\lambda(V)$ .

**Theorem 8.** For any partition  $\lambda \vdash n$ ,

$$\mathbb{S}(S^\lambda) \cong S^\lambda(V).$$

*Proof.* By applying the (exact) functor  $\mathbb{S}(-)$  to the short exact sequence

$$0 \rightarrow Q^\lambda \rightarrow \widetilde{M}^\lambda \rightarrow S^\lambda \rightarrow 0$$

we get the short exact sequence

$$0 \rightarrow \mathbb{S}(Q^\lambda) \rightarrow \mathbb{S}(\widetilde{M}^\lambda) \rightarrow \mathbb{S}(S^\lambda) \rightarrow 0e$$

and find that

$$\mathbb{S}(S^\lambda) \cong \mathbb{S}(\widetilde{M}^\lambda)/\mathbb{S}(Q^\lambda) \cong \widetilde{M}^\lambda(V)/Q^\lambda(V) \cong S^\lambda(V),$$

proving the assertion.  $\square$

**Remark 9.** If  $\ell(\lambda) > m$ , i.e. if the Young diagram of  $\lambda$  has more than  $m = \dim(V)$  rows, then  $\mathbb{S}(S^\lambda) = S^\lambda(V) = 0$ . If  $\ell(\lambda) \leq m$  then  $\mathbb{S}(S^\lambda) = S^\lambda(V)$  is again an irreducible representation.

**Corollary 10.** For any  $n \geq 0$ ,  $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}$  where  $f^\lambda$  denotes the number of standard Young tableaux of shape  $\lambda$ .

*Proof.* We find with Theorem 7 and Maschke's theorem that

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$$

with

$$\begin{aligned} f^\lambda &= \text{multiplicity of } S^\lambda \text{ in } \mathbb{C}[S_n] \\ &= \text{dimension of } S^\lambda \\ &= \text{number of standard Young tableaux of shape } \lambda \end{aligned}$$

where the second equality follows (for example) from the Artin–Wedderburn theorem. It follows that

$$\begin{aligned}
V^{\otimes n} &= V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] \\
&\cong \mathbb{S}(\mathbb{C}[S_n]) \\
&\cong \mathbb{S} \left( \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda} \right) \\
&\cong \bigoplus_{\lambda \vdash n} \mathbb{S}(S^\lambda)^{\oplus f^\lambda} \\
&\cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}
\end{aligned}$$

as claimed.  $\square$

## 4 Translation into ring language

### 4.1 The Grothendieck ring of $\mathrm{GL}(V)$

Recall that the Grothendieck group  $K_0(\mathcal{A})$  of a semisimple abelian category  $\mathcal{A}$  is generated by the isomorphism classes of objects of  $\mathcal{A}$  subject to the relation  $[A] + [B] = [A \oplus B]$  for any two objects  $A, B \in \mathcal{A}$ . Then in particular

$$[A] + [B] = [A \oplus B] \quad (1)$$

for all  $A, B \in \mathcal{A}$ , and if  $\mathcal{A}$  semisimple then these relations suffice to define  $K_0(\mathrm{GL}(V))$ .

Let  $\mathcal{A}$  be the semisimple category of finite-dimensional polynomial representations of  $\mathrm{GL}(V)$ . We abbreviate  $K_0(\mathrm{GL}(V)) := K_0(\mathcal{A})$ . This becomes a ring when endowed with the multiplication

$$[V] \cdot [W] = [V \otimes W].$$

It follows from last week's classification of irreducible finite-dimensional polynomial  $\mathrm{GL}(V)$ -representations that  $K_0(\mathrm{GL}(V))$  admits a  $\mathbb{Z}$ -basis given by the isomorphism classes  $[S^\lambda(V)]$  where  $\lambda$  ranges through all partitions with  $\ell(\lambda) \leq m$ .

The character of a  $\mathrm{GL}(V)$ -representation results in a well-defined ring homomorphism

$$\mathrm{Char}: K_0(\mathrm{GL}(V)) \rightarrow \Lambda(m)$$

where  $\Lambda(m)$  denotes the ring of symmetric polynomials in the variables  $x_1, \dots, x_m$ .

### 4.2 The representation ring of the symmetric groups

For every  $n \geq 0$  let  $R_n$  be the Grothendieck group of  $S_n$ -**rep**, the category of finite-dimensional  $S_n$ -representations. We can define on  $R := \bigoplus_{n \geq 0} R_n$  a multiplication via

$$[E] \cdot [F] = [E \circ F].$$

This makes  $R$  into a commutative graded ring.

