

# Differential Graded Hopf Algebras I

## 1. Conventions and Notations

In the following  $k$  denotes an arbitrary field. All vector spaces, algebras, tensor products, etc. are over  $k$ , unless otherwise stated. All occurring maps are linear unless otherwise stated. We abbreviate “differential graded” by “dg”.

A **dg-vector space** is the same as a chain complex. of vector spaces, a **dg-subspace** the same as a chain subcomplex. We write  $|v|$  for the degree of an element  $v$ , which is then assumed to be homogeneous. We always regard graded objects as dg-objects with zero differential. We regard  $k$  as a dg-vector space concentrated in degree 0.

If  $V, W$  are dg-vector spaces then  $V \otimes W$  is a dg-vector space with

$$|v \otimes w| = |v| + |w|, \quad d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w).$$

The **twist map**  $\tau: V \otimes W \rightarrow W \otimes V$  given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

is an isomorphism of dg-vector spaces.<sup>1</sup> We use the Koszul **sign convention**: Whenever homogeneous  $x, y$  are swapped the sign  $(-1)^{|x||y|}$  is introduced. This results in a well-defined  $S_n$ -action on  $V^{\otimes n}$  via homomorphisms of dg-vector spaces, given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1, \dots, v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for homogeneous  $v_i$ , where  $\varepsilon_{v_1, \dots, v_n}(\sigma)$  is the **Koszul sign**. (See Appendix A.2.)

## 2. Differential Graded Algebras

**Definition 2.1.**

- (1) A **dg-algebra** is a dg-vector space  $A$  together with homomorphisms of dg-vector spaces  $m: A \otimes A \rightarrow A$  and  $u: k \rightarrow A$  that make the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\text{id} \otimes m} & A \otimes A \\ m \otimes \text{id} \downarrow & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array} \quad \begin{array}{ccc} k \otimes A & \xleftarrow{\sim} & A \xrightarrow{\sim} A \otimes k \\ u \otimes \text{id} \downarrow & & \parallel \downarrow \text{id} \otimes u \\ A \otimes A & \xrightarrow{m} & A \xleftarrow{m} A \otimes A \end{array}$$

<sup>1</sup>The naive twist map  $v \otimes w \mapsto w \otimes v$  is not a homomorphism of dg-vector spaces.

(2) The dg-algebra  $A$  is **graded commutative** if the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow m & \swarrow m \\ & A & \end{array}$$

(3) A **dg-ideal** in a dg-algebra  $A$  is a dg-subspace that is also an ideal.<sup>2</sup>

**Remark 2.2.** A dg-algebra is the same as a graded algebra  $A$  (in particular  $|1| = 0$ ) together with a differential  $d$  satisfying  $d(1) = 0$  and the **graded Leibniz rule**

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b). \quad (1)$$

(See Appendix A.3 for further remarks.)

**Examples 2.3.** (See Appendix A.4 for the explicit calculations and further examples.)

- (1) Every algebra  $A$  is a dg-algebra concentrated in degree 0, in particular  $A = k$ .
- (2) If  $V$  is a dg-vector space then  $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$  is again a dg-vector space with

$$\begin{aligned} |v_1 \cdots v_n| &= |v_1| + \cdots + |v_n|, \\ d(v_1 \cdots v_n) &= \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n. \end{aligned}$$

This makes  $T(V)$  into a dg-algebra, with multiplication given by concatenation

$$(v_1 \cdots v_i) \cdot (v_{i+1} \cdots v_n) = v_1 \cdots v_n.$$

The inclusion  $V \rightarrow T(V)$  is a homomorphism of dg-vector spaces and if  $f: V \rightarrow A$  is any homomorphism of dg-vector spaces into a dg-algebra  $A$  then  $f$  extends uniquely to a homomorphism of dg-algebras  $F: T(V) \rightarrow A$ :

$$\begin{array}{ccc} T(V) & \xrightarrow{F} & A \\ \uparrow & \nearrow f & \\ V & & \end{array}$$

The dg-algebra  $T(V)$  is the **dg-tensor algebra** on  $V$ .

**Proposition 2.4** (Constructions with dg-algebras). Let  $A, B$  be a dg-algebras.

- (1) The tensor product  $A \otimes B$  becomes a dg-algebra with

$$1_{A \otimes B} = 1_A \otimes 1_B \quad \text{and} \quad m_{A \otimes B} = (m_A \otimes m_B) \circ (\text{id} \otimes \tau \otimes \text{id}),$$

$$\text{i.e. } (a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2.$$

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<sup>2</sup>By an “ideal” we always mean a two-sided ideal.

(2) The dg-algebra  $A^{\text{op}}$  is given by  $u_{A^{\text{op}}} = u_A$  and  $m_{A^{\text{op}}} = m_A \circ \tau$ , i.e.

$$1_A = 1_{A^{\text{op}}} \quad \text{and} \quad a \cdot_{\text{op}} b = (-1)^{|a||b|} b \cdot a.$$

(3) If  $I$  is a dg-ideal in  $A$  then  $A/I$  inherits the structure of a dg-algebra

(4) If  $A$  is a dg-algebra then  $Z(A)$  is a graded subalgebra of  $A$ ,  $B(A)$  is a graded ideal in  $Z(A)$  and  $H(A)$  is hence a graded algebra.

*Proof.* See Appendix A.5.  $\square$

**Lemma 2.5.** An ideal  $I$  in a dg-algebra  $A$  is a dg-ideal if and only if  $I$  is generated by homogeneous elements  $x_\alpha$  with  $d(x_\alpha) \in I$  for every  $\alpha$ .

*Proof.* See Appendix A.6.  $\square$

**Definition 2.6.** The **graded commutator** in a dg-algebra  $A$  is the unique bilinear extension of

$$[a, b] := ab - (-1)^{|a||b|} ba.$$

(See Appendix A.7 for a remark.)

**Example 2.7.** Let  $V$  be a dg-vector space. The ideal

$$I := ([v, w] \mid v, w \in V \text{ are homogeneous})$$

is a dg-ideal in  $T(V)$ , and the quotient  $\Lambda(V) := T(V)/I$  is the **dg-symmetric algebra** on  $V$ . (See Appendix A.8 for the explicit calculations and further remarks about  $\Lambda(V)$ .)

### 3. Differential Graded Coalgebras

**Definition 3.1.**

(1) A **dg-coalgebra** is a dg-vector space  $C$  together with homomorphisms of dg-vector spaces  $\Delta: C \rightarrow C \otimes C$  and  $\varepsilon: C \rightarrow k$  that make the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \text{id} \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes C \otimes C \end{array} \quad \begin{array}{ccccc} C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C \\ \varepsilon \otimes \text{id} \downarrow & & \parallel & & \downarrow \text{id} \otimes \varepsilon \\ k \otimes C & \xrightarrow{\sim} & C & \xleftarrow{\sim} & C \otimes k \end{array}$$

(2) The dg-coalgebra  $C$  is **graded cocommutative** if the following diagram commutes:

$$\begin{array}{ccc} & C & \\ \Delta \swarrow & & \searrow \Delta \\ C \otimes C & \xrightarrow{\tau} & C \otimes C \end{array}$$

(3) A **dg-coideal** in a dg-coalgebra  $C$  is a dg-subspace that is a coideal.<sup>3</sup>

**Remark 3.2.** A dg-coalgebra is the same as a graded coalgebra  $C$  together with a differential  $d$  such that  $\varepsilon$  vanishes on  $B_0(C)$  and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)}). \quad (2)$$

(See Appendix A.9 for further remarks.)

**Example 3.3.** For any dg-vector space  $V$  the induced dg-vector space  $T(V)$  becomes a dg-coalgebra with the deconcatination

$$\begin{aligned} \Delta: T(V) &\rightarrow T(V) \otimes T(V), \quad v_1 \cdots v_n \mapsto \sum_{i=0}^n v_1 \cdots v_i \otimes v_{i+1} \cdots v_n, \\ \varepsilon: T(V) &\rightarrow k, \quad v_1 \cdots v_n \mapsto \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(See Appendix A.10 for the explicit calculations.)

**Proposition 3.4** (Constructions with dg-coalgebras). Let  $C, D$  be dg-coalgebras.

(1) The tensor product  $C \otimes D$  is again a dg-coalgebra with

$$\begin{aligned} \varepsilon_{C \otimes D}(c \otimes d) &= \varepsilon(c)\varepsilon(d), \\ \Delta_{C \otimes D}(c \otimes d) &= \sum_{(c), (d)} (-1)^{|c_{(2)}||d_{(1)}|} (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}). \end{aligned}$$

(2) If  $I$  is a dg-coideal in  $C$  then  $C/I$  inherits a dg-coalgebra structure.

(3) If  $C$  is a dg-coalgebra then  $Z(C)$  is a graded subcoalgebra of  $C$ ,  $B(C)$  is a graded coideal in  $Z(C)$  and  $H(C)$  is hence a graded coalgebra.

*Proof.* See Appendix A.11. □

## 4. Differential Graded Bialgebras

**Definition 4.1.**

- (1) A **dg-bialgebra** is a tuple  $(B, m, u, \Delta, \varepsilon)$  so that  $(B, m, u)$  is a dg-algebra,  $(B, \Delta, \varepsilon)$  is a dg-coalgebra and  $\Delta, \varepsilon$  are homomorphisms of dg-algebras. (See Appendix A.12 for remarks about this definition.)
- (2) A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

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<sup>3</sup>By “coideal” we always mean a two-sided coideal.

