统计信号处理

第十一章

简单假设检验 I
（确定信号的检测）

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内容概要

- 一、匹配滤波器
- 二、广义匹配滤波器
- 三、二元通信
- 四、多元通信
- 五、小结

一、匹配滤波器

两类假设：

$$H_0: x[n]=w[n], n=0,1,\ldots,N-1$$

$$H_1: x[n]=s[n]+w[n], n=0,1,\ldots,N-1$$

其中信号$s[n]$是已知的，$w[n]$是均值为零、方差为σ^2的高斯白噪声。如何判断是否存在信号？

采用NP准则，若似然比

$$L(\mathbf{x})=\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}>\gamma$$

则判 H_1

$$p(\mathbf{x};H_1)=\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n])^2\right\}$$

$$p(\mathbf{x};H_0)=\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}$$

$$L(\mathbf{x})=\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}=\exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}s^2[n]-2\sum_{n=0}^{N-1}x[n]s[n]\right)\right\}$$

$$l(\mathbf{x})=\ln L(\mathbf{x})=-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}s^2[n]-2\sum_{n=0}^{N-1}x[n]s[n]\right)$$

$$L(x)=\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}>\gamma \quad\Rightarrow\quad l(x)>\ln\gamma$$

$$\underbrace{\sum_{n=0}^{N-1}x[n]s[n]}_{T(\mathbf{x})}>\underbrace{\sigma^2\ln\gamma+\frac{1}{2}\sum_{n=0}^{N-1}s^2[n]}_{\gamma'}$$

$$T(\mathbf{x})=\sum_{n=0}^{N-1}x[n]s[n]>\gamma'$$

- 相关器(correlator)
- 仿形相关器(replica-correlator)

● 另一种解读

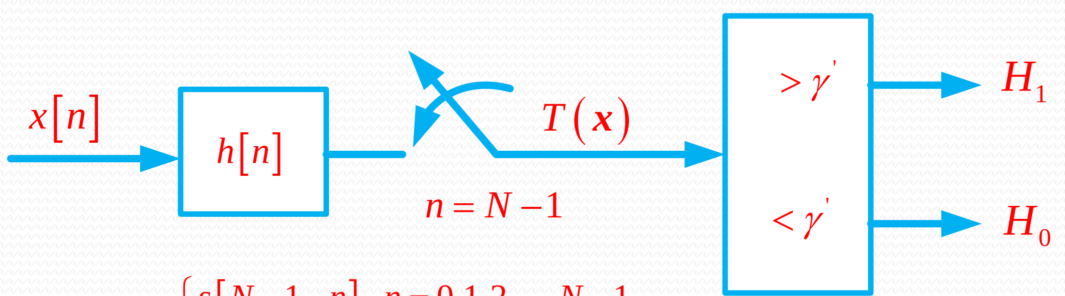
$$T(\boldsymbol{x})=\sum_{n=0}^{N-1}x[n]s[n]>\gamma'$$

- 对冲击响应为 $h[n]$（$0\le n\le N-1$ 时非零）的FIR滤波器，其输出为

$$y[n]=\sum_{k=0}^{n}x[k]h[n-k] \Rightarrow y[N-1]=\sum_{k=0}^{N-1}x[k]h[N-1-k]$$

- 若取 $h[n]=s[N-1-n]$

$$y[N-1]=\sum_{k=0}^{N-1}x[k]h[N-1-k]=\sum_{k=0}^{N-1}x[k]s[k]$$



匹配滤波器 (matched-filter)

$$h[n]=\begin{cases}s[N-1-n], & n=0,1,2,\ldots,N-1\\ 0, & others\end{cases}$$

● 从输出信噪比(output SNR)的角度

$$y[N-1]=\sum_{k=0}^{N-1}x[k]h[N-1-k]$$

输出信噪比(output SNR)：

$$\eta=\frac{E^2\left(y[N-1];H_1\right)}{\mathrm{var}\left(y[N-1];H_0\right)}=\frac{\left\{E\left(\sum_{k=0}^{N-1}\left(s[k]+w[k]\right)h[N-1-k]\right)\right\}^2}{E\left\{\left(\sum_{k=0}^{N-1}w[k]h[N-1-k]-0\right)^2\right\}}=\frac{\left(\sum_{k=0}^{N-1}s[k]h[N-1-k]\right)^2}{E\left\{\left(\sum_{k=0}^{N-1}w[k]h[N-1-k]\right)^2\right\}}$$

令 $\boldsymbol{s}=\left[s[0],s[1],\ldots,s[N-1]\right]^T$，$\boldsymbol{h}=\left[h[N-1],h[N-2],\ldots,h[0]\right]^T$

$\boldsymbol{w}=\left[w[0],w[1],\ldots,w[N-1]\right]^T$

$$\Rightarrow\quad \eta=\frac{\left(\boldsymbol{h}^T\boldsymbol{s}\right)^2}{E\left\{\left(\boldsymbol{h}^T\boldsymbol{w}\right)^2\right\}}$$

$$\eta=\frac{\left(\boldsymbol{h}^T\boldsymbol{s}\right)^2}{E\left\{\left(\boldsymbol{h}^T\boldsymbol{w}\right)^2\right\}}=\frac{\left(\boldsymbol{h}^T\boldsymbol{s}\right)^2}{E\left\{\boldsymbol{h}^T\boldsymbol{w}\boldsymbol{w}^T\boldsymbol{h}\right\}}=\frac{\left(\boldsymbol{h}^T\boldsymbol{s}\right)^2}{\boldsymbol{h}^T E\left\{\boldsymbol{w}\boldsymbol{w}^T\right\}\boldsymbol{h}}=\frac{\left(\boldsymbol{h}^T\boldsymbol{s}\right)^2}{\sigma^2\boldsymbol{h}^T\boldsymbol{h}}$$

