

Graduate Seminar on Topology

Differential Graded Hopf Algebras I

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In the following k denotes a field. All vector spaces, all kinds of algebras, all tensor products, etc. are over k , unless otherwise stated. All occurring maps are linear unless otherwise stated. We will sometime assume additional constraints on the characteristic of k , but will make this explicit when it occurs.

1 Preliminary Notions and Notations

A **graded vector space** is a vector space V together with a grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$. The elements $v \in V_n$ are **homogeneous** of **degree** $|v| = n$.

Whenever we write $|v|$ the element v is assumed to be homogeneous.

A map $f: V \rightarrow W$ between graded vector spaces is **graded** of **degree** $|f| = d$ if $f(V_n) \subseteq V_{n+d}$ for all n . A **differential** on V is a map $V \rightarrow V$ of degree -1 . A **dg-vector space** is a graded vector space together with a differential, i.e. a chain complex. A dg-subspace is a chain subcomplex and the usual notions of quotients, direct sums and products and morphisms apply. We will always regard graded objects as differential graded objects with zero differential.

graded = differential graded with $d = 0$

Hence every statement about dg-objects entails a statement about graded objects.

If V and W are graded vector spaces then $V \otimes W$ is again a graded vector space with $|v \otimes w| = |v| + |w|$, i.e. $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. The **twist map** $\tau: V \otimes W \rightarrow W \otimes V$ is given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

We hence adhere to the Koszul-Quillen **sign convention**:

Whenever elements x, y are swapped, the sign $(-1)^{|x||y|}$ is introduced.

If $f: V \rightarrow V'$ and $g: W \rightarrow W'$ are graded maps then $f \otimes g: V \otimes W \rightarrow V' \otimes W'$ is the graded map of degree $|f \otimes g| = |f| + |g|$ with

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

If V, W are dg-vector space then $V \otimes W$ is a dg-vector space with $d_{V \otimes W} = d \otimes \text{id} + \text{id} \otimes d$; more explicitly,

$$d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w).$$

Higher tensor products are defined inductively. The twist map τ is an isomorphism of dg-vector spaces.¹ We regard the ground field k as a dg-vector space concentrated in degree 0. Then the natural isomorphism $k \otimes V \cong V$ and $V \otimes k \cong V$ are isomorphism of dg-vector spaces. The dg-vector space $\text{Hom}(V, W)$ is given by

$$\text{Hom}(V, W)_n = \{\text{graded maps } V \rightarrow W \text{ of degree } n\}$$

with differential

$$d(f) = d \circ f - (-1)^{|f|} f \circ d.$$

If V, W are dg-vector spaces then the **algebraic Künneth isomorphism** is the natural isomorphism of graded vector spaces

$$H(V \otimes W) \cong H(V) \otimes H(W), \quad [v \otimes w] \mapsto [v] \otimes [w].$$

2 Differential Graded Algebra

Definition 2.1. A **differential graded algebra** or **dg-algebra** is a dg-vector space A together with morphisms of dg-vector spaces $m: A \otimes A \rightarrow A$ and $u: k \rightarrow A$ that make the diagrams

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\text{id} \otimes m} & A \otimes A \\ m \otimes \text{id} \downarrow & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array} \quad \begin{array}{ccc} k \otimes A & \xleftarrow{\sim} & A \xrightarrow{\sim} A \otimes k \\ u \otimes \text{id} \downarrow & & \parallel \downarrow \text{id} \otimes u \\ A \otimes A & \xrightarrow{m} & A \xleftarrow{m} A \otimes A \end{array}$$

commute. The dg-algebra A is **graded commutative** if the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow m & \swarrow m \\ & A & \end{array}$$

commutes. A **morphism** of dg-algebras $f: A \rightarrow B$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ m \downarrow & & \downarrow m \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} & k & \\ u \swarrow & & \searrow u \\ A & \xrightarrow{f} & B \end{array}$$

¹The naive twist map $v \otimes w \mapsto w \otimes v$ is not a morphism of dg-vector spaces.

Definition 2.2. A graded map $\delta: A \rightarrow A$ for a graded algebra A is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \text{id} + \text{id} \otimes \delta);$$

more explicitly,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|}a\delta(b).$$

Remark 2.3.