**Remark 4.2.** The compatibility of the multiplication and comultiplication of  $B$  means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)}.$$

**Warning 4.3.** A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication  $\Delta: B \rightarrow B \otimes B$  is a homomorphism of dg-algebras into  $B \otimes B'$  but not necessarily an algebra homomorphism into the sign-less tensor product  $B \otimes_k B$ . We will see an explicit counterexample in Example 5.7.

**Proposition 4.4** (Constructions with dg-bialgebras). Let  $B, \mathcal{B}$  be dg-bialgebras.

- (1) If  $I$  is a dg-biideal in  $B$  then  $B/I$  inherits a dg-bialgebra structure.
- (2) The cycles  $Z(\mathcal{B})$  form a graded sub-bialgebra of  $\mathcal{B}$ ,  $B(\mathcal{B})$  is a graded biideal in  $Z(\mathcal{B})$  and  $H(\mathcal{B})$  is hence a graded bialgebra.

*Proof.* See Appendix A.13 □

## 5. Differential Graded Hopf Algebras

**Definition 5.1.**

- (1) An **antipode** for a dg-bialgebra  $H$  is a homomorphism of dg-vector spaces

$$S: H \rightarrow H$$

that makes the following diagram commute:

$$\begin{array}{ccccc} & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\ \Delta \nearrow & & & & \searrow m \\ H & \xrightarrow{\varepsilon} & k & \xrightarrow{u} & H \\ \Delta \searrow & & & & \nearrow m \\ & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H & \end{array} \quad (3)$$

If  $H$  admits an antipode then it is a **dg-Hopf algebra**.

- (2) A **dg-Hopf ideal** in a dg-Hopf algebra  $H$  is a dg-biideal  $I$  with  $S(I) \subseteq I$ .

**Warning 5.2.** A dg-Hopf algebra need not have an underlying Hopf algebra structure.

**Remark 5.3.** Let  $H$  be a dg-Hopf algebra.

- (1) The commutativity of the diagram (3) means more explicitly that

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \quad \text{and} \quad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H.$$

(No additional signs occur because  $|S| = 0$ .)

- (2) One can again characterize  $S$  using the convolution product on  $\text{Hom}_k(C, A)$  (see Appendix A.14). This then shows in particular the uniqueness of  $S$ .

**Proposition 5.4** (Constructions with dg-Hopf algebras). Let  $H, \mathcal{H}$  be dg-Hopf algebras.

- (1) If  $I$  is a dg-Hopf ideal in  $H$  then  $H/I$  inherits a dg-Hopf algebra structure.  
(2) The graded bialgebra  $H(\mathcal{H})$  is a graded Hopf algebra with antipode  $H(S_{\mathcal{H}})$ .

*Proof.* See Appendix A.15. □

**Example 5.5.** Let  $V$  be a dg-vector space. The maps

$$\begin{aligned} V &\rightarrow T(V) \otimes T(V), & v &\mapsto v \otimes 1 + 1 \otimes v, \\ V &\rightarrow k, & v &\mapsto 0, \\ V &\rightarrow T(V)^{\text{op}}, & v &\mapsto -v \end{aligned}$$

are homomorphisms of dg-vector spaces and thus induce homomorphisms of dg-algebras

$$\begin{aligned} \Delta: T(V) &\rightarrow T(V) \otimes T(V), \\ \varepsilon: T(V) &\rightarrow k, \\ S: T(V) &\rightarrow T(V)^{\text{op}}. \end{aligned}$$

These homomorphisms are explicitly given by

$$\begin{aligned} \Delta(v_1 \cdots v_n) &= \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}, \\ \varepsilon(v_1 \cdots v_n) &= \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \\ S(v_1 \cdots v_n) &= (-1)^{\sum_{1 \leq i < j \leq n} |v_i| |v_j|} (-1)^n v_n \cdots v_1 \end{aligned}$$

for homogeneous  $v_i$ , where  $S$  is viewed as a map  $T(V) \rightarrow T(V)$  and  $\text{Sh}(p, q) \subseteq S_{p+q}$  denotes the set of  $p$ - $q$ -shuffles. These maps make  $T(V)$  into a dg-Hopf algebra. (See Appendix A.16 for the explicit calculations.)

**Example 5.6** (Quotients of dg-Hopf algebras). Let  $V$  be a dg-vector space. The dg-algebra  $\Lambda(V) = T(V)/I$  from Example 2.7 inherits from  $T(V)$  the structure of a dg-Hopf algebra because the dg-ideal  $I$  is a dg-Hopf ideal in  $T(V)$  (see Appendix A.17).

**Example 5.7** (Exterior Algebra). Let  $V$  be a vector space. We regard  $V$  as a dg-vector space concentrated in degree 1. Then  $\Lambda(V) = \bigwedge(V)$  as graded algebras whence  $\bigwedge(V)$  is a graded Hopf algebra. But for  $\text{char } k \neq 2$  there exists no bialgebra structure on  $\bigwedge(V)$  (see Appendix A.18).

**Example 5.8** (Homology of dg-Hopf algebras). Let  $V$  be a dg-vector space.

- (1) The inclusion  $V \rightarrow T(V)$  is a homomorphism of dg-vector spaces and thus induces a homomorphism of graded vector spaces  $H(V) \rightarrow H(T(V))$ , which in turn induces a homomorphism of graded algebras

$$\alpha: T(H(V)) \rightarrow H(T(V)), \quad [v_1] \cdots [v_n] \mapsto [v_1 \cdots v_n]$$

where  $v_1, \dots, v_n \in Z(V)$ . We see on representatives that  $\alpha$  is a homomorphism of graded Hopf algebras. We can write  $\alpha$  as

$$H(T(V)) = H\left(\bigoplus_{d \geq 0} V^{\otimes d}\right) \cong \bigoplus_{d \geq 0} H(V^{\otimes d}) \cong \bigoplus_{d \geq 0} H(V)^{\otimes d} = T(H(V))$$

which shows that  $\alpha$  is an isomorphism.

- (2) If  $\text{char}(k) = 0$  then also  $H(\Lambda(V)) \cong \Lambda(H(V))$ : We get again a canonical homomorphism of graded algebras

$$\beta: \Lambda(H(V)) \rightarrow H(\Lambda(V)), \quad [v_1] \cdots [v_n] \mapsto [v_1 \cdots v_n]$$

where  $v_1, \dots, v_n \in Z(V)$ . The symmetrization map

$$s: \Lambda(V) \rightarrow T(V), \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n)$$

is a section for the projection  $p: T(V) \rightarrow \Lambda(V)$  and a homomorphism of dg-vector spaces (see Appendix A.19). Together with the projection  $\tilde{p}: T(H(V)) \rightarrow \Lambda(H(V))$  and symmetrization map  $\tilde{s}: \Lambda(H(V)) \rightarrow T(H(V))$  we have the following diagram:

$$\begin{array}{ccc} T(H(V)) & \xrightleftharpoons[\alpha^{-1}]{\alpha} & H(T(V)) \\ \tilde{p} \updownarrow \tilde{s} & & H(p) \updownarrow H(s) \\ \Lambda(H(V)) & \xrightleftharpoons[\beta']{\beta} & H(\Lambda(V)) \end{array}$$

We have  $\beta = H(p) \circ \alpha \circ \tilde{s}$ , and  $\beta' := \tilde{p} \circ \alpha^{-1} \circ H(s)$  is an inverse to  $\beta$  (see Appendix A.19). This shows that  $\beta$  is an isomorphism.

## 6. Differential Graded Lie Algebras

Let  $\text{char}(k) = 0$ .

**Definition 6.1.**

- (1) A **dg-Lie algebra** is a dg-vector space  $\mathfrak{g}$  together with a homomorphism of dg-vector spaces  $[-, -]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $[-, -]$  is **graded skew symmetric** in the sense that the diagram

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\tau} & \mathfrak{g} \otimes \mathfrak{g} \\ & \searrow \quad \swarrow & \\ & [-, -] \quad -[-, -] & \\ & \mathfrak{g} & \end{array}$$

commutes, and such that  $[x, -]$  is for every homogeneous  $x$  a graded derivation.

- (2) A dg-Lie ideal in a dg-Lie algebra  $\mathfrak{g}$  is a dg-subspace with  $[\mathfrak{g}, I] \subseteq I$ .

**Remark 6.2.** That  $\mathfrak{g}$  is a dg-Lie algebra means that

$$\begin{aligned} [\mathfrak{g}_i, \mathfrak{g}_j] &\subseteq \mathfrak{g}_{i+j}, \\ [x, y] &= -(-1)^{|x||y|}[y, x], \\ [x, [y, z]] &= [[x, y], z] + (-1)^{|x||y|}[y, [x, z]], \\ d([x, y]) &= [d(x), y] + (-1)^{|x|}[x, d(y)]. \end{aligned} \tag{4}$$

We can rewrite (4) as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x, [y, z]] = 0.$$

**Warning 6.3.** A dg-Lie algebra need not have an underlying Lie algebra structure.

**Example 6.4.**

- (1) Every dg-algebra  $A$  is a dg-Lie algebra when endowed with the graded commutator.  
(2) In any dg-bialgebra  $B$  the subspace of primitive elements,

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\},$$

is a dg-Lie subalgebra of  $B$ .

