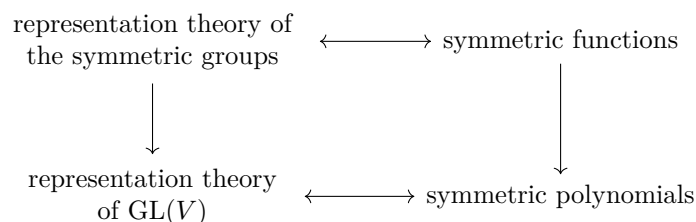


More on the Representation Theory of $\mathrm{GL}(V)$

We're working over the fixed ground field \mathbb{C} . We abbreviate $\otimes_{\mathbb{C}}$ as \otimes . We denote by V some m -dimensional vector space. The set of natural numbers \mathbb{N} contains 0. We write $\lambda \vdash n$ to mean that λ is a partition of n .

In this talk we will continue to explain the following connections:



1 Recalling the last talk

A **semistandard Young tableau** of shape λ assigns to each box of the Young diagram a natural number $1, 2, 3, \dots$, weakly increasing in each row and strictly increasing in each column.

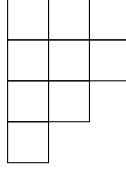
1.1 The irreducible representations $S^\lambda(V)$

We have previously seen that the isomorphism classes of irreducible finite-dimensional polynomial representations of $\mathrm{GL}(V)$ are indexed by the set of all partitions. For every partition λ with $\lambda = (\lambda_1, \dots, \lambda_t)$ let

$$\widetilde{M}^\lambda(V) = \bigwedge^{\tilde{\lambda}_1}(V) \otimes \cdots \otimes \bigwedge^{\tilde{\lambda}_n}(V).$$

where $\tilde{\lambda}$ is the transposed of λ . The tensor factor $\bigwedge^{\tilde{\lambda}_j}(V)$ hence comes from the j -th column of the Young diagram associated to λ , which has height $\tilde{\lambda}_j$. Note that $\widetilde{M}^\lambda(V) = 0$ if $\ell(\lambda) > m$, i.e. if the Young diagram of λ has more than m rows.

Example 1. The Young diagram of $\lambda = (3, 3, 2, 1)$ is given by



and therefore

$$\widetilde{M}^\lambda(V) = \bigwedge^4(V) \otimes \bigwedge^3(V) \otimes \bigwedge^2(V).$$

The irreducible polynomial representation of $\mathrm{GL}(V)$ associated to the partition λ given by

$$S^\lambda(V) = \widetilde{M}^\lambda(V) / Q^\lambda(V)$$

for a certain submodule Q_λ of \widetilde{M}_λ . This is the **Schur module** associated to λ .

Example 2.

- (1) For $\lambda = (1, \dots, 1) \vdash n$ we have $S^\lambda(V) = \bigwedge^n(V)$.
- (2) For $\lambda = (n)$ we have $S^\lambda(V) = \mathrm{Sym}^n(V)$.

1.2 Characters

The polynomial $\mathrm{GL}(V)$ -representations can also be described by their characters: Using a basis e_1, \dots, e_m of V we may identify $\mathrm{GL}(V)$ with $\mathrm{GL}_m(\mathbb{C})$. For any finite-dimensional polynomial representation $\rho: \mathrm{GL}(V) \rightarrow \mathrm{GL}(W)$ the **weight space** W_β for the **weight** $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$ is given by

$$W_\beta = \{w \in W \mid \mathrm{diag}(x_1, \dots, x_m) \cdot w = x_1^{\beta_1} \cdots x_m^{\beta_m} w \text{ for all } x_1, \dots, x_m \in \mathbb{C}^\times\}.$$

The representation W decomposes into weight spaces in the sense that $W = \bigoplus_{\beta \in \mathbb{N}^m} W_\beta$ and its **character** is the polynomial $\mathrm{Char}(W) \in \mathbb{Z}[x_1, \dots, x_m]$ given by

$$\mathrm{Char}(W)(x_1, \dots, x_m) = \mathrm{tr}(\rho(\mathrm{diag}(x_1, \dots, x_m))) = \sum_{\beta \in \mathbb{N}^m} \dim(W_\beta) x^\beta.$$

The character $\mathrm{Char}(W)$ is actually a symmetric polynomial, and the representation W is (up to isomorphism) uniquely determined by its character.

Example 3.

- (1) The basis $e_{i_1} \wedge \cdots \wedge e_{i_n}$ of $\bigwedge^n(V)$ with $i_1 < \cdots < i_n$ consists of weight vectors, and its character given by is

$$\mathrm{Char}\left(\bigwedge^n(V)\right)(x_1, \dots, x_m) = \sum_{1 \leq i_1 < \cdots < i_n \leq m} x_{i_1} \cdots x_{i_n} = e_n(x_1, \dots, x_m),$$

the n -th elementary symmetric polynomial in m variables.

