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Graduate Seminar on Topology

Differential Graded Hopf Algebras I*

Introducing Signs

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In the following k denotes a field. All vector spaces, all kinds of algebras, all tensor products, etc. are over k, unless otherwise stated. All occuring maps are linear unless otherwise stated. We will sometime assume additional constraints on the characteristic of k, but will make this explicit when it occurs.

1 Notations

Combine the subsections into a shorter one.

1.1 Graded Vector Spaces

A **grading** on a vector space V is a decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$. A **graded vector space** V is a vector space together with a grading. An element $v \in V_n$ is **homogeneous** of **degree** $|v| = n$.

Whenever we write $|v|$ the element v is assumed to be homogeneous.

A linear subspace U of V is a **graded subspace** if $U = \bigoplus_{n \in \mathbb{Z}} U_n$ for some linear subspaces $U_n \subseteq V_n$. The quotient $V/U = \bigoplus_{n \in \mathbb{Z}} V_n/U_n$ is then again a graded vector space. If $(V^\alpha)_\alpha$ is a collection of graded vector spaces then their **(direct) sum** is the graded vector space $\bigoplus_\alpha V^\alpha$ with $(\bigoplus_\alpha V^\alpha)_n = \bigoplus_\alpha V^\alpha_n$. If V and W are graded vector space then $\mathrm{gHom}(V,W)$ is the graded vector space with

$$\mathrm{gHom}(V,W)_n \coloneqq \{\text{maps } f\colon V \to W \text{ of degree } n\}\,.^1$$

A map $f\colon V \to W$ between graded vector spaces is **graded** of **degree** $|f| = d$ if $f(V_n) \subseteq W_{n+d}$ for all n. A **morphism of graded vector spaces** is a graded map of degree 0.

*Available online at `jendrikstelzner.de/dg_hopf_extended.pdf`.

[1] The spaces $\mathrm{gHom}(V,W)_n$ are linearly independent in $\mathrm{Hom}_k(V,W)$, i.e. the sum $\sum_n \mathrm{gHom}(V,W)_n$ is direct. We can thus regard $\mathrm{gHom}(V,W)$ as a subspace of $\mathrm{Hom}_k(V,W)$.

The **tensor product** $V \otimes W$ of two graded vector spaces V, W is the vector space $V \otimes W$ together with the grading $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. The grading on the higher tensor products $V^1 \otimes \cdots \otimes V^t$ are defined inductively as

$$(V^1 \otimes \cdots V^t)_n = \bigoplus_{n_1+\cdots+n_t=n} V^1_{n_1} \otimes \cdots \otimes V^t_{n_t} .$$

The **twist map** $\tau \colon V \otimes W \to W \otimes V$ is the isomorphism of graded vector spaces

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v .$$

We hence adhere to the Koszul-Quillen **sign convention**:

Whenever two homegeneous elements x, y are swapped,
the sign $(-1)^{|x||y|}$ is introduced.

If $f \colon V \to W$ and $g \colon V' \to W'$ are graded maps then the map $f \otimes g \colon V \otimes W \to V \otimes W$ is the graded map

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) .$$

1.2 Differential Graded Vector Spaces

A **differential** on a graded vector space V is a linear map $d \colon V \to V$ of degree -1 with $d^2 = 0$. A **differential graded vector space** or **dg-vector space** is a graded vector space V together with a differential.[2] A **morphism** of dg-vector space $f \colon V \to W$ is a morphism of graded vector spaces with $d \circ f = f \circ d$.

A graded subspace U of a dg-vector space V is a **dg-subspace** if $d(U) \subseteq U$. The graded vector space V/U then inherits a differential from V, that is given on representatives by d. If $(V^\alpha)_\alpha$ is a collection of dg-vector space then $\bigoplus_\alpha V^\alpha$ is a dg-vector space with differential $d_{(\bigoplus_\alpha V^\alpha)} = \bigoplus_\alpha d_{V^\alpha}$. If V and W are dg-vector space then $\mathrm{dgHom}(V, W)$ is the graded vector space $\mathrm{gHom}(V, W)$ together with the differential

$$d_{\mathrm{dgHom}(V,W)}(f) = d \circ f - (-1)^{|f|} f \circ d .$$

If V and W are dg-vector space then $V \otimes W$ inherits the differential

$$d_{(V \otimes W)} = d \otimes \mathrm{id} + \mathrm{id} \otimes d$$

more explicitely

$$d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w) .$$

The twist map $\tau \colon V \otimes W \to W \otimes V$ is an isomorphism of dg-vector space.[3]

If V is a dg-vector space then $\mathrm{Z}(V) \coloneqq \ker d$ and $\mathrm{B}(V) \coloneqq \operatorname{im} d$ are graded subspaces with $\mathrm{Z}(V) \subseteq \mathrm{B}(V)$. The graded vector space $\mathrm{H}(V) \coloneqq \mathrm{Z}(V)/\mathrm{B}(V)$ is the **homology** of V. There exists a natural isomorphism of graded vector spaces

$$\mathrm{H}(V \otimes W) \cong \mathrm{H}(V) \otimes \mathrm{H}(W)$$

[2] A dg-vector space is hence the same as a chain complex.
[3] The naive twist map $v \otimes w \mapsto w \otimes v$ is not a morphism of dg-vector space.

that is on representatives given by $[v \otimes w] \leftarrow\!\shortmid [v] \otimes [w]$, called the **algebraic Künneth isomorphism**.

