Differential Graded Hopf Algebras I

1. Conventions and Notations

In the following k denotes an arbitrary field. All vector spaces, algebras, tensor products, etc. are over k, unless otherwise stated. All occuring maps are linear unless otherwise stated. We abbreviate “differential graded” by “dg”.

A **dg-vector space** is the same as a chain complex. of vector spaces, a **dg-subspace** the same as a chain subcomplex. We write $|v|$ for the degree of an element v, which is then assumed to be homogeneous. We always regard graded objects as dg-objects with zero differential. We regard k as a dg-vector space concentrated in degree 0.

If V, W are dg-vector spaces then $V \otimes W$ is a dg-vector space with

$$|v \otimes w| = |v| + |w|\,, \qquad d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w)\,.$$

The **twist map** $\tau\colon V \otimes W \to W \otimes V$ given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

is an isomorphism of dg-vector spaces.[1] We use the Koszul **sign convention**: Whenever homogeneous x, y are swapped the sign $(-1)^{|x||y|}$ is introduced. This results in a well-defined S_n-action on $V^{\otimes n}$ via homomorphisms of dg-vector spaces, given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1,\dots,v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for homogeneous v_i, where $\varepsilon_{v_1,\dots,v_n}(\sigma)$ is the **Koszul sign**. (See Appendix A.2.)

2. Differential Graded Algebras

Definition 2.1.

(1) A **dg-algebra** is a dg-vector space A together with homomorphisms of dg-vector spaces $m\colon A \otimes A \to A$ and $u\colon k \to A$ that make the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} k \otimes A \arrow[d, "u\otimes\mathrm{id}"'] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & A \otimes k \arrow[d, "\mathrm{id}\otimes u"] \\ A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

[1] The naive twist map $v \otimes w \mapsto w \otimes v$ is not a homomorphism of dg-vector spaces.

(2) The dg-algebra A is **graded commutative** if the following diagram commutes:

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

(3) A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.[2]

Remark 2.2. A dg-algebra is the same as a graded algebra A (in particular $|1| = 0$) together with a differential d satisfying $d(1) = 0$ and the **graded Leibniz rule**

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)\,. \tag{1}$$

(See Appendix A.3 for further remarks.)

Examples 2.3. (See Appendix A.4 for the explicit calculations and further examples.)

(1) Every algebra A is a dg-algebra concentrated in degree 0, in particular $A = k$.

(2) If V is a dg-vector space then $\mathrm{T}(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ is again a dg-vector space with

$$\begin{aligned} |v_1 \cdots v_n| &= |v_1| + \cdots + |v_n|\,, \\ d(v_1 \cdots v_n) &= \sum_{i=1}^{n} (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n\,. \end{aligned}$$

This makes $\mathrm{T}(V)$ into a dg-algebra, with multiplication given by concatination

$$(v_1 \cdots v_i) \cdot (v_{i+1} \cdots v_n) = v_1 \cdots v_n\,.$$

The inclusion $V \to \mathrm{T}(V)$ is a homomorphism of dg-vector spaces and if $f\colon V \to A$ is any homomorphism of dg-vector spaces into a dg-algebra A then f extends uniquely to a homomorphism of dg-algebras $F\colon\ \mathrm{T}(V) \to A$:

$$\begin{tikzcd} \mathrm{T}(V) \arrow[r, dashed, "F"] & A \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

The dg-algebra $\mathrm{T}(V)$ is the **dg-tensor algebra** on V.

Proposition 2.4 (Constructions with dg-algebras)**.** Let A, B be a dg-algebras.

(1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$1_{A \otimes B} = 1_A \otimes 1_B \qquad \text{and} \qquad m_{A \otimes B} = (m_A \otimes m_B) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})\,,$$

i.e. $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2$.

[2]By an "ideal" we always mean a two-sided ideal.

(2) The dg-algebra A^{op} is given by $u_{A^{\mathrm{op}}} = u_A$ and $m_{A^{\mathrm{op}}} = m_A \circ \tau$, i.e.

$$1_A = 1_{A^{\mathrm{op}}} \qquad \text{and} \qquad a \cdot_{\mathrm{op}} b = (-1)^{|a||b|} b \cdot a\,.$$

(3) If I is a dg-ideal in A then A/I inherits the structure of a dg-algebra

(4) If A is a dg-algebra then $\mathrm{Z}(A)$ is a graded subalgebra of A, $\mathrm{B}(A)$ is a graded ideal in $\mathrm{Z}(A)$ and $\mathrm{H}(A)$ is hence a graded algebra.

Proof. See Appendix A.5. □

Lemma 2.5. An ideal I in a dg-algebra A is a dg-ideal if and only if I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α.

Proof. See Appendix A.6. □

Definition 2.6. The **graded commutator** in a dg-algebra A is the unique bilinear extension of

$$[a, b] \coloneqq ab - (-1)^{|a||b|} ba\,.$$

(See Appendix A.7 for a remark.)

Example 2.7. Let V be a dg-vector space. The ideal

$$I \coloneqq \big([v, w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-ideal in $\mathrm{T}(V)$, and the quotient $\Lambda(V) \coloneqq \mathrm{T}(V)/I$ is the **dg-symmetric algebra** on V. (See Appendix A.8 for the explicit calculations and further remarks about $\Lambda(V)$.)

3. Differential Graded Coalgebras

Definition 3.1.

(1) A **dg-coalgebra** is a dg-vector space C together with homomorphisms of dg-vector spaces $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ that make the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \otimes C \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ k \otimes C \arrow[r, "\sim"] & C & C \otimes k \arrow[l, "\sim"'] \end{tikzcd}$$

(2) The dg-coalgebra C is **graded cocommutative** if the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

(3) A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.[3]

Remark 3.2. A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that ε vanishes on $\mathrm{B}_0(C)$ and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)}) . \tag{2}$$

(See Appendix A.9 for further remarks.)

Example 3.3. For any dg-vector space V the induced dg-vector space $\mathrm{T}(V)$ becomes a dg-coalgebra with the deconcatination

$$\Delta \colon \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V) , \quad v_1 \cdots v_n \mapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n ,$$

$$\varepsilon \colon \mathrm{T}(V) \to k , \quad v_1 \cdots v_n \mapsto \begin{cases} 1 & \text{if } n = 0 , \\ 0 & \text{otherwise} . \end{cases}$$

(See Appendix A.10 for the explicit calculations.)

Proposition 3.4 (Constructions with dg-coalgebras)**.** Let C, D be dg-coalgebras.

(1) The tensor product $C \otimes D$ is again a dg-coalgebra with

$$\begin{aligned} \varepsilon_{C \otimes D}(c \otimes d) &= \varepsilon(c)\varepsilon(d) , \\ \Delta_{C \otimes D}(c \otimes d) &= \sum_{(c),(d)} (-1)^{|c_{(2)}||d_{(1)}|} (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}) . \end{aligned}$$

(2) If I is a dg-coideal in C then C/I inherits a dg-coalgebra structure.

(3) If C is a dg-coalgebra then $\mathrm{Z}(C)$ is a graded subcoalgebra of C, $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ and $\mathrm{H}(C)$ is hence a graded coalgebra.

Proof. See Appendix A.11. □

4. Differential Graded Bialgebras

Definition 4.1.

(1) A **dg-bialgebra** is a tuple $(B, m, u, \Delta, \varepsilon)$ so that (B, m, u) is a dg-algebra, (B, Δ, ε) is a dg-coalgebra and Δ, ε are homomorphisms of dg-algebras. (See Appendix A.12 for remarks about this definition.)

(2) A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

[3] By "coideal" we always mean a two-sided coideal.

Remark 4.2. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)} \,.$$

Warning 4.3. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta\colon B \to B \otimes B$ is a homomorphism of dg-algebras into $B \otimes B$ but not necessarily an algebra homomorphism into the sign-less tensor product $B \otimes_k B$. We will see an explicit counterexample in Example 5.7.

Proposition 4.4 (Constructions with dg-bialgebras)**.** Let B, $\mathcal{B}$ be dg-bialgebras.

(1) If I is a dg-biideal in B then B/I inherits a dg-bialgebra structure.

(2) The cycles $\mathrm{Z}(\mathcal{B})$ form a graded sub-bialgebra of $\mathcal{B}$, $\mathrm{B}(\mathcal{B})$ is a graded biideal in $\mathrm{Z}(\mathcal{B})$ and $\mathrm{H}(\mathcal{B})$ is hence a graded bialgebra.

Proof. See Appendix A.13 □

5. Differential Graded Hopf Algebras

Definition 5.1.

(1) An **antipode** for a dg-bialgebra H is a homomorphism of dg-vector spaces

$$S\colon H \to H$$

that makes the following diagram commute:

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "S\otimes\mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & k \arrow[r, "u"] & H \\
 & H \otimes H \arrow[r, "\mathrm{id}\otimes S"'] & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd} \tag{3}$$

If H admits an antipode then it is a **dg-Hopf algebra**.