(See Appendix A.20 for explicit calculations and another example.)

**Lemma 6.5.** Let  $\mathfrak{g}$  be a dg-Lie algebra.

- (1) If  $I$  is a dg-Lie ideal in  $\mathfrak{g}$  then  $\mathfrak{g}/I$  inherits a dg-Lie algebra structure.



- (2) The cycles  $Z(\mathfrak{g})$  form a graded Lie subalgebra of  $\mathfrak{g}$ ,  $B(\mathfrak{g})$  is a graded Lie ideal in  $Z(\mathfrak{g})$  and  $H(\mathfrak{g})$  is thus a graded Lie algebra.

*Proof.* See Appendix A.21. □

**Definition 6.6.** The **universal enveloping dg-algebra** of a dg-Lie algebra  $\mathfrak{g}$  is

$$U(\mathfrak{g}) = T(\mathfrak{g}) / ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous}).$$

**Proposition 6.7.**

- (1) The composition  $i: \mathfrak{g} \rightarrow T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is a homomorphism of dg-Lie algebras.  
(2) If  $A$  is any dg-algebra and  $f: \mathfrak{g} \rightarrow A$  a homomorphism of dg-Lie algebras there exists a unique homomorphism of dg-algebras  $F: U(\mathfrak{g}) \rightarrow A$  that extends  $f$ :

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\quad F \quad} & A \\ \uparrow i & \nearrow f & \\ \mathfrak{g} & & \end{array}$$

- (3) The universal enveloping dg-algebra  $U(\mathfrak{g})$  inherits from  $T(\mathfrak{g})$  the structure of a dg-Hopf algebra.

*Proof.* See Appendix A.22. □

We will now show that  $H(U(\mathfrak{g})) \cong U(H(\mathfrak{g}))$ . For this we need a version of the Poincaré–Birkhoff–Witt theorem (PBW theorem) for dg-Lie algebras and their universal enveloping dg-algebras, which we formulate in Appendix A.23. We will also blackbox the following consequences of the PBW theorem.

**Corollary 6.8** (of the PBW theorem). Let  $\mathfrak{g}$  be a dg-Lie algebra.

- (1) The canonical map  $\mathfrak{g} \rightarrow U(\mathfrak{g})$  is injective.  
(2) The dg-Lie algebra  $\mathfrak{g}$  can be retrieved from  $U(\mathfrak{g})$  as  $\mathbb{P}(U(\mathfrak{g})) = \mathfrak{g}$ .  
(3) If  $s: \Lambda(\mathfrak{g}) \rightarrow T(\mathfrak{g})$  denotes the symmetrization map from Example 5.8 then

$$e: \Lambda(\mathfrak{g}) \xrightarrow{s} T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

is an isomorphism of dg-vector spaces (and even of dg-coalgebra). □

**Example 6.9** (Homology of  $U(\mathfrak{g})$ ). The inclusion  $\mathfrak{g} \rightarrow U(\mathfrak{g})$  is a homomorphism of dg-Lie algebra and so induces a homomorphism of graded Lie algebras  $H(\mathfrak{g}) \rightarrow H(U(\mathfrak{g}))$ , which in turn induces a homomorphism of graded algebras

$$\gamma: U(H(\mathfrak{g})) \rightarrow H(U(\mathfrak{g})), \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

for  $x_1, \dots, x_n \in Z(\mathfrak{g})$ . We see on representatives that this is a homomorphism of dg-Hopf algebras. It is an isomorphism: We denote the isomorphisms of dg-vector spaces  $\Lambda(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  and  $\Lambda(H(\mathfrak{g})) \rightarrow U(H(\mathfrak{g}))$  from Corollary 6.8 by  $e$  and  $\tilde{e}$ . Together with the isomorphism of graded algebras

$$\beta: \Lambda(H(\mathfrak{g})) \rightarrow H(\Lambda(\mathfrak{g})), \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

from Example 5.8 we get the following commutative diagram:

$$\begin{array}{ccc} \Lambda(H(\mathfrak{g})) & \xrightarrow[\tilde{e}]{\sim} & U(H(\mathfrak{g})) \\ \beta \downarrow \sim & & \downarrow \gamma \\ H(\Lambda(\mathfrak{g})) & \xrightarrow[H(e)]{\sim} & H(U(\mathfrak{g})) \end{array}$$

The arrows  $e$ ,  $H(e)$ ,  $\beta$  are isomorphisms, hence  $\gamma$  is one.

**Remark 6.10.**

- (1) If  $\mathcal{H}$  is a dg-Hopf algebra then  $H(\mathbb{P}(\mathcal{H})) \cong \mathbb{P}(H(\mathcal{H}))$ . (This statement can be found without proof in [Lod92, Theorem A.9].)
- (2) If  $H$  is a graded cocommutative connected<sup>4</sup> dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that  $H \cong U(\mathbb{P}(H))$ . Together with Corollary 6.8 this results in an equivalence between the categories of dg-Lie algebras and cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B, Theorem 4.5].

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<sup>4</sup>The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

## A. Calculations, Proofs and Remarks

### A.1. More Conventions and Notations

A map  $f: V \rightarrow W$  is **graded** of **degree**  $d = |f|$  if  $f(V_n) \subseteq V_{n+d}$  for all  $n$ . The differential  $d$  is a graded map of degree  $-1$ . If  $f: V \rightarrow V'$ ,  $g: W \rightarrow W'$  are graded maps then  $f \otimes g: V \otimes V' \rightarrow W \otimes W'$  is the graded map of degree  $|f \otimes g| = |f| + |g|$  given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

If  $f, g$  are homomorphisms of dg-vector spaces then so is  $f \otimes g$ .

If  $V, W$  are dg-vector spaces then  $\text{Hom}(V, W)$  is the dg-vector space with

$$\begin{aligned} \text{Hom}(V, W)_n &= \{\text{graded maps } V \rightarrow W \text{ of degree } n\}, \\ d(f) &= d \circ f - (-1)^{|f|} f \circ d. \end{aligned}$$

The spaces  $\text{Hom}(V, W)_n$  are linearly independent in  $\text{Hom}_k(V, W)$ , in the sense that the sum  $\sum_n \text{Hom}(V, W)_n$  is direct. We therefore regard  $\text{Hom}(V, W) = \bigoplus_n \text{Hom}(V, W)_n$  as a linear subspace of  $\text{Hom}_k(V, W)$ .

### A.2. The Koszul Sign

We have for every  $i = 1, \dots, n-1$  a twist map

$$\begin{aligned} \tau_i: V^{\otimes n} &\rightarrow V^{\otimes n}, \\ v_1 \otimes \dots \otimes v_n &\mapsto v_1 \otimes \dots \otimes \tau(v_i \otimes v_{i+1}) \otimes \dots \otimes v_n \\ &\mapsto (-1)^{|v_i||v_{i+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n. \end{aligned}$$

The group  $S_n$  is generated by the simple reflections  $\sigma_1, \dots, \sigma_{n-1}$  with relations

$$\begin{aligned} \sigma_i^2 &= 1 & \text{for } i = 1, \dots, n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i-j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, n-2. \end{aligned}$$

We check that the twist maps  $\tau_1, \dots, \tau_{n-1}$  satisfy these relations, which shows that  $S_n$  acts on  $V^{\otimes n}$  such that  $s_i$  acts via  $\tau_i$ : We have

$$\tau_i^2(v_1 \otimes \dots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} \tau_i(v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n) = v_1 \otimes \dots \otimes v_n$$

and thus  $\tau_i^2 = 1$ . If  $|i-j| \geq 2$  then

$$\begin{aligned} &\tau_i \tau_j(v_1 \otimes \dots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_j||v_{j+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_{j+1} \otimes v_j \otimes \dots \otimes v_n \\ &= \tau_j \tau_i(v_1 \otimes \dots \otimes v_n) \end{aligned}$$

and thus  $\tau_i \tau_j = \tau_j \tau_i$ . We also have

$$\begin{aligned}
& \tau_i \tau_{i+1} \tau_i (v_1 \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|} \tau_i \tau_{i+1} (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|+|v_i||v_{i+2}|} \tau_i (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_{i+2} \otimes v_i \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|+|v_i||v_{i+2}|+|v_{i+1}||v_{i+2}|} v_1 \otimes \cdots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n
\end{aligned}$$

and similarly

$$\begin{aligned}
& \tau_{i+1} \tau_i \tau_{i+1} (v_1 \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_{i+1}||v_{i+2}|} \tau_{i+1} \tau_i (v_1 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes v_{i+1} \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+2}|+|v_{i+1}||v_{i+2}|} \tau_{i+1} (v_1 \otimes \cdots \otimes v_{i+2} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|+|v_i||v_{i+2}|+|v_{i+1}||v_{i+2}|} v_1 \otimes \cdots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n.
\end{aligned}$$

Therefore  $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ . We now have the desired action of  $S_n$  on  $V^{\otimes n}$ . The twist maps  $\tau_i$  are homomorphisms of dg-vector spaces whence  $S_n$  acts by homomorphisms of dg-vector spaces.

