

Differential Graded Hopf Algebras I

In the following k denotes an arbitrary field. All vector spaces, algebras, tensor products, etc. are over k , unless otherwise stated. All occurring maps are linear unless otherwise stated. We abbreviate “differential graded” by “dg”.

1. Preliminary Notions and Notations

A **dg-vector space** V is the same as a chain complex, a **dg-subspace** the same as a chain subcomplex. We write $|v|$ for the degree of v and assume v to be homogeneous. We always regard graded objects as dg-objects with zero differential. We regard k as a dg-vector space concentrated in degree 0.

If V, W are dg-vector spaces then $V \otimes W$ is a dg-vector space with

$$|v \otimes w| = |v| + |w|, \quad d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w).$$

The **twist map** $\tau: V \otimes W \rightarrow W \otimes V$ given by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

is an isomorphism of dg-vector spaces.¹ We use the Koszul-Quillen **sign convention**: Whenever x, y are swapped the sign $(-1)^{|x||y|}$ is introduced. This results in a well-defined S_n -action on $V^{\otimes n}$ via homomorphisms of dg-vector spaces, given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1, \dots, v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

for homogeneous v_i , where $\varepsilon_{v_1, \dots, v_n}(\sigma)$ is the **Koszul sign**. (See Appendix A.1.)

A map $f: V \rightarrow W$ is **graded** of **degree** $d = |f|$ if $f(V_n) \subseteq W_{n+d}$ for all n , and $\text{Hom}(V, W)$ is the dg-vector space with

$$\begin{aligned} \text{Hom}(V, W)_n &= \{\text{graded maps } V \rightarrow W \text{ of degree } n\}, \\ d(f) &= d \circ f - (-1)^{|f|} f \circ d. \end{aligned}$$

The differential d is a graded map of degree -1 . If $f: V \rightarrow V', g: W \rightarrow W'$ are graded maps then $f \otimes g: V \otimes V' \rightarrow W \otimes W'$ is the graded map of degree $|f \otimes g| = |f| + |g|$ given by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

¹The naive twist map $v \otimes w \mapsto w \otimes v$ is not a homomorphism of dg-vector spaces.

2. Differential Graded Algebras

Definition 2.1.

- (1) A **dg-algebra** is a dg-vector space A together with homomorphisms of dg-vector spaces $m: A \otimes A \rightarrow A$ and $u: k \rightarrow A$ that make the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\text{id} \otimes m} & A \otimes A \\ m \otimes \text{id} \downarrow & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array} \quad \begin{array}{ccccc} k \otimes A & \xleftarrow{\sim} & A & \xrightarrow{\sim} & A \otimes k \\ u \otimes \text{id} \downarrow & & \parallel & & \downarrow \text{id} \otimes u \\ A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A \end{array}$$

- (2) The dg-algebra A is **graded commutative** if the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow m & \swarrow m \\ & A & \end{array}$$

- (3) A **homomorphism** of dg-algebras $f: A \rightarrow B$ is a homomorphism of dg-vector spaces such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ m \downarrow & & \downarrow m \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} & k & \\ u \swarrow & & \searrow u \\ A & \xrightarrow{f} & B \end{array}$$

- (4) A **dg-ideal** in a dg-algebra A is a dg-subspace that is also an ideal.²

Remark 2.2. A dg-algebra is the same as a graded algebra A (in particular $|1| = 0$) together with a differential d such that $d(1) = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b). \quad (1)$$

(See Appendix A.2 for further remarks.)

Examples 2.3. (See Appendix A.3 for the explicit calculations)

- (1) Every algebra A is a dg-algebra concentrated in degree 0, in particular $A = k$.
- (2) For any dg-vector space V the algebra structure of $\text{End}_k(V)$ restricts to a dg-algebra structure on $\text{End}(V) = \text{Hom}(V, V)$.
- (3) If V is a dg-vector space then $T(V) = \bigoplus_{d \geq 0} V^{\otimes d}$ is again a dg-vector space with

$$\begin{aligned} |v_1 \cdots v_n| &= |v_1| + \cdots + |v_n|, \\ d(v_1 \cdots v_n) &= \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n. \end{aligned}$$

²By an “ideal” we always mean a two-sided ideal.

This makes $T(V)$ into a dg-algebra, with multiplication given by concatenation

$$(v_1 \cdots v_i) \cdot (v_{i+1} \cdots v_n) = v_1 \cdots v_n.$$

The inclusion $V \rightarrow T(V)$ is a homomorphism of dg-vector spaces and if $f: V \rightarrow A$ is any homomorphism of dg-vector spaces into a dg-algebra A then f extends uniquely to a homomorphism of dg-algebras $F: T(V) \rightarrow A$:

$$\begin{array}{ccc} T(V) & \xrightarrow{F} & A \\ \uparrow & \nearrow f & \\ V & & \end{array}$$

The dg-algebra $T(V)$ is the **dg-tensor algebra** on V .

Proposition 2.4 (Constructions with dg-algebras). Let A, B be a dg-algebras.

- (1) The tensor product $A \otimes B$ becomes a dg-algebra with

$$1_{A \otimes B} = 1_A \otimes 1_B \quad \text{and} \quad m_{A \otimes B} = (m_A \otimes m_B) \circ (\text{id} \otimes \tau \otimes \text{id}),$$

$$\text{i.e. } (a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2.$$

- (2) The dg-algebra A^{op} is given by $u_{A^{\text{op}}} = u_A$ and $m_{A^{\text{op}}} = m_A \circ \tau$, i.e.

$$1_A = 1_{A^{\text{op}}} \quad \text{and} \quad a \cdot_{\text{op}} b = (-1)^{|a||b|} b \cdot a.$$

- (3) If I is a dg-ideal in A then A/I inherits the structure of a dg-algebra
(4) If A is a dg-algebra then $Z(A)$ is a graded subalgebra of A , $B(A)$ is a graded ideal in $Z(A)$ and $H(A)$ is hence a graded algebra.

Proof. See Appendix A.4. □

Lemma 2.5. An ideal I in a dg-algebra A is a dg-ideal if and only if I is generated by homogeneous elements x_α with $d(x_\alpha) \in I$ for every α .

Proof. See Appendix A.5. □

Definition 2.6. The **graded commutator** in a dg-algebra A is the unique bilinear extension of

$$[a, b] := ab - (-1)^{|a||b|} ba.$$

(See Appendix A.6 for a remark.)

Example 2.7. Let V be a dg-vector space. The ideal

$$I := ([v, w] \mid v, w \in V \text{ are homogeneous})$$

is a dg-ideal in $T(V)$, and the quotient $\Lambda(V) := T(V)/I$ is the **dg-tensor algebra** on V . (See Appendix A.7 for the explicit calculations and further remarks about $\Lambda(V)$.)

3. Differential Graded Coalgebras

Definition 3.1.

- (1) A **dg-coalgebra** is a dg-vector space C together with homomorphisms of dg-vector spaces $\Delta: C \rightarrow C \otimes C$ and $\varepsilon: C \rightarrow k$ that make the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \text{id} \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes C \otimes C \end{array} \quad \begin{array}{ccccc} C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C \\ \varepsilon \otimes \text{id} \downarrow & & \parallel & & \downarrow \text{id} \otimes \varepsilon \\ k \otimes C & \xrightarrow{\sim} & C & \xleftarrow{\sim} & C \otimes k \end{array}$$

- (2) The dg-coalgebra C is **graded cocommutative** if the following diagram commutes:

$$\begin{array}{ccc} & C & \\ \Delta \swarrow & & \searrow \Delta \\ C \otimes C & \xrightarrow{\tau} & C \otimes C \end{array}$$

- (3) A **homomorphism** of dg-coalgebra $f: C \rightarrow D$ is a homomorphism of dg-vector spaces such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \Delta \downarrow & & \downarrow \Delta \\ C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \end{array} \quad \begin{array}{ccc} C & \xrightarrow{f} & D \\ \varepsilon \searrow & & \swarrow \varepsilon \\ & k & \end{array}$$

- (4) A **dg-coideal** in a dg-coalgebra C is a dg-subspace that is a coideal.³

Remark 3.2. A dg-coalgebra is the same as a graded coalgebra C together with a differential d such that ε vanishes on $B_0(C)$ and

$$\Delta(d(c)) = \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|} c_{(1)} \otimes d(c_{(2)}). \quad (2)$$

(See Appendix A.8 for further remarks.)

Example 3.3. For any dg-vector space V the induced dg-vector space $T(V)$ becomes a dg-coalgebra with the deconcatination

$$\begin{aligned} \Delta: T(V) &\rightarrow T(V) \otimes T(V), \quad v_1 \cdots v_n \mapsto \sum_{i=0}^n v_1 \cdots v_i \otimes v_{i+1} \cdots v_n, \\ \varepsilon: T(V) &\rightarrow k, \quad v_1 \cdots v_n \mapsto \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(See Appendix A.9 for the explicit calculations.)

³By “coideal” we always mean a two-sided coideal.

Proposition 3.4. Let C, D be dg-coalgebras.

(1) The tensor product $C \otimes D$ is again a dg-coalgebra with

$$\Delta_{C \otimes D}(c \otimes d) = \sum_{(c), (d)} (-1)^{|c_{(2)}||d_{(1)}|} (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)}).$$

(2) We have a co-opposite dg-coalgebra C^{cop} .

(3) If I is a dg-coideal in C then C/I inherits a dg-coalgebra structure.

(4) If C is a dg-coalgebra then $Z(C)$ is a graded subcoalgebra of C , $B(C)$ is a graded coideal in $Z(C)$ and $H(C)$ is hence a graded coalgebra.

Proof. See Appendix A.10. □

4. Differential Graded Bialgebras

Lemma 4.1. Let B be a dg-vector space, (B, m, u) a dg-algebra and (B, Δ, ε) a dg-coalgebra. Then the following are equivalent:

(1) Δ and ε are homomorphisms of dg-algebras.