(2) A **dg-Hopf ideal** in a dg-Hopf algebra H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.2. A dg-Hopf algebra need not have an underlying Hopf algebra structure.

Remark 5.3. Let H be a dg-Hopf algebra.

(1) The commutativity of the diagram (3) means more explicitely that

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \qquad \text{and} \qquad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H \,.$$

(No additional signs occur because $|S| = 0$.)

(2) One can again characterize S using the convolution product on $\operatorname{Hom}_k(C, A)$ (see Appendix A.14). This then shows in particular the uniqueness of S.

Proposition 5.4 (Constructions with dg-Hopf algebras)**.** Let H, $\mathcal{H}$ be dg-Hopf algebras.

(1) If I is a dg-Hopf ideal in H then H/I inherits a dg-Hopf algebra structure.

(2) The graded bialgebra $\mathrm{H}(\mathcal{H})$ is a graded Hopf algebra with antipode $\mathrm{H}(S_{\mathcal{H}})$.

Proof. See Appendix A.15. □

Example 5.5. Let V be a dg-vector space. The maps

$$\begin{aligned} &V \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, && v \mapsto v \otimes 1 + 1 \otimes v\,, \\ &V \to k\,, && v \mapsto 0\,, \\ &V \to \mathrm{T}(V)^{\mathrm{op}}\,, && v \mapsto -v \end{aligned}$$

are homomorphisms of dg-vector spaces and thus induce homomorphisms of dg-algebras

$$\begin{aligned} \Delta\colon\ & \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \\ \varepsilon\colon\ & \mathrm{T}(V) \to k\,, \\ S\colon\ & \mathrm{T}(V) \to \mathrm{T}(V)^{\mathrm{op}}\,. \end{aligned}$$

These homomorphisms are explicitely given by

$$\begin{aligned} \Delta(v_1 \cdots v_n) &= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} \varepsilon_{v_1,\dots,v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}\,, \\ \varepsilon(v_1 \cdots v_n) &= \begin{cases} 1 & \text{if } n = 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \\ S(v_1 \cdots v_n) &= (-1)^{\sum_{1 \leq i < j \leq n} |v_i||v_j|} (-1)^n v_n \cdots v_1 \end{aligned}$$

for homogeneous v_i, where S is viewed as a map $\mathrm{T}(V) \to \mathrm{T}(V)$ and $\mathrm{Sh}(p, q) \subseteq \mathrm{S}_{p+q}$ denotes the set of p-q-shuffles. These maps make $\mathrm{T}(V)$ into a dg-Hopf algebra. (See Appendix A.16 for the explicit calculations.)

Example 5.6 (Quotients of dg-Hopf algebras)**.** Let V be a dg-vector space. The dg-algebra $\Lambda(V) = \mathrm{T}(V)/I$ from Example 2.7 inherits from $\mathrm{T}(V)$ the structure of a dg-Hopf algebra because the dg-ideal I is a dg-Hopf ideal in $\mathrm{T}(V)$ (see Appendix A.17).

Example 5.7 (Exterior Algebra)**.** Let V be a vector space. We regard V as a dg-vector space concentrated in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\operatorname{char} k \neq 2$ and $V \neq 0$ there exists no bialgebra structure on $\bigwedge(V)$ (see Appendix A.18).

Example 5.8 (Homology of dg-Hopf algebras)**.** Let V be a dg-vector space.

(1) The inclusion $V \to \mathrm{T}(V)$ is a homomorphism of dg-vector spaces and thus induces a homomorphism of graded vector spaces $\mathrm{H}(V) \to \mathrm{H}(\mathrm{T}(V))$, which in turn induces a homomorphism of graded algebras

$$\alpha\colon\ \mathrm{T}(\mathrm{H}(V)) \to \mathrm{H}(\mathrm{T}(V))\,, \quad [v_1]\cdots[v_n] \mapsto [v_1\cdots v_n]$$

where $v_1, \ldots, v_n \in \mathrm{Z}(V)$. We see on representatives that α is a homomorphism of graded Hopf algebras. We can write α as

$$\mathrm{H}(\mathrm{T}(V)) = \mathrm{H}\left(\bigoplus_{d\geq 0} V^{\otimes d}\right) \cong \bigoplus_{d\geq 0} \mathrm{H}\big(V^{\otimes d}\big) \cong \bigoplus_{d\geq 0} \mathrm{H}(V)^{\otimes d} = \mathrm{T}(\mathrm{H}(V))$$

which shows that α is an isomorphism.

(2) If $\mathrm{char}(k) = 0$ then also $\mathrm{H}(\Lambda(V)) \cong \Lambda(\mathrm{H}(V))$: We get again a canonical homomorphism of graded Hopf algebras

$$\beta\colon\ \Lambda(\mathrm{H}(V)) \to \mathrm{H}(\Lambda(V))\,, \quad [v_1]\cdots[v_n] \mapsto [v_1\cdots v_n]$$

where $v_1, \ldots, v_n \in \mathrm{Z}(V)$. The symmetrization map

$$s\colon\ \Lambda(V) \to \mathrm{T}(V)\,, \quad v_1\cdots v_n \mapsto \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n} \sigma\cdot(v_1\otimes\cdots\otimes v_n)$$

is a section for the projection $p\colon\ \mathrm{T}(V) \to \Lambda(V)$ and a homomorphism of dg-vector spaces (see Appendix A.19). Together with the projection $\tilde{p}\colon\ \mathrm{T}(\mathrm{H}(V)) \to \Lambda(\mathrm{H}(V))$ and symmetrization map $\tilde{s}\colon\ \Lambda(\mathrm{H}(V)) \to \mathrm{T}(\mathrm{H}(V))$ we have the following diagram:

$$\begin{tikzcd}
\mathrm{T}(\mathrm{H}(V)) \arrow[r, bend left, "\alpha"] \arrow[d, bend right, "\tilde{p}"'] & \mathrm{H}(\mathrm{T}(V)) \arrow[l, bend left, "\alpha^{-1}"] \arrow[d, bend right, "\mathrm{H}(p)"'] \\
\Lambda(\mathrm{H}(V)) \arrow[r, dashed, bend left, "\beta"] \arrow[u, bend right, "\tilde{s}"'] & \mathrm{H}(\Lambda(V)) \arrow[l, dashed, bend left, "\beta'"] \arrow[u, bend right, "\mathrm{H}(s)"']
\end{tikzcd}$$

We have $\beta = \mathrm{H}(p)\circ\alpha\circ\tilde{s}$, and $\beta' \coloneqq \tilde{p}\circ\alpha^{-1}\circ\mathrm{H}(s)$ is an inverse to β (see Appendix A.19). This shows that β is an isomorphism.

6. Differential Graded Lie Algebras

Let $\mathrm{char}(k) = 0$.

Definition 6.1.

(1) A **dg-Lie algebra** is a dg-vector space $\mathfrak{g}$ together with a homomorphism of dg-vector spaces $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is **graded skew symmmetric** in the sense that the diagram

$$\begin{tikzcd} \mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\tau"] \arrow[dr, "{[-,-]}"'] & & \mathfrak{g} \otimes \mathfrak{g} \arrow[dl, "{-[-,-]}"] \\ & \mathfrak{g} & \end{tikzcd}$$

commutes, and such that $[x,-]$ is for every homogeneous x a graded derivation.

(2) A dg-Lie ideal in a dg-Lie algebra $\mathfrak{g}$ is a dg-subspace with $[\mathfrak{g}, I] \subseteq I$.

Remark 6.2. That $\mathfrak{g}$ is a dg-Lie algebra means that

$$\begin{aligned} [\mathfrak{g}_i, \mathfrak{g}_j] &\subseteq \mathfrak{g}_{i+j}\,, \\ [x,y] &= -(-1)^{|x||y|}[y,x]\,, \\ [x,[y,z]] &= [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]\,, \\ d([x,y]) &= [d(x),y] + (-1)^{|x|}[x,d(y)]\,. \end{aligned} \tag{4}$$

We can rewrite (4) as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x,[y,z]] = 0\,.$$

Warning 6.3. A dg-Lie algebra need not have an underlying Lie algebra structure.

Example 6.4.

(1) Every dg-algebra A is a dg-Lie algebra when endowed with the graded commutator.

(2) In any dg-bialgebra B the subspace of primitive elements,

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}\,,$$

is a dg-Lie subalgebra of B.

(See Appendix A.20 for explicit calculations and another example.)

Lemma 6.5. Let $\mathfrak{g}$ be a dg-Lie algebra.

(1) If I is a dg-Lie ideal in $\mathfrak{g}$ then $\mathfrak{g}/I$ inherits a dg-Lie algebra structure.

(2) The cycles $\mathrm{Z}(\mathfrak{g})$ form a graded Lie subalgebra of $\mathfrak{g}$, $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$ and $\mathrm{H}(\mathfrak{g})$ is thus a graded Lie algebra.

