Jendrik Stelzner `jendrikstelzner.de/seminars/dg_hopf.pdf` May 20, 2019

Differential Graded Hopf Algebras I

In the following k denotes a field. All vector spaces, algebras, tensor products, etc. are over k, unless otherwise stated. All occuring maps are linear unless otherwise stated. Additional constraints on $\mathrm{char}(k)$ are made explicit when used.

1. Preliminary Notions and Notations

A **graded vector space** is a vector space V together with a **grading** $V = \bigoplus_{n \in \mathbb{Z}} V_n$. The elements $v \in V_n$ are **homogeneous** of **degree** $|v| = n$.

$$\boxed{\text{Whenever we write } |v| \text{ the element } v \text{ is assumed to be homogeneous.}}$$

A map $f \colon V \to W$ between graded vector spaces is **graded** of **degree** $d = |f|$ if $f(V_n) \subseteq V_{n+d}$ for all n. A **differential** on V is a map $V \to V$ of degree -1 with $d^2 = 0$. A **dg-vector space** is a graded vector space together with a differential, i.e. a chain complex; the usual definitions and assertions about chain complexes apply. A **dg-subspace** is a chain subcomplex. We always regard graded objects as differential graded objects with zero differential.

$$\boxed{\text{graded} \longleftrightarrow \text{differential graded with } d = 0}$$

If V, W are graded vector spaces then $V \otimes W$ is also one with $|v \otimes w| = |v| + |w|$, i.e. $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. The **twist map** $\tau \colon V \otimes W \to W \otimes V$ is given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v \,.$$

We hence adhere to the Koszul-Quillen **sign convention**:

$$\boxed{\text{Whenever homogeneous } x, y \text{ are swapped the sign } (-1)^{|x||y|} \text{ is introduced.}}$$

If V, W are dg-vector spaces then $\mathrm{Hom}(V, W)$ is the dg-vector space with

$$\begin{aligned} \mathrm{Hom}(V,W)_n &= \{\text{graded maps } V \to W \text{ of degree } n\} \,, \\ d(f) &= d \circ f - (-1)^{|f|} f \circ d \,. \end{aligned}$$

If $f \colon V \to V'$, $g \colon W \to W'$ are graded maps then $f \otimes g \colon V \otimes V' \to W \otimes W'$ is given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) \,;$$

in particular $|f \otimes g| = |f| + |g|$. If V, W are dg-vector spaces then $V \otimes W$ is a dg-vector space with $d_{V\otimes W} = d_V \otimes \mathrm{id} + \mathrm{id} \otimes d_W$; more explicitely,

$$d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w)\,.$$

Higher tensor products are defined inductively. The twist map τ is an isomorphism of dg-vector spaces.[1] We regard k as a dg-vector space concentrated in degree 0.

2. Differential Graded Algebras

Definition 2.1. A **differential graded algebra** or **dg-algebra** is a dg-vector space A together with morphisms of dg-vector spaces $m\colon A \otimes A \to A$ and $u\colon k \to A$ that make the algebra diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes \mathrm{id}"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[d, "u\otimes \mathrm{id}"'] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & A \otimes k \arrow[d, "\mathrm{id}\otimes u"] \\ A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

commute. The dg-algebra A is **graded commutative** if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

commutes. A **morphism** of dg-algebras $f\colon A \to B$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & B \otimes B \arrow[d, "m"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} & k \arrow[dl, "u"'] \arrow[dr, "u"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

Definition 2.2. A graded map $\delta\colon A \to A$ for a graded algebra A is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\,;$$

more explicitely,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b)\,.$$

Remark 2.3.

(1) A dg-algebra is the same as a graded algebra A (in particular $|1| = 0$) together with a differential d such that $d(1) = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)\,,$$

i.e. such that d is a graded derivation (of degree -1).

[1] The naive twist map $v \otimes w \mapsto w \otimes v$ is not a morphism of dg-vector spaces.

(2) The graded commutativity of A means $ab = (-1)^{|a||b|}ba$. If $|a|$ or $|b|$ is even then $ab = ba$; if $|a|$ is odd and $\mathrm{char}(k) \neq 2$ then $a^2 = 0$.

(3) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

Examples 2.4.

(1) Every algebra A is a dg-algebra concentrated in degree 0, in particular $A = k$.