Without sign the action of  $S_n$  on  $V^{\otimes n}$  would be given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

(so that the tensor factor  $v_i$  is moved to the  $\sigma(i)$ -th position). The above action of  $S_n$  on  $V^{\otimes n}$  is hence given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1, \dots, v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

with signs  $\varepsilon_{v_1, \dots, v_n}(\sigma) \in \{1, -1\}$ .

### A.3. Remark 2.2

- (1) If  $A$  is a graded algebra then a graded map  $\delta: A \rightarrow A$  is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \text{id} + \text{id} \otimes \delta);$$

more explicitly,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b).$$

The compatibility condition (1) in the definition of a dg-algebra thus states that the differential  $d$  is a derivation for  $A$ .

- (2) We see that there are two equivalent ways to make a graded vector space into a dg-algebra:

$$\begin{array}{ccc}
\text{graded} & \xrightarrow{\text{multiplication}} & \text{graded} \\
\text{vector spaces} & & \text{algebras} \\
\downarrow \text{differential} & & \downarrow \text{differential} \\
\text{dg-vector spaces} & \xrightarrow{\text{multiplication}} & \text{dg-algebras}
\end{array}$$

- (3) The graded commutativity of  $A$  means  $ab = (-1)^{|a||b|}ba$ . If  $|a|$  is even or  $|b|$  is even then  $ab = ba$ ; if  $|a|$  is odd then  $a^2 = -a^2$  and thus  $a^2 = 0$  if  $\text{char}(k) \neq 2$ .
- (4) A homomorphism  $f$  of dg-algebras is the same as a homomorphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since  $|f| = 0$ .)

#### A.4. Examples 2.3

- (2) It remains to check the compatibility of the multiplication and dg-structure of  $T(V)$ : It holds that  $1_{T(V)} \in T(V)_0$  with  $d(1_{T(V)}) = 0$ . Furthermore

$$\begin{aligned} |v_1 \cdots v_n \cdot w_1 \cdots w_m| &= |v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_m| \\ &= |v_1 \cdots v_n| + |w_1 \cdots w_m| \end{aligned}$$

and

$$\begin{aligned} & d(v_1 \cdots v_n \cdot w_1 \cdots w_m) \\ &= \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \cdot w_1 \cdots w_m \\ &\quad + \sum_{j=1}^m (-1)^{|v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_{j-1}|} v_1 \cdots v_n \cdot w_1 \cdots d(w_j) \cdots w_m \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1| + \cdots + |v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m) \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1 \cdots v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m). \end{aligned}$$

This shows that  $T(V)$  is indeed a dg-algebra.

Let  $A$  be another dg-algebra and  $f: V \rightarrow A$  a homomorphism of dg-vector spaces and let  $F: T(V) \rightarrow A$  be the unique extension of  $f$  to an algebra homomorphism, given by  $F(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$ . The algebra homomorphism  $F$  is a homomorphism of graded algebras because

$$\begin{aligned} |F(v_1 \cdots v_n)| &= |f(v_1) \cdots f(v_n)| \\ &= |f(v_1)| + \cdots + |f(v_n)| \\ &= |v_1| + \cdots + |v_n| \\ &= |v_1 \cdots v_n|. \end{aligned}$$

It is also a homomorphism of dg-vector spaces because

$$\begin{aligned}
d(F(v_1 \cdots v_n)) &= d(f(v_1) \cdots f(v_n)) \\
&= \sum_{i=1}^n (-1)^{|f(v_1)| + \cdots + |f(v_{i-1})|} f(v_1) \cdots d(f(v_i)) \cdots f(v_n) \\
&= \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} f(v_1) \cdots f(d(v_i)) \cdots f(v_n) \\
&= F\left(\sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n\right) \\
&= F(d(v_1 \cdots v_n)).
\end{aligned}$$

- (3) For any dg-vector space  $V$  the algebra structure of  $\text{End}_k(V)$  restricts to a dg-algebra structure on  $\text{End}(V) = \text{Hom}(V, V)$ :

It holds that  $\text{id}_V \in \text{End}(V)_0$  and if  $f, g \in \text{End}(V)$  are graded maps then  $f \circ g$  is again a graded map. Therefore  $\text{End}(V)$  is a subalgebra of  $\text{End}_k(V)$ . If  $f, g \in \text{End}(V)$  are homogeneous then  $|f \circ g| = |f| + |g|$  so  $\text{End}(V)$  is a graded algebra. We see from

$$\begin{aligned}
d(f \circ g) &= d \circ f \circ g - (-1)^{|f \circ g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f| + |g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|} f \circ d \circ g + (-1)^{|f|} f \circ d \circ g - (-1)^{|f| + |g|} f \circ g \circ d \\
&= (d \circ f - (-1)^{|f|} d \circ f) \circ g + (-1)^{|f|} f \circ (d \circ g - (-1)^{|g|} g \circ d) \\
&= d(f) \circ g + (-1)^{|f|} f \circ d(g)
\end{aligned}$$

and

$$d(\text{id}_V) = d \circ \text{id}_V - \text{id}_V \circ d = d - d = 0$$

that  $\text{End}(V)$  is a dg-algebra.

### A.5. Proposition 2.4

- (3) The quotient  $A/I$  is a dg-vector space and an algebra and the compatibility of these structures can be checked on representatives.
- (4) The cycles  $Z(A)$  form a graded subspace with  $1 \in Z(A)$  and if  $a, b \in Z(A)$  are homogeneous then

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b) = 0$$

and hence  $ab \in Z(A)$ . The boundaries  $B(A)$  form a graded subspace and if  $a \in Z(A)$  and  $b \in B(B)$  are homogeneous with  $b = d(a')$  then

$$b \cdot a = d(a') \cdot a = d(a \cdot a') - (-1)^{|a|} a' \cdot d(a) = d(a \cdot a')$$

and hence  $ba \in B(A)$ . Similarly  $ab \in B(A)$ .

**Warning A.1.** If  $A \otimes_k B$  is the sign-less tensor product with  $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$  then  $A \otimes B \neq A \otimes_k B$  as algebras, i.e. the underlying algebra of  $A \otimes B$  is not the tensor product of the underlying algebras of  $A$  and  $B$ . The underlying algebra of  $A^{\text{op}}$  is similarly not the opposite of the underlying algebra of  $A$ .

### A.6. Lemma 2.5

That  $I$  is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, IX, 2.5] or [Bou89, II.§11.3]. It remains to show that  $d(I) \subseteq I$  if  $d(x_\alpha) \in I$  for every  $\alpha$ : The ideal  $I$  is spanned by  $ax_\alpha b$  with  $a, b \in A$  homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|}ad(x_\alpha)b + (-1)^{|a|+|x_\alpha|}ax_\alpha d(b) \in I$$

since  $x_\alpha, d(x_\alpha) \in I$ .

### A.7. Definition 2.6

We have for homogeneous  $a, b$  that  $[a, b] = 0$  if and only if  $a, b$  graded commute with each other. If  $A$  is a dg-algebra and  $|a|$  is even then  $[a, a] = 0$ . But if  $|a|$  is odd then  $[a, a] = 2a^2$ . This means in particular that the graded commutator of an element with itself does not necessarily vanish (because not every element need to graded-commute with itself).

### A.8. Example 2.7

- (1) The ideal  $I$  is a dg-ideal as the generators  $[v, w]$  are homogeneous and (by Example 6.4)

$$d([v, w]) = [d(v), w] + (-1)^{|v|}[v, d(w)] \in I.$$

- (2) If  $S$  is a graded commutative dg-algebra,  $f: V \rightarrow S$  a homomorphism of dg-vector spaces then  $f$  extends uniquely to a homomorphism of dg-algebras  $F: \Lambda(V) \rightarrow S$ :

$$\begin{array}{ccc} \Lambda(V) & \xrightarrow{\quad F \quad} & S \\ \uparrow & \nearrow f & \\ V & & \end{array}$$

- (3) Let  $A$  and  $B$  be two dg-algebras. If  $C$  is any other dg-algebra and if  $f: A \rightarrow C$  and  $g: B \rightarrow C$  are two homomorphisms of dg-algebras whose images graded-commute, in the sense that

$$f(a)g(b) = (-1)^{|a||b|}g(b)f(a)$$

for all  $a \in A, b \in B$ , then the linear map

$$\varphi: A \otimes B \rightarrow C, \quad a \otimes b \mapsto f(a)g(b)$$

is again a homomorphism of dg-algebras. The inclusions  $i: A \rightarrow A \otimes B$ ,  $a \mapsto a \otimes 1$  and  $j: B \rightarrow A \otimes B$ ,  $b \mapsto 1 \otimes b$  are homomorphisms of dg-algebras. For every homomorphism of dg-algebras  $\varphi: A \otimes B \rightarrow C$  the compositions  $\varphi \circ i: A \rightarrow A \otimes B$  and  $\varphi \circ j: B \rightarrow A \otimes B$  are again homomorphisms of dg-algebras. This gives a one-to-one correspondence

$$\begin{aligned} \left\{ \begin{array}{l} \text{homomorphisms of dg-algebras} \\ f: A \rightarrow C, g: B \rightarrow C \\ \text{whose images graded-commute} \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} \text{homomorphisms of dg-algebras} \\ \varphi: A \otimes B \rightarrow C \end{array} \right\}, \\ (f, g) &\longmapsto (a \otimes b \mapsto f(a)g(b)), \\ (\varphi \circ i, \varphi \circ j) &\longleftarrow \varphi. \end{aligned}$$

