# Problem 10

a) Demonstrate that the repeated extrapolation in the trapezoidal rule formula from $R_{i,1}$ and $R_{i+1,1}$, gives the repeated Simpson's rule with step $h_i$ in $R_{i,i}$

The repeated trapezoidal rule is

$$R_{n,1} = \frac{h_n}{2}\left[f(a) + 2\sum_{i=1}^{2^{n-1}-1} f(a+ih_n) + f(b)\right]$$

where $h_n = \frac{b-a}{2^{n-1}}$

Richardson extrapolation formula to improve the approximation

$$R_{n,2} = \frac{4R_{n,1} - R_{n-1,1}}{3}$$

Applying trapezoidal rule approximations

$$R_{n-1,1} = \frac{h_{n-1}}{2}\left[f(a) + 2\sum_{i=1}^{2^{n-2}-1} f(a+ih_{n-1}) + f(b)\right]$$

$$R_{n,1} = \frac{h_n}{2}\left[f(a) + 2\sum_{i=1}^{2^{n-1}-1} f(a+ih_n) + f(b)\right]$$

Since $h_n = \frac{h_{n-1}}{2}$ we can express the sums in $R_{n-1,1}$ in terms of $h_n$

$$R_{n-1,1} = \frac{h_{n-1}}{2}\left[f(a) + 2\sum_{i=1}^{2^{n-2}-1} f(a+i2h_n) + f(b)\right]$$

Now we need to apply the extrapolation formula, substituting $R_{n,1}$ and $R_{n-1,1}$

$R_{n,2} =$

$$= \frac{4\left[\frac{h_n}{2}\left[f(a) + 2\sum_{i=1}^{2^{n-1}-1} f(a+ih_n) + f(b)\right]\right] - \frac{h_{n-1}}{2}\left[f(a) + 2\sum_{i=1}^{2^{n-2}-1} f(a+i2h_n) + f(b)\right]}{3}$$

$$= \frac{4\left[\frac{h_m}{2}\left[f(a) + 2\sum_{i=1}^{2^{m-1}-1} f(a+ih_m) + f(b)\right]\right] - \frac{2h_m}{2}\left[f(a) + 2\sum_{i=1}^{2^{m-2}-1} f(a+i2h_m) + f(b)\right]}{3}$$

$$= \frac{2h_m\left[f(a) + 2\sum_{i=1}^{2^{m-1}-1} f(a+ih_m) + f(b)\right] - h_m\left[f(a) + 2\sum_{i=1}^{2^{m-2}-1} f(a+i2h_m) + f(b)\right]}{3}$$

$$= \frac{h_m\left[2f(a) + 4\sum_{i=1}^{2^{m-1}-1} f(a+ih_m) + 2f(b) - f(a) + 2\sum_{i=1}^{2^{m-2}-1} f(a+i2h_m) - f(b)\right]}{3}$$

$$= \frac{h_m}{3}\left[f(a) + f(b) + 4\sum_{i=1}^{2^{m-1}-1} f(a+ih_m) + 2\sum_{i=2}^{2^{m-2}-1} f(a+(2i+1)h_m)\right]$$

now it can be observed that this matches the composite Simpson's rule formula

$$R_{m,2} = S_{m,1}$$