Differential Graded Hopf Algebras I

Jendrik Stelzner

In the following k denotes a field. All vector spaces, all kinds of algebras, all tensor products, etc. are over k, unless otherwise stated. All occuring maps are linear unless otherwise stated. We will sometime assume additional constraints on the characteristic of k, but will make this explicit when it occurs.

1 Preliminary Notions and Notations

A **graded vector space** is a vector space V together with a grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$. The elements $v \in V_n$ are **homogeneous** of **degree** $|v| = n$.

$$\boxed{\text{Whenever we write } |v| \text{ the element } v \text{ is assumed to be homogeneous.}}$$

A map $f \colon V \to W$ between graded vector spaces is **graded** of **degree** $|f| = d$ if $f(V_n) \subseteq V_{n+d}$ for all n. A **differential** on V is a map $V \to V$ of degree -1 with $d^2 = 0$. A **dg-vector space** is a graded vector space together with a differential, i.e. a chain complex. A dg-subspace is a chain subcomplex and the usual notions of quotients, direct sums and products and morphisms apply. We will always regard graded objects as differential graded objects with zero differential.

$$\boxed{\text{graded} \iff \text{differential graded with } d = 0}$$

Hence every statement about dg-objects entails a statement about graded objects.

If V and W are graded vector spaces then $V \otimes W$ is again a graded vector space with $|v \otimes w| = |v| + |w|$, i.e. $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. The **twist map** $\tau \colon V \otimes W \to W \otimes V$ is given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v\,.$$

We hence adhere to the Koszul-Quillen **sign convention**:

$$\boxed{\text{Whenever elements } x,\, y \text{ are swapped, the sign } (-1)^{|x||y|} \text{ is introduced.}}$$

If $f\colon V \to V'$ and $g\colon W \to W'$ are graded maps then $f \otimes g\colon V \otimes V' \to W \otimes W'$ is the graded map of degree $|f \otimes g| = |f| + |g|$ with

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)\,.$$

If V, W are dg-vector space then $V \otimes W$ is a dg-vector space with $d_{V\otimes W} = d \otimes \mathrm{id} + \mathrm{id} \otimes d$; more explicitely,

$$d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w)\,.$$

Higher tensor products are defined inductively. The twist map τ is an isomorphism of dg-vector spaces.[1] We regard the ground field k as a dg-vector space concentrated in degree 0. Then the natural isomorphism $k \otimes V \cong V$ and $V \otimes k \cong V$ are isomorphism of dg-vector spaces. The dg-vector space $\mathrm{Hom}(V, W)$ is given by

$$\mathrm{Hom}(V, W)_n = \{\text{graded maps } V \to W \text{ of degree } n\}$$

with differential

$$d(f) = d \circ f - (-1)^{|f|} f \circ d\,.$$

If V, W are dg-vector spaces then the **algebraic Künneth isomorphism** is the natural isomorphism of graded vector spaces

$$\mathrm{H}(V \otimes W) \cong \mathrm{H}(V) \otimes \mathrm{H}(W)\,, \quad [v \otimes w] \leftarrow\!\shortmid [v] \otimes [w]\,.$$

2 Differential Graded Algebras

Definition 2.1. A **differential graded algebra** or **dg-algebra** is a dg-vector space A together with morphisms of dg-vector spaces $m\colon A \otimes A \to A$ and $u\colon k \to A$ that make the diagrams

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes A \arrow[d, "u\otimes\mathrm{id}"'] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & A \otimes k \arrow[d, "\mathrm{id}\otimes u"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}$$

commute. The dg-algebra A is **graded commutative** if the diagram

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

commutes. A **morphism** of dg-algebras $f\colon A \to B$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & B \otimes B \arrow[d, "m"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& k \arrow[dl, "u"'] \arrow[dr, "u"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}$$

[1] The naive twist map $v \otimes w \mapsto w \otimes v$ is not a morphism of dg-vector spaces.

Definition 2.2. A graded map $\delta\colon A \to A$ for a graded algebra A is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\,;$$

more explicitely,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b)\,.$$

Remark 2.3.