(2) m and u are homomorphisms of dg-coalgebras.

Proof. The same diagrammatic proof as in the non-dg case (as seen in the second talk). □

Definition 4.2.

(1) If the conditions of Lemma 4.1 are satisfied then $(B, \mu, u, \Delta, \varepsilon)$ is a **dg-bialgebra**.

(2) A map $f: B \rightarrow C$ is a **homomorphism** of dg-bialgebras if it is both a homomorphism of dg-algebras and of dg-coalgebras.

(3) A **dg-biideal** is a dg-subspace that is both a dg-ideal and a dg-coideal.

Remark 4.3. The compatibility of the multiplication and comultiplication of B means

$$\Delta(bc) = \sum_{(b), (c)} (-1)^{|b_{(2)}||c_{(1)}|} b_{(1)}c_{(1)} \otimes b_{(2)}c_{(2)}.$$

Warning 4.4. A dg-bialgebra does in general *not* have an underlying bialgebra structure: The comultiplication $\Delta: B \rightarrow B \otimes B$ is a homomorphism of dg-algebras into $B \otimes B'$ but not necessarily an algebra homomorphism into the sign-less tensor product $B \otimes_k B$. We will see an explicit counterexample in Example 5.8.

Proposition 4.5. Let B, \mathcal{B} be dg-bialgebras.

(1) If I is a dg-biideal in B then B/I inherits a dg-bialgebra structure.

- (2) The cycles $Z(\mathcal{B})$ form a graded sub-bialgebra of \mathcal{B} , $B(\mathcal{B})$ is a graded biideal in $Z(\mathcal{B})$ and $H(\mathcal{B})$ is hence a graded bialgebra.

Proof. See Appendix A.11 \square

Definition 4.6. In a dg-bialgebra B , $x \in B$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$, and

$$\mathbb{P}(B) := \{x \in B \mid x \text{ is primitive.}\}$$

Lemma 4.7. If $x, y \in B$ are primitive then $[x, y]$ is again primitive.

Proof. See Example 6.5. \square

5. Differential Graded Hopf Algebras

Lemma 5.1. If C is a dg-coalgebra and A is a dg-algebra then the convolution product

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C$$

on $\text{Hom}_k(C, A)$ makes $\text{Hom}(C, A)$ into a dg-algebra.

Proof. See Appendix A.12. \square

Definition 5.2.

- (1) An **antipode** for a dg-bialgebra H is a convolution-inverse S to id_H in $\text{Hom}(H, H)$. If H admits an antipode then it is a **dg-Hopf algebra**.
- (2) A **homomorphism** of dg-Hopf algebras is a homomorphism of dg-bialgebras.
- (3) A **dg-Hopf ideal** in a dg-Hopf algebra H is a dg-biideal I with $S(I) \subseteq I$.

Warning 5.3. A dg-Hopf algebra need not have an underlying Hopf algebra structure.

Remark 5.4. The antipode of a dg-Hopf algebra H is the the unique homomorphism of dg-vector spaces $S: H \rightarrow H$ that makes the diagram

$$\begin{array}{ccccc}
 & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\
 \Delta \nearrow & & & & \searrow m \\
 H & \xrightarrow{\varepsilon} & k & \xrightarrow{u} & H \\
 \Delta \searrow & & & & \nearrow m \\
 & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H &
 \end{array} \tag{3}$$

commute. (See Appendix A.13.) This means more explicitly that

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \quad \text{and} \quad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H.$$

(No additional signs occur because $|S| = 0$.)

Proposition 5.5. Let H, \mathcal{H} be dg-Hopf algebras.

- (1) If I is a dg-Hopf ideal in H then H/I inherits a dg-Hopf algebra structure.
- (2) The graded bialgebra $H(\mathcal{H})$ is a graded Hopf algebra with antipode $H(S_{\mathcal{H}})$.

Proof. See Appendix A.14. □

Example 5.6. Let V be a dg-vector space. The maps

$$\begin{aligned} V &\rightarrow T(V) \otimes T(V), & v &\mapsto v \otimes 1 + 1 \otimes v, \\ V &\rightarrow k, & v &\mapsto 0, \\ V &\rightarrow T(V)^{\text{op}}, & v &\mapsto -v \end{aligned}$$

are homomorphisms of dg-vector spaces and thus induce homomorphisms of dg-algebras

$$\begin{aligned} \Delta: T(V) &\rightarrow T(V) \otimes T(V), \\ \varepsilon: T(V) &\rightarrow k, \\ S: T(V) &\rightarrow T(V)^{\text{op}}. \end{aligned}$$

The maps Δ and ε are explicitly given by

$$\begin{aligned} \Delta(v_1 \cdots v_n) &= \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}, \\ \varepsilon(v_1 \cdots v_n) &= \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \\ S(v_1 \cdots v_n) &= (-1)^{\sum_{1 \leq i < j \leq n} |v_i| |v_j|} (-1)^n v_n \cdots v_1 \end{aligned}$$

for homogeneous v_i , where S is viewed as a map $T(V) \rightarrow T(V)$ and $\text{Sh}(p, q)$ denotes the set of p - q -shuffles. These maps make $T(V)$ a dg-Hopf algebra. (See Appendix A.15 for the explicit calculations.)

Example 5.7 (Quotients of dg-Hopf algebras). Let V be a dg-vector space. The dg-algebra $\Lambda(V) = T(V)/I$ from Example 2.7 inherits from $T(V)$ the structure of a dg-Hopf algebra because the dg-ideal I is a dg-Hopf ideal in $T(V)$ (see Appendix A.16).

Example 5.8 (Exterior Algebra). Let V be a vector space. We regard V as a dg-vector space concentrated in degree 1. Then $\Lambda(V) = \bigwedge(V)$ as graded algebras whence $\bigwedge(V)$ is a graded Hopf algebra. But for $\text{char } k \neq 2$ there exists no bialgebra structure on $\Lambda := \bigwedge(V)$ (see Appendix A.17).

Example 5.9 (Homology of dg-Hopf algebras). Let V be a dg-vector space.

- (1) The inclusion $V \rightarrow T(V)$ is a homomorphism of dg-vector spaces and thus induces a homomorphism of graded vector spaces $H(V) \rightarrow H(T(V))$, which in turn induces a homomorphism of graded algebras

$$\alpha: T(H(V)) \rightarrow H(T(V)), \quad [v_1] \cdots [v_n] \mapsto [v_1 \cdots v_n]$$

where $v_1, \dots, v_n \in Z(V)$. We see on representatives that α is a homomorphism of graded Hopf algebras. We can write α as

$$H(T(V)) = H\left(\bigoplus_{d \geq 0} V^{\otimes d}\right) \cong \bigoplus_{d \geq 0} H(V^{\otimes d}) \cong \bigoplus_{d \geq 0} H(V)^{\otimes d} = T(H(V))$$

which shows that α is an isomorphism.

- (2) If $\text{char}(k) = 0$ then also $H(\Lambda(V)) \cong \Lambda(H(V))$: We get again a canonical homomorphism of graded algebras

$$\beta: \Lambda(H(V)) \rightarrow H(\Lambda(V)), \quad [v_1] \cdots [v_n] \mapsto [v_1 \cdots v_n]$$

where $v_1, \dots, v_n \in Z(V)$. The symmetrization map

$$s: \Lambda(V) \rightarrow T(V), \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n)$$

is a section for the projection $p: T(V) \rightarrow \Lambda(V)$ and a homomorphism of dg-vector spaces (see Appendix A.18). Together with the projection $\tilde{p}: T(H(V)) \rightarrow \Lambda(H(V))$ and symmetrization map $\tilde{s}: \Lambda(H(V)) \rightarrow T(H(V))$ we have the following diagram:

$$\begin{array}{ccc} T(H(V)) & \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\alpha^{-1}} \end{array} & H(T(V)) \\ \tilde{p} \updownarrow \tilde{s} & & H(p) \updownarrow H(s) \\ \Lambda(H(V)) & \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\beta'} \end{array} & H(\Lambda(V)) \end{array}$$

We have $\beta = H(p) \circ \alpha \circ \tilde{s}$, and $\beta' := \tilde{p} \circ \alpha^{-1} \circ H(s)$ is an inverse to β (see Appendix A.18). This shows that β is an isomorphism.

6. Differential Graded Lie Algebras

Let $\text{char}(k) = 0$.

Recall 6.1. A Lie algebra is a vector space \mathfrak{g} together with a map $[-, -]: \mathfrak{g} \otimes_k \mathfrak{g} \rightarrow \mathfrak{g}$ such that $[-, -]$ is skew-symmetric and $[x, -]: \mathfrak{g} \rightarrow \mathfrak{g}$ is a derivation for every $x \in \mathfrak{g}$; the last assertion is equivalent to the Jacobi identity $\sum_{\text{cyclic}} [x, [y, z]] = 0$.

Definition 6.2.

- (1) A **dg-Lie algebra** is a dg-vector space \mathfrak{g} together with a homomorphism of dg-vector spaces $[-, -]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ such that $[-, -]$ is **graded skew symmetric**

in the sense that the diagram

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\tau} & \mathfrak{g} \otimes \mathfrak{g} \\ & \searrow \quad \swarrow & \\ & [-, -] \quad -[-, -] & \\ & \mathfrak{g} & \end{array}$$

commutes, and such that $[x, -]$ is for every homogeneous x a derivation of degree $|x|$.

- (2) A **homomorphism** of dg-Lie algebras $f: \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of dg-vector spaces that makes the following diagram commute:

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{f \otimes f} & \mathfrak{h} \otimes \mathfrak{h} \\ [-, -] \downarrow & & \downarrow [-, -] \\ \mathfrak{g} & \xrightarrow{f} & \mathfrak{h} \end{array}$$

- (3) A dg-Lie ideal in a dg-Lie algebra \mathfrak{g} is a dg-subspace that is a Lie ideal.