(2) We see similarly that

$$\text{Char}(\text{Sym}^n(V))(x_1, \dots, x_m) = \sum_{1 \leq i_1 \leq \dots \leq i_n \leq m} x_{i_1} \cdots x_{i_n} = h_n(x_1, \dots, x_m),$$

the n -th complete homogeneous symmetric polynomial in m variables.

2 From S_n to $\text{GL}(V)$

We denote for any group G by $\mathbb{1}_G$ the trivial representation of G . We denote for $n \geq 1$ by \mathbb{U}_{S_n} the sign representation of the symmetric group S_n .

For every $n \geq 0$ the tensor power $V^{\otimes n}$ is again a $\text{GL}(V)$ -representation, and it is also a right S_n -representation via

$$(e_1 \otimes \cdots \otimes e_n) \cdot \sigma = e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}.$$

It follows that for every S_n -representation E we get a $\text{GL}(V)$ -representation

$$\mathbb{S}(E) := V^{\otimes n} \otimes_{\mathbb{C}[S_n]} E.$$

This construction results an exact (additive) functor

$$\mathbb{S}: S_n\text{-rep} \rightarrow \text{GL}(V)\text{-rep},$$

where $S_n\text{-rep}$ is the category of finite-dimensional S_n -representations and $\text{GL}(V)\text{-rep}$ denotes the category of finite-dimensional polynomial $\text{GL}(V)$ -representations. The functor \mathbb{S} is the **Schur functor**. It depends on both n and V , but we will omit this in our notation.

Example 4.

(1) We have that $\mathbb{1}_{S_n} \cong \mathbb{C}[S_n] / \langle \sigma - 1 \mid \sigma \in S_n \rangle_{\mathbb{C}}$ and thus

$$\begin{aligned} \mathbb{S}(\mathbb{1}_{S_n}) &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] / \langle \sigma - 1 \mid \sigma \in S_n \rangle_{\mathbb{C}} \\ &\cong V^{\otimes n} / \langle x - x \cdot \sigma \mid t \in V^{\otimes n} \rangle_{\mathbb{C}} \\ &\cong \text{Sym}^n(V). \end{aligned}$$

(2) We find similarly that $\mathbb{S}(\mathbb{U}_{S_n}) \cong \bigwedge^n(V)$.

If V_1, \dots, V_t are representations of the groups S_{n_1}, \dots, S_{n_t} then for $n = n_1 + \dots + n_t$,

$$V_1 \circ \cdots \circ V_t := \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n_1} \times \cdots \times S_{n_t}]} (V_1 \boxtimes \cdots \boxtimes V_t)$$

is a representation of S_n .

Lemma 5. Let V_i , V and W be representations of symmetric groups. Then

(1) $(V_1 \circ \cdots \circ V_s) \circ \cdots \circ (W_1 \circ \cdots \circ W_t) \cong V_1 \circ \cdots \circ W_t$,

(2) $V \circ W \cong W \circ V$.

Lemma 6. If E and F are representations of symmetric groups S_a and S_b then

$$\mathbb{S}(E \circ F) \cong \mathbb{S}(E) \otimes \mathbb{S}(F).$$

Proof. We find that

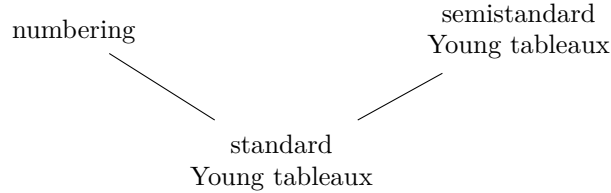
$$\begin{aligned} \mathbb{S}(E \circ F) &= V^{\otimes(a+b)} \otimes_{\mathbb{C}[S_{a+b}]} \mathbb{C}[S_{a+b}] \otimes_{\mathbb{C}[S_a \times S_b]} (E \boxtimes F) \\ &\cong V^{\otimes(a+b)} \otimes_{\mathbb{C}[S_a \times S_b]} (E \boxtimes F) \\ &\cong (V^{\otimes a} \boxtimes V^{\otimes b}) \otimes_{\mathbb{C}[S_a] \otimes \mathbb{C}[S_b]} (E \boxtimes F) \\ &\cong (V^{\otimes a} \otimes_{\mathbb{C}[S_a]} E) \otimes (V^{\otimes b} \otimes_{\mathbb{C}[S_b]} F) \\ &= \mathbb{S}(E) \otimes \mathbb{S}(F) \end{aligned}$$

as claimed. \square

3 Some representation theory of S_n

For better use of the Schur functor \mathbb{S} we need some understand of the representation theory of the symmetric group S_n , where $n \geq 0$. In this section we always assume that $\lambda \vdash n$ unless otherwise specified.

