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Differential Graded Hopf Algebras I

In the following k denotes a field. All vector spaces, algebras, tensor products, etc. are over k, unless otherwise stated. All occuring maps are linear unless otherwise stated. Additional constraints on $\mathrm{char}(k)$ are made explicit when used.

1. Preliminary Notions and Notations

A **dg-vector space** V is the same as a chain complex, a **dg-subspace** the same as a chain subcomplex. The elements $v \in V_n$ are **homogeneous** of **degree** $|v| = n$. Whenever we write $|v|$ the element v is assumed to be homogeneous. We always regard graded objects as differential graded objects with zero differential. We regard k as a dg-vector space concentrated in degree 0.

If V, W are dg-vector spaces then $V \otimes W$ is a dg-vector space with

$$|v \otimes w| = |v| + |w| \,, \qquad d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w) \,.$$

The **twist map** $\tau \colon V \otimes W \to W \otimes V$ given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

is an isomorphism of dg-vector spaces.[1] We use the Koszul-Quillen **sign convention**: Whenever homogeneous x, y are swapped the sign $(-1)^{|x||y|}$ is introduced. This results in an S_n-action on $V^{\otimes n}$ via morphisms of dg-vector spaces, given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1,\dots,v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \,.$$

for homogeneous v_i. Here $\varepsilon_{v_1,\dots,v_n}(\sigma)$ is the **Koszul sign**. (See Appendix A.1.)

A map $f \colon V \to W$ is **graded** of **degree** $d = |f|$ if $f(V_n) \subseteq V_{n+d}$ for all n, and $\mathrm{Hom}(V,W)$ is the dg-vector spaces with

$$\mathrm{Hom}(V,W)_n = \{\text{graded maps } V \to W \text{ of degree } n\} \,,$$
$$d(f) = d \circ f - (-1)^{|f|} f \circ d \,.$$

The differential d is a graded map of degree -1. If $f \colon V \to V'$, $g \colon W \to W'$ are graded maps then $f \otimes g \colon V \otimes V' \to W \otimes W'$ is the graded map given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w) \,.$$

In particular $|f \otimes g| = |f| + |g|$.

[1] The naive twist map $v \otimes w \mapsto w \otimes v$ is not a morphism of dg-vector spaces.

2. Differential Graded Algebras

Definition 2.1.

(1) A **differential graded algebra** or **dg-algebra** is a dg-vector space A together with morphisms of dg-vector spaces $m\colon A \otimes A \to A$ and $u\colon k \to A$ that make the following diagrams commute:

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes A \arrow[d, "u\otimes\mathrm{id}"'] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & A \otimes k \arrow[d, "\mathrm{id}\otimes u"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}
$$

(2) The dg-algebra A is **graded commutative** if the following diagram commutes:

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
$$

(3) A **morphism** of dg-algebras $f\colon A \to B$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$
\begin{tikzcd}
A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] & B \otimes B \arrow[d, "m"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& k \arrow[dl, "u"'] \arrow[dr, "u"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}
$$

(4) A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.[2]

Definition 2.2. A graded map $\delta\colon A \to A$ for a graded algebra A is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\,;$$

more explicitely,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b)\,.$$

Remark 2.3. A dg-algebra is the same as a graded algebra A (in particular $|1| = 0$) together with a differential d such that $d(1) = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)\,,$$

i.e. such that d is a graded derivation. (See Appendix A.2 for further remarks.)

Examples 2.4. (See Appendix A.3 for the explicit calculations)

[2]By an “ideal” we always mean a two-sided ideal.

(1) Every algebra A is a dg-algebra concentrated in degree 0, in particular $A = k$.

(2) For any dg-vector space V the algebra structure of $\mathrm{End}_k(V)$ restricts to a dg-algebra structure on $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

(3) If V is a dg-vector space then $\mathrm{T}(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space with

$$|v_1 \cdots v_n| = |v_1| + \cdots + |v_n| ,$$
$$d(v_1 \cdots v_n) = \sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n .$$

This makes $\mathrm{T}(V)$ into a dg-algebra, with multiplication given by concatination

$$(v_1 \cdots v_i) \cdot (v_{i+1} \cdots v_n) = v_1 \cdots v_n .$$

The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector spaces and if $f \colon V \to A$ is any morphism of dg-vector spaces into a dg-algebra A then f extends uniquely to a morphism of dg-algebras $F \colon \mathrm{T}(V) \to A$:

$$\begin{tikzcd} \mathrm{T}(V) \arrow[r, dashed, "F"] & A \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

The dg-algebra $\mathrm{T}(V)$ is the **differential graded tensor algebra** on V.

Proposition 2.5 (Constructions with dg-algebras)**.** Let A, B be a dg-algebras.

(1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$m_{A \otimes B} \colon A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m} A \otimes B$$
$$u_{A \otimes B} \colon k \xrightarrow{\sim} k \otimes k \xrightarrow{u \otimes u} A \otimes B .$$

More explicitely, $1_{A \otimes B} = 1_A \otimes 1_B$ and $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2$.

(2) The dg-algebra A^{op} is given by $u_{A^{\mathrm{op}}} = u_A$ and $m_{A^{\mathrm{op}}} = m_A \circ \tau$. If $\cdot$ denotes the multiplication in A and $*$ the multiplication in A^{op} then more explicitely

$$1_A = 1_{A^{\mathrm{op}}} , \qquad a * b = (-1)^{|a||b|} b \cdot a .$$

(3) If I is a dg-ideal in A then A/I inherits the structure of a dg-algebra

(4) If A is a dg-algebra then $\mathrm{Z}(A)$ is a graded subalgebra of A, $\mathrm{B}(A)$ is a graded ideal in $\mathrm{Z}(A)$ and $\mathrm{H}(A)$ is hence a graded algebra.

Proof. See Appendix A.4. □

Warning 2.6. If $A \otimes_k B$ is the sign-less tensor product with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ then $A \otimes B \neq A \otimes_k B$ as algebras, i.e. the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B. The underlying algebra of A^{op} is similarly not the opposite of the underlying algebra of A.

Lemma 2.7. An ideal I in a dg-algebra A is a dg-ideal if and only if I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α.

Proof. See Appendix A.5. $\square$

Definition 2.8. The **graded commutator** in a dg-algebra A is the unique bilinear extension of

$$[a, b] \coloneqq ab - (-1)^{|a||b|} ba \,.$$

Example 2.9. Let V be a dg-vector space. The ideal

$$I \coloneqq \big([v, w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-ideal in $\mathrm{T}(V)$, and the quotient $\Lambda(V) \coloneqq \mathrm{T}(V)/I$ is the **differential graded symmetric algebra** on V. (See Appendix A.6 for the explicit calculations and further remarks about $\Lambda(V)$.)

3. Differential Graded Coalgebras

Definition 3.1.

(1) A **differential graded coalgebra** or **dg-coalgebra** is a dg-vector space C together with morphisms of dg-vector spaces $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ that make the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[d, "\varepsilon \otimes \mathrm{id}"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \otimes C \arrow[d, "\mathrm{id} \otimes \varepsilon"] \\ k \otimes C \arrow[r, "\sim"] & C & C \otimes k \arrow[l, "\sim"'] \end{tikzcd}$$

(2) The dg-coalgebra C is **graded cocommutative** if the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau"] & & C \otimes C \end{tikzcd}$$

(3) A **morphism** of dg-coalgebra $f\colon C \to D$ is a morphism of dg-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & D \arrow[d, "\Delta"] \\ C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}$$

(4) A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.[3]

Remark 3.2. A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that ε vanishes on $\mathrm{B}_0(C)$ and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)}) \,.$$

(See Appendix A.7 for further remarks.)

Example 3.3. For any dg-vector space V the induced dg-vector space $\mathrm{T}(V)$ becomes a dg-coalgebra with the deconcatination

$$\Delta\colon\ \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, \quad v_1 \cdots v_n \mapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n \,,$$

$$\varepsilon\colon\ \mathrm{T}(V) \to k\,, \quad v_1 \cdots v_n \mapsto \begin{cases} 1 & \text{if } n = 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(See Appendix A.8 for the explicit calculations.)

Proposition 3.4. Let C, D be dg-coalgebras.

(1) The tensor product $C \otimes D$ is again a dg-coalgebra with

$$\Delta_{C\otimes D}(c \otimes d) = \sum_{(c),(d)} (-1)^{|c_{(2)}||d_{(1)}|} (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}) \,.$$

(2) We have a co-opposite dg-coalgebra C^{cop}.

(3) If I is a dg-coideal in C then C/I inherits a dg-coalgebra structure.

(4) If C is a dg-coalgebra then $\mathrm{Z}(C)$ is a graded subcoalgebra of C, $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ and $\mathrm{H}(C)$ is hence a graded coalgebra.

Proof. See Appendix A.9. □

4. Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, (B, m, u) a dg-algebra and (B, Δ, ε) a dg-coalgebra. Then the following are equivalent:

(1) Δ and ε are morphisms of dg-algebras.

(2) m and u are morphisms of dg-coalgebras.