- (1) A dg-algebra is the same as a graded algebra A together with a differential d such that $d(1) = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|}a \cdot d(b),$$

i.e. such that d is a graded derivation (of degree -1).

$$\begin{array}{ccc} \text{graded} & \xrightarrow{\text{multiplication}} & \text{graded} \\ \text{vector spaces} & & \text{algebras} \\ \text{differential} \downarrow & & \downarrow \text{differential} \\ \text{dg-vector spaces} & \xrightarrow{\text{multiplication}} & \text{dg-algebras} \end{array}$$

- (2) The graded commutativity of A means $ab = (-1)^{|a||b|}ba$. If $|a|$ or $|b|$ is even then $ab = ba$; if $|a|$ is odd and $\text{char}(k) \neq 2$ then $a^2 = 0$.
- (3) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

Examples 2.4.

- (1) Every algebra A is a dg-algebra concentrated in degree 0. This holds in particular for $A = k$.
- (2) If V is any dg-vector space then the algebra structure of $\text{End}_k(V)$ restricts to a dg-algebra structure on $\text{End}(V) = \text{Hom}(V, V)$.
- (3) If V is a dg-vector space then $T(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space with

$$|v_1 \cdots v_n| = |v_1| + \cdots + |v_n|$$

and

$$d(v_1 \cdots v_n) = \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_i|} v_1 \cdots d(v_i) \cdots v_n.$$

This makes the tensor $T(V)$ into a dg-algebra. The inclusion $V \rightarrow T(V)$ is a morphism of dg-vector spaces and if A is any other dg-algebra and $f: V \rightarrow A$

any morphism of dg-vector spaces then f extends uniquely to a morphism of dg-algebras $F: T(V) \rightarrow A$:

$$\begin{array}{ccc} T(V) & \xrightarrow{\quad F \quad} & A \\ \uparrow & \nearrow f & \\ V & & \end{array}$$

- (4) If V is any vector space then the symmetric algebra $S(V)$ is a graded algebra and a commutative algebra, but not a graded commutative algebra. The exterior algebra $\bigwedge(V)$ is a graded algebra, it is in general not a commutative algebra (unless $\dim V \leq 1$), but it is a graded commutative algebra.

Lemma 2.5. Let A, B be dg-algebras.

- (1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$\begin{aligned} m_{A \otimes B}: A \otimes B \otimes A \otimes B &\xrightarrow{\text{id} \otimes \tau \otimes \text{id}} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m} A \otimes B \\ u_{A \otimes B}: k &\xrightarrow{\sim} k \otimes k \xrightarrow{u \otimes u} A \otimes B. \end{aligned}$$

More explicitly, $1_{A \otimes B} = 1_A \otimes 1_B$ and $(a \otimes b)(a' \otimes b') = (-1)^{|a'| |b|} aa' \otimes bb'$.

- (2) If $f: A \rightarrow A'$ and $g: B \rightarrow B'$ are morphism of dg-algebras then so is $f \otimes g$.
(3) The twist map $\tau: A \otimes B \rightarrow B \otimes A$ is a morphism of dg-algebras.
(4) If $A = (A, m, u)$ then $A^{\text{op}} = (A, m^{\text{op}}, u)$ with $m^{\text{op}} = m \circ \tau$ is again a dg-algebra. \square

Warning 2.6. If A, B are dg-algebras then the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B . The underlying algebra of A^{op} is not the opposite of the underlying algebra of A . (Both thanks to signs.)

Definition 2.7. A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.

Lemma 2.8. For an ideal I in a dg-algebra A the following conditions are equivalent:

- (1) I is a dg-ideal.
(2) I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α .

Proof. That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, pp. IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α : The ideal I is spanned by $ax_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|} ad(x_\alpha)b + (-1)^{|a|+|x_\alpha|} ax_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$. \square

Lemma 2.9. If I is a dg-ideal in a dg-algebra A then A/I inherits the structure of a dg-algebra such that the projection $A \rightarrow A/I$ is a morphism of dg-algebras. \square