(1) A dg-algebra is the same as a graded algebra A together with a differential d such that $d(1) = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)\,,$$

i.e. such that d is a graded derivation (of degree -1).

$$\begin{tikzcd}
\begin{array}{c}\text{graded}\\ \text{vector spaces}\end{array} \arrow[r, "\text{multiplication}"] \arrow[d, "\text{differential}"'] & \begin{array}{c}\text{graded}\\ \text{algebras}\end{array} \arrow[d, "\text{differential}"] \\
\text{dg-vector spaces} \arrow[r, "\text{multiplication}"'] & \text{dg-algebras}
\end{tikzcd}$$

(2) The graded commutativity of A means $ab = (-1)^{|a||b|}ba$. If $|a|$ or $|b|$ is even then $ab = ba$; if $|a|$ is odd and $\mathrm{char}(k) \neq 2$ then $a^2 = 0$.

(3) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

Examples 2.4.

(1) Every algebra A is a dg-algebra concentrated in degree 0. This holds in particular for $A = k$.

(2) If V is any dg-vector space then the algebra structure of $\mathrm{End}_k(V)$ restricts to a dg-algebra structure on $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

(3) If V is a dg-vector space then $\mathrm{T}(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space with

$$|v_1 \cdots v_n| = |v_1| + \cdots + |v_n|$$

and

$$d(v_1 \cdots v_n) = \sum_{i=1}^{n} (-1)^{|v_1| + \cdots + |v_i|} v_1 \cdots d(v_i) \cdots v_n\,.$$

This makes the tensor $\mathrm{T}(V)$ into a dg-algebra. The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector spaces and if A is any other dg-algebra and $f\colon V \to A$

any morphism of dg-vector spaces then f extends uniquely to a morphism of dg-algebras $F\colon \mathrm{T}(V) \to A$:

$$\begin{tikzcd} \mathrm{T}(V) \arrow[r, dashed, "F"] & A \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

(4) If V is any vector space then the symmetric algebra $\mathrm{S}(V)$ is a graded algebra and a commutative algebra, but not a graded commutative algebra. The exterior algebra $\bigwedge(V)$ is a graded algebra, it is in general not a commutative algebra (unless $\dim V \leq 1$), but it is a graded commutative algebra.

Lemma 2.5. Let A, B be dg-algebras.

(1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$m_{A\otimes B}\colon A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{m\otimes m} A \otimes B$$
$$u_{A\otimes B}\colon k \xrightarrow{\sim} k \otimes k \xrightarrow{u\otimes u} A \otimes B\,.$$

More explicitely, $1_{A\otimes B} = 1_A \otimes 1_B$ and $(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$.

(2) If $f\colon A \to A'$ and $g\colon B \to B'$ are morphism of dg-algebras then so is $f \otimes g$.

(3) The twist map $\tau\colon A \otimes B \to B \otimes A$ is a morphism of dg-algebras.

(4) If $A = (A, m, u)$ then $A^{\mathrm{op}} = (A, m^{\mathrm{op}}, u)$ with $m^{\mathrm{op}} = m \circ \tau$ is again a dg-algebra. $\square$

Warning 2.6. If A, B are dg-algebras then the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B. The underlying algebra of A^{op} is not the opposite of the underlying algebra of A. (Both thanks to signs.)

Definition 2.7. A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.

Lemma 2.8. For an ideal I is a dg-algebra A the following conditions are equivalent:

(1) I is a dg-ideal.

(2) I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α.

Proof. That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, pp. IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α: The ideal I is spanned by $ax_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|} ad(x_\alpha)b + (-1)^{|a|+|x_\alpha|} ax_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$. $\square$

Lemma 2.9. If I is a dg-ideal in a dg-algebra A then A/I inherits the structure of a dg-algebra such that the projection $A \to A/I$ is a morphism of dg-algebras. $\square$

Definition 2.10. If A is a dg-algebra then the **dg-comutator** is given by

$$[a,b] \coloneqq ab - (-1)^{|a||b|} ba\,.$$

Example 2.11. Let V be a dg-vector space. The ideal

$$I \coloneqq \big([v,w] \,\big|\, v,w \in V \text{ are homogeneous}\big)$$

is a dg-ideal in $\mathrm{T}(V)$ since the generators $[v,w]$ are homogeneous with

$$d([v,w]) = [d(v),w] + (-1)^{|v|}[v,d(w)] \in I\,.$$

The dg-algebra $\Lambda(V) \coloneqq \mathrm{T}(V)/I$ is the **differential graded symmetric algebra** on V. If S is any other graded symmetric dg-algebra and $f\colon V \to S$ any morphism of dg-vector space then f extends uniquely to a morphism of dg-algebras $F\colon\ \Lambda(V) \to S$:

$$\begin{tikzcd}
\Lambda(V) \arrow[r, dashed, "F"] & S \\
V \arrow[u] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proposition 2.12. If A is a dg-algebra then $\mathrm{Z}(A)$ is a graded subalgebra of A, $\mathrm{B}(A)$ is a graded ideal in $\mathrm{Z}(A)$ and $\mathrm{H}(A)$ is hence a graded algebra.

