Linear Conic Optimization Part III

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Duality Theory of Linear Conic Programs

Content

- Definition of LCoP and LCoD
- Conjugate Duality Theory
- Deriving LCoD from LCoP
- Conic Duality Theorems for LCoP
- Duality Theorems of LP, SOCP and SDP

Linear Conic Programs

$$\begin{array}{rl} \min & c \bullet x \\ s.t. & a^i \bullet x = b_i, i = 1, \ldots, m \\ & x \in K \end{array} \qquad \text{(LCoP)}$$

where c, $a^i, b_i, i = 1, 2, \ldots, m$ are given coefficients, and K is a closed convex cone, such as $\mathbb{R}^n_+$, $\mathcal{L}^n$, $\mathcal{S}^n_+$, etc.

$$\begin{array}{rl} \max & b^T y \\ s.t. & \sum_{i=1}^m y_i a^i + s = c \\ & s \in K^*,\ y \in \mathbb{R}^m \end{array} \qquad \text{(LCoD)}$$

where K^* is the dual cone of K.

Conjugate Duality Theory

Conjugate Program

$$
\begin{array}{ll} \inf & f(x) \\ s.t. & x \in \mathcal{X} \cap K \end{array} \qquad \text{(CP)}
$$

where $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ and K is a cone in $\mathbb{R}^n$.

Conjugate Dual

$$
\begin{array}{ll} \inf & f^*(y) \\ s.t. & y \in \mathcal{Y} \cap K^* \end{array} \qquad \text{(CD)}
$$

where $f^* : \mathcal{Y}$ is the conjugate transform of $f : \mathcal{X}$ and K^* is the dual cone of K.

- feas(*) denotes the feasible domain of problem (*)
- opt(*) denotes the optimal solution set of problem (*)
- v(*) denotes the optimal value of problem (*)

How to Get the Dual?–LP

$$\begin{array}{rl} \min & c^T x \\ s.t. & Ax = b \\ & x \in \mathbb{R}^n_+ \end{array}$$

$$\mathcal{X} = \{x \in \mathbb{R}^n | Ax = b\}, \mathcal{K} = \mathbb{R}^n_+.$$

$$Ax = b \Leftrightarrow (B, N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \Leftrightarrow x_B = B^{-1}(b - N x_N).$$

$$\begin{aligned} f^*(z) = & \sup_{x\in\mathcal{X}} (x^T z - c^T x) = \sup_{x\in\mathcal{X}} (z-c)_B^T x_B + (z-c)_N^T x_N \\ = & \sup_{x_N \in \mathbb{R}^{n-m}} (z-c)_B^T B^{-1} b + [(z-c)_N - N^T (B^{-1})^T (z-c)_B]^T x_N \\ = & \begin{cases} (z-c)_B^T B^{-1} b, & (z-c)_N - N^T (B^{-1})^T (z-c)_B = 0 \\ +\infty, & otherwise \end{cases} \end{aligned}$$

$$\mathcal{Y} = \{z \in \mathbb{R}^n | (z-c)_N - N^T (B^{-1})^T (z-c)_B = 0\}, \mathcal{K}^* = \mathbb{R}^n_+.$$

$$(z-c)_N - N^T (B^{-1})^T (z-c)_B = 0 \Leftrightarrow \begin{pmatrix} (z-c)_B \\ (z-c)_N \end{pmatrix} - (B, N)^T (B^{-1})^T (z-c)_B = 0.$$

How to Get the Dual?–LP

Let $w = (B^{-1})^T (z - c)_B$.

$$\begin{array}{rl} \inf & b^T w \\ s.t. & w = (B^{-1})^T (z-c)_B \\ & z - c - A^T w = 0 \\ & z \in \mathbb{R}^n_+, w \in \mathbb{R}^m. \end{array}$$

$w = (B^{-1})^T (z - c)_B$ is redundant. Let $y = -w$.

$$\begin{array}{rl} -\max & b^T y \\ s.t. & A^T y + z = c \\ & z \in \mathbb{R}^n_+, y \in \mathbb{R}^m. \end{array}$$

Note: There is a negative sign for the dual problems.

Conjugate Duality Theory

Theorem (Conjugate duality theorem/KKT duality theorem)

If $x \in \text{feas}(\text{CP})$ and $y \in \text{feas}(\text{CD})$, then

$$0 \leq x \bullet y \leq f(x) + f^*(y)$$

with the equality holding if and only if

$$x \bullet y = 0 \text{ and } y \in \partial f(x),$$

in which case

$$x \in \text{opt}(\text{CP}) \text{ and } y \in \text{opt}(\text{CD}).$$

Proof

The inequality follows from Fenchel's inequality and the definition of dual cone. The rest follows easily.