Remark 1.1. Every graded vector space can be regarded as a dg-vector space with zero differential. We will therefore state most of the following definitions and propositions only for the differential graded case, which then always entails a graded version of the statement.

We regard the ground field k as a dg-vector space concentrated in degree 0. Then the natural isomorphism $k \otimes V \cong V$ and $V \otimes k \cong V$ are isomorphism of dg-vector space.

2 Differential Graded Algebra

Definition 2.1. A **differential graded algebra** or **dg-algebra** is a dg-vector space A together with morphisms of dg-vector space $m\colon A \otimes A \to A$ and $u\colon k \to A$ that make the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[d, "u\otimes \mathrm{id}"'] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & A \otimes k \arrow[d, "\mathrm{id}\otimes u"] \\ A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

commute. The dg-algebra A is **graded commutative** if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

commutes. A **morphism** of dg-algebras $f\colon A \to B$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & B \otimes B \arrow[d, "m"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} & k \arrow[dl, "u"'] \arrow[dr, "u"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

Definition 2.2. If A is a dg-algebra then a graded map $\delta\colon A \to A$ is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\,;$$

more explicitely,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b)\,.$$

Remark 2.3.

(1) A dg-algebra is the same as a graded algebra A together with a differential d such that $d(1) = 0$ and
$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)\,,$$
i.e. such that d is a graded derivation (of degree -1).

(2) The commutativity of A means that $ab = (-1)^{|a||b|} ba$. If $|a|$ is odd and $\mathrm{char}(k) \neq 2$ then $a^2 = 0$.

(3) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

Examples 2.4.

(1) Every algebra A is a dg-algebra concentrated in degree 0. This holds in particular for $A = k$.

(2) If V is any dg-vector space then the algebra structure of $\mathrm{End}_k(V)$ restricts to a dg-algebra structure on $\mathrm{dgEnd}(V)$.

(3) If V is a dg-vector space then $\mathrm{T}(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space, with
$$|v_1 \cdots v_n| = |v_1| + \cdots + |v_n|$$
and
$$d(v_1 \cdots v_n) = \sum_{i=1}^{n} (-1)^{|v_1| + \cdots + |v_i|} v_1 \cdots d(v_i) \cdots v_n\,.$$
This makes the tensor $\mathrm{T}(V)$ into a dg-algebra that is denoted by $\mathrm{T}(V)$ The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector space and if A is any other dg-algebra and $f \colon V \to A$ any morphism of dg-vector spaces then f extends uniquely to a morphism of dg-algebras $F \colon \mathrm{T}(V) \to A$:
$$\begin{tikzcd} \mathrm{T}(V) \arrow[r, dashed, "F"] & A \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

Add the shuffle dg algebra.

(4) If V is any vector space then the symmetric algebra $\mathrm{S}(V)$ is a graded algebra and a commutative algebra, but not a graded commutative algebra. The exterior algebra $\bigwedge(V)$ is a graded algebra, it is in general not a commutative algebra (unless $\dim V \leq 1$), but it is a graded commutative algebra.

Lemma 2.5. Let A, B be dg-algebra.

(1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$m_{A\otimes B} \colon A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{m\otimes m} A \otimes B$$
$$u_{A\otimes B} \colon k \xrightarrow{\sim} k \otimes k \xrightarrow{u\otimes u} A \otimes B\,.$$

More explicitely, $1_{A\otimes B} = 1_A \otimes 1_B$ and $(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$.

(2) If f and g are morphism of dg-algebras then so is $f \otimes g$.

(3) The twist map $\tau \colon A \otimes B \to B \otimes A$ is a morphism of dg-algebras.

(4) If $A = (A, m, u)$ then $A^{\mathrm{op}} = (A, m^{\mathrm{op}}, u)$ with $m^{\mathrm{op}} = m \circ \tau$ is again a dg-algebra. □

Warning 2.6. If A, B are dg-algebras then the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B. The underlying algebra of A^{op} is not the opposite of the underlying algebra of A. (Both thanks to signs.)

Definition 2.7. A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.

Lemma 2.8. For an ideal I is a dg-algebra A the following conditions are equivalent:

(1) I is a dg-ideal.

(2) I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α.