(2) For any dg-vector space V the algebra structure of $\mathrm{End}_k(V)$ restricts to a dg-algebra structure on $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

(3) If V is a dg-vector space then $\mathrm{T}(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space with

$$|v_1 \cdots v_n| = |v_1| + \cdots + |v_n| \,,$$
$$d(v_1 \cdots v_n) = \sum_{i=1}^{n} (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \,.$$

This makes $\mathrm{T}(V)$ into a dg-algebra, with multiplication given by concatination

$$(v_1 \cdots v_n) \cdot (v_{n+1} \cdots v_m) = v_1 \cdots v_m \,.$$

The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector spaces and if $f\colon V \to A$ is any morphism of dg-vector spaces into a dg-algebra A then f extends uniquely to a morphism of dg-algebras $F\colon \mathrm{T}(V) \to A$:

$$\begin{tikzcd} \mathrm{T}(V) \arrow[r, dashed, "F"] & A \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

The dg-algebra $\mathrm{T}(V)$ is the **differential graded tensor algebra** on V.

Lemma 2.5. Let A, B be dg-algebras.

(1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$m_{A \otimes B}\colon A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m} A \otimes B$$
$$u_{A \otimes B}\colon k \xrightarrow{\sim} k \otimes k \xrightarrow{u \otimes u} A \otimes B \,.$$

More explicitely, $1_{A \otimes B} = 1_A \otimes 1_B$ and $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2$.

(2) If $f\colon A \to A'$ and $g\colon B \to B'$ are morphism of dg-algebras then so is $f \otimes g$.

(3) The twist map $\tau\colon A \otimes B \to B \otimes A$ is an isomorphism of dg-algebras.

(4) The dg-algebra A^{op} is given by $u_{A^{\mathrm{op}}} = u_A$ and $m^{\mathrm{op}} = m_A \circ \tau$. If $\cdot$ denotes the multiplication in A and $*$ the multiplication in A^{op} then more explicitely

$$1_A = 1_{A^{\mathrm{op}}}\,, \qquad a * b = (-1)^{|a||b|} b \cdot a\,. \qquad \square$$

Warning 2.6. If $A \otimes_k B$ is the non-dg tensor product with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ then $A \otimes B \neq A \otimes_k B$ as algebras, i.e. the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B. The underlying algebra of A^{op} is similarly not the opposite of the underlying algebra of A.

Definition 2.7. A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.[2]

Lemma 2.8. If I is a dg-ideal in A then A/I inherits the structure of a dg-algebra. $\square$

Lemma 2.9. An ideal I is a dg-algebra A is a dg-ideal if and only if I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α. (Being a dg-ideal can be checked on homegenous generators.)

Proof. That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, pp. IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α: The ideal I is spanned by $a x_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(a x_\alpha b) = d(a) x_\alpha b + (-1)^{|a|} a d(x_\alpha) b + (-1)^{|a|+|x_\alpha|} a x_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$. $\square$

Definition 2.10. The **dg-comutator** in a dg-algebra A is the bilinear extension of

$$[a, b] \coloneqq ab - (-1)^{|a||b|} ba\,.$$

Example 2.11. Let V be a dg-vector space. The ideal

$$I \coloneqq \big([v, w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-ideal in $\mathrm{T}(V)$ since the generators $[v, w]$ are homogeneous with

$$d([v, w]) = [d(v), w] + (-1)^{|v|}[v, d(w)] \in I\,.$$

The dg-algebra $\Lambda(V) \coloneqq \mathrm{T}(V)/I$ is the **differential graded symmetric algebra** on V. If S is any other graded symmetric dg-algebra and $f\colon V \to S$ any morphism of dg-vector space then f extends uniquely to a morphism of dg-algebras $F\colon \Lambda(V) \to S$:

$$\begin{tikzcd} \Lambda(V) \arrow[r, dashed, "F"] & S \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 2.12. If A is a dg-algebra then $\mathrm{Z}(A)$ is a graded subalgebra of A, $\mathrm{B}(A)$ is a graded ideal in $\mathrm{Z}(A)$ and $\mathrm{H}(A)$ is hence a graded algebra. $\square$

[2]By an ideal we always mean a two-sided ideal.