The category  $S_n\text{-rep}$  is semisimple by Maschke's theorem and it follows from Theorem 7 that  $R_n$  admits a  $\mathbb{Z}$ -basis given by the isomorphism classes  $[S^\lambda]$  with  $\lambda \vdash n$ . The ring  $R$  hence admits a  $\mathbb{Z}$ -basis given by the isomorphism classes  $[S^\lambda]$  where  $\lambda$  ranges through all partitions (of all natural numbers).

The additivity of the Schur functor(s)  $\mathbb{S}: S_n\text{-rep} \rightarrow \mathcal{A}$  gives for every  $n \geq 0$  a group homomorphism  $R_n \rightarrow K_0(\text{GL}(V))$ , which together give a group homomorphism  $R \rightarrow K_0(\text{GL}(V))$ . It follows from Lemma 6 that this is a ring homomorphism. By abuse of notation we denote this homomorphism again by  $\mathbb{S}$ .

We have argued in Remark 9 that the kernel of  $\mathbb{S}: R \rightarrow K_0(\text{GL}(V))$  is spanned by those  $[S^\lambda]$  with  $\ell(\lambda) > m$ , whereas all other basis vector  $[S^\lambda]$  with  $\ell(\lambda) \leq m$  are mapped bijectively onto the basis  $[S^\lambda(V)]$  of  $K_0(\text{GL}(V))$ .

### 4.3 The ring of symmetric functions

For every  $k \geq 0$  let  $\Lambda(k)$  denote the ring of symmetric polynomials, a subring of  $\mathbb{Z}[x_1, \dots, x_k]$ .

When dealing with symmetric polynomials it often happens that the number of variables,  $k$ , does not matter: One has a family  $(f_k)_{k \geq 0}$  of symmetric polynomials  $f_k \in \Lambda(k)$  such that  $f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1})$  for every  $k \geq 1$ , and for every  $k \geq 1$  an identity involving  $f_k$  which reduces for  $x_k \rightarrow 0$  to the identity involving  $f_{k-1}$ .

To formalize this phenomenon we introduce the **ring of symmetric functions**  $\Lambda$ : For every degree  $n \geq 0$  we set

$$\Lambda_n := \left\{ (f_k)_{k \geq 0} \left| \begin{array}{l} f_k \in \Lambda_n(k) \text{ with} \\ f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1}) \\ \text{for every } k \geq 1 \end{array} \right. \right\}$$

where  $\Lambda_n(k) \rightarrow \Lambda_n(k-1)$  is the group homomorphism given by setting  $x_k \rightarrow 0$ . We combine these groups into a graded ring  $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$  with multiplication given by

$$(f_k)_{k \geq 0} \cdot (g_k)_{k \geq 0} := (f_k g_k)_{k \geq 0}.$$

**Remark 11.** We have  $\Lambda = \lim(\Lambda(0) \leftarrow \Lambda(1) \leftarrow \Lambda(2) \leftarrow \Lambda(3) \leftarrow \dots)$  in the category of graded rings.

**Example 12.** For every degree  $n \geq 0$  we have an element

$$e_n := (e_n(), e_n(x_1), e_n(x_1, x_2), e_n(x_1, x_2, x_3), \dots) \in \Lambda_n$$

and similarly elements  $h_n, p_n \in \Lambda_n$ . We get for every partition  $\lambda = (\lambda_1, \dots, \lambda_t)$  an induced element

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_t} \in \Lambda_{|\lambda|}$$

and similarly elements  $h_\lambda, p_\lambda \in \Lambda_{|\lambda|}$ .



**Example 13** (Schur polynomials and Schur functions). Let  $\lambda$  be a partition. For every semistandard Young tableaux  $T$  of shape  $\lambda$  with entries in  $\{1, \dots, k\}$  let

$$x_T := \prod_{i \in T} x_i.$$

The **Schur polynomial**  $s_\lambda(x_1, \dots, x_k)$  is defined as

$$s_\lambda(x_1, \dots, x_k) = \sum_{T} x_T \in \mathbb{Z}[x_1, \dots, x_n]$$

where  $T$  ranges through the semistandard Young tableaux of shape  $\lambda$  with entries in  $\{1, \dots, k\}$ .