Proof. The same diagramatic proof as in the non-dg case (see talk 2). □

[3] By "coideal" we always mean a two-sided coideal.

Definition 4.2.

(1) If the conditions of Lemma 4.1 are satisfied then $(B, \mu, u, \Delta, \varepsilon)$ is a dg-bialgebra.

(2) A map $f\colon B \to C$ is a **morphism** of dg-bialgebras if it is both a morphism of dg-algebras and of dg-coalgebras.

(3) A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b),(c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)} \,.$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta\colon B \to B \otimes B$ is a morphism of dg-algebras into $B \otimes B'$ but not necessarily an algebra morphism into the sign-less tensor product $B \otimes_k B$. We will see an explicit counterexample in Example 5.8.

Proposition 4.5. Let B, $\mathcal{B}$ be dg-bialgebras.

(1) If I is a dg-biideal in B then B/I inherits a dg-bialgebra structure.

(2) The cycles $\mathrm{Z}(\mathcal{B})$ form a graded sub-bialgebra of $\mathcal{B}$, $\mathrm{B}(\mathcal{B})$ is a graded biideal in $\mathrm{Z}(\mathcal{B})$ and $\mathrm{H}(\mathcal{B})$ is hence a graded bialgebra.

Proof. See Appendix A.10 □

Definition 4.6. In a dg-bialgebra B, $x \in B$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$, and

$$\mathbb{P}(B) \coloneqq \{x \in B \mid x \text{ is primitive}\,.\}$$

Lemma 4.7. If $x, y \in B$ are primitive then $[x, y]$ is again primitive.

Proof. See Example 6.5. □

5. Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C$$

on $\mathrm{Hom}_k(C, A)$ makes $\mathrm{Hom}(C, A)$ into a dg-algebra.

Proof. See Appendix A.11. □

Definition 5.2.

(1) An **antipode** for a dg-bialgebra H is a convolution-inverse S to id_H in $\mathrm{Hom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**.

(2) A **morphism** of dg-Hopf algebras is a morphism of dg-bialgebras.

(3) A **dg-Hopf ideal** in a dg-Hopf algebra H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra need not have an underlying Hopf algebra structure.

Remark 5.4. The antipode of a dg-Hopf algebra H is the the unique morphism of dg-vector spaces $S\colon H \to H$ that makes the diagram

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "S\otimes\mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & k \arrow[r, "u"] & & H \\
 & H \otimes H \arrow[r, "\mathrm{id}\otimes S"'] & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd} \tag{1}$$

commute. (See Appendix A.12.) This means more explicitely that

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \qquad \text{and} \qquad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H \,.$$

(No additional signs occur because $|S| = 0$.)

Proposition 5.5. Let H, $\mathcal{H}$ be dg-Hopf algebras.

(1) If I is a dg-Hopf ideal in H then H/I inherits a dg-Hopf algebra structure.

(2) The graded bialgebra $\mathrm{H}(\mathcal{H})$ is a graded Hopf algebra with antipode $\mathrm{H}(S_{\mathcal{H}})$.

Proof. See Appendix A.13. □

Example 5.6. Let V be a dg-vector space. The maps

$$\begin{aligned}
&V \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,, && v \mapsto v \otimes 1 + 1 \otimes v\,,\\
&V \to k\,, && v \mapsto 0\,,\\
&V \to \mathrm{T}(V)^{\mathrm{op}}\,, && v \mapsto -v
\end{aligned}$$

are morphisms of dg-vector spaces and hence induces a morphism of dg-algebras

$$\begin{aligned}
\Delta\colon\ & \mathrm{T}(V) \to \mathrm{T}(V) \otimes \mathrm{T}(V)\,,\\
\varepsilon\colon\ & \mathrm{T}(V) \to k\,,\\
S\colon\ & \mathrm{T}(V) \to \mathrm{T}(V)^{\mathrm{op}}
\end{aligned}$$

The maps Δ and ε are explicitely given by

$$\begin{aligned}
\Delta(v_1 \cdots v_n) &= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} \varepsilon_{v_1,\dots,v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}\,,\\
\varepsilon(v_1 \cdots v_n) &= \begin{cases} 1 & \text{if } n = 0\,,\\ 0 & \text{otherwise}\,,\end{cases}\\
S(v_1 \cdots v_n) &= (-1)^{\sum_{1 \leq i < j \leq n} |v_i||v_j|} (-1)^n v_n \cdots v_1
\end{aligned}$$

for homogeneous v_i, when S is viewed as a map $\mathrm{T}(V) \to \mathrm{T}(V)$. These maps make $\mathrm{T}(V)$ is a dg-Hopf algebra. (See Appendix A.14 for the explicit calculations.)

Example 5.7 (Quotients of dg-Hopf algebras)**.** Let V be a dg-vector space. The dg-algebra $\Lambda(V) = \mathrm{T}(V)/I$ from Example 2.9 inherits from $\mathrm{T}(V)$ the structure of a dg-Hopf algebra because the dg-ideal

$$I = \big([v, w] \,\big|\, v, w \in V \text{ are homogeneous}\big)$$

is a dg-Hopf ideal in $\mathrm{T}(V)$ (see Appendix A.15).

Example 5.8 (Exterior Algebra)**.** Let V be a vector space. We regard V as a dg-vector space concentrated in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\operatorname{char} k \neq 2$ there exists no bialgebra structure on $\Lambda \coloneqq \bigwedge(V)$ (see Appendix A.16).

Example 5.9 (Homology of dg-Hopf algebras)**.** Let V be a dg-vector space.

(1) The inclusion $V \to \mathrm{T}(V)$ is a morphism of dg-vector spaces and thus induces a morphism of graded vector spaces $\mathrm{H}(V) \to \mathrm{H}(\mathrm{T}(V))$, which in turn induces a morphism of graded algebras

$$\alpha\colon\ \mathrm{T}(\mathrm{H}(V)) \to \mathrm{H}(\mathrm{T}(V))\,, \quad [v_1] \cdots [v_n] \mapsto [v_1 \cdots v_n]$$

where $v_1, \dots, v_n \in \mathrm{Z}(V)$. We see on representatives that α is a morphism of graded Hopf algebras. We can write α as

$$\mathrm{H}(\mathrm{T}(V)) = \mathrm{H}\left(\bigoplus_{d \geq 0} V^{\otimes d}\right) \cong \bigoplus_{d \geq 0} \mathrm{H}\big(V^{\otimes d}\big) \cong \bigoplus_{d \geq 0} \mathrm{H}(V)^{\otimes d} = \mathrm{T}(\mathrm{H}(V))$$

which shows that α is an isomorphism.

(2) If $\operatorname{char}(k) = 0$ then also $\mathrm{H}(\Lambda(V)) \cong \Lambda(\mathrm{H}(V))$: We get again a canonical morphism of graded algebras

$$\beta\colon\ \Lambda(\mathrm{H}(V)) \to \mathrm{H}(\Lambda(V))\,, \quad [v_1] \cdots [v_n] \mapsto [v_1 \cdots v_n]$$

where $v_1, \dots, v_n \in \mathrm{Z}(V)$. The symmetrization map

$$\begin{aligned} s\colon\ \Lambda(V) \to \mathrm{T}(V)\,, \quad v_1 \cdots v_n \mapsto{}& \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n) \\ ={}& \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \end{aligned}$$

is a section for the projection $p\colon\ \mathrm{T}(V) \to \Lambda(V)$ and a morphism of dg-vector spaces (see Appendix A.17). (The equality holds for homogeneous v_i.) Together with the projection $\tilde{p}\colon \mathrm{T}(\mathrm{H}(V)) \to \Lambda(\mathrm{H}(V))$ and symmetrization map $\tilde{s}\colon \Lambda(\mathrm{H}(V)) \to \mathrm{T}(\mathrm{H}(V))$ we have the following diagram:

$$\begin{tikzcd}
\mathrm{T}(\mathrm{H}(V)) \arrow[r, bend left, "\alpha"] \arrow[d, shift right, "\tilde{p}"'] & \mathrm{H}(\mathrm{T}(V)) \arrow[l, bend left, "\alpha^{-1}"] \arrow[d, shift right, "\mathrm{H}(p)"'] \\
\Lambda(\mathrm{H}(V)) \arrow[r, dashed, bend left, "\beta"] \arrow[u, shift right, "\tilde{s}"'] & \mathrm{H}(\Lambda(V)) \arrow[l, dashed, bend left, "\beta'"] \arrow[u, shift right, "\mathrm{H}(s)"']
\end{tikzcd}$$

We have $\beta = \mathrm{H}(p) \circ \alpha \circ \tilde{s}$, and $\beta' := \tilde{p} \circ \alpha^{-1} \circ \mathrm{H}(s)$ is an inverse to β (see Appendix A.17). This shows that β is an isomorphism.

6. Differential Graded Lie Algebras

Let $\mathrm{char}(k) = 0$.

Recall 6.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a map $[-,-]\colon \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is skew-symmetric and for every $x \in \mathfrak{g}$ the map $[x,-]\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation; the last assertion is equivalent to the Jacobi identity $\sum_{\text{cyclic}}[x,[y,z]] = 0$.