利用Cauchy-Schwarz不等式：$\left(\boldsymbol{h}^T\boldsymbol{s}\right)^2\le\left(\boldsymbol{h}^T\boldsymbol{h}\right)\left(\boldsymbol{s}^T\boldsymbol{s}\right)$

$$\eta\le\frac{\boldsymbol{s}^T\boldsymbol{s}}{\sigma^2}$$

当且仅当$\boldsymbol{h}=c\boldsymbol{s}$ 时取等号

即 $h[n]=cs[N-1-n]$

$$y[N-1]=\sum_{k=0}^{N-1}x[k]h[N-1-k]$$

$$\eta=\frac{\left(\sum_{k=0}^{N-1}h[N-1-k]s[k]\right)^2}{E\left\{\left(\sum_{k=0}^{N-1}h[N-1-k]w[k]\right)^2\right\}}$$

——成比例的系数不会影响检测性能！

——匹配滤波使输出信噪比最大！

● 匹配滤波器的性能

$$T(\boldsymbol{x})=\sum_{n=0}^{N-1}x[n]s[n]$$

- 在 H_0 假设下

$$E\left(T(\boldsymbol{x});H_0\right)=E\left(\sum_{n=0}^{N-1}w[n]s[n]\right)=0$$

$\sum_{n=0}^{N-1}s^2[n]=\varepsilon$ ——信号功率

$$\operatorname{var}\left(T(\boldsymbol{x});H_0\right)=\operatorname{var}\left(\sum_{n=0}^{N-1}w[n]s[n]\right)=\sum_{n=0}^{N-1}\operatorname{var}\left(w[n]\right)s^2[n]=\sum_{n=0}^{N-1}\sigma^2s^2[n]=\sigma^2\varepsilon$$

- 在 H_1 假设下

$$E\left(T(\boldsymbol{x});H_1\right)=E\left(\sum_{n=0}^{N-1}(s[n]+w[n])s[n]\right)=E\left(\sum_{n=0}^{N-1}s^2[n]\right)=\varepsilon$$

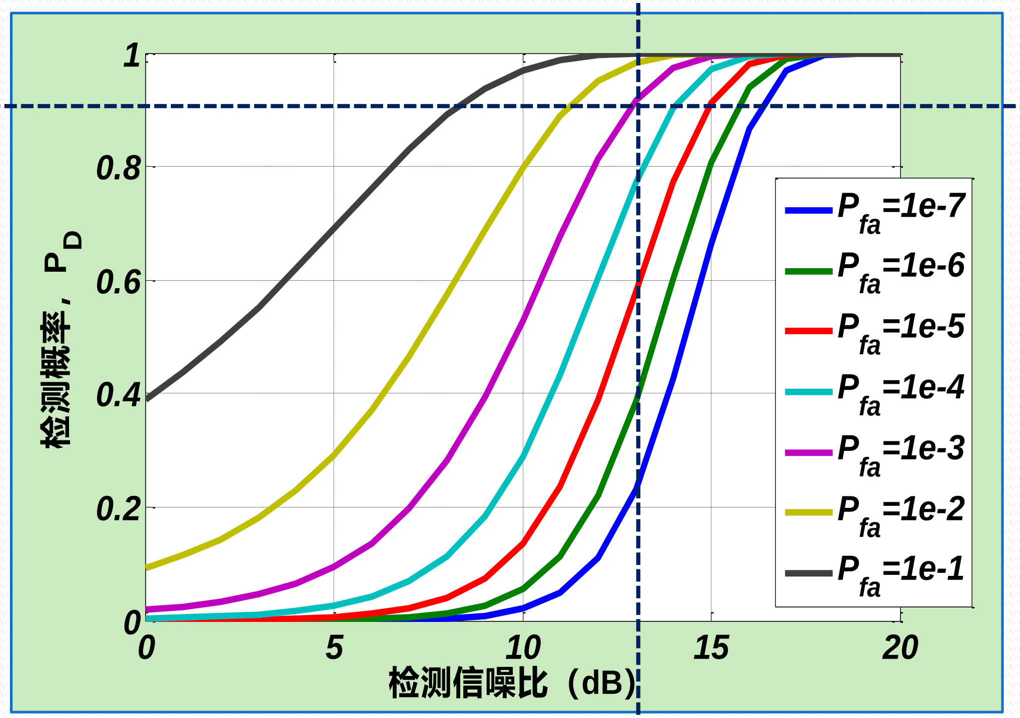
$$\operatorname{var}\left(T(\boldsymbol{x});H_1\right)=\operatorname{var}\left(\sum_{n=0}^{N-1}(s[n]+w[n])s[n]\right)=\operatorname{var}\left(\sum_{n=0}^{N-1}w[n]s[n]\right)=\sigma^2\varepsilon$$

$$T(\boldsymbol{x}) \sim \begin{cases} N\left(0, \sigma^2 \varepsilon\right), & H_0 \\ N\left(\varepsilon, \sigma^2 \varepsilon\right), & H_1 \end{cases}$$

$$\Rightarrow \left\{ \begin{array}{l} P_{fa} = \Pr\left(T(\boldsymbol{x}) > \gamma'; H_0\right) = Q\left(\gamma' / \sqrt{\sigma^2 \varepsilon}\right) \Rightarrow \gamma' = \sqrt{\sigma^2 \varepsilon} Q^{-1}\left(P_{fa}\right) \\ P_D = \Pr\left(T(\boldsymbol{x}) > \gamma'; H_1\right) = Q\left(\left(\gamma' - \varepsilon\right) / \sqrt{\sigma^2 \varepsilon}\right) \end{array} \right\}$$

$$\Rightarrow P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \frac{\varepsilon}{\sqrt{\sigma^2 \varepsilon}}\right)$$

$$= Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{\varepsilon}{\sigma^2}}\right)$$