Conjugate Duality Theory

Theorem (Weak duality theorem)

If both CP and CD are feasible, then

(i) $v(\mathrm{CP})$ is finite and

$$v(\mathrm{CP}) + f^*(y) \geq 0, \forall\, y \in \mathrm{feas}(\mathrm{CD});$$

(ii) $v(\mathrm{CD})$ is finite and

$$v(\mathrm{CP}) + v(\mathrm{CD}) \geq 0.$$

Proof

This theorem follows from the previous KKT duality theorem.

Conjugate Duality Theory

Theorem (Fenchel's theorem/Strong duality theorem)

Suppose that $f : \mathcal{X}$ and K are closed and convex. If $v(\mathrm{CD})$ is finite and one of the following conditions holds:

(i) $\mathrm{ri}(K^*) \cap \mathrm{ri}(\mathcal{Y}) \neq \emptyset$,

(ii) both K^* and $\mathcal{Y}$ are polyhedrons,

then

$$v(\mathrm{CP}) + v(\mathrm{CD}) = 0 \text{ and } \mathrm{opt}(\mathrm{CP}) \neq \emptyset.$$

Similarly, if $v(\mathrm{CP})$ is finite and one of the following conditions holds:

(i) $\mathrm{ri}(K) \cap \mathrm{ri}(\mathcal{X}) \neq \emptyset$,

(ii) both K and $\mathcal{X}$ are polyhedrons,

then

$$v(\mathrm{CP}) + v(\mathrm{CD}) = 0 \text{ and } \mathrm{opt}(\mathrm{CD}) \neq \emptyset.$$

Proof: See Xing and Fang's book "Introduction to Linear Conic Optimization" Theorem 4.23 and Theorem 4.24.

Deriving LCoD from LCoP

LCoP

$$\begin{array}{ll} \min & c \bullet x \\ s.t. & a^i \bullet x = b_i, i = 1, \ldots, m \\ & x \in K \end{array} \qquad \text{(LCoP)}$$

Deriving LCoD in the framework of conjugate program.

Deriving LCoD from LCoP

LCoP as CP

Variables: $u^T = (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{m+1}$;

$f(u) = u_0$;

$\mathcal{X} = \{u \in \mathbb{R}^{m+1} | u_i = b_i, i = 1, \ldots, m\}$;

$K_0 = \{u \in \mathbb{R}^{m+1} | u_0 = c \bullet x, u_i = a^i \bullet x, x \in K, i = 1, \ldots, m\}$.

$$\begin{array}{ll} \inf & f(u) \\ s.t. & u \in \mathcal{X} \cap K_0 \end{array}$$

Deriving LCoD from LCoP

Corresponding CD

Variables: $v^T = (v_0, v_1, \ldots, v_m) \in \mathbb{R}^{m+1}$;

$$
\begin{aligned}
f^*(v) &= \sup_{u \in \mathcal{X}} \{u \bullet v - f(u)\} < +\infty \\
&= \sup_{u_0 \in \mathbb{R}} \{(v_0 - 1)u_0 + \sum_{i=1}^{m} b_i v_i\}
\end{aligned}
$$

Hence

$f^*(v) = \sum_{i=1}^m b_i v_i$;

$\mathcal{Y} = \{v \in \mathbb{R}^{m+1} | v_0 = 1\}$;

Deriving LCoD from LCoP

Corresponding CD

Moreover,

$$\begin{aligned} K_0^* &= \{v \in \mathbb{R}^{m+1} | v \bullet u \geq 0, \forall u \in K_0\} \\ &= \{v \in \mathbb{R}^{m+1} | (v_0 c + \textstyle\sum_{i=1}^m v_i a^i) \bullet x \geq 0, \forall x \in K\} \\ &= \{v \in \mathbb{R}^{m+1} | v_0 c + \textstyle\sum_{i=1}^m v_i a^i \in K^*\}. \end{aligned}$$

$$\mathcal{Y} \cap K_0^* = \{v \in \mathbb{R}^{m+1} | c + \sum_{i=1}^m v_i a^i = s, s \in K^*\}.$$

$$\begin{array}{ll} \inf & \sum_{i=1}^m b_i v_i \\ s.t. & c + \sum_{i=1}^m v_i a^i = s \\ & s \in K^* \end{array}$$

Deriving LCoD from LCoP

CD to LCoD

Define variables: $y = -(v_1, \ldots, v_m)^T$, we have

$$\begin{array}{ll} \max & b^T y \\ s.t. & \sum_{i=1}^m y_i a^i + s = c \\ & s \in K^*, \; y \in \mathbb{R}^m \end{array} \qquad \text{(LCoD)}$$

Therefore, the duality theorems of conjugate programs may apply to LCoP.