Proof. That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, pp. IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α: The ideal I is spanned by $ax_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|} ad(x_\alpha)b + (-1)^{|a|+|x_\alpha|} ax_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$. □

Lemma 2.9. If I is a dg-ideal in a dg-algebra then A/I inherits the structure of a dg-algebra such that the projection $A \to A/I$ is a morphism of dg-algebras. □

Definition 2.10. If A is a dg-algebra then the **dg-comutator** of $a, b \in A$ is

$$[a, b] \coloneqq ab - (-1)^{|a||b|} ba\,.$$

Example 2.11. Let V be a dg-vector space. The ideal

$$I \coloneqq \big([v, w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-ideal in $\mathrm{T}(V)$ since the generators $[v, w]$ are homogeneous with

$$d([v, w]) = [d(v), w] + (-1)^{|v|} [v, d(w)] \in I\,.$$

The dg-algebra $\mathrm{S}(V) \coloneqq \mathrm{T}(V)/I$ is the **differential graded symmetric algebra** on V.

Proposition 2.12. If A is a dg-algebra then $\mathrm{Z}(A)$ is a graded subalgebra of A, $\mathrm{B}(A)$ is a graded ideal in $\mathrm{Z}(A)$ and $\mathrm{H}(A)$ is hence a graded algebra.

3 Different Graded Coalgebras

Definition 3.1. A **differential graded coalgebra** or **dg-coalgebra** is a dg-vector space C together with morphisms of dg-vector space $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ that make the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ k \otimes C \arrow[r, "\sim"] & C & C \otimes k \arrow[l, "\sim"'] \end{tikzcd}$$

commute. The dg-coalgebra C is **graded cocommutative** if the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

A **morphism** of dg-coalgebra $f\colon C \to D$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & D \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}$$

Definition 3.2. If C is a dg-coalgebra then a graded map $\omega\colon C \to C$ is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \mathrm{id} + \mathrm{id} \otimes \omega) \circ \Delta\,;$$

more explicitely,

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)})\,.$$

Remark 3.3.

(1) A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)})\,,$$

i.e. such that d is a graded coderivation of degree -1.

(2) The cocommutativity of C means that

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}\,.$$

(3) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.

(4) Every coalgebra C is a dg-coalgebra centered in degree 0. This holds in particular for $C = k$.

Example 3.4. Let V be a dg-vector space. Then $\mathrm{T}(V)$ becomes a dg-coalgebra with the deconcatination

$$\Delta\colon \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v_1 \cdots v_n \mapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n\,,$$
$$\varepsilon\colon \mathrm{T}(V) \to k\,, \quad v_1 \cdots v_n \mapsto \delta_{n0}\,.$$

Lemma 3.5. Let C, D be dg-coalgebras.

(1) The tensor product $C \otimes D$ becomes a dg-coalgebra with

$$\Delta_{C\otimes D}\colon C \otimes D \xrightarrow{\Delta\otimes\Delta} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} C \otimes D \otimes C \otimes D$$
$$\varepsilon_{C\otimes D}\colon C \otimes D \xrightarrow{\varepsilon\otimes\varepsilon} k \otimes k \xrightarrow{\sim} k$$

(2) If f and g are morphism of dg-coalgebras then so is $f \otimes g$.

(3) The twist map $\tau\colon C \otimes D \to D \otimes C$ is a morphism of dg-coalgebras.

(4) If $C = (C, \Delta, \varepsilon)$ then $C^{\mathrm{cop}} = (C, \Delta^{\mathrm{cop}}, \varepsilon)$ with $\Delta^{\mathrm{op}} = \tau \circ \Delta$ is again a dg-coalgebra.

Warning 3.6. If C, D are dg-coalgebras then the underlying coalgebra of $C \otimes D$ is not the tensor product of the underlying coalgebras of C and D. The underlying coalgebra of C^{op} is not the coopposite of the underlying coalgebra of C. (Again both thanks to signs.)

Definition 3.7. A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.

Lemma 3.8. If I is a dg-coideal in a dg-coalgebra C then C/I inherits the structure of a dg-coalgebra such that the projection $C \to C/I$ is a morphism of dg-coalgebra. $\square$

Proposition 3.9. If C is a dg-coalgebra then $\mathrm{Z}(C)$ is a graded subcoalgebra of A, $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ and $\mathrm{H}(C)$ is hence a graded coalgebra. $\square$

4 Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, let (B, m, u) be a dg-algebra and let (B, Δ, ε) be a dg-coalgebra. Then the following conditions are equivalent:

(1) Δ and ε are morphisms of dg-algebras.

(2) m and u are morphisms of dg-coalgebras.