- ✓ 对1e-3虚警率，0.9以上检测概率，所需最低信噪比约为**13dB**

检测性能：$P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{\varepsilon}{\sigma^2}}\right)$

——检测性能只与信号能量 ε 有关，与信号波形无关

- 如何改善性能？

✓ 增大 ε

$$\varepsilon = \sum_{n=0}^{N-1} s^2[n]$$

✓ 增大 $s[n]$

$$H_1 : x[n] = s[n] + w[n], n = 0,1,\ldots, N-1$$

- 增大信号发射功率 √

$$T(\mathbf{x}) = \sum_{n=0}^{N-1}\left(s[n]+w[n]\right)s[n]$$

- 增大本地信号功率 ×

✓ 增大 N √

——增加信号预检测积分时间

——实际系统改善性能常用的方式

- H_0 时

$$E\left(T(\mathbf{x});H_0\right) = E\left(\sum_{n=0}^{N-1} w[n]s[n]\right) = 0$$

$$\mathrm{var}\left(T(\mathbf{x});H_0\right) = \mathrm{var}\left(\sum_{n=0}^{N-1} w[n]s[n]\right) = \sum_{n=0}^{N-1}\mathrm{var}\left(w[n]\right)s^2[n] = \sum_{n=0}^{N-1}\sigma^2 s^2[n] = \sigma^2\varepsilon$$

- H_1 时

$$E\left(T(\mathbf{x});H_1\right) = E\left(\sum_{n=0}^{N-1}\left(s[n]+w[n]\right)s[n]\right) = E\left(\sum_{n=0}^{N-1} s^2[n]\right) = \varepsilon$$

处理增益

$$\mathrm{var}\left(T(\mathbf{x});H_1\right) = \mathrm{var}\left(\sum_{n=0}^{N-1}\left(s[n]+w[n]\right)s[n]\right) = \sum_{n=0}^{N-1}\mathrm{var}\left(w[n]\right)s^2[n] = \sigma^2\varepsilon$$

二、广义匹配滤波器

两类假设：

$$H_0: x[n]=w[n], n=0,1,\ldots,N-1$$

$$H_1: x[n]=s[n]+w[n], n=0,1,\ldots,N-1$$

其中信号$s[n]$是已知的，$w[n]$是零均值高斯有色噪声，且$\mathbf{w}\sim N(0,\mathbf{C})$如何判断是否存在信号？

采用NP准则，若似然比

$$L(\mathbf{x})=\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}>\gamma$$

则判 H_1

$$p(\mathbf{x};H_1)=\frac{1}{(2\pi)^{N/2}\det^{1/2}(\mathbf{C})}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{s})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{s})\right\}$$

$$p(\mathbf{x};H_0)=\frac{1}{(2\pi)^{N/2}\det^{1/2}(\mathbf{C})}\exp\left\{-\frac{1}{2}\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right\}$$

$$l(\mathbf{x})=\ln\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}=-\frac{1}{2}\left((\mathbf{x}-\mathbf{s})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{s})-\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right)=\mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}-\frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$$

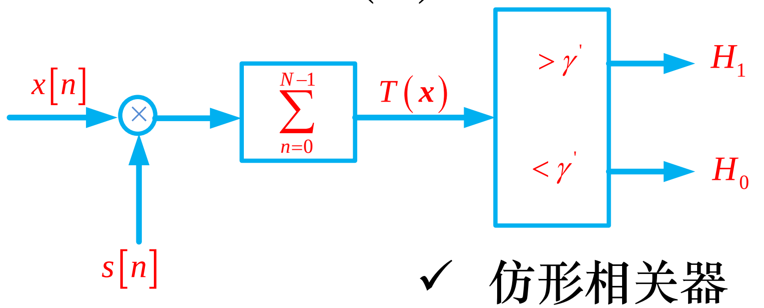
$$l(x)=\ln\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}>\ln\gamma$$

$$\underbrace{\mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}}_{T(\mathbf{x})}>\underbrace{\ln\gamma+\frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}_{\gamma'}$$

$$T(\mathbf{x})=\mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}>\gamma'$$

- 修正后的信号
- 广义仿形相关器
- 广义匹配滤波器

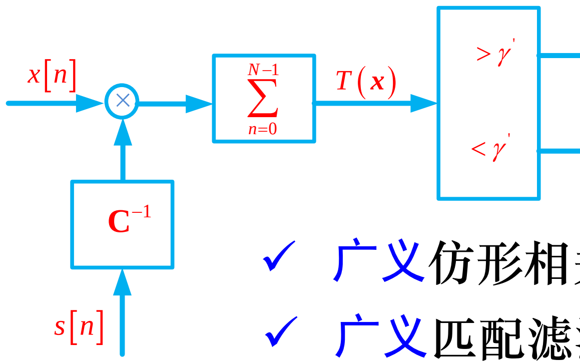
高斯白噪声时：$T(\mathbf{x}) = \mathbf{x}^T\mathbf{s}$