Conic Duality Theorems for LCoP

Theorem (Weak duality theorem)

If both LCoP and LCoD are feasible, then

$$c \bullet x \geq b^T y, \forall x \in \text{feas(LCoP)}, \ (y, s) \in \text{feas(LCoD)}.$$

Theorem (Strong duality theorem)

(i) If $\text{feas(LCoP)} \cap \text{int}(K) \neq \emptyset$ and $v(\text{LCoP})$ is finite, then there exists $(y^*, s^*) \in \text{feas(LCoD)}$ such that $b^T y^* = v(\text{LCoP})$.

(ii) If $\text{feas(LCoD)} \cap \text{int}(K^*) \neq \emptyset$ and $v(\text{LCoD})$ is finite, then there exists $x^* \in \text{feas(LCoP)}$ such that $c \bullet x = v(\text{LCoD})$.

Proof: Applications of Fenchel's theorem/Strong duality theorem.

Conic Duality Theorems for LCoP

Theorem (KKT duality theorem)

If $\mathrm{feas(LCoP)}$ and $\mathrm{feas(LCoD)}$ are both nonempty and $\mathrm{feas(LCoP)} \cap \mathrm{int}(K) \neq \emptyset$, then x^* is optimal for LCoP if and only if the following conditions hold:

(i) $x^* \in \mathrm{feas(LCoP)}$;

(ii) There exists $(y^*, s^*) \in \mathrm{feas(LCoD)}$;

(iii) $c \bullet x^* = b^T y^*$ (or equivalently $x^* \bullet s^* = c \bullet x^* - b^T y^* = 0$).

Proof: $\Longrightarrow$ follows from strong duality theorem.

$\Longleftarrow$ is obvious.

Linear Program (LP)

$$\begin{array}{ll} \min & c^T x \\ s.t. & Ax = b \\ & x \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LP)}$$

$$\begin{array}{ll} \max & b^T y \\ s.t. & A^T y + s = c \\ & s \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LD)}$$

Linear Program (LP)

Theorem (LP duality theorem)

(i) If either LP or LD is unbounded, then the other one is infeasible.

(ii) If either $v(\mathrm{LP})$ or $v(\mathrm{LD})$ is finite, then there exist $x^* \in \mathrm{feas}(\mathrm{LP})$ and $(y^*, s^*) \in \mathrm{feas}(\mathrm{LD})$ such that $v(\mathrm{LP}) = c^T x^* = b^T y^* = v(\mathrm{LD})$.

(iii) If LP is feasible and $v(\mathrm{LP})$ is finite, then x^* is optimal for LP if and only if the following conditions hold:

(a) $Ax^* = b$, $x^* \geq_{\mathbb{R}^n_+} 0$;

(b) there exists (y^*, s^*) satisfying $A^T y^* + s^* = c$, $s \geq_{\mathbb{R}^n_+} 0$;

(c) $(x^*)^T s^* = c^T x^* - b^T y^* = 0$.

Second Order Cone Program (SOCP)

$$
\begin{array}{rl}
\min & c^T x \\
s.t. & Ax = b \\
 & x \geq_K 0
\end{array} \qquad \text{(SOCP)}
$$

where $K = \mathcal{L}^{n_1} \times \cdots \times \mathcal{L}^{n_r} = \{x \in \mathbb{R}^n | n_1 + \cdots + n_r = n,\ (x_1, ..., x_{n_1})^T \in \mathcal{L}^{n_1}\ , ...,\ (x_{n-n_r+1}, ..., x_n)^T \in \mathcal{L}^{n_r}\}, n_i \geq 1, i = 1, 2, \ldots, r$.

$$
\begin{array}{rl}
\max & b^T y \\
s.t. & A^T y + s = c \\
 & s \geq_K 0
\end{array} \qquad \text{(SOCD)}
$$

Second Order Cone Program (SOCP)

Theorem (SOCP duality theorem)

(i) If either SOCP or SOCD is unbounded, then the other one is infeasible.

(ii) If there exists a feasible solution $\bar{x}$ such that $\bar{x} \in \mathrm{int}(K)$, and $v(\mathrm{SOCP})$ is finite, then there exist $(y^*, s^*) \in \mathrm{feas}(\mathrm{SOCD})$ such that $v(\mathrm{SOCP}) = b^T y^* = v(\mathrm{SOCD})$.