Proof. The same diagramatic proof as in the ungraded case. ☐

Definition 4.2. A **dg-bialgebra** is a quintuple $(B, \mu, u, \Delta, \varepsilon)$ such that the equivalent conditions of Lemma 4.1 are satisfied. A map $f \colon B \to C$ is a **morphism** of dg-bialgebras if it is both a morphism of dg-algebras and of dg-coalgebras. A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)}$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure because $\Delta \colon B \to B \otimes B$ is a morphism of dg-algebras, but not necessarily one of algebras.

$$\begin{tikzcd}
\text{algebras} & & \text{dg-algebras} \arrow[ll, dashed] & \\
& \text{bialgebras} \arrow[ul] \arrow[dl] & & \text{dg-bialgebras} \arrow[ll, dashed, "/" marking] \arrow[ul] \arrow[dl] \\
\text{coalgebras} & & \text{dg-coalgebras} \arrow[ll, dashed] &
\end{tikzcd}$$

Lemma 4.5. If B is a dg-bialgebra then B^{op}, B^{cop} and $B^{\text{op,cop}}$ are again dg-bialgebras. ☐

Lemma 4.6. If I is a dg-biideal in a dg-bialgebra B then B/I inherits from B the structure of a dg-bialgebra such that the projection $B \to B/I$ is a morphism of dg-bialgebra. ☐

Proposition 4.7. If B is a dg-bialgebra then $\mathrm{Z}(B)$ is a graded sub-bialgebra of B, $\mathrm{B}(B)$ is a graded biideal in $\mathrm{Z}(A)$ and $\mathrm{H}(B)$ is hence a graded bialgebra. ☐

Definition 4.8. If B is a dg-bialgebra then $x \in B$ is primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Lemma 4.9. If B is a dg-bialgebra and $x, y \in B$ are primitive then $[x, y]$ is again primitive. ☐

5 Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product on $\mathrm{Hom}_k(C, A)$ makes $\mathrm{dgHom}(C, A)$ into a dg-algebra. ☐

Dual of dg-coalgebra is a dg-algebra.

Definition 5.2. An **antipode** for a dg-bialgebra H is an inverse S to id_H with respect to the convolution product of $\mathrm{dgHom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**. A **morphism** of dg-Hopf algebra is a morphism of dg-bialgebras. A **dg-Hopf ideal** in H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra does not in general have an underlying Hopf algebra structure.

Remark 5.4. Let H be a dg-Hopf algebra.

(1) The antipode of H is unique.

(2) The antipode S of H is the the unique morphism of dg-vector spaces that makes the following diagram commute:

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "u"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd} \tag{1}$$

This means more explicitely that

$$\sum_{(c)} S(c_{(1)})c_{(2)} = \varepsilon(c)1_H \qquad \text{and} \qquad \sum_{(c)} c_{(1)}S(c_{(2)}) = \varepsilon(c)1_H \,.$$

(No additional signs occur because $|S| = 0$.)

Check that the antipode is an antimorphism.

Lemma 5.5. If I is a dg-Hopf ideal in a dg-Hopf algebra H then H/I inherits from H the structure of a dg-Hopf algebra such that the projection $H \to H/I$ is a morphism of dg-Hopf algebras.

Example 5.6. Let V be a dg-vector space.

(1) The map
$$V \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v \mapsto v \otimes 1 + 1 \otimes v$$
is a morphism of dg-vector space and hence induces a morphism of dg-algebras
$$\Delta\colon\ \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,.$$
The zero map $V \to 0$ induces a morphism of dg-algebras $\varepsilon\colon\ \mathrm{T}(V) \to \mathrm{T}(0) = k$. These make $\mathrm{T}(V)$ into a dg-bialgebra; the necessary diagrams can be checked on the algebra generators V of $\mathrm{T}(V)$. The comultiplication Δ and ε are explicitely given by
$$\begin{aligned}
\Delta(v_1 \cdots v_n) &= \Delta(v_1) \cdots \Delta(v_n) \\
&= (v_1 \otimes 1 + 1 \otimes v_1) \cdots (v_n \otimes 1 + 1 \otimes v_n) \\
&= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} (-1)^{n_p(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}
\end{aligned}$$

where

$$n_p(\sigma) = \sum \left\{ |v_i||v_j| \,\middle|\, 1 \le i \le p,\, p+1 \le j \le n,\, \sigma(i) > \sigma(j) \right\},$$

and the counit is given by

$$\varepsilon(v_1 \cdots v_n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The map

$$V \to \mathrm{T}(V)\,, \quad v \mapsto -v$$

is a morphism of dg-vector spaces and hence induces a morphism of dg-vector spaces

$$S\colon\ \mathrm{T}(V) \to \mathrm{T}(V)^{\mathrm{op}}\,.$$

As a map $S\colon\ \mathrm{T}(V) \to \mathrm{T}(V)$ this is given by

$$S(v_1 \cdots v_n) = (-1)^{n + \sum_{1 \le i < j \le n} |v_i||v_j|} v_n \cdots v_1\,.$$

It can now be checked on monomials that S is an antipode for $\mathrm{T}(V)$, making it a dg-Hopf algebra.[4] This is the **differential graded tensor algebra** on V.

Find an argument to check this only on algebra generators.