✓ 仿形相关器

✓ 匹配滤波器

高斯有色噪声时：$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}$



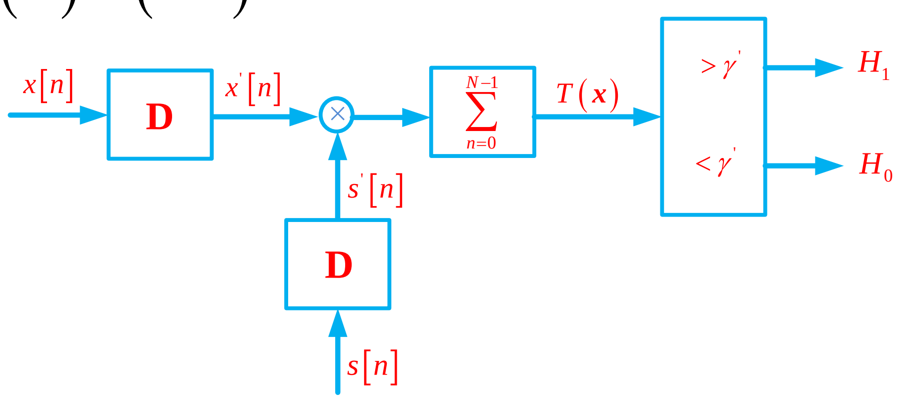
✓ 广义仿形相关器

✓ 广义匹配滤波器

令 $\mathbf{C}^{-1} = \mathbf{D}^T\mathbf{D}$ $\quad T(\mathbf{x}) = (\mathbf{D}\mathbf{x})^T\mathbf{D}\mathbf{s}$ ——匹配滤波的思想

D 称为预白化矩阵(prewhitening matrix)

$T(\mathbf{x}) = (\mathbf{D}\mathbf{x})^T\mathbf{D}\mathbf{s}$



$\mathbf{x}=\mathbf{s}+\mathbf{w}$

$\mathbf{D}\mathbf{x}=\mathbf{D}\mathbf{s}+\mathbf{D}\mathbf{w}$

• 广义匹配滤波器的性能

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s}$$

- 在 H_0 假设下

$$E\left(T(\mathbf{x}); H_0\right) = E\left(\mathbf{w}^T \mathbf{C}^{-1} \mathbf{s}\right) = 0$$

$$\operatorname{var}\left(T(\mathbf{x}); H_0\right) = \operatorname{var}\left(\mathbf{w}^T \mathbf{C}^{-1} \mathbf{s}\right) = E\left(\left(\mathbf{w}^T \mathbf{C}^{-1} \mathbf{s} - 0\right)^2\right) = E\left(\mathbf{s}^T \mathbf{C}^{-1} \mathbf{w} \mathbf{w}^T \mathbf{C}^{-1} \mathbf{s}\right) = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}$$

- 在 H_1 假设下

$$E\left(T(\mathbf{x}); H_1\right) = E\left(\left(\mathbf{s} + \mathbf{w}\right)^T \mathbf{C}^{-1} \mathbf{s}\right) = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}$$

$$\operatorname{var}\left(T(\mathbf{x}); H_1\right) = \operatorname{var}\left(\left(\mathbf{s} + \mathbf{w}\right)^T \mathbf{C}^{-1} \mathbf{s}\right) = \operatorname{var}\left(\mathbf{w}^T \mathbf{C}^{-1} \mathbf{s}\right) = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}$$

$$T(\mathbf{x}) \sim \begin{cases} N\left(0, \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}\right), & H_0 \\ N\left(\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}, \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}\right), & H_1 \end{cases}$$

$$\left.\begin{aligned} &P_{fa} = \Pr\left(T(\mathbf{x}) > \gamma^{'}; H_0\right) = Q\left(\frac{\gamma^{'}}{\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}}\right) \Rightarrow \gamma^{'} = \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\,Q^{-1}\left(P_{fa}\right) \\ &P_D = \Pr\left(T(\mathbf{x}) > \gamma^{'}; H_1\right) = Q\left(\frac{\gamma^{'} - \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}{\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}}\right) \end{aligned}\right\}$$

$$\Rightarrow P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right)$$

——检测性能不仅与信号能量有关，而且与信号波形有关

——不同于高斯白噪声时的情况

• 信号优化设计

$$\left.\begin{aligned} &\max_{\mathbf{s}} \left\{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}\right\} \\ &s.t \quad \mathbf{s}^T\mathbf{s} = \varepsilon \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &J = \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} + \lambda\left(\varepsilon - \mathbf{s}^T\mathbf{s}\right) \\ &\frac{\partial J}{\partial \mathbf{s}} = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &\underline{\mathbf{C}^{-1}\mathbf{s} = \lambda\mathbf{s}} \\ &\mathbf{s}^T\mathbf{s} = \varepsilon \end{aligned}\right\}$$

✓ $\mathbf{s}$ 为特征向量

✓ λ 为特征值

$$\Rightarrow \underline{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} = \lambda\varepsilon}$$

选择信号为 $\mathbf{C}^{-1}$ 最大特征值所对应的特征向量

● 一般线性模型下信号检测

两类假设：

$$H_0: \mathbf{x} = \mathbf{w}$$

$$H_1: \mathbf{x} = \mathbf{H}\boldsymbol{\theta}_1 + \mathbf{w}$$

其中观测矩阵 $\mathbf{H}$ 和参数矢量 $\boldsymbol{\theta}_1$ 是已知的，$\mathbf{w}$ 是零均值高斯噪声，且 $\mathbf{w} \sim N(0, \mathbf{C})$