(4) It follows for any two dg-vector spaces  $V$  and  $W$  that

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

since we have for every dg-algebra  $A$  natural bijections

$$\begin{aligned} &\{\text{homomorphisms of dg-algebras } \Lambda(V \oplus W) \rightarrow A\} \\ &\cong \{\text{homomorphisms of dg-vector spaces } V \oplus W \rightarrow A\} \\ &\cong \{(f, g) \mid \text{homomorphisms of dg-vector spaces } f: V \rightarrow A, g: W \rightarrow A\} \\ &\cong \{(\varphi, \psi) \mid \text{homomorphisms of dg-algebras } \varphi: \Lambda(V) \rightarrow A, \psi: \Lambda(W) \rightarrow A\} \\ &\cong \{\text{homomorphisms of dg-algebras } \Lambda(V) \otimes \Lambda(W) \rightarrow A\}. \end{aligned}$$

More explicitly, the inclusions  $V \rightarrow V \oplus W$  and  $W \rightarrow V \oplus W$  induce homomorphisms of dg-algebras  $\Lambda(V) \rightarrow \Lambda(V \oplus W)$  and  $\Lambda(W) \rightarrow \Lambda(V \oplus W)$  that give an isomorphism of dg-algebras

$$\Lambda(V) \otimes \Lambda(W) \xrightarrow{\sim} \Lambda(V \oplus W), \quad v_1 \cdots v_n \otimes w_1 \cdots w_m \mapsto v_1 \cdots v_n w_1 \cdots w_m.$$

(5) Let  $V$  be a graded vector space.

If  $V$  is concentrated in even degrees then  $\Lambda(V) = S(V)$  and if  $V$  is concentrated in odd degrees then  $\Lambda(V) = \bigwedge(V)$ , with the grading of  $\Lambda(V)$  and  $\bigwedge(V)$  induced by the one of  $V$ .

We have  $V = V_{\text{even}} \oplus V_{\text{odd}}$  as graded vector spaces where  $V_{\text{even}} = \bigoplus_n V_{2n}$  and  $V_{\text{odd}} = \bigoplus_n V_{2n+1}$ , and hence

$$\Lambda(V) = \Lambda(V_{\text{even}} \oplus V_{\text{odd}}) \cong \Lambda(V_{\text{even}}) \otimes \Lambda(V_{\text{odd}}) = S(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$$

The graded algebra  $S(V_{\text{even}})$  is concentrated in even degree and so it follows that in the tensor product  $S(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$  the simple tensors (strictly) commute, i.e.  $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$ . Hence

$$\Lambda(V) \cong S(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}})$$

where  $\otimes_k$  denotes the sign-less tensor product.



- (6) Let  $\text{char}(k) \neq 2$  and let  $V$  be a dg-vector space with basis  $(x_\alpha)_{\alpha \in A}$  consisting of homogeneous elements such that  $(A, \leq)$  is linearly ordered. Then  $\Lambda(V)$  admits as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \quad \text{where } t \geq 0, \alpha_1 < \cdots < \alpha_t, n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.}^5$$

To see this we use the above decomposition

$$\Lambda(V) \cong S(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}}) \quad (5)$$

as graded algebras: We split up the given basis  $(x_\alpha)_{\alpha \in A}$  of  $V$  into a basis  $(x_\alpha)_{\alpha \in A'}$  of  $V_{\text{even}}$  and  $(x_\alpha)_{\alpha \in A''}$  of  $V_{\text{odd}}$  (since all  $x_\alpha$  are homogeneous). Then  $S(V_{\text{even}})$  has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \quad \text{where } r \geq 0, \alpha_1 < \cdots < \alpha_r \text{ and } n_i \geq 1,$$

and  $\bigwedge(V_{\text{odd}})$  has as a basis the ordered wedges

$$x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_s} \quad \text{where } s \geq 0, \alpha_1 < \cdots < \alpha_s.$$

It follows that with (5) that  $\Lambda(V)$  admits the basis

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \cdot x_{\beta_1} \cdots x_{\beta_s} \quad \text{where } \begin{cases} r, s \geq 0, n_i \geq 1, \\ \alpha_1 < \cdots < \alpha_r, \\ \beta_1 < \cdots < \beta_s, \\ |x_{\alpha_i}| \text{ even, } |x_{\beta_j}| \text{ odd.} \end{cases}$$

We can now rearrange these basis vectors into the desired form because the factors  $x_{\alpha_i}^{n_i}$  and  $x_{\beta_j}$  commute.

### A.9. Remark 3.2

- (1) If  $C$  is a graded coalgebra then a graded map  $\omega: C \rightarrow C$  is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \text{id} + \text{id} \otimes \omega) \circ \Delta.$$

This means more explicitly that

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)}).$$

The compability (2) means that the differential  $d$  (which is a graded map of degree  $|-d| = -1$ ) is a coderivation.

- (2) The graded cocommutativity of  $C$  means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}.$$

- (3) A homomorphism of dg-coalgebras is the same as a homomorphism of the underlying graded coalgebras that commutes with the differentials.

- (4) Every coalgebra  $C$  is a dg-coalgebra centered in degree 0, in particular  $C = k$ .

<sup>5</sup>The condition  $n_i = 1$  for  $|x_{\alpha_i}|$  odd comes from the equality  $\alpha_i^2 = [\alpha_i, \alpha_i]/2$ .

### A.10. Example 3.3

We have seen in the first talk that  $(T(C), \Delta, \varepsilon)$  is a coalgebra. We have for every  $i = 0, \dots, n$  that

$$\begin{aligned} |v_1 \cdots v_i \otimes v_{i+1} \cdots v_n| &= |v_1 \cdots v_i| + |v_{i+1} \cdots v_n| \\ &= |v_1| + \cdots + |v_i| + |v_{i+1}| + \cdots + |v_n| \\ &= |v_1| + \cdots + |v_n|, \end{aligned}$$

so we have a graded coalgebra. We also have

$$\begin{aligned} & d(\Delta(v_1 \cdots v_n)) \\ &= \sum_{i=0}^n d(v_1 \cdots v_i \otimes v_{i+1} \cdots v_n) \\ &= \sum_{i=0}^n (d(v_1 \cdots v_i) \otimes v_{i+1} \cdots v_n + (-1)^{|v_1 \cdots v_i|} v_1 \cdots v_i \otimes d(v_{i+1} \cdots v_n)) \\ &= \sum_{i=0}^n \left( \sum_{j=1}^i (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\ &\quad \left. + (-1)^{|v_1 \cdots v_i|} \sum_{j=i+1}^n (-1)^{|v_{i+1}| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\ &= \sum_{i=0}^n \left( \sum_{j=1}^i (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\ &\quad \left. + \sum_{j=i+1}^n (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\ &= \Delta \left( \sum_{j=1}^n (-1)^{|v_1| + \cdots + |v_j|} v_1 \otimes \cdots \otimes d(v_j) \otimes \cdots \otimes v_n \right) \\ &= \Delta(d(v_1 \cdots v_n)) \end{aligned}$$

which shows that  $\Delta$  is a homomorphism of dg-vector spaces.

### A.11. Proposition 3.4

(3) The quotient  $C/I$  is a dg-vector space and a coalgebra, and the compatibility of these structures can be checked on representatives.

(4) If  $c \in Z(C)$  then

$$d(\Delta(c)) = \Delta(d(c)) = \Delta(0) = 0$$

because  $\Delta$  is a homomorphism of dg-vector spaces, and hence

$$\Delta(c) \in Z(C \otimes C) = Z(C) \otimes Z(C).$$

This shows that  $Z(C)$  is a subcoalgebra of  $C$ . It is also a graded subspace of  $C$  and hence a graded subcoalgebra.

For  $b \in B(C)$  with  $b = d(c)$  we have

$$\begin{aligned}\Delta(b) &= \Delta(d(c)) = d(\Delta(c)) = d\left(\sum_{(c)} c_{(1)} \otimes c_{(2)}\right) \\ &= \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|c_{(1)} \otimes d(c_{(2)})} \in B(C) \otimes C + C \otimes B(C).\end{aligned}$$

We also have

$$\varepsilon(b) = \varepsilon(d(c)) = d(\varepsilon(c)) = 0.$$

This shows that  $B(C)$  is a coideal in  $C$ . It follows from the upcoming lemma that  $B$  is also a coideal in  $Z(C)$ . Then  $B(C)$  is a graded coideal in  $Z(C)$  because  $B(C)$  is a graded subspace of  $Z(C)$ .

**Lemma A.2.** Let  $C$  be a coalgebra and let  $B$  be a subcoalgebra of  $C$ . If  $I$  is a coideal in  $C$  with  $I \subseteq C$  then  $I$  is also a coideal in  $B$ .

*Proof.* It follows from the inclusions  $I \subseteq B \subseteq C$  that

$$(C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B.$$

Hence

$$\Delta(I) = \Delta(I) \cap \Delta(B) \subseteq (C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B.$$

Also  $\varepsilon_B(I) = \varepsilon_C(I) = 0$ . □

#### A.12. Definition 4.1

One can also could equivalently require  $m, u$  to be homomorphisms of dg-coalgebras:

**Lemma A.3.** Let  $B$  be a dg-vector space,  $(B, m, u)$  a dg-algebra and  $(B, \Delta, \varepsilon)$  a dg-coalgebra. Then the following conditions are equivalent:

- (1)  $\Delta$  and  $\varepsilon$  are homomorphisms of dg-algebras.
- (2)  $m$  and  $u$  are homomorphisms of dg-coalgebras.