(2) The dg-algebra $\mathrm{S}(V) = \mathrm{T}(V)/I$ from Example 2.11 inherits from $\mathrm{T}(V)$ the structure of a dg-Hopf algebra because the dg-ideal

$$I = \big([v, w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-Hopf ideal in $\mathrm{T}(V)$ since

$$\begin{aligned} \varepsilon([v, w]) &= 0\,, \\ \Delta([v, w]) &= [v, w] \otimes 1 + 1 \otimes [v, w] \in I \otimes \mathrm{T}(V) + \mathrm{T}(V) \otimes I \\ S([v, w]) &= -[v, w] \in I\,. \end{aligned}$$

Remark 5.7. Let V, W be a dg-vector space.

(1) The inclusions $V, W \to V \oplus W$ induce morphisms of dg-Hopf algebras

$$\mathrm{S}(V), \mathrm{S}(W) \to \mathrm{S}(V \oplus W)$$

which give an isomorphism of dg-Hopf algebras $\mathrm{S}(V) \otimes \mathrm{S}(W) \to \mathrm{S}(V \oplus W)$.

Let now $\mathrm{char}(k) \ne 2$.

(2) If V is concentrated in even degrees then $\mathrm{S}(V) = \mathrm{S}(V)$ and if V is concentrated in odd degrees then $\mathrm{S}(V) = \bigwedge(V)$, both with the gradings induced by V.

[4] In the resulting expressions the terms for $v_1 \cdots v_p \otimes v_{p+1} \cdots v_n$ and $v_2 \cdots v_p \otimes v_1 v_{p+1} \cdots v_n$ because of signs.

(3) If $V_{\mathrm{even}} = \bigoplus_{n\in\mathbb{Z}} V_{2n}$ and $V_{\mathrm{odd}} = \bigoplus_{n\in\mathbb{Z}} V_{2n+1}$ then

$$\mathrm{S}(V) = \mathrm{S}(V_{\mathrm{even}} \oplus V_{\mathrm{odd}}) \cong \mathrm{S}(V_{\mathrm{even}}) \otimes \mathrm{S}(V_{\mathrm{odd}}) \cong \mathrm{S}(V_{\mathrm{even}}) \otimes \bigwedge(V_{\mathrm{odd}})\,.$$

Moreover, the tensor factors $\mathrm{S}(V_{\mathrm{even}})$ and $\bigwedge(V_{\mathrm{odd}})$ strictly commute in $\mathrm{S}(V)$ (i.e. $xy = yx$) so that

$$\mathrm{S}(V) \cong \mathrm{S}(V_{\mathrm{even}}) \otimes_k \bigwedge(V_{\mathrm{odd}})$$

as algebras, where $\otimes_k$ denotes "tensor product without signs".

Example 5.8 (Exterior Algebra)**.** Let V be a vector space. We may regard V as a dg-vector space centered in degree 1. Then $\mathrm{S}(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a dg-Hopf algebra. But for $\operatorname{char} k \neq 2$ there exists no bialgebra structure on $\Lambda \coloneqq \bigwedge(V)$. Suppose otherwise.

Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker\varepsilon = \bigoplus_{d\geq 1} \bigwedge^n(V) =: I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{\Lambda \otimes I} \qquad \text{and} \qquad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes \Lambda}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}\,.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2}\,,$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}\,.$$

But $v^2 = 0$, hence

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}\,.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

Proposition 5.9. If H is a dg-Hopf algebra with antipode S then the graded bialgebra $\mathrm{H}(H)$ is a graded Hopf algebra with antipode induced by S. □

Example 5.10.

(1) If V is a dg-vector space then

$$\mathrm{H}(\mathrm{T}(V)) = \mathrm{H}\left(\bigoplus_{d\geq 0} V^{\otimes d}\right) \cong \bigoplus_{d\geq 0} \mathrm{H}\big(V^{\otimes d}\big) \cong \bigoplus_{d\geq 0} \mathrm{H}(V)^{\otimes d} = \mathrm{T}(\mathrm{H}(V))$$

as graded vector spaces by the algebraic Künneth isomorphism. We see on representatives that this is already an isomorphism of graded Hopf algebras.

(2)

For $\mathrm{char}(k) = 0$ for the graded commutative algebra using the symmetrization operator.

6 Chevalley–Eilenberg

For this section we fix a Lie algebra $\mathfrak{g}$. The algebra morphism $\varepsilon\colon \mathrm{U}(\mathfrak{g}) \to k = \mathrm{End}_k(k)$ makes the ground field k into a symmetric $\mathrm{U}(\mathfrak{g})$-bimodule.