NP检测器：$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{H}\boldsymbol{\theta}_1 \Rightarrow T(\mathbf{x}) = \left\{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}\right\}^T \boldsymbol{\theta}_1 \Rightarrow$

$$T(\mathbf{x}) = \left\{\left(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}\right)\left(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}\right\}^T \boldsymbol{\theta}_1$$

$\boldsymbol{\theta}_1$的MVU估计为：$\hat{\boldsymbol{\theta}}_1 = \left(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}$

$$\Rightarrow T(\mathbf{x}) = \hat{\boldsymbol{\theta}}_1^T\left(\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}\right)\boldsymbol{\theta}_1 = \hat{\boldsymbol{\theta}}_1^T\mathbf{C}_{\hat{\theta}_1}^{-1}\boldsymbol{\theta}_1$$

对比 $T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}$

MVU　协方差阵　真值

相同思路：

“先估计，再检测”

——估计与检测间联系

——检测理论中广泛应用

三、二元通信

二元通信信号检测问题：

$$H_0: x[n] = s_0[n] + w[n], \quad n = 0,1,\ldots,N-1$$

$$H_1: x[n] = s_1[n] + w[n], \quad n = 0,1,\ldots,N-1$$

其中信号$s_0[n], s_1[n]$假定是已知的，$w[n]$是均值为零、方差为σ^2的高斯白噪声。如何使错误概率最小化？

为使错误概率最小

$$\frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} > \frac{P(H_0)}{P(H_1)} = \gamma \text{ 时，判 } H_1$$

最小错误概率判决准则

若先验概率相同

$$p(\mathbf{x}|H_1) > p(\mathbf{x}|H_0)$$

最大似然判决准则

$$p(\mathbf{x} \mid H_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left(x[n] - s_i[n] \right)^2 \right\}$$

我们判使

$$D_i^2 = \sum_{n=0}^{N-1} \left(x[n] - s_i[n] \right)^2 = \|\mathbf{x} - \mathbf{s}_i\|^2$$

- ✓ 最小距离接收机
- ✓ 最小距离分类器（模式识别）

最小的假设 H_i 成立

$$D_i^2 = \sum_{n=0}^{N-1} x^2[n] + \underline{\sum_{n=0}^{N-1} s_i^2[n] - 2\sum_{n=0}^{N-1} x[n] s_i[n]}$$

$$T_i(\mathbf{x}) = \sum_{n=0}^{N-1} x[n] s_i[n] - \frac{1}{2} \underbrace{\sum_{n=0}^{N-1} s_i^2[n]}_{\varepsilon_i}$$

$$\boxed{T_i(\mathbf{x}) = \sum_{n=0}^{N-1} x[n] s_i[n] - \frac{1}{2} \varepsilon_i}$$

——匹配滤波的思想

——与谁匹配得好就判给谁

二元通信的性能

错误概率

$$P_e = \Pr\{判H_0, H_1为真\} + \Pr\{判H_1, H_0为真\}$$

$$= P(H_0, H_1) + P(H_1, H_0)$$

$$= P(H_0 | H_1) P(H_1) + P(H_1 | H_0) P(H_0)$$

若先验概率相同

$$P_e = \frac{1}{2}\{P(H_0 | H_1) + P(H_1 | H_0)\}$$

$$= \frac{1}{2}\{\Pr(T_0(\boldsymbol{x}) - T_1(\boldsymbol{x}) > 0 | H_1) + \Pr(T_1(\boldsymbol{x}) - T_0(\boldsymbol{x}) > 0 | H_0)\}$$

令 $T(\boldsymbol{x}) = T_1(\boldsymbol{x}) - T_0(\boldsymbol{x})$

$$P_e = \frac{1}{2}\{\Pr(T(\boldsymbol{x}) < 0 | H_1) + \Pr(T(\boldsymbol{x}) > 0 | H_0)\}$$

$$T(\boldsymbol{x}) = T_1(\boldsymbol{x}) - T_0(\boldsymbol{x})$$

$$= \left\{\sum_{n=0}^{N-1} x[n]s_1[n] - \frac{1}{2}\sum_{n=0}^{N-1} s_1^2[n]\right\} - \left\{\sum_{n=0}^{N-1} x[n]s_0[n] - \frac{1}{2}\sum_{n=0}^{N-1} s_0^2[n]\right\}$$

$$= \sum_{n=0}^{N-1} x[n]\left(s_1[n] - s_0[n]\right) - \frac{1}{2}\left(\sum_{n=0}^{N-1} s_1^2[n] - \sum_{n=0}^{N-1} s_0^2[n]\right)$$

- 在 H_0 假设下

$$E\{T(\boldsymbol{x}) \mid H_0\} = E\left\{\sum_{n=0}^{N-1}\left(s_0[n] + w[n]\right)\left(s_1[n] - s_0[n]\right) - \frac{1}{2}\left(\sum_{n=0}^{N-1} s_1^2[n] - \sum_{n=0}^{N-1} s_0^2[n]\right)\right\}$$

$$= -\frac{1}{2}\sum_{n=0}^{N-1}\left(s_1[n] - s_0[n]\right)^2 \quad = -\frac{1}{2}\left\|\boldsymbol{s}_1 - \boldsymbol{s}_0\right\|^2$$

$$\operatorname{var}\{T(\boldsymbol{x}) \mid H_0\} = \operatorname{var}\left\{\sum_{n=0}^{N-1}\left(s_0[n] + w[n]\right)\left(s_1[n] - s_0[n]\right) - \frac{1}{2}\left(\sum_{n=0}^{N-1} s_1^2[n] - \sum_{n=0}^{N-1} s_0^2[n]\right)\right\}$$