Definition 2.10. If A is a dg-algebra then the **dg-comutator** is given by

$$[a, b] := ab - (-1)^{|a||b|}ba.$$

Example 2.11. Let V be a dg-vector space. The ideal

$$I := ([v, w] \mid v, w \in V \text{ are homogeneous})$$

is a dg-ideal in $T(V)$ since the generators $[v, w]$ are homogeneous with

$$d([v, w]) = [d(v), w] + (-1)^{|v|}[v, d(w)] \in I.$$

The dg-algebra $\Lambda(V) := T(V)/I$ is the **differential graded symmetric algebra** on V . If S is any other graded symmetric dg-algebra and $f: V \rightarrow S$ any morphism of dg-vector space then f extends uniquely to a morphism of dg-algebras $F: \Lambda(V) \rightarrow S$:

$$\begin{array}{ccc} \Lambda(V) & \xrightarrow{F} & S \\ \uparrow & \nearrow f & \\ V & & \end{array}$$

Proposition 2.12. If A is a dg-algebra then $Z(A)$ is a graded subalgebra of A , $B(A)$ is a graded ideal in $Z(A)$ and $H(A)$ is hence a graded algebra.

3 Differential Graded Coalgebras

Definition 3.1. A **differential graded coalgebra** or **dg-coalgebra** is a dg-vector space C together with morphisms of dg-vector space $\Delta: C \rightarrow C \otimes C$ and $\varepsilon: C \rightarrow k$ that make the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \text{id} \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes C \otimes C \end{array} \quad \begin{array}{ccccc} C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C \\ \varepsilon \otimes \text{id} \downarrow & & \parallel & & \downarrow \varepsilon \otimes \text{id} \\ k \otimes C & \xrightarrow{\sim} & C & \xleftarrow{\sim} & C \otimes k \end{array}$$

commute. The dg-coalgebra C is **graded cocommutative** if the following diagram commutes:

$$\begin{array}{ccc} & C & \\ \Delta \swarrow & & \searrow \Delta \\ C \otimes C & \xrightarrow{\tau} & C \otimes C \end{array}$$

A **morphism** of dg-coalgebra $f: C \rightarrow D$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \Delta \downarrow & & \downarrow \Delta \\ C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \end{array} \quad \begin{array}{ccc} C & \xrightarrow{f} & D \\ \varepsilon \searrow & & \swarrow \varepsilon \\ & k & \end{array}$$

Definition 3.2. A graded map $\omega: C \rightarrow C$ of a graded coalgebra is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \text{id} + \text{id} \otimes \omega) \circ \Delta;$$

more explicitly,

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)}).$$

Remark 3.3.

- (1) A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that d vanishes on the zero boundaries and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)}),$$

i.e. such that d is a graded coderivation of degree -1 .

- (2) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}.$$

- (3) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.
- (4) Every coalgebra C is a dg-coalgebra centered in degree 0. This holds in particular for $C = k$.

Example 3.4. Let V be a dg-vector space. Then the induced dg-vector space $T(V)$ becomes a dg-coalgebra with the deconcatenation

$$\begin{aligned} \Delta: T(V) &\rightarrow T(V) \otimes T(V), \quad v_1 \cdots v_n \mapsto \sum_{i=0}^n v_1 \cdots v_i \otimes v_{i+1} \cdots v_n, \\ \varepsilon: T(V) &\rightarrow k, \quad v_1 \cdots v_n \mapsto \delta_{n0}. \end{aligned}$$

Lemma 3.5. Let C, D be dg-coalgebras.

- (1) The tensor product $C \otimes D$ becomes a dg-coalgebra with

$$\begin{aligned} \Delta_{C \otimes D}: C \otimes D &\xrightarrow{\Delta \otimes \Delta} C \otimes C \otimes D \otimes D \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} C \otimes D \otimes C \otimes D \\ \varepsilon_{C \otimes D}: C \otimes D &\xrightarrow{\varepsilon \otimes \varepsilon} k \otimes k \xrightarrow{\sim} k \end{aligned}$$

- (2) If $f: C \rightarrow C'$ and $g: D \rightarrow D'$ are morphism of dg-coalgebras then so is $f \otimes g$.
- (3) The twist map $\tau: C \otimes D \rightarrow D \otimes C$ is a morphism of dg-coalgebras.
- (4) If $C = (C, \Delta, \varepsilon)$ then $C^{\text{cop}} = (C, \Delta^{\text{cop}}, \varepsilon)$ with $\Delta^{\text{op}} = \tau \circ \Delta$ is again a dg-coalgebra.