Remark 6.3. That \mathfrak{g} is a dg-Lie algebra means that

$$\begin{aligned} [\mathfrak{g}_i, \mathfrak{g}_j] &\subseteq \mathfrak{g}_{i+j}, \\ [x, y] &= -(-1)^{|x||y|}[y, x], \\ [x, [y, z]] &= [[x, y], z] + (-1)^{|x||y|}[y, [x, z]], \\ d([x, y]) &= [d(x), y] + (-1)^{|x|}[x, d(y)]. \end{aligned} \tag{4}$$

We can rewrite (4) as the **graded Jacobi identity**

$$\sum_{\text{cyclic}} (-1)^{|x||z|}[x, [y, z]] = 0.$$

Warning 6.4. A dg-Lie algebra need not have an underlying Lie algebra structure.

Example 6.5.

- (1) Every dg-algebra A becomes a dg-Lie algebra with the graded commutator.
- (2) In any dg-bialgebra B the subspace of primitive elements

$$\mathbb{P}(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is a dg-Lie subalgebra of B .

(See Appendix A.19 for explicit calculations and another example.)

Lemma 6.6. Let \mathfrak{g} be a dg-Lie algebra.

- (1) If I is a dg-Lie ideal in \mathfrak{g} then \mathfrak{g}/I inherits the a dg-Lie algebra structure.

- (2) The cycles $Z(\mathfrak{g})$ form a graded Lie subalgebra of \mathfrak{g} , $B(\mathfrak{g})$ is a graded Lie ideal in $Z(\mathfrak{g})$ and $H(\mathfrak{g})$ is thus a graded Lie algebra.

Proof. See Appendix A.20. □

Definition 6.7. The **universal enveloping algebra** of a dg-Lie algebra \mathfrak{g} is

$$U(\mathfrak{g}) = T(\mathfrak{g}) / ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \mid x, y \in \mathfrak{g} \text{ homogeneous}).$$

Proposition 6.8.

- (1) The composition $i: \mathfrak{g} \rightarrow T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is a homomorphism of dg-Lie algebras.
- (2) If A is any dg-algebra and $f: \mathfrak{g} \rightarrow A$ a homomorphism of dg-Lie algebras there exists a unique homomorphism of dg-algebras $F: U(\mathfrak{g}) \rightarrow A$ that extends f :

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\quad F \quad} & A \\ \uparrow i & \nearrow f & \\ \mathfrak{g} & & \end{array}$$

- (3) The universal enveloping algebra $U(\mathfrak{g})$ inherits from $T(\mathfrak{g})$ the structure of a dg-Hopf algebra.

Proof. See Appendix A.21. □

We will now show that $H(U(\mathfrak{g})) \cong U(H(\mathfrak{g}))$. For this we need a version of the Poincaré–Birkhoff–Witt theorem (PBW theorem) for dg-Lie algebras, which we formulate in Appendix A.22. We will also blackbox the following consequences of the PBW theorem.

Corollary 6.9 (of the PBW theorem). Let \mathfrak{g} be a dg-Lie algebra.

- (1) The canonical map $\mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective.
- (2) It holds that $\mathbb{P}(U(\mathfrak{g})) = \mathfrak{g}$.
- (3) If $s: \Lambda(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ denotes the symmetrization map from Example 5.9 then

$$e: \Lambda(\mathfrak{g}) \xrightarrow{\quad s \quad} T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

is an isomorphism of dg-vector spaces (and even of dg-coalgebra). □

Example 6.10 (Homology of $U(\mathfrak{g})$). The inclusion $\mathfrak{g} \rightarrow U(\mathfrak{g})$ is a homomorphism of dg-Lie algebra and so induces a homomorphism of graded Lie algebras $H(\mathfrak{g}) \rightarrow H(U(\mathfrak{g}))$, which in turn induces a homomorphism of graded algebras

$$\gamma: U(H(\mathfrak{g})) \rightarrow H(U(\mathfrak{g})), \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

for $x_1, \dots, x_n \in Z(\mathfrak{g})$. We see on representatives that this is a homomorphism of dg-Hopf algebras, and it is an isomorphism: We denote the isomorphisms $\Lambda(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ and $\Lambda(H(\mathfrak{g})) \rightarrow U(H(\mathfrak{g}))$ from Corollary 6.9 by e and \tilde{e} . With the isomorphism of graded algebras

$$\beta: \Lambda(H(\mathfrak{g})) \rightarrow H(\Lambda(\mathfrak{g})), \quad [x_1] \cdots [x_n] \mapsto [x_1 \cdots x_n]$$

from Example 5.9 we get the following commutative diagram:

$$\begin{array}{ccc} \Lambda(H(\mathfrak{g})) & \xrightarrow[\tilde{e}]{\sim} & U(H(\mathfrak{g})) \\ \beta \downarrow \sim & & \downarrow \gamma \\ H(\Lambda(\mathfrak{g})) & \xrightarrow[H(e)]{\sim} & H(U(\mathfrak{g})) \end{array}$$

The arrows e , $H(e)$, β are isomorphisms, hence γ is one.

Remark 6.11.

- (1) If \mathcal{H} is a dg-Hopf algebra then $H(\mathbb{P}(\mathcal{H})) \cong \mathbb{P}(H(\mathcal{H}))$. (This statement can be found without proof in [Lod92, Theorem A.9].)
- (2) If H is a graded cocommutative connected⁴ dg-Hopf algebra then a version of the Cartier–Milnor–Moore theorem asserts that $H \cong U(\mathbb{P}(H))$. Together with Corollary 6.9 this results in an equivalence between the categories of dg-Lie algebras and cocommutative connected dg-Hopf algebras, see [Qui69, Appendix B, Theorem 4.5].

⁴The connectedness is defined in terms of the underlying dg-coalgebra, not that of the dg-algebra.

A. Calculations, Proofs and Remarks

A.1. The Koszul Sign

We have for every $i = 1, \dots, n-1$ a twist map

$$\begin{aligned}\tau_i: V^{\otimes n} &\rightarrow V^{\otimes n}, \\ v_1 \otimes \dots \otimes v_n &\mapsto v_1 \otimes \dots \otimes \tau(v_i \otimes v_{i+1}) \otimes \dots \otimes v_n \\ &\mapsto (-1)^{|v_i||v_{i+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n.\end{aligned}$$

The group S_n is generated by the simple reflections $\sigma_1, \dots, \sigma_{n-1}$ with relations

$$\begin{aligned}\sigma_i^2 &= 1 && \text{for } i = 1, \dots, n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for } |i-j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \dots, n-2.\end{aligned}$$

We check that the twist maps $\tau_1, \dots, \tau_{n-1}$ satisfy these relations, which shows that S_n acts on $V^{\otimes n}$ such that s_i acts via τ_i : We have

$$\tau_i^2(v_1 \otimes \dots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} \tau_i(v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n) = v_1 \otimes \dots \otimes v_n$$

and thus $\tau_i^2 = 1$. If $|i-j| \geq 2$ then

$$\begin{aligned}\tau_i \tau_j(v_1 \otimes \dots \otimes v_n) &= (-1)^{|v_i||v_{i+1}| + |v_j||v_{j+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_{j+1} \otimes v_j \otimes \dots \otimes v_n \\ &= \tau_j \tau_i(v_1 \otimes \dots \otimes v_n)\end{aligned}$$

and thus $\tau_i \tau_j = \tau_j \tau_i$. We also have

$$\begin{aligned}\tau_i \tau_{i+1} \tau_i(v_1 \otimes \dots \otimes v_n) &= (-1)^{|v_i||v_{i+1}|} \tau_i \tau_{i+1}(v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \dots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_i||v_{i+2}|} \tau_i(v_1 \otimes \dots \otimes v_{i+1} \otimes v_{i+2} \otimes v_i \otimes \dots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_i||v_{i+2}| + |v_{i+1}||v_{i+2}|} v_1 \otimes \dots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n\end{aligned}$$

and similarly

$$\begin{aligned}\tau_{i+1} \tau_i \tau_{i+1}(v_1 \otimes \dots \otimes v_n) &= (-1)^{|v_{i+1}||v_{i+2}|} \tau_{i+1} \tau_i(v_1 \otimes \dots \otimes v_i \otimes v_{i+2} \otimes v_{i+1} \otimes \dots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+2}| + |v_{i+1}||v_{i+2}|} \tau_{i+1}(v_1 \otimes \dots \otimes v_{i+2} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_n) \\ &= (-1)^{|v_i||v_{i+1}| + |v_i||v_{i+2}| + |v_{i+1}||v_{i+2}|} v_1 \otimes \dots \otimes v_{i+2} \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_n.\end{aligned}$$

Therefore $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$. We now have the desired action of S_n on $V^{\otimes n}$. The twist maps τ_i are morphisms of dg-vector spaces whence S_n acts by morphisms of dg-vector spaces.

Without sign the action of S_n on $V^{\otimes n}$ would be given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

(so that the tensor factor v_i is moved to the $\sigma(i)$ -th position). The above action of S_n on $V^{\otimes n}$ is hence given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \varepsilon_{v_1, \dots, v_n}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

with signs $\varepsilon_{v_1, \dots, v_n}(\sigma) \in \{1, -1\}$.

A.2. Remark 2.2

- (1) If A is a graded algebra then a graded map $\delta: A \rightarrow A$ is a **derivation** if

$$\delta \circ m = m \circ (\delta \otimes \text{id} + \text{id} \otimes \delta);$$

more explicitly,

$$\delta(ab) = \delta(a)b + (-1)^{|\delta||a|} a\delta(b).$$

The compatibility condition (1) in the definition of a dg-algebra thus states that the differential d is a derivation for A .

