Jendrik Stelzner jendrikstelzner.de/seminars/dg_hopf.pdf May 20, 2019

Differential Graded Hopf Algebras I

In the following k denotes a field. All vector spaces, algebras, tensor products, etc. are over k, unless otherwise stated. All occuring maps are linear unless otherwise stated. Additional constraints on $\mathrm{char}(k)$ are made explicit when used.

1. Preliminary Notions and Notations

A **graded vector space** is a vector space V together with a **grading** $V = \bigoplus_{n \in \mathbb{Z}} V_n$. The elements $v \in V_n$ are **homogeneous** of **degree** $|v| = n$.

$$\boxed{\text{Whenever we write } |v| \text{ the element } v \text{ is assumed to be homogeneous.}}$$

A map $f\colon V \to W$ between graded vector spaces is **graded** of **degree** $d = |f|$ if $f(V_n) \subseteq V_{n+d}$ for all n. A **differential** on V is a map $V \to V$ of degree -1 with $d^2 = 0$. A **dg-vector space** is a graded vector space together with a differential, i.e. a chain complex; the usual definitions and assertions about chain complexes apply. A **dg-subspace** is a chain subcomplex. We always regard graded objects as differential graded objects with zero differential.

$$\boxed{\text{graded} \longleftrightarrow \text{differential graded with } d = 0}$$

If V, W are graded vector spaces then $V \otimes W$ is also one with $|v \otimes w| = |v| + |w|$, i.e. $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. The **twist map** $\tau\colon V \otimes W \to W \otimes V$ is given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v \,.$$

We hence adhere to the Koszul-Quillen **sign convention**:

$$\boxed{\text{Whenever homogeneous } x, y \text{ are swapped the sign } (-1)^{|x||y|} \text{ is introduced.}}$$

If V, W are dg-vector spaces then $\mathrm{Hom}(V, W)$ is the dg-vector space with

$$\begin{aligned} \mathrm{Hom}(V,W)_n &= \{\text{graded maps } V \to W \text{ of degree } n\}\,, \\ d(f) &= d \circ f - (-1)^{|f|} f \circ d\,. \end{aligned}$$

If $f\colon V \to V'$, $g\colon W \to W'$ are graded maps then $f \otimes g\colon V \otimes V' \to W \otimes W'$ is given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)\,;$$

in particular $|f \otimes g| = |f| + |g|$. If V, W are dg-vector spaces then $V \otimes W$ is a dg-vector space with $d_{V \otimes W} = d_V \otimes \mathrm{id} + \mathrm{id} \otimes d_W$; more explicitely,

$$d(v \otimes w) = d(v) \otimes w + (-1)^{|d||v|} v \otimes d(w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w)\,.$$

Higher tensor products $V_1 \otimes \cdots \otimes V_n$ are defined inductively. The twist map τ is an isomorphism of dg-vector spaces.[1] We regard k as a dg-vector space concentrated in degree 0.

2. Differential Graded Algebras

Definition 2.1. A **differential graded algebra** or **dg-algebra** is a dg-vector space A together with morphisms of dg-vector spaces $m\colon A \otimes A \to A$ and $u\colon k \to A$ that make the algebra diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[d, "u\otimes\mathrm{id}"'] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & A \otimes k \arrow[d, "\mathrm{id}\otimes u"] \\ A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

commute. The dg-algebra A is **graded commutative** if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

commutes. A **morphism** of dg-algebras $f\colon A \to B$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & B \otimes B \arrow[d, "m"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} & k \arrow[dl, "u"'] \arrow[dr, "u"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

Definition 2.2. A graded map $\delta\colon A \to A$ for a graded algebra A is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\,;$$

more explicitely,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b)\,.$$

[1] The naive twist map $v \otimes w \mapsto w \otimes v$ is not a morphism of dg-vector spaces.

Remark 2.3.

(1) A dg-algebra is the same as a graded algebra A (in particular $|1| = 0$) together with a differential d such that $d(1) = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)\,,$$

i.e. such that d is a graded derivation (of degree -1).

(2) The graded commutativity of A means $ab = (-1)^{|a||b|} ba$. If $|a|$ is even or $|b|$ is even then $ab = ba$; if $|a|$ is odd then $a^2 = -a^2$ and thus $a^2 = 0$ if $\mathrm{char}(k) \neq 2$.

(3) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

Examples 2.4.

(1) Every algebra A is a dg-algebra concentrated in degree 0, in particular $A = k$.

(2) For any dg-vector space V the algebra structure of $\mathrm{End}_k(V)$ restricts to a dg-algebra structure on $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

(3) If V is a dg-vector space then $\mathrm{T}(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space with

$$\begin{aligned} |v_1 \cdots v_n| &= |v_1| + \cdots + |v_n|\,, \\ d(v_1 \cdots v_n) &= \sum_{i=1}^{n} (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n\,. \end{aligned}$$

This makes $\mathrm{T}(V)$ into a dg-algebra, with multiplication given by concatination

$$(v_1 \cdots v_i) \cdot (v_{i+1} \cdots v_n) = v_1 \cdots v_n\,.$$

The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector spaces and if $f \colon V \to A$ is any morphism of dg-vector spaces into a dg-algebra A then f extends uniquely to a morphism of dg-algebras $F \colon\ \mathrm{T}(V) \to A$:

$$\begin{tikzcd} \mathrm{T}(V) \arrow[r, dashed, "F"] & A \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

The dg-algebra $\mathrm{T}(V)$ is the **differential graded tensor algebra** on V.

Lemma 2.5. Let A, B be dg-algebras.

(1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$m_{A\otimes B}\colon A\otimes B\otimes A\otimes B \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} A\otimes A\otimes B\otimes B \xrightarrow{m\otimes m} A\otimes B$$
$$u_{A\otimes B}\colon k \xrightarrow{\sim} k\otimes k \xrightarrow{u\otimes u} A\otimes B\,.$$

More explicitely, $1_{A\otimes B} = 1_A \otimes 1_B$ and $(a_1\otimes b_1)(a_2\otimes b_2) = (-1)^{|a_2||b_1|}a_1a_2\otimes b_1b_2$.

(2) The twist map $\tau\colon A\otimes B \to B\otimes A$ is an isomorphism of dg-algebras and if $f\colon A\to A'$ and $g\colon B\to B'$ are morphism of dg-algebras then so is $f\otimes g\colon A\otimes B\to A'\otimes B'$.

(3) The dg-algebra A^{op} is given by $u_{A^{\mathrm{op}}} = u_A$ and $m^{\mathrm{op}} = m_A\circ\tau$. If $\cdot$ denotes the multiplication in A and $*$ the multiplication in A^{op} then more explicitely

$$1_A = 1_{A^{\mathrm{op}}}\,,\qquad a*b = (-1)^{|a||b|}b\cdot a\,. \qquad \square$$

Warning 2.6. If $A\otimes_k B$ is the non-dg tensor product with $(a\otimes b)(a'\otimes b') = aa'\otimes bb'$ then $A\otimes B \neq A\otimes_k B$ as algebras, i.e. the underlying algebra of $A\otimes B$ is not the tensor product of the underlying algebras of A and B. The underlying algebra of A^{op} is similarly not the opposite of the underlying algebra of A.