3. Differential Graded Coalgebras

Definition 3.1. A **differential graded coalgebra** or **dg-coalgebra** is a dg-vector space C together with morphisms of dg-vector space $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ that make the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ k \otimes C \arrow[r, "\sim"] & C & C \otimes k \arrow[l, "\sim"'] \end{tikzcd}$$

commute. The dg-coalgebra C is **graded cocommutative** if the diagram

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

commutes. A **morphism** of dg-coalgebra $f\colon C \to D$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & D \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}$$

Definition 3.2. A graded map $\omega\colon C \to C$ of a graded coalgebra is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \mathrm{id} + \mathrm{id} \otimes \omega) \circ \Delta\,;$$

more explicitely,

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)})\,.$$

Remark 3.3.

(1) A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that d vanishes on $\mathrm{B}_0(C)$ and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)})\,,$$

i.e. such that d is a graded coderivation of degree -1.

(2) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}\,.$$

(3) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.

(4) Every coalgebra C is a dg-coalgebra centered in degree 0, in particular $C = k$.

Lemma 3.4. Let C, D be dg-coalgebras.

(1) The tensor product $C \otimes D$ becomes a dg-coalgebra with

$$\Delta_{C\otimes D} \colon C \otimes D \xrightarrow{\Delta\otimes\Delta} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} C \otimes D \otimes C \otimes D$$
$$\varepsilon_{C\otimes D} \colon C \otimes D \xrightarrow{\varepsilon\otimes\varepsilon} k \otimes k \xrightarrow{\sim} k$$

(2) If $f \colon C \to C'$ and $g \colon D \to D'$ are morphism of dg-coalgebras then so is $f \otimes g$.

(3) The twist map $\tau \colon C \otimes D \to D \otimes C$ is a morphism of dg-coalgebras.

(4) If $C = (C, \Delta, \varepsilon)$ then $C^{\mathrm{cop}} = (C, \Delta^{\mathrm{cop}}, \varepsilon)$ with $\Delta^{\mathrm{op}} = \tau \circ \Delta$ is again a dg-coalgebra.

Warning 3.5. If $C \otimes_k D$ is the non-dg tensor product then $C \otimes D \neq C \otimes_k D$ as coalgebras, i.e. the underlying coalgebra of $C \otimes D$ is not the tensor product of the underlying coalgebras of C and D. The underlying coalgebra of C^{op} is similarly not the coopposite of the underlying coalgebra of C.

Definition 3.6. A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.

Lemma 3.7. If I is a dg-coideal in C then C/I inherits a dg-coalgebra structure. $\square$

Proposition 3.8. If C is a dg-coalgebra then $\mathrm{Z}(C)$ is a graded subcoalgebra of A, $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ and $\mathrm{H}(C)$ is hence a graded coalgebra. $\square$

4. Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, (B, m, u) a dg-algebra and (B, Δ, ε) a dg-coalgebra. Then the following are equivalent:

(1) Δ and ε are morphisms of dg-algebras.

(2) m and u are morphisms of dg-coalgebras. $\square$

Definition 4.2. A **dg-bialgebra** is a quintuple $(B, \mu, u, \Delta, \varepsilon)$ such that the equivalent conditions of Lemma 4.1 are satisfied. A map $f \colon B \to C$ is a **morphism** of dg-bialgebras if it is both a morphism of dg-algebras and of dg-coalgebras. A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)}$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta\colon B \to B \otimes B$ is a morphism of dg-algebras where the algebra structure on $B \otimes B$ is given by $(b \otimes b') \cdot (b'' \otimes b''') = (-1)^{|b'||b''|} bb'' \otimes b'b'''$. But it is in general not an algebra homomorphism with respect to the multiplication $(b \otimes b') \cdot (b'' \otimes b''') = bb'' \otimes b'b'''$.

$$\begin{tikzcd}
\text{algebras} & & \text{dg-algebras} \arrow[ll, dashed] & \\
& \text{bialgebras} \arrow[ul] \arrow[dl] & & \text{dg-bialgebras} \arrow[ul] \arrow[dl] \arrow[ll, dashed, "/" marking] \\
\text{coalgebras} & & \text{dg-coalgebras} \arrow[ll, dashed] &
\end{tikzcd}$$

We will see an explicit counterexample in Example 5.7.