3 Differential Graded Coalgebras

Definition 3.1. A **differential graded coalgebra** or **dg-coalgebra** is a dg-vector space C together with morphisms of dg-vector space $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ that make the diagrams

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\
C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\
k \otimes C \arrow[r, "\sim"] & C & C \otimes k \arrow[l, "\sim"']
\end{tikzcd}$$

commute. The dg-coalgebra C is **graded cocommutative** if the following diagram commutes:

$$\begin{tikzcd}
& C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[rr, "\tau"] & & C \otimes C
\end{tikzcd}$$

A **morphism** of dg-coalgebra $f\colon C \to D$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta"'] & D \arrow[d, "\Delta"] \\
C \otimes C \arrow[r, "f\otimes f"] & D \otimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\
& k &
\end{tikzcd}$$

Definition 3.2. A graded map $\omega\colon C \to C$ of a graded coalgebra is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \mathrm{id} + \mathrm{id} \otimes \omega) \circ \Delta\,;$$

more explicitely,

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)})\,.$$

Remark 3.3.

(1) A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that d vanishes on the zero boundaries and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)})\,,$$

i.e. such that d is a graded coderivation of degree -1.

(2) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}\,.$$

(3) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.

(4) Every coalgebra C is a dg-coalgebra centered in degree 0. This holds in particular for $C = k$.

Example 3.4. Let V be a dg-vector space. Then the induced dg-vector space $\mathrm{T}(V)$ becomes a dg-coalgebra with the deconcatination

$$\Delta\colon\ \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v_1 \cdots v_n \mapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n\,,$$
$$\varepsilon\colon\ \mathrm{T}(V) \to k\,, \quad v_1 \cdots v_n \mapsto \delta_{n0}\,.$$

Lemma 3.5. Let C, D be dg-coalgebras.

(1) The tensor product $C \otimes D$ becomes a dg-coalgebra with

$$\Delta_{C\otimes D}\colon C \otimes D \xrightarrow{\Delta\otimes\Delta} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} C \otimes D \otimes C \otimes D$$
$$\varepsilon_{C\otimes D}\colon C \otimes D \xrightarrow{\varepsilon\otimes\varepsilon} k \otimes k \xrightarrow{\sim} k$$

(2) If $f\colon C \to C'$ and $g\colon D \to D'$ are morphism of dg-coalgebras then so is $f \otimes g$.

(3) The twist map $\tau\colon C \otimes D \to D \otimes C$ is a morphism of dg-coalgebras.

(4) If $C = (C, \Delta, \varepsilon)$ then $C^{\mathrm{cop}} = (C, \Delta^{\mathrm{cop}}, \varepsilon)$ with $\Delta^{\mathrm{op}} = \tau \circ \Delta$ is again a dg-coalgebra.

Warning 3.6. If C, D are dg-coalgebras then the underlying coalgebra of $C \otimes D$ is not the tensor product of the underlying coalgebras of C and D. The underlying coalgebra of C^{op} is not the coopposite of the underlying coalgebra of C. (Again both thanks to signs.)

Definition 3.7. A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.

Lemma 3.8. If I is a dg-coideal in a dg-coalgebra C then C/I inherits the structure of a dg-coalgebra such that the projection $C \to C/I$ is a morphism of dg-coalgebra. $\square$

Proposition 3.9. If C is a dg-coalgebra then $\mathrm{Z}(C)$ is a graded subcoalgebra of A, $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ and $\mathrm{H}(C)$ is hence a graded coalgebra. $\square$

4 Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, let (B, m, u) be a dg-algebra and let (B, Δ, ε) be a dg-coalgebra. Then the following conditions are equivalent:

(1) Δ and ε are morphisms of dg-algebras.