6.1 The Chevalley–Eilenberg Complex

Definition 6.1. The **Chevalley-Eilenberg complex** of $\mathfrak{g}$ is in degree n given by $\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n(\mathfrak{g})$ and the differential d_{CE} is given by

$$\begin{aligned} &d_{\mathrm{CE}}(u \otimes x_1 \wedge \cdots \wedge x_n) \\ = &\sum_{i=1}^{n} (-1)^i u x_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n \\ &+ \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n \,. \end{aligned}$$

Remark 6.2. If we set $[u, x] \coloneqq ux$ and $(x_0, x_1, \ldots, x_n) \coloneqq x_0 \otimes x_1 \wedge \cdots \wedge x_n$ then

$$d_{\mathrm{CE}}(x_0, x_1, \ldots, x_n) = \sum_{0 \leq i < j \leq n} (-1)^{j+1} (x_0, x_1, \ldots, [x_i, x_j], \ldots, \widehat{x_j}, \ldots, x_n)$$

where $[x_i, x_j]$ appears in the i-th position.

Theorem 6.3. The Chevalley-Eilenberg complex together with the counit $\varepsilon\colon \mathrm{U}(\mathfrak{g}) \to k$ is a projective resolutions of k as a left $\mathrm{U}(\mathfrak{g})$-module.

Proof. A proof due to Koszul using spectrac sequences can be found in [Wei94, Theorem 7.7.2]. A more elementary proof can be found in [HS94, p. VII.4]. $\square$

Remark 6.4. The Lie algebra cohomology of $\mathfrak{g}$ with values in a left $\mathrm{U}(\mathfrak{g})$-module V is given by

$$\mathrm{H}_{\mathrm{Lie}}(\mathfrak{g}, V) \coloneqq \mathrm{Ext}_{\mathrm{U}(\mathfrak{g})}(k, V)$$

and the Lie algebra homology of $\mathfrak{g}$ with values in a a right $\mathrm{U}(\mathfrak{g})$-module V is given by

$$\mathrm{H}^{\mathrm{Lie}}(\mathfrak{g}, V) \coloneqq \mathrm{Tor}^{\mathrm{U}(\mathfrak{g})}(V, k) \,.$$

The Chevalley-Eilenberg complex can be used to compute these: The Lie algebra cohomology of $\mathfrak{g}$ is the cohomology of the cochain complex

$$\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}\left(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}), V\right) \cong \mathrm{Hom}_k\left(\bigwedge(\mathfrak{g}), V\right)$$

that is in degree n given by

$$\mathrm{Hom}_k\left(\bigwedge^n(\mathfrak{g}), V\right) \cong \{\text{alternating multilinear maps } \mathfrak{g}^{\times n} \to V\}$$

and has the differential

$$d(\omega)(x_1,\dots,x_n) = \sum_{i=1}^{n}(-1)^i x_i\omega(x_1,\dots,\widehat{x_i},\dots,x_n) + \sum_{1\le i<j\le n}(-1)^{i+j-1}\omega([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_n)\,.$$

The Lie algebra homology of $\mathfrak{g}$ is the homology of the chain complex

$$V\otimes_{\mathrm{U}(\mathfrak{g})}\mathrm{U}(\mathfrak{g})\otimes_k \bigwedge(\mathfrak{g}) \cong V\otimes_k \bigwedge(\mathfrak{g})$$

that has the differential

$$\begin{aligned} &d(v\otimes x_1\wedge\cdots\wedge x_n)\\ =&\sum_{i=1}^{n}(-1)^i(v\cdot x_i)\otimes x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_n\\ &+\sum_{1\le i<j\le n}(-1)^{i+j-1}v\otimes[x_i,x_j]\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge\widehat{x_j}\wedge\cdots\wedge x_n\,. \end{aligned}$$

6.2 The Chevalley–Eilenberg Coalgebra

For $V=\mathfrak{g}$ we get a chain complex $\bigwedge\mathfrak{g}$ with Chevalley–Eilenberg differential

$$d_{\mathrm{CE}}(x_1\wedge\cdots\wedge x_n) = \sum_{1\le i<j\le n}(-1)^{i+j-1}[x_i,x_j]\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge\widehat{x_j}\wedge\cdots\wedge x_n\,.$$

We observe that the differential $\bigwedge^2\mathfrak{g}\to\mathfrak{g}$ is precisely the Lie bracket $[-,-]$. The composition $\bigwedge^3\mathfrak{g}\to\bigwedge^2\mathfrak{g}\to\mathfrak{g}$ is given by

$$\begin{aligned} x\wedge y\wedge z &\mapsto [x,y]\wedge z - [x,z]\wedge y + [y,z]\wedge x\\ &\mapsto [[x,y],z]-[[x,z],y]+[[y,z],x]\\ &= -\Big([z,[x,y]]+[y,[z,x]]+[x,[y,z]]\Big) \end{aligned}$$

We can regard $\bigwedge\mathfrak{g}$ as a graded coalgebra as in Example 5.8, i.e. such that V consists of primitive elements.

Proposition 6.5. The Chevalley–Eilenberg differential makes $\bigwedge\mathfrak{g}$ into a dg-coalgebra, and it is the unique extension of the Lie bracket to a coderivation of $\bigwedge\mathfrak{g}$. Any coderivation on $\bigwedge\mathfrak{g}$ comes from a Lie algebra structure of $\mathfrak{g}$. This gives a one-to-one correspondence between Lie brackets on $\mathfrak{g}$ and coderivations on $\bigwedge\mathfrak{g}$.