A **numbering** of a Young tableaux Y of shape $\lambda \vdash n$ fills in the boxes of Y with the numbers $1, \dots, n$. A **standard Young tableaux** is a semistandard Young tableaux that is also a numbering. This means that we assign to each box of the Young diagram of λ precisely one integer $1, 2, \dots, n$ such that both rows and columns are (necessarily strictly) increasing.



3.1 The representation M^λ

Two numbering T and T' of a Young diagram of shape λ are **row-equivalent** if they have the same entries in each row. A **row tabloid** is an equivalence class of row-equivalent numbering. A row tabloid can be represented as follows:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline 4 & 6 & 5 \\ \hline 8 & 7 & \\ \hline \end{array}$$

Every numbering T defines a row tabloid $[T]$.

The group S_n acts transitive on the set of numberings of shape λ , which induces a transitive group action on the set of row tabloids of shape λ . The **row group** $R(T)$ of a numbering T is given by all permutations $\sigma \in S_n$ that act row-wise on T , i.e. the stabilizer of the associated tabloid $[T]$.

It follows that S_n acts linearly on M^λ , the free vector space on the set of row tabloids of shape λ , and that for any numbering T of shape λ ,

$$\begin{aligned} M^\lambda &\cong \mathbb{C}[S_n]/(p-1 \mid p \in R(T)) \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}[R(T)]/(p-1 \mid p \in R(T)) \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{1}_{R(T)} \end{aligned}$$

If T is the “horizontal standard numbering”, e.g.

1	2	3	4	5
6	7	8	9	

for $\lambda = (5, 4)$, then $R(T) = S_{\lambda_1} \times \cdots \times S_{\lambda_t}$ where $\lambda = (\lambda_1, \dots, \lambda_t)$ and thus

$$\begin{aligned} M^\lambda &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{1}_{R(T)} \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda_1} \times \cdots \times S_{\lambda_t}]} \mathbb{1}_{S_{\lambda_1} \times \cdots \times S_{\lambda_t}} \\ &\cong \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{\lambda_1} \times \cdots \times S_{\lambda_t}]} (\mathbb{1}_{S_{\lambda_1}} \boxtimes \cdots \boxtimes \mathbb{1}_{S_{\lambda_t}}) \\ &= \mathbb{1}_{S_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{S_{\lambda_t}}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{S}(M^\lambda) &\cong \mathbb{S}(\mathbb{1}_{S_{\lambda_1}} \circ \cdots \circ \mathbb{1}_{S_{\lambda_t}}) \\ &\cong \mathbb{S}(\mathbb{1}_{S_{\lambda_1}}) \otimes \cdots \otimes \mathbb{S}(\mathbb{1}_{S_{\lambda_t}}) \\ &\cong \text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_t}(V) \\ &=: M^\lambda(V), \end{aligned}$$

or alternatively that

$$\begin{aligned} \mathbb{S}(M^\lambda) &= V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n]/(p-1 \mid p \in R(T)) \\ &\cong V^{\otimes n}/(x - x.p \mid p \in R(T)) \\ &\cong V^{\otimes n}/(x - x.p \mid p \in S_{\lambda_1} \times \cdots \times S_{\lambda_t}) \\ &\cong \text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_t}(V). \end{aligned}$$

3.2 The representation \widetilde{M}^λ

We can alter the construction of M^λ in two ways: Working with column instead of rows and introducing an alternating sign.

We denote by $C(T)$ the column group of a numbering T , i.e. all permutations that act column-wise on T . We let \widetilde{M}^λ be the free vector generated by the numbering T of shape λ subject to the relations $T = \text{sgn}(\sigma)\sigma.T$ for $\sigma \in C(T)$. The resulting vector space generators $[T]$ of \widetilde{M}^λ may be visualized as follows:

$$\begin{vmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{vmatrix} = - \begin{vmatrix} 2 & 4 \\ 1 & 5 \\ 3 & \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \\ 3 & \end{vmatrix}$$

The group S_n acts on \widetilde{M}^λ via $\sigma.[T] = [\sigma.T]$ for any numbering T of shape λ , and

$$\begin{aligned} \widetilde{M}^\lambda &\cong \mathbb{C}[S_n]/(q - \text{sgn}(q)1 \mid q \in C(T)) \\ &\cong \mathbb{U}_{\lambda_1} \circ \cdots \circ \mathbb{U}_{\lambda_t}. \end{aligned}$$