*Proof.* The same diagrammatic proof as in the non-dg case (as seen in the second talk). □

#### A.13. Proposition 4.4

- (1) It follows from Proposition 2.4 and Proposition 3.4 that  $B/I$  is a dg-algebra and dg-coalgebra. The compatibility can be checked on representatives.
- (2) It follows from Proposition 2.4 and Proposition 3.4 that  $H(B)$  is again a dg-algebra and dg-coalgebra, and the compatibility of these structures can be checked on representatives.

### A.14. Remark 5.3

If  $C$  is a dg-coalgebra and  $A$  is a dg-algebra then the convolution product

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C$$

on  $\text{Hom}_k(C, A)$  makes  $\text{Hom}(C, A)$  into a dg-algebra:

We have  $1_{\text{Hom}_k(C, A)} = u \circ \epsilon \in \text{Hom}(C, A)_0$  because both  $u_A$  and  $\epsilon_C$  are homomorphisms of dg-vector spaces and thus of degree 0. If  $f, g \in \text{Hom}(C, A)$  are graded maps then  $f \otimes g$  is again a graded map and thus

$$f * g = m \circ (f \otimes g) \circ \Delta$$

is a graded map as a composition of graded maps. This shows that  $\text{Hom}(C, A)$  is a subalgebra of  $\text{Hom}_k(C, A)$ .

We have

$$|f * g| = |m \circ (f \otimes g) \circ \Delta| = |m| + (|f| + |g|) + |\Delta| = |f| + |g|$$

so  $\text{Hom}(C, A)$  is a graded algebra with respect to the convolution product.

Furthermore

$$\begin{aligned} & d(f * g) \\ &= d \circ (f * g) - (-1)^{|f * g|} (f * g) \circ d \\ &= d \circ m \circ (f \otimes g) \otimes \Delta - (-1)^{|f| + |g|} m \circ (f \otimes g) \circ \Delta \circ d \\ &= m \circ d_{A \otimes A} \circ (f \otimes g) \otimes \Delta - (-1)^{|f| + |g|} m \circ (f \otimes g) \circ d_{C \otimes C} \circ \Delta \\ &= m \circ (d \otimes 1 + 1 \otimes d) \circ (f \otimes g) \otimes \Delta \\ &\quad - (-1)^{|f| + |g|} m \circ (f \otimes g) \circ (d \otimes 1 + 1 \otimes d) \circ \Delta \\ &= m \circ (d \otimes \text{id}) \circ (f \otimes g) \otimes \Delta + m \circ (\text{id} \otimes d) \circ (f \otimes g) \otimes \Delta \\ &\quad - (-1)^{|f| + |g|} m \circ (f \otimes g) \circ (d \otimes \text{id}) \circ \Delta \\ &\quad - (-1)^{|f| + |g|} m \circ (f \otimes g) \circ (\text{id} \otimes d) \circ \Delta \\ &= m \circ ((d \circ f) \otimes g) \otimes \Delta + (-1)^{|f|} m \circ (f \otimes (d \circ g)) \otimes \Delta \\ &\quad - (-1)^{|f|} m \circ ((f \circ d) \otimes g) \circ \Delta - (-1)^{|f| + |g|} m \circ (f \otimes (g \circ d)) \circ \Delta \\ &= m \circ ((d \circ f - (-1)^{|f|} f \circ d) \otimes g) \otimes \Delta \\ &\quad + (-1)^{|f|} m \circ (f \otimes (d \circ g - (-1)^{|g|} g \circ d)) \otimes \Delta \\ &= m \circ (d(f) \otimes g) \circ \Delta + (-1)^{|f|} m \circ (f \otimes d(g)) \otimes \Delta \\ &= d(f) * g + (-1)^{|f|} f * d(g) \end{aligned}$$

because  $m$  and  $\Delta$  are commute with the differentials. Hence  $\text{Hom}(C, A)$  is a dg-algebra with respect to the convolution product.

Now we need to explain why an inverse to  $\text{id}_H$  in  $\text{Hom}(H, H)$  with respect to the convolution product  $*$  is again a homomorphism of dg-vector spaces. For this we use the following result:

**Lemma A.4.** Let  $A$  be a dg-algebra and let  $a \in A$  be a homogeneous unit.

- (1) The inverse  $a^{-1}$  is homogeneous of degree  $|a^{-1}| = -|a|$ .
- (2) If  $a$  is a cycle then so is  $a^{-1}$ .

*Proof.*

- (1) Let  $d = |a|$  and let  $a^{-1} = \sum_n a'_n$  be the homogeneous decomposition of  $a^{-1}$ . It follows from  $1 = ab = \sum_n aa'_n$  that in degree zero,  $1 = aa'_{-d}$ . Thus  $a'_{-d}$  is the inverse of  $a$ , i.e.  $a^{-1} = a'_{-d} \in A_{-d}$ .
- (2) It follows from

$$0 = d(1) = d(aa^{-1}) = d(a)a^{-1} + (-1)^{|a|}ad(a^{-1})$$

that  $(-1)^{|a|}ad(a^{-1}) = 0$  because  $d(a) = 0$ . Hence  $d(a^{-1}) = 0$  as  $a$  is a unit.  $\square$

The space  $Z_0(\text{Hom}(V, W))$  consists of the homomorphism of dg-vector spaces  $V \rightarrow W$ . It hence follows from Lemma A.4 that if  $f \in Z_0(\text{Hom}(V, W))$  admits an inverse  $g$  with respect to the convolution product that again  $g \in Z_0(\text{Hom}(V, W))$ .

#### A.15. Proposition 5.4

- (1) It follows from Proposition 4.4 that  $H$  is a dg-bialgebra and the condition  $S(I) \subseteq I$  ensures that  $S$  induces a homomorphism of dg-vector spaces  $\bar{S}: H/I \rightarrow H/I$ . The antipode condition for  $\bar{S}$  can now be checked on representatives.
- (2) The homology  $H(\mathcal{H})$  is a dg-bialgebra by Proposition 4.4 and that  $H(S_{\mathcal{H}})$  is an antipode can be checked on representatives.

#### A.16. Example 5.5

The dg-coalgebra diagrams for  $(T(V), \Delta, \varepsilon)$  can be checked on algebra generators of  $T(V)$  because all arrows in these diagrams are homomorphisms of dg-algebras. It hence suffices to check these diagrams for elements of  $V$ , where this is straightforward.

It remains to check the equalities

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \quad \text{and} \quad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H$$

for the monomials  $h = v_1 \cdots v_n$ . If  $n = 0$  then  $h = 1$  and both equalities hold, so we consider in the following the case  $n \geq 1$ . Then  $\varepsilon(v_1 \cdots v_n) = 0$  so we have to show that in the sums  $\sum_{(h)} S(h_{(1)})h_{(2)}$  and  $\sum_{(h)} h_{(1)}S(h_{(2)})$  all terms cancel out. We consider for simplicity only the sum  $\sum_{(h)} S(h_{(1)})h_{(2)}$ .<sup>6</sup> We have

$$\Delta(v_1 \cdots v_n) = \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}. \quad (6)$$

<sup>6</sup>The author hasn't actually checked the other sum.

Here

$$S(v_{\sigma(1)} \cdots v_{\sigma(p)}) = (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} v_{\sigma(p)} \cdots v_{\sigma(1)}$$

and thus

$$\begin{aligned} & (m \circ (S \otimes \text{id}) \circ \Delta)(v_1 \cdots v_n) \\ &= \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} \\ & \quad \cdot v_{\sigma(p)} \cdots v_{\sigma(1)} v_{\sigma(p+1)} \cdots v_{\sigma(n)}. \end{aligned} \tag{7}$$

We see that in (6) any two terms of the form

$$w_1 w_2 \cdots w_i \otimes w_{i+1} \cdots w_n \quad \text{and} \quad w_2 \cdots w_i \otimes w_1 w_{i+1} \cdots w_n$$

give in (7) the up to sign same term  $w_i \cdots w_2 w_1 w_{i+1} \cdots w_n$ . We now check that the signs differ, so that in (7) both terms cancel out. This then shows that the sum (7) becomes zero.