$$= \operatorname{var}\left\{\sum_{n=0}^{N-1} w[n]\left(s_1[n] - s_0[n]\right)\right\} \quad = \sum_{n=0}^{N-1}\left(s_1[n] - s_0[n]\right)^2 \operatorname{var}\left(w[n]\right)$$

$$= \sigma^2\left\|\boldsymbol{s}_1 - \boldsymbol{s}_0\right\|^2$$

- 在 H_1 假设下

$$E\{T(\mathbf{x})|H_1\}=\frac{1}{2}\|\mathbf{s}_1-\mathbf{s}_0\|^2$$

$$\operatorname{var}\{T(\mathbf{x})|H_1\}=\sigma^2\|\mathbf{s}_1-\mathbf{s}_0\|^2$$

$$\Rightarrow \quad T(\mathbf{x})\sim\begin{cases}N\left(-\frac{1}{2}\|\mathbf{s}_1-\mathbf{s}_0\|^2,\sigma^2\|\mathbf{s}_1-\mathbf{s}_0\|^2\right), & H_0\\ N\left(\frac{1}{2}\|\mathbf{s}_1-\mathbf{s}_0\|^2,\sigma^2\|\mathbf{s}_1-\mathbf{s}_0\|^2\right), & H_1\end{cases}$$

因此，错误概率为：

$$P_e=\frac{1}{2}\left\{\Pr\left(T(\mathbf{x})<0|H_1\right)+\Pr\left(T(\mathbf{x})>0|H_0\right)\right\}=\frac{1}{2}\left\{1-\Pr\left(T(\mathbf{x})>0|H_1\right)+\Pr\left(T(\mathbf{x})>0|H_0\right)\right\}$$

$$=\frac{1}{2}Q\left(\frac{\frac{1}{2}\|\mathbf{s}_1-\mathbf{s}_0\|^2}{\sqrt{\sigma^2\|\mathbf{s}_1-\mathbf{s}_0\|^2}}\right)+\frac{1}{2}Q\left(\frac{\frac{1}{2}\|\mathbf{s}_1-\mathbf{s}_0\|^2}{\sqrt{\sigma^2\|\mathbf{s}_1-\mathbf{s}_0\|^2}}\right)=Q\left(\frac{\frac{1}{2}\|\mathbf{s}_1-\mathbf{s}_0\|^2}{\sqrt{\sigma^2\|\mathbf{s}_1-\mathbf{s}_0\|^2}}\right)=Q\left(\frac{1}{2}\sqrt{\frac{\|\mathbf{s}_1-\mathbf{s}_0\|^2}{\sigma^2}}\right)$$

$$\|\mathbf{s}_1-\mathbf{s}_0\|^2=(\mathbf{s}_1-\mathbf{s}_0)^T(\mathbf{s}_1-\mathbf{s}_0)$$

$$=\underbrace{\mathbf{s}_1^T\mathbf{s}_1+\mathbf{s}_0^T\mathbf{s}_0}_{2\bar{\varepsilon}}-2\mathbf{s}_1^T\mathbf{s}_0$$

平均信号能量 $\bar{\varepsilon}=\left(\mathbf{s}_1^T\mathbf{s}_1+\mathbf{s}_0^T\mathbf{s}_0\right)/2$

$$=2\bar{\varepsilon}(1-\rho_s) \longrightarrow P_e=Q\left(\sqrt{\frac{\bar{\varepsilon}(1-\rho_s)}{2\sigma^2}}\right)$$

信号相关系数 $\rho_s=\dfrac{\mathbf{s}_1^T\mathbf{s}_0}{\left(\mathbf{s}_1^T\mathbf{s}_1+\mathbf{s}_0^T\mathbf{s}_0\right)/2}$

$$P_e = Q\left(\sqrt{\frac{\bar{\varepsilon}(1-\rho_s)}{2\sigma^2}}\right), \rho_s = \frac{\mathbf{s}_1^T\mathbf{s}_0}{\left(\mathbf{s}_1^T\mathbf{s}_1 + \mathbf{s}_0^T\mathbf{s}_0\right)/2}$$

- 信号相关系数越小，越易区分不同信号，错误概率越小
- 在信号功率一定情况下，为使 P_e 最小，应尽量减小相关系数

➢ 相移键控(PSK)

$$s_0[n] = A\cos(2\pi f_0 n),\quad n = 0,1,\ldots,N-1$$

$$s_1[n] = -A\cos(2\pi f_0 n),\quad n = 0,1,\ldots,N-1$$

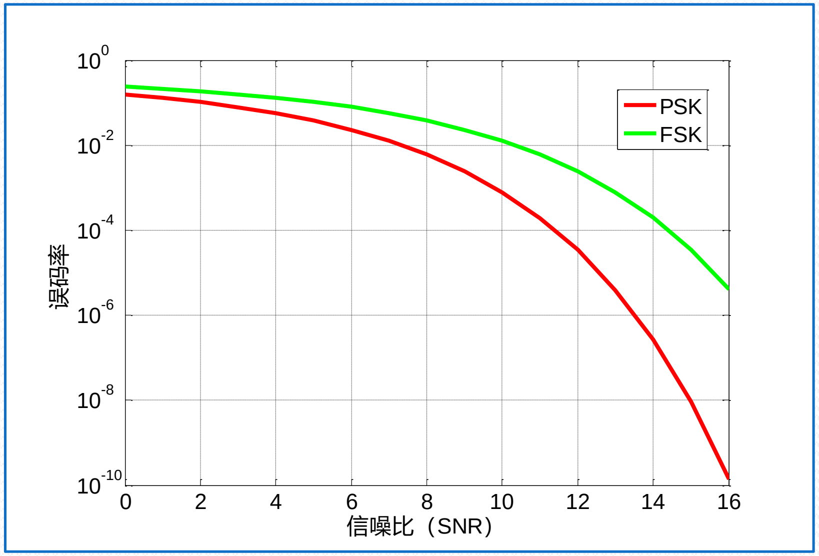
$$P_e = Q\left(\sqrt{\frac{\bar{\varepsilon}}{\sigma^2}}\right)$$