It follows that

$$\mathbb{S}(\widetilde{M}^\lambda) \cong \bigwedge^{\lambda_1}(V) \otimes \cdots \otimes \bigwedge^{\lambda_t}(V) = \widetilde{M}^\lambda(V).$$

3.3 Specht modules

The irreducible representations of the symmetric group S_n can be indexed by the partitions $\lambda \vdash n$ and constructed as follows:

If T is any numbering of shape λ then its **Young centralizer** is the element

$$c_T := \sum_{q \in C(T)} \text{sgn}(q)q \in \mathbb{C}[S_n].$$

The **Specht module** S^λ is the subspace of M^λ spanned by the elements $v_T := c_T \cdot [T]$ as T ranges through the numbering of shape λ . It holds that $\sigma \cdot v_T = v_{\sigma.T}$ for every $\sigma \in S_n$ and numbering T of shape λ whence S^λ is a subrepresentation of M^λ .

Theorem 7 (Classification of irreducible representations of S_n).

- (1) The representations S^λ with $\lambda \vdash n$ are pairwise non-isomorphic irreducible representations of the symmetric group S_n , and every irreducible representation of S_n is of this form.
- (2) The elements v_T where T ranges through the standard Young tableaux of shape λ form a basis of S^λ . Hence

$$\dim S^\lambda = \text{number of standard Young tableaux of shape } \lambda.$$

One can also construct the Specht module S^λ as a quotient of the representation \widetilde{M}^λ : The map

$$\widetilde{M}^\lambda \rightarrow S^\lambda \quad [T] \mapsto v_T$$

is a well-defined surjective homomorphism of S_n -representations. Its kernel Q^λ can be described similarly to $Q^\lambda(V)$, and it follows that under the isomorphism $\mathbb{S}(\widetilde{M}^\lambda) \cong \widetilde{M}^\lambda(V)$ it follows that $\mathbb{S}(Q^\lambda) \cong Q^\lambda(V)$.

Theorem 8. For any partition $\lambda \vdash n$,

$$\mathbb{S}(S^\lambda) \cong S^\lambda(V).$$

Proof. By applying the (exact) functor $\mathbb{S}(-)$ to the short exact sequence

$$0 \rightarrow Q^\lambda \rightarrow \widetilde{M}^\lambda \rightarrow S^\lambda \rightarrow 0$$

we get the short exact sequence

$$0 \rightarrow \mathbb{S}(Q^\lambda) \rightarrow \mathbb{S}(\widetilde{M}^\lambda) \rightarrow \mathbb{S}(S^\lambda) \rightarrow 0$$

and find that

$$\mathbb{S}(S^\lambda) \cong \mathbb{S}(\widetilde{M}^\lambda)/\mathbb{S}(Q^\lambda) \cong \widetilde{M}^\lambda(V)/Q^\lambda(V) \cong S^\lambda(V),$$

proving the assertion. \square

Remark 9. If $\ell(\lambda) > m$, i.e. if the Young diagram of λ has more than $m = \dim(V)$ rows, then $\mathbb{S}(S^\lambda) = S^\lambda(V) = 0$. If $\ell(\lambda) \leq m$ then $\mathbb{S}(S^\lambda) = S^\lambda(V)$ is again an irreducible representation.

Corollary 10. For any $n \geq 0$, $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f_\lambda}$ where f_λ denotes the number of standard Young tableaux of shape λ .

Proof. We find with Theorem 7 and Maschke's theorem that

$$\mathbb{C}[\mathbb{S}_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f_\lambda}$$

with

$$\begin{aligned} f_\lambda &= \text{multiplicity of } S_\lambda \text{ in } \mathbb{C}[\mathbb{S}_n] \\ &= \text{dimension of } S_\lambda \\ &= \text{number of standard Young tableaux of shape } \lambda \end{aligned}$$

where the second equality follows (for example) from the Artin–Wedderburn theorem. It follows that

$$\begin{aligned} V^{\otimes n} &= V^{\otimes n} \otimes_{\mathbb{C}[\mathbb{S}_n]} \mathbb{C}[\mathbb{S}_n] \\ &\cong \mathbb{S}(\mathbb{C}[\mathbb{S}_n]) \\ &\cong \mathbb{S}\left(\bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f_\lambda}\right) \\ &\cong \bigoplus_{\lambda \vdash n} \mathbb{S}(S^\lambda)^{\oplus f_\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f_\lambda} \end{aligned}$$

as claimed. \square

Example 11. The partitions of $n = 2$ are $\lambda = (1, 1)$ and $\mu = (2)$. Each of those partitions admits precisely one standard Young tableau:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