Lemma 4.5. If I is a dg-biideal in B then B/I inherits a dg-bialgebra structure. $\square$

Proposition 4.6. If B is a dg-bialgebra then $\mathrm{Z}(B)$ is a graded sub-bialgebra of B, $\mathrm{B}(B)$ is a graded biideal in $\mathrm{Z}(B)$ and $\mathrm{H}(B)$ is hence a graded bialgebra. $\square$

Definition 4.7. If B is a dg-bialgebra then $x \in B$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Lemma 4.8. If $x, y \in B$ are primitive then $[x, y]$ is again primitive. $\square$

5. Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product on $\mathrm{Hom}_k(C, A)$ makes $\mathrm{Hom}(C, A)$ into a dg-algebra. $\square$

Definition 5.2. An **antipode** for a dg-bialgebra H is an inverse S to id_H with respect to the convolution product of $\mathrm{Hom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**. A **morphism** of dg-Hopf algebras is a morphism of dg-bialgebras. A **dg-Hopf ideal** in H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra need not have an underlying Hopf algebra structure.

Remark 5.4. The antipode of a dg-Hopf algebra H is the the unique morphism of dg-vector spaces $S\colon H \to H$ that makes the diagram

$$\begin{tikzcd}
& H \otimes H \arrow[r, "S \otimes \mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & k \arrow[r, "u"] & H \\
& H \otimes H \arrow[r, "\mathrm{id} \otimes S"'] & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd} \tag{1}$$

commute. This means more explicitely that

$$\sum_{(c)} S(c_{(1)})c_{(2)} = \varepsilon(c)1_H \qquad \text{and} \qquad \sum_{(c)} c_{(1)}S(c_{(2)}) = \varepsilon(c)1_H \,.$$

(No additional signs occur because $|S| = 0$.)

Lemma 5.5. If I is a dg-Hopf ideal in H then H/I a dg-Hopf algebra structure. ☐

Example 5.6. Let V be a dg-vector space.

(1) The map

$$V \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v \mapsto v \otimes 1 + 1 \otimes v$$

is a morphism of dg-vector spaces and hence induces a morphism of dg-algebras

$$\Delta\colon \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,.$$

The zero map $V \to 0$ induces a morphism of dg-algebras

$$\varepsilon\colon \mathrm{T}(V) \to \mathrm{T}(0) = k\,.$$

These maps make $\mathrm{T}(V)$ into a dg-bialgebra; the necessary diagrams can be checked on the algebra generators V of $\mathrm{T}(V)$ because all arrows occuring in the bialgebra diagrams are morphisms of dg-algebras. The maps Δ and ε are explicitely given by

$$\begin{aligned}\Delta(v_1 \cdots v_n) &= \Delta(v_1) \cdots \Delta(v_n) \\ &= (v_1 \otimes 1 + 1 \otimes v_1) \cdots (v_n \otimes 1 + 1 \otimes v_n) \\ &= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} (-1)^{n_p(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}\end{aligned}$$

where

$$n_p(\sigma) = \sum \Big\{ |v_i||v_j| \;\Big|\; 1 \leq i \leq p,\ p+1 \leq j \leq n,\ \sigma(i) > \sigma(j) \Big\}\,,$$

and

$$\varepsilon(v_1 \cdots v_n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The map

$$V \to \mathrm{T}(V)^{\mathrm{op}}\,, \quad v \mapsto -v$$

is a morphism of dg-vector spaces and hence induces a morphism of dg-algebras

$$S\colon \mathrm{T}(V) \to \mathrm{T}(V)^{\mathrm{op}}\,.$$

As a map $S\colon \mathrm{T}(V) \to \mathrm{T}(V)$ this is given by

$$S(v_1 \cdots v_n) = (-1)^{\sum_{1 \leq i < j \leq n} |v_i||v_j|} (-1)^n v_n \cdots v_1\,.$$

It can now be checked on the monomials $v_1 \cdots v_n$ that S is an antipode for $\mathrm{T}(V)$, making it a dg-Hopf algebra.