Proof. See Appendix A.21. □

Definition 6.6. The **universal enveloping dg-algebra** of a dg-Lie algebra $\mathfrak{g}$ is

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/\big([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous}\big)\,.$$

Proposition 6.7.

(1) The composition $i\colon \mathfrak{g} \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ is a homomorphism of dg-Lie algebras.

(2) If A is any dg-algebra and $f\colon \mathfrak{g} \to A$ a homomorphism of dg-Lie algebras there exists a unique homomorphism of dg-algebras $F\colon\ \mathrm{U}(\mathfrak{g}) \to A$ that extends f:

$$\begin{tikzcd} \mathrm{U}(\mathfrak{g}) \arrow[r, dashed, "F"] & A \\ \mathfrak{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

(3) The universal enveloping dg-algebra $\mathrm{U}(\mathfrak{g})$ inherits from $\mathrm{T}(\mathfrak{g})$ the structure of a dg-Hopf algebra.

Proof. See Appendix A.22. □

We will now show that $\mathrm{H}(\mathrm{U}(\mathfrak{g})) \cong \mathrm{U}(\mathrm{H}(\mathfrak{g}))$. For this we need a version of the Poincaré–Birkhoff–Witt theorem (PBW theorem) for dg-Lie algebras and their universal enveloping dg-algebras, which we formulate in Appendix A.23. We will also blackbox the following consequences of the PBW theorem.

Corollary 6.8 (of the PBW theorem)**.** Let $\mathfrak{g}$ be a dg-Lie algebra.

(1) The canonical map $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is injective.

(2) The dg-Lie algebra $\mathfrak{g}$ can be retrieved from $\mathrm{U}(\mathfrak{g})$ as $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$.

(3) If $s\colon\ \Lambda(\mathfrak{g}) \to \mathrm{T}(\mathfrak{g})$ denotes the symmetrization map from Example 5.8 then

$$e\colon\ \Lambda(\mathfrak{g}) \xrightarrow{s} \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$$

is an isomorphism of dg-vector spaces (and even of dg-coalgebra). □

Example 6.9 (Homology of $\mathrm{U}(\mathfrak{g})$)**.** The inclusion $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is a homomorphism of dg-Lie algebra and so induces a homomorphism of graded Lie algebras $\mathrm{H}(\mathfrak{g}) \to \mathrm{H}(\mathrm{U}(\mathfrak{g}))$, which in turn induces a homomorphism of graded algebras

$$\gamma\colon\ \mathrm{U}(\mathrm{H}(\mathfrak{g})) \to \mathrm{H}(\mathrm{U}(\mathfrak{g}))\,, \quad [x_1]\cdots[x_n] \mapsto [x_1\cdots x_n]$$

for $x_1, \ldots, x_n \in \mathrm{Z}(\mathfrak{g})$. We see on representatives that this is a homomorphism of dg-Hopf algebras. It is an isomorphism: We denote the isomorphisms of dg-vector spaces $\Lambda(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ and $\Lambda(\mathrm{H}(\mathfrak{g})) \to \mathrm{U}(\mathrm{H}(\mathfrak{g}))$ from Corollary 6.8 by e and $\tilde{e}$. Together with the isomorphism of graded algebras

$$\beta\colon\ \Lambda(\mathrm{H}(\mathfrak{g})) \to \mathrm{H}(\Lambda(\mathfrak{g}))\,, \quad [x_1]\cdots[x_n] \mapsto [x_1\cdots x_n]$$

from Example 5.8 we get the following commutative diagram:

$$\begin{tikzcd}
\Lambda(\mathrm{H}(\mathfrak{g})) \arrow[r, "\sim", "\tilde{e}"'] \arrow[d, "\beta"', "\sim"] & \mathrm{U}(\mathrm{H}(\mathfrak{g})) \arrow[d, dashed, "\gamma"] \\
\mathrm{H}(\Lambda(\mathfrak{g})) \arrow[r, "\sim", "\mathrm{H}(e)"'] & \mathrm{H}(\mathrm{U}(\mathfrak{g}))
\end{tikzcd}$$

The arrows e, $\mathrm{H}(e)$, β are isomorphisms, hence γ is one.

Remark 6.10.

(1) If $\mathcal{H}$ is a dg-Hopf algebra then $\mathrm{H}(\mathbb{P}(\mathcal{H})) \cong \mathbb{P}(\mathrm{H}(\mathcal{H}))$. (This statement can be found without proof in [Lod92, Theorem A.9].)

(2) If H is a graded cocommutative connected[4] dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong \mathrm{U}(\mathbb{P}(H))$. Together with Corollary 6.8 this results in an equivalence between the categories of dg-Lie algebras and graded cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B,Theorem 4.5].

[4] The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

A. Calculations, Proofs and Remarks

A.1. More Conventions and Notations

A map $f\colon V \to W$ is **graded** of **degree** $d = |f|$ if $f(V_n) \subseteq V_{n+d}$ for all n. The differential d is a graded map of degree -1. If $f\colon V \to V'$, $g\colon W \to W'$ are graded maps then $f \otimes g\colon V \otimes V' \to W \otimes W'$ is the graded map of degree $|f \otimes g| = |f| + |g|$ given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) \,.$$

If f, g are homomorphisms of dg-vector spaces then so is $f \otimes g$.

If V, W are dg-vector spaces then $\operatorname{Hom}(V, W)$ is the dg-vector space with

$$\begin{aligned} \operatorname{Hom}(V,W)_n &= \{\text{graded maps } V \to W \text{ of degree } n\} \,, \\ d(f) &= d \circ f - (-1)^{|f|} f \circ d \,. \end{aligned}$$

The spaces $\operatorname{Hom}(V,W)_n$ are linearly independent in $\operatorname{Hom}_k(V,W)$, in the sense that the sum $\sum_n \operatorname{Hom}(V,W)_n$ is direct. We therefore regard $\operatorname{Hom}(V,W) = \bigoplus_n \operatorname{Hom}(V,W)_n$ as a linear subspace of $\operatorname{Hom}_k(V,W)$.

A.2. The Koszul Sign

We have for every $i = 1, \dots, n-1$ a twist map

$$\begin{aligned} \tau_i\colon V^{\otimes n} &\to V^{\otimes n} \,, \\ v_1 \otimes \cdots \otimes v_n &\mapsto v_1 \otimes \cdots \otimes \tau(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n \\ &\mapsto (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \,. \end{aligned}$$

The group S_n is generated by the simple reflections $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\begin{aligned} \sigma_i^2 &= 1 && \text{for } i = 1, \dots, n-1 \,, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for } |i-j| \geq 2 \,, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \dots, n-2 \,. \end{aligned}$$

We check that the twist maps $\tau_1, \dots, \tau_{n-1}$ satisfy these relations, which shows that S_n acts on $V^{\otimes n}$ such that s_i acts via τ_i: We have

$$\tau_i^2(v_1 \otimes \cdots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} \tau_i(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots v_n) = v_1 \otimes \cdots \otimes v_n$$

and thus $\tau_i^2 = 1$. If $|i-j| \geq 2$ then

$$\begin{aligned} &\tau_i \tau_j (v_1 \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_j||v_{j+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes v_n \\ &= \tau_j \tau_i (v_1 \otimes \cdots \otimes v_n) \end{aligned}$$

and thus $\tau_i\tau_j = \tau_j\tau_i$. We also have

$$
\begin{aligned}
&\tau_i\tau_{i+1}\tau_i(v_1 \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|}\tau_i\tau_{i+1}(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|+|v_i||v_{i+2}|}\tau_i(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_{i+2} \otimes v_i \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|+|v_i||v_{i+2}|+|v_{i+1}||v_{i+2}|}v_1 \otimes \cdots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\tau_{i+1}\tau_i\tau_{i+1}(v_1 \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_{i+1}||v_{i+2}|}\tau_{i+1}\tau_i(v_1 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes v_{i+1} \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+2}|+|v_{i+1}||v_{i+2}|}\tau_{i+1}(v_1 \otimes \cdots \otimes v_{i+2} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) \\
&= (-1)^{|v_i||v_{i+1}|+|v_i||v_{i+2}|+|v_{i+1}||v_{i+2}|}v_1 \otimes \cdots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \,.
\end{aligned}
$$

Therefore $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$. We now have the desired action of S_n on $V^{\otimes n}$. The twist maps τ_i are homomorphisms of dg-vector spaces whence S_n acts by homomorphisms of dg-vector spaces.

Without sign the action of S_n on $V^{\otimes n}$ would be given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

(so that the tensor factor v_i it moved to the $\sigma(i)$-th position). The above action of S_n on $V^{\otimes n}$ is hence given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1,\dots,v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

with signs $\varepsilon_{v_1,\dots,v_n}(\sigma) \in \{1,-1\}$.

A.3. Remark 2.2

(1) If A is a graded algebra then a graded map $\delta\colon A \to A$ is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\,;$$

more explicitely,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|}a\delta(b)\,.$$

The compatibility condition (1) in the definition of a dg-algebra thus states that the differential d is a derivation for A.