Warning 3.6. If C, D are dg-coalgebras then the underlying coalgebra of $C \otimes D$ is not the tensor product of the underlying coalgebras of C and D . The underlying coalgebra of C^{op} is not the coopposite of the underlying coalgebra of C . (Again both thanks to signs.)

Definition 3.7. A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.

Lemma 3.8. If I is a dg-coideal in a dg-coalgebra C then C/I inherits the structure of a dg-coalgebra such that the projection $C \rightarrow C/I$ is a morphism of dg-coalgebras. \square

Proposition 3.9. If C is a dg-coalgebra then $Z(C)$ is a graded subcoalgebra of A , $B(C)$ is a graded coideal in $Z(C)$ and $H(C)$ is hence a graded coalgebra. \square

4 Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, let (B, m, u) be a dg-algebra and let (B, Δ, ε) be a dg-coalgebra. Then the following conditions are equivalent:

- (1) Δ and ε are morphisms of dg-algebras.
- (2) m and u are morphisms of dg-coalgebras.

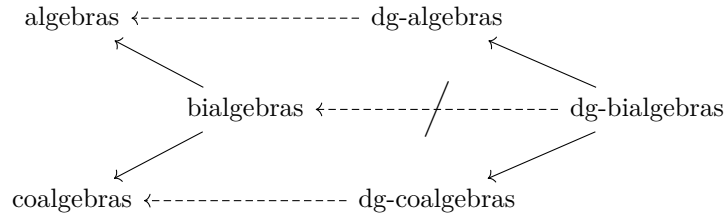
Proof. The same diagrammatic proof as in the non-dg case. \square

Definition 4.2. A **dg-bialgebra** is a quintuple $(B, \mu, u, \Delta, \varepsilon)$ such that the equivalent conditions of Lemma 4.1 are satisfied. A map $f: B \rightarrow C$ is a **morphism** of dg-bialgebras if it is both a morphism of dg-algebras and of dg-coalgebras. A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)}$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta: B \rightarrow B \otimes B$ is a morphism of dg-algebras where the algebra structure on $B \otimes B$ is given by $(b \otimes b') \cdot (b'' \otimes b''') = (-1)^{|b'||b''|} bb'' \otimes b'b'''$. But it is in general not an algebra homomorphism with respect to the multiplication $(b \otimes b') \cdot (b'' \otimes b''') = bb'' \otimes b'b'''$.



We will see an explicit counterexample in Example 5.7.

Lemma 4.5. If B is a dg-bialgebra then B^{op} , B^{cop} and $B^{\text{op,cop}}$ are again dg-bialgebras. \square

Lemma 4.6. If I is a dg-biideal in a dg-bialgebra B then B/I inherits from B the structure of a dg-bialgebra such that the projection $B \rightarrow B/I$ is a morphism of dg-bialgebras. \square

Proposition 4.7. If B is a dg-bialgebra then $Z(B)$ is a graded sub-bialgebra of B , $B(B)$ is a graded biideal in $Z(B)$ and $H(B)$ is hence a graded bialgebra. \square

Definition 4.8. If B is a dg-bialgebra then $x \in B$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Lemma 4.9. If B is a dg-bialgebra and $x, y \in B$ are primitive then $[x, y]$ is again primitive. \square

5 Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product on $\text{Hom}_k(C, A)$ makes $\text{Hom}(C, A)$ into a dg-algebra. \square

Definition 5.2. An **antipode** for a dg-bialgebra H is an inverse S to id_H with respect to the convolution product of $\text{Hom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**. A **morphism** of dg-Hopf algebras is a morphism of dg-bialgebras. A **dg-Hopf ideal** in H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra does in general not have an underlying Hopf algebra structure.

Remark 5.4. Let H be a dg-Hopf algebra.

- (1) The antipode of H is unique.
- (2) The antipode of H is the the unique morphism of dg-vector spaces $S: H \rightarrow H$ that makes the following diagram commute:

$$\begin{array}{ccccc}
 & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\
 \Delta \nearrow & & & & \searrow m \\
 H & \xrightarrow{\varepsilon} & k & \xrightarrow{u} & H \\
 \Delta \searrow & & & & \nearrow m \\
 & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H &
 \end{array} \tag{1}$$

This means more explicitly that

$$\sum_{(c)} S(c_{(1)})c_{(2)} = \varepsilon(c)1_H \quad \text{and} \quad \sum_{(c)} c_{(1)}S(c_{(2)}) = \varepsilon(c)1_H.$$

(No additional signs occur because $|S| = 0$.)