(2) The dg-algebra $\Lambda(V) = \mathrm{T}(V)/I$ from Example 2.11 inherits from $\mathrm{T}(V)$ the structure of a dg-Hopf algebra because the dg-ideal

$$I = \big([v,w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-Hopf ideal in $\mathrm{T}(V)$, since

$$\begin{aligned}
\varepsilon([v,w]) &= 0\,, \\
\Delta([v,w]) &= [v,w] \otimes 1 + 1 \otimes [v,w] \in I \otimes \mathrm{T}(V) + \mathrm{T}(V) \otimes I\,, \\
S([v,w]) &= -[v,w] \in I\,.
\end{aligned}$$

For the computation of Δ we use that v, w are primitive in $\mathrm{T}(V)$ and $[v,w]$ is therefore again primitive.

Example 5.7 (Exterior Algebra)**.** Let V be a vector space. We regard V as a dg-vector space concentrated in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\operatorname{char} k \neq 2$ there exists no bialgebra structure on $\Lambda \coloneqq \bigwedge(V)$; we prove this in Appendix A.1.

Proposition 5.8. If $\mathcal{H}$ is a dg-Hopf algebra with antipode S then the graded bialgebra $\mathrm{H}(\mathcal{H})$ is a graded Hopf algebra with antipode induced by S. $\square$

Example 5.9. If V is a dg-vector space then

$$\mathrm{H}(\mathrm{T}(V)) = \mathrm{H}\left(\bigoplus_{d\geq 0} V^{\otimes d}\right) \cong \bigoplus_{d\geq 0} \mathrm{H}\big(V^{\otimes d}\big) \cong \bigoplus_{d\geq 0} \mathrm{H}(V)^{\otimes d} = \mathrm{T}(\mathrm{H}(V))$$

as graded vector spaces by the algebraic Künneth isomorphism. We see on representatives that this is already an isomorphism of graded Hopf algebras.

6. Differential Graded Lie Algebras

Let $\operatorname{char}(k) \neq 2$.

Recall 6.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a map $[-,-]\colon \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is skew-symmetric and for every $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation; the last assertion is equivalent to the Jacobi identity $\sum_{\text{cyclic}}[x,[y,z]] = 0$.

Definition 6.2. A **dg-Lie algebra** is a dg-vector space $\mathfrak{g}$ together with a morphism $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is **graded skew symmmetric**, i.e. such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\tau"] \arrow[dr, "{[-,-]}"'] & & \mathfrak{g} \otimes \mathfrak{g} \arrow[dl, "{-[-,-]}"] \\
& \mathfrak{g} &
\end{tikzcd}$$

commutes, and such that $[x,-]$ is for every x a derivation of degree $|x|$.

Remark 6.3. Let $\mathfrak{g}$ be a dg-Lie algebra. Then $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j and

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

and

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] \tag{2}$$

and

$$d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)] \,.$$

We can rewrite (2) as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x, [y, z]] = 0 \,.$$

Warning 6.4. A dg-Lie algebra need not have an underlying Lie algebra structure.

Example 6.5.

(1) Every dg-algebra A becomes a dg-Lie algebra with the dg-comutator.

(2) For a graded algebra A then the graded subspace $\mathrm{Der}(A) \subseteq \mathrm{End}(A)$ given by

$$\mathrm{Der}(A)_n \coloneqq \{\text{derivations of } A \text{ of degree } n\} \subseteq \mathrm{End}(A)_n$$

is a dg-Lie subalgebra of $\mathrm{End}(A)$.

(3) In any dg-bialgebra B the subspace of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B.

Lemma 6.6. If $\mathfrak{g}$ is a dg-Lie algebra then $\mathrm{Z}(\mathfrak{g})$ is a graded Lie subalgebra of $\mathfrak{g}$, $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$ and $\mathrm{H}(\mathfrak{g})$ is thus an graded Lie algebra.

Definition 6.7. The **universal enveloping algebra** of a dg-Lie algebra $\mathfrak{g}$ is a dg-algebra $\mathrm{U}(\mathfrak{g})$ together with a morphism of dg-Lie algebras $i\colon \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ such that for every other dg-algebra A and every morphism of dg-Lie algebras $f\colon \mathfrak{g} \to A$ there exists a unique morphism of dg-algebras $F\colon\ \mathrm{U}(\mathfrak{g}) \to A$ that extends f:

$$\begin{tikzcd} \mathrm{U}(\mathfrak{g}) \arrow[r, dashed, "F"] & A \\ \mathfrak{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 6.8. Every dg-Lie algebra $\mathfrak{g}$ admits a universal enveloping algebra. It is unique up to unique isomorphism and can be constructed as