Definition 6.2.

(1) A **dg-Lie algebra** is a dg-vector space $\mathfrak{g}$ together with a morphism $[-,-]\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ such that $[-,-]$ is **graded skew symmmetric**, i.e. such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{g} \arrow[rr, "\tau"] \arrow[dr, "{[-,-]}"'] & & \mathfrak{g} \otimes \mathfrak{g} \arrow[dl, "{-[-,-]}"] \\
& \mathfrak{g} &
\end{tikzcd}$$

commutes, and such that $[x,-]$ is for every homogeneous x a derivation of degree $|x|$.

(2) A dg-Lie ideal in a dg-Lie algebra $\mathfrak{g}$ is a dg-subspace that is a Lie ideal.

Remark 6.3. That $\mathfrak{g}$ is a dg-Lie algebra means that

$$\begin{aligned}
[\mathfrak{g}_i, \mathfrak{g}_j] &\subseteq \mathfrak{g}_{i+j}\,, \\
[x,y] &= -(-1)^{|x||y|}[y,x]\,, \\
[x,[y,z]] &= [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]\,, \\
d([x,y]) &= [d(x),y] + (-1)^{|x|}[x,d(y)]\,.
\end{aligned} \tag{2}$$

We can rewrite (2) as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x,[y,z]] = 0\,.$$

Warning 6.4. A dg-Lie algebra need not have an underlying Lie algebra structure.

Example 6.5.

(1) Every dg-algebra A becomes a dg-Lie algebra with the graded commutator.

(2) For a graded algebra A the graded subspace $\mathrm{Der}(A) \subseteq \mathrm{End}(A)$ given by

$$\mathrm{Der}(A)_n \coloneqq \{\text{derivations of } A \text{ of degree } n\} \subseteq \mathrm{End}(A)_n$$

is a dg-Lie subalgebra of $\mathrm{End}(A)$.

(3) In any dg-bialgebra B the subspace of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B.

(See Appendix A.18 for explicit calculations.)

Lemma 6.6. Let $\mathfrak{g}$ be a dg-Lie algebra.

(1) If I is a dg-Lie ideal in $\mathfrak{g}$ then $\mathfrak{g}/I$ inherits the a dg-Lie algebra structure.

(2) The cycles $\mathrm{Z}(\mathfrak{g})$ form a graded Lie subalgebra of $\mathfrak{g}$, $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$ and $\mathrm{H}(\mathfrak{g})$ is thus a graded Lie algebra.

Proof. See Appendix A.19. ☐

Definition 6.7. The **universal enveloping algebra** of a dg-Lie algebra $\mathfrak{g}$ is a dg-algebra $\mathrm{U}(\mathfrak{g})$ together with a morphism of dg-Lie algebras $i\colon \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ such that for every other dg-algebra A and every morphism of dg-Lie algebras $f\colon \mathfrak{g} \to A$ there exists a unique morphism of dg-algebras $F\colon\ \mathrm{U}(\mathfrak{g}) \to A$ that extends f:

$$\begin{tikzcd} \mathrm{U}(\mathfrak{g}) \arrow[r, dashed, "F"] & A \\ \mathfrak{g} \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 6.8. Every dg-Lie algebra $\mathfrak{g}$ admits a universal enveloping algebra. It is unique up to unique isomorphism and can be constructed as

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/\big([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous}\big)$$

together with the composition $i\colon \mathfrak{g} \to \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$. It inherits from $\mathrm{T}(\mathfrak{g})$ the structure of a dg-Hopf algebra. ☐

Proof. See Appendix A.20. ☐

We will now show that $\mathrm{H}(\mathrm{U}(\mathfrak{g})) \cong \mathrm{U}(\mathrm{H}(\mathfrak{g}))$. For this we need a version of the Poincaré–Birkhoff–Witt theorem (PBW theorem) for dg-Lie algebras, which we formulate in Appendix A.21. We will not attempt to prove this theorem here, and instead refer to [Qui69, Appendix B,Theorem 2.3] and [FHT01, §21(a)]. We will also blackbox the following consequences of the PBW theorem.

Corollary 6.9 (of the PBW theorem)**.** Let $\mathfrak{g}$ be a dg-Lie algebra.

(1) The canonical map $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is injective.

(2) It holds that $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$.

(3) If $s\colon \Lambda(\mathfrak{g}) \to \mathrm{T}(\mathfrak{g})$ denotes the symmetrization map from Example 5.9 then the composition

$$e\colon \Lambda(\mathfrak{g}) \xrightarrow{s} \mathrm{T}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})\,, \quad x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon_{x_1,\dots,x_n}(\sigma^{-1}) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is an isomorphism of dg-vector spaces (and even of dg-coalgebra). □

Example 6.10 (Homology of $\mathrm{U}(\mathfrak{g})$])**.** The inclusion $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is a morphism of dg-Lie algebra and thus induces a morphism of graded Lie algebras $\mathrm{H}(\mathfrak{g}) \to \mathrm{H}(\mathrm{U}(\mathfrak{g}))$, which is turn induces a morphism of graded algebras

$$\gamma\colon \mathrm{U}(\mathrm{H}(\mathfrak{g})) \to \mathrm{H}(\mathrm{U}(\mathfrak{g}))\,, \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

for $x_1, \dots, x_n \in \mathrm{Z}(\mathfrak{g})$. We see on representatives that this is a morphism of dg-Hopf algebras, and it is an isomorphism:

We denote the isomorphisms $\Lambda(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ and $\Lambda(\mathrm{H}(\mathfrak{g})) \to \mathrm{U}(\mathrm{H}(\mathfrak{g}))$ from Corollary 6.9 by e and $\tilde{e}$. With the isomorphism of graded algebras

$$\beta\colon \Lambda(\mathrm{H}(\mathfrak{g})) \to \mathrm{H}(\Lambda(\mathfrak{g}))\,, \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

from Example 5.9 we get the following commutative diagram:

$$\begin{tikzcd}
\Lambda(\mathrm{H}(\mathfrak{g})) \arrow[r, "\sim", "\tilde{e}"'] \arrow[d, "\beta"', "\sim"] & \mathrm{U}(\mathrm{H}(\mathfrak{g})) \arrow[d, dashed, "\gamma"] \\
\mathrm{H}(\Lambda(\mathfrak{g})) \arrow[r, "\sim", "\mathrm{H}(e)"'] & \mathrm{H}(\mathrm{U}(\mathfrak{g}))
\end{tikzcd}$$

The arrows e, $\mathrm{H}(e)$, β are isomorphisms, hence γ is one.

Remark 6.11.

(1) If $\mathcal{H}$ is a dg-Hopf algebra then $\mathrm{H}(\mathbb{P}(\mathcal{H})) \cong \mathbb{P}(\mathrm{H}(\mathcal{H}))$. (This statement can be found without proof in [Lod92, Theorem A.9].)

(2) If H is a graded cocommutative connected[4] dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong \mathrm{U}(\mathbb{P}(H))$. Together with Corollary 6.9 this results in an equivalence between the categories of dg-Lie algebras and cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B,Theorem 4.5].

[4]The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

A. Calculations, Proofs and Remarks

A.1. The Koszul Sign

We have for every $i = 1, \dots, n-1$ a twist map

$$\begin{aligned} \tau_i \colon V^{\otimes n} &\to V^{\otimes n}\,, \\ v_1 \otimes \cdots \otimes v_n &\mapsto v_1 \otimes \cdots \otimes \tau(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n \\ &\mapsto (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n\,. \end{aligned}$$

The group S_n is generated by the simple reflections $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\begin{aligned} \sigma_i^2 &= 1 && \text{for } i = 1, \dots, n-1\,, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for } |i-j| \geq 2\,, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \dots, n-2\,. \end{aligned}$$

We check that the twist maps $\tau_1, \dots, \tau_{n-1}$ satisfy these relations, which shows that S_n acts on $V^{\otimes n}$ such that s_i acts via τ_i: We have

$$\tau_i^2(v_1 \otimes \cdots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} \tau_i(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots v_n) = v_1 \otimes \cdots \otimes v_n$$

and thus $\tau_i^2 = 1$. If $|i-j| \geq 2$ then

$$\begin{aligned} &\tau_i \tau_j (v_1 \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_j||v_{j+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_{j+1} \otimes v_j \otimes \cdots \otimes v_n \\ &= \tau_j \tau_i (v_1 \otimes \cdots \otimes v_n) \end{aligned}$$

and thus $\tau_i \tau_j = \tau_j \tau_i$. We also have

$$\begin{aligned} &\tau_i \tau_{i+1} \tau_i (v_1 \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}|} \tau_i \tau_{i+1} (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_i||v_{i+2}|} \tau_i (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_{i+2} \otimes v_i \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_i||v_{i+2}| + |v_{i+1}||v_{i+2}|} v_1 \otimes \cdots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \end{aligned}$$

and similarly

$$\begin{aligned} &\tau_{i+1} \tau_i \tau_{i+1} (v_1 \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_{i+1}||v_{i+2}|} \tau_{i+1} \tau_i (v_1 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes v_{i+1} \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+2}| + |v_{i+1}||v_{i+2}|} \tau_{i+1} (v_1 \otimes \cdots \otimes v_{i+2} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_i||v_{i+2}| + |v_{i+1}||v_{i+2}|} v_1 \otimes \cdots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n\,. \end{aligned}$$

Therefore $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$. We now have the desired action of S_n on $V^{\otimes n}$. The twist maps τ_i are morphisms of dg-vector spaces whence S_n acts by morphisms of dg-vector spaces.