(2) We see that there are two equivalent ways to make a graded vector space into a dg-algebra:

$$
\begin{tikzcd}
\begin{array}{c}\text{graded}\\ \text{vector spaces}\end{array} \arrow[r, "\text{multiplication}"] \arrow[d, "\text{differential}"'] & \begin{array}{c}\text{graded}\\ \text{algebras}\end{array} \arrow[d, "\text{differential}"] \\
\text{dg-vector spaces} \arrow[r, "\text{multiplication}"'] & \text{dg-algebras}
\end{tikzcd}
$$

(3) The graded commutativity of A means $ab = (-1)^{|a||b|}ba$. If $|a|$ is even or $|b|$ is even then $ab = ba$; if $|a|$ is odd then $a^2 = -a^2$ and thus $a^2 = 0$ if $\operatorname{char}(k) \neq 2$.

(4) A homomorphism f of dg-algebras is the same as a homomorphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

A.4. Examples 2.3

(2) It remains to check the compatibility of the multiplication and dg-structure of $\mathrm{T}(V)$: It holds that $1_{\mathrm{T}(V)} \in \mathrm{T}(V)_0$ with $d(1_{\mathrm{T}(V)}) = 0$. Furthermore

$$\begin{aligned} |v_1 \cdots v_n \cdot w_1 \cdots w_m| &= |v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_m| \\ &= |v_1 \cdots v_n| + |w_1 \cdots w_m| \end{aligned}$$

and

$$\begin{aligned} &d(v_1 \cdots v_n \cdot w_1 \cdots w_m) \\ &= \sum_{i=1}^{n} (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \cdot w_1 \cdots w_m \\ &\quad + \sum_{j=1}^{m} (-1)^{|v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_{j-1}|} v_1 \cdots v_n \cdot w_1 \cdots d(w_j) \cdots w_m \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1| + \cdots + |v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m) \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1 \cdots v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m)\,. \end{aligned}$$

This shows that $\mathrm{T}(V)$ is indeed a dg-algebra.

Let A be another dg-algebra and $f\colon V \to A$ a homomorphism of dg-vector spaces an let $F\colon \mathrm{T}(V) \to A$ be the unique extension of f to an algebra homomorphism, given by $F(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$. The algebra homomorphism F is a homomorphism of graded algebras because

$$\begin{aligned} |F(v_1 \cdots v_n)| &= |f(v_1) \cdots f(v_n)| \\ &= |f(v_1)| + \cdots + |f(v_n)| \\ &= |v_1| + \cdots + |v_n| \\ &= |v_1 \cdots v_n|\,. \end{aligned}$$

It is also a homomorphism of dg-vector spaces because

$$
\begin{aligned}
d(F(v_1 \cdots v_n)) &= d(f(v_1) \cdots f(v_n)) \\
&= \sum_{i=1}^{n} (-1)^{|f(v_1)|+\cdots+|f(v_{i-1})|} f(v_1) \cdots d(f(v_i)) \cdots f(v_n) \\
&= \sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} f(v_1) \cdots f(d(v_i)) \cdots f(v_n) \\
&= F\left(\sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \right) \\
&= F(d(v_1 \cdots v_n)) .
\end{aligned}
$$

(3) For any dg-vector space V the algebra structure of $\mathrm{End}_k(V)$ restricts to a dg-algebra structure on $\mathrm{End}(V) = \mathrm{Hom}(V, V)$:

It holds that $\mathrm{id}_V \in \mathrm{End}(V)_0$ and if $f, g \in \mathrm{End}(V)$ are graded maps then $f \circ g$ is again a graded map Therefore $\mathrm{End}(V)$ is a subalgebra of $\mathrm{End}_k(V)$. If $f, g \in \mathrm{End}(V)$ are homogeneous then $|f \circ g| = |f| + |g|$ so $\mathrm{End}(V)$ is a graded algebra. We see from

$$
\begin{aligned}
d(f \circ g) &= d \circ f \circ g - (-1)^{|f \circ g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|} f \circ d \circ g + (-1)^{|f|} f \circ d \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\
&= (d \circ f - (-1)^{|f|} d \circ f) \circ g + (-1)^{|f|} f \circ (d \circ g - (-1)^{|g|} g \circ d) \\
&= d(f) \circ g + (-1)^{|f|} f \circ d(g)
\end{aligned}
$$

and

$$
d(\mathrm{id}_V) = d \circ \mathrm{id}_V - \mathrm{id}_V \circ d = d - d = 0
$$

that $\mathrm{End}(V)$ is a dg-algebra.

A.5. Proposition 2.4

(3) The quotient A/I is a dg-vector space and an algebra and the compatibility of these structures can be checked on representatives.

(4) The cycles $\mathrm{Z}(A)$ form a graded subspace with $1 \in \mathrm{Z}(A)$ and if $a, b \in \mathrm{Z}(A)$ are homogeneous then

$$
d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b) = 0
$$

and hence $ab \in \mathrm{Z}(A)$. The boundaries $\mathrm{B}(A)$ form a graded subspace and if $a \in \mathrm{Z}(A)$ and $b \in \mathrm{B}(B)$ are homogeneous with $b = d(a')$ then

$$
b \cdot a = d(a') \cdot a = d(a \cdot a') - (-1)^{|a|} a' \cdot d(a) = d(a \cdot a')
$$

and hence $ba \in \mathrm{B}(A)$. Simlarly $ab \in \mathrm{B}(A)$.

Warning A.1. If $A \otimes_k B$ is the sign-less tensor product with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ then $A \otimes B \neq A \otimes_k B$ as algebras, i.e. the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B. The underlying algebra of A^{op} is similarly not the opposite of the underlying algebra of A.

A.6. Lemma 2.5

That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α: The ideal I is spanned by $ax_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|} a d(x_\alpha) b + (-1)^{|a|+|x_\alpha|} a x_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$.

A.7. Definition 2.6

We have for homogeneous a, b that $[a, b] = 0$ if and only if a, b graded commute with each other. If A is a dg-algebra and $|a|$ is even then $[a, a] = 0$. But if $|a|$ is odd then $[a, a] = 2a^2$. This means in particular that the graded commutator of an element with itself does not necessarily vanish (because not every element need to graded-commute with itself).

A.8. Example 2.7

(1) The ideal I is a dg-ideal as the generators $[v, w]$ are homogeneous and (by Example 6.4)

$$d([v, w]) = [d(v), w] + (-1)^{|v|}[v, d(w)] \in I\,.$$

(2) If S is a graded commutative dg-algebra, $f\colon V \to S$ a homomorphism of dg-vector spaces then f extends uniquely to a homomorphism of dg-algebras $F\colon\ \Lambda(V) \to S$:

$$\begin{tikzcd} \Lambda(V) \arrow[r, dashed, "F"] & S \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

(3) Let A and B be two dg-algebras. If C is any other dg-algebra and if $f\colon A \to C$ and $g\colon B \to C$ are two homomorphisms of dg-algebras whose images graded-commute, in the sense that

$$f(a)g(b) = (-1)^{|a||b|} g(b)f(a)$$

for all $a \in A$, $b \in B$, then the linear map

$$\varphi\colon A \otimes B \to C\,, \quad a \otimes b \mapsto f(a)g(b)$$

is again a homomorphism of dg-algebras. The inclusions $i\colon A \to A \otimes B$, $a \mapsto a \otimes 1$ and $j\colon B\colon B \to A \otimes B$, $b \mapsto 1 \otimes b$ are homomorphisms of dg-algebras. For every homomorphism of dg-algebras $\varphi\colon A \otimes B \to C$ the compositions $\varphi \circ i\colon A \to A \otimes B$ and $\varphi\colon j\colon B \to A \otimes B$ are again homomorphisms of dg-algebras. This gives a one-to-one correspondence

$$\left\{\begin{matrix}\text{homomorphisms of dg-algebras} \\ f\colon A \to C,\ g\colon B \to C \\ \text{whose images graded-commute}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homomorphisms of dg-algebras} \\ \varphi\colon A \otimes B \to C\end{matrix}\right\},$$

$$\begin{aligned}(f, g) &\longmapsto (a \otimes b \mapsto f(a)g(b)), \\ (\varphi \circ i, \varphi \circ j) &\longleftarrow\!\shortmid \varphi .\end{aligned}$$

(4) It follows for any two dg-vector spaces V and W that

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

since we have for every dg-algebra A natural bijections

$$\begin{aligned} &\{\text{homomorphisms of dg-algebras } \Lambda(V \oplus W) \to A\} \\ \cong\ &\{\text{homomorphisms of dg-vector spaces } V \oplus W \to A\} \\ \cong\ &\{(f, g) \mid \text{homomorphisms of dg-vector spaces } f\colon V \to A,\ g\colon W \to A\} \\ \cong\ &\{(\varphi, \psi) \mid \text{homomorphisms of dg-algebras } \varphi\colon\ \Lambda(V) \to A,\ \psi\colon\ \Lambda(W) \to A\} \\ \cong\ &\{\text{homomorphisms of dg-algebras } \Lambda(V) \otimes \Lambda(W) \to A\} . \end{aligned}$$

More explicitely, the inclusions $V \to V \oplus W$ and $W \to V \oplus W$ induce homomorphisms of dg-algebras $\Lambda(V) \to \Lambda(V \oplus W)$ and $\Lambda(W) \to \Lambda(V \oplus W)$ that give an isohomomorphism of dg-algebras

$$\Lambda(V) \otimes \Lambda(W) \xrightarrow{\sim} \Lambda(V \oplus W), \quad v_1 \cdots v_n \otimes w_1 \cdots w_m \mapsto v_1 \cdots v_n w_1 \cdots w_m .$$

(5) Let V be a graded vector space.