- (2) We see that there are two equivalent ways to make a graded vector space into a dg-algebra:

$$\begin{array}{ccc} \text{graded vector spaces} & \xrightarrow{\text{multiplication}} & \text{graded algebras} \\ \text{differential} \downarrow & & \downarrow \text{differential} \\ \text{dg-vector spaces} & \xrightarrow{\text{multiplication}} & \text{dg-algebras} \end{array}$$

- (3) The graded commutativity of A means $ab = (-1)^{|a||b|}ba$. If $|a|$ is even or $|b|$ is even then $ab = ba$; if $|a|$ is odd then $a^2 = -a^2$ and thus $a^2 = 0$ if $\text{char}(k) \neq 2$.
- (4) A morphism f of dg-algebras is the same as a morphism of the underlying graded algebras that commutes with the differentials. (No additional signs occur since $|f| = 0$.)

A.3. Examples 2.3

- (2) It holds that $\text{id}_V \in \text{End}(V)_0$ and if $f, g \in \text{End}(V)$ are graded maps then $f \circ g$ is again a graded map. Therefore $\text{End}(V)$ is a subalgebra of $\text{End}_k(V)$. If $f, g \in \text{End}(V)$

are homogeneous then $|f \circ g| = |f| + |g|$ so $\text{End}(V)$ is a graded algebra. We see from

$$\begin{aligned}
d(f \circ g) &= d \circ f \circ g - (-1)^{|f \circ g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\
&= d \circ f \circ g - (-1)^{|f|} f \circ d \circ g + (-1)^{|f|} f \circ d \circ g - (-1)^{|f|+|g|} f \circ g \circ d \\
&= (d \circ f - (-1)^{|f|} d \circ f) \circ g + (-1)^{|f|} f \circ (d \circ g - (-1)^{|g|} g \circ d) \\
&= d(f) \circ g + (-1)^{|f|} f \circ d(g)
\end{aligned}$$

and

$$d(\text{id}_V) = d \circ \text{id}_V - \text{id}_V \circ d = d - d = 0$$

that $\text{End}(V)$ is a dg-algebra.

- (3) It remains to check the compatibility of the multiplication and dg-structure of $T(V)$: It holds that $1_{T(V)} \in T(V)_0$ with $d(1_{T(V)}) = 0$. Furthermore

$$\begin{aligned}
|v_1 \cdots v_n \cdot w_1 \cdots w_m| &= |v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_m| \\
&= |v_1 \cdots v_n| + |w_1 \cdots w_m|
\end{aligned}$$

and

$$\begin{aligned}
&d(v_1 \cdots v_n \cdot w_1 \cdots w_m) \\
&= \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n \cdot w_1 \cdots w_m \\
&\quad + \sum_{j=1}^m (-1)^{|v_1| + \cdots + |v_n| + |w_1| + \cdots + |w_{j-1}|} v_1 \cdots v_n \cdot w_1 \cdots d(w_j) \cdots w_m \\
&= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1| + \cdots + |v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m) \\
&= d(v_1 \cdots v_n) \cdot w_1 \cdots w_m + (-1)^{|v_1 \cdots v_n|} v_1 \cdots v_n \cdot d(w_1 \cdots w_m).
\end{aligned}$$

This shows that $T(V)$ is indeed a dg-algebra.

Let A be another dg-algebra and $f: V \rightarrow A$ a morphism of dg-vector spaces and let $F: T(V) \rightarrow A$ be the unique extension of f to an algebra morphism, given by $F(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$. The algebra morphism F is a morphism of graded algebras because

$$\begin{aligned}
|F(v_1 \cdots v_n)| &= |f(v_1) \cdots f(v_n)| \\
&= |f(v_1)| + \cdots + |f(v_n)| \\
&= |v_1| + \cdots + |v_n| \\
&= |v_1 \cdots v_n|.
\end{aligned}$$

It is also a morphism of dg-vector spaces because

$$\begin{aligned}
d(F(v_1 \cdots v_n)) &= d(f(v_1) \cdots f(v_n)) \\
&= \sum_{i=1}^n (-1)^{|f(v_1)| + \cdots + |f(v_{i-1})|} f(v_1) \cdots d(f(v_i)) \cdots f(v_n) \\
&= \sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} f(v_1) \cdots f(d(v_i)) \cdots f(v_n) \\
&= F\left(\sum_{i=1}^n (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots d(v_i) \cdots v_n\right) \\
&= F(d(v_1 \cdots v_n)).
\end{aligned}$$

A.4. Proposition 2.4

- (3) The quotient A/I is a dg-vector space and an algebra and the compatibility of these structures can be checked on representatives.
- (4) The cycles $Z(A)$ form a graded subspace with $1 \in Z(A)$ and if $a, b \in Z(A)$ are homogeneous then

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b) = 0$$

and hence $ab \in Z(A)$. The boundaries $B(A)$ form a graded subspace and if $a \in Z(A)$ and $b \in B(A)$ are homogeneous with $b = d(a')$ then

$$b \cdot a = d(a') \cdot a = d(a \cdot a') - (-1)^{|a|} a' \cdot d(a) = d(a \cdot a')$$

and hence $ba \in B(A)$. Similarly $ab \in B(A)$.

Warning A.1. If $A \otimes_k B$ is the sign-less tensor product with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ then $A \otimes B \neq A \otimes_k B$ as algebras, i.e. the underlying algebra of $A \otimes B$ is not the tensor product of the underlying algebras of A and B . The underlying algebra of A^{op} is similarly not the opposite of the underlying algebra of A .

A.5. Lemma 2.5

That I is a graded ideal if and only if it is generated by homogeneous elements is well-known, see [Lan02, IX, 2.5] or [Bou89, II.§11.3]. It remains to show that $d(I) \subseteq I$ if $d(x_\alpha) \in I$ for every α : The ideal I is spanned by $ax_\alpha b$ with $a, b \in A$ homogeneous, and

$$d(ax_\alpha b) = d(a)x_\alpha b + (-1)^{|a|} ad(x_\alpha)b + (-1)^{|a|+|x_\alpha|} ax_\alpha d(b) \in I$$

since $x_\alpha, d(x_\alpha) \in I$.

A.6. Definition 2.6

If A is a dg-algebra and $|a|$ is even then $[a, a] = 0$. But if $|a|$ is odd then $[a, a] = 2a^2$. This means in particular that the graded commutator of an element with itself does not necessarily vanish (because not every element need to graded-commute with itself).

A.7. Example 2.7

- (1) The ideal I is a dg-ideal as the generators $[v, w]$ are homogeneous and (by Example 6.5)

$$d([v, w]) = [d(v), w] + (-1)^{|v|}[v, d(w)] \in I.$$

- (2) If S is any graded commutative dg-algebra and $f: V \rightarrow S$ a morphism of dg-vector spaces then f extends uniquely to a morphism of dg-algebras $F: \Lambda(V) \rightarrow S$:

$$\begin{array}{ccc} \Lambda(V) & \xrightarrow{\quad F \quad} & S \\ \uparrow & \nearrow f & \\ V & & \end{array}$$

- (3) Let A and B be two dg-algebras. If C is any other dg-algebra and if $f: A \rightarrow C$ and $g: B \rightarrow C$ are two morphisms of dg-algebras whose images graded-commute, in the sense that

$$f(a)g(b) = (-1)^{|a||b|}g(b)f(a)$$

for all $a \in A, b \in B$, then the linear map

$$\varphi: A \otimes B \rightarrow C, \quad a \otimes b \mapsto f(a)g(b)$$

is again a morphism of dg-algebras. The inclusions $i: A \rightarrow A \otimes B, a \mapsto a \otimes 1$ and $j: B \rightarrow A \otimes B, b \mapsto 1 \otimes b$ are morphisms of dg-algebras. For every morphism of dg-algebras $\varphi: A \otimes B \rightarrow C$ the compositions $\varphi \circ i: A \rightarrow C$ and $\varphi \circ j: B \rightarrow C$ are again morphisms of dg-algebras. This gives a one-to-one correspondence

$$\begin{aligned} \left\{ \begin{array}{l} \text{morphisms of dg-algebras} \\ f: A \rightarrow C, g: B \rightarrow C \\ \text{whose images graded-commute} \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} \text{morphisms of dg-algebras} \\ \varphi: A \otimes B \rightarrow C \end{array} \right\}, \\ (f, g) &\longmapsto (a \otimes b \mapsto f(a)g(b)), \\ (\varphi \circ i, \varphi \circ j) &\longleftarrow \varphi. \end{aligned}$$

- (4) It follows for any two dg-vector spaces V and W that

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

since we have for every dg-algebra A natural bijections

$$\begin{aligned}
& \{\text{morphisms of dg-algebras } \Lambda(V \oplus W) \rightarrow A\} \\
& \cong \{\text{morphisms of dg-vector spaces } V \oplus W \rightarrow A\} \\
& \cong \{(f, g) \mid \text{morphisms of dg-vector spaces } f: V \rightarrow A, g: W \rightarrow A\} \\
& \cong \{(\varphi, \psi) \mid \text{morphisms of dg-algebras } \varphi: \Lambda(V) \rightarrow A, \psi: \Lambda(W) \rightarrow A\} \\
& \cong \{\text{morphisms of dg-algebras } \Lambda(V) \otimes \Lambda(W) \rightarrow A\}.
\end{aligned}$$

More explicitly, the inclusions $V \rightarrow V \oplus W$ and $W \rightarrow V \oplus W$ induce morphisms of dg-algebras $\Lambda(V) \rightarrow \Lambda(V \oplus W)$ and $\Lambda(W) \rightarrow \Lambda(V \oplus W)$ that give an isomorphism of dg-algebras

$$\Lambda(V) \otimes \Lambda(W) \xrightarrow{\sim} \Lambda(V \oplus W), \quad v_1 \cdots v_n \otimes w_1 \cdots w_m \mapsto v_1 \cdots v_n w_1 \cdots w_m.$$

(5) Let V be a graded vector space.