Without sign the action of S_n on $V^{\otimes n}$ would be given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

(so that the tensor factor v_i it moved to the $\sigma(i)$-th position). The above action of S_n on $V^{\otimes n}$ is hence given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1,\dots,v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

with signs $\varepsilon_{v_1,\dots,v_n}(\sigma) \in \{1, -1\}$.

A.2. Remark 2.3

(1) We see that there are two equivalent ways to make a graded vector space into a dg-algebra:

$$\begin{tikzcd}
\begin{array}{c}\text{graded}\\ \text{vector spaces}\end{array} \arrow[r, "\text{multiplication}"] \arrow[d, "\text{differential}"'] & \begin{array}{c}\text{graded}\\ \text{algebras}\end{array} \arrow[d, "\text{differential}"] \\
\text{dg-vector spaces} \arrow[r, "\text{multiplication}"'] & \text{dg-algebras}
\end{tikzcd}$$

(2) The graded commutativity of A means $ab = (-1)^{|a||b|}ba$. If $|a|$ is even or $|b|$ is even then $ab = ba$; if $|a|$ is odd then $a^2 = -a^2$ and thus $a^2 = 0$ if $\operatorname{char}(k) \neq 2$.

(3) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

A.3. Examples 2.4

(2) It holds that $\mathrm{id}_V \in \operatorname{End}(V)_0$ and if $f, g \in \operatorname{End}(V)$ are graded maps then $f \circ g$ is again a graded map Therefore $\operatorname{End}(V)$ is a subalgebra of $\operatorname{End}_k(V)$. If $f, g \in \operatorname{End}(V)$ are homogeneous then $|f \circ g| = |f| + |g|$ so $\operatorname{End}(V)$ is a graded algebra. We see from

$$\begin{aligned}
d(f \circ g) &= d \circ f \circ g - (-1)^{|f \circ g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|} f \circ d \circ g + (-1)^{|f|} f \circ d \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\
&= (d \circ f - (-1)^{|f|} d \circ f) \circ g + (-1)^{|f|} f \circ (d \circ g - (-1)^{|g|} g \circ d) \\
&= d(f) \circ g + (-1)^{|f|} f \circ d(g)
\end{aligned}$$

and

$$d(\mathrm{id}_V) = d \circ \mathrm{id}_V - \mathrm{id}_V \circ d = d - d = 0$$

that $\operatorname{End}(V)$ is a dg-algebra.

(3) It remains to check the compatibility of the multiplication and dg-structure of $\mathrm{T}(V)$: It holds that $1_{\mathrm{T}(V)} \in \mathrm{T}(V)_0$ with $d(1_{\mathrm{T}(V)}) = 0$. Furthermore

$$\begin{aligned}|v_1 \cdots v_n \cdot w_1 \cdots w_m| &= |v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_m| \\ &= |v_1 \cdots v_n| + |w_1 \cdots w_m|\end{aligned}$$

and

$$\begin{aligned}&d(v_1 \cdots v_n \cdot w_1 \cdots w_m) \\ &= \sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \cdot w_1 \cdots w_m \\ &\quad + \sum_{j=1}^{m} (-1)^{|v_1|+\cdots+|v_n|+|w_1|+\cdots+|w_{j-1}|} v_1 \cdots v_n \cdot w_1 \cdots d(w_j) \cdots w_m \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1|+\cdots+|v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m) \\ &= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1 \cdots v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m)\,.\end{aligned}$$

This shows that $\mathrm{T}(V)$ is indeed a dg-algebra.

Let A be another dg-algebra and $f\colon V \to A$ a morphism of dg-vector spaces an let $F\colon \mathrm{T}(V) \to A$ be the unique extension of f to an algebra morphism, given by $F(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$. The algebra morphism F is a morphism of graded algebras because

$$\begin{aligned}|F(v_1 \cdots v_n)| &= |f(v_1) \cdots f(v_n)| \\ &= |f(v_1)| + \cdots + |f(v_n)| \\ &= |v_1| + \cdots + |v_n| \\ &= |v_1 \cdots v_n|\,.\end{aligned}$$

It is also a morphism of dg-vector spaces because

$$\begin{aligned}d(F(v_1 \cdots v_n)) &= d(f(v_1) \cdots f(v_n)) \\ &= \sum_{i=1}^{n} (-1)^{|f(v_1)|+\cdots+|f(v_{i-1})|} f(v_1) \cdots d(f(v_i)) \cdots f(v_n) \\ &= \sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} f(v_1) \cdots f(d(v_i)) \cdots f(v_n) \\ &= F\left(\sum_{i=1}^{n} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n\right) \\ &= F(d(v_1 \cdots v_n))\,.\end{aligned}$$

A.4. Proposition 2.5

(3) The quotient A/I is a dg-vector space and an algebra and the compatibility of these structures can be checked on representatives.

(4) The cycles $\mathrm{Z}(A)$ form a graded subspace with $1 \in \mathrm{Z}(A)$ and if $a, b \in \mathrm{Z}(A)$ are homogeneous then

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b) = 0$$

and hence $ab \in \mathrm{Z}(A)$. The boundaries $\mathrm{B}(A)$ form a graded subspace and if $a \in \mathrm{Z}(A)$ and $b \in \mathrm{B}(B)$ are homogeneous with $b = d(a')$ then

$$b \cdot a = d(a') \cdot a = d(a \cdot a') - (-1)^{|a|} a' \cdot d(a) = d(a \cdot a')$$

and hence $ba \in \mathrm{B}(A)$. Simlarly $ab \in \mathrm{B}(A)$.

A.5. Lemma 2.7

That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α: The ideal I is spanned by $a x_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(a x_\alpha b) = d(a) x_\alpha b + (-1)^{|a|} a d(x_\alpha) b + (-1)^{|a|+|x_\alpha|} a x_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$.

A.6. Example 2.9

(1) The ideal I is a dg-ideal as the generators $[v, w]$ are homogeneous and (by Example 6.5)

$$d([v, w]) = [d(v), w] + (-1)^{|v|} [v, d(w)] \in I\,.$$

(2) If S is any graded commutative dg-algebra and $f\colon V \to S$ a morphism of dg-vector spaces then f extends uniquely to a morphism of dg-algebras $F\colon \Lambda(V) \to S$:

$$\begin{tikzcd} \Lambda(V) \arrow[r, dashed, "F"] & S \\ V \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

(3) Let A and B be two dg-algebras. If C is any other dg-algebra and if $f\colon A \to C$ and $g\colon B \to C$ are two morphisms of dg-algebras whose images graded-commute, in the sense that

$$f(a) g(b) = (-1)^{|a||b|} g(b) f(a)$$

for all $a \in A$, $b \in B$, then the linear map

$$\varphi\colon A \otimes B \to C\,, \quad a \otimes b \mapsto f(a) g(b)$$

is again a morphism of dg-algebras. The inclusions $i\colon A \to A \otimes B$, $a \mapsto a \otimes 1$ and $j\colon B\colon B \to A \otimes B$, $b \mapsto 1 \otimes b$ are morphisms of dg-algebras. For every morphism of dg-algebras $\varphi\colon A \otimes B \to C$ the compositions $\varphi \circ i\colon A \to A \otimes B$ and $\varphi\colon j\colon B \to A \otimes B$ are again morphisms of dg-algebras. This gives a one-to-one correspondence

$$\left\{\begin{matrix}\text{morphisms of dg-algebras}\\ f\colon A \to C,\ g\colon B \to C\\ \text{whose images graded-commute}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{morphisms of dg-algebras}\\ \varphi\colon A \otimes B \to C\end{matrix}\right\},$$

$$\begin{aligned}(f,g) &\longmapsto (a \otimes b \mapsto f(a)g(b))\,,\\ (\varphi \circ i, \varphi \circ j) &\longleftarrow\!\shortmid\ \varphi\,.\end{aligned}$$

(4) It follows for any two dg-vector spaces V and W that

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

since we have for every dg-algebra A natural bijections

$$\begin{aligned}&\{\text{morphisms of dg-algebras } \Lambda(V \oplus W) \to A\}\\ \cong\ &\{\text{morphisms of dg-vector spaces } V \oplus W \to A\}\\ \cong\ &\{(f,g) \mid \text{morphisms of dg-vector spaces } f\colon V \to A,\ g\colon W \to A\}\\ \cong\ &\{(\varphi,\psi) \mid \text{morphisms of dg-algebras } \varphi\colon\ \Lambda(V) \to A,\ \psi\colon\ \Lambda(W) \to A\}\\ \cong\ &\{\text{morphisms of dg-algebras } \Lambda(V) \otimes \Lambda(W) \to A\}\,.\end{aligned}$$