(2) m and u are morphisms of dg-coalgebras.

Proof. The same diagramatic proof as in the non-dg case. $\square$

Definition 4.2. A **dg-bialgebra** is a quintuple $(B, \mu, u, \Delta, \varepsilon)$ such that the equivalent conditions of Lemma 4.1 are satisfied. A map $f \colon B \to C$ is a **morphism** of dg-bialgebras if it is both a morphism of dg-algebras and of dg-coalgebras. A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)} c_{(1)} \otimes b_{(2)} c_{(2)}$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta \colon B \to B \otimes B$ is a morphism of dg-algebras where the algebra structure on $B \otimes B$ is given by $(b \otimes b') \cdot (b'' \otimes b''') = (-1)^{|b'||b''|} bb'' \otimes b'b'''$. But it is in general not an algebra homomorphism with respect to the multiplication $(b \otimes b') \cdot (b'' \otimes b''') = bb'' \otimes b'b'''$.

$$\begin{tikzcd}
\text{algebras} & & \text{dg-algebras} \arrow[ll, dashed] & \\
& \text{bialgebras} \arrow[ul] \arrow[dl] & & \text{dg-bialgebras} \arrow[ll, dashed, "/" marking] \arrow[ul] \arrow[dl] \\
\text{coalgebras} & & \text{dg-coalgebras} \arrow[ll, dashed] &
\end{tikzcd}$$

We will see an explicit counterexample in Example 5.7.

Lemma 4.5. If B is a dg-bialgebra then B^{op}, B^{cop} and $B^{\mathrm{op,cop}}$ are again dg-bialgebras. $\square$

Lemma 4.6. If I is a dg-biideal in a dg-bialgebra B then B/I inherits from B the structure of a dg-bialgebra such that the projection $B \to B/I$ is a morphism of dg-bialgebra. $\square$

Proposition 4.7. If B is a dg-bialgebra then $\mathrm{Z}(B)$ is a graded sub-bialgebra of B, $\mathrm{B}(B)$ is a graded biideal in $\mathrm{Z}(B)$ and $\mathrm{H}(B)$ is hence a graded bialgebra. $\square$

Definition 4.8. If B is a dg-bialgebra then $x \in B$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Lemma 4.9. If B is a dg-bialgebra and $x, y \in B$ are primitive then $[x, y]$ is again primitive. $\square$

5 Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product on $\mathrm{Hom}_k(C, A)$ makes $\mathrm{Hom}(C, A)$ into a dg-algebra. $\square$

Definition 5.2. An **antipode** for a dg-bialgebra H is an inverse S to id_H with respect to the convolution product of $\mathrm{Hom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**. A **morphism** of dg-Hopf algebras is a morphism of dg-bialgebras. A **dg-Hopf ideal** in H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra does in general not have an underlying Hopf algebra structure.

Remark 5.4. Let H be a dg-Hopf algebra.

(1) The antipode of H is unique.

(2) The antipode of H is the the unique morphism of dg-vector spaces $S\colon H \to H$ that makes the following diagram commute:

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "S \otimes \mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[r, "u"] & H \\
 & H \otimes H \arrow[r, "\mathrm{id} \otimes S"'] & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd} \tag{1}$$

This means more explicitely that

$$\sum_{(c)} S(c_{(1)}) c_{(2)} = \varepsilon(c) 1_H \qquad \text{and} \qquad \sum_{(c)} c_{(1)} S(c_{(2)}) = \varepsilon(c) 1_H \,.$$

(No additional signs occur because $|S| = 0$.)

Lemma 5.5. If I is a dg-Hopf ideal in a dg-Hopf algebra H then H/I inherits from H the structure of a dg-Hopf algebra such that the projection $H \to H/I$ is a morphism of dg-Hopf algebras.

Example 5.6. Let V be a dg-vector space.