If V is concentrated in even degrees then $\Lambda(V) = S(V)$ and if V is concentrated in odd degrees then $\Lambda(V) = \bigwedge(V)$, with the grading of $\Lambda(V)$ and $\bigwedge(V)$ induced by the one of V .

We have $V = V_{\text{even}} \oplus V_{\text{odd}}$ as graded vector spaces where $V_{\text{even}} = \bigoplus_n V_{2n}$ and $V_{\text{odd}} = \bigoplus_n V_{2n+1}$, and hence

$$\Lambda(V) = \Lambda(V_{\text{even}} \oplus V_{\text{odd}}) \cong \Lambda(V_{\text{even}}) \otimes \Lambda(V_{\text{odd}}) = S(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$$

The graded algebra $S(V_{\text{even}})$ is concentrated in even degree and so it follows that in the tensor product $S(V_{\text{even}}) \otimes \bigwedge(V_{\text{odd}})$ the simple tensors (strictly) commute, i.e. $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$. Hence

$$\Lambda(V) \cong S(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}})$$

where \otimes_k denotes the sign-less tensor product.

(6) Let $\text{char}(k) \neq 2$ and let V be a dg-vector space with basis $(x_\alpha)_{\alpha \in A}$ consisting of homogeneous elements such that (A, \leq) is linearly ordered. Then $\Lambda(V)$ admits as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \quad \text{where } t \geq 0, \alpha_1 < \cdots < \alpha_t, n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.}^5$$

To see this we use the above decomposition

$$\Lambda(V) \cong S(V_{\text{even}}) \otimes_k \bigwedge(V_{\text{odd}}) \tag{5}$$

as graded algebras: We split up the given basis $(x_\alpha)_{\alpha \in A}$ of V into a basis $(x_\alpha)_{\alpha \in A'}$ of V_{even} and $(x_\alpha)_{\alpha \in A''}$ of V_{odd} (since all x_α are homogeneous). Then $S(V_{\text{even}})$ has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \quad \text{where } r \geq 0, \alpha_1 < \cdots < \alpha_r \text{ and } n_i \geq 1,$$

⁵The condition $n_i = 1$ for $|x_{\alpha_i}|$ odd comes from the equality $\alpha_i^2 = [\alpha_i, \alpha_i]/2$.

and $\bigwedge(V_{\text{odd}})$ has as a basis the ordered wedges

$$x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_s} \quad \text{where } s \geq 0, \alpha_1 < \cdots < \alpha_s.$$

It follows that with (5) that $\Lambda(V)$ admits the basis

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_r}^{n_r} \cdot x_{\beta_1} \cdots x_{\beta_s} \quad \text{where } \begin{cases} r, s \geq 0, n_i \geq 1, \\ \alpha_1 < \cdots < \alpha_r, \\ \beta_1 < \cdots < \beta_s, \\ |x_{\alpha_i}| \text{ even, } |x_{\beta_j}| \text{ odd.} \end{cases}$$

We can now rearrange these basis vectors into the desired form because the factors $x_{\alpha_i}^{n_i}$ and x_{β_j} commute.

A.8. Remark 3.2

(1) If C is a graded coalgebra then a graded map $\omega: C \rightarrow C$ is a **coderivation** if

$$\Delta \circ \omega = (\omega \otimes \text{id} + \text{id} \otimes \omega) \circ \Delta.$$

This means more explicitly that

$$\Delta(\omega(c)) = \sum_{(c)} \omega(c_{(1)}) \otimes c_{(2)} + (-1)^{|\omega||c_{(1)}|} c_{(1)} \otimes \omega(c_{(2)}).$$

The compability (2) means that the differential d (which is a graded map of degree $|-d| = -1$) is a coderivation.

(2) The graded cocommutativity of C means

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} (-1)^{|c_{(1)}||c_{(2)}|} c_{(2)} \otimes c_{(1)}.$$

(3) A morphism of dg-coalgebras is the same as a morphism of the underlying graded coalgebras that commutes with the differentials.

(4) Every coalgebra C is a dg-coalgebra centered in degree 0, in particular $C = k$.

A.9. Example 3.3

We have seen in the first talk that $(T(C), \Delta, \varepsilon)$ is a coalgebra. We have for every $i = 0, \dots, n$ that

$$\begin{aligned} |v_1 \cdots v_i \otimes v_{i+1} \cdots v_n| &= |v_1 \cdots v_i| + |v_{i+1} \cdots v_n| \\ &= |v_1| + \cdots + |v_i| + |v_{i+1}| + \cdots + |v_n| \\ &= |v_1| + \cdots + |v_n|, \end{aligned}$$

so we have a graded coalgebra. We also have

$$\begin{aligned}
& d(\Delta(v_1 \cdots v_n)) \\
&= \sum_{i=0}^n d(v_1 \cdots v_i \otimes v_{i+1} \cdots v_n) \\
&= \sum_{i=0}^n (d(v_1 \cdots v_i) \otimes v_{i+1} \cdots v_n + (-1)^{|v_1 \cdots v_i|} v_1 \cdots v_i \otimes d(v_{i+1} \cdots v_n)) \\
&= \sum_{i=0}^n \left(\sum_{j=1}^i (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\
&\quad \left. + (-1)^{|v_1 \cdots v_i|} \sum_{j=i+1}^n (-1)^{|v_{i+1}| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\
&= \sum_{i=0}^n \left(\sum_{j=1}^i (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots d(v_j) \cdots v_i \otimes v_{i+1} \cdots v_n \right. \\
&\quad \left. + \sum_{j=i+1}^n (-1)^{|v_1| + \cdots + |v_{j-1}|} v_1 \cdots v_i \otimes v_{i+1} \cdots d(v_j) \cdots v_n \right) \\
&= \Delta \left(\sum_{j=1}^n (-1)^{|v_1| + \cdots + |v_j|} v_1 \otimes \cdots \otimes d(v_j) \otimes \cdots \otimes v_n \right) \\
&= \Delta(d(v_1 \cdots v_n))
\end{aligned}$$

which shows that Δ is a morphism of dg-vector spaces.

A.10. Proposition 3.4

- (3) The quotient C/I is a dg-vector space and a coalgebra, and the compatibility of these structures can be checked on representatives.
- (4) If $c \in Z(C)$ then

$$d(\Delta(c)) = \Delta(d(c)) = \Delta(0) = 0$$

because Δ is a morphism of dg-vector spaces, and hence

$$\Delta(c) \in Z(C \otimes C) = Z(C) \otimes Z(C).$$

This shows that $Z(C)$ is a subcoalgebra of C . It is also a graded subspace of C and hence a graded subcoalgebra.

For $b \in B(C)$ with $b = d(c)$ we have

$$\begin{aligned}
\Delta(b) &= \Delta(d(c)) = d(\Delta(c)) = d\left(\sum_{(c)} c_{(1)} \otimes c_{(2)}\right) \\
&= \sum_{(c)} d(c_{(1)}) \otimes c_{(2)} + (-1)^{|c_{(1)}|c_{(1)} \otimes d(c_{(2)})} \in B(C) \otimes C + C \otimes B(C).
\end{aligned}$$

We also have

$$\varepsilon(b) = \varepsilon(d(c)) = d(\varepsilon(c)) = 0.$$

This shows that $B(C)$ is a coideal in C . It follows from the upcoming lemma that B is also a coideal in $Z(C)$. Then $B(C)$ is a graded coideal in $Z(C)$ because $B(C)$ is a graded subspace of $Z(C)$.

Lemma A.2. Let C be a coalgebra and let B be a subcoalgebra of C . If I is a coideal in C with $I \subseteq C$ then I is also a coideal in B .

Proof. It follows from the inclusions $I \subseteq B \subseteq C$ that

$$(C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B.$$

Hence

$$\Delta(I) = \Delta(I) \cap \Delta(B) \subseteq (C \otimes I + I \otimes C) \cap (B \otimes B) = B \otimes I + I \otimes B.$$

Also $\varepsilon_B(I) = \varepsilon_C(I) = 0$. □

A.11. Proposition 4.5

- (1) It follows from Proposition 2.4 and Proposition 3.4 that B/I is a dg-algebra and dg-coalgebra. The compatibility can be checked on representatives.
- (2) It follows from Proposition 2.4 and Proposition 3.4 that $H(\mathcal{B})$ is again a dg-algebra and dg-coalgebra, and the compatibility of these structures can be checked on representatives.

A.12. Lemma 5.1

We have $1_{\text{Hom}_k(C,A)} = u \circ \epsilon \in \text{Hom}(C, A)_0$ because both u_A and ϵ_C are morphisms of dg-vector spaces and thus of degree 0. If $f, g \in \text{Hom}(C, A)$ are graded maps then $f \otimes g$ is again a graded map and thus

$$f * g = m \circ (f \otimes g) \circ \Delta$$

is a graded map as a composition of graded maps. This shows that $\text{Hom}(C, A)$ is a subalgebra of $\text{Hom}_k(C, A)$.

We have

$$|f * g| = |m \circ (f \otimes g) \circ \Delta| = |m| + (|f| + |g|) + |\Delta| = |f| + |g|$$

so $\text{Hom}(C, A)$ is a graded algebra with respect to the convolution product.