More explicitely, the inclusions $V \to V \oplus W$ and $W \to V \oplus W$ induce morphisms of dg-algebras $\Lambda(V) \to \Lambda(V \oplus W)$ and $\Lambda(W) \to \Lambda(V \oplus W)$ that give an isomorphism of dg-algebras

$$\Lambda(V) \otimes \Lambda(W) \xrightarrow{\sim} \Lambda(V \oplus W)\,, \quad v_1 \cdots v_n \otimes w_1 \cdots w_m \mapsto v_1 \cdots v_n w_1 \cdots w_m\,.$$

(5) Let V be a graded vector space. If V is concentrated in even degrees then $\Lambda(V) = \mathrm{S}(V)$ and if V is concentrated in odd degrees then $\Lambda(V) = \bigwedge(V)$, with the grading of $\Lambda(V)$ and $\bigwedge(V)$ induced by the one of V. We have $V = V_{\text{even}} \oplus V_{\text{odd}}$ as graded vector spaces where $V_{\text{even}} = \bigoplus_n V_{2n}$ and $V_{\text{odd}} = \bigoplus_n V_{2n+1}$, and hence

$$\Lambda(V) = \Lambda(V_{\text{even}} \oplus V_{\text{odd}}) \cong \Lambda(V_{\text{even}}) \otimes \Lambda(V_{\text{odd}}) = \mathrm{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$$

The graded algebra $\mathrm{S}(V_{\text{even}})$ is concentrated in even degree and so it follows that in the tensor product $\mathrm{S}(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$ the simple tensors (strictly) commute, i.e. $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$. Hence

$$\Lambda(V) \cong \mathrm{S}(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}})$$

where $\otimes_k$ denotes the sign-less tensor product.

(6) Let $\mathrm{char}(k) \neq 2$ and let V be a dg-vector space with basis $(x_\alpha)_{\alpha\in A}$ consisting of homogeneous elements such that $(A, \leq)$ is linearly ordered. Then $\Lambda(V)$ admits as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t,\ n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.}^5$$

To see this we use the above decomposition

$$\Lambda(V) \cong \mathrm{S}(V_{\mathrm{even}}) \otimes_k \bigwedge(V_{\mathrm{odd}}) \tag{3}$$

as graded algebras: We may split up the given basis $(x_\alpha)_{\alpha\in A}$ of V into a basis $(x_\alpha)_{\alpha\in A'}$ of V_{even} and $(x_\alpha)_{\alpha\in A''}$ of V_{odd} (since all x_α are homogeneous). Then $\mathrm{S}(V_{\mathrm{even}})$ has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \qquad \text{where } r \geq 0,\ \alpha_1 < \cdots < \alpha_r \text{ and } n_i \geq 1\,,$$

and $\bigwedge(V_{\mathrm{odd}})$ has as a basis the ordered wedges

$$x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_s} \qquad \text{where } s \geq 0,\ \alpha_1 < \cdots < \alpha_s\,.$$

It follows that with (3) that $\Lambda(V)$ admits the basis

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \cdot x_{\beta_1} \cdots x_{\beta_s} \qquad \text{where} \begin{cases} r, s \geq 0,\ n_i \geq 1, \\ \alpha_1 < \cdots < \alpha_r, \\ \beta_1 < \cdots < \beta_s, \\ |x_{\alpha_i}| \text{ even, } |x_{\beta_j}| \text{ odd.} \end{cases}$$

We can now rearrange these basis vectors into the desired form becaus the factors $x_{\alpha_i}^{n_i}$ and x_{β_j} commute.

A.7. Remark 3.2

(1) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}\,.$$

(2) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.

(3) Every coalgebra C is a dg-coalgebra centered in degree 0, in particular $C = k$.

[5] The condition $n_i = 1$ for $|x_{\alpha_i}|$ odd commes from the equality $\alpha_i^2 = [\alpha_i, \alpha_i]/2$.

A.8. Example 3.3

We have seen in a previous talk that $(\mathrm{T}(C), \Delta, \varepsilon)$ is a coalgebra. We have for every $i = 0, \ldots, n$ that

$$\begin{aligned}|v_1 \cdots v_i \otimes v_{i+1} \cdots v_n| &= |v_1 \cdots v_i| + |v_{i+1} \cdots v_n| \\ &= |v_1| + \cdots + |v_i| + |v_{i+1}| + \cdots + |v_n| \\ &= |v_1| + \cdots + |v_n| \,,\end{aligned}$$

so we have a graded coalgebra. We also have

$$\begin{aligned}
&d(\Delta(v_1 \cdots v_n)) \\
&= \sum_{i=0}^{n} d(v_1 \cdots v_i \otimes v_{i+1} \cdots v_n) \\
&= \sum_{i=0}^{n} \big(d(v_1 \cdots v_i) \otimes v_{i+1} \cdots v_n + (-1)^{|v_1 \cdots v_i|} v_1 \cdots v_i \otimes d(v_{i+1} \cdots v_n) \big) \\
&= \sum_{i=0}^{n} \left(\sum_{j=1}^{i} (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\
&\qquad \left. + (-1)^{|v_1 \cdots v_i|} \sum_{j=i+1}^{n} (-1)^{|v_{i+1}| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\
&= \sum_{i=0}^{n} \left(\sum_{j=1}^{i} (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\
&\qquad \left. + \sum_{j=i+1}^{n} (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\
&= \Delta\left(\sum_{j=1}^{n} (-1)^{|v_1| + \cdots + |v_j|} v_1 \otimes \cdots \otimes d(v_j) \otimes \cdots \otimes v_n \right) \\
&= \Delta(d(v_1 \cdots v_n))
\end{aligned}$$

which shows that Δ is a morphism of dg-vector spaces.

A.9. Proposition 3.4

(3) The quotient C/I is a dg-vector space and a coalgebra, and the compatibility of these structures can be checked on representatives.

(4) If $c \in \mathrm{Z}(C)$ then

$$d(\Delta(c)) = \Delta(d(c)) = \Delta(0) = 0$$

because Δ is a morphism of dg-vector spaces, and hence

$$\Delta(c) \in \mathrm{Z}(C \otimes C) = \mathrm{Z}(C) \otimes \mathrm{Z}(C) \,.$$

This shows that $\mathrm{Z}(C)$ is a subcoalgebra of C. It is also a graded subspace of C and hence a graded subcoalgebra.

For $b \in \mathrm{B}(C)$ with $b = d(c)$ we have

$$\begin{aligned}\Delta(b) = \Delta(d(c)) = d(\Delta(c)) &= d\Bigg(\sum_{(c)} c_{(1)} \otimes c_{(2)}\Bigg) \\ &= \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|c_{(1)} \otimes d(c_{(2)})} \in \mathrm{B}(C) \otimes C + C \otimes \mathrm{B}(C)\,.\end{aligned}$$

We also have

$$\varepsilon(b) = \varepsilon(d(c)) = d(\varepsilon(c)) = 0\,.$$

This shows that $\mathrm{B}(C)$ is a coideal in C. It follows from the upcoming lemma that B is also a coideal in $\mathrm{Z}(C)$. Then $\mathrm{B}(C)$ is a graded coideal in $\mathrm{Z}(C)$ because $\mathrm{B}(C)$ is a graded subspace of $\mathrm{Z}(C)$.

Lemma A.1. Let C be a coalgebra and let B be a subcoalgebra of C. If I is a coideal in C with $I \subseteq C$ then I is also a coideal in B.

Proof. It follows from the inclusions $I \subseteq B \subseteq C$ that

$$(C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B\,.$$

Hence

$$\Delta(I) = \Delta(I) \cap \Delta(B) \subseteq (C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B\,.$$

Also $\varepsilon_B(I) = \varepsilon_C(I) = 0$. □

A.10. Proposition 4.5

(1) It follows from Proposition 2.5 and Proposition 3.4 that B/I is a dg-algebra and dg-coalgebra. The compatibility can be checked on representatives.

(2) It follows from Proposition 2.5 and Proposition 3.4 that $\mathrm{H}(\mathcal{B})$ is again a dg-algebra and dg-coalgebra, and the compatibility of these structures can be checked on representatives.

A.11. Lemma 5.1

We have $1_{\mathrm{Hom}_k(C,A)} = u \circ \epsilon \in \mathrm{Hom}(C,A)_0$ because both u_A and ϵ_C are morphisms of dg-vector spaces and thus of degree 0. If $f, g \in \mathrm{Hom}(C,A)$ are graded maps then $f \otimes g$ is again a graded map and thus

$$f * g = m \circ (f \otimes g) \circ \Delta$$

is a graded map as a composition of graded maps. This shows that $\mathrm{Hom}(C,A)$ is a subalgebra of $\mathrm{Hom}_k(C,A)$.

We have

$$|f * g| = |m \circ (f \otimes g) \circ \Delta| = |m| + (|f| + |g|) + |\Delta| = |f| + |g|$$

so $\mathrm{Hom}(C, A)$ is a graded algebra with respect to the convolution product.

