More on the Representation Theory of $\mathrm{GL}(V)$

We're working over the fixed ground field $\mathbb{C}$. We abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$.

A **partition** of a natural number $n \geq 0$ is a tuple of positive natural numbers $\lambda = (\lambda_1, \dots, \lambda_t)$ with $\lambda_1 \geq \cdots \geq \lambda_t$ and $\lambda_1 + \cdots + \lambda_t = n$. We write $|\lambda| \coloneqq n$ and denote by $\ell(\lambda) \coloneqq t$ the **length** of λ. We observe that the length $\ell(\lambda)$ coincides with the number of rows of the Young diagram of shape λ. We write $\lambda \vdash n$ to mean that λ is a partition of n.

In this talk we will continue to explain the following connections:

$$
\begin{array}{ccc}
\text{represenation theory of the symmetric groups} & \longleftrightarrow & \text{symmetric functions} \\
\downarrow & & \downarrow \\
\text{representation theory of } \mathrm{GL}(V) & \longleftrightarrow & \text{symmetric polynomials}
\end{array}
$$

0 Recalling the last talk

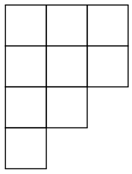
A **filling** of shape λ assigns to each box of the Young diagram of λ is positive natural number, or more generally elements of some set.

In the last talk we have seen that the isomorphism classes of polynomial representations of $\mathrm{GL}(V)$ are indexed by the set of all partitions, Par. For every partition λ with $\lambda = (\lambda_1, \dots, \lambda_t)$ let

$$\widetilde{M}^\lambda(V) = \bigwedge^{\tilde{\lambda}_1}(V) \otimes \cdots \otimes \bigwedge^{\tilde{\lambda}_n}(V)\,.$$

where $\tilde{\lambda}$ is the transposed of λ. The tensor factors $\bigwedge^{\tilde{\lambda}_j}(V)$ hence comes from the j-th column of (the Young diagram of) λ, which has height $\tilde{\lambda}_j$. Note that $\widetilde{M}^\lambda(V) = 0$ if $\ell\,\lambda) > m$, i.e.

Example 1. The Young diagram of $\lambda = (3,3,2,1)$ is given by



and therefore

$$\widetilde{M}^\lambda(V) = \bigwedge^4(V) \otimes \bigwedge^3(V) \otimes \bigwedge^2(V)\,.$$

If T is a filling of shape λ with vectors $v_1, \ldots, v_n \in V$ then we get an associated element $\tilde{v}_T$ of $\widetilde{M}^\lambda(V)$.

Example 2. For

$$T = \begin{array}{|c|c|} \hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

the associated element is given by

$$\tilde{v}_T = (v_1 \wedge v_2 \wedge v_3) \otimes (v_4 \wedge v_5)\,.$$

The submodule $Q^\lambda(V)$ of $\widetilde{M}^\lambda(V)$ is by definition generated by all differences

$$\tilde{v}_T - \sum_{T'} \tilde{v}_{T'} \tag{1}$$

where we run through all fillings T' of λ that arise as follows: We fix columns i and j of T with $i < j$ and a fix a set of entries Y of the j-th column. We take any subset X of the i-th column of T. Then we interchange the entries of X and Y while maintaining their vertical orders.

Example 3. For $\lambda = (2,2,2)$ we choose for the filling

$$T = \begin{array}{|c|c|} \hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & v_6 \\ \hline \end{array}$$

the first two entries of the second column, so that $i = 1$ and $j = 2$. We then get the following resulting fillings:

$$\begin{array}{|c|c|} \hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & v_6 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & v_6 \\ \hline \end{array}\,, \begin{array}{|c|c|} \hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & v_6 \\ \hline \end{array}\,, \begin{array}{|c|c|} \hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & v_6 \\ \hline \end{array} \longrightarrow \underbrace{\begin{array}{|c|c|} \hline v_4 & v_1 \\ \hline v_5 & v_2 \\ \hline v_3 & v_6 \\ \hline \end{array}}_{=:T'_1}\,, \underbrace{\begin{array}{|c|c|} \hline v_4 & v_1 \\ \hline v_2 & v_3 \\ \hline v_5 & v_6 \\ \hline \end{array}}_{=:T'_2}\,, \underbrace{\begin{array}{|c|c|} \hline v_1 & v_2 \\ \hline v_4 & v_3 \\ \hline v_5 & v_6 \\ \hline \end{array}}_{=:T'_3}$$

We hence quotient out the relation

$$\tilde{v}_T - \tilde{v}_{T_1'} - \tilde{v}_{T_2'} - \tilde{v}_{T_3'}\,.$$

The irreducible polynomial representation of $\mathrm{GL}(V)$ associated to a partition λ is now given by

$$S^\lambda(V) = \widetilde{M}^\lambda(V)/Q^\lambda(V)$$

Example 4.

(1) For $\lambda = (1,\ldots,1) \vdash n$ we have $S^\lambda(V) = \bigwedge^n(V)$.