(1) The map
$$V \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v \mapsto v \otimes 1 + 1 \otimes v$$
is a morphism of dg-vector space and hence induces a morphism of dg-algebras
$$\Delta\colon\ \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,.$$
The zero map $V \to 0$ induces a morphism of dg-algebras $\varepsilon\colon\ \mathrm{T}(V) \to \mathrm{T}(0) = k$. These make $\mathrm{T}(V)$ into a dg-bialgebra; the necessary diagrams can be checked on the algebra generators V of $\mathrm{T}(V)$ because all arrows occuring in the bialgebra diagrams are now morphisms of dg-algebras. The comultiplication Δ and ε are explicitely given by
$$\begin{aligned}\Delta(v_1 \cdots v_n) &= \Delta(v_1) \cdots \Delta(v_n) \\ &= (v_1 \otimes 1 + 1 \otimes v_1) \cdots (v_n \otimes 1 + 1 \otimes v_n) \\ &= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} (-1)^{n_p(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}\end{aligned}$$
where
$$n_p(\sigma) = \sum \left\{ |v_i||v_j| \;\middle|\; 1 \leq i \leq p,\, p+1 \leq j \leq n,\, \sigma(i) > \sigma(j) \right\},$$
and the counit is given by
$$\varepsilon(v_1 \cdots v_n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$
The map
$$V \to \mathrm{T}(V)^{\mathrm{op}}\,, \quad v \mapsto -v$$
is a morphism of dg-vector spaces and hence induces a morphism of dg-algebras
$$S\colon\ \mathrm{T}(V) \to \mathrm{T}(V)^{\mathrm{op}}\,.$$
As a map $S\colon\ \mathrm{T}(V) \to \mathrm{T}(V)$ this is given by
$$S(v_1 \cdots v_n) = (-1)^{\sum_{1 \leq i < j \leq n} |v_i||v_j|} (-1)^n v_n \cdots v_1\,.$$
It can now be checked on monomials that S is an antipode for $\mathrm{T}(V)$, making it a dg-Hopf algebra.[2] This is the **differential graded tensor algebra** on V.

[2]In the resulting expressions the terms for $v_1 \cdots v_p \otimes v_{p+1} \cdots v_n$ and $v_2 \cdots v_p \otimes v_1 v_{p+1} \cdots v_n$ cancel out because of signs.

(2) The dg-algebra $\Lambda(V) = \mathrm{T}(V)/I$ from Example 2.11 inherits from $\mathrm{T}(V)$ the structure of a dg-Hopf algebra because the dg-ideal

$$I = \big([v,w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-Hopf ideal in $\mathrm{T}(V)$ since

$$\begin{aligned} \varepsilon([v,w]) &= 0\,, \\ \Delta([v,w]) &= [v,w] \otimes 1 + 1 \otimes [v,w] \in I \otimes \mathrm{T}(V) + \mathrm{T}(V) \otimes I\,, \\ S([v,w]) &= -[v,w] \in I\,. \end{aligned}$$

For the computation of Δ we use that v, w are primitive in $\mathrm{T}(V)$ and $[v,w]$ is therefore again primitive.

Example 5.7 (Exterior Algebra)**.** Let V be a vector space. We may regard V as a dg-vector space centered in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\operatorname{char} k \neq 2$ there exists no bialgebra structure on $\Lambda \coloneqq \bigwedge(V)$. Suppose otherwise.

Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker \varepsilon = \bigoplus_{d \geq 1} \bigwedge^n(V) =: I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{\Lambda \otimes I} \qquad \text{and} \qquad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes \Lambda}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}\,.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2}\,,$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}\,.$$

Now $v^2 = 0$, hence

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}\,.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

Proposition 5.8. If $\mathcal{H}$ is a dg-Hopf algebra with antipode S then the graded bialgebra $\mathrm{H}(\mathcal{H})$ is a graded Hopf algebra with antipode induced by S. □

Example 5.9. If V is a dg-vector space then

$$\mathrm{H}(\mathrm{T}(V)) = \mathrm{H}\Bigg(\bigoplus_{d \geq 0} V^{\otimes d}\Bigg) \cong \bigoplus_{d \geq 0} \mathrm{H}\big(V^{\otimes d}\big) \cong \bigoplus_{d \geq 0} \mathrm{H}(V)^{\otimes d} = \mathrm{T}(\mathrm{H}(V))$$

as graded vector spaces by the algebraic Künneth isomorphism. We see on representatives that this is already an isomorphism of graded Hopf algebras.

6 Differential Graded Lie Algebras

Let $\mathrm{char}(k) \neq 2$.

Recall 6.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a map $[-,-]\colon \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is skew-symmetric and for every $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation; the last assertion is equivalent to the Jacobi identity $\sum_{\text{cyclic}}[x,[y,z]] = 0$.