Definition 2.7. A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.[2]

Lemma 2.8. If I is a dg-ideal in A then A/I inherits the structure of a dg-algebra.

Proof. The quotient A/I is a dg-vector space and an algebra and the compatibility of these structures can be checked on representatives. $\square$

Lemma 2.9. An ideal I in a dg-algebra A is a dg-ideal if and only if I is generated by homogeneous elements x_α with $d(x_\alpha)\in I$ for every α. (Being a dg-ideal can be checked on homegenous generators.)

Proof. That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, pp. IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I)\subseteq I$ if $d(x_\alpha)\in I$ for every α: The ideal I is spanned by $ax_\alpha b$ with $a,b\in A$ homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|}ad(x_\alpha)b + (-1)^{|a|+|x_\alpha|}ax_\alpha d(b)\in I$$

since $x_\alpha, d(x_\alpha)\in I$. $\square$

Definition 2.10. The **graded commutator** in a dg-algebra A is the unique bilinear extension of

$$[a,b] \coloneqq ab - (-1)^{|a||b|}ba\,.$$

Warning 2.11. If $|a|$ is even then $[a,a]=0$ but if $|a|$ is odd then $[a,a]=2a^2$.

[2]By an ideal we always mean a two-sided ideal.

Example 2.12. Let V be a dg-vector space. The ideal

$$I \coloneqq \big([v,w] \,\big|\, v,w \in V \text{ are homogeneous}\big)$$

is a dg-ideal in $\mathrm{T}(V)$ since the generators $[v,w]$ are homogeneous with (by Example 6.5)

$$d([v,w]) = [d(v),w] + (-1)^{|v|}[v,d(w)] \in I\,.$$

The dg-algebra $\Lambda(V) \coloneqq \mathrm{T}(V)/I$ is the **differential graded symmetric algebra** on V. If S is any graded commutative dg-algebra and $f\colon V \to S$ a morphism of dg-vector spaces then f extends uniquely to a morphism of dg-algebras $F\colon \Lambda(V) \to S$:

$$\begin{tikzcd}
\Lambda(V) \arrow[r, dashed, "F"] & S \\
V \arrow[u] \arrow[ur, "f"'] &
\end{tikzcd}$$

Remark 2.13. If V is a graded vector space with decomposition $V_{\text{even}} = \bigoplus_{n\in\mathbb{Z}} V_{2n}$ and $V_{\text{odd}} = \bigoplus_{n\in\mathbb{Z}} V_{2n+1}$ then the inclusions $V_{\text{even}}, V_{\text{odd}} \to V$ induce an isomorphism of graded vector spaces

$$\mathrm{S}(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}}) \xrightarrow{\sim} \Lambda(V)$$

where $\otimes_k$ denotes the sign-less tensor product $(a\otimes b)(a'\otimes b') = aa' \otimes bb'$. (See Appendix A.2 for more details.)

Corollary 2.14. Let $\operatorname{char}(k) \neq 2$ and let V be a dg-vector space with basis $(x_\alpha)_{\alpha\in A}$ consisting of homogeneous elements such that $(A,\leq)$ is linearly ordered. Then $\Lambda(V)$ admits as a basis the ordered monomials

$$x_{\alpha_1}^{n_1}\cdots x_{\alpha_t}^{n_t} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t,\ n_i \geq 0 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.}^3 \qquad \square$$

Proposition 2.15. If A is a dg-algebra then $\mathrm{Z}(A)$ is a graded subalgebra of A, $\mathrm{B}(A)$ is a graded ideal in $\mathrm{Z}(A)$ and $\mathrm{H}(A)$ is hence a graded algebra.

Proof. The cycles $\mathrm{Z}(A)$ is a graded subspace with $1 \in \mathrm{Z}(A)$ and if $a,b \in \mathrm{Z}(A)$ are homogeneous then

$$d(a\cdot b) = d(a)\cdot b + (-1)^{|a|} a \cdot d(b) = 0$$

and hence $ab \in \mathrm{Z}(A)$. The boundaries $\mathrm{B}(A)$ is a graded subspace and if $a \in \mathrm{Z}(A)$ and $b \in \mathrm{B}(B)$ are homogeneous with $b = d(a')$ then

$$a\cdot b = d(a\cdot a') - (-1)^{|a|} d(a)\cdot a' = d(a\cdot a')$$

and hence $ab \in \mathrm{B}(A)$. $\square$

[3]The condition $n_i = 1$ for $|x_{\alpha_i}|$ odd commes from the equality $\alpha_i^2 = [\alpha_i,\alpha_i]/2$.

3. Differential Graded Coalgebras

Definition 3.1. A **differential graded coalgebra** or **dg-coalgebra** is a dg-vector space C together with morphisms of dg-vector space $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ that make the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ k \otimes C \arrow[r, "\sim"] & C & C \otimes k \arrow[l, "\sim"'] \end{tikzcd}$$

commute. The dg-coalgebra C is **graded cocommutative** if the diagram

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

commutes. A **morphism** of dg-coalgebra $f\colon C \to D$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & D \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}$$

Definition 3.2. A graded map $\omega\colon C \to C$ of a graded coalgebra is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \mathrm{id} + \mathrm{id} \otimes \omega) \circ \Delta\,;$$

more explicitely,

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)})\,.$$

Remark 3.3.

(1) A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that d vanishes on $\mathrm{B}_0(C)$ and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)})\,,$$

i.e. such that d is a graded coderivation of degree -1.

(2) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}\,.$$

(3) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.

(4) Every coalgebra C is a dg-coalgebra centered in degree 0, in particular $C = k$.

Remark 3.4. One can now define tensor product of dg-coalgebras and the opposite of a dg-coalgebra. If C, D are dg-coalgebras then

$$\Delta_{C\otimes D}(c\otimes d) = \sum_{(c),(d)} (-1)^{|c_{(2)}||d_{(1)}|}(c_{(1)}\otimes d_{(1)})\otimes(c_{(2)}\otimes d_{(2)})\,.$$

Definition 3.5. A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.

Lemma 3.6. If I is a dg-coideal in C then C/I inherits a dg-coalgebra structure.

Proof. The quotient C/I is a dg-vector spaces and a coalgebra, and the compatibility of these structures can be checked on representatives. □

Proposition 3.7. If C is a dg-coalgebra then $\mathrm{Z}(C)$ is a graded subcoalgebra of C, $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ and $\mathrm{H}(C)$ is hence a graded coalgebra.

Proof. See Appendix A.3. □

4. Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, (B, m, u) a dg-algebra and (B, Δ, ε) a dg-coalgebra. Then the following are equivalent:

(1) Δ and ε are morphisms of dg-algebras.

(2) m and u are morphisms of dg-coalgebras.