(2) For $\lambda = (n)$ we have $S^\lambda(V) = \mathrm{Sym}^n(V)$.

Remark 5. The submodule $Q^\lambda(V)$ is already generated by those generators (1) in whose construction the columns i and j are adjacent, i.e. such that $j = i+1$.

The polynomial $\mathrm{GL}(V)$-representations can also be described by their characters: Using a basis $e_1,\ldots,e_d$ of V we may identify $\mathrm{GL}(V)$ with $\mathrm{GL}_m(\mathbb{C})$. For any polynomial representation $\rho\colon\ \mathrm{GL}(V) \to \mathrm{GL}(W)$ the weight space W_β with $\beta = (\beta_1,\ldots,\beta_m) \in \mathbb{N}^m$ is given by

$$W_\beta = \{w \in W \mid \mathrm{diag}(x_1,\ldots,x_d).w = x_1^{\beta_1}\cdots x_m^{\beta_m} w \text{ for all } x_1,\ldots,x_m \in \mathbb{C}^\times\}\,.$$

The representation W decomposes into weight spaces in the sense that $W = \bigoplus_{\beta\in\mathbb{N}^d} W_\beta$ and its character is the polynomial

$$\mathrm{Char}(\mathbb{S}(x_1,\ldots,x_m)) = \mathrm{tr}(\rho(\mathrm{diag}(x_1,\ldots,x_m))) = \sum_{\beta\in\mathbb{Z}^m} \dim(W_\beta)x^\beta \in \mathbb{Z}[x_1,\ldots,x_m]\,.$$

The character $\mathrm{Char}(W)$ is actually a symmetric polynomial, and the representation W is (up to isomorphism) uniquely determined by its character.

Example 6.

(1) The basis $e_{i_1}\wedge\cdots\wedge e_{i_n}$ of $\bigwedge^n(V)$ with $i_1 < \cdots < i_n$ consists of weight vectors, and its character is

$$\mathrm{Char}(\bigwedge^n(V))(x_1,\ldots,x_m) = \sum_{1\le i_1<\cdots<i_n\le m} x_{i_1}\cdots x_{i_n} = e_n(x_1,\ldots,x_m)\,,$$

the n-th elementary symmetric polynomial in m variables.

(2) We see similarly that

$$\mathrm{Char}(\mathrm{Sym}^n(V))(x_1,\ldots,x_m) = \sum_{1\le i_1\le\cdots\le i_n\le m} x_{i_1}\cdots x_{i_n} = h_n(x_1,\ldots,x_m)\,,$$

the n-th complete homogeneous symmetric polynomial in m variables.

1 From S_n to $\mathrm{GL}(V)$

We denote for any group G by $\mathbb{1}_G$ the trivial representation of G. We denote for $n \geq 1$ by $\mathbb{U}_n$ the sign representation of the symmetric group S_n.

For every $n \geq 0$ the tensor power $V^{\otimes n}$ is again a $\mathrm{GL}(V)$-representation, and it is also a right S_n-representation via

$$(e_1 \otimes \cdots \otimes e_n).\sigma = e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)} .$$

It follows that for every S_n-representation E we get a new $\mathrm{GL}(V)$-representation

$$\mathbb{S}(E) \coloneqq V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} E$$

This construction results an exact (additive) functor

$$\mathbb{S}\colon\ \mathrm{S}_n\,\textbf{-rep} \to \mathrm{GL}(V)\textbf{-rep}\,,$$

the **Schur functor**.

Example 7.

(1) We have that $\mathbb{1}_{\mathrm{S}_n} \cong \mathbb{C}[\mathrm{S}_n]/\langle \sigma - 1 \mid \sigma \in \mathrm{S}_n \rangle_{\mathbb{C}}$ and thus

$$\begin{aligned}\mathbb{S}(\mathbb{1}_{\mathrm{S}_n}) &\cong V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} \mathbb{C}[\mathrm{S}_n]/\langle \sigma - 1 \mid \sigma \in \mathrm{S}_n \rangle_{\mathbb{C}} \\ &\cong V^{\otimes n}/\langle x - x.\sigma \mid t \in V^{\otimes n} \rangle_{\mathbb{C}} \\ &\cong \mathrm{Sym}^n(V)\,.\end{aligned}$$

(2) We find similarly that $\mathbb{S}(\mathbb{U}_n) \cong \bigwedge^n(V)$.

If $V_1, \dots, V_t$ are representations of the groups $\mathrm{S}_{n_1}, \dots, \mathrm{S}_{n_t}$ then for $n = n_1 + \cdots + n_t$,

$$V_1 \circ \cdots \circ V_t \coloneqq \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[\mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_t}]} (V_1 \otimes \cdots \otimes V_t)$$

is a representation of S_n.

Lemma 8. Let V_i, V and W be representations of symmetric groups. Then

(1) $(V_1 \circ \cdots \circ V_s) \circ \cdots \circ (W_1 \circ \cdots \circ W_t) \cong V_1 \circ \cdots \circ W_t$,

(2) $V \circ W \cong W \circ V$.