It follows that $f_\lambda = f_\mu = 1$ and therefore

$$V \otimes V = V^{\otimes 2} \cong S^\lambda(V)^{\oplus f_\lambda} \oplus S^\mu(V)^{\oplus f_\mu} \cong \text{Sym}^2(V) \oplus \bigwedge^2(V).$$

Remark 12. The Schur functor $\mathbb{S} = V^{\otimes n} \otimes_{\mathbb{C}[S_n]} (-)$ admits a right adjoint given by $\mathbb{S}' = \text{Hom}_{\text{GL}(V)}(V^{\otimes n}, -)$. We have for every partition λ with $\lambda \vdash n$ and $\ell(\lambda) \leq m$ and every finite-dimensional polynomial $\text{GL}(V)$ -representation W that

$$\begin{aligned} \text{multiplicity of } S^\lambda \text{ in } \mathbb{S}'(W) &= \dim \text{Hom}_{\mathbb{S}_n}(S^\lambda, \mathbb{S}'(W)) \\ &= \dim \text{Hom}_{\text{GL}(V)}(\mathbb{S}(S^\lambda), W) \\ &= \dim \text{Hom}_{\text{GL}(V)}(S^\lambda(V), W) \\ &= \text{multiplicity of } S^\lambda(V) \text{ in } W \end{aligned}$$

We see in particular that $\mathbb{S}'(S^\lambda(V)) \cong S^\lambda$.

4 Translation into rings

4.1 The Grothendieck ring of $\text{GL}(V)$

Recall that the Grothendieck group $K_0(\mathcal{A})$ of an abelian category \mathcal{A} is generated by the isomorphism classes of objects of \mathcal{A} subject to the relation $[B] = [A] + [C]$ for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} . Then in particular

$$[A] + [B] = [A \oplus B] \tag{1}$$

for all $A, B \in \mathcal{A}$. If the abelian category \mathcal{A} is semisimple, i.e. if every short exact sequence in \mathcal{A} splits, then the relations (1) suffice to construct $K_0(\mathcal{A})$.

Let \mathcal{A} be the semisimple category of finite-dimensional polynomial representations of $\text{GL}(V)$. We abbreviate $K_0(\text{GL}(V)) := K_0(\mathcal{A})$. This becomes a ring when endowed with the multiplication

$$[V] \cdot [W] = [V \otimes W].$$

It follows from last week's classification of irreducible finite-dimensional polynomial $\text{GL}(V)$ -representations that $K_0(\text{GL}(V))$ admits a \mathbb{Z} -basis given by the isomorphism classes $[S^\lambda(V)]$ where λ ranges through all partitions with $\ell(\lambda) \leq m$.

The character of a $\text{GL}(V)$ -representation results in a well-defined ring homomorphism

$$\text{Char}: K_0(\text{GL}(V)) \rightarrow \Lambda(m)$$

where $\Lambda(m)$ denotes the ring of symmetric polynomials in the variables x_1, \dots, x_m .

4.2 The representation ring of S_n

For every $n \geq 0$ let R_n be the Grothendieck group of $S_n\text{-rep}$, the category of finite-dimensional S_n -representations. We can define on $R := \bigoplus_{n \geq 0} R_n$ a multiplication via

$$[E] \cdot [F] = [E \circ F].$$

Note that $1_R = [\mathbb{1}_{S_0}]$ is multiplicative neutral and that this multiplication is associative and commutative by Lemma 5. The multiplication is also distributive and $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$. Hence R becomes a graded ring.

The category $S_n\text{-rep}$ is semisimple by Maschke's theorem and it follows from Theorem 7 that R_n admits a \mathbb{Z} -basis given by the isomorphism classes $[S^\lambda]$ with $\lambda \vdash n$. The ring R hence admits a \mathbb{Z} -basis given by the isomorphism classes $[S^\lambda]$ where λ ranges through all partitions (of all natural numbers).

The additivity of the Schur functor(s) $\mathbb{S}: S_n\text{-rep} \rightarrow \mathcal{A}$ gives for every $n \geq 0$ a group homomorphism $R_n \rightarrow K_0(\text{GL}(V))$, which together give a group homomorphism $R \rightarrow K_0(\text{GL}(V))$. It follows from Lemma 6 that this is a ring homomorphism. By abuse of notation we denote this homomorphism again by \mathbb{S} .

We have argued in Remark 9 that the kernel of $\mathbb{S}: R \rightarrow K_0(\text{GL}(V))$ is spanned by those $[S^\lambda]$ with $\ell(\lambda) > m$, whereas all other basis vector $[S^\lambda]$ with $\ell(\lambda) \leq m$ are mapped bijectively onto the basis $[S^\lambda(V)]$ of $K_0(\text{GL}(V))$.