Proof. The same diagramatic proof as in the non-dg case. □

Definition 4.2. If the conditions of Lemma 4.1 are satisfied then $(B, \mu, u, \Delta, \varepsilon)$ is a dg-bialgebra. A map $f\colon B \to C$ is a **morphism** of dg-bialgebras if it is both a morphism of dg-algebras and of dg-coalgebras. A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)}\otimes b_{(2)}c_{(2)}$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta\colon B \to B\otimes B$ is a morphism of dg-algebras into $B\otimes B'$ but not necessarily an algebra morphism into the sign-less tensor product $B\otimes_k B$. We will see an explicit counterexample in Example 5.7.

Lemma 4.5. If I is a dg-biideal in B then B/I inherits a dg-bialgebra structure.

Proof. It follows from Lemma 2.8 and Lemma 3.6 that B/I is a dg-algebra and dg-coalgebra. The compatibility can be checked on representatives. $\square$

Proposition 4.6. If B is a dg-bialgebra then $\mathrm{Z}(B)$ is a graded sub-bialgebra of B, $\mathrm{B}(B)$ is a graded biideal in $\mathrm{Z}(B)$ and $\mathrm{H}(B)$ is hence a graded bialgebra.

Proof. It follows from Proposition 2.15 and Proposition 3.7 that B is again a dg-algebra and dg-coalgebra, and the compatibility of these structures can be checked on representatives. $\square$

Definition 4.7. If B is a dg-bialgebra then $x \in B$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Lemma 4.8. If $x, y \in B$ are primitive then $[x, y]$ is again primitive.

Proof. See Example 6.5. $\square$

5. Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product on $\mathrm{Hom}_k(C, A)$ makes $\mathrm{Hom}(C, A)$ into a dg-algebra. $\square$

Proof. See Appendix A.4. $\square$

Definition 5.2. An **antipode** for a dg-bialgebra H is an inverse S to id_H with respect to the convolution product of $\mathrm{Hom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**. A **morphism** of dg-Hopf algebras is a morphism of dg-bialgebras. A **dg-Hopf ideal** in H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra need not have an underlying Hopf algebra structure.

Remark 5.4. The antipode of a dg-Hopf algebra H is the the unique morphism of dg-vector spaces $S\colon H \to H$ that makes the diagram

$$\begin{tikzcd} & H \otimes H \arrow[r, "S\otimes\mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[r, "u"] & H \\ & H \otimes H \arrow[r, "\mathrm{id}\otimes S"'] & H \otimes H \arrow[ur, "m"'] & \end{tikzcd} \tag{1}$$

commute. This means more explicitely that

$$\sum_{(c)} S(c_{(1)})c_{(2)} = \varepsilon(c)1_H \qquad \text{and} \qquad \sum_{(c)} c_{(1)}S(c_{(2)}) = \varepsilon(c)1_H\,.$$

(No additional signs occur because $|S| = 0$.)

Lemma 5.5. If I is a dg-Hopf ideal in H then H/I a dg-Hopf algebra structure. $\square$

Example 5.6. Let V be a dg-vector space.

(1) Every Hopf algebra can be regarded as a dg-Hopf algebra concentrated in degree 0.

(2) The map
$$V \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v \mapsto v \otimes 1 + 1 \otimes v$$
is a morphism of dg-vector spaces and hence induces a morphism of dg-algebras
$$\Delta\colon\ \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,.$$
The zero map $V \to 0$ induces a morphism of dg-algebras
$$\varepsilon\colon\ \mathrm{T}(V) \to \mathrm{T}(0) = k\,.$$
These maps make $\mathrm{T}(V)$ into a dg-bialgebra; the necessary diagrams can be checked on the algebra generators V of $\mathrm{T}(V)$ because all arrows occuring in the bialgebra diagrams are morphisms of dg-algebras. The maps Δ and ε are explicitely given by
$$\begin{aligned}\Delta(v_1 \cdots v_n) &= \Delta(v_1) \cdots \Delta(v_n) \\ &= (v_1 \otimes 1 + 1 \otimes v_1) \cdots (v_n \otimes 1 + 1 \otimes v_n) \\ &= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} (-1)^{n_p(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}\end{aligned}$$
where
$$n_p(\sigma) = \sum \Big\{ |v_i||v_j| \,\Big|\, 1 \leq i \leq p,\ p+1 \leq j \leq n,\ \sigma(i) > \sigma(j) \Big\}\,,$$
and
$$\varepsilon(v_1 \cdots v_n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$
The map
$$V \to \mathrm{T}(V)^{\mathrm{op}}\,, \quad v \mapsto -v$$
is a morphism of dg-vector spaces and hence induces a morphism of dg-algebras
$$S\colon\ \mathrm{T}(V) \to \mathrm{T}(V)^{\mathrm{op}}\,.$$
As a map $S\colon\ \mathrm{T}(V) \to \mathrm{T}(V)$ this is given by
$$S(v_1 \cdots v_n) = (-1)^{\sum_{1 \leq i < j \leq n} |v_i||v_j|} (-1)^n v_n \cdots v_1\,.$$
It can now be checked on the monomials $v_1 \cdots v_n$ that S is an antipode for $\mathrm{T}(V)$, making it a dg-Hopf algebra.

(3) The dg-algebra $\Lambda(V) = \mathrm{T}(V)/I$ from Example 2.12 inherits from $\mathrm{T}(V)$ the structure of a dg-Hopf algebra because the dg-ideal

$$I = \big([v,w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-Hopf ideal in $\mathrm{T}(V)$, since

$$\begin{aligned}\varepsilon([v,w]) &= 0\,,\\ \Delta([v,w]) &= [v,w] \otimes 1 + 1 \otimes [v,w] \in I \otimes \mathrm{T}(V) + \mathrm{T}(V) \otimes I\,,\\ S([v,w]) &= -[v,w] \in I\,.\end{aligned}$$

For the computation of Δ we use that v, w are primitive in $\mathrm{T}(V)$ and $[v,w]$ is therefore again primitive.

Example 5.7 (Exterior Algebra)**.** Let V be a vector space. We regard V as a dg-vector space concentrated in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\operatorname{char} k \neq 2$ there exists no bialgebra structure on $\Lambda \coloneqq \bigwedge(V)$, see Appendix A.5.

Proposition 5.8. If $\mathcal{H}$ is a dg-Hopf algebra with antipode S then the graded bialgebra $\mathrm{H}(\mathcal{H})$ is a graded Hopf algebra with antipode induced by S. □

Example 5.9. Let V be a dg-vector space.

(1) The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector spaces and thus induces a morphism of graded vector spaces $\mathrm{H}(V) \to \mathrm{H}(\mathrm{T}(V))$, which in turn induces a morphism of graded algebras

$$\alpha\colon\ \mathrm{T}(\mathrm{H}(V)) \to \mathrm{H}(\mathrm{T}(V))\,, \quad [v_1]\cdots[v_n] \mapsto [v_1 \cdots v_n]$$

where $v_1, \dots, v_n \in \mathrm{Z}(V)$. We see on representatives that α is a morphism of graded Hopf algebras and from

$$\mathrm{H}(\mathrm{T}(V)) = \mathrm{H}\left(\bigoplus_{d \geq 0} V^{\otimes d}\right) \cong \bigoplus_{d \geq 0} \mathrm{H}\big(V^{\otimes d}\big) \cong \bigoplus_{d \geq 0} \mathrm{H}(V)^{\otimes d} = \mathrm{T}(\mathrm{H}(V))$$

that α is an isomorphism.