Lemma 9. If E and F are representations of symmetric groups S_a and S_b then

$$\mathbb{S}(E \circ F) \cong \mathbb{S}(V) \otimes \mathbb{S}(W)\,.$$

Proof. We find that

$$\begin{aligned}\mathbb{S}(V \circ W) &= E^{\otimes(a+b)} \otimes_{\mathbb{C}[\mathrm{S}_{a+b}]} \mathbb{C}[\mathrm{S}_{a+b}] \otimes_{\mathbb{C}[\mathrm{S}_a \times \mathrm{S}_b]} (V \otimes W) \\ &\cong E^{\otimes(a+b)} \otimes_{\mathbb{C}[\mathrm{S}_a \times \mathrm{S}_b]} (V \otimes W) \\ &\cong (E^{\otimes a} \otimes E^{\otimes b}) \otimes_{\mathbb{C}[\mathrm{S}_a] \otimes \mathbb{C}[\mathrm{S}_b]} (V \otimes W) \\ &\cong (E^{\otimes a} \otimes_{\mathbb{C}[\mathrm{S}_a]} V) \otimes (E^{\otimes b} \otimes_{\mathbb{C}[\mathrm{S}_b]} W) \\ &= \mathbb{S}(V) \otimes \mathbb{S}(W)\end{aligned}$$

as claimed. □

2 Some representation theory of S_n

A **numbering** of a Young tableaux Y of shape $\lambda \vdash n$ fills in the boxes of Y with the numbers $1, \dots, n$.

2.1 The representation M^λ

Two numbering T and T' of a Young diagram of shape λ are **row-equivalent** if they have the same entries in each row. A **row tabloid** is an equivalence class of row-equivalent numbering. A row tabloid can be represented as follows:

$$\begin{array}{ccc}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \cline{1-2}\end{array} \;=\; \begin{array}{ccc}\hline 2 & 1 & 3 \\ \hline 4 & 6 & 5 \\ \hline 8 & 7 & \\ \cline{1-2}\end{array}$$

Every numbering T defines a tabloid $[T]$.

The group S_n acts transitive on the set of numberings of shape $\lambda \vdash n$, which induces a transitive group action on the set of row tabloids of shape λ. The **row group** $R(T)$ of a numbering T is given by all permutations $\sigma \in \mathrm{S}_n$ that act row-wise on T, i.e. the stabilizer of the associated tabloid $[T]$.

It follows that S_n acts linearly in M^λ, the free vector space on the set of Young tabloids of shape λ, and that for any numbering T of shape λ,

$$\begin{aligned} M^\lambda &\cong \mathbb{C}[\mathrm{S}_n]/(p-1 \mid p \in R(T)) \\ &\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}[R(T)]/(p-1 \mid p \in R(T)) \\ &\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{1}_{R(T)} \end{aligned}$$

If T is the "horizontal standard numbering", e.g.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & \\ \cline{1-4}\end{array}$$

for $\lambda = (5,4)$, then $R(T) = \mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}$ where $\lambda = (\lambda_1, \dots, \lambda_t)$ and thus

$$\begin{aligned} M^\lambda &\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{1}_{R(T)} \\ &\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[\mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}]} \mathbb{1}_{\mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}} \\ &\cong \mathbb{C}[\mathrm{S}_n] \otimes_{\mathbb{C}[\mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}]} (\mathbb{1}_{\mathrm{S}_{\lambda_1}} \otimes \cdots \otimes \mathbb{1}_{\mathrm{S}_{\lambda_t}}) \\ &= \mathbb{1}_{\mathrm{S}_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{\mathrm{S}_{\lambda_t}} . \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{S}(M^\lambda) &\cong \mathbb{S}(\mathbb{1}_{\mathrm{S}_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{\mathrm{S}_{\lambda_t}}) \\ &\cong \mathbb{S}(\mathbb{1}_{\mathrm{S}_{\lambda_1}}) \otimes \cdots \otimes \mathbb{S}(\mathbb{1}_{\mathrm{S}_{\lambda_t}}) \\ &\cong \mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_t}(E) , \end{aligned}$$

or alternatively that

$$\begin{aligned}\mathbb{S}(M^\lambda) &= V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} \mathbb{C}[\mathrm{S}_n]/(p-1 \mid p \in R(T)) \\ &\cong V^{\otimes n}/(x - x.p \mid p \in R(T)) \\ &\cong V^{\otimes n}/(x - x.p \mid p \in \mathrm{S}_{\lambda_1} \times \cdots \times \mathrm{S}_{\lambda_t}) \\ &\cong \mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_t}(E)\,.\end{aligned}$$

2.2 The representation $\widetilde{M}^\lambda$

We can alter the construction of M^λ in two ways: Working with column instead of rows and introducing an alternating sign:

We denote by $C(T)$ the column group of a numbering T, i.e. all permutations that act column-wise on T. We let $\widetilde{M}$ be the free vector generated by the numbering T of shape $\lambda \vdash n$ subject to the relations $T = \mathrm{sgn}(\sigma)\sigma.T$ for $\sigma \in C(T)$. The resulting generators $[T]$ of $\widetilde{M}^\lambda$ may be visualized as follows:

$$\left|\begin{array}{c|c} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}\right| = -\left|\begin{array}{c|c} 2 & 4 \\ 1 & 5 \\ 3 & \end{array}\right| = \left|\begin{array}{c|c} 2 & 5 \\ 1 & 4 \\ 3 & \end{array}\right|$$

The group S_n acts on $\widetilde{M}^\lambda$ via $\sigma.[T] = [\sigma.T]$ for any numbering T of shape λ, and

$$\begin{aligned}\widetilde{M}^\lambda &\cong \mathbb{C}[\mathrm{S}_n]/(q - \mathrm{sgn}(q)1 \mid q \in C(T)) \\ &\cong \mathbb{U}_{\lambda_1} \circ \cdots \circ \mathbb{U}_{\lambda_t}\,.\end{aligned}$$

It follows that

$$\mathbb{S}(\widetilde{M}^\lambda) \cong \bigwedge^{\lambda_1}(E) \otimes \cdots \otimes \bigwedge^{\lambda_t}(E) = \widetilde{M}^\lambda(E)\,.$$

Example 10. Let $\lambda = (2,2,1)$. Let $v_1, \dots, v_5 \in E$ and consider an arbitrary filling

$$T = \begin{array}{|c|c|}\hline \sigma(1) & \sigma(4) \\ \hline \sigma(2) & \sigma(5) \\ \hline \sigma(3) & \\ \cline{1-1}\end{array}$$

of shape λ, where $\sigma \in \mathrm{S}_5$. In $\mathbb{S}(\widetilde{M}^\lambda) = V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} \widetilde{M}^\lambda$ we get an element

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_5) \otimes [T] &= (v_1 \otimes \cdots \otimes v_5) \otimes [\sigma.T_0] \\ &= ((v_1 \otimes \cdots \otimes v_5).\sigma) \otimes [T_0] \\ &= v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(5)} \otimes [T_0]\end{aligned}$$

where

$$T_0 = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

is the "vertical standard numbering". By identifying $\widetilde{M}^\lambda$ with $\mathbb{C}[\mathrm{S}_n]/(q-1 \mid q \in C(T))$ via $[\tau] \mapsto \tau.[T_0] = [\tau.T_0]$ and using that $C(T) = \mathrm{S}_3 \times \mathrm{S}_2$ we find that the corresponding element of $\widetilde{M}^\lambda(V) = \bigwedge^3(V) \otimes \bigwedge^2(V)$ is given by

$$\left(v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)}\right) \otimes \left(v_{\sigma(4)} \wedge v_{\sigma(5)}\right)$$

Note that this is precisely the element $\tilde{v}_{\widetilde{T}}$ for the filling $\tilde{T}$ given as follows:

$$\widetilde{T} = \begin{array}{|c|c|} \hline v_{\sigma(1)} & v_{\sigma(4)} \\ \hline v_{\sigma(2)} & v_{\sigma(5)} \\ \hline v_{\sigma(3)} & \\ \cline{1-1} \end{array}$$

2.3 Specht modules

The irreducible representations of the symmetric group S_n can be indexed by the partitions $\lambda \in \mathrm{Par}(n)$ and constructed as follows:

If T is any numbering of shape λ then its **Young centralizer** is the element

$$c_T \coloneqq \sum_{q \in C(T)} \operatorname{sgn}(q) q \in \mathbb{C}[\mathrm{S}_n] \,.$$

The **Specht module** S^λ is the linear subspace of M^λ spanned by $v_T \coloneqq c_T \cdot [T]$ as T runs through the numbering of shape λ. It holds that $\sigma \cdot v_T = v_{\sigma \cdot T}$ for every $\sigma \in \mathrm{S}_n$ whence S^λ is a subrepresentation of M^λ.

Theorem 11 (Classification of irreducible representations of S_n)**.**

(1) The representations S^λ with $\lambda \vdash n$ are pairwise non-isomorphic irreducible representations of the group S_n, and every irreducible representation of S_n is of this form.

(2) The elements v_T, where T runs through the standard Young tableaux of shape λ, form a basis of S^λ. Hence

$$\dim S^\lambda = \text{number of standard Young tableaux of shape } \lambda \,.$$

One can also construct the Spect module S^λ as as a quotient of the representation $\widetilde{M}^\lambda$: The map

$$\widetilde{M}^\lambda \to S^\lambda \quad [T] \mapsto v_T$$

is a well-defined surjective homomorphism of S_n-representations. Its kernel Q^λ is generated by the differences

$$[T] - \sum_{T'} [T']$$

where $[T']$ runs through all numberings of λ that arise from T in the following way: We fix a column $j = 2, \dots, \lambda_1$ and fix a set Y of entries the j-th of T. Then fory any equinumerous subset X of the $(j-1)$-th column of T we get such a T' by interchanging the entries of X and Y while presvering their vertical ordering. We find the following:

Theorem 12. For any partition λ,

$$\mathbb{S}(S^\lambda) \cong S^\lambda(V) .$$

Proof. By applying the (exact) functor $\mathbb{S}(-)$ to the short exact sequence

$$0 \to Q^\lambda \to \widetilde{M}^\lambda \to S^\lambda \to 0$$

we get the short exact sequence

$$0 \to \mathbb{S}(Q^\lambda) \to \mathbb{S}(\widetilde{M}^\lambda) \to \mathbb{S}(S^\lambda) \to 0 .$$

We have previously seen that $\mathbb{S}(\widetilde{M}^\lambda) \cong \widetilde{M}^\lambda(V)$. We see from the calculations showcased in Example 10 and the explicit description of Q^λ and $Q^\lambda(V)$ that the image of $\mathbb{S}(Q^\lambda)$ in $\mathbb{S}(\widetilde{M}^\lambda)$ corresponds precisely to $Q^\lambda(V)$. Hence

$$\mathbb{S}(S^\lambda) \cong \mathbb{S}(\widetilde{M}^\lambda)/\mathbb{S}(Q^\lambda) \cong \widetilde{M}^\lambda(V)/Q^\lambda(V) \cong S^\lambda(V) .$$

This proves the assertion. $\square$

Remark 13.

(1) One can similarly construct the irreducible representations of S_n as subrepresentations $\widetilde{S}^\lambda$ of M^λ. Then $\widetilde{S}^\lambda$ is a quotient of M^λ and the compositions $S^\lambda \to M^\lambda \to \widetilde{S}^\lambda$ and $\widetilde{S}^\lambda \to \widetilde{M}^\lambda \to S^\lambda$ are isomorphisms. Hence $S^\lambda \cong \tilde{S}^\lambda$.

(2) It then follows that $\mathbb{S}(\widetilde{S}^\lambda) \cong \mathbb{S}(S^\lambda) \cong S^\lambda(V)$. If $\lambda = (\lambda_1, \dots, \lambda_t)$ then it follows from $\mathbb{S}(M^\lambda) \cong \mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_t}(E)$ similarly to above that the Schur modules $S^\lambda(V)$ can also be constructed as quotients of $\mathrm{Sym}^{\lambda_1}(E) \otimes \cdots \mathrm{Sym}^{\lambda_t}(E)$.

(3) If $\ell(\lambda) > m$, i.e. if the Young diagram of λ constrists of more than m rows, then $\mathbb{S}(S^\lambda) = S^\lambda(V) = 0$. If $\ell(\lambda) \leq m$ then $\mathbb{S}(S^\lambda) = S^\lambda(V)$ is again an irreducible representation.

Corollary 14. For any $n \geq 0$, $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f_\lambda}$ where f_λ denotes the number of standard Young tableaux of shape λ.

Proof. We find with Theorem 11 that

$$\mathbb{C}[\mathrm{S}_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}.$$

with

$$\begin{aligned} f_\lambda &= \text{multiplicity of } S_\lambda \text{ in } \mathbb{C}[\mathrm{S}_n] \\ &= \text{dimension of } S_\lambda \\ &= \text{number of standard Young tableaux of shape } \lambda \end{aligned}$$

where the second equality follows from the Artin–Wedderburn theorem. It follows that

$$\begin{aligned} V^{\otimes n} &= V^{\otimes n} \otimes_{\mathbb{C}[\mathrm{S}_n]} \mathbb{C}[\mathrm{S}_n] \\ &\cong \mathbb{S}(\mathbb{C}[\mathrm{S}_n]) \\ &\cong \mathbb{S}\left(\bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda} \right) \\ &\cong \bigoplus_{\lambda \vdash n} \mathbb{S}(S^\lambda)^{\oplus f^\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda} \end{aligned}$$

as claimed. □

Example 15. The partitions of $n = 2$ are $\lambda_1 = (1, 1)$ and $\lambda_2 = (2)$. Each of those partitions admits precisely one standard Young tableau:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

It follows that

$$V \otimes V = V^{\otimes 2} \cong S^{\lambda_1}(V)^{\oplus 1} \oplus S^{\lambda_2}(V)^{\oplus 1} \cong \mathrm{Sym}^2(V) \oplus \bigwedge^2(V).$$

3 Translation into rings

3.1 The Grothendieck ring of $\mathrm{GL}(V)$

Recall that the Grothendieck groups of an abelian category $\mathcal{A}$ is generated by the isomorphism classes of objects of $\mathcal{A}$ subject to the relation $[B] = [A] + [C]$ for every short exact sequence $0 \to A \to B \to C \to 0$. Then in particular $[A] + [B] = [A \oplus B]$ for all $A, B \in \mathcal{A}$. If the abelian category $\mathcal{A}$ is semisimple, i.e. if every short exact sequence in $\mathcal{A}$ splits, then these conditions already suffice.