4.3 The ring of symmetric functions

For every $k \geq 0$ let $\Lambda(k)$ denote the ring of symmetric polynomials, a subring of $\mathbb{Z}[x_1, \dots, x_k]$.

When dealing with symmetric polynomials it often happens that the number of variables, k , does not matter: One has a family $(f_k)_{k \geq 0}$ of symmetric polynomials $f_k \in \Lambda(k)$ such that $f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1})$ for every $k \geq 1$, and for every $k \geq 1$ an identity involving f_k which reduces for $x_k \rightarrow 0$ to the identity involving f_{k-1} .

Example 13. The elementary symmetric polynomials $e_n(x_1, \dots, x_k)$ and the completely symmetric polynomials $h_n(x_1, \dots, x_k)$ are for every $k \geq 1$ related by the formulas

$$\sum_{i=0}^s (-1)^i h_{s-i}(x_1, \dots, x_k) e_i(x_1, \dots, x_k) = 0$$

where $s \geq 0$. The identities in $k-1$ variables follows from the corresponding ones in k variables by setting $x_k \rightarrow 0$.

To formalize this phenomenon we introduce the **ring of symmetric functions** Λ : For every degree $n \geq 0$ we set

$$\begin{aligned} \Lambda_n &:= \left\{ (f_k)_{k \geq 0} \left| \begin{array}{l} f_k \in \Lambda_n(k) \text{ with} \\ f_k(x_1, \dots, x_{k-1}, 0) = f_{k-1}(x_1, \dots, x_{k-1}) \\ \text{for every } k \geq 1 \end{array} \right. \right\} \\ &= \lim(\Lambda_n(0) \leftarrow \Lambda_n(1) \leftarrow \Lambda_n(2) \leftarrow \Lambda_n(3) \leftarrow \dots) \end{aligned}$$

where $\Lambda_n(k) \rightarrow \Lambda_n(k-1)$ is the group homomorphism given by setting $x_k \rightarrow 0$. We combine these groups into a graded ring $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ with multiplication given by

$$(f_k)_{k \geq 0} \cdot (g_k)_{k \geq 0} := (f_k g_k)_{k \geq 0}.$$

Example 14. For every $n \geq 0$ we have an element

$$e_n := (e_n(), e_n(x_1), e_n(x_1, x_2), e_n(x_1, x_2, x_3), \dots) \in \Lambda_n$$

and similarly elements $h_n, p_n \in \Lambda_n$. We get for every partition $\lambda = (\lambda_1, \dots, \lambda_t)$ an induced element

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_t} \in \Lambda_{|\lambda|}$$

and similarly elements $h_\lambda, p_\lambda \in \Lambda_{|\lambda|}$.

Remark 15.

(1) We have that

$$\Lambda = \lim(\Lambda(0) \leftarrow \Lambda(1) \leftarrow \Lambda(2) \leftarrow \Lambda(3) \leftarrow \dots)$$

in the category of graded rings.

(2) The elements of the ring Λ are not functions, despite its name.

Example 16 (Schur polynomials and Schur functions). Let λ be a partition. For every semistandard Young tableaux T of shape λ with entries in $\{1, \dots, k\}$ let

$$x_T := \prod_{i \in T} x_i.$$

The **Schur polynomial** $s_\lambda(x_1, \dots, x_k)$ is defined as

$$s_\lambda(x_1, \dots, x_k) = \sum_{T'} x_{T'} \in \mathbb{Z}[x_1, \dots, x_n]$$

where T' ranges through the semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$.

If for example $\lambda = (2, 2)$ and $k = 3$ then the semistandard Young tableaux are as follows:

1	1	1	1	1	2	2	2
2	2	2	3	3	2	3	3

The Schur polynomial $s_{(2,2)}(x_1, x_2, x_3)$ is therefore given by

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1^2 x_3^2 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_2^2 x_3^2.$$

We observe the following:

(1) The Schur polynomial $s_\lambda(x_1, \dots, x_k)$ is homogeneous of degree $|\lambda|$.

- (2) If $\ell(\lambda) > k$, i.e. if the Young diagram of λ has more than k rows, then the Schur polynomial $s_\lambda(x_1, \dots, x_k)$ vanishes since there exist no semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$. (We don't have enough entries to make the first column strictly increasing, which is required for a semistandard Young tableaux.)
- (3) The Specht module $S^\lambda(V)$ has a basis e_T where T ranges through the semistandard Young tableaux with entries in $\{1, \dots, m\}$. Each e_T is a weight vector with corresponding weight x_T . Hence

$$s_\lambda(x_1, \dots, x_m) = \text{Char}(S^\lambda(V))(x_1, \dots, x_m).$$

This shows in particular that $s_\lambda(x_1, \dots, x_m)$ is a symmetric polynomial. We also see again that $s_\lambda(x_1, \dots, x_m) = 0$ if $\ell(\lambda) > m$ since then $S^\lambda(V) = 0$.