Furthermore

$$\begin{aligned}
&d(f * g) \\
&= d \circ (f * g) - (-1)^{|f*g|}(f * g) \circ d \\
&= d \circ m \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ \Delta \circ d \\
&= m \circ d_{A \otimes A} \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ d_{C \otimes C} \circ \Delta \\
&= m \circ (d \otimes 1 + 1 \otimes d) \circ (f \otimes g) \otimes \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes 1 + 1 \otimes d) \circ \Delta \\
&= m \circ (d \otimes \mathrm{id}) \circ (f \otimes g) \otimes \Delta + m \circ (\mathrm{id} \otimes d) \circ (f \otimes g) \otimes \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes \mathrm{id}) \circ \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (\mathrm{id} \otimes d) \circ \Delta \\
&= m \circ ((d \circ f) \otimes g) \otimes \Delta + (-1)^{|f|} m \circ (f \otimes (d \circ g)) \otimes \Delta \\
&\quad - (-1)^{|f|} m \circ ((f \circ d) \otimes g) \circ \Delta - (-1)^{|f|+|g|} m \circ (f \otimes (g \circ d)) \circ \Delta \\
&= m \circ ((d \circ f - (-1)^{|f|} f \circ d) \otimes g) \otimes \Delta \\
&\quad + (-1)^{|f|} m \circ (f \otimes (d \circ g - (-1)^{|g|} g \circ d)) \otimes \Delta \\
&= m \circ (d(f) \otimes g) \circ \Delta + (-1)^{|f|} m \circ (f \otimes d(g)) \otimes \Delta \\
&= d(f) * g + (-1)^{|f|} f * d(g)
\end{aligned}$$

because m and Δ are commute with the differentials. Hence $\mathrm{Hom}(C, A)$ is a dg-algebra with respect to the convolution product.

A.12. Remark 5.4

We need to explain why an inverse to id_H in $\mathrm{Hom}(H, H)$ with respect to the convolution product $*$ is again a morphism of dg-vector spaces. For this we use the following result:

Lemma A.2. Let A be a dg-algebra and let $a \in A$ be a homogeneous unit.

(1) The inverse a^{-1} is homogeneous of degree $|a^{-1}| = -|a|$.

(2) If a is a cycle then so is a^{-1}.

Proof.

(1) Let $d = |a|$ and let $a^{-1} = \sum_n a'_n$ be the homogeneous decomposition of a^{-1}. It follows from $1 = ab = \sum_n aa'_n$ that in degree zero, $1 = aa'_{-d}$. Thus a'_{-d} is the inverse of a, i.e. $a^{-1} = a'_{-d} \in A_{-d}$.

(2) It follows from
$$0 = d(1) = d(aa^{-1}) = d(a)a^{-1} + (-1)^{|a|} a d(a^{-1})$$
that $(-1)^{|a|} a d(a^{-1}) = 0$ because $d(a) = 0$. Hence $d(a^{-1}) = 0$ as a is a unit. □

The space $\mathrm{Z}_0(\mathrm{Hom}(V, W))$ consists of the morphism of dg-vector spaces $V \to W$. It hence follows from Lemma A.2 that if $f \in \mathrm{Z}_0(\mathrm{Hom}(V, W))$ admits an inverse g with respect to the convolution product that again $g \in \mathrm{Z}_0(\mathrm{Hom}(V, W))$.

A.13. Proposition 5.5

(1) It follows from Proposition 4.5 that H is a dg-bialgebra and the condition $S(I) \subseteq I$ ensures that S induces a morphism of dg-vector spaces $\overline{S} \colon H/I \to H/I$. The antipode condition for $\overline{S}$ can now be checked on representatives.

(2) The homology $\mathrm{H}(\mathcal{H})$ is a dg-bialgebra by Proposition 4.5 and that $\mathrm{H}(S_{\mathcal{H}})$ is an antipode can be checked on representatives.

A.14. Example 5.6

The dg-coalgebra diagrams for $(\mathrm{T}(V), \Delta, \varepsilon)$ can be checked on algebra generators of $\mathrm{T}(V)$ because all arrows in these diagrams are morphisms of dg-algebras. It hence sufficies to check these diagrams for elements of V, where this is straightforward.

It remains to check the equalities
$$\sum_{(h)} S(h_{(1)}) h_{(2)} = \varepsilon(h) 1_H \quad \text{and} \quad \sum_{(h)} h_{(1)} S(h_{(2)}) = \varepsilon(h) 1_H$$
for the monomials $h = v_1 \cdots v_n$. If $n = 0$ then $h = 1$ and both equalities hold, so we consider in the following the case $n \geq 1$. Then $\varepsilon(v_1 \cdots v_n) = 0$ so we have to show that in the sums $\sum_{(h)} S(h_{(1)}) h_{(2)}$ and $\sum_{(h)} h_{(1)} S(h_{(2)})$ all terms cancel out. We consider for simplicity only the sum $\sum_{(h)} S(h_{(1)}) h_{(2)}$.[6] We have
$$\Delta(v_1 \cdots v_n) = \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)} \,. \tag{4}$$
Here
$$S(v_{\sigma(1)} \cdots v_{\sigma(p)}) = (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} v_{\sigma(p)} \cdots v_{\sigma(1)}$$

[6] The author hasn't actually checked the other sum.

and thus

$$
\begin{aligned}
&(m \circ (S \otimes \mathrm{id}) \circ \Delta)(v_1 \cdots v_n) \\
&= \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} \varepsilon_{v_1,\dots,v_n}(\sigma^{-1})(-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|} \\
&\qquad\qquad \cdot v_{\sigma(p)} \cdots v_{\sigma(1)} v_{\sigma(p+1)} \cdots v_{\sigma(n)} \,.
\end{aligned} \tag{5}
$$

We see that in (4) any two terms of the form

$$
w_1 w_2 \cdots w_i \otimes w_{i+1} \cdots w_n \quad \text{and} \quad w_2 \cdots w_i \otimes w_1 w_{i+1} \cdots w_n
$$

give in (5) the up to sign same term $w_i \cdots w_2 w_1 w_{i+1} \cdots w_n$. We now check that the signs differ, so that in (5) both terms cancel out. This then shows that in $(m\circ(S\otimes\mathrm{id})\circ\Delta)(v_1 \cdots v_n)$ the two terms cancel out, so that the overall sum becomes zero.

For $1 \le p \le n$ and $\sigma \in \mathrm{Sh}(p, n-p)$ with $\sigma(p) < \sigma(1)$ the term associated to $v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{(p+1)} \cdots v_{\sigma(n)}$ is given by

$$
v_{\sigma(2)} \cdots v_{\sigma(p)} \otimes v_{\sigma(1)} v_{\sigma(p+1)} \cdots v_{\sigma(n)} = v_{\tau(1)} \cdots v_{\tau(p-1)} \otimes v_{\tau(p)} \cdots v_{\tau(n)}
$$

for the permuation $\omega \in \mathrm{Sh}(p-1, n-p+1)$ given by

$$
\omega = \sigma \circ (1\,2\,\cdots\,p)\,,
$$

i.e.

$$
\omega(i) = \begin{cases} \sigma(i+1) & \text{if } 1 \le i \le p-1\,, \\ \sigma(1) & \text{if } i = p\,, \\ \sigma(i) & \text{if } p+1 \le i \le n\,. \end{cases}
$$

We see from the Koszul sign rule that the signs $\varepsilon_{v_1,\dots,v_n}(\sigma^{-1})$ and $\varepsilon_{v_1,\dots,v_n}(\omega^{-1})$ differ by the factor $(-1)^{|v_{\sigma(1)}||v_{\sigma(2)}|+\cdots+|v_{\sigma(1)}||v_{\sigma(p)}|}$. Therefore

$$
\begin{aligned}
&\varepsilon_{v_1,\dots,v_n}(\sigma^{-1})(-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|} \\
&= \varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^{|v_{\sigma(1)}||v_{\sigma(2)}|+\cdots+|v_{\sigma(1)}||v_{\sigma(p)}|}(-1)^p(-1)^{\sum_{1\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|} \\
&= \varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^p(-1)^{\sum_{2\le i<j\le p}|v_{\sigma(i)}||v_{\sigma(j)}|} \\
&= \varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^p(-1)^{\sum_{1\le i<j\le p-1}|v_{\omega(i)}||v_{\omega(j)}|} \\
&= -\,\varepsilon_{v_1,\dots,v_n}(\omega^{-1})(-1)^{p-1}(-1)^{\sum_{1\le i<j\le p-1}|v_{\omega(i)}||v_{\omega(j)}|}\,.
\end{aligned}
$$

Thus the signs differ as claimed.