Find a proper reference.

6.3 The Chevalley–Eilenberg Algebra

Write this, check duality with CE-coalgebra.

7 Differential Graded Lie Algebras

Let $\mathrm{char}(k) \neq 2$.

Recall 7.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a map $[-,-]\colon \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is skew-symmetric and for every $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation; this is equivalent to the Jacobi identity $\sum_{\text{cyclic}}[x,[y,z]] = 0$.

Definition 7.2. A **dg-Lie algebra** is a dg-vector space $\mathfrak{g}$ together with a morphism $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is skew symmmetric in the sense that the diagram

$$\begin{tikzcd} \mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\tau"] \arrow[dr, "{[-,-]}"'] & & \mathfrak{g} \otimes \mathfrak{g} \arrow[dl, "{[-,-]}"] \\ & \mathfrak{g} & \end{tikzcd}$$

commutes, and that for every homogeneous $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation (of degree $|x|$).

Remark 7.3. Let $\mathfrak{g}$ be a dg-Lie algebra. We have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j,

$$[x,y] = (-1)^{|x||y|}[y,x]$$

and

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]] \tag{2}$$

and

$$d([x,y]) = [d(x),y] + (-1)^{|x|}[x,d(y)]\,.$$

Condition (2) can be rewritten by the skew-symmetry of $[-,-]$ as

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x,[y,z]] = 0\,.$$

Warning 7.4. A dg-Lie algebra does in general not have an underlying Lie algebra structure.

Example 7.5.

(1) Every dg-algebra A becomes a dg-Lie algebra with respect to the dg-comutator

$$[a,b] \coloneqq ab - (-1)^{|a||b|}ba\,.$$

(2) If A is a graded algebra then the graded subspace $\mathrm{gDer}(A)$ of $\mathrm{gEnd}(A)$ given by

$$\mathrm{gDer}(A)_n = \{\text{derivations of } A \text{ of degree } n\}$$

is a dg-Lie subalgebra of $\mathrm{dgEnd}(A)$.

(3) If B is a dg-bialgebra then the set of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B.

Lemma 7.6. If $\mathfrak{g}$ is a dg-Lie algebra then $\mathrm{Z}(\mathfrak{g})$ is a graded Lie subalgebra of $\mathfrak{g}$, $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$ and $\mathrm{H}(\mathfrak{g})$ is thus an graded Lie algebra.

Definition 7.7. The **universal enveloping algebra** of a dg-Lie algebra $\mathfrak{g}$ is a dg-algebra $\mathrm{U}(\mathfrak{g})$ together with a morphism of dg-Lie algebras $i\colon \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ such that for every other dg-algebra A and every morphism of dg-Lie algebras $f\colon \mathfrak{g} \to A$ there exists a unique morphism of dg-algebras $F\colon \mathrm{U}(\mathfrak{g}) \to A$ that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{U}(\mathfrak{g}) \arrow[r, dashed, "F"] & A \\ \mathfrak{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 7.8. For every dg-Lie algebra $\mathfrak{g}$ a universal enveloping algebra exists. It is unique up to unique isomorphism and can be constructed as

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g}) / \big([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous}\big)$$

together with the composition $i\colon \mathfrak{g} \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$. It inherits from $\mathrm{T}(\mathfrak{g})$ the structure of a dg-Hopf algebra. $\square$

Proof. We check that the given ideal I is a dg-Hopf ideal. It is generated by homegenous elements which satisfy

$$\begin{aligned}
&d([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= d([x,y]_{\mathrm{T}(\mathfrak{g})}) - d([x,y]_{\mathfrak{g}}) \\
&= [d(x),y]_{\mathrm{T}(\mathfrak{g})} + (-1)^{|x|}[x,d(y)]_{\mathrm{T}(\mathfrak{g})} + [d(x),y]_{\mathfrak{g}} + (-1)^{|x|}[x,d(y)]_{\mathfrak{g}} \\
&= \Big([d(x),y]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}}\Big) + (-1)^{|v|}\Big([x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [x,d(y)]_{\mathfrak{g}}\Big) \\
&\in I
\end{aligned}$$

so I is a dg-ideal. Also

$$\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = \varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})}) - \varepsilon([x,y]_{\mathfrak{g}}) = 0 - 0 = 0$$

and

$$\begin{aligned}
&\Delta([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= \Delta([x,y]_{\mathrm{T}(\mathfrak{g})}) - \Delta([x,y]_{\mathfrak{g}})) \\
&= [x,y]_{\mathrm{T}(\mathfrak{g})} \otimes 1 + 1 \otimes [x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x,y]_{\mathfrak{g}} \\
&= ([x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}}) \otimes 1 - 1 \otimes ([x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}}) \\
&\in I \otimes \mathrm{T}(\mathfrak{g}) + \mathrm{T}(\mathfrak{g}) \otimes I\,.
\end{aligned}$$

and finally

$$S([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = S([x,y]_{\mathrm{T}(\mathfrak{g})}) - S([x,y]_{\mathfrak{g}}) = -[x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}} \in I\,.$$

The assertion follows. □

Remark 7.9. Let $\mathfrak{g}$, $\mathfrak{h}$ be a dg-Lie algebras.