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g}) / \big([x, y]_{\mathrm{T}(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous}\big)$$

together with the composition $i\colon \mathfrak{g} \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$. It inherits from $\mathrm{T}(\mathfrak{g})$ the structure of a dg-Hopf algebra. $\square$

Proof. We check that the given ideal I is a dg-Hopf ideal. It is generated by homegenous elements which satisfy

$$\begin{aligned}
&d([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= d([x,y]_{\mathrm{T}(\mathfrak{g})}) - d([x,y]_{\mathfrak{g}}) \\
&= [d(x),y]_{\mathrm{T}(\mathfrak{g})} + (-1)^{|x|}[x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}} - (-1)^{|x|}[x,d(y)]_{\mathfrak{g}} \\
&= \left([d(x),y]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}}\right) + (-1)^{|x|}\left([x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [x,d(y)]_{\mathfrak{g}}\right) \\
&\in I
\end{aligned}$$

so it is a dg-ideal. Also

$$\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = \varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})}) - \varepsilon([x,y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are homogoneous of degree ≥ 1,

$$\begin{aligned}
&\Delta([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= \Delta([x,y]_{\mathrm{T}(\mathfrak{g})}) - \Delta([x,y]_{\mathfrak{g}})) \\
&= [x,y]_{\mathrm{T}(\mathfrak{g})} \otimes 1 + 1 \otimes [x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x,y]_{\mathfrak{g}} \\
&= ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&\in I \otimes \mathrm{T}(\mathfrak{g}) + \mathrm{T}(\mathfrak{g}) \otimes I
\end{aligned}$$

since both $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are primitive, and finally

$$S([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = S([x,y]_{\mathrm{T}(\mathfrak{g})}) - S([x,y]_{\mathfrak{g}}) = -[x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}} \in I\,.$$

Thus the dg-ideal I is already a dg-Hopf ideal. □

Remark 6.9. Let $\mathfrak{g}$, $\mathfrak{h}$ be a dg-Lie algebras.

(1) The famous Poincaré–Birkhoff–Witt theorem generalizes to the universal enveloping algebras of dg-Lie algebras. It can be expressed as an isomorphism of dg-coalgebra $\Lambda(\mathfrak{g}) \cong \mathrm{U}(\mathfrak{g})$ and shows that $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$. Details on this can be found in [Qui69, Appendix B, Theorem 2.3] and [FHT01, §21(a)].

(2) It holds that $\mathrm{H}(\mathrm{U}(\mathfrak{g})) \cong \mathrm{U}(\mathrm{H}(\mathfrak{g}))$, see [Qui69, Appendix B, Proposition 2.1] or [FHT01, Theorem 21.7].

(3) If H is a graded cocommutative connected[3] dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong \mathrm{U}(\mathbb{P}(H))$, which results in an equivalence between the categories of dg-Lie algebras and cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B, Theorem 4.5].

[3]The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

A. Calculations and Proofs

A.1. Example 5.7

Suppose that there exists a bialgebra structure on $\bigwedge(V)$. Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker \varepsilon = \bigoplus_{d \geq 1} \bigwedge^n(V) =: I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{\Lambda \otimes I} \qquad \text{and} \qquad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes \Lambda}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2},$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

Now $v^2 = 0$, hence

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

References

[Bou89] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics. Springer-Verlag Berlin Heidelberg New York, 1989, pp. xxiii+709. ISBN: 3-540-64243-9.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory*. Graduate Texts in Mathematics 205. Springer-Verlag New York, 2001, pp. xxxiii+539. ISBN: 978-0-387-95068-6. DOI: `10.1007/978-1-4613-0105-9`.

[Lan02] Serge Lang. *Algebra*. Graduate Texts in Mathematics 211. Springer-Verlag New York, 2002, pp. xv+914. ISBN: 978-0-387-95385-4. DOI: `10.1007/978-1-4613-0041-0`.

[MO18] David E Speyer. *When is the exterior algebra a Hopf algebra?* November 30, 2018. URL: `https://mathoverflow.net/q/316544` (visited on May 7, 2019).

[Qui69] Daniel Quillen. "Rational homotopy theory". In: *Ann. of Math. (2)* 90 (1969), pp. 205–295. ISSN: 0003-486X. DOI: `10.2307/1970725`.