Let $\mathcal{A}$ be the semisimple category of finite dimensional polynomial representations of $\mathrm{GL}(V)$. We denote the Grothendieck group of $\mathcal{A}$ by $\mathrm{K}_0(\mathrm{GL}(V))$. This becomes a ring when endowed with the multiplication

$$[V] \cdot [W] = [V \otimes W] \,.$$

It follows from last week's classification of irreducible finite dimensional polynomial $\mathrm{GL}(V)$-representations that $\mathrm{K}_0(\mathrm{GL}(V))$ has a basis given by $[S^\lambda(V)]$ where λ ranges through all partitions with $\ell\,\lambda) \leq m$.

3.2 The representation ring of S_n

For every $n \geq 0$ let R_n be the Grothendieck group of S_n-**rep**, the category of finite dimensional S_n-representations. We can define on $R \coloneqq \bigoplus_{n \geq 0} R_n$ a multiplication via

$$[E] \cdot [F] = [E \circ F] \,.$$

Note that $[\mathbb{1}_{S_0}]$ is multiplicative neutral and that this multiplicaton is associative and commutative by Lemma 8. The multiplication is also distributive and $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$. Hence R becomes a graded ring. The category S_n-**rep** is semisimple by Maschke's theorem and it follows from Theorem 11 that R_n has as a $\mathbb{Z}$-basis given by $[S^\lambda]$ for $\lambda \vdash n$. Hence R has a $\mathbb{Z}$-basis given by $[S^\lambda]$ where λ ranges through all partitions.

For every $n \geq 0$ the additivity of the Schur functor $\mathbb{S}\colon S_n\text{-}\mathbf{rep} \to \mathcal{A}$ gives a group homomorphism $R_n \to \mathrm{K}_0(\mathrm{GL}(V))$, which together give a group homomorphism $R \to \mathrm{K}_0(\mathrm{GL}(V))$. It follows from Lemma 9 that this is a ring homomorphism. By abuse of notation we denote this homomorphism again by $\mathbb{S}$.

We have argued in part (3) of Remark 13 that the kernel of $\mathbb{S}\colon R \to \mathrm{K}_0(\mathrm{GL}(V))$ is spanned by those $[S^\lambda]$ with $\ell\,\lambda) > m$, whereas all other basis vector $[S^\lambda]$ with $\ell\,\lambda) \leq m$ are mapped bijectively onto the basis $[S^\lambda(V)]$ of $\mathrm{K}_0(\mathrm{GL}(V))$.

3.3 The ring of symmetric functions

For every $k \geq 0$ let $\Lambda(k)$ denote the ring of symmetric polynomials, a subring of $\mathbb{Z}[x_1, \dots, x_k]$.

When dealing with symmetric polynomials it often happens that the number of variables, k, does not matter: One has a family $(f_k)_{k \geq 0}$ of symmetric polynomials $f_k \in \Lambda(k)$ such that $f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1})$ for every $k \geq 1$, and an identity involving f_n which reduces for $x_k \to 0$ to the identity involving f_{k-1}.

Example 16. The elementary symmetric polynomials $e_n(x_1, \dots, x_k)$ and the completely symmetric polynomials $h_n(x_1, \dots, x_k)$ are for every $k \geq 1$ related by the formula

$$\sum_{i=0}^{s} (-1)^i h_{s-i}(x_1, \dots, x_k) e_i(x_1, \dots, x_k) = 0 \,.$$

for every $s \geq 0$. This identity in $k-1$ variables follows from the one in k variables by setting $x_k \to 0$.

To formalize this phenomenon we introduce the **ring of symmetric functions** Λ: For every degree $n \geq 0$ we set

$$\begin{aligned}\Lambda_n &\coloneqq \left\{ (f_k)_{k\geq 0} \;\middle|\; \begin{array}{c} f_k \in \Lambda_n(k) \text{ with} \\ f_k(x_1,\dots,x_{k-1},0) = f_{k-1}(x_1,\dots,x_{k-1}) \\ \text{for every } k \geq 1 \end{array} \right\} \\ &= \lim(\Lambda_n(0) \leftarrow \Lambda_n(1) \leftarrow \Lambda_n(2) \leftarrow \Lambda_n(3) \leftarrow \cdots)\end{aligned}$$

where $\Lambda_n(k) \to \Lambda_n(k-1)$ is the group homomorphism given by $x_k \to 0$. We combine these groups into a graded ring $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ with multiplication given by

$$(f_k)_{k\geq 0} \cdot (g_k)_{k\geq 0} \coloneqq (f_k g_k)_{k\geq 0} \,.$$

Example 17. For every $n \geq 0$ we have an element

$$e_n \coloneqq (e_n(), e_n(x_1), e_n(x_1,x_2), e_n(x_1,x_2,x_3), \dots) \in \Lambda_n$$

and similarly elements $h_n, p_n \in \Lambda_n$. We get for every partition $\lambda = (\lambda_1, \dots, \lambda_t)$ an induced element

$$e_\lambda \coloneqq e_{\lambda_1} \cdots e_{\lambda_t} \in \Lambda_{|\lambda|}$$

and similarly elements $h_\lambda, p_\lambda 0 \in \Lambda_{|\lambda|}$.

Remark 18.