We observe the following:

- (1) The Schur polynomial  $s_\lambda(x_1, \dots, x_k)$  is homogeneous of degree  $|\lambda|$ .
- (2) If  $\ell(\lambda) > k$ , i.e. if the Young diagram of  $\lambda$  has more than  $k$  rows, then the Schur polynomial  $s_\lambda(x_1, \dots, x_k)$  vanishes since there exist no semistandard Young tableaux of shape  $\lambda$  with entries in  $\{1, \dots, k\}$ . (We don't have enough entries to make the first column strictly increasing, which is required for a semistandard Young tableaux.)
- (3) The Specht module  $S^\lambda(V)$  has a basis  $e_T$  where  $T$  ranges through the semistandard Young tableaux with entries in  $\{1, \dots, m\}$ . Each  $e_T$  is a weight vector with corresponding weight  $x_T$ . Hence

$$s_\lambda(x_1, \dots, x_m) = \text{Char}(S^\lambda(V))(x_1, \dots, x_m).$$

This shows in particular that  $s_\lambda(x_1, \dots, x_m)$  is a symmetric polynomial. We also see again that  $s_\lambda(x_1, \dots, x_m) = 0$  if  $\ell(\lambda) > m$  since then  $S^\lambda(V) = 0$ .

- (4) It holds that  $s_\lambda(x_1, \dots, x_{k-1}, 0) = s_\lambda(x_1, \dots, x_{k-1})$ .

We find that we get a well-defined element  $s_\lambda \in \Lambda_{|\lambda|}$ , the **Schur function** associated to  $\lambda$ .

**Proposition 14.** For every  $k \geq 0$  the ring of symmetric polynomials  $\Lambda(k)$  has the Schur polynomials  $s_\lambda(x_1, \dots, x_k)$  with  $\ell(\lambda) \leq k$  as a  $\mathbb{Z}$ -basis.

**Corollary 15.** The ring homomorphism  $\text{Char}: K_0(\text{GL}(V)) \rightarrow \Lambda(m)$  is an isomorphism.

*Proof.* The basis  $[S^\lambda(V)]$  of  $K_0(\text{GL}(V))$  is mapped to the basis  $s_\lambda(x_1, \dots, x_m)$  of  $\Lambda(m)$ , both indexed by the partitions  $\lambda$  with  $\ell(\lambda) \leq m$ .  $\square$

We have for every  $k \geq 0$  a homomorphism of graded rings

$$\Lambda \rightarrow \Lambda(k), \quad f \mapsto f(x_1, \dots, x_k)$$

that assigns to  $f$  the entailed symmetric polynomial in  $k$  variables. An equality  $f = g$  holds in  $\Lambda$  if and only if for every  $k \geq 0$  the equality  $f(x_1, \dots, x_k) = g(x_1, \dots, x_k)$  hold.

**Proposition 16.**

- (1) The symmetric functions  $e_1, e_2, \dots$  generate  $\Lambda$  and are algebraically independent.
- (2) The symmetric functions  $h_1, h_2, \dots$  generate  $\Lambda$  and are algebraically independent.
- (3) The monomials  $e_\lambda$  where  $\lambda$  ranges through all partitions form a  $\mathbb{Z}$ -basis of  $\Lambda$ .
- (4) The monomials  $h_\lambda$  where  $\lambda$  ranges through all partitions form a  $\mathbb{Z}$ -basis of  $\Lambda$ .
- (5) The symmetric functions  $s_\lambda$  where  $\lambda$  ranges through all partitions form a  $\mathbb{Z}$ -basis of  $\Lambda$ .

**Theorem 17.** Let  $\Phi: \Lambda \rightarrow R$  be the unique additive group homomorphism that maps the basis element  $e_\lambda$  to the element  $[\widetilde{M}^\lambda]$  where  $\lambda$  ranges through all partitions.

- (1) The map  $\Phi$  is an isomorphism of rings.
- (2) It holds that  $\Phi(h_\lambda) = [M^\lambda]$ .
- (3) It holds that  $\Phi(s_\lambda) = [S^\lambda]$ .

The multiplicity of  $\Phi$  stems from the identity  $\widetilde{M}^\lambda = \mathbb{U}_{\lambda_1} \circ \dots \circ \mathbb{U}_{\lambda_t}$  for  $\lambda = (\lambda_1, \dots, \lambda_t)$ .

## 5 Conclusion

**Corollary 18.** The composition

$$\Lambda \xrightarrow{\Phi^{-1}} R \xrightarrow{\mathbb{S}} K_0(\mathrm{GL}(V)) \xrightarrow{\mathrm{Char}} \Lambda(m)$$

is given by  $f \mapsto f(x_1, \dots, x_m)$ .

