Calculus II

1 Parametric Equations and Polar Coordinates

1.1 Parametrization of Plane Curves

1.1.1 Parametric Equations

If x and y are given as functions

$$x = f(t), y = g(t)$$

over interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve. The equations are parametric equations for the curve.

1.2 Calculus with Parametric Curves

1.2.1 Parametric Formula for dy/dx

If all three derivative exists and $dx/dt \neq 0$ then,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

1.2.2 Parametric Formula for d^2y/dx^2

If y is twice differentiable functions of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

1.2.3 Length of Parametrically Defined Curve

If curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of** C is the definite integral

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

1.2.4 Areas of Surfaces of Revolution

If a smooth curve $x = f(t), y = g(t)$, $a \le t \le b$, is traversed exactly once as t increase from $t = a$ to $t = b$, then the areas of surfaces generated by revolving the curve about each axes are as follows.

- Revolving about the x-axis $(0 \le y)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- Revolving about the y-axis $(0 \le x)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

1.3 Polar Coordinates

1.3.1 Definition of Polar Coordinates

First, we fix point O be an **origin**(called **pole**). Then each point P can be assigning to it a **polar coordinate pair** $P(r, \theta)$ where r is directed distance between O and P, θ is directed angle between ray OP and initial ray (start from O and direction is positive part of x-axis).

1.3.2 Relations between Polar and Cartesian

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

1.4 Graphing Polar Coordinate Equations

1.4.1 Symmetry

- Symmetry about x-axis: (r, θ) is symmetric with $(r, -\theta)$ and $(-r, \pi - \theta)$.
- Symmetry about y-axis: (r, θ) is symmetric with $(r, \pi - \theta)$ and $(-r, -\theta)$.
- Symmetry about origin: (r, θ) is symmetric with $(-r, \theta)$ and $(r, \theta + \pi)$.

1.4.2 Slope

The slope of the given function of polar coordinate $r = f(\theta)$ in the Cartesian xy-plane:

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $dx/d\theta \neq 0$ at (r, θ).

1.5 Areas and Lengths in Polar Coordinates

1.5.1 Area

- Area of the region between Origin and the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

- Area of the region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$:

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 d\theta = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) d\theta$$

1.5.2 Length

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of curve is:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

1.6 Conic Curves in Cartesian Coordinates

1.6.1 Parabola

- Focus on the x-axis:

  $$y^2 = 4px$$

  where p is focal length.

- Focus on the y-axis:

  $$x^2 = 4py$$

  where p is focal length.

1.6.2 Ellipse

- Foci on the x-axis : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 (a > b)$

  Center-to-focus distance : $c = \sqrt{a^2 - b^2}$

  Foci : $(\pm c, 0)$

- Foci on the y-axis : $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 (a > b)$

  Center-to-focus distance : $c = \sqrt{a^2 - b^2}$

  Foci : $(0, \pm c)$

1.6.3 Hyperbola

- Foci on the x-axis : $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

  Center-to-focus distance : $c = \sqrt{a^2 + b^2}$

  Foci : $(\pm c, 0)$

  Asymptotes : $y = \pm \dfrac{b}{a} x$

- Foci on the y-axis : $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

  Center-to-focus distance : $c = \sqrt{a^2 + b^2}$

  Foci : $(0, \pm c)$

  Asymptotes : $y = \pm \dfrac{a}{b} x$

1.7 Conic Curves in Polar Coordinates

1.7.1 Eccentricity

- Ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $(a > b)$:

$$e = \frac{\sqrt{a^2 - b^2}}{a} < 1$$

- Hyperbola $(x^2/a^2) - (y^2/b^2) = 1$:

$$e = \frac{\sqrt{a^2 + b^2}}{a} > 1$$

- Eccentricity of parabola is $e = 1$ and eccentricity of circle is $e = 0$.

1.7.2 Polar Equation of Conic Curves

$$r = \frac{ke}{1 + e\cos\theta}$$

where $x = k > 0$ is the vertical directrix.

1.7.3 Polar Equation of Conic Curves with Semimajor Axis a

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

1.7.4 Polar Equation of Lines

If the point $P_0\left(r_0, \theta_0\right)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then the equation of L is

$$r\cos\left(\theta - \theta_0\right) = r_0.$$

1.7.5 Polar Equation of Circles

If the center lies on x-axis,

$$r = 2a\cos\theta$$

If the center lies on y-axis,

$$r = 2a\sin\theta$$

If the center lies on Origin,

$$r = a$$

2 Vectors and the Geometry of Space

2.1 Three-Dimensional Coordinate Systems

2.1.1 Distance

The **distance** between $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$ is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

2.1.2 The Standard Equation of Sphere of Radius a and center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

2.2 Vectors

2.2.1 Component Form

The vector represented by the directed segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is $|\overrightarrow{AB}|$. Two vectors are equal if they have a same length and same direction.

If v is a **n-dimensional** vector with initial point at the origin, then the **component form** of $\mathbf{v}$ is

$$\langle v_1, v_2, \ldots, v_n \rangle$$

and **magnitude** or **length** of $\mathbf{v}$ is

$$\sqrt{\sum_{i=1}^{n} {v_i}^2} = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$$

2.2.2 Vector Algebra Operations

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors and k a scalar.