If V is concentrated in even degrees then $\Lambda(V) = \mathrm{S}(V)$ and if V is concentrated in odd degrees then $\Lambda(V) = \bigwedge(V)$, with the grading of $\Lambda(V)$ and $\bigwedge(V)$ induced by the one of V.

We have $V = V_{\text{even}} \oplus V_{\text{odd}}$ as graded vector spaces where $V_{\text{even}} = \bigoplus_n V_{2n}$ and $V_{\text{odd}} = \bigoplus_n V_{2n+1}$, and hence

$$\Lambda(V) = \Lambda(V_{\text{even}} \oplus V_{\text{odd}}) \cong \Lambda(V_{\text{even}}) \otimes \Lambda(V_{\text{odd}}) = \mathrm{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$$

The graded algebra $\mathrm{S}(V_{\text{even}})$ is concentrated in even degree and so it follows that in the tensor product $\mathrm{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$ the simple tensors (strictly) commute, i.e. $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$. Hence

$$\Lambda(V) \cong \mathrm{S}(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}})$$

where $\otimes_k$ denotes the sign-less tensor product.

(6) Let $\mathrm{char}(k) \neq 2$ and let V be a dg-vector space with basis $(x_\alpha)_{\alpha\in A}$ consisting of homogeneous elements such that $(A, \leq)$ is linearly ordered. Then $\Lambda(V)$ admits as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t,\ n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.}$$ [5]

To see this we use the above decomposition

$$\Lambda(V) \cong \mathrm{S}(V_{\mathrm{even}}) \otimes_k \bigwedge(V_{\mathrm{odd}}) \tag{5}$$

as graded algebras: We split up the given basis $(x_\alpha)_{\alpha\in A}$ of V into a basis $(x_\alpha)_{\alpha\in A'}$ of V_{even} and $(x_\alpha)_{\alpha\in A''}$ of V_{odd} (since all x_α are homogeneous). Then $\mathrm{S}(V_{\mathrm{even}})$ has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \qquad \text{where } r \geq 0,\ \alpha_1 < \cdots < \alpha_r \text{ and } n_i \geq 1\,,$$

and $\bigwedge(V_{\mathrm{odd}})$ has as a basis the ordered wedges

$$x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_s} \qquad \text{where } s \geq 0,\ \alpha_1 < \cdots < \alpha_s\,.$$

It follows that with (5) that $\Lambda(V)$ admits the basis

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \cdot x_{\beta_1} \cdots x_{\beta_s} \qquad \text{where } \begin{cases} r, s \geq 0,\ n_i \geq 1, \\ \alpha_1 < \cdots < \alpha_r, \\ \beta_1 < \cdots < \beta_s, \\ |x_{\alpha_i}| \text{ even, } |x_{\beta_j}| \text{ odd.} \end{cases}$$

We can now rearrange these basis vectors into the desired form becaus the factors $x_{\alpha_i}^{n_i}$ and x_{β_j} commute.

A.9. Remark 3.2

(1) If C is a graded coalgebra then a graded map $\omega\colon C \to C$ is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \mathrm{id} + \mathrm{id} \otimes \omega) \circ \Delta\,.$$

This means more explicitely that

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)})\,.$$

The compability (2) means that the differential d (which is a graded map of degree $|-d| = -1$) is a coderivation.

(2) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}\,.$$

(3) A homomorphism of dg-coalgebras is the same as a homomorphism of the underlying graded coalgebras that commutes with the differentials.

(4) Every coalgebra C is a dg-coalgebra centered in degree 0, in particular $C = k$.

[5] The condition $n_i = 1$ for $|x_{\alpha_i}|$ odd commes from the equality $\alpha_i^2 = [\alpha_i, \alpha_i]/2$.

A.10. Example 3.3

We have seen in the first talk that $(\mathrm{T}(C), \Delta, \varepsilon)$ is a coalgebra. We have for every $i = 0, \dots, n$ that

$$\begin{aligned}|v_1 \cdots v_i \otimes v_{i+1} \cdots v_n| &= |v_1 \cdots v_i| + |v_{i+1} \cdots v_n| \\ &= |v_1| + \cdots + |v_i| + |v_{i+1}| + \cdots + |v_n| \\ &= |v_1| + \cdots + |v_n| \,,\end{aligned}$$

so we have a graded coalgebra. We also have

$$\begin{aligned}& d(\Delta(v_1 \cdots v_n)) \\ &= \sum_{i=0}^{n} d(v_1 \cdots v_i \otimes v_{i+1} \cdots v_n) \\ &= \sum_{i=0}^{n} \bigl(d(v_1 \cdots v_i) \otimes v_{i+1} \cdots v_n + (-1)^{|v_1 \cdots v_i|} v_1 \cdots v_i \otimes d(v_{i+1} \cdots v_n) \bigr) \\ &= \sum_{i=0}^{n} \left(\sum_{j=1}^{i} (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\ &\qquad \left. + (-1)^{|v_1 \cdots v_i|} \sum_{j=i+1}^{n} (-1)^{|v_{i+1}| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\ &= \sum_{i=0}^{n} \left(\sum_{j=1}^{i} (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\ &\qquad \left. + \sum_{j=i+1}^{n} (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\ &= \Delta \left(\sum_{j=1}^{n} (-1)^{|v_1| + \cdots + |v_j|} v_1 \otimes \cdots \otimes d(v_j) \otimes \cdots \otimes v_n \right) \\ &= \Delta(d(v_1 \cdots v_n))\end{aligned}$$

which shows that Δ is a homomorphism of dg-vector spaces.

A.11. Proposition 3.4

(3) The quotient C/I is a dg-vector space and a coalgebra, and the compatibility of these structures can be checked on representatives.

(4) If $c \in \mathrm{Z}(C)$ then

$$d(\Delta(c)) = \Delta(d(c)) = \Delta(0) = 0$$

because Δ is a homomorphism of dg-vector spaces, and hence

$$\Delta(c) \in \mathrm{Z}(C \otimes C) = \mathrm{Z}(C) \otimes \mathrm{Z}(C) \,.$$

This shows that $\mathrm{Z}(C)$ is a subcoalgebra of C. It is also a graded subspace of C and hence a graded subcoalgebra.

For $b \in \mathrm{B}(C)$ with $b = d(c)$ we have

$$\begin{aligned}\Delta(b) = \Delta(d(c)) = d(\Delta(c)) &= d\Bigg(\sum_{(c)} c_{(1)} \otimes c_{(2)}\Bigg) \\ &= \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|c_{(1)} \otimes d(c_{(2)})} \in \mathrm{B}(C) \otimes C + C \otimes \mathrm{B}(C)\,.\end{aligned}$$

We also have

$$\varepsilon(b) = \varepsilon(d(c)) = d(\varepsilon(c)) = 0\,.$$

This shows that $\mathrm{B}(C)$ is a coideal in C. It follows from the upcoming lemma that B is also a coideal in $\mathrm{Z}(C)$. Then $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ because $\mathrm{B}(C)$ is a graded subspace of $\mathrm{Z}(C)$.

Lemma A.2. Let C be a coalgebra and let B be a subcoalgebra of C. If I is a coideal in C with $I \subseteq C$ then I is also a coideal in B.

Proof. It follows from the inclusions $I \subseteq B \subseteq C$ that

$$(C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B\,.$$

Hence

$$\Delta(I) = \Delta(I) \cap \Delta(B) \subseteq (C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B\,.$$

Also $\varepsilon_B(I) = \varepsilon_C(I) = 0$. □

A.12. Definition 4.1

One can also could equivalently require m, u to be homomorphisms of dg-coalgebras:

Lemma A.3. Let B be a dg-vector space, (B, m, u) a dg-algebra and (B, Δ, ε) a dg-coalgebra. Then the following conditions are equivalent:

(1) Δ and ε are homomorphisms of dg-algebras.

(2) m and u are homomorphisms of dg-coalgebras.

Proof. The same diagramatic proof as in the non-dg case (as seen in the second talk). □

A.13. Proposition 4.4

(1) It follows from Proposition 2.4 and Proposition 3.4 that B/I is a dg-algebra and dg-coalgebra. The compatibility can be checked on representatives.