*Proof.* The assertion holds for the basis elements  $s_\lambda$  of  $\Lambda$  as

$$s_\lambda \mapsto [S^\lambda] \mapsto [\mathbb{S}(S^\lambda)] = [S^\lambda(V)] \mapsto s_\lambda(x_1, \dots, x_m).$$

The general assertion follow by additivity of all occuring maps. □

We have thus finally arrived at the following commutative diagram of rings:

$$\begin{array}{ccc} R & \xleftarrow{\sim} & \Lambda \\ \mathbb{S} \downarrow & & \downarrow f \mapsto f(x_1, \dots, x_m) \\ K_0(\mathrm{GL}(V)) & \xrightarrow[\mathrm{Char}]{\sim} & \Lambda(m) \end{array}$$

We have the following special cases of this diagram:

$$\begin{array}{ccc}
[S^\lambda] & \longleftrightarrow & s_\lambda \\
\downarrow & & \downarrow \\
[S^\lambda(V)] & \longleftrightarrow & s_\lambda(x_1, \dots, x_n) \\
\\ 
[\widetilde{M}^\lambda] & \longleftrightarrow & e_{\bar{\lambda}} & \quad & [M^\lambda] & \longleftrightarrow & h_\lambda \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
[\widetilde{M}^\lambda(V)] & \longleftrightarrow & e_{\bar{\lambda}}(x_1, \dots, x_n) & \quad & [M^\lambda(V)] & \longleftrightarrow & h_\lambda(x_1, \dots, x_n)
\end{array}$$

We can now use these correspondences to translate between problems about the representation theory of  $S_n$ , the representation theory of  $GL(V)$ , and the combinatorics of symmetric polynomials.

**Example 19.**

- (1) For every  $n \geq 0$  there exists unique natural numbers  $f^\lambda$  for  $\lambda \vdash n$  such that one and thus all of the following conditions hold:

- a)  $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$ ,
- b)  $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}$ ,
- c)  $(x_1 + \dots + x_m)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda(x_1, \dots, x_m)$  for every  $m \geq 0$ ,
- d)  $e_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$  in  $\Lambda$ .

We have already seen that it follows from the first description that  $f^\lambda$  is given by the number of standard Young tableaux of shape  $\lambda$ .

- (2) For every partition  $\lambda$  there exist unique natural numbers  $K_{\mu, \lambda}$  for  $\mu \triangleleft \lambda$  such that one and thus all of the following conditions hold:

- a)  $M^\lambda \cong S^\lambda \oplus \bigoplus_{\mu \triangleleft \lambda} (S^\mu)^{\oplus K_{\mu, \lambda}}$ ,
- b)  $M^\lambda(V) \cong S^\lambda(V) \oplus \bigoplus_{\mu \triangleleft \lambda} S^\mu(V)^{\oplus K_{\mu, \lambda}}$ ,
- c)  $h_\lambda(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m) + \sum_{\mu \triangleleft \lambda} K_{\mu, \lambda} s_\mu(x_1, \dots, x_m)$  for all  $m \geq 0$ ,
- d)  $h_\lambda = s_\lambda + \sum_{\mu \triangleleft \lambda} K_{\mu, \lambda} s_\mu$ .

The numbers  $K_{\mu, \lambda}$  are the **Kostka numbers**.

- (3) For any two partitions  $\lambda$  and  $\mu$  there exist natural numbers  $c_{\lambda, \mu}^\nu$  for  $\nu \vdash |\lambda| + |\mu|$  such that one and thus all of the following conditions hold:

- a)  $S^\lambda \circ S^\mu \cong \sum_{\nu} (S^\nu)^{\oplus c_{\lambda, \mu}^\nu}$ ,
- b)  $S^\lambda(V) \otimes S^\mu(V) \cong \sum_{\nu} S^\nu(V)^{\oplus c_{\lambda, \mu}^\nu}$ ,
- c)  $s_\lambda(x_1, \dots, x_m) s_\mu(x_1, \dots, x_m) = \sum_{\nu} c_{\lambda, \mu}^\nu s_\nu(x_1, \dots, x_m)$ ,
- d)  $s_\lambda s_\mu = \sum_{\nu} c_{\lambda, \mu}^\nu s_\nu$ .

The numbers  $c_{\lambda, \mu}^\nu$  are the **Littlewood–Richardson coefficients**.