- (4) It holds that $s_\lambda(x_1, \dots, x_{k-1}, 0) = s_\lambda(x_1, \dots, x_{k-1})$.

We find that we get a well-defined element $s_\lambda \in \Lambda_{|\lambda|}$, the **Schur function**.

Proposition 17. For every $k \geq 0$ the ring of symmetric polynomials $\Lambda(k)$ has the Schur polynomials $s_\lambda(x_1, \dots, x_k)$ with $\ell(\lambda) \leq k$ as a basis.

Corollary 18. The ring homomorphism $\text{Char}: K_0(\text{GL}(V)) \rightarrow \Lambda(m)$ is an isomorphism.

Proof. The basis $[S^\lambda(V)]$ of $K_0(\text{GL}(V))$ is mapped to the basis $s_\lambda(x_1, \dots, x_m)$ of $\Lambda(m)$, both indexed with $\ell(\lambda) \leq m$. \square

We have for every $k \geq 0$ a homomorphism of graded rings

$$\Lambda \rightarrow \Lambda(k), \quad f \mapsto f(x_1, \dots, x_k)$$

that assigns to f the entailed symmetric polynomial in k variables. An equality $f = g$ holds in Λ if and only if for every $k \geq 0$ the equality $f(x_1, \dots, x_k) = g(x_1, \dots, x_k)$ hold.

Example 19. The equality $\sum_{i=0}^s (-1)^i h_{s-i} e_i = 0$ with $s \geq 0$ hold in Λ .

Proposition 20.

- (1) The symmetric functions e_1, e_2, \dots generate Λ and are algebraically independent.
- (2) The monomials e_λ where λ ranges through all partitions form a \mathbb{Z} -basis of Λ .
- (3) The symmetric functions h_1, h_2, \dots generate Λ and are algebraically independent.
- (4) The monomials h_λ where λ ranges through all partitions form a \mathbb{Z} -basis of Λ .
- (5) The symmetric functions s_λ where λ ranges through all partitions form a \mathbb{Z} -basis of Λ .

The mapping $e_i(x_1, \dots, x_k) \mapsto e_i(x_1, \dots, x_{k+1})$ with $i = 0, \dots, k$ extends to an embedding of rings $\Lambda(k) \rightarrow \Lambda(k+1)$, and the above shows that one can regard Λ is the resulting colimit, i.e. as the polynomial ring $\mathbb{Z}[e_1, e_2, e_3, \dots]$. Similarly for h_i instead of e_i .

Theorem 21. Let $\Phi: \Lambda \rightarrow R$ be the unique additive group homomorphism that maps the basis element e_λ to the element $[\widetilde{M}^\lambda]$ where λ ranges through all partitions.

- (1) The map Φ is an isomorphism of rings.
- (2) It holds that $\Phi(h_\lambda) = [M^\lambda]$.
- (3) It holds that $\Phi(s_\lambda) = [S^\lambda]$.

The multiplicity of Φ stems from the identity $\widetilde{M}^\lambda = \mathbb{U}_{\lambda_1} \circ \dots \circ \mathbb{U}_{\lambda_t}$ for $\lambda = (\lambda_1, \dots, \lambda_t)$.

5 Representations and symmetric polynomials

Corollary 22. The composition

$$\Lambda \xrightarrow{\Phi^{-1}} R \xrightarrow{\mathbb{S}} K_0(\mathrm{GL}(V)) \xrightarrow{\mathrm{Char}} \Lambda(m)$$

is given by $f \mapsto f(x_1, \dots, x_m)$.

Proof. The assertion holds for the basis elements s_λ of Λ as

$$s_\lambda \mapsto [S^\lambda] \mapsto [\mathbb{S}(S^\lambda)] = [S^\lambda(V)] \mapsto s_\lambda(x_1, \dots, x_m).$$

The general assertion follow by additivity of all occuring maps. □

We have thus finally arrived at the following commutative diagram of rings:

$$\begin{array}{ccc} R & \xleftarrow{\sim} & \Lambda \\ \mathbb{S} \downarrow & & \downarrow f \mapsto f(x_1, \dots, x_m) \\ K_0(\mathrm{GL}(V)) & \xrightarrow[\mathrm{Char}]{\sim} & \Lambda(m) \end{array}$$