Lemma 5.5. If I is a dg-Hopf ideal in a dg-Hopf algebra H then H/I inherits from H the structure of a dg-Hopf algebra such that the projection $H \rightarrow H/I$ is a morphism of dg-Hopf algebras.

Example 5.6. Let V be a dg-vector space.

(1) The map

$$V \rightarrow T(V) \otimes T(V), \quad v \mapsto v \otimes 1 + 1 \otimes v$$

is a morphism of dg-vector space and hence induces a morphism of dg-algebras

$$\Delta: T(V) \rightarrow T(V) \otimes T(V).$$

The zero map $V \rightarrow 0$ induces a morphism of dg-algebras $\varepsilon: T(V) \rightarrow T(0) = k$. These make $T(V)$ into a dg-bialgebra; the necessary diagrams can be checked on the algebra generators V of $T(V)$ because all arrows occuring in the bialgebra diagrams are now morphisms of dg-algebras. The comultiplication Δ and ε are explicitly given by

$$\begin{aligned} \Delta(v_1 \cdots v_n) &= \Delta(v_1) \cdots \Delta(v_n) \\ &= (v_1 \otimes 1 + 1 \otimes v_1) \cdots (v_n \otimes 1 + 1 \otimes v_n) \\ &= \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} (-1)^{n_p(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)} \end{aligned}$$

where

$$n_p(\sigma) = \sum \left\{ |v_i| |v_j| \mid 1 \leq i \leq p, p+1 \leq j \leq n, \sigma(i) > \sigma(j) \right\},$$

and the counit is given by

$$\varepsilon(v_1 \cdots v_n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The map

$$V \rightarrow T(V)^{\text{op}}, \quad v \mapsto -v$$

is a morphism of dg-vector spaces and hence induces a morphism of dg-algebras

$$S: T(V) \rightarrow T(V)^{\text{op}}.$$

As a map $S: T(V) \rightarrow T(V)$ this is given by

$$S(v_1 \cdots v_n) = (-1)^{\sum_{1 \leq i < j \leq n} |v_i| |v_j|} (-1)^n v_n \cdots v_1.$$

It can now be checked on monomials that S is an antipode for $T(V)$, making it a dg-Hopf algebra.² This is the **differential graded tensor algebra** on V .

²In the resulting expressions the terms for $v_1 \cdots v_p \otimes v_{p+1} \cdots v_n$ and $v_2 \cdots v_p \otimes v_1 v_{p+1} \cdots v_n$ cancel out because of signs.

- (2) The dg-algebra $\Lambda(V) = T(V)/I$ from Example 2.11 inherits from $T(V)$ the structure of a dg-Hopf algebra because the dg-ideal

$$I = ([v, w] \mid v, w \in V \text{ are homogeneous})$$

is a dg-Hopf ideal in $T(V)$ since

$$\begin{aligned}\varepsilon([v, w]) &= 0, \\ \Delta([v, w]) &= [v, w] \otimes 1 + 1 \otimes [v, w] \in I \otimes T(V) + T(V) \otimes I, \\ S([v, w]) &= -[v, w] \in I.\end{aligned}$$

For the computation of Δ we use that v, w are primitive in $T(V)$ and $[v, w]$ is therefore again primitive.

Example 5.7 (Exterior Algebra). Let V be a vector space. We may regard V as a dg-vector space centered in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\text{char } k \neq 2$ there exists no bialgebra structure on $\Lambda := \bigwedge(V)$. Suppose otherwise.

Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker \varepsilon = \bigoplus_{d \geq 1} \bigwedge^d(V) =: I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{I \otimes I} \quad \text{and} \quad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes I}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2},$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

Now $v^2 = 0$, hence

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

Proposition 5.8. If \mathcal{H} is a dg-Hopf algebra with antipode S then the graded bialgebra $H(\mathcal{H})$ is a graded Hopf algebra with antipode induced by S . \square

Example 5.9. If V is a dg-vector space then

$$H(T(V)) = H\left(\bigoplus_{d \geq 0} V^{\otimes d}\right) \cong \bigoplus_{d \geq 0} H(V^{\otimes d}) \cong \bigoplus_{d \geq 0} H(V)^{\otimes d} = T(H(V))$$

as graded vector spaces by the algebraic Künneth isomorphism. We see on representatives that this is already an isomorphism of graded Hopf algebras.