➢ 频移键控(FSK)

$$s_0[n] = A\cos(2\pi f_0 n),\quad n = 0,1,\ldots,N-1$$

$$s_1[n] = A\cos(2\pi f_1 n),\quad n = 0,1,\ldots,N-1$$

$$P_e = Q\left(\sqrt{\frac{\bar{\varepsilon}}{2\sigma^2}}\right)$$



四、多元通信

多元通信信号检测问题：

$$H_i : x[n] = s_i[n] + w[n],\ \ n = 0,1,\ldots,N-1$$

其中信号 $s_i[n]$ 假定是已知的，$w[n]$ 是均值为零、方差为 σ^2 的高斯白噪声。如何使错误概率最小化？

为使错误概率最小

$$\max_i P(H_i \mid \mathbf{x})$$ **最大后验概率判决准则**

若先验概率相同

$$\max_i P(\mathbf{x} \mid H_i)$$ **最大似然判决准则**

$$p(\mathbf{x}|H_i)=\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left\{-\frac{1}{2\sigma^2}\underline{\sum_{n=0}^{N-1}(x[n]-s_i[n])^2}\right\}$$

我们判使 $D_i^2=\sum_{n=0}^{N-1}(x[n]-s_i[n])^2$ 最小的假设 H_i 成立

$$D_i^2=\sum_{n=0}^{N-1}x^2[n]+\underline{\sum_{n=0}^{N-1}s_i^2[n]-2\sum_{n=0}^{N-1}x[n]s_i[n]}$$

$$\Rightarrow\quad T_i(\mathbf{x})=\sum_{n=0}^{N-1}x[n]s_i[n]-\frac{1}{2}\underbrace{\sum_{n=0}^{N-1}s_i^2[n]}_{\varepsilon_i}$$

$$T_i(\mathbf{x})=\sum_{n=0}^{N-1}x[n]s_i[n]-\frac{1}{2}\varepsilon_i$$

当各信号能量相等时，有

$$T_i(\mathbf{x})=\sum_{n=0}^{N-1}x[n]s_i[n]-\frac{1}{2}\varepsilon$$

● 错误概率

$$P_e=\sum_{j=0}^{M-1}\sum_{\substack{i=0\\i\neq j}}^{M-1}P\left(H_i,H_j\right)=\sum_{j=0}^{M-1}\sum_{\substack{i=0\\i\neq j}}^{M-1}P\left(H_i\mid H_j\right)P\left(H_j\right)$$

当先验概率相同时，有

$$P_e=\frac{1}{M}\sum_{j=0}^{M-1}\sum_{\substack{i=0\\i\neq j}}^{M-1}P\left(H_i\mid H_j\right)$$

$$=\frac{1}{M}\sum_{j=0}^{M-1}\Pr\left(T_j<\max\left\{T_0,\ldots T_{j-1},T_{j+1},\ldots,T_{M-1}\right\}\mid H_j\right)$$

$T_j\left(\boldsymbol{x}\right)$简记为 T_j

$$=\Pr\left(T_j<\max\left\{T_0,\ldots T_{j-1},T_{j+1},\ldots,T_{M-1}\right\}\mid H_j\right)$$

$$=\Pr\left(T_0<\max\left\{T_1,T_2,\ldots,T_{M-1}\right\}\mid H_0\right)$$

$$=1-\Pr\left(\max\left\{T_1,T_2,\ldots,T_{M-1}\right\}<T_0\mid H_0\right)$$

$$=1-\Pr\left(T_1<T_0,T_2<T_0,\ldots,T_{M-1}<T_0\mid H_0\right)$$

$$=1-\int_{-\infty}^{+\infty}\Pr\left(T_1<t,T_2<t,\ldots,T_{M-1}<t\mid T_0=t,H_0\right)p_{T_0}\left(t\right)dt$$

● 一种特例——若信号是正交的

$$\operatorname{cov}\left(T_i(\boldsymbol{x}),T_j(\boldsymbol{x})\mid H_l\right)=E\left\{\left(T_i(\boldsymbol{x})-E\left(T_i(\boldsymbol{x})\right)\right)\left(T_j(\boldsymbol{x})-E\left(T_j(\boldsymbol{x})\right)\right)\mid H_l\right\}$$

$$T_i(\boldsymbol{x})=\sum_{n=0}^{N-1}x[n]s_i[n]-\frac{1}{2}\varepsilon \Rightarrow E\left\{T_i(\boldsymbol{x})\mid H_l\right\}=E\left(\sum_{n=0}^{N-1}x[n]s_i[n]-\frac{1}{2}\varepsilon\mid H_l\right)$$

$$=E\left(\sum_{n=0}^{N-1}\left(s_l[n]+w[n]\right)s_i[n]-\frac{1}{2}\varepsilon\mid H_l\right)$$

$$=\sum_{n=0}^{N-1}s_l[n]s_i[n]-\frac{1}{2}\varepsilon$$

$$\Rightarrow \left\{T_i(\boldsymbol{x})-E\left(T_i(\boldsymbol{x})\right)\right\}\mid H_l=\sum_{n=0}^{N-1}\left(s_l[n]+w[n]\right)s_i[n]-\sum_{n=0}^{N-1}s_l[n]s_i[n]=\sum_{n=0}^{N-1}w[n]s_i[n]$$