We have the following special cases of this diagram:

$$\begin{array}{ccc} [S^\lambda] & \xleftarrow{\quad} & s_\lambda \\ \downarrow & & \downarrow \\ [S^\lambda(V)] & \longleftrightarrow & s_\lambda(x_1, \dots, x_n) \end{array}$$

$$\begin{array}{ccc} [\widetilde{M}^\lambda] & \xleftarrow{\quad} & e_\lambda \\ \downarrow & & \downarrow \\ [\widetilde{M}^\lambda(V)] & \longleftrightarrow & e_\lambda(x_1, \dots, x_n) \end{array} \quad \begin{array}{ccc} [M^\lambda] & \xleftarrow{\quad} & h_\lambda \\ \downarrow & & \downarrow \\ [M^\lambda(V)] & \longleftrightarrow & h_\lambda(x_1, \dots, x_n) \end{array}$$

We can now use these correspondences to translate between problems about the representation theory of S_n , the representation theory of $\mathrm{GL}(V)$, and the combinatorics of symmetric polynomials.

Example 23.

- (1) For every $n \geq 0$ there exists unique natural numbers f^λ for $\lambda \vdash n$ such that one and thus all of the following conditions hold:

- a) $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus f^\lambda}$,
- b) $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda(V)^{\oplus f^\lambda}$,
- c) $(x_1 + \dots + x_m)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda(x_1, \dots, x_m)$ for every $m \geq 0$,
- d) $e_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$ in Λ .

We have already seen that it follows from the first description that f^λ is given by the number of standard Young tableaux of shape λ .

- (2) For every partition λ there exist unique natural numbers $K_{\mu, \lambda}$ for $\mu \triangleleft \lambda$ such that one and thus all of the following conditions hold:

- a) $M^\lambda \cong S^\lambda \oplus \bigoplus_{\mu \triangleleft \lambda} (S^\mu)^{\oplus K_{\mu, \lambda}}$,
- b) $M^\lambda(V) \cong S^\lambda(V) \oplus \bigoplus_{\mu \triangleleft \lambda} S^\mu(V)^{\oplus K_{\mu, \lambda}}$,
- c) $h_\lambda(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m) + \sum_{\mu \triangleleft \lambda} K_{\mu, \lambda} s_\mu(x_1, \dots, x_m)$ for all $m \geq 0$,
- d) $h_\lambda = s_\lambda + \sum_{\mu \triangleleft \lambda} K_{\mu, \lambda} s_\mu$.

The numbers $K_{\mu, \lambda}$ are the **Kostka numbers**.

- (3) For every partition λ there exist unique natural numbers $K_{\tilde{\mu}, \lambda}$ such that one and thus all of the following conditions hold:

- a) $\widetilde{M}^\lambda \cong S^{\bar{\lambda}} \oplus \bigoplus_{\tilde{\mu} \triangleleft \lambda} (S^\mu)^{\oplus K_{\tilde{\mu}, \lambda}}$,
- b) $\widetilde{M}^\lambda(V) \cong S^{\bar{\lambda}}(V) \oplus \bigoplus_{\tilde{\mu} \triangleleft \lambda} S^\mu(V)^{\oplus K_{\tilde{\mu}, \lambda}}$,
- c) $e_\lambda(x_1, \dots, x_m) = s_{\bar{\lambda}}(x_1, \dots, x_m) + \sum_{\tilde{\mu} \triangleleft \lambda} K_{\tilde{\mu}, \lambda} s_\mu(x_1, \dots, x_m)$,
- d) $e_\lambda = s_{\bar{\lambda}} + \sum_{\tilde{\mu} \triangleleft \lambda} K_{\tilde{\mu}, \lambda} s_\mu$.

The numbers $K_{\mu, \lambda}$ are again the Kostka numbers as above.

- (4) For any two partitions λ and μ there exist natural numbers $c_{\lambda, \mu}^\nu$ for $\nu \vdash |\lambda| + |\mu|$ such that one and thus all of the following conditions hold:

- a) $S^\lambda \circ S^\mu \cong \sum_\nu (S^\nu)^{\oplus c_{\lambda, \mu}^\nu}$,
- b) $S^\lambda(V) \otimes S^\mu(V) \cong \sum_\nu S^\nu(V)^{\oplus c_{\lambda, \mu}^\nu}$,
- c) $s_\lambda(x_1, \dots, x_m) s_\mu(x_1, \dots, x_m) = \sum_\nu c_{\lambda, \mu}^\nu s_\nu(x_1, \dots, x_m)$,
- d) $s_\lambda s_\mu = \sum_\nu c_{\lambda, \mu}^\nu s_\nu$.

The numbers $c_{\lambda, \mu}^\nu$ are the **Littlewood–Richardson coefficients**.