(1) We have that

$$\Lambda = \lim(\Lambda(0) \leftarrow \Lambda(1) \leftarrow \Lambda(2) \leftarrow \Lambda(3) \leftarrow \cdots)$$

in the category of graded rings.

(2) The elements of the ring Λ are not functions, despite its name.

Example 19 (Schur polynomials and Schur functions)**.** Let λ be a partition. For every semistandard Young tableaux T^1 of shape λ with entries in $\{1, \dots, k\}$ let

$$x_T \coloneqq \prod_{i \in T} x_i \,.$$

The **Schur polynomial** $s_\lambda(x_1, \dots, x_k)$ is defined as

$$s_\lambda(x_1,\dots,x_k) = \sum_{T'} x_{T'} \in \mathbb{Z}[x_1,\dots,x_n]$$

where T' ranges through the semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$.

[1]This means that T is a filling that is weakly increasing in each row but strictly increasing in each column

If for example $\lambda = (2, 2)$ and $m = 3$ then the semistandard Young tableaux are as follows:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$$

The Schur polynomial $s_{(2,2)}(x_1, x_2, x_3)$ is therefore given by

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1^2 x_3^2 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_2^2 x_3^3 \,.$$

We observe the following:

(1) The Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ is homogeneous of degree $|\lambda|$.

(2) If $\ell(\lambda) > k$, i.e. if the Young diagram of λ has more than k rows, then the Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ vanishes since there exist no semistandard Young tableaux of shape λ with entries in $\{1, \ldots, k\}$. (We don't have enough entries to make the first column strictly increasing, which is required for a semistandard Young tableaux.)

(3) If $e_1, \ldots, e_k$ is a basis of V then the Specht module $S^\lambda(V)$ has a basis e_T where T ranges through the semistandard Young tableaux with entries in $\{1, \ldots, k\}$. Each e_T is a weight vector with corresponding weight x_T. Hence

$$s_\lambda(x_1, \ldots, x_k) = \operatorname{Char}(S^\lambda(V)) \,.$$

This shows in particular that that $s_\lambda(x_1, \ldots, x_k)$ is a symmetric polynomial. We also see again that $s_\lambda(x_1, \ldots, x_k) = 0$ if $\ell(\lambda) > k$.

(4) It holds that $s_\lambda(x_1, \ldots, x_{k-1}, 0) = s_\lambda(x_1, \ldots, x_{k-1})$.

We find that we get a well-defined element $s_\lambda \in \Lambda_{|\lambda|}$, the **Schur function**.

We have for every $k \geq 0$ a homomorphism of graded rings

$$\Lambda \to \Lambda(k) \,, \quad f \mapsto f(x_1, \ldots, x_k)$$

that assigns to f the entailed symmetric polynomial in k variables. An equality $f = g$ holds in Λ if and only if for every $k \geq 0$ the equality $f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k)$ hold.

Example 20. For every $s \geq 0$ the equality $\sum_{i=0}^{s} (-1)^i h_{s-i} e_i = 0$ holds in Λ.

Proposition 21. For every $k \geq 0$ the ring of symmetric polynomials $\Lambda(k)$ has the Schur polynomials $s_\lambda(x_1, \ldots, x_k)$ with $\ell(\lambda) \leq k$ as a basis.

Proposition 22.

(1) The symmetric functions $e_1, e_2, \ldots$ generate Λ and are algebraically independent.

(2) The monomials e_λ where λ ranges through all partitions is a $\mathbb{Z}$-basis of Λ.

(3) The symmetric functions $h_1, h_2, \dots$ generate Λ and are algebraically independent.

(4) The monomials h_λ where λ ranges through all partitions is a $\mathbb{Z}$-basis of Λ.

(5) The symmetric functions s_λ where λ ranges through all partitions forms a $\mathbb{Z}$-basis of Λ.

The mapping $e_i(x_1, \dots, x_k) \mapsto e_i(x_1, \dots, x_{k+1})$ with $i = 0, \dots, k$ gives an embedding of rings $\Lambda(k) \to \Lambda(k+1)$, and the above shows that one can regard Λ is the resulting colimit, i.e. as the polynomial ring $\mathbb{Z}[e_1, e_2, e_3, \dots]$. Similarly for h_i instead of e_i.

Theorem 23. Let $\Phi\colon \Lambda \to R$ be the unique additive group homomorphism that maps the basis element e_λ to the element $[\widetilde{M}]$.

(1) The map Φ is an isomorphism of rings.

(2) It holds that $\Phi(h_\lambda) = [M^\lambda]$.

(3) It holds that $\Phi(s_\lambda) = [S^\lambda]$.

4 The goal

Corollary 24. The composition

$$\Lambda \xrightarrow{\Phi^{-1}} R \xrightarrow{\mathbb{S}} \mathrm{K}_0(\mathrm{GL}(V)) \xrightarrow{\mathrm{Char}} \Lambda(m)$$

is given by $f \mapsto f(x_1, \dots, x_m)$.