Definition 6.2. A **dg-Lie algebra** is a dg-vector space $\mathfrak{g}$ together with a morphism $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is **graded skew symmmetric** in the sense that the diagram

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\tau"] \arrow[dr, "{[-,-]}"'] & & \mathfrak{g} \otimes \mathfrak{g} \arrow[dl, "{-[-,-]}"] \\
& \mathfrak{g} &
\end{tikzcd}$$

commutes, and that for every homogeneous $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation (of degree $|x|$).

Remark 6.3. Let $\mathfrak{g}$ be a dg-Lie algebra. Then $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j and

$$[x,y] = -(-1)^{|x||y|}[y,x]$$

and

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]] \tag{2}$$

and

$$d([x,y]) = [d(x),y] + (-1)^{|x|}[x,d(y)]\,.$$

Condition (2) can be rewritten by the graded skew-symmetry of $[-,-]$ as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x,[y,z]] = 0\,.$$

Warning 6.4. A dg-Lie algebra does in general not have an underlying Lie algebra structure.

Example 6.5.

(1) Every dg-algebra A becomes a dg-Lie algebra with respect to the dg-comutator.

(2) If A is a graded algebra then the graded subspace $\mathrm{Der}(A)$ of $\mathrm{End}(A)$ given by

$$\mathrm{Der}(A)_n = \{\text{derivations of } A \text{ of degree } n\}$$

is a dg-Lie subalgebra of $\mathrm{End}(A)$.

(3) If B is a dg-bialgebra then the set of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B.

Lemma 6.6. If $\mathfrak{g}$ is a dg-Lie algebra then $\mathrm{Z}(\mathfrak{g})$ is a graded Lie subalgebra of $\mathfrak{g}$, $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$ and $\mathrm{H}(\mathfrak{g})$ is thus an graded Lie algebra.

Definition 6.7. The **universal enveloping algebra** of a dg-Lie algebra $\mathfrak{g}$ is a dg-algebra $\mathrm{U}(\mathfrak{g})$ together with a morphism of dg-Lie algebras $i\colon \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ such that for every other dg-algebra A and every morphism of dg-Lie algebras $f\colon \mathfrak{g} \to A$ there exists a unique morphism of dg-algebras $F\colon\ \mathrm{U}(\mathfrak{g}) \to A$ that extends f, i.e. that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{U}(\mathfrak{g}) \arrow[r, dashed, "F"] & A \\ \mathfrak{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 6.8. Every dg-Lie algebra $\mathfrak{g}$ admits a universal enveloping algebra. It is unique up to unique isomorphism and can be constructed as

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/\big([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \mid x,y \in \mathfrak{g} \text{ homogeneous}\big)$$

together with the composition $i\colon \mathfrak{g} \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$. It inherits from $\mathrm{T}(\mathfrak{g})$ the structure of a dg-Hopf algebra. □

Proof. We check that the given ideal I is a dg-Hopf ideal. It is generated by homegenous elements which satisfy

$$\begin{aligned} &d([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\ &= d([x,y]_{\mathrm{T}(\mathfrak{g})}) - d([x,y]_{\mathfrak{g}}) \\ &= [d(x),y]_{\mathrm{T}(\mathfrak{g})} + (-1)^{|x|}[x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}} - (-1)^{|x|}[x,d(y)]_{\mathfrak{g}} \\ &= \Big([d(x),y]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}}\Big) + (-1)^{|x|}\Big([x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [x,d(y)]_{\mathfrak{g}}\Big) \\ &\in I \end{aligned}$$

so I is a dg-ideal. Also

$$\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = \varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})}) - \varepsilon([x,y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are homogoneous of degree ≥ 1,

$$\begin{aligned} &\Delta([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\ &= \Delta([x,y]_{\mathrm{T}(\mathfrak{g})}) - \Delta([x,y]_{\mathfrak{g}})) \\ &= [x,y]_{\mathrm{T}(\mathfrak{g})} \otimes 1 + 1 \otimes [x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x,y]_{\mathfrak{g}} \\ &= ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\ &\in I \otimes \mathrm{T}(\mathfrak{g}) + \mathrm{T}(\mathfrak{g}) \otimes I \end{aligned}$$

and finally

$$S([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = S([x,y]_{\mathrm{T}(\mathfrak{g})}) - S([x,y]_{\mathfrak{g}}) = -[x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}} \in I\,.$$

Thus the dg-ideal I is already a dg-Hopf ideal. □

Remark 6.9. Let $\mathfrak{g}$, $\mathfrak{h}$ be a dg-Lie algebras.

