# Test Linear System Direct Method

## Problem 1

### I Crammer Method for $n=3$

For the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

In order to find $x_i$ we need to solve $x_i = \frac{\det(A_i)}{\det(A)}$, $i=\overline{1,3}$, where $A_i$ is the matrix $A$ with the $i$-th column replaced by the constant column.

Computation of a determinant of a matrix for $n=3$

Having $D = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the $\det(D)$ is

$$\det(D) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\cdot e\cdot i + d\cdot h\cdot c + g\cdot b\cdot f - b\cdot d\cdot i - g\cdot e\cdot c - a\cdot h\cdot f$$

$\Rightarrow$ we have 12 multiplication, 2 additions and 3 substractions resulting in 17 operations for the computation of a determinant of order 3

We need to compute 4 determinants so the total operations are 68.

Taking into account the last 3 divisions needed for each $x_i$ $\Big\} \Rightarrow$

$\Rightarrow$ For Crammer's method with $n=3$ we have 71 flops

## II Crammer Method $n=4$

Taking the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$$

In order to find $x_i$ we need to solve $x_i = \dfrac{\det(A_i)}{\det(A)}$, $i=\overline{1,3}$, where $A_i$ is the matrix $A$ with the i-th column replaced by the constant column.

Computation of a determinant of a matrix for $n=4$:

Having $D = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$ the $\det(D)$ is

$$\det(D) = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}$$

$$= (-1)^{1+1} a \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} + (-1)^{2+1} b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} +$$

$$+ (-1)^{3+1} c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} + (-1)^{4+1} d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

From I we know that computing a determinant for a matrix of order 3 is 17 flops. Besides we still have 8 multiplications and 3 additions.

Therefore computing one determinant of order 4 takes $17 \times 4 + 8 + 3 = 79$ flops.

For Cammer's method with $n = 4$ we have 5 determinants and 4 divisions resulting in a total of $5 \times 79 + 4 = 399$ flops

## III Gaussian method for $n = 3$

Taking the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

We have the augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

We want to obtain zeros under the main diagonal. For each of the 3 elements we need 1 multiplication and 1 addition $\Rightarrow 2 \times 3 \Rightarrow 6$ flops.

After these computations we have obtained the echelon form of the system.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ \qquad\qquad a_{22}x_2 + a_{23}x_3 = b_2 \\ \qquad\qquad\qquad\quad a_{33}x_3 = b_3 \end{cases}$$

- for $x_3$ we have 1 division = 1 flop
- for $x_2$ we have $\dfrac{b_2 - a_{23}x_3}{a_{22}}$, 1 division, 1 multiplication and 1 subtraction = 3 flops
- for $x_1$ we have $\dfrac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$, 1 division, 2 multiplication and 2 subtraction = 5 flops

For the Gaussian method with $n = 3$

$$6 + 1 + 3 + 5 = 15 \text{ flops}$$

## IV Gaussian method

Taking the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$$

We have the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right)$$

Following the logic from III for each of the elements from below the main diagonal we need 1 multiplication and 1 addition, Therefore we have for the 6 elements 12 flops.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ \qquad a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ \qquad\qquad a_{33}x_3 + a_{34}x_4 = b_3 \\ \qquad\qquad\qquad a_{44}x_4 = b_4 \end{cases}$$

- for $x_4$ we have 1 division = 1 flop
- for $x_3$ we have $\dfrac{b_3 - a_{34}x_4}{a_{33}} = 3$ flops
- for $x_2$ we have $\dfrac{b_2 - a_{23}x_3 - a_{24}x_4}{a_{22}} = 5$ flops
- for $x_1$ we have $\dfrac{b_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4}{a_{11}} = 7$ flops

For the Gaussian method with $n=4$ we have :

$$12 + 1 + 3 + 5 + 7 = 28 \text{ flops}$$

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Crammer's method
- $n = 3 \Rightarrow 71$ flops
- $n = 4 \Rightarrow 399$ flops

Gaussian method
- $n = 3 \Rightarrow 15$ flops
- $n = 4 \Rightarrow 28$ flops

In conclusion the Gaussian elimination is less complex than Crammer's method.