(iii) If there exists a feasible solution $(\bar{y}, \bar{s})$ such that $\bar{s} \in \mathrm{int}(K)$, and $v(\mathrm{SOCD})$ is finite, then there exist $x^* \in \mathrm{feas}(\mathrm{SOCP})$ such that $v(\mathrm{SOCP}) = c^T x^* = v(\mathrm{SOCD})$.

Second Order Cone Program (SOCP)

Theorem (SOCP duality theorem)

(iv) If both SOCP and SOCD are feasible, and there exists a feasible solution $\bar{x}$ such that $\bar{x} \in \mathrm{int}(K)$, then x^* is optimal for SOCP if and only if the following conditions hold:

(a) $Ax^* = b$, $x^* \geq_K 0$;

(b) there exists (y^*, s^*) satisfying $A^T y^* + s^* = c$, $s^* \geq_K 0$;

(c) $(x^*)^T s^* = c^T x^* - b^T y^* = 0$.

Difference between LP and SOCP (interior feasible solution):

$$\begin{array}{ll} \min & -x_2 \\ s.t. & x_1 - x_3 = 0 \\ & x \in \mathcal{L}^3 \end{array} \qquad \begin{array}{ll} \max & 0 \cdot y \\ s.t. & \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - y \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -y \\ -1 \\ y \end{bmatrix} \in \mathcal{L}^3 \end{array}$$

$v(SOCP) = 0$ but SOCD is infeasible.

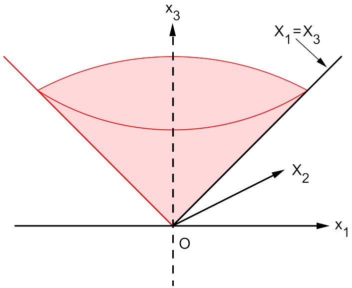

Figure: Feasible domain is a ray $x_1 = x_3$ in hyperplane $x_2 = 0$. No feasible interior point.

Second Order Cone Program (SOCP)

Finite nonzero duality gap:

$$
\begin{array}{rl}
\min & -x_2 \\
s.t. & x_1 + x_3 - x_4 + x_5 = 0 \\
 & x_2 + x_4 = 1 \\
 & x \in \mathcal{L}^3 \times \mathcal{L}^2
\end{array}
$$

$$
\begin{array}{rllll}
\max & & y_2 & & \\
 & y_1 & & +s_1 & = 0 \\
 & & y_2 & +s_2 & = -1 \\
s.t. & y_1 & & +s_3 & = 0 \\
 & -y_1 & +y_2 & +s_4 & = 0 \\
 & y_1 & & +s_5 & = 0 \\
 & & & s \in \mathcal{L}^3 \times \mathcal{L}^2 &
\end{array}
$$

$$
x^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad y^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad s^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

$$
v(\mathrm{SOCP}) = 0 \neq -1 = v(\mathrm{SOCD})
$$

Second Order Cone Program (SOCP)

Zero duality gap with non-attainable value:

$$
\begin{array}{llll}
\min & x_1 & & \\
s.t. & -x_2 & -x_3 & = 0 \\
 & x_2 & & = -1 \\
 & & x \in \mathcal{L}^3 &
\end{array}
\qquad
\begin{array}{lllll}
\max & & -y_2 & & \\
s.t. & & & s_1 & = 1 \\
 & -y_1 & +y_2 & +s_2 & = 0 \\
 & -y_1 & & +s_3 & = 0 \\
 & & s \in \mathcal{L}^3 & &
\end{array}
$$

$$
x^* = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad v(SOCD) = 0 \text{ but not attainable.}
$$

Let $y_1 = k, y_2 = \frac{1}{k}, k \geq 1.$

$$
\sqrt{1 + (y_1 - y_2)^2} = \sqrt{k^2 - 1 + \frac{1}{k^2}} \leq k.
$$

Positive Semidefinite Program (SDP)

$$\begin{aligned} \min \quad & C \bullet X \\ s.t. \quad & \mathcal{A}X = b \\ & X \succeq 0 \end{aligned} \qquad \text{(SDP)}$$

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & \mathcal{A}^* y + S = C \\ & S \succeq 0 \end{aligned} \qquad \text{(SDD)}$$

Note:

$$\mathcal{A}^* y = \sum_{i=1}^{m} y_i A_i$$

Positive Semidefinite Program (SDP)

Theorem (SDP duality theorem)

(i) If either SDP or SDD is unbounded, then the other one is infeasible.

(ii) If there exists a feasible solution $\bar{X}$ such that $\bar{X} \succ 0$, and $v(\mathrm{SDP})$ is finite, then there exist $(y^*, S^*) \in \mathrm{feas}(\mathrm{SDD})$ such that $v(\mathrm{