Furthermore

$$\begin{aligned} & d(f * g) \\ &= d \circ (f * g) - (-1)^{|f * g|} (f * g) \circ d \end{aligned}$$

$$\begin{aligned}
&= d \circ m \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ \Delta \circ d \\
&= m \circ d_{A \otimes A} \circ (f \otimes g) \otimes \Delta - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ d_{C \otimes C} \circ \Delta \\
&= m \circ (d \otimes 1 + 1 \otimes d) \circ (f \otimes g) \otimes \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes 1 + 1 \otimes d) \circ \Delta \\
&= m \circ (d \otimes \text{id}) \circ (f \otimes g) \otimes \Delta + m \circ (\text{id} \otimes d) \circ (f \otimes g) \otimes \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (d \otimes \text{id}) \circ \Delta \\
&\quad - (-1)^{|f|+|g|} m \circ (f \otimes g) \circ (\text{id} \otimes d) \circ \Delta \\
&= m \circ ((d \circ f) \otimes g) \otimes \Delta + (-1)^{|f|} m \circ (f \otimes (d \circ g)) \otimes \Delta \\
&\quad - (-1)^{|f|} m \circ ((f \circ d) \otimes g) \circ \Delta - (-1)^{|f|+|g|} m \circ (f \otimes (g \circ d)) \circ \Delta \\
&= m \circ ((d \circ f - (-1)^{|f|} f \circ d) \otimes g) \otimes \Delta \\
&\quad + (-1)^{|f|} m \circ (f \otimes (d \circ g - (-1)^{|g|} g \circ d)) \otimes \Delta \\
&= m \circ (d(f) \otimes g) \circ \Delta + (-1)^{|f|} m \circ (f \otimes d(g)) \otimes \Delta \\
&= d(f) * g + (-1)^{|f|} f * d(g)
\end{aligned}$$

because m and Δ are commute with the differentials. Hence $\text{Hom}(C, A)$ is a dg-algebra with respect to the convolution product.

A.13. Remark 5.4

We need to explain why an inverse to id_H in $\text{Hom}(H, H)$ with respect to the convolution product $*$ is again a morphism of dg-vector spaces. For this we use the following result:

Lemma A.3. Let A be a dg-algebra and let $a \in A$ be a homogeneous unit.

- (1) The inverse a^{-1} is homogeneous of degree $|a^{-1}| = -|a|$.
- (2) If a is a cycle then so is a^{-1} .

Proof.

- (1) Let $d = |a|$ and let $a^{-1} = \sum_n a'_n$ be the homogeneous decomposition of a^{-1} . It follows from $1 = ab = \sum_n aa'_n$ that in degree zero, $1 = aa'_{-d}$. Thus a'_{-d} is the inverse of a , i.e. $a^{-1} = a'_{-d} \in A_{-d}$.
- (2) It follows from

$$0 = d(1) = d(aa^{-1}) = d(a)a^{-1} + (-1)^{|a|} ad(a^{-1})$$

that $(-1)^{|a|} ad(a^{-1}) = 0$ because $d(a) = 0$. Hence $d(a^{-1}) = 0$ as a is a unit. \square

The space $Z_0(\text{Hom}(V, W))$ consists of the morphism of dg-vector spaces $V \rightarrow W$. It hence follows from Lemma A.3 that if $f \in Z_0(\text{Hom}(V, W))$ admits an inverse g with respect to the convolution product that again $g \in Z_0(\text{Hom}(V, W))$.

A.14. Proposition 5.5

- (1) It follows from Proposition 4.5 that H is a dg-bialgebra and the condition $S(I) \subseteq I$ ensures that S induces a morphism of dg-vector spaces $\bar{S}: H/I \rightarrow H/I$. The antipode condition for \bar{S} can now be checked on representatives.
- (2) The homology $H(\mathcal{H})$ is a dg-bialgebra by Proposition 4.5 and that $H(S_{\mathcal{H}})$ is an antipode can be checked on representatives.

A.15. Example 5.6

The dg-coalgebra diagrams for $(T(V), \Delta, \varepsilon)$ can be checked on algebra generators of $T(V)$ because all arrows in these diagrams are morphisms of dg-algebras. It hence suffices to check these diagrams for elements of V , where this is straightforward.

It remains to check the equalities

$$\sum_{(h)} S(h_{(1)})h_{(2)} = \varepsilon(h)1_H \quad \text{and} \quad \sum_{(h)} h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H$$

for the monomials $h = v_1 \cdots v_n$. If $n = 0$ then $h = 1$ and both equalities hold, so we consider in the following the case $n \geq 1$. Then $\varepsilon(v_1 \cdots v_n) = 0$ so we have to show that in the sums $\sum_{(h)} S(h_{(1)})h_{(2)}$ and $\sum_{(h)} h_{(1)}S(h_{(2)})$ all terms cancel out. We consider for simplicity only the sum $\sum_{(h)} S(h_{(1)})h_{(2)}$.⁶ We have

$$\Delta(v_1 \cdots v_n) = \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \cdots v_{\sigma(n)}. \quad (6)$$

Here

$$S(v_{\sigma(1)} \cdots v_{\sigma(p)}) = (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} v_{\sigma(p)} \cdots v_{\sigma(1)}$$

and thus

$$\begin{aligned} & (m \circ (S \otimes \text{id}) \circ \Delta)(v_1 \cdots v_n) \\ &= \sum_{p=0}^n \sum_{\sigma \in \text{Sh}(p, n-p)} \varepsilon_{v_1, \dots, v_n}(\sigma^{-1}) (-1)^p (-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}| |v_{\sigma(j)}|} \\ & \quad \cdot v_{\sigma(p)} \cdots v_{\sigma(1)} v_{\sigma(p+1)} \cdots v_{\sigma(n)}. \end{aligned} \quad (7)$$

We see that in (6) any two terms of the form

$$w_1 w_2 \cdots w_i \otimes w_{i+1} \cdots w_n \quad \text{and} \quad w_2 \cdots w_i \otimes w_1 w_{i+1} \cdots w_n$$

give in (7) the up to sign same term $w_i \cdots w_2 w_1 w_{i+1} \cdots w_n$. We now check that the signs differ, so that in (7) both terms cancel out. This then shows that the sum (7) becomes zero.

⁶The author hasn't actually checked the other sum.

For $1 \leq p \leq n$ and $\sigma \in \text{Sh}(p, n-p)$ with $\sigma(p) < \sigma(1)$ the term associated to $v_{\sigma(1)} \cdots v_{\sigma(p)} \otimes v_{(p+1)} \cdots v_{\sigma(n)}$ is given by

$$v_{\sigma(2)} \cdots v_{\sigma(p)} \otimes v_{\sigma(1)} v_{\sigma(p+1)} \cdots v_{\sigma(n)} = v_{\tau(1)} \cdots v_{\tau(p-1)} \otimes v_{\tau(p)} \cdots v_{\tau(n)}$$

for the permutation $\omega \in \text{Sh}(p-1, n-p+1)$ given by

$$\omega = \sigma \circ (1\ 2 \cdots p),$$

i.e.

$$\omega(i) = \begin{cases} \sigma(i+1) & \text{if } 1 \leq i \leq p-1, \\ \sigma(1) & \text{if } i = p, \\ \sigma(i) & \text{if } p+1 \leq i \leq n. \end{cases}$$

We see from the Koszul sign rule that the signs $\varepsilon_{v_1, \dots, v_n}(\sigma^{-1})$ and $\varepsilon_{v_1, \dots, v_n}(\omega^{-1})$ differ by the factor $(-1)^{|v_{\sigma(1)}||v_{\sigma(2)}| + \cdots + |v_{\sigma(1)}||v_{\sigma(p)}|}$. Therefore

$$\begin{aligned} & \varepsilon_{v_1, \dots, v_n}(\sigma^{-1})(-1)^p(-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}||v_{\sigma(j)}|} \\ &= \varepsilon_{v_1, \dots, v_n}(\omega^{-1})(-1)^{|v_{\sigma(1)}||v_{\sigma(2)}| + \cdots + |v_{\sigma(1)}||v_{\sigma(p)}|}(-1)^p(-1)^{\sum_{1 \leq i < j \leq p} |v_{\sigma(i)}||v_{\sigma(j)}|} \\ &= \varepsilon_{v_1, \dots, v_n}(\omega^{-1})(-1)^p(-1)^{\sum_{2 \leq i < j \leq p} |v_{\sigma(i)}||v_{\sigma(j)}|} \\ &= \varepsilon_{v_1, \dots, v_n}(\omega^{-1})(-1)^p(-1)^{\sum_{1 \leq i < j \leq p-1} |v_{\omega(i)}||v_{\omega(j)}|} \\ &= -\varepsilon_{v_1, \dots, v_n}(\omega^{-1})(-1)^{p-1}(-1)^{\sum_{1 \leq i < j \leq p-1} |v_{\omega(i)}||v_{\omega(j)}|}. \end{aligned}$$

Thus the signs differ as claimed.