$$\Rightarrow \operatorname{cov}\left(T_i(\boldsymbol{x}),T_j(\boldsymbol{x})\mid H_l\right)=E\left(\sum_{n=0}^{N-1}w[n]s_i[n]\sum_{m=0}^{N-1}w[m]s_j[m]\right)$$

$$=E\left(\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}w[n]w[m]s_i[n]s_j[m]\right)$$

$$\mathrm{cov}\left(T_i(\boldsymbol{x}),T_j(\boldsymbol{x})\,|\,H_l\right)=E\left(\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}w[n]w[m]s_i[n]s_j[m]\right)$$

$$=\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}E\left(w[n]w[m]\right)s_i[n]s_j[m]$$

$$=\sigma^2\sum_{n=0}^{N-1}s_i[n]s_j[n]$$

若信号是正交的，即对 $i\neq j$ 有 $\sum_{n=0}^{N-1}s_i[n]s_j[n]=0$ ，此时

$$\mathrm{cov}\left(T_i(\boldsymbol{x}),T_j(\boldsymbol{x})\,|\,H_l\right)=0$$

这说明 $T_i(\boldsymbol{x}),T_j(\boldsymbol{x})$ 不相关。又由于$T_i(\boldsymbol{x})$是高斯的，因此意味着 $T_i(\boldsymbol{x}),T_j(\boldsymbol{x})$ **相互独立**

$$P_e=1-\int_{-\infty}^{+\infty}\Pr\left(T_1<t,T_2<t,\ldots,T_{M-1}<t\,|\,T_0=t,H_0\right)p_{T_0}(t)\,dt$$

$$P_e=1-\int_{-\infty}^{+\infty}\prod_{j=1}^{M-1}\Pr\left(T_j<t\,|\,H_0\right)p_{T_0}(t)\,dt$$

在 H_0 条件下

$$T_j = \sum_{n=0}^{N-1} x[n]s_j[n] - \frac{1}{2}\varepsilon = \sum_{n=0}^{N-1}\left(s_0[n]+w[n]\right)s_j[n] - \frac{1}{2}\varepsilon \sim \begin{cases} N\left(\dfrac{\varepsilon}{2},\sigma^2\varepsilon\right), & j=0 \\ N\left(-\dfrac{\varepsilon}{2},\sigma^2\varepsilon\right), & j\neq 0 \end{cases}$$

$$\boxed{P_e = 1-\int_{-\infty}^{+\infty}\prod_{j=1}^{M-1}\Pr\left(T_j<t\,|\,H_0\right)p_{T_0}(t)\,dt}$$

$$P_e = 1-\int_{-\infty}^{+\infty}\Phi^{M-1}\left(\underbrace{\frac{t+\varepsilon/2}{\sqrt{\sigma^2\varepsilon}}}_{u}\right)\frac{1}{\sqrt{2\pi\sigma^2\varepsilon}}\exp\left\{-\frac{1}{2\sigma^2\varepsilon}\left(t-\frac{\varepsilon}{2}\right)^2\right\}dt$$

$$=1-\int_{-\infty}^{+\infty}\Phi^{M-1}(u)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(u-\sqrt{\frac{\varepsilon}{\sigma^2}}\right)^2\right\}du$$

- 错误概率，随信噪比 ε/σ^2 增加而减小
- 错误概率，随 “元” M 增加而增大——因需区分更多信号

● 无误码数据传输的极限

分组数据传输：以 L 位表示一组数据，其无误码传输的极限？

离散化后信号能量可表示为 $\varepsilon = \sum_{n=0}^{N-1} s^2[n\Delta] = \frac{LPT}{\Delta}$

P ：发射功率

T ：每“位”持续时间

Δ ：采样间隔

$$P_e = 1 - \int_{-\infty}^{+\infty} \Phi^{M-1}(u) \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(u - \sqrt{\frac{\varepsilon}{\sigma^2}} \right)^2 \right\} du$$

$$P_e = 1 - \int_{-\infty}^{+\infty} \Phi^{M-1}(u) \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(u - \sqrt{\frac{LPT}{\Delta\sigma^2}} \right)^2 \right\} du$$

可证明， 当 $\frac{PT}{\Delta\sigma^2} > 2\ln 2$ 时，$P_e \to 0$

$$\frac{PT}{\Delta\sigma^2} > 2\ln 2 \Rightarrow \frac{P\frac{1}{R}}{\Delta\sigma^2} > 2\ln 2 \Rightarrow R < \frac{P}{2\sigma^2\Delta\ln 2} = \frac{P}{2N_0 B\Delta\ln 2} = \frac{P}{2N_0 B\frac{1}{2B}\ln 2}$$

香农信道容量 $C = \lim_{B\to\infty} B\log_2\left(1 + \frac{P}{N_0 B}\right) = \frac{P}{N_0 \ln 2}$ ⟺ $= \frac{P}{N_0 \ln 2}$

五、小结

- 高斯噪声中已知信号检测问题

  - NP准则
    - **匹配滤波器**（高斯白噪声） ——匹配滤波的思想！
    - 广义匹配滤波器（高斯色噪声）

  - 最小错误概率判决准则 →(先验概率相同)→ 最大似然判决准则 → **最小距离判决（接收/分类/…)**
    - 二元通信时
    - 多元通信时

- 重点：具体应用，及与其他课程间的联系——知识体系