A.15. Example 5.7

We have

$$\begin{aligned}\varepsilon([v,w]) &= \varepsilon\big(vw-(-1)^{|v||w|}wv\big)\\ &= \varepsilon(vw)-(-1)^{|v||w|}wv\\ &= \varepsilon(v)\varepsilon(w)-(-1)^{|v||w|}\varepsilon(w)\varepsilon(v)\\ &= 0\end{aligned}$$

as $\varepsilon(v)=\varepsilon(w)=0$. The elements v and w are primitive whence $[v,w]$ is primitive. Therefore

$$\Delta([v,w]) = [v,w]\otimes 1 + 1\otimes [v,w] \in I\otimes \mathrm{T}(V) + \mathrm{T}(V)\otimes I\,.$$

Also

$$\begin{aligned}S([v,w]) &= S\big(vw-(-1)^{|v||w|}wv\big)\\ &= S(vw)-(-1)^{|v||w|}S(wv)\\ &= (-1)^{|v||w|}wv - vw\\ &= -\big(vw-(-1)^{|v||w|}wv\big)\\ &= -[v,w]\\ &\in I\,.\end{aligned}$$

A.16. Example 5.8

Suppose that there exists a bialgebra structure on $\bigwedge(V)$. Then $\varepsilon(v)^2=\varepsilon(v^2)=0$ and thus $\varepsilon(v)=0$ for all $v\in V$, so $\ker\varepsilon = \bigoplus_{d\geq 1}\bigwedge^n(V) =: I$. Let $v\in V$. Then by the counital axiom,

$$\Delta(v)\equiv v\otimes 1 \pmod{\Lambda\otimes I} \qquad \text{and} \qquad \Delta(v)\equiv 1\otimes v \pmod{I\otimes\Lambda}$$

and thus

$$\Delta(v)\equiv v\otimes 1 + 1\otimes v \pmod{I\otimes I}\,.$$

It follows that

$$\Delta(v^2)\equiv (v\otimes 1+1\otimes v)^2 \pmod{(v\otimes 1)(I\otimes I)+(1\otimes v)(I\otimes I)+(I\otimes I)^2}\,,$$

and therefore

$$\Delta(v^2)\equiv v^2\otimes 1 + 2v\otimes v + 1\otimes v^2 \pmod{I\otimes I^2 + I^2\otimes I}\,.$$

Now $v^2=0$, hence

$$2v\otimes v\equiv 0 \pmod{I\otimes I^2+I^2\otimes I}\,.$$

But $2\neq 0$ and $v\neq 0$ hence $2v\otimes v\neq 0$ while $v\otimes v\notin I\otimes I^2+I^2\otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

A.17. Example 5.9

(1) The action of S_n on $V^{\otimes n}$ is by morphism of dg-vector spaces as mentioned in Section 1 and shown in Appendix A.1. The symmetrization map

$$\tilde{s}\colon \mathrm{T}(V) \to \mathrm{T}(V)\,, \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n)$$

therefore results in a morphism of dg-vector spaces $\tilde{s}\colon \mathrm{T}(V) \to \mathrm{T}(V)$.[7] It follows that the factored map $s\colon \Lambda(V) \to \mathrm{T}(V)$ is again a morphism of dg-vector spaces.

(2) We observe that the diagrams

$$\begin{tikzcd} \mathrm{T}(\mathrm{H}(V)) \arrow[r, "\alpha"] \arrow[d, "\tilde{p}"'] & \mathrm{H}(\mathrm{T}(V)) \arrow[d, "\mathrm{H}(p)"] \\ \Lambda(\mathrm{H}(V)) \arrow[r, "\beta"] & \mathrm{H}(\Lambda(V)) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{T}(\mathrm{H}(V)) \arrow[r, "\alpha"] & \mathrm{H}(\mathrm{T}(V)) \\ \Lambda(\mathrm{H}(V)) \arrow[r, "\beta"] \arrow[u, "\tilde{s}"] & \mathrm{H}(\Lambda(V)) \arrow[u, "\mathrm{H}(s)"'] \end{tikzcd}$$

commute. Indeed, for representatives $v_1, \ldots, v_n \in \mathrm{Z}(V)$ the first diagram gives

$$\begin{tikzcd} {[v_1] \otimes \cdots \otimes [v_n]} \arrow[r, mapsto] \arrow[d, mapsto] & {[v_1 \otimes \cdots \otimes v_n]} \arrow[d, mapsto] \\ {[v_1] \cdots [v_n]} \arrow[r, mapsto] & {[v_1 \cdots v_n]} \end{tikzcd}$$

and the second diagram is given as follows:

$$\begin{tikzcd} \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1})[v_{\sigma(1)}] \otimes \cdots \otimes [v_{\sigma(n)}] \arrow[r, mapsto] & \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1})[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}] \\ {[v_1] \cdots [v_n]} \arrow[r, mapsto] \arrow[u, mapsto] & {[v_1 \cdots v_n]} \arrow[u, mapsto] \end{tikzcd}$$

It follows that

$$\beta\beta' = \beta\tilde{p}\alpha^{-1}\,\mathrm{H}(s) = \mathrm{H}(p)\alpha\alpha^{-1}\,\mathrm{H}(s) = \mathrm{H}(p)\,\mathrm{H}(s) = \mathrm{id}_{\mathrm{H}(\Lambda(V))}$$

and similarly

$$\beta'\beta = \tilde{p}\alpha^{-1}\,\mathrm{H}(s)\beta = \tilde{p}\alpha^{-1}\alpha\tilde{s} = \tilde{p}\tilde{s} = \mathrm{id}_{\Lambda(\mathrm{H}(V))}$$

A.18. Example 6.5

(1) If $a, b \in A$ are homogeneous then $[a, b] = ab - (-1)^{|a||b|}ba$ is homogeneous of degree $|a| + |b|$, so $[A_i, A_j] \subseteq A_{i+j}$ for all i, j. Also

$$[a, b] = ab - (-1)^{|a||b|}ba = -(-1)^{|a||b|}\big(ba - (-1)^{|a||b|}ab\big) = -(-1)^{|a||b|}[b, a]$$

[7] This map is a projection of $\mathrm{T}(V)$ on its dg-subspace of graded symmetric tensors.

and

$$
\begin{aligned}
d([a,b]) &= d\big(ab - (-1)^{|a||b|}ba\big) \\
&= d(ab) - (-1)^{|a||b|}d(ba) \\
&= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||b|}\big(d(b)a + (-1)^{|b|}bd(a)\big) \\
&= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||b|}d(b)a - (-1)^{|a||b|+|b|}bd(a) \\
&= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||d(b)|+|a|}d(b)a - (-1)^{|d(a)||b|}bd(a) \\
&= d(a)b - (-1)^{|d(a)||b|}bd(a) + (-1)^{|a|}\big(ad(b) - (-1)^{|a||d(b)|}d(b)a\big) \\
&= [d(a),b] + (-1)^{|a|}[a,d(b)]\,.
\end{aligned}
$$

We check the graded Jacobi identity for homogeneous $a, b, c \in A$. We have

$$
\begin{aligned}
[a,[b,c]] &= \big[a, bc - (-1)^{|b||c|}cb\big] \\
&= [a,bc] - (-1)^{|b||c|}[a,cb] \\
&= abc - (-1)^{|a||bc|}bca - (-1)^{|b||c|}\big(acb - (-1)^{|a||cb|}cba\big) \\
&= abc - (-1)^{|a||bc|}bca - (-1)^{|b||c|}acb + (-1)^{|a||cb|+|b||c|}cba \\
&= abc - (-1)^{|a|(|b|+|c|)}bca - (-1)^{|b||c|}acb + (-1)^{|a|(|b|+|c|)+|b||c|}cba \\
&= abc - (-1)^{|a||b|+|a||c|}bca - (-1)^{|b||c|}acb + (-1)^{|a||b|+|a||c|+|b||c|}cba
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(-1)^{|a||c|}[a,[b,c]] = (-1)^{|a||c|}abc &- (-1)^{|a||b|}bca \\
&- (-1)^{|a||c|+|b||c|}acb + (-1)^{|a||b|+|b||c|}cba\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{\text{cyclic}} (-1)^{|a||c|}[a,[b,c]] &= \sum_{\text{cyclic}} (-1)^{|a||c|}abc - \sum_{\text{cyclic}} (-1)^{|a||b|}bca \\
&\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|}acb + \sum_{\text{cyclic}} (-1)^{|a||b|+|b||c|}cba \\
&= \sum_{\text{cyclic}} (-1)^{|b||a|}bca - \sum_{\text{cyclic}} (-1)^{|a||b|}bca \\
&\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|}acb + \sum_{\text{cyclic}} (-1)^{|b||c|+|c||a|}acb \\
&= 0\,.
\end{aligned}
$$