A.16. Example 5.7

We have

$$\begin{aligned} \varepsilon([v, w]) &= \varepsilon(vw - (-1)^{|v||w|}wv) \\ &= \varepsilon(vw) - (-1)^{|v||w|}\varepsilon(wv) \\ &= \varepsilon(v)\varepsilon(w) - (-1)^{|v||w|}\varepsilon(w)\varepsilon(v) \\ &= 0 \end{aligned}$$

as $\varepsilon(v) = \varepsilon(w) = 0$. The elements v and w are primitive whence $[v, w]$ is primitive. Therefore

$$\Delta([v, w]) = [v, w] \otimes 1 + 1 \otimes [v, w] \in I \otimes T(V) + T(V) \otimes I.$$

Also

$$\begin{aligned}
S([v, w]) &= S(vw - (-1)^{|v||w|}wv) \\
&= S(vw) - (-1)^{|v||w|}S(wv) \\
&= (-1)^{|v||w|}wv - vw \\
&= -(vw - (-1)^{|v||w|}wv) \\
&= -[v, w] \\
&\in I.
\end{aligned}$$

A.17. Example 5.8

Suppose that there exists a bialgebra structure on $\bigwedge(V)$. Then $\varepsilon(v)^2 = \varepsilon(v^2) = 0$ and thus $\varepsilon(v) = 0$ for all $v \in V$, so $\ker \varepsilon = \bigoplus_{d \geq 1} \bigwedge^d(V) =: I$. Let $v \in V$. Then by the counital axiom,

$$\Delta(v) \equiv v \otimes 1 \pmod{\Lambda \otimes I} \quad \text{and} \quad \Delta(v) \equiv 1 \otimes v \pmod{I \otimes \Lambda}$$

and thus

$$\Delta(v) \equiv v \otimes 1 + 1 \otimes v \pmod{I \otimes I}.$$

It follows that

$$\Delta(v^2) \equiv (v \otimes 1 + 1 \otimes v)^2 \pmod{(v \otimes 1)(I \otimes I) + (1 \otimes v)(I \otimes I) + (I \otimes I)^2},$$

and therefore

$$\Delta(v^2) \equiv v^2 \otimes 1 + 2v \otimes v + 1 \otimes v^2 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

Now $v^2 = 0$, hence

$$2v \otimes v \equiv 0 \pmod{I \otimes I^2 + I^2 \otimes I}.$$

But $2 \neq 0$ and $v \neq 0$ hence $2v \otimes v \neq 0$ while $v \otimes v \notin I \otimes I^2 + I^2 \otimes I$, a contradiction. (This proof is taken from [MO18] and partially from [Bou89, III.§11.3]).

A.18. Example 5.9

- (1) The action of S_n on $V^{\otimes n}$ is by morphism of dg-vector spaces as mentioned in Section 1 and shown in Appendix A.1. The symmetrization map

$$\tilde{s}: T(V) \rightarrow T(V), \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (v_1 \otimes \cdots \otimes v_n)$$

therefore results in a morphism of dg-vector spaces $\tilde{s}: T(V) \rightarrow T(V)$.⁷ It follows that the factored map $s: \Lambda(V) \rightarrow T(V)$ is again a morphism of dg-vector spaces.

⁷This map is a projection of $T(V)$ on its dg-subspace of graded symmetric tensors.

(2) We observe that the diagrams

$$\begin{array}{ccc} \mathrm{T}(\mathrm{H}(V)) & \xrightarrow{\alpha} & \mathrm{H}(\mathrm{T}(V)) \\ \bar{p} \downarrow & & \downarrow \mathrm{H}(p) \\ \Lambda(\mathrm{H}(V)) & \xrightarrow{\beta} & \mathrm{H}(\Lambda(V)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{T}(\mathrm{H}(V)) & \xrightarrow{\alpha} & \mathrm{H}(\mathrm{T}(V)) \\ \bar{s} \uparrow & & \uparrow \mathrm{H}(s) \\ \Lambda(\mathrm{H}(V)) & \xrightarrow{\beta} & \mathrm{H}(\Lambda(V)) \end{array}$$

commute. Indeed, for representatives $v_1, \dots, v_n \in \mathrm{Z}(V)$ the first diagram gives

$$\begin{array}{ccc} [v_1] \otimes \cdots \otimes [v_n] & \longmapsto & [v_1 \otimes \cdots \otimes v_n] \\ \downarrow & & \downarrow \\ [v_1] \cdots [v_n] & \longmapsto & [v_1 \cdots v_n] \end{array}$$

and the second diagram is given as follows:

$$\begin{array}{ccc} \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) [v_{\sigma(1)}] \otimes \cdots \otimes [v_{\sigma(n)}] & \longmapsto & \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) [v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}] \\ \uparrow & & \uparrow \\ [v_1] \cdots [v_n] & \longmapsto & [v_1 \cdots v_n] \end{array}$$

It follows that

$$\beta\beta' = \beta\tilde{p}\alpha^{-1}\mathrm{H}(s) = \mathrm{H}(p)\alpha\alpha^{-1}\mathrm{H}(s) = \mathrm{H}(p)\mathrm{H}(s) = \mathrm{id}_{\mathrm{H}(\Lambda(V))}$$

and similarly

$$\beta'\beta = \tilde{p}\alpha^{-1}\mathrm{H}(s)\beta = \tilde{p}\alpha^{-1}\alpha\tilde{s} = \tilde{p}\tilde{s} = \mathrm{id}_{\Lambda(\mathrm{H}(V))}$$

A.19. Example 6.5

(1) If $a, b \in A$ are homogeneous then $[a, b] = ab - (-1)^{|a||b|}ba$ is homogeneous of degree $|a| + |b|$, so $[A_i, A_j] \subseteq A_{i+j}$ for all i, j . Also

$$[a, b] = ab - (-1)^{|a||b|}ba = -(-1)^{|a||b|}(ba - (-1)^{|a||b|}ab) = -(-1)^{|a||b|}[b, a]$$

and

$$\begin{aligned} d([a, b]) &= d(ab - (-1)^{|a||b|}ba) \\ &= d(ab) - (-1)^{|a||b|}d(ba) \\ &= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||b|}(d(b)a + (-1)^{|b|}bd(a)) \\ &= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||b|}d(b)a - (-1)^{|a||b|+|b|}bd(a) \\ &= d(a)b + (-1)^{|a|}ad(b) - (-1)^{|a||d(b)|+|a|}d(b)a - (-1)^{|d(a)||b|}bd(a) \\ &= d(a)b - (-1)^{|d(a)||b|}bd(a) + (-1)^{|a|}(ad(b) - (-1)^{|a||d(b)|}d(b)a) \end{aligned}$$

$$= [d(a), b] + (-1)^{|a|} [a, d(b)].$$

We check the graded Jacobi identity for homogeneous $a, b, c \in A$. We have

$$\begin{aligned} [a, [b, c]] &= [a, bc - (-1)^{|b||c|} cb] \\ &= [a, bc] - (-1)^{|b||c|} [a, cb] \\ &= abc - (-1)^{|a||bc|} bca - (-1)^{|b||c|} (acb - (-1)^{|a||cb|} cba) \\ &= abc - (-1)^{|a||bc|} bca - (-1)^{|b||c|} acb + (-1)^{|a||cb|+|b||c|} cba \\ &= abc - (-1)^{|a|(|b|+|c|)} bca - (-1)^{|b||c|} acb + (-1)^{|a|(|b|+|c|)+|b||c|} cba \\ &= abc - (-1)^{|a||b|+|a||c|} bca - (-1)^{|b||c|} acb + (-1)^{|a||b|+|a||c|+|b||c|} cba \end{aligned}$$

and therefore

$$\begin{aligned} (-1)^{|a||c|} [a, [b, c]] &= (-1)^{|a||c|} abc - (-1)^{|a||b|} bca \\ &\quad - (-1)^{|a||c|+|b||c|} acb + (-1)^{|a||b|+|b||c|} cba. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{\text{cyclic}} (-1)^{|a||c|} [a, [b, c]] &= \sum_{\text{cyclic}} (-1)^{|a||c|} abc - \sum_{\text{cyclic}} (-1)^{|a||b|} bca \\ &\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|} acb + \sum_{\text{cyclic}} (-1)^{|a||b|+|b||c|} cba \\ &= \sum_{\text{cyclic}} (-1)^{|b||a|} bca - \sum_{\text{cyclic}} (-1)^{|a||b|} bca \\ &\quad - \sum_{\text{cyclic}} (-1)^{|a||c|+|b||c|} acb + \sum_{\text{cyclic}} (-1)^{|b||c|+|c||a|} acb \\ &= 0. \end{aligned}$$

(2) If $a \in \mathbb{P}(B)$ with homogeneous decomposition $a = \sum_n a_n$ then

$$\Delta(a) = \Delta\left(\sum_n a_n\right) = \sum_n \Delta(a_n)$$

but also

$$\Delta(a) = a \otimes 1 + 1 \otimes a = \sum_n (a_n \otimes 1 + 1 \otimes a_n).$$

By comparing homogeneous components we see that $\Delta(a_n) = a_n \otimes 1 + 1 \otimes a_n$ for all n . This means that all homogeneous components a_n are again primitive, which

shows that $\mathbb{P}(B)$ is a graded subspace of B . If $a \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(d(a)) &= d(\Delta(a)) \\
&= d(a \otimes 1 + 1 \otimes a) \\
&= d(a \otimes 1) + d(1 \otimes a) \\
&= d(a) \otimes 1 + (-1)^{|a|} a \otimes d(1) + d(1) \otimes a + (-1)^{|1|} 1 \otimes d(a) \\
&= d(a) \otimes 1 + 1 \otimes d(a)
\end{aligned}$$

because $|1| = 0$ and $d(1) = 0$. Therefore $\mathbb{P}(B)$ is a dg-subspace of B .

If $a, b \in \mathbb{P}(B)$ then

$$\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b) \\
&= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) \\
&= (a \otimes 1)(b \otimes 1) + (a \otimes 1)(1 \otimes b) + (1 \otimes a)(b \otimes 1) + (1 \otimes a)(1 \otimes b) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a + 1 \otimes ab.
\end{aligned}$$

If a, b are homogeneous then it follows that

$$\begin{aligned}
\Delta([a, b]) &= \Delta(ab - (-1)^{|a||b|}ba) \\
&= \Delta(ab) - (-1)^{|a||b|}\Delta(ba) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|}b \otimes a + 1 \otimes ab \\
&\quad - (-1)^{|a||b|}(ba \otimes 1 + b \otimes a + (-1)^{|a||b|}a \otimes b + 1 \otimes ba) \\
&= ab \otimes 1 + a \otimes b + (-1)^{|a||b|}b \otimes a + 1 \otimes ab \\
&\quad - (-1)^{|a||b|}ba \otimes 1 - (-1)^{|a||b|}b \otimes a - a \otimes b - (-1)^{|a||b|}1 \otimes ba \\
&= (ab - (-1)^{|a||b|}ba) \otimes 1 + 1 \otimes (ab - (-1)^{|a||b|}ba) \\
&= [a, b] \otimes 1 + 1 \otimes [a, b]
\end{aligned}$$

which shows that $[a, b] \in \mathbb{P}(B)$. Thus $\mathbb{P}(B)$ is a dg-Lie subalgebra of B .