(2) It follows from Proposition 2.4 and Proposition 3.4 that $\mathrm{H}(\mathcal{B})$ is again a dg-algebra and dg-coalgebra, and the compatibility of these structures can be checked on representatives.

A.14. Remark 5.3

If C is a dg-coalgebra and A is a dg-algebra then the convolution product

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C$$

on $\mathrm{Hom}_k(C, A)$ makes $\mathrm{Hom}(C, A)$ into a dg-algebra:

We have $1_{\mathrm{Hom}_k(C,A)} = u \circ \epsilon \in \mathrm{Hom}(C, A)_0$ because both u_A and ϵ_C are homomorphisms of dg-vector spaces and thus of degree 0. If $f, g \in \mathrm{Hom}(C, A)$ are graded maps then $f \otimes g$ is again a graded map and thus

$$f * g = m \circ (f \otimes g) \circ \Delta$$

is a graded map as a composition of graded maps. This shows that $\mathrm{Hom}(C, A)$ is a subalgebra of $\mathrm{Hom}_k(C, A)$.

We have

$$|f * g| = |m \circ (f \otimes g) \circ \Delta| = |m| + (|f| + |g|) + |\Delta| = |f| + |g|$$

so $\mathrm{Hom}(C, A)$ is a graded algebra with respect to the convolution product.

Furthermore

$$\begin{aligned}
&d(f * g) \\
&= d \circ (f * g) - (-1)^{|f*g|}(f * g) \circ d \\
&= d \circ m \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ \Delta \circ d \\
&= m \circ d_{A \otimes A} \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ d_{C \otimes C} \circ \Delta \\
&= m \circ (d \otimes 1 + 1 \otimes d) \circ (f \otimes g) \otimes \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes 1 + 1 \otimes d) \circ \Delta \\
&= m \circ (d \otimes \mathrm{id}) \circ (f \otimes g) \otimes \Delta + m \circ (\mathrm{id} \otimes d) \circ (f \otimes g) \otimes \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes \mathrm{id}) \circ \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (\mathrm{id} \otimes d) \circ \Delta \\
&= m \circ ((d \circ f) \otimes g) \otimes \Delta + (-1)^{|f|} m \circ (f \otimes (d \circ g)) \otimes \Delta \\
&\quad - (-1)^{|f|} m \circ ((f \circ d) \otimes g) \circ \Delta - (-1)^{|f|+|g|} m \circ (f \otimes (g \circ d)) \circ \Delta \\
&= m \circ ((d \circ f - (-1)^{|f|} f \circ d) \otimes g) \otimes \Delta \\
&\quad + (-1)^{|f|} m \circ (f \otimes (d \circ g - (-1)^{|g|} g \circ d)) \otimes \Delta \\
&= m \circ (d(f) \otimes g) \circ \Delta + (-1)^{|f|} m \circ (f \otimes d(g)) \otimes \Delta \\
&= d(f) * g + (-1)^{|f|} f * d(g)
\end{aligned}$$

because m and Δ are commute with the differentials. Hence $\mathrm{Hom}(C, A)$ is a dg-algebra with respect to the convolution product.

Now we need to explain why an inverse to id_H in $\mathrm{Hom}(H, H)$ with respect to the convolution product $*$ is again a homomorphism of dg-vector spaces. For this we use the following result:

Lemma A.4. Let A be a dg-algebra and let $a \in A$ be a homogeneous unit.

(1) The inverse a^{-1} is homogeneous of degree $|a^{-1}| = -|a|$.

(2) If a is a cycle then so is a^{-1}.

Proof.

(1) Let $d = |a|$ and let $a^{-1} = \sum_n a'_n$ be the homogeneous decomposition of a^{-1}. It follows from $1 = ab = \sum_n aa'_n$ that in degree zero, $1 = aa'_{-d}$. Thus a'_{-d} is the inverse of a, i.e. $a^{-1} = a'_{-d} \in A_{-d}$.

(2) It follows from

$$0 = d(1) = d(aa^{-1}) = d(a)a^{-1} + (-1)^{|a|}ad(a^{-1})$$

that $(-1)^{|a|}ad(a^{-1}) = 0$ because $d(a) = 0$. Hence $d(a^{-1}) = 0$ as a is a unit. □

The space $\mathrm{Z}_0(\mathrm{Hom}(V, W))$ consists of the homomorphism of dg-vector spaces $V \to W$. It hence follows from Lemma A.4 that if $f \in \mathrm{Z}_0(\mathrm{Hom}(V, W))$ admits an inverse g with respect to the convolution product that again $g \in \mathrm{Z}_0(\mathrm{Hom}(V, W))$.

A.15. Proposition 5.4

(1) It follows from Proposition 4.4 that H is a dg-bialgebra and the condition $S(I) \subseteq I$ ensures that S induces a homomorphism of dg-vector spaces $\overline{S} \colon H/I \to H/I$. The antipode condition for $\overline{S}$ can now be checked on representatives.

(2) The homology $\mathrm{H}(\mathcal{H})$ is a dg-bialgebra by Proposition 4.4 and that $\mathrm{H}(S_{\mathcal{H}})$ is an antipode can be checked on representatives.

A.16. Example 5.5

The dg-coalgebra diagrams for $(\mathrm{T}(V), \Delta, \varepsilon)$ can be checked on algebra generators of $\mathrm{T}(V)$ because all arrows in these diagrams are homomorphisms of dg-algebras. It hence sufficies to check these diagrams for elements of V, where this is straightforward.

It remains to check the equalities

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \quad \text{and} \quad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H$$

for the monomials $h = v_1 \cdots v_n$. If $n = 0$ then $h = 1$ and both equalities hold, so we consider in the following the case $n \geq 1$. Then $\varepsilon(v_1 \cdots v_n) = 0$ so we have to show that in the sums $\sum_{(h)} S(h_{(1)})h_{(2)}$ and $\sum_{(h)} h_{(1)}S(h_{(2)})$ all terms cancel out. We consider for simplicity only the sum $\sum_{(h)} S(h_{(1)})h_{(2)}$.[6] We have

$$\Delta(v_1 \cdots v_n) = \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} \varepsilon_{v_1,\dots,v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}\,. \tag{6}$$

[6] The author hasn't actually checked the other sum.

Here

$$S(v_{\sigma(1)}\cdots v_{\sigma(p)}) = (-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|}v_{\sigma(p)}\cdots v_{\sigma(1)}$$

and thus

$$\begin{aligned}
&(m\circ(S\otimes \mathrm{id})\circ\Delta)(v_1\cdots v_n)\\
&=\sum_{p=0}^{n}\sum_{\sigma\in \mathrm{Sh}(p,n-p)}\varepsilon_{v_1,\dots,v_n}(\sigma^{-1})(-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|}\\
&\qquad\qquad\cdot v_{\sigma(p)}\cdots v_{\sigma(1)}v_{\sigma(p+1)}\cdots v_{\sigma(n)}\,. \qquad (7)
\end{aligned}$$

We see that in (6) any two terms of the form

$$w_1w_2\cdots w_i\otimes w_{i+1}\cdots w_n \quad\text{and}\quad w_2\cdots w_i\otimes w_1w_{i+1}\cdots w_n$$

give in (7) the up to sign same term $w_i\cdots w_2w_1w_{i+1}\cdots w_n$. We now check that the signs differ, so that in (7) both terms cancel out. This then shows that the sum (7) becomes zero.

For $1\le p\le n$ and $\sigma\in\mathrm{Sh}(p,n-p)$ with $\sigma(p)<\sigma(1)$ the term associated to $v_{\sigma(1)}\cdots v_{\sigma(p)}\otimes v_{(p+1)}\cdots v_{\sigma(n)}$ is given by

$$v_{\sigma(2)}\cdots v_{\sigma(p)}\otimes v_{\sigma(1)}v_{\sigma(p+1)}\cdots v_{\sigma(n)} = v_{\tau(1)}\cdots v_{\tau(p-1)}\otimes v_{\tau(p)}\cdots v_{\tau(n)}$$

for the permuation $\omega\in\mathrm{Sh}(p-1,n-p+1)$ given by

$$\omega = \sigma\circ(1\,2\,\cdots\,p)\,,$$

i.e.

$$\omega(i) = \begin{cases}\sigma(i+1) & \text{if } 1\le i\le p-1\,,\\ \sigma(1) & \text{if } i=p\,,\\ \sigma(i) & \text{if } p+1\le i\le n\,.\end{cases}$$

We see from the Koszul sign rule that the signs $\varepsilon_{v_1,\dots,v_n}(\sigma^{-1})$ and $\varepsilon_{v_1,\dots,v_n}(\omega^{-1})$ differ by the factor $(-1)^{|v_{\sigma(1)}||v_{\sigma(2)}|+\cdots+|v_{\sigma(1)}||v_{\sigma(p)}|}$. Therefore

$$\begin{aligned}
&\varepsilon_{v_1,\dots,v_n}(\sigma^{-1})(-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|}\\
&=\varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^{|v_{\sigma(1)}||v_{\sigma(2)}|+\cdots+|v_{\sigma(1)}||v_{\sigma(p)}|}(-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|}\\
&=\varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^p(-1)^{\sum_{2\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|}\\
&=\varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^p(-1)^{\sum_{1\le i<j\le p-1}|v_{\omega(i)}||v_{\omega(j)}|}\\
&=-\,\varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^{p-1}(-1)^{\sum_{1\le i<j\le p-1}|v_{\omega(i)}||v_{\omega(j)}|}\,.
\end{aligned}$$

Thus the signs differ as claimed.