(1) The product $\mathfrak{g} \times \mathfrak{h}$ is again a dg-Lie algebra with

$$[(x, y), (x', y')] = ([x, x'], [y, y']) .$$

The inclusions $\mathfrak{g}, \mathfrak{h} \to \mathfrak{g} \times \mathfrak{h}$ induce morphisms of dg-Hopf algebras

$$\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{h}) \to \mathrm{U}(\mathfrak{g} \times \mathfrak{h})$$

that results in an isomorphism of dg-Hopf algebras

$$\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{h}) \cong \mathrm{U}(\mathfrak{g} \times \mathfrak{h}) .$$

(2) The Hopf algebra structure of $\mathrm{U}(\mathfrak{g})$ is induced from underlying morphisms of dg-Lie algebras: The diagonal morphism $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, $v \mapsto (v, v)$ induces the comultiplication

$$\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g} \times \mathfrak{g}) \cong \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$$

the morphism $\mathfrak{g} \to 0$ induced the counit

$$\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(0) = k$$

and the morphism $\mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$, $v \mapsto -v$ induces the antipode

$$\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}^{\mathrm{op}}) = \mathrm{U}(\mathfrak{g})^{\mathrm{op}}$$

(3) The famous Poincaré–Birkhoff–Witt theorem generalizes to the universal enveloping algebras of dg-Lie algebras. It can be expressed as an isomorphism of dg-coalgebra $\Lambda(\mathfrak{g}) \cong \mathrm{U}(\mathfrak{g})$ and show that $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$. This can be found in [Qui69, Appendix B, Theorem 2.3] and [FHT01, §21(a)] for more details on this.

(4) It holds that $\mathrm{H}(\mathrm{U}(\mathfrak{g})) \cong \mathrm{U}(\mathrm{H}(\mathfrak{g}))$, see [Qui69, Appendix B, Proposition 2.1] or [FHT01, Theorem 21.7].

(5) If H is a graded cocommutative connected[3] dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong \mathrm{U}(\mathbb{P}(H))$, which results in an equivalence between the categories of dg-Lie algebras and graded cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B, Theorem 4.5].

7 Homology of the Primitive Part

Theorem 7.1 ([Lod92, Theorem A.9])**.** Let $\mathcal{H}$ be a dg-Hopf algebra. The inclusion $\mathbb{P}(\mathcal{H}) \to \mathcal{H}$ is a morphism of dg-Lie algebras and thus induced a morphism of graded Lie algebras $\mathrm{H}(\mathbb{P}(\mathcal{H})) \to \mathrm{H}(\mathcal{H})$. This morphism restricts to an isomorphism of graded Lie algebras $\mathrm{H}(\mathbb{P}(\mathcal{H})) \to \mathbb{P}(\mathrm{H}(\mathcal{H}))$.

[3] The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

References

[Bou89] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg New York, 1989, pp. xxiii+709. ISBN: 3-540-64243-9.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory*. Graduate Texts in Mathematics 205. Springer-Verlag New York, 2001, pp. xxxiii+539. ISBN: 978-0-387-95068-6. DOI: 10.1007/978-1-4613-0105-9.

[Lan02] Serge Lang. *Algebra*. Graduate Texts in Mathematics 211. Springer-Verlag New York, 2002, pp. xv+914. ISBN: 978-0-387-95385-4. DOI: 10.1007/978-1-4613-0041-0.

[Lod92] Jean-Lois Loday. *Cyclic Homology*. Grundlagen der mathematischen Wissenschaften 301. Springer Verlag Berlin-Heidelberg, 1992, pp. xix+516. ISBN: 978-3-662-21741-2. DOI: 10.1007/978-3-662-21739-9.

[MO18] David E Speyer. *When is the exterior algebra a Hopf algebra?* November 30, 2018. URL: https://mathoverflow.net/q/316544 (visited on May 7, 2019).

[Qui69] Daniel Quillen. “Rational homotopy theory”. In: *Ann. of Math. (2)* 90 (1969), pp. 205–295. ISSN: 0003-486X. DOI: 10.2307/1970725.