(2) If $\operatorname{char}(k) = 0$ then also $\mathrm{H}(\Lambda(V)) \cong \Lambda(\mathrm{H}(V))$: We get again a canonical morphism of graded algebras

$$\beta\colon\ \Lambda(\mathrm{H}(V)) \to \mathrm{H}(\Lambda(V))\,, \quad [v_1]\cdots[v_n] \mapsto [v_1 \cdots v_n]$$

where $v_1, \dots, v_n \in \mathrm{Z}(V)$. The symmetrization map $s\colon\ \Lambda(V) \to \mathrm{T}(V)$ given by

$$s_n\colon\ \Lambda(V)_n \to \mathrm{T}(V)_n\,, \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{n(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

with $n(\sigma) = \sum\{|v_i||v_j| \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$ is a section for the projection $p\colon \mathrm{T}(V) \to \Lambda(V)$. The map s is a morphism of dg-vector spaces and we have the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(\mathrm{T}(V)) \arrow[r, bend left, "\alpha_n"] \arrow[d, bend right, "\mathrm{H}_n(p)"'] & \mathrm{T}(\mathrm{H}(V))_n \arrow[l, bend left, "\alpha_n^{-1}"] \arrow[d, bend right, "p_n"'] \\
\mathrm{H}_n(\Lambda(V)) \arrow[r, dashed, bend left, "\beta_n"] \arrow[u, bend right, "\mathrm{H}_n(s)"'] & \Lambda(\mathrm{H}(V))_n \arrow[l, dashed, bend left, "\beta'_n"] \arrow[u, bend right, "s_n"']
\end{tikzcd}$$

Here $\tilde{p}\colon \mathrm{T}(\mathrm{H}(V)) \to \Lambda(\mathrm{H}(V))$ and $\tilde{s}\colon \Lambda(\mathrm{H}(V)) \to \mathrm{T}(\mathrm{H}(V))$ denote the projection and section. We have $\beta_n = p_n \circ \alpha_n \circ \mathrm{H}_n(s)$. The composition $\beta'_n \coloneqq \mathrm{H}_n(p) \circ \alpha_n^{-1} \circ s_n$ is given by

$$\beta'_n([v_1] \cdots [v_n]) = [v_1 \cdots v_n]$$

and hence an inverse to β_n. This shows that β_n is an isomorphism.

6. Differential Graded Lie Algebras

Let $\mathrm{char}(k) = 0$.

Recall 6.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a map $[-,-]\colon \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is skew-symmetric and for every $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation; the last assertion is equivalent to the Jacobi identity $\sum_{\text{cyclic}}[x,[y,z]] = 0$.

Definition 6.2. A **dg-Lie algebra** is a dg-vector space $\mathfrak{g}$ together with a morphism $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is **graded skew symmmetric**, i.e. such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\tau"] \arrow[dr, "{[-,-]}"'] & & \mathfrak{g} \otimes \mathfrak{g} \arrow[dl, "{-[-,-]}"] \\
& \mathfrak{g} &
\end{tikzcd}$$

commutes, and such that $[x,-]$ is for every x a derivation of degree $|x|$.

Remark 6.3. Let $\mathfrak{g}$ be a dg-Lie algebra. Then $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j and

$$\begin{aligned}
[x,y] &= -(-1)^{|x||y|}[y,x]\,, \\
[x,[y,z]] &= [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]\,, \\
d([x,y]) &= [d(x),y] + (-1)^{|x|}[x,d(y)]\,.
\end{aligned} \tag{2}$$

We can rewrite (2) as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x,[y,z]] = 0\,.$$

Warning 6.4.

(1) A dg-Lie algebra need not have an underlying Lie algebra structure.

(2) It may happen that $[x,x] \neq 0$. If $|x|$ is even then $[x,x] = -[x,x]$ and thus $[x,x] = 0$ but if $|x|$ is odd then this may not hold.

Example 6.5.

(1) Every dg-algebra A becomes a dg-Lie algebra with the graded commutator.

(2) For a graded algebra A then the graded subspace $\mathrm{Der}(A) \subseteq \mathrm{End}(A)$ given by

$$\mathrm{Der}(A)_n \coloneqq \{\text{derivations of } A \text{ of degree } n\} \subseteq \mathrm{End}(A)_n$$

is a dg-Lie subalgebra of $\mathrm{End}(A)$.

(3) In any dg-bialgebra B the subspace of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B.

(See Appendix A.6 for explicit calculations.)

Lemma 6.6. If $\mathfrak{g}$ is a dg-Lie algebra then $\mathrm{Z}(\mathfrak{g})$ is a graded Lie subalgebra of $\mathfrak{g}$, $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$ and $\mathrm{H}(\mathfrak{g})$ is thus an graded Lie algebra.

Definition 6.7. The **universal enveloping algebra** of a dg-Lie algebra $\mathfrak{g}$ is a dg-algebra $\mathrm{U}(\mathfrak{g})$ together with a morphism of dg-Lie algebras $i\colon \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ such that for every other dg-algebra A and every morphism of dg-Lie algebras $f\colon \mathfrak{g} \to A$ there exists a unique morphism of dg-algebras $F\colon \mathrm{U}(\mathfrak{g}) \to A$ that extends f:

$$\begin{tikzcd}
\mathrm{U}(\mathfrak{g}) \arrow[r, dashed, "F"] & A \\
\mathfrak{g} \arrow[u, "i"] \arrow[ur, "f"']
\end{tikzcd}$$

Proposition 6.8. Every dg-Lie algebra $\mathfrak{g}$ admits a universal enveloping algebra. It is unique up to unique isomorphism and can be constructed as

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g}) / \big([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous} \big)$$

together with the composition $i\colon \mathfrak{g} \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$. It inherits from $\mathrm{T}(\mathfrak{g})$ the structure of a dg-Hopf algebra. $\square$