For  $1 \leq p \leq n$  and  $\sigma \in \text{Sh}(p, n-p)$  with  $\sigma(p) < \sigma(1)$  the term associated to  $v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}$  is given by

$$v_{\sigma(2)} \cdots v_{\sigma(p)} \otimes v_{\sigma(1)} v_{\sigma(p+1)} \cdots v_{\sigma(n)} = v_{\tau(1)} \cdots v_{\tau(p-1)} \otimes v_{\tau(p)} \cdots v_{\tau(n)}$$

for the permutation  $\omega \in \text{Sh}(p-1, n-p+1)$  given by

$$\omega = \sigma \circ (1 \, 2 \cdots p),$$

i.e.

$$\omega(i) = \begin{cases} \sigma(i+1) & \text{if } 1 \leq i \leq p-1, \\ \sigma(1) & \text{if } i = p, \\ \sigma(i) & \text{if } p+1 \leq i \leq n. \end{cases}$$

We see from the Koszul sign rule that the signs  $\varepsilon_{v_1, \dots, v_n}(\sigma^{-1})$  and  $\varepsilon_{v_1, \dots, v_n}(\omega^{-1})$  differ by the factor  $(-1)^{|v_{\sigma(1)}| |v_{\sigma(2)}| + \cdots + |v_{\sigma(1)}| |v_{\sigma(p)}|}$ . Therefore

$$\begin{aligned} & \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} \\ &= \varepsilon_{v_1, \dots, v_n}(\omega^{-1}) (-1)^{|v_{\sigma(1)}| |v_{\sigma(2)}| + \cdots + |v_{\sigma(1)}| |v_{\sigma(p)}|} (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} \\ &= \varepsilon_{v_1, \dots, v_n}(\omega^{-1}) (-1)^p (-1)^{\sum_{2 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} \\ &= \varepsilon_{v_1, \dots, v_n}(\omega^{-1}) (-1)^p (-1)^{\sum_{1 \leq i < j \leq p-1} |v_{\omega(i)}| |v_{\omega(j)}|} \\ &= - \varepsilon_{v_1, \dots, v_n}(\omega^{-1}) (-1)^{p-1} (-1)^{\sum_{1 \leq i < j \leq p-1} |v_{\omega(i)}| |v_{\omega(j)}|}. \end{aligned}$$

Thus the signs differ as claimed.

### A.17. Example 5.6

We have

$$\begin{aligned}
\varepsilon([v, w]) &= \varepsilon(vw - (-1)^{|v||w|}wv) \\
&= \varepsilon(vw) - (-1)^{|v||w|}\varepsilon(wv) \\
&= \varepsilon(v)\varepsilon(w) - (-1)^{|v||w|}\varepsilon(w)\varepsilon(v) \\
&= 0
\end{aligned}$$

as  $\varepsilon(v) = \varepsilon(w) = 0$ . The elements  $v$  and  $w$  are primitive whence  $[v, w]$  is primitive. Therefore

$$\Delta([v, w]) = [v, w] \otimes 1 + 1 \otimes [v, w] \in I \otimes T(V) + T(V) \otimes I.$$

Also

$$\begin{aligned}
S([v, w]) &= S(vw - (-1)^{|v||w|}wv) \\
&= S(vw) - (-1)^{|v||w|}S(wv) \\
&= (-1)^{|v||w|}wv - vw \\
&= -(vw - (-1)^{|v||w|}wv) \\
&= -[v, w] \\
&\in I.
\end{aligned}$$

### A.18. Example 5.7

Suppose that there exists a bialgebra structure on  $E := \bigwedge(V)$ . Then  $\varepsilon(v)^2 = \varepsilon(v^2) = 0$  and thus  $\varepsilon(v) = 0$  for all  $v \in V$ , so  $\ker \varepsilon = \bigoplus_{n \geq 1} E_n =: I$ . Let  $v \in V$ . Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{E \otimes I} \quad \text{and} \quad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes E}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2},$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

Now  $v^2 = 0$  and thus

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

But  $2 \neq 0$  and  $v \neq 0$  hence  $2v \otimes v \neq 0$  while  $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$ , a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

### A.19. Example 5.8

- (1) The action of  $S_n$  on  $V^{\otimes n}$  is by homomorphism of dg-vector spaces as mentioned in Section 1 and shown in Appendix A.2. The symmetrization map

$$\tilde{s}: T(V) \rightarrow T(V), \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n)$$

therefore results in a homomorphism of dg-vector spaces  $\tilde{s}: T(V) \rightarrow T(V)$ .<sup>7</sup> It follows that the factored map  $s: \Lambda(V) \rightarrow T(V)$  is again a homomorphism of dg-vector spaces.

- (2) We observe that the diagrams

$$\begin{array}{ccc} T(H(V)) & \xrightarrow{\alpha} & H(T(V)) \\ \tilde{p} \downarrow & & \downarrow H(p) \\ \Lambda(H(V)) & \xrightarrow{\beta} & H(\Lambda(V)) \end{array} \quad \text{and} \quad \begin{array}{ccc} T(H(V)) & \xrightarrow{\alpha} & H(T(V)) \\ \tilde{s} \uparrow & & \uparrow H(s) \\ \Lambda(H(V)) & \xrightarrow{\beta} & H(\Lambda(V)) \end{array}$$

commute. Indeed, for representatives  $v_1, \dots, v_n \in Z(V)$  the first diagram gives

$$\begin{array}{ccc} [v_1] \otimes \cdots \otimes [v_n] & \longmapsto & [v_1 \otimes \cdots \otimes v_n] \\ \downarrow & & \downarrow \\ [v_1] \cdots [v_n] & \longmapsto & [v_1 \cdots v_n] \end{array}$$

and the second diagram is given as follows:

$$\begin{array}{ccc} \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) [v_{\sigma(1)}] \otimes \cdots \otimes [v_{\sigma(n)}] & \longmapsto & \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) [v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}] \\ \uparrow & & \uparrow \\ [v_1] \cdots [v_n] & \longmapsto & [v_1 \cdots v_n] \end{array}$$

It follows that

$$\beta\beta' = \beta\tilde{p}\alpha^{-1}H(s) = H(p)\alpha\alpha^{-1}H(s) = H(p)H(s) = \text{id}_{H(\Lambda(V))}$$

and similarly

$$\beta'\beta = \tilde{p}\alpha^{-1}H(s)\beta = \tilde{p}\alpha^{-1}\alpha\tilde{s} = \tilde{p}\tilde{s} = \text{id}_{\Lambda(H(V))}$$

### A.20. Example 6.4

- (1) If  $a, b \in A$  are homogeneous then  $[a, b] = ab - (-1)^{|a||b|}ba$  is homogeneous of degree  $|a| + |b|$ , so  $[A_i, A_j] \subseteq A_{i+j}$  for all  $i, j$ . Also

$$[a, b] = ab - (-1)^{|a||b|}ba = -(-1)^{|a||b|}(ba - (-1)^{|a||b|}ab) = -(-1)^{|a||b|}[b, a]$$

<sup>7</sup>This map is a projection of  $T(V)$  on its dg-subspace of graded symmetric tensors.



and

$$\begin{aligned}
d([a, b]) &= d(ab - (-1)^{|a||b|}ba) \\
&= d(ab) - (-1)^{|a||b|}d(ba) \\
&= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||b|}(d(b)a + (-1)^{|b|}bd(a)) \\
&= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||b|}d(b)a - (-1)^{|a||b|+|b|}bd(a) \\
&= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||d(b)|+|a|}d(b)a - (-1)^{|d(a)||b|}bd(a) \\
&= d(a)b - (-1)^{|d(a)||b|}bd(a) + (-1)^{|a|}(ad(b) - (-1)^{|a||d(b)|}d(b)a) \\
&= [d(a), b] + (-1)^{|a|}[a, d(b)].
\end{aligned}$$

We check the graded Jacobi identity for homogeneous  $a, b, c \in A$ . We have

$$\begin{aligned}
[a, [b, c]] &= [a, bc - (-1)^{|b||c|}cb] \\
&= [a, bc] - (-1)^{|b||c|}[a, cb] \\
&= abc - (-1)^{|a||bc|}bca - (-1)^{|b||c|}(acb - (-1)^{|a||cb|}cba) \\
&= abc - (-1)^{|a||bc|}bca - (-1)^{|b||c|}acb + (-1)^{|a||cb|+|b||c|}cba \\
&= abc - (-1)^{|a|(|b|+|c|)}bca - (-1)^{|b||c|}acb + (-1)^{|a|(|b|+|c|)+|b||c|}cba \\
&= abc - (-1)^{|a||b|+|a||c|}bca - (-1)^{|b||c|}acb + (-1)^{|a||b|+|a||c|+|b||c|}cba
\end{aligned}$$

and therefore

$$\begin{aligned}
(-1)^{|a||c|}[a, [b, c]] &= (-1)^{|a||c|}abc - (-1)^{|a||b|}bca \\
&\quad - (-1)^{|a||c|+|b||c|}acb + (-1)^{|a||b|+|b||c|}cba.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{\text{cyclic}} (-1)^{|a||c|}[a, [b, c]] &= \sum_{\text{cyclic}} (-1)^{|a||c|}abc - \sum_{\text{cyclic}} (-1)^{|a||b|}bca \\
&\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|}acb + \sum_{\text{cyclic}} (-1)^{|a||b|+|b||c|}cba \\
&= \sum_{\text{cyclic}} (-1)^{|b||a|}bca - \sum_{\text{cyclic}} (-1)^{|a||b|}bca \\
&\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|}acb + \sum_{\text{cyclic}} (-1)^{|b||c|+|c||a|}acb \\
&= 0.
\end{aligned}$$

(2) If  $a \in \mathbb{P}(B)$  with homogeneous decomposition  $a = \sum_n a_n$  then

$$\Delta(a) = \Delta\left(\sum_n a_n\right) = \sum_n \Delta(a_n)$$

but also

$$\Delta(a) = a \otimes 1 + 1 \otimes a = \sum_n (a_n \otimes 1 + 1 \otimes a_n).$$

By comparing homogeneous components we see that  $\Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n$  for all  $n$ . This means that all homogeneous components  $a_n$  are again primitive, which shows that  $\mathbb{P}(B)$  is a graded subspace of  $B$ . If  $a \in \mathbb{P}(B)$  then

$$\begin{aligned} \Delta(d(a)) &= d(\Delta(a)) \\ &= d(a \otimes 1 + 1 \otimes a) \\ &= d(a \otimes 1) + d(1 \otimes a) \\ &= d(a) \otimes 1 + (-1)^{|a|} a \otimes d(1) + d(1) \otimes a + (-1)^{|1|} 1 \otimes d(a) \\ &= d(a) \otimes 1 + 1 \otimes d(a) \end{aligned}$$

because  $|1| = 0$  and  $d(1) = 0$ . Therefore  $\mathbb{P}(B)$  is a dg-subspace of  $B$ .