Proof. The assertion holds for the basis elements s_λ of Λ as

$$s_\lambda \mapsto [S^\lambda] \mapsto [\mathbb{S}(S^\lambda)] = [S^\lambda(V)] \mapsto s_\lambda(x_1, \dots, x_m)\,.$$

The general assertion follow by additivity of all occuring maps. $\square$

We have thus finally arrived at the following commutative diagram of rings:

$$\begin{tikzcd}
R \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathbb{S}"'] & \Lambda \arrow[d, "f \mapsto f(x_1,\dots,x_m)"] \\
\mathrm{K}_0(\mathrm{GL}(V)) \arrow[r, "\sim", "\mathrm{Char}"'] & \Lambda(m)
\end{tikzcd}$$

We have the following special cases of this diagram:

$$\begin{tikzcd}
{[S^\lambda]} \arrow[r, leftrightarrow] \arrow[d] & s_\lambda \arrow[d] \\
{[S^\lambda(V)]} \arrow[r, leftrightarrow] & s_\lambda(x_1, \dots, x_n)
\end{tikzcd}$$

$$\begin{tikzcd}
{[\widetilde{M}^\lambda]} \arrow[r, leftrightarrow] \arrow[d] & e_\lambda \arrow[d] \\
{[\widetilde{M}^\lambda(V)]} \arrow[r, leftrightarrow] & e_\lambda(x_1, \dots, x_n)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[M^\lambda]} \arrow[r, leftrightarrow] \arrow[d] & h_\lambda \arrow[d] \\
{[M^\lambda(V)]} \arrow[r, leftrightarrow] & h_\lambda(x_1, \dots, x_n)
\end{tikzcd}$$

We can now use these correspondeces to translate between problems about the representation theory of S_n, the representation theory of $\mathrm{GL}(V)$, and the combinatorics of symmetric polynomials.

Example 25.

(1) For every $n \geq 0$ there exists unique natural numbers f^λ for $\lambda \vdash n$ such that one and thus all of the following conditions hold:

 a) $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$,
 b) $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}$,
 c) $(x_1 + \cdots + x_m)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda(x_1, \dots, x_m)$ for every $m \geq 0$,
 d) $e_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$ in Λ.

 We have already seen from the first description that f^λ is the number of standard Young tableaux of shape λ.

(2) For every partition λ there exist unique natural numbers $K_{\mu,\lambda}$ such that one and thus all of the following conditions hold:

 a) $M^\lambda \cong S^\lambda \oplus \bigoplus_{\mu \lhd \lambda} (S^\mu)^{\oplus K_{\mu,\lambda}}$,
 b) $M^\lambda(V) \cong S^\lambda(V) \oplus \bigoplus_{\mu \lhd \lambda} S^\mu(V)^{\oplus K_{\mu,\lambda}}$,
 c) $h_\lambda(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m) + \sum_{\mu \lhd \lambda} K_{\mu,\lambda} s_\mu(x_1, \dots, x_m)$,
 d) $h_\lambda = s_\lambda + \sum_{\mu \lhd \lambda} K_{\mu,\lambda} s_\mu$.

 The numbers $K_{\mu,\lambda}$ are the **Kostka numbers**.

(3) For every partition λ there exist unique natural numbers $\tilde{K}_{\mu,\lambda}$ such that one and thus all of the following conditions hold:

 a) $\widetilde{M}^\lambda \cong S^\lambda \oplus \bigoplus_{\mu \lhd \lambda} (S^\mu)^{\oplus \tilde{K}_{\mu,\lambda}}$,
 b) $\widetilde{M}^\lambda(V) \cong S^\lambda(V) \oplus \bigoplus_{\mu \lhd \lambda} S^\mu(V)^{\oplus \tilde{K}_{\mu,\lambda}}$,
 c) $e_\lambda(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m) + \sum_{\mu \lhd \lambda} \tilde{K}_{\mu,\lambda} s_\mu(x_1, \dots, x_m)$,

d) $e_\lambda = s_\lambda + \sum_{\mu \lhd \lambda} \tilde{K}_{\mu,\lambda} s_\mu$.

The numbers $\tilde{K}_{\mu,\lambda}$ are again the Kostka numbers, connected to the above via $\tilde{K}_{\mu,\lambda} = K_{\tilde{\mu},\lambda}$ where $\tilde{\mu}$ denotes the transposed of μ.

(4) For any two partitions λ ad μ there exist natural numbers $c^\nu_{\lambda,\mu}$ such that one and thus all of the following conditions holds:

a) $S^\lambda \circ S^\mu \cong \sum_\nu (S^\nu)^{\oplus c^\nu_{\lambda,\mu}}$,

b) $S^\lambda(V) \otimes S^\mu(V) \cong \sum_\nu S^\nu(V)^{\oplus c^\nu_{\lambda,\mu}}$,

c) $s_\lambda(x_1, \dots, x_m) s_\mu(x_1, \dots, x_m) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x_1, \dots, x_m)$,

d) $s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu} s_\nu$.

The numbers $c^\nu_{\lambda,\mu}$ are the **Littlewood–Richardson coefficients**.