Proof. We check that the given ideal I is a dg-Hopf ideal. It is generated by homegenous elements which satisfy

$$\begin{aligned}
&d([x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}})\\
&=d([x,y]_{\mathrm{T}(\mathfrak{g})})-d([x,y]_{\mathfrak{g}})\\
&=[d(x),y]_{\mathrm{T}(\mathfrak{g})}+(-1)^{|x|}[x,d(y)]_{\mathrm{T}(\mathfrak{g})}-[d(x),y]_{\mathfrak{g}}-(-1)^{|x|}[x,d(y)]_{\mathfrak{g}}\\
&=\Big([d(x),y]_{\mathrm{T}(\mathfrak{g})}-[d(x),y]_{\mathfrak{g}}\Big)+(-1)^{|x|}\Big([x,d(y)]_{\mathrm{T}(\mathfrak{g})}-[x,d(y)]_{\mathfrak{g}}\Big)\\
&\in I
\end{aligned}$$

so it is a dg-ideal. Also

$$\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}})=\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})})-\varepsilon([x,y]_{\mathfrak{g}})=0-0=0$$

because $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are homogoneous of degree ≥ 1,

$$\begin{aligned}
&\Delta([x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}})\\
&=\Delta([x,y]_{\mathrm{T}(\mathfrak{g})})-\Delta([x,y]_{\mathfrak{g}}))\\
&=[x,y]_{\mathrm{T}(\mathfrak{g})}\otimes 1+1\otimes[x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}}\otimes 1-1\otimes[x,y]_{\mathfrak{g}}\\
&=([x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}})\otimes 1+1\otimes([x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}})\\
&\in I\otimes\mathrm{T}(\mathfrak{g})+\mathrm{T}(\mathfrak{g})\otimes I
\end{aligned}$$

since both $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are primitive, and finally

$$S([x,y]_{\mathrm{T}(\mathfrak{g})}-[x,y]_{\mathfrak{g}})=S([x,y]_{\mathrm{T}(\mathfrak{g})})-S([x,y]_{\mathfrak{g}})=-[x,y]_{\mathrm{T}(\mathfrak{g})}+[x,y]_{\mathfrak{g}}\in I\,.$$

Thus the dg-ideal I is already a dg-Hopf ideal. □

We will now show that $\mathrm{H}(\mathrm{U}(\mathfrak{g}))\cong\mathrm{U}(\mathrm{H}(\mathfrak{g}))$. For this we need a version of the Poincaré–Birkhoff–Witt theorem (PBW theorem) for dg-Lie algebra; we will not prove this, but refer to [Qui69, Appendix B, Theorem 2.3] and [FHT01, §21(a)]

Recall 6.9. If $\mathfrak{g}$ is a Lie algebra with basis $(x_\alpha)_{\alpha\in A}$ where $(A,\leq)$ is linearly ordered then the PBW theorem asserts that $\mathrm{U}(\mathfrak{g})$ has as a basis all ordered monomials

$$x_{\alpha_1}^{n_1}\cdots x_{\alpha_t}^{n_t}\qquad\text{where } t\geq 0,\ \alpha_1<\cdots<\alpha_t \text{ and } n_i\geq 0\,.$$

This shows in particular that the Lie algebra homomorphism $\mathfrak{g}\to\mathrm{U}(\mathfrak{g})$ is injective, and it also follows that $\mathbb{P}(\mathrm{U}(\mathfrak{g}))=\mathfrak{g}$. Moreover, $\operatorname{gr}\mathrm{U}(\mathfrak{g})\cong\mathrm{S}(\mathfrak{g})$ where $\operatorname{gr}\mathrm{U}(\mathfrak{g})$ denotes the associated graded for the standard filtration of $\mathrm{U}(\mathfrak{g})$.

Theorem 6.10 (dg-PBW theorem)**.** Let $\mathfrak{g}$ be a dg-Lie algebra with basis $(x_\alpha)_{\alpha\in A}$ consisting of homogeneous elements such that $(A,\leq)$ is linearly ordered. Then $\mathrm{U}(\mathfrak{g})$ has as a basis all ordered monomials

$$x_{\alpha_1}\cdots x_{\alpha_n}\qquad\text{where } t\geq 0,\ \alpha_1<\cdots<\alpha_t,\ n_i\geq 0 \text{ and } n_i=1 \text{ if } |x_{\alpha_i}| \text{ is odd.}$$ □

Corollary 6.11. Let $\mathfrak{g}$ be a dg-Lie algebra.

(1) The canonical map $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is injective.

(2) If $s\colon \Lambda(\mathfrak{g}) \to \mathrm{T}(\mathfrak{g})$ denotes the symmetrization map from Example 5.9 then the composition

$$e\colon \Lambda(\mathfrak{g}) \xrightarrow{s} \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})\,, \quad x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{n(\sigma)} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is an isomorphism of dg-vector spaces (and even of dg-coalgebra). □

Corollary 6.12 ([Qui69, Appendix B])**.** The inclusion $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is a morphism of dg-Lie algebra and thus induces a morphism of graded Lie algebras $\mathrm{H}(\mathfrak{g}) \to \mathrm{H}(\mathrm{U}(\mathfrak{g}))$, which is turn induces a morphism of graded algebras

$$\gamma\colon \mathrm{U}(\mathrm{H}(\mathfrak{g})) \to \mathrm{H}(\mathrm{U}(\mathfrak{g}))\,, \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

for $x_1, \ldots, x_n \in \mathrm{Z}(\mathfrak{g})$. Then γ is an isomorphism of graded Hopf algebras.

Proof. We denote the isomorphisms $\Lambda(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ and $\Lambda(\mathrm{H}(\mathfrak{g})) \to \mathrm{U}(\mathrm{H}(\mathfrak{g}))$ from Corollary 6.11 by e. With the isomorphism of graded algebras

$$\beta\colon \Lambda(\mathrm{H}(\mathfrak{g})) \to \mathrm{H}(\Lambda(\mathfrak{g}))\,, \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

where $x_1, \ldots, x_n \in \mathrm{Z}(\mathfrak{g})$ from Example 5.9 we get the following commutative diagram:

$$\begin{tikzcd} \Lambda(\mathrm{H}(\mathfrak{g})) \arrow[r, "\sim", "e"'] \arrow[d, "\beta"', "\sim"] & \mathrm{U}(\mathrm{H}(\mathfrak{g})) \arrow[d, dashed, "\gamma"] \\ \mathrm{H}(\Lambda(\mathfrak{g})) \arrow[r, "\sim", "\mathrm{H}(e)"'] & \mathrm{H}(\mathrm{U}(\mathfrak{g})) \end{tikzcd}$$

The arrows e, $\mathrm{H}(e)$, β are isomorphisms, hence γ is one. □

Remark 6.13.

(1) If $\mathcal{H}$ is a dg-Hopf algebra then $\mathrm{H}(\mathbb{P}(\mathcal{H})) \cong \mathbb{P}(\mathrm{H}(\mathcal{H}))$. (This statement can be found without proof in [Lod92, Theorem A.9].)

(2) If H is a graded cocommutative connected[4] dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong \mathrm{U}(\mathbb{P}(H))$, which results in an equivalence between the categories of dg-Lie algebras and cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B, Theorem 4.5].

[4]The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

A. Calculations and Proofs

A.1. Examples 2.4

(2) It holds that $\mathrm{id}_V \in \mathrm{End}(V)_0$ and if $f, g \in \mathrm{End}(V)$ are graded maps then $f \circ g$ is again a graded map Therefore $\mathrm{End}(V)$ is a subalgebra of $\mathrm{End}_k(V)$. If $f, g \in \mathrm{End}(V)$ are homogeneous then $|f \circ g| = |f| + |g|$ so $\mathrm{End}(V)$ is a graded algebra. We see from

$$\begin{aligned} d(f \circ g) &= d \circ f \circ g - (-1)^{|f \circ g|} f \circ g \circ d \\ &= d \circ f \circ g - (-1)^{|f|+|g| f \circ g \circ d} \\ &= d \circ f \circ g - (-1)^{|f|} f \circ d \circ g + (-1)^{|f|} f \circ d \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\ &= (d \circ f - (-1)^{|f|} d \circ f) \circ g + (-1)^{|f|} f \circ (d \circ g - (-1)^{|g|} g \circ d) \\ &= d(f) \circ g + (-1)^{|f|} f \circ d(g) \end{aligned}$$

and

$$d(\mathrm{id}_V) = d \circ \mathrm{id}_V - \mathrm{id}_V \circ d = d - d = 0$$

that $\mathrm{End}(V)$ is a dg-algebra.