A.17. Example 5.6

We have

$$\begin{aligned}\varepsilon([v,w]) &= \varepsilon\big(vw - (-1)^{|v||w|}wv\big)\\ &= \varepsilon(vw) - (-1)^{|v||w|}wv\\ &= \varepsilon(v)\varepsilon(w) - (-1)^{|v||w|}\varepsilon(w)\varepsilon(v)\\ &= 0\end{aligned}$$

as $\varepsilon(v) = \varepsilon(w) = 0$. The elements v and w are primitive whence $[v,w]$ is primitive. Therefore

$$\Delta([v,w]) = [v,w] \otimes 1 + 1 \otimes [v,w] \in I \otimes \mathrm{T}(V) + \mathrm{T}(V) \otimes I\,.$$

Also

$$\begin{aligned}S([v,w]) &= S\big(vw - (-1)^{|v||w|}wv\big)\\ &= S(vw) - (-1)^{|v||w|}S(wv)\\ &= (-1)^{|v||w|}wv - vw\\ &= -\big(vw - (-1)^{|v||w|}wv\big)\\ &= -[v,w]\\ &\in I\,.\end{aligned}$$

A.18. Example 5.7

Suppose that there exists a bialgebra structure on $E \coloneqq \bigwedge(V)$. Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker\varepsilon = \bigoplus_{n\geq 1} E_n \eqqcolon I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{E \otimes I} \qquad \text{and} \qquad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes E}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}\,.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2}\,,$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}\,.$$

Now $v^2 = 0$ and thus

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}\,.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

A.19. Example 5.8

(1) The action of S_n on $V^{\otimes n}$ is by homomorphism of dg-vector spaces as mentioned in Section 1 and shown in Appendix A.2. The symmetrization map

$$\tilde{s} \colon \mathrm{T}(V) \to \mathrm{T}(V)\,, \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n)$$

therefore results in a homomorphism of dg-vector spaces $\tilde{s} \colon \mathrm{T}(V) \to \mathrm{T}(V)$.[7] It follows that the factored map $s \colon \Lambda(V) \to \mathrm{T}(V)$ is again a homomorphism of dg-vector spaces.

(2) We observe that the diagrams

$$\begin{tikzcd} \mathrm{T}(\mathrm{H}(V)) \arrow[r, "\alpha"] \arrow[d, "\tilde{p}"'] & \mathrm{H}(\mathrm{T}(V)) \arrow[d, "\mathrm{H}(p)"] \\ \Lambda(\mathrm{H}(V)) \arrow[r, "\beta"] & \mathrm{H}(\Lambda(V)) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{T}(\mathrm{H}(V)) \arrow[r, "\alpha"] & \mathrm{H}(\mathrm{T}(V)) \\ \Lambda(\mathrm{H}(V)) \arrow[r, "\beta"] \arrow[u, "\tilde{s}"] & \mathrm{H}(\Lambda(V)) \arrow[u, "\mathrm{H}(s)"'] \end{tikzcd}$$

commute. Indeed, for representatives $v_1, \dots, v_n \in \mathrm{Z}(V)$ the first diagram gives

$$\begin{tikzcd} {[v_1] \otimes \cdots \otimes [v_n]} \arrow[r, mapsto] \arrow[d, mapsto] & {[v_1 \otimes \cdots \otimes v_n]} \arrow[d, mapsto] \\ {[v_1] \cdots [v_n]} \arrow[r, mapsto] & {[v_1 \cdots v_n]} \end{tikzcd}$$

and the second diagram is given as follows:

$$\begin{tikzcd} \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_n} \varepsilon(\sigma^{-1})[v_{\sigma(1)}] \otimes \cdots \otimes [v_{\sigma(n)}] \arrow[r, mapsto] & \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_n} \varepsilon(\sigma^{-1})[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}] \\ {[v_1] \cdots [v_n]} \arrow[r, mapsto] \arrow[u, mapsto] & {[v_1 \cdots v_n]} \arrow[u, mapsto] \end{tikzcd}$$

It follows that

$$\beta\beta' = \beta\tilde{p}\alpha^{-1}\,\mathrm{H}(s) = \mathrm{H}(p)\alpha\alpha^{-1}\,\mathrm{H}(s) = \mathrm{H}(p)\,\mathrm{H}(s) = \mathrm{id}_{\mathrm{H}(\Lambda(V))}$$

and similarly

$$\beta'\beta = \tilde{p}\alpha^{-1}\,\mathrm{H}(s)\beta = \tilde{p}\alpha^{-1}\alpha\tilde{s} = \tilde{p}\tilde{s} = \mathrm{id}_{\Lambda(\mathrm{H}(V))}$$

A.20. Example 6.4

(1) If $a, b \in A$ are homogeneous then $[a,b] = ab - (-1)^{|a||b|}ba$ is homogeneous of degree $|a| + |b|$, so $[A_i, A_j] \subseteq A_{i+j}$ for all i, j. Also

$$[a,b] = ab - (-1)^{|a||b|}ba = -(-1)^{|a||b|}\big(ba - (-1)^{|a||b|}ab\big) = -(-1)^{|a||b|}[b,a]$$

[7] This map is a projection of $\mathrm{T}(V)$ on its dg-subspace of graded symmetric tensors.

and

$$
\begin{aligned}
d([a,b]) &= d\big(ab-(-1)^{|a||b|}ba\big)\\
&= d(ab)-(-1)^{|a||b|}d(ba)\\
&= d(a)b+(-1)^{|a|}ad(b)-(-1)^{|a||b|}\big(d(b)a+(-1)^{|b|}bd(a)\big)\\
&= d(a)b+(-1)^{|a|}ad(b)-(-1)^{|a||b|}d(b)a-(-1)^{|a||b|+|b|}bd(a)\\
&= d(a)b+(-1)^{|a|}ad(b)-(-1)^{|a||d(b)|+|a|}d(b)a-(-1)^{|d(a)||b|}bd(a)\\
&= d(a)b-(-1)^{|d(a)||b|}bd(a)+(-1)^{|a|}\big(ad(b)-(-1)^{|a||d(b)|}d(b)a\big)\\
&= [d(a),b]+(-1)^{|a|}[a,d(b)]\,.
\end{aligned}
$$

We check the graded Jacobi identity for homogeneous $a,b,c\in A$. We have

$$
\begin{aligned}
[a,[b,c]] &= \big[a,bc-(-1)^{|b||c|}cb\big]\\
&= [a,bc]-(-1)^{|b||c|}[a,cb]\\
&= abc-(-1)^{|a||bc|}bca-(-1)^{|b||c|}\big(acb-(-1)^{|a||cb|}cba\big)\\
&= abc-(-1)^{|a||bc|}bca-(-1)^{|b||c|}acb+(-1)^{|a||cb|+|b||c|}cba\\
&= abc-(-1)^{|a|(|b|+|c|)}bca-(-1)^{|b||c|}acb+(-1)^{|a|(|b|+|c|)+|b||c|}cba\\
&= abc-(-1)^{|a||b|+|a||c|}bca-(-1)^{|b||c|}acb+(-1)^{|a||b|+|a||c|+|b||c|}cba
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(-1)^{|a||c|}[a,[b,c]] = (-1)^{|a||c|}abc &- (-1)^{|a||b|}bca\\
&-(-1)^{|a||c|+|b||c|}acb+(-1)^{|a||b|+|b||c|}cba\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{\text{cyclic}}(-1)^{|a||c|}[a,[b,c]] &= \sum_{\text{cyclic}}(-1)^{|a||c|}abc-\sum_{\text{cyclic}}(-1)^{|a||b|}bca\\
&\quad -\sum_{\text{cyclic}}(-1)^{|a||c|+|b||c|}acb+\sum_{\text{cyclic}}(-1)^{|a||b|+|b||c|}cba\\
&= \sum_{\text{cyclic}}(-1)^{|b||a|}bca-\sum_{\text{cyclic}}(-1)^{|a||b|}bca\\
&\quad -\sum_{\text{cyclic}}(-1)^{|a||c|+|b||c|}acb+\sum_{\text{cyclic}}(-1)^{|b||c|+|c||a|}acb\\
&= 0\,.
\end{aligned}
$$

(2) If $a\in\mathbb{P}(B)$ with homogeneous decomposition $a=\sum_n a_n$ then

$$
\Delta(a)=\Delta\left(\sum_n a_n\right)=\sum_n\Delta(a_n)
$$

but also

$$\Delta(a) = a \otimes 1 + 1 \otimes a = \sum_n (a_n \otimes 1 + 1 \otimes a_n)\,.$$

By comparing homogeneous components we see that $\Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n$ for all n. This means that all homogeneous components a_n are again primitive, which shows that $\mathbb{P}(B)$ is a graded subspace of B. If $a \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(d(a)) &= d(\Delta(a)) \\
&= d(a \otimes 1 + 1 \otimes a) \\
&= d(a \otimes 1) + d(1 \otimes a) \\
&= d(a) \otimes 1 + (-1)^{|a|} a \otimes d(1) + d(1) \otimes a + (-1)^{|1|} 1 \otimes d(a) \\
&= d(a) \otimes 1 + 1 \otimes d(a)
\end{aligned}$$

because $|1| = 0$ and $d(1) = 0$. Therefore $\mathbb{P}(B)$ is a dg-subspace of B.