6 Differential Graded Lie Algebras

Let $\text{char}(k) \neq 2$.

Recall 6.1. A Lie algebra is a vector space \mathfrak{g} together with a map $[-, -]: \mathfrak{g} \otimes_k \mathfrak{g} \rightarrow \mathfrak{g}$ such that $[-, -]$ is skew-symmetric and for every $x \in \mathfrak{g}$ the map $[x, -]: \mathfrak{g} \rightarrow \mathfrak{g}$ is a derivation; the last assertion is equivalent to the Jacobi identity $\sum_{\text{cyclic}} [x, [y, z]] = 0$.

Definition 6.2. A **dg-Lie algebra** is a dg-vector space \mathfrak{g} together with a morphism $[-, -]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ such that $[-, -]$ is **graded skew symmetric** in the sense that the diagram

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\tau} & \mathfrak{g} \otimes \mathfrak{g} \\ & \searrow \quad \swarrow & \\ & [-, -] \quad -[-, -] & \\ & \mathfrak{g} & \end{array}$$

commutes, and that for every homogeneous $x \in \mathfrak{g}$ the map $[x, -]: \mathfrak{g} \rightarrow \mathfrak{g}$ is a derivation (of degree $|x|$).

Remark 6.3. Let \mathfrak{g} be a dg-Lie algebra. Then $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j and

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

and

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] \quad (2)$$

and

$$d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)].$$

Condition (2) can be rewritten by the graded skew-symmetry of $[-, -]$ as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x, [y, z]] = 0.$$

Warning 6.4. A dg-Lie algebra does in general not have an underlying Lie algebra structure.

Example 6.5.

- (1) Every dg-algebra A becomes a dg-Lie algebra with respect to the dg-comutator.
- (2) If A is a graded algebra then the graded subspace $\text{Der}(A)$ of $\text{End}(A)$ given by

$$\text{Der}(A)_n = \{\text{derivations of } A \text{ of degree } n\}$$

is a dg-Lie subalgebra of $\text{End}(A)$.

- (3) If B is a dg-bialgebra then the set of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B .

Lemma 6.6. If \mathfrak{g} is a dg-Lie algebra then $Z(\mathfrak{g})$ is a graded Lie subalgebra of \mathfrak{g} , $B(\mathfrak{g})$ is a graded Lie ideal in $Z(\mathfrak{g})$ and $H(\mathfrak{g})$ is thus an graded Lie algebra.

Definition 6.7. The **universal enveloping algebra** of a dg-Lie algebra \mathfrak{g} is a dg-algebra $U(\mathfrak{g})$ together with a morphism of dg-Lie algebras $i: \mathfrak{g} \rightarrow U(\mathfrak{g})$ such that for every other dg-algebra A and every morphism of dg-Lie algebras $f: \mathfrak{g} \rightarrow A$ there exists a unique morphism of dg-algebras $F: U(\mathfrak{g}) \rightarrow A$ that extends f , i.e. that makes the following diagram commute:

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\quad F \quad} & A \\ i \uparrow & \nearrow f & \\ \mathfrak{g} & & \end{array}$$

Proposition 6.8. Every dg-Lie algebra \mathfrak{g} admits a universal enveloping algebra. It is unique up to unique isomorphism and can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous})$$

together with the composition $i: \mathfrak{g} \rightarrow T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. It inherits from $T(\mathfrak{g})$ the structure of a dg-Hopf algebra. \square