(2) The subspace $\mathrm{Der}(A)$ is by construction a graded subspace of $\mathrm{End}(A)$. Let δ, ε be graded derivations. Then for all homogeneous $a, b \in A$,

$$
(\delta\varepsilon)(ab) = \delta(\varepsilon(ab))
$$

$$
\begin{aligned}
&= \delta(\varepsilon(a)b + (-1)^{|\varepsilon||a|}a\varepsilon(b)) \\
&= \delta(\varepsilon(a)b) + (-1)^{|\varepsilon||a|}\delta(a\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\big(\delta(a)\varepsilon(b) + (-1)^{|\delta||a|}a\delta(\varepsilon(b))\big) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{(|\varepsilon|+|a|)|\delta|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b))
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) &= (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) + (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b))
\end{aligned}
$$

and therefore

$$
\begin{aligned}
[\delta, \varepsilon](ab) &= (\delta\varepsilon - (-1)^{|\delta||\varepsilon|}\varepsilon\delta)(ab) \\
&= (\delta\varepsilon)(ab) - (-1)^{|\delta||\varepsilon|}(\varepsilon\delta)(ab) \\
&= \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) \\
&\quad + (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) \\
&\quad - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b - (-1)^{|\varepsilon||a|}\delta(a)\varepsilon(b) \\
&\quad - (-1)^{|\delta||\varepsilon|+|\delta||a|}\varepsilon(a)\delta(b) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|}a\varepsilon(\delta(b)) \\
&= \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|}\varepsilon(\delta(a))b \\
&\quad + (-1)^{|\delta||a|+|\varepsilon||a|}\big(a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|}a\varepsilon(\delta(b))\big) \\
&= [\delta, \varepsilon](a)b + (-1)^{|[\delta,\varepsilon]||a|}a[\delta, \varepsilon](b)\,.
\end{aligned}
$$

This shows that $[\delta, \varepsilon] \in \mathrm{Der}(A)$, so that $\mathrm{Der}(A)$ is a graded Lie subalgebra of $\mathrm{End}(A)$. If $\delta \in \mathrm{Der}(A)$ is homogeneous then

$$
d(\delta) = d \circ \delta - (-1)^{|\delta|}\delta \circ d = [d, \delta]
$$

is again a graded derivation, and hence $\mathrm{Der}(A)$ is a dg-subspace of $\mathrm{End}(A)$.

(3) If $a \in \mathbb{P}(B)$ with homogeneous decomposition $a = \sum_n a_n$ then

$$
\Delta(a) = \Delta\left(\sum_n a_n\right) = \sum_n \Delta(a_n)
$$

but also

$$\Delta(a) = a \otimes 1 + 1 \otimes a = \sum_n (a_n \otimes 1 + 1 \otimes a_n)\,.$$

By comparing homogeneous components we see that $\Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n$ for all n. This means that all homogeneous components a_n are again primitive, which shows that $\mathbb{P}(B)$ is a graded subspace of B. If $a \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(d(a)) &= d(\Delta(a)) \\
&= d(a \otimes 1 + 1 \otimes a) \\
&= d(a \otimes 1) + d(1 \otimes a) \\
&= d(a) \otimes 1 + (-1)^{|a|} a \otimes d(1) + d(1) \otimes a + (-1)^{|1|} 1 \otimes d(a) \\
&= d(a) \otimes 1 + 1 \otimes d(a)
\end{aligned}$$

because $|1| = 0$ and $d(1) = 0$. Therefore $\mathbb{P}(B)$ is a dg-subspace of B.

If $a, b \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b) \\
&= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) \\
&= (a \otimes 1)(b \otimes 1) + (a \otimes 1)(1 \otimes b) + (1 \otimes a)(b \otimes 1) + (1 \otimes a)(1 \otimes b) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab\,.
\end{aligned}$$

If a, b are homogeneous then it follows that

$$\begin{aligned}
\Delta([a,b]) &= \Delta\bigl(ab - (-1)^{|a||b|} ba\bigr) \\
&= \Delta(ab) - (-1)^{|a||b|} \Delta(ba) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab \\
&\quad - (-1)^{|a||b|} \bigl(ba \otimes 1 + b \otimes a + (-1)^{|a||b|} a \otimes b + 1 \otimes ba\bigr) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab \\
&\quad - (-1)^{|a||b|} ba \otimes 1 - (-1)^{|a||b|} b \otimes a - a \otimes b - (-1)^{|a||b|} 1 \otimes ba \\
&= \bigl(ab - (-1)^{|a||b|} ba\bigr) \otimes 1 + 1 \otimes \bigl(ab - (-1)^{|a||b|} ba\bigr) \\
&= [a,b] \otimes 1 + 1 \otimes [a,b]
\end{aligned}$$

which shows that $[a,b] \in \mathbb{P}(B)$. Thus $\mathbb{P}(B)$ is a dg-Lie subalgebra of B.

A.19. Lemma 6.6

(1) The quotient $\mathfrak{g}/I$ is again a dg-vector spaces and a Lie algebra. The compatibility of these structures can be checked on generators.

(2) The cycles $\mathrm{Z}(\mathfrak{g})$ form a graded subspace of $\mathfrak{g}$. For homogeneous $x, y \in \mathrm{Z}(\mathfrak{g})$,

$$d([x,y]) = [d(x), y] + (-1)^{|x|}[x, d(y)] = [0, y] + (-1)^{|x|}[x, 0] = 0\,,$$

so $\mathrm{Z}(\mathfrak{g})$ is indeed a graded Lie subalgebra of $\mathfrak{g}$. The boundaries $\mathrm{B}(\mathfrak{g})$ form a graded subspace of $\mathrm{Z}(\mathfrak{g})$. If $x \in \mathrm{B}(\mathfrak{g})$ with $x = d(x')$, where $x' \in \mathfrak{g}$ is homogeneous, then for every $y \in \mathrm{Z}(\mathfrak{g})$,

$$[x, y] = [d(x'), y] = d([x', y]) - (-1)^{|x'|}[x', \underbrace{d(y)}_{=0}] = d([x', y]) \in \mathrm{B}(\mathfrak{g})\,.$$

Thus $\mathrm{B}(\mathfrak{g})$ is a graded Lie ideal in $\mathrm{Z}(\mathfrak{g})$.

A.20. Proposition 6.8

We check that the given ideal I is a dg-Hopf ideal. It is generated by homogenous elements which satisfy

$$\begin{aligned}
&d([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= d([x,y]_{\mathrm{T}(\mathfrak{g})}) - d([x,y]_{\mathfrak{g}}) \\
&= [d(x),y]_{\mathrm{T}(\mathfrak{g})} + (-1)^{|x|}[x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}} - (-1)^{|x|}[x,d(y)]_{\mathfrak{g}} \\
&= \Big([d(x),y]_{\mathrm{T}(\mathfrak{g})} - [d(x),y]_{\mathfrak{g}}\Big) + (-1)^{|x|}\Big([x,d(y)]_{\mathrm{T}(\mathfrak{g})} - [x,d(y)]_{\mathfrak{g}}\Big) \in I
\end{aligned}$$

so it is a dg-ideal. Also

$$\varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = \varepsilon([x,y]_{\mathrm{T}(\mathfrak{g})}) - \varepsilon([x,y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are homogoneous of degree ≥ 1,

$$\begin{aligned}
&\Delta([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&= \Delta([x,y]_{\mathrm{T}(\mathfrak{g})}) - \Delta([x,y]_{\mathfrak{g}})) \\
&= [x,y]_{\mathrm{T}(\mathfrak{g})} \otimes 1 + 1 \otimes [x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x,y]_{\mathfrak{g}} \\
&= ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) \\
&\in I \otimes \mathrm{T}(\mathfrak{g}) + \mathrm{T}(\mathfrak{g}) \otimes I
\end{aligned}$$

since both $[x,y]_{\mathrm{T}(\mathfrak{g})}$ and $[x,y]_{\mathfrak{g}}$ are primitive, and finally

$$S([x,y]_{\mathrm{T}(\mathfrak{g})} - [x,y]_{\mathfrak{g}}) = S([x,y]_{\mathrm{T}(\mathfrak{g})}) - S([x,y]_{\mathfrak{g}}) = -[x,y]_{\mathrm{T}(\mathfrak{g})} + [x,y]_{\mathfrak{g}} \in I\,.$$

Thus the dg-ideal I is already a dg-Hopf ideal.

A.21. The Poincaré–Birkhoff–Witt theorem

Recall A.3. If $\mathfrak{g}$ is a Lie algebra with basis $(x_\alpha)_{\alpha \in A}$ where $(A, \leq)$ is linearly ordered then the PBW theorem asserts that $\mathrm{U}(\mathfrak{g})$ has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t \text{ and } n_i \geq 1\,.$$

This shows in particular that the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is injective, and it also follows that $\mathbb{P}(\mathrm{U}(\mathfrak{g})) = \mathfrak{g}$. Moreover, $\operatorname{gr} \mathrm{U}(\mathfrak{g}) \cong \mathrm{S}(\mathfrak{g})$ where $\operatorname{gr} \mathrm{U}(\mathfrak{g})$ denotes the associated graded for the standard filtration of $\mathrm{U}(\mathfrak{g})$.

Theorem A.4 (dg-PBW theorem)**.** Let $\mathfrak{g}$ be a dg-Lie algebra with basis $(x_\alpha)_{\alpha\in A}$ consisting of homogeneous elements such that $(A, \leq)$ is linearly ordered. Then $\mathrm{U}(\mathfrak{g})$ has as a basis all ordered monomials

$$x_{\alpha_1} \cdots x_{\alpha_n} \qquad \text{where } t \geq 0,\ \alpha_1 < \cdots < \alpha_t,\ n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.} \qquad \square$$

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