If  $a, b \in \mathbb{P}(B)$  then

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ &= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) \\ &= (a \otimes 1)(b \otimes 1) + (a \otimes 1)(1 \otimes b) + (1 \otimes a)(b \otimes 1) + (1 \otimes a)(1 \otimes b) \\ &= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab. \end{aligned}$$

If  $a, b$  are homogeneous then it follows that

$$\begin{aligned} \Delta([a, b]) &= \Delta(ab - (-1)^{|a||b|}ba) \\ &= \Delta(ab) - (-1)^{|a||b|}\Delta(ba) \\ &= ab \otimes 1 + a \otimes b + (-1)^{|a||b|}b \otimes a + 1 \otimes ab \\ &\quad - (-1)^{|a||b|}(ba \otimes 1 + b \otimes a + (-1)^{|a||b|}a \otimes b + 1 \otimes ba) \\ &= ab \otimes 1 + a \otimes b + (-1)^{|a||b|}b \otimes a + 1 \otimes ab \\ &\quad - (-1)^{|a||b|}ba \otimes 1 - (-1)^{|a||b|}b \otimes a - a \otimes b - (-1)^{|a||b|}1 \otimes ba \\ &= (ab - (-1)^{|a||b|}ba) \otimes 1 + 1 \otimes (ab - (-1)^{|a||b|}ba) \\ &= [a, b] \otimes 1 + 1 \otimes [a, b] \end{aligned}$$

which shows that  $[a, b] \in \mathbb{P}(B)$ . Thus  $\mathbb{P}(B)$  is a dg-Lie subalgebra of  $B$ .

(3) If  $A$  is a graded algebra, then the graded subspace  $\text{Der}(A) \subseteq \text{End}(A)$  given by

$$\text{Der}(A)_n := \{\text{derivations of } A \text{ of degree } n\} \subseteq \text{End}(A)_n$$

is a dg-Lie subalgebra of  $\text{End}(A)$ :

Let  $\delta, \varepsilon$  be graded derivations. Then for all homogeneous  $a, b \in A$ ,

$$(\delta\varepsilon)(ab) = \delta(\varepsilon(ab))$$

$$\begin{aligned}
&= \delta(\varepsilon(a)b + (-1)^{|\varepsilon||a|}a\varepsilon(b)) \\
&= \delta(\varepsilon(a)b) + (-1)^{|\varepsilon||a|}\delta(a\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}(\delta(a)\varepsilon(b) + (-1)^{|\delta||a|}a\delta(\varepsilon(b))) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{(|\varepsilon|+|a|)|\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b))
\end{aligned}$$

It follows that

$$\begin{aligned}
(-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) &= (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) + (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b))
\end{aligned}$$

and therefore

$$\begin{aligned}
[\delta, \varepsilon](ab) &= (\delta\varepsilon - (-1)^{|\delta||\varepsilon|}\varepsilon\delta)(ab) \\
&= (\delta\varepsilon)(ab) - (-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&\quad - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b - (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad - (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}(a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|}a\varepsilon(\delta(b))) \\
&= [\delta, \varepsilon](a)b + (-1)^{|\delta, \varepsilon||a|}a[\delta, \varepsilon](b).
\end{aligned}$$

This shows that  $[\delta, \varepsilon] \in \text{Der}(A)$ , so that  $\text{Der}(A)$  is a graded Lie subalgebra of  $\text{End}(A)$ . If  $\delta \in \text{Der}(A)$  is homogeneous then

$$d(\delta) = d \circ \delta - (-1)^{|\delta|}\delta \circ d = [d, \delta]$$

is again a graded derivation, and hence  $\text{Der}(A)$  is a dg-subspace of  $\text{End}(A)$ .

### A.21. Lemma 6.5

- (1) The quotient  $\mathfrak{g}/I$  is again a dg-vector spaces and a Lie algebra. The compatibility of these structures can be checked on generators.
- (2) The cycles  $Z(\mathfrak{g})$  form a graded subspace of  $\mathfrak{g}$ . For homogeneous  $x, y \in Z(\mathfrak{g})$ ,

$$d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)] = [0, y] + (-1)^{|x|}[x, 0] = 0,$$

so  $Z(\mathfrak{g})$  is indeed a graded Lie subalgebra of  $\mathfrak{g}$ . The boundaries  $B(\mathfrak{g})$  form a graded subspace of  $Z(\mathfrak{g})$ . If  $x \in B(\mathfrak{g})$  with  $x = d(x')$ , where  $x' \in \mathfrak{g}$  is homogeneous, then for every  $y \in Z(\mathfrak{g})$ ,

$$[x, y] = [d(x'), y] = d([x', y]) - (-1)^{|x'|}[x', \underbrace{d(y)}_{=0}] = d([x', y]) \in B(\mathfrak{g}).$$

Thus  $B(\mathfrak{g})$  is a graded Lie ideal in  $Z(\mathfrak{g})$ .

### A.22. Proposition 6.7

- (1) This follows from the choice of ideal  $I$ .
- (2) This is a combination of the universal properties of the dg-tensor algebra and that of the quotient dg-algebra.
- (3) We check that the given ideal  $I$  is a dg-Hopf ideal. It is generated by homogenous elements which satisfy

$$\begin{aligned} & d([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &= d([x, y]_{T(\mathfrak{g})}) - d([x, y]_{\mathfrak{g}}) \\ &= [d(x), y]_{T(\mathfrak{g})} + (-1)^{|x|}[x, d(y)]_{T(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} - (-1)^{|x|}[x, d(y)]_{\mathfrak{g}} \\ &= \left( [d(x), y]_{T(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} \right) + (-1)^{|x|} \left( [x, d(y)]_{T(\mathfrak{g})} - [x, d(y)]_{\mathfrak{g}} \right) \in I \end{aligned}$$

so it is a dg-ideal. Also

$$\varepsilon([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) = \varepsilon([x, y]_{T(\mathfrak{g})}) - \varepsilon([x, y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because  $[x, y]_{T(\mathfrak{g})}$  and  $[x, y]_{\mathfrak{g}}$  are homogeneous of degree  $\geq 1$ ,

$$\begin{aligned} & \Delta([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &= \Delta([x, y]_{T(\mathfrak{g})}) - \Delta([x, y]_{\mathfrak{g}}) \\ &= [x, y]_{T(\mathfrak{g})} \otimes 1 + 1 \otimes [x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x, y]_{\mathfrak{g}} \\ &= ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &\in I \otimes T(\mathfrak{g}) + T(\mathfrak{g}) \otimes I \end{aligned}$$

since both  $[x, y]_{T(\mathfrak{g})}$  and  $[x, y]_{\mathfrak{g}}$  are primitive, and finally

$$S([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) = S([x, y]_{T(\mathfrak{g})}) - S([x, y]_{\mathfrak{g}}) = -[x, y]_{T(\mathfrak{g})} + [x, y]_{\mathfrak{g}} \in I.$$

Thus the dg-ideal  $I$  is already a dg-Hopf ideal.

### A.23. The Poincaré–Birkhoff–Witt theorem

**Recall A.5.** If  $\mathfrak{g}$  is a Lie algebra with basis  $(x_\alpha)_{\alpha \in A}$  where  $(A, \leq)$  is linearly ordered then the PBW theorem asserts that  $U(\mathfrak{g})$  has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \quad \text{where } t \geq 0, \alpha_1 < \cdots < \alpha_t \text{ and } n_i \geq 1.$$

This shows in particular that the Lie algebra homomorphism  $\mathfrak{g} \rightarrow U(\mathfrak{g})$  is injective, and it also follows that  $\mathbb{P}(U(\mathfrak{g})) = \mathfrak{g}$ . Moreover,  $\text{gr } U(\mathfrak{g}) \cong S(\mathfrak{g})$  where  $\text{gr } U(\mathfrak{g})$  denotes the associated graded for the standard filtration of  $U(\mathfrak{g})$ .

**Theorem A.6** (dg-PBW theorem). Let  $\mathfrak{g}$  be a dg-Lie algebra with basis  $(x_\alpha)_{\alpha \in A}$  consisting of homogeneous elements such that  $(A, \leq)$  is linearly ordered. Then  $U(\mathfrak{g})$  has as a basis all ordered monomials

$$x_{\alpha_1} \cdots x_{\alpha_n} \quad \text{where } t \geq 0, \alpha_1 < \cdots < \alpha_t, n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.} \quad \square$$

We will not attempt to prove this theorem here, and instead refer to [Qui69, Appendix B, Theorem 2.3] and [FHT01, §21(a)].

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