Addition:	$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
Scalar Multiplication:	$k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

Let $\mathbf{u}$, $\mathbf{v}$ are vectors and a, b, c are scalars then following properties are satisfied.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5. $\mathbf{0u} = \mathbf{0}$

6. $\mathbf{1u} = \mathbf{u}$

7. $a(b\mathbf{u}) = (ab)\mathbf{u}$

8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

2.2.3 Unit Vectors

A unit vector is the vector of **length 1**.

If $\mathbf{v} \neq \mathbf{0}$ then,

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called direction of $\mathbf{v}$.

2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses v as its length and direction.

2.3 The Dot Product

2.3.1 Dot Product

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u||v| \cos\theta$$

where θ is angle between $\mathbf{u}$ and $\mathbf{v}$.

If $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal**, then

$$\mathbf{u} \cdot \mathbf{v} = 0$$

2.3.2 Properties of Dot Product

If **u**, **v** and **w** are vector and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot w$

4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

5. $\mathbf{0} \cdot \mathbf{u} = 0$

The vector projection of **u** onto **v** is the vector

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$

The scalar component of **u** in the direction **v** is the scalar

$$|\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

2.4 The Cross Product

2.4.1 The Cross Product of Two Vectors in Space

The **cross product** $\mathbf{u} \times \mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}| \sin\theta)\, \mathbf{n}$$

where unit vector **n** is perpendicular to a plane consists **u** and **v** selected by the **right-hand rule**.

Nonzero vectors **u** and **v** are parallel if and only if $\mathbf{u} \times \mathbf{v} = 0$.

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$ **2.** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ **4.** $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$ **6.** $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

2.4.2 Calculating the Cross Product

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors then,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

2.5 Lines and Planes in the Space

2.5.1 Lines and Line Segments in Space

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty$$

where $\mathbf{r}$ is the position vector of a point on L

A standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

2.5.2 Distance from a Point to a Line in Space

Distance from a point S to a line through a P parallel to $\mathbf{v}$ is

$$d = \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|}$$

2.5.3 An Equation for a Plane in Space

The plane through $P_0\left(x_0, y_0, z_0\right)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

2.5.4 The Distance from a Point to a Plane

If P is a point on the plane with normal $\mathbf{n}$, then the distance from any point S to a plane is the length of vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$:

$$d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

2.5.5 Angle between Two Planes

The angle between two planes with normal $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$

3 Vector-Valued Functions and Motion in Space

3.1 Curves in Space and Their Tangents

3.1.1 Limits and Continuity

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function with domain D and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has a limit $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous over its interval domain.

3.1.2 Derivatives and Motion

The derivative of vector function $\mathbf{r}(t) = \mathbf{f}(t)\,\mathbf{i} + \mathbf{g}(t)\,\mathbf{j} + \mathbf{h}(t)\,\mathbf{k}$ is

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

And the velocity of vector function $\mathbf{r}(t)$ is $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$, speed is $|\mathbf{v}|$ and acceleration is $\mathbf{a}(t) = \dfrac{d^2 r}{dt^2}$.

If $\mathbf{u}$ and $\mathbf{v}$ are vector function, then

$$\frac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$
$$\frac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

3.2 Arc Length in Space

3.2.1 Arc Length Along a Space Curve

The **length** of smooth curve $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b |\mathbf{v}|\,dt.$$

3.2.2 Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x(\tau)]^2 + [y(\tau)]^2 + [x(\tau)]^2}\,d\tau = \int_{t_0}^{t} |v(\tau)|\,d\tau$$

3.3 Curvature and Normal Vectors of a Curve

3.3.1 Curvature of a Plane Curve

If the $\mathbf{T} = \dfrac{d\mathbf{r}}{ds}$ is a unit vector of a smooth curve $\mathbf{r}$, then the curvature κ of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

3.3.2 Normal Vector of a Curve

At a point where $\kappa \neq 0$, the **principal unit normal vector** for a smooth curve is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

3.3.3 Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at a point P where $\kappa \neq 0$ is circle in a plane of curve that

1. is tangent to the curve at P
2. has same curvature the curve has at P
3. has center that lies toward the concave side of the curve

And radius of curvature is $\rho = \dfrac{1}{\kappa}$, center of osculating circle is

$$\overrightarrow{OP} + \rho \mathbf{N}$$

3.4 Tangential and Normal Components of Acceleration

3.4.1 Binormal Vector

If $\mathbf{T}$ is tangential unit vector and $\mathbf{N}$ is normal unit vector of curve on a point, then **binormal vector B** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

And we call together $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ the **Fernet Frame** or **TNB Frame**.

3.4.2 Tangential and Normal Components of Acceleration

If the acceleration vector is written as

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

then

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} |\mathbf{v}| \qquad \text{and} \qquad a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa |\mathbf{v}|^2$$

are the **tangential** and **normal** scalar components of acceleration.

3.4.3 Torsion

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

3.4.4 Formulas for Computing Curvature and Torsion

- Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

- Formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if} \quad \mathbf{v} \times \mathbf{a} \neq \mathbf{0})$$

3.5 Velocity and Acceleration in Polar Coordinates

3.5.1 Motion in Polar and Cylindrical Coordinates

$$\begin{aligned} &Position: && \mathbf{r} = r\mathbf{u}_r + z\mathbf{k} \\ &Velocity: && \mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \\ &Acceleration: && \mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{u}_\theta + \ddot{z}\mathbf{k} \end{aligned}$$