(3) It remains to check the compatibility of the multiplication and dg-structure of $\mathrm{T}(V)$: It holds that $1_{\mathrm{T}(V)} \in \mathrm{T}(V)_0$ with $d(1_{\mathrm{T}(V)}) = 0$. Furthermore

$$\begin{aligned} |v_1 \cdots v_n \cdot w_1 \cdots w_m| &= |v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_m| \\ &= |v_1 \cdots v_n| + |w_1 \cdots w_m| \end{aligned}$$

and

$$\begin{aligned} & d(v_1 \cdots v_n \cdot w_1 \cdots w_m) \\ &= \sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \cdot w_1 \cdots w_m \\ &\quad + \sum_{j=1}^{m} (-1)^{|v_1|+\cdots+|v_n|+|w_1|+\cdots+|w_{j-1}|} v_1 \cdots v_n \cdot w_1 \cdots d(w_j) \cdots w_m \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1|+\cdots+|v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m) \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1 \cdots v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m)\,. \end{aligned}$$

This shows that $\mathrm{T}(V)$ is indeed a dg-algebra.

Let A be another dg-algebra and $f\colon V \to A$ a morphism of dg-vector spaces an let $F\colon \mathrm{T}(V) \to A$ be the unique extension of f to an algebra morphism, given by $F(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$. The algebra morphism F is a morphism of graded algebras because

$$\begin{aligned} |F(v_1 \cdots v_n)| &= |f(v_1) \cdots f(v_n)| \\ &= |f(v_1)| + \cdots + |f(v_n)| \\ &= |v_1| + \cdots + |v_n| \\ &= |v_1 \cdots v_n|\,. \end{aligned}$$

It is also a morphism of dg-vector spaces because

$$\begin{aligned}
d(F(v_1\cdots v_n)) &= d(f(v_1)\cdots f(v_n)) \\
&= \sum_{i=1}^{n}(-1)^{|f(v_1)|+\cdots+|f(v_{i-1})|} f(v_1)\cdots d(f(v_i))\cdots f(v_n) \\
&= \sum_{i=1}^{n}(-1)^{|v_1|+\cdots+|v_{i-1}|} f(v_1)\cdots f(d(v_i))\cdots f(v_n) \\
&= F\left(\sum_{i=1}^{n}(-1)^{|v_1|+\cdots+|v_{i-1}|} v_1\cdots d(v_i)\cdots v_n\right) \\
&= F(d(v_1\cdots v_n))\,.
\end{aligned}$$

A.2. Remark 2.13

Let A and B be two dg-algebras. If C is any other dg-algebra and if $f\colon A\to C$ and $g\colon B\to C$ are two morphisms of dg-algebras with

$$f(a)g(b) = (-1)^{|a||b|}g(b)f(a)$$

for all $a\in A$, $b\in B$ then the linear map

$$\varphi\colon A\otimes B\to C\,,\quad a\otimes b\mapsto f(a)g(b)$$

is again a morphism of dg-algebras. The inclusions $i\colon A\to A\otimes B$, $a\mapsto a\otimes 1$ and $j\colon B\colon B\to A\otimes B$, $b\mapsto 1\otimes b$ are morphisms of dg-algebras. For every morphism of dg-algebras $\varphi\colon A\otimes B\to C$ the compositions $\varphi\circ i\colon A\to A\otimes B$ and $\varphi\colon j\colon B\to A\otimes B$ are again morphisms of dg-algebras. This gives a one-to-one correspondence

$$\begin{aligned}
\left\{\begin{matrix}\text{morphisms of dg-algebras}\\ f\colon A\to C,\, g\colon B\to C\end{matrix}\right\} &\longleftrightarrow \left\{\begin{matrix}\text{morphisms of dg-algebras}\\ \varphi\colon A\otimes B\to C.\end{matrix}\right\}, \\
(f,g) &\longmapsto (a\otimes b\mapsto f(a)g(b))\,, \\
(\varphi\circ i,\varphi\circ j) &\longleftarrow\!\shortmid\ \varphi\,.
\end{aligned}$$

It follows for any two dg-vector spaces V and W that

$$\Lambda(V\oplus W)\cong\Lambda(V)\otimes\Lambda(W)$$

since we have for every dg-algebra A natural bijections

$$\begin{aligned}
&\{\text{morphisms of dg-algebras } \Lambda(V\oplus W)\to A\} \\
\cong\ &\{\text{morphisms of dg-vector spaces } V\oplus W\to A\} \\
\cong\ &\{(f,g)\mid \text{morphisms of dg-vector spaces } f\colon V\to A,\, g\colon W\to A\} \\
\cong\ &\{(\varphi,\psi)\mid \text{morphisms of dg-algebras } \varphi\colon\ \Lambda(V)\to A,\, \psi\colon\ \Lambda(W)\to A\} \\
\cong\ &\{\text{morphisms of dg-algebras } \Lambda(V)\otimes\Lambda(W)\to A\}\,.
\end{aligned}$$

More explicitely, the inclusions $V \to V \oplus W$ and $W \to V \oplus W$ induce morphisms of dg-algebras $\Lambda(V) \to \Lambda(V \oplus W)$ and $\Lambda(W) \to \Lambda(V \oplus W)$ that give an isomorphism of dg-algebras

$$\Lambda(V) \otimes \Lambda(W) \xrightarrow{\sim} \Lambda(V \oplus W)\,, \quad v_1 \cdots v_n \otimes w_1 \cdots w_m \mapsto v_1 \cdots v_n w_1 \cdots w_m\,.$$

Let V be a graded vector space. If V is concentrated in even degrees then $\Lambda(V) = \mathrm{S}(V)$ and if V is concentrated in odd degrees then $\Lambda(V) = \bigwedge(V)$, with the grading of $\Lambda(V)$ and $\bigwedge(V)$ induced by the one of V. We have $V = V_{\text{even}} \oplus V_{\text{odd}}$ as graded vector spaces where $V_{\text{even}} = \bigoplus_n V_{2n}$ and $V_{\text{odd}} = \bigoplus_n V_{2n+1}$, and hence

$$\Lambda(V) = \Lambda(V_{\text{even}} \oplus V_{\text{odd}}) \cong \Lambda(V_{\text{even}}) \otimes \Lambda(V_{\text{odd}}) = \mathrm{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$$

The graded algebra $\mathrm{S}(V_{\text{even}})$ is concentrated in even degree and so it follows that in the tensor product $\mathrm{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$ the simple tensors (strictly) commute, i.e. $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$. Hence

$$\Lambda(V) \cong \mathrm{S}(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}})$$

where $\otimes_k$ denotes the sign-less tensor product.