If $a, b \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b) \\
&= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) \\
&= (a \otimes 1)(b \otimes 1) + (a \otimes 1)(1 \otimes b) + (1 \otimes a)(b \otimes 1) + (1 \otimes a)(1 \otimes b) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab\,.
\end{aligned}$$

If a, b are homogeneous then it follows that

$$\begin{aligned}
\Delta([a,b]) &= \Delta\big(ab - (-1)^{|a||b|} ba\big) \\
&= \Delta(ab) - (-1)^{|a||b|} \Delta(ba) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab \\
&\quad - (-1)^{|a||b|} \big(ba \otimes 1 + b \otimes a + (-1)^{|a||b|} a \otimes b + 1 \otimes ba\big) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab \\
&\quad - (-1)^{|a||b|} ba \otimes 1 - (-1)^{|a||b|} b \otimes a - a \otimes b - (-1)^{|a||b|} 1 \otimes ba \\
&= \big(ab - (-1)^{|a||b|} ba\big) \otimes 1 + 1 \otimes \big(ab - (-1)^{|a||b|} ba\big) \\
&= [a,b] \otimes 1 + 1 \otimes [a,b]
\end{aligned}$$

which shows that $[a,b] \in \mathbb{P}(B)$. Thus $\mathbb{P}(B)$ is a dg-Lie subalgebra of B.

(3) If A is a graded algebra, then the graded subspace $\operatorname{Der}(A) \subseteq \operatorname{End}(A)$ given by

$$\operatorname{Der}(A)_n \coloneqq \{\text{derivations of } A \text{ of degree } n\} \subseteq \operatorname{End}(A)_n$$

is a dg-Lie subalgebra of $\operatorname{End}(A)$:

Let δ, ε be graded derivations. Then for all homogeneous $a, b \in A$,

$$(\delta\varepsilon)(ab) = \delta(\varepsilon(ab))$$

$$
\begin{aligned}
&= \delta(\varepsilon(a)b + (-1)^{|\varepsilon||a|}a\varepsilon(b)) \\
&= \delta(\varepsilon(a)b) + (-1)^{|\varepsilon||a|}\delta(a\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\big(\delta(a)\varepsilon(b) + (-1)^{|\delta||a|}a\delta(\varepsilon(b))\big) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{(|\varepsilon|+|a|)|\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b))
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) &= (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) + (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b))
\end{aligned}
$$

and therefore

$$
\begin{aligned}
[\delta,\varepsilon](ab) &= (\delta\varepsilon - (-1)^{|\delta||\varepsilon|}\varepsilon\delta)(ab) \\
&= (\delta\varepsilon)(ab) - (-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&\quad - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b - (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad - (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}\big(a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|}a\varepsilon(\delta(b))\big) \\
&= [\delta,\varepsilon](a)b + (-1)^{|[\delta,\varepsilon]||a|}a[\delta,\varepsilon](b)\,.
\end{aligned}
$$

This shows that $[\delta,\varepsilon] \in \mathrm{Der}(A)$, so that $\mathrm{Der}(A)$ is a graded Lie subalgebra of $\mathrm{End}(A)$. If $\delta \in \mathrm{Der}(A)$ is homogeneous then

$$
d(\delta) = d \circ \delta - (-1)^{|\delta|}\delta \circ d = [d,\delta]
$$

is again a graded derivation, and hence $\mathrm{Der}(A)$ is a dg-subspace of $\mathrm{End}(A)$.

A.21. Lemma 6.5

(1) The quotient $\mathfrak{g}/I$ is again a dg-vector spaces and a Lie algebra. The compatibility of these structures can be checked on generators.

(2) The cycles $\mathrm{Z}(\mathfrak{g})$ form a graded subspace of $\mathfrak{g}$. For homogeneous $x, y \in \mathrm{Z}(\mathfrak{g})$,

$$d([x,y]) = [d(x), y] + (-1)^{|x|}[x, d(y)] = [0, y] + (-1)^{|x|}[x, 0] = 0\,,$$

so $\mathrm{Z}(\mathfrak{g})$ is indeed a graded Lie subalgebra of $\mathfrak{g}$. The boundaries $\mathrm{B}(\mathfrak{g})$ form a graded subspace of $\mathrm{Z}(\mathfrak{g})$. If $x \in \mathrm{B}(\mathfrak{g})$ with $x = d(x')$, where $x' \in \mathfrak{g}$ is homogeneous, then for every $y \in \mathrm{Z}(\mathfrak{g})$,

$$[x, y] = [d(x'), y] = d([x', y]) - (-1)^{|x'|}[x', \underbrace{d(y)}_{=0}] = d([x', y]) \in \mathrm{B}(\mathfrak{g})\,.$$

Thus $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$.

A.22. Proposition 6.7

(1) This follows from the choice of ideal I.

(2) This is a combination of the universal properties of the dg-tensor algebra and that of the quotient dg-algebra.

(3) We check that the given ideal I is a dg-Hopf ideal. It is generated by homogenous elements which satisfy

$$\begin{aligned}
&d([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= d([x,y]_{\mathrm{T}(\mathfrak{g})}) - d([x,y]_{\mathfrak{g}}) \\
&= [d(x), y]_{\mathrm{T}(\mathfrak{g})} + (-1)^{|x|}[x, d(y)]_{\mathrm{T}(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} - (-1)^{|x|}[x, d(y)]_{\mathfrak{g}} \\
&= \Big([d(x), y]_{\mathrm{T}(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}}\Big) + (-1)^{|x|}\Big([x, d(y)]_{\mathrm{T}(\mathfrak{g})} - [x, d(y)]_{\mathfrak{g}}\Big) \in I
\end{aligned}$$

so it is a dg-ideal. Also

$$\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = \varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})}) - \varepsilon([x,y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are homogoneous of degree ≥ 1,

$$\begin{aligned}
&\Delta([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= \Delta([x,y]_{\mathrm{T}(\mathfrak{g})}) - \Delta([x,y]_{\mathfrak{g}})) \\
&= [x,y]_{\mathrm{T}(\mathfrak{g})} \otimes 1 + 1 \otimes [x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x,y]_{\mathfrak{g}} \\
&= ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&\in I \otimes \mathrm{T}(\mathfrak{g}) + \mathrm{T}(\mathfrak{g}) \otimes I
\end{aligned}$$

since both $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are primitive, and finally

$$S([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = S([x,y]_{\mathrm{T}(\mathfrak{g})}) - S([x,y]_{\mathfrak{g}}) = -[x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}} \in I\,.$$

Thus the dg-ideal I is already a dg-Hopf ideal.

A.23. The Poincaré–Birkhoff–Witt theorem

Recall A.5. If $\mathfrak{g}$ is a Lie algebra with basis $(x_\alpha)_{\alpha\in A}$ where $(A, \leq)$ is linearly ordered then the PBW theorem asserts that $\mathrm{U}(\mathfrak{g})$ has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t \text{ and } n_i \geq 1\,.$$

This shows in particular that the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is injective, and it also follows that $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$. Moreover, $\operatorname{gr}\mathrm{U}(\mathfrak{g}) \cong \mathrm{S}(\mathfrak{g})$ where $\operatorname{gr}\mathrm{U}(\mathfrak{g})$ denotes the associated graded for the standard filtration of $\mathrm{U}(\mathfrak{g})$.

Theorem A.6 (dg-PBW theorem)**.** Let $\mathfrak{g}$ be a dg-Lie algebra with basis $(x_\alpha)_{\alpha\in A}$ consisting of homogeneous elements such that $(A, \leq)$ is linearly ordered. Then $\mathrm{U}(\mathfrak{g})$ has as a basis all ordered monomials

$$x_{\alpha_1} \cdots x_{\alpha_n} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t,\ n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.} \qquad \square$$

We will not attempt to prove this theorem here, and instead refer to [Qui69, Appendix B,Theorem 2.3] and [FHT01, §21(a)].

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