Proof. We check that the given ideal I is a dg-Hopf ideal. It is generated by homegeneous elements which satisfy

$$\begin{aligned} & d([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &= d([x, y]_{T(\mathfrak{g})}) - d([x, y]_{\mathfrak{g}}) \\ &= [d(x), y]_{T(\mathfrak{g})} + (-1)^{|x|} [x, d(y)]_{T(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} - (-1)^{|x|} [x, d(y)]_{\mathfrak{g}} \\ &= \left([d(x), y]_{T(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} \right) + (-1)^{|x|} \left([x, d(y)]_{T(\mathfrak{g})} - [x, d(y)]_{\mathfrak{g}} \right) \\ &\in I \end{aligned}$$

so I is a dg-ideal. Also

$$\varepsilon([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) = \varepsilon([x, y]_{T(\mathfrak{g})}) - \varepsilon([x, y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because $[x, y]_{T(\mathfrak{g})}$ and $[x, y]_{\mathfrak{g}}$ are homogeneuous of degree ≥ 1 ,

$$\begin{aligned} & \Delta([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &= \Delta([x, y]_{T(\mathfrak{g})}) - \Delta([x, y]_{\mathfrak{g}}) \\ &= [x, y]_{T(\mathfrak{g})} \otimes 1 + 1 \otimes [x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x, y]_{\mathfrak{g}} \\ &= ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &\in I \otimes T(\mathfrak{g}) + T(\mathfrak{g}) \otimes I \end{aligned}$$

and finally

$$S([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) = S([x, y]_{T(\mathfrak{g})}) - S([x, y]_{\mathfrak{g}}) = -[x, y]_{T(\mathfrak{g})} + [x, y]_{\mathfrak{g}} \in I.$$

Thus the dg-ideal I is already a dg-Hopf ideal. \square

Remark 6.9. Let $\mathfrak{g}, \mathfrak{h}$ be a dg-Lie algebras.

- (1) The product $\mathfrak{g} \times \mathfrak{h}$ is again a dg-Lie algebra with

$$[(x, y), (x', y')] = ([x, x'], [y, y']).$$

The inclusions $\mathfrak{g}, \mathfrak{h} \rightarrow \mathfrak{g} \times \mathfrak{h}$ induce morphisms of dg-Hopf algebras

$$U(\mathfrak{g}), U(\mathfrak{h}) \rightarrow U(\mathfrak{g} \times \mathfrak{h})$$

that results in an isomorphism of dg-Hopf algebras

$$U(\mathfrak{g}) \otimes U(\mathfrak{h}) \cong U(\mathfrak{g} \times \mathfrak{h}).$$

- (2) The Hopf algebra structure of $U(\mathfrak{g})$ is induced from underlying morphisms of dg-Lie algebras: The diagonal morphism $\mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$, $v \mapsto (v, v)$ induces the comultiplication

$$U(\mathfrak{g}) \rightarrow U(\mathfrak{g} \times \mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

the morphism $\mathfrak{g} \rightarrow 0$ induced the counit

$$U(\mathfrak{g}) \rightarrow U(0) = k$$

and the morphism $\mathfrak{g} \rightarrow \mathfrak{g}^{\text{op}}$, $v \mapsto -v$ induces the antipode

$$U(\mathfrak{g}) \rightarrow U(\mathfrak{g}^{\text{op}}) = U(\mathfrak{g})^{\text{op}}$$

- (3) The famous Poincaré–Birkhoff–Witt theorem generalizes to the universal enveloping algebras of dg-Lie algebras. It can be expressed as an isomorphism of dg-coalgebra $\Lambda(\mathfrak{g}) \cong U(\mathfrak{g})$ and show that $\mathbb{P}(U(\mathfrak{g})) = \mathfrak{g}$. This can be found in [Qui69, Appendix B, Theorem 2.3] and [FHT01, §21(a)] for more details on this.
- (4) It holds that $H(U(\mathfrak{g})) \cong U(H(\mathfrak{g}))$, see [Qui69, Appendix B, Proposition 2.1] or [FHT01, Theorem 21.7].
- (5) If H is a graded cocommutative connected³ dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong U(\mathbb{P}(H))$, which results in an equivalence between the categories of dg-Lie algebras and graded cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B, Theorem 4.5].

7 Homology of the Primitive Part

Theorem 7.1 ([Lod92, Theorem A.9]). Let \mathcal{H} be a dg-Hopf algebra. The inclusion $\mathbb{P}(\mathcal{H}) \rightarrow \mathcal{H}$ is a morphism of dg-Lie algebras and thus induced a morphism of graded Lie algebras $H(\mathbb{P}(\mathcal{H})) \rightarrow H(\mathcal{H})$. This morphism restricts to an isomorphism of graded Lie algebras $H(\mathbb{P}(\mathcal{H})) \rightarrow \mathbb{P}(H(\mathcal{H}))$.

³The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

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