A.3. Proposition 3.7

If $c \in \mathrm{Z}(C)$ then

$$d(\Delta(c)) = \Delta(d(c)) = \Delta(0) = 0$$

because Δ is a morphism of dg-vector spaces, and hence

$$\Delta(c) \in \mathrm{Z}(C \otimes C) = \mathrm{Z}(C) \otimes \mathrm{Z}(C)\,.$$

This shows that $\mathrm{Z}(C)$ is a subcoalgebra of C. It is also a graded subspace of C and hence a graded subcoalgebra.

For $b \in \mathrm{B}(C)$ with $b = d(c)$ we have

$$\begin{aligned}\Delta(b) = \Delta(d(c)) = d(\Delta(c)) &= d\Bigg(\sum_{(c)} c_{(1)} \otimes c_{(2)}\Bigg) \\ &= \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}| c_{(1)} \otimes d(c_{(2)})} \in \mathrm{B}(C) \otimes C + C \otimes \mathrm{B}(C)\,.\end{aligned}$$

We also have

$$\varepsilon(b) = \varepsilon(d(c)) = d(\varepsilon(c)) = 0\,.$$

This shows that $\mathrm{B}(C)$ is a coideal in C. It follows from the upcoming lemma that B is also a coideal in $\mathrm{Z}(C)$. Then $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ because $\mathrm{B}(C)$ is a graded subspace of $\mathrm{Z}(C)$.

Lemma A.1. Let C be a coalgebra and let B be a subcoalgebra of C. If I is a coideal in C with $I \subseteq C$ then I is also a coideal in B.

Proof. It follows from the inclusions $I \subseteq B \subseteq C$ that

$$(C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B\,.$$

Hence

$$\Delta(I) = \Delta(I) \cap \Delta(B) \subseteq (C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B\,.$$

Also $\varepsilon_B(I) = \varepsilon_C(I) = 0$. □

A.4. Lemma 5.1

We have $1_{\mathrm{Hom}_k(C,A)} = u \circ \epsilon \in \mathrm{Hom}(C, A)_0$ because both u_A and ϵ_C are morphisms of dg-vector spaces and thus of degree 0. If $f, g \in \mathrm{Hom}(C, A)$ are graded maps then $f \otimes g$ is again a graded map and thus

$$f * g = m \circ (f \otimes g) \circ \Delta$$

is a graded map as a composition of graded maps. This shows that $\mathrm{Hom}(C, A)$ is a subalgebra of $\mathrm{Hom}_k(C, A)$.

We have

$$|f * g| = |m \circ (f \otimes g) \circ \Delta| = |m| + (|f| + |g|) + |\Delta| = |f| + |g|$$

so $\mathrm{Hom}(C, A)$ is a graded algebra with respect to the convolution product.

Furthermore

$$\begin{aligned}
&d(f * g)\\
&= d \circ (f * g) - (-1)^{|f*g|}(f * g) \circ d\\
&= d \circ m \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ \Delta \circ d\\
&= m \circ d_{A\otimes A} \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ d_{C\otimes C} \circ \Delta\\
&= m \circ (d \otimes 1 + 1 \otimes d) \circ (f \otimes g) \otimes \Delta\\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes 1 + 1 \otimes d) \circ \Delta\\
&= m \circ (d \otimes \mathrm{id}) \circ (f \otimes g) \otimes \Delta + m \circ (\mathrm{id} \otimes d) \circ (f \otimes g) \otimes \Delta\\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes \mathrm{id}) \circ \Delta\\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (\mathrm{id} \otimes d) \circ \Delta\\
&= m \circ ((d \circ f) \otimes g) \otimes \Delta + (-1)^{|f|} m \circ (f \otimes (d \circ g)) \otimes \Delta\\
&\quad - (-1)^{|f|} m \circ ((f \circ d) \otimes g) \circ \Delta - (-1)^{|f|+|g|} m \circ (f \otimes (g \circ d)) \circ \Delta\\
&= m \circ ((d \circ f - (-1)^{|f|} f \circ d) \otimes g) \otimes \Delta\\
&\quad + (-1)^{|f|} m \circ (f \otimes (d \circ g - (-1)^{|g|} g \circ d)) \otimes \Delta\\
&= m \circ (d(f) \otimes g) \circ \Delta + (-1)^{|f|} m \circ (f \otimes d(g)) \otimes \Delta\\
&= d(f) * g + (-1)^{|f|} f * d(g)
\end{aligned}$$

because m and Δ are commute with the differentials. Hence $\mathrm{Hom}(C, A)$ is a dg-algebra with respect to the convolution product.

A.5. Example 5.7

Suppose that there exists a bialgebra structure on $\bigwedge(V)$. Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker \varepsilon = \bigoplus_{d \geq 1} \bigwedge^n(V) =: I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{\Lambda \otimes I} \qquad \text{and} \qquad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes \Lambda}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2},$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

Now $v^2 = 0$, hence

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

A.6. Example 6.5

(1) If $a, b \in A$ are homogeneous then $[a, b] = ab - (-1)^{|a||b|} ba$ is homogeneous of degree $|a| + |b|$, so $[A_i, A_j] \subseteq A_{i+j}$ for all i, j. Also

$$[a, b] = ab - (-1)^{|a||b|} ba = -(-1)^{|a||b|}\big(ba - (-1)^{|a||b|} ab\big) = -(-1)^{|a||b|}[b, a]$$

and

$$\begin{aligned} d([a, b]) &= d\big(ab - (-1)^{|a||b|} ba\big) \\ &= d(ab) - (-1)^{|a||b|} d(ba) \\ &= d(a)b + (-1)^{|a|} a d(b) - (-1)^{|a||b|}\big(d(b)a + (-1)^{|b|} b d(a)\big) \\ &= d(a)b + (-1)^{|a|} a d(b) - (-1)^{|a||b|} d(b)a - (-1)^{|a||b|+|b|} b d(a) \\ &= d(a)b + (-1)^{|a|} a d(b) - (-1)^{|a||d(b)|+|a|} d(b)a - (-1)^{|d(a)||b|} b d(a) \\ &= d(a)b - (-1)^{|d(a)||b|} b d(a) + (-1)^{|a|}\big(a d(b) - (-1)^{|a||d(b)|} d(b) a\big) \\ &= [d(a), b] + (-1)^{|a|}[a, d(b)]. \end{aligned}$$