(1) The product $\mathfrak{g} \times \mathfrak{h}$ is again a dg-Lie algebra with

$$[(x,y),(x',y')] = ([x,x'],[y,y'])\,.$$

The inclusions $\mathfrak{g}, \mathfrak{h} \to \mathfrak{g} \times \mathfrak{h}$ induce morphisms of dg-Hopf algebras

$$\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{h}) \to \mathrm{U}(\mathfrak{g} \times \mathfrak{h})$$

that results in an isomorphism of dg-Hopf algebras $\mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{h}) \to \mathrm{U}(\mathfrak{g} \times \mathfrak{h})$.

(2) The Hopf algebra structure of $\mathrm{U}(\mathfrak{g})$ is induced from underlying morphisms of dg-Lie algebras: The diagonal morphism $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, $v \mapsto (v,v)$ induces the comultiplication $\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g} \times \mathfrak{g}) \cong \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$, the morphism $\mathfrak{g} \to 0$ induced the counit $\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(0) = k$ and the morphism $\mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$, $v \mapsto -v$ induces the antipode $\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}^{\mathrm{op}}) = \mathrm{U}(\mathfrak{g})^{\mathrm{op}}$.

(3) The famous Poincaré–Birkhoff–Witt theorem generalizes to dg-Lie algebras. It can be expressed as an isomorphism of dg-coalgebra $\mathrm{S}(\mathfrak{g}) \cong \mathrm{U}(\mathfrak{g})$ and show that $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$. See [Qui69, B, Theorem 2.3] and [FHT01, §21 (a)] for more details on this.

(4) It holds that $\mathrm{H}(\mathrm{U}(\mathfrak{g})) \cong \mathrm{U}(\mathrm{H}(\mathfrak{g}))$, see [Qui69, B, Proposition 2.1] or [FHT01, Theorem 21.7].

(5) If H is a graded cocommutative connected[5] dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong \mathrm{U}(\mathbb{P}(H))$, which results in an equivalence between the categories of dg-Lie algebras and graded cocommutative connected dg-Hopf algebras, see [Qui69, B, Theorem 4.5].

8 Homology of the Primitive Part

Theorem 8.1 ([Lod92, Theorem A.9])**.** Let $\mathcal{H}$ be a dg-Hopf algebra. The inclusion $\mathbb{P}(\mathcal{H}) \to \mathcal{H}$ is a morphism of dg-Lie algebras and thus induced a morphism of graded Lie algebras $\mathrm{H}(\mathbb{P}(\mathcal{H})) \to \mathrm{H}(\mathcal{H})$. This morphism restricts to an isomorphism of graded Lie algebras $\mathrm{H}(\mathbb{P}(\mathcal{H})) \to \mathbb{P}(\mathrm{H}(\mathcal{H}))$.

Find a proof.

[5] The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

References

[Bou89] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg New York, 1989, pp. xxiii+709. ISBN: 3-540-64243-9.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory*. Graduate Texts in Mathematics 205. Springer-Verlag New York, 2001, pp. xxxiii+539. ISBN: 978-0-387-95068-6. DOI: 10.1007/978-1-4613-0105-9.

[HS94] Peter J. Hilton and Urs Stammbach. *A Course in Homological Algebra*. Graduate Texts in Mathematics 4. Springer-Verlag New York, 1994, pp. xii+366. ISBN: 978-0-387-94823-2. DOI: 10.1007/978-1-4419-8566-8.

[Lan02] Serge Lang. *Algebra*. Graduate Texts in Mathematics 211. Springer-Verlag New York, 2002, pp. xv+914. ISBN: 978-0-387-95385-4. DOI: 10.1007/978-1-4613-0041-0.

[Lod92] Jean-Lois Loday. *Cyclic Homology*. Grundlagen der mathematischen Wissenschaften 301. Springer Verlag Berlin-Heidelberg, 1992, pp. xix+516. ISBN: 978-3-662-21741-2. DOI: 10.1007/978-3-662-21739-9.

[MO18] David E Speyer. *When is the exterior algebra a Hopf algebra?* November 30, 2018. URL: https://mathoverflow.net/q/316544 (visited on May 7, 2019).

[Qui69] Daniel Quillen. "Rational homotopy theory". In: *Ann. of Math. (2)* 90 (1969), pp. 205–295. ISSN: 0003-486X. DOI: 10.2307/1970725.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, 1994, pp. xiv+450. ISBN: 978-0-521-55987-1. DOI: 10.1017/CBO9781139644136.