(3) If A is a graded algebra, then the graded subspace $\text{Der}(A) \subseteq \text{End}(A)$ given by

$$\text{Der}(A)_n := \{\text{derivations of } A \text{ of degree } n\} \subseteq \text{End}(A)_n$$

is a dg-Lie subalgebra of $\text{End}(A)$:

Let δ, ε be graded derivations. Then for all homogeneous $a, b \in A$,

$$\begin{aligned}
(\delta\varepsilon)(ab) &= \delta(\varepsilon(ab)) \\
&= \delta(\varepsilon(a)b + (-1)^{|\varepsilon||a|}a\varepsilon(b)) \\
&= \delta(\varepsilon(a)b) + (-1)^{|\varepsilon||a|}\delta(a\varepsilon(b)) \\
&= \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|}\varepsilon(a)\delta(b)
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{|\varepsilon||a|} (\delta(a)\varepsilon(b) + (-1)^{|\delta||a|} a\delta(\varepsilon(b))) \\
& = \delta(\varepsilon(a))b + (-1)^{|\varepsilon(a)||\delta|} \varepsilon(a)\delta(b) \\
& \quad + (-1)^{|\varepsilon||a|} \delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|} a\delta(\varepsilon(b)) \\
& = \delta(\varepsilon(a))b + (-1)^{(|\varepsilon|+|a|)|\delta|} \varepsilon(a)\delta(b) \\
& \quad + (-1)^{|\varepsilon||a|} \delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|} a\delta(\varepsilon(b)) \\
& = \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|} \varepsilon(a)\delta(b) \\
& \quad + (-1)^{|\varepsilon||a|} \delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|} a\delta(\varepsilon(b))
\end{aligned}$$

It follows that

$$\begin{aligned}
(-1)^{|\delta||\varepsilon|} (\varepsilon\delta)(ab) & = (-1)^{|\delta||\varepsilon|} \varepsilon(\delta(a))b + (-1)^{|\varepsilon||a|} \delta(a)\varepsilon(b) \\
& \quad + (-1)^{|\delta||\varepsilon|+|\delta||a|} \varepsilon(a)\delta(b) + (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|} a\varepsilon(\delta(b))
\end{aligned}$$

and therefore

$$\begin{aligned}
[\delta, \varepsilon](ab) & = (\delta\varepsilon - (-1)^{|\delta||\varepsilon|} \varepsilon\delta)(ab) \\
& = (\delta\varepsilon)(ab) - (-1)^{|\delta||\varepsilon|} (\varepsilon\delta)(ab) \\
& = \delta(\varepsilon(a))b + (-1)^{|\delta||\varepsilon|+|\delta||a|} \varepsilon(a)\delta(b) \\
& \quad + (-1)^{|\varepsilon||a|} \delta(a)\varepsilon(b) + (-1)^{|\delta||a|+|\varepsilon||a|} a\delta(\varepsilon(b)) \\
& \quad - (-1)^{|\delta||\varepsilon|} \varepsilon(\delta(a))b - (-1)^{|\varepsilon||a|} \delta(a)\varepsilon(b) \\
& \quad - (-1)^{|\delta||\varepsilon|+|\delta||a|} \varepsilon(a)\delta(b) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|} a\varepsilon(\delta(b)) \\
& = \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|} \varepsilon(\delta(a))b \\
& \quad + (-1)^{|\delta||a|+|\varepsilon||a|} a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|+|\delta||a|+|\varepsilon||a|} a\varepsilon(\delta(b)) \\
& = \delta(\varepsilon(a))b - (-1)^{|\delta||\varepsilon|} \varepsilon(\delta(a))b \\
& \quad + (-1)^{|\delta||a|+|\varepsilon||a|} (a\delta(\varepsilon(b)) - (-1)^{|\delta||\varepsilon|} a\varepsilon(\delta(b))) \\
& = [\delta, \varepsilon](a)b + (-1)^{|\delta, \varepsilon||a|} a[\delta, \varepsilon](b).
\end{aligned}$$

This shows that $[\delta, \varepsilon] \in \text{Der}(A)$, so that $\text{Der}(A)$ is a graded Lie subalgebra of $\text{End}(A)$. If $\delta \in \text{Der}(A)$ is homogeneous then

$$d(\delta) = d \circ \delta - (-1)^{|\delta|} \delta \circ d = [d, \delta]$$

is again a graded derivation, and hence $\text{Der}(A)$ is a dg-subspace of $\text{End}(A)$.

A.20. Lemma 6.6

- (1) The quotient \mathfrak{g}/I is again a dg-vector spaces and a Lie algebra. The compatibility of these structures can be checked on generators.

- (2) The cycles $Z(\mathfrak{g})$ form a graded subspace of \mathfrak{g} . For homogeneous $x, y \in Z(\mathfrak{g})$,

$$d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)] = [0, y] + (-1)^{|x|}[x, 0] = 0,$$

so $Z(\mathfrak{g})$ is indeed a graded Lie subalgebra of \mathfrak{g} . The boundaries $B(\mathfrak{g})$ form a graded subspace of $Z(\mathfrak{g})$. If $x \in B(\mathfrak{g})$ with $x = d(x')$, where $x' \in \mathfrak{g}$ is homogeneous, then for every $y \in Z(\mathfrak{g})$,

$$[x, y] = [d(x'), y] = d([x', y]) - (-1)^{|x'|}[x', \underbrace{d(y)}_{=0}] = d([x', y]) \in B(\mathfrak{g}).$$

Thus $B(\mathfrak{g})$ is a graded Lie ideal in $Z(\mathfrak{g})$.

A.21. Proposition 6.8

- (1) This follows from the choice of ideal I .
- (2) This is a combination of the universal properties of the dg-tensor algebra and that of the quotient dg-algebra.
- (3) We check that the given ideal I is a dg-Hopf ideal. It is generated by homogenous elements which satisfy

$$\begin{aligned} & d([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &= d([x, y]_{T(\mathfrak{g})}) - d([x, y]_{\mathfrak{g}}) \\ &= [d(x), y]_{T(\mathfrak{g})} + (-1)^{|x|}[x, d(y)]_{T(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} - (-1)^{|x|}[x, d(y)]_{\mathfrak{g}} \\ &= \left([d(x), y]_{T(\mathfrak{g})} - [d(x), y]_{\mathfrak{g}} \right) + (-1)^{|x|} \left([x, d(y)]_{T(\mathfrak{g})} - [x, d(y)]_{\mathfrak{g}} \right) \in I \end{aligned}$$

so it is a dg-ideal. Also

$$\varepsilon([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) = \varepsilon([x, y]_{T(\mathfrak{g})}) - \varepsilon([x, y]_{\mathfrak{g}}) = 0 - 0 = 0$$

because $[x, y]_{T(\mathfrak{g})}$ and $[x, y]_{\mathfrak{g}}$ are homogeneous of degree ≥ 1 ,

$$\begin{aligned} & \Delta([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &= \Delta([x, y]_{T(\mathfrak{g})}) - \Delta([x, y]_{\mathfrak{g}}) \\ &= [x, y]_{T(\mathfrak{g})} \otimes 1 + 1 \otimes [x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}} \otimes 1 - 1 \otimes [x, y]_{\mathfrak{g}} \\ &= ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \otimes 1 + 1 \otimes ([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) \\ &\in I \otimes T(\mathfrak{g}) + T(\mathfrak{g}) \otimes I \end{aligned}$$

since both $[x, y]_{T(\mathfrak{g})}$ and $[x, y]_{\mathfrak{g}}$ are primitive, and finally

$$S([x, y]_{T(\mathfrak{g})} - [x, y]_{\mathfrak{g}}) = S([x, y]_{T(\mathfrak{g})}) - S([x, y]_{\mathfrak{g}}) = -[x, y]_{T(\mathfrak{g})} + [x, y]_{\mathfrak{g}} \in I.$$

Thus the dg-ideal I is already a dg-Hopf ideal.

A.22. The Poincaré–Birkhoff–Witt theorem

Recall A.4. If \mathfrak{g} is a Lie algebra with basis $(x_\alpha)_{\alpha \in A}$ where (A, \leq) is linearly ordered then the PBW theorem asserts that $U(\mathfrak{g})$ has as a basis the ordered monomials

$$x_{\alpha_1}^{n_1} \cdots x_{\alpha_t}^{n_t} \quad \text{where } t \geq 0, \alpha_1 < \cdots < \alpha_t \text{ and } n_i \geq 1.$$

This shows in particular that the Lie algebra homomorphism $\mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective, and it also follows that $\mathbb{P}(U(\mathfrak{g})) = \mathfrak{g}$. Moreover, $\text{gr } U(\mathfrak{g}) \cong S(\mathfrak{g})$ where $\text{gr } U(\mathfrak{g})$ denotes the associated graded for the standard filtration of $U(\mathfrak{g})$.

Theorem A.5 (dg-PBW theorem). Let \mathfrak{g} be a dg-Lie algebra with basis $(x_\alpha)_{\alpha \in A}$ consisting of homogeneous elements such that (A, \leq) is linearly ordered. Then $U(\mathfrak{g})$ has as a basis all ordered monomials

$$x_{\alpha_1} \cdots x_{\alpha_n} \quad \text{where } t \geq 0, \alpha_1 < \cdots < \alpha_t, n_i \geq 1 \text{ and } n_i = 1 \text{ if } |x_{\alpha_i}| \text{ is odd.} \quad \square$$

We will not attempt to prove this theorem here, and instead refer to [Qui69, Appendix B, Theorem 2.3] and [FHT01, §21(a)].

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