We check the graded Jacobi identity for homogeneous $a, b, c \in A$. We have

$$\begin{aligned} [a, [b, c]] &= \big[a, bc - (-1)^{|b||c|} cb\big] \\ &= [a, bc] - (-1)^{|b||c|}[a, cb] \end{aligned}$$

$$
\begin{aligned}
&= abc - (-1)^{|a||bc|}bca - (-1)^{|b||c|}\big(acb - (-1)^{|a||cb|}cba\big) \\
&= abc - (-1)^{|a||bc|}bca - (-1)^{|b||c|}acb + (-1)^{|a||cb|+|b||c|}cba \\
&= abc - (-1)^{|a|(|b|+|c|)}bca - (-1)^{|b||c|}acb + (-1)^{|a|(|b|+|c|)+|b||c|}cba \\
&= abc - (-1)^{|a||b|+|a||c|}bca - (-1)^{|b||c|}acb + (-1)^{|a||b|+|a||c|+|b||c|}cba
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(-1)^{|a||c|}[a,[b,c]] = (-1)^{|a||c|}abc &- (-1)^{|a||b|}bca \\
&- (-1)^{|a||c|+|b||c|}acb + (-1)^{|a||b|+|b||c|}cba\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{\text{cyclic}} (-1)^{|a||c|}[a,[b,c]] &= \sum_{\text{cyclic}} (-1)^{|a||c|}abc - \sum_{\text{cyclic}} (-1)^{|a||b|}bca \\
&\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|}acb + \sum_{\text{cyclic}} (-1)^{|a||b|+|b||c|}cba \\
&= \sum_{\text{cyclic}} (-1)^{|b||a|}bca - \sum_{\text{cyclic}} (-1)^{|a||b|}bca \\
&\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|}acb + \sum_{\text{cyclic}} (-1)^{|b||c|+|c||a|}acb \\
&= 0\,.
\end{aligned}
$$

(2) The subspace $\mathrm{Der}(A)$ is by construction a graded subspace of $\mathrm{End}(A)$. Let δ, ε be graded derivations. Then for all homogeneous $a, b \in A$,

$$
\begin{aligned}
(\delta\varepsilon)(ab) &= \delta(\varepsilon(ab)) \\
&= \delta(\varepsilon(a)b + (-1)^{|\varepsilon||a|}a\varepsilon(b)) \\
&= \delta(\varepsilon(a)b) + (-1)^{|\varepsilon||a|}\delta(a\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\big(\delta(a)\varepsilon(b) + (-1)^{|\delta||a|}a\delta(\varepsilon(b))\big) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{(|\varepsilon|+|a|)|\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b))
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) &= (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) + (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b))
\end{aligned}
$$

and therefore

$$\begin{aligned}
[\delta,\varepsilon](ab) &= (\delta\varepsilon - (-1)^{|\delta||\varepsilon|}\varepsilon\delta)(ab) \\
&= (\delta\varepsilon)(ab) - (-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&\quad - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b - (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad - (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}\big(a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|}a\varepsilon(\delta(b))\big) \\
&= [\delta,\varepsilon](a)b + (-1)^{|[\delta,\varepsilon]||a|}a[\delta,\varepsilon](b)\,.
\end{aligned}$$

This shows that $[\delta,\varepsilon] \in \mathrm{Der}(A)$, so that $\mathrm{Der}(A)$ is a graded Lie subalgebra of $\mathrm{End}(A)$. If $\delta \in \mathrm{Der}(A)$ is homogeneous then

$$d(\delta) = d \circ \delta - (-1)^{|\delta|}\delta \circ d = [d,\delta]$$

is again a graded derivation, and hence $\mathrm{Der}(A)$ is a dg-subspace of $\mathrm{End}(A)$.

(3) If $a \in \mathbb{P}(B)$ with homogeneous decomposition $a = \sum_n a_n$ then

$$\Delta(a) = \Delta\left(\sum_n a_n\right) = \sum_n \Delta(a_n)$$

but also

$$\Delta(a) = a \otimes 1 + 1 \otimes a = \sum_n (a_n \otimes 1 + 1 \otimes a_n)\,.$$

By comparing homogeneous components we see that $\Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n$ for all n. This means that all homogeneous components a_n are again primitive, which shows that $\mathbb{P}(B)$ is a graded subspace of B. If $a \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(d(a)) &= d(\Delta(a)) \\
&= d(a \otimes 1 + 1 \otimes a) \\
&= d(a \otimes 1) + d(1 \otimes a) \\
&= d(a) \otimes 1 + (-1)^{|a|}a \otimes d(1) + d(1) \otimes a + (-1)^{|1|}1 \otimes d(a) \\
&= d(a) \otimes 1 + 1 \otimes d(a)
\end{aligned}$$

because $|1| = 0$ and $d(1) = 0$. Therefore $\mathbb{P}(B)$ is a dg-subspace of B.

If $a, b \in \mathbb{P}(B)$ then

$$\begin{aligned}\Delta(ab) &= \Delta(a)\Delta(b)\\ &= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b)\\ &= (a \otimes 1)(b \otimes 1) + (a \otimes 1)(1 \otimes b) + (1 \otimes a)(b \otimes 1) + (1 \otimes a)(1 \otimes b)\\ &= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab\,.\end{aligned}$$

If a, b are homogeneous then it follows that

$$\begin{aligned}\Delta([a, b]) &= \Delta\big(ab - (-1)^{|a||b|} ba\big)\\ &= \Delta(ab) - (-1)^{|a||b|} \Delta(ba)\\ &= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab\\ &\quad - (-1)^{|a||b|} \big(ba \otimes 1 + b \otimes a + (-1)^{|a||b|} a \otimes b + 1 \otimes ba\big)\\ &= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab\\ &\quad - (-1)^{|a||b|} ba \otimes 1 - (-1)^{|a||b|} b \otimes a - a \otimes b - (-1)^{|a||b|} 1 \otimes ba\\ &= \big(ab - (-1)^{|a||b|} ba\big) \otimes 1 + 1 \otimes \big(ab - (-1)^{|a||b|} ba\big)\\ &= [a, b] \otimes 1 + 1 \otimes [a, b]\end{aligned}$$

which shows that $[a, b] \in \mathbb{P}(B)$. Thus $\mathbb{P}(B)$ is a dg-Lie subalgebra of B.

References

[Bou89] Nicolas Bourbaki. *Algebra I. Chapters 1–3.* Elements of Mathematics. Springer-Verlag Berlin Heidelberg New York, 1989, pp. xxiii+709. ISBN: 3-540-64243-9.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory.* Graduate Texts in Mathematics 205. Springer-Verlag New York, 2001, pp. xxxiii+539. ISBN: 978-0-387-95068-6. DOI: 10.1007/978-1-4613-0105-9.

[Lan02] Serge Lang. *Algebra.* Graduate Texts in Mathematics 211. Springer-Verlag New York, 2002, pp. xv+914. ISBN: 978-0-387-95385-4. DOI: 10.1007/978-1-4613-0041-0.

[Lod92] Jean-Lois Loday. *Cyclic Homology.* Grundlagen der mathematischen Wissenschaften 301. Springer Verlag Berlin-Heidelberg, 1992, pp. xix+516. ISBN: 978-3-662-21741-2. DOI: 10.1007/978-3-662-21739-9.

[MO18] David E Speyer. *When is the exterior algebra a Hopf algebra?* November 30, 2018. URL: https://mathoverflow.net/q/316544 (visited on May 7, 2019).

[Qui69] Daniel Quillen. “Rational homotopy theory”. In: *Ann. of Math. (2)* 90 (1969), pp. 205–295. ISSN: 0003-486X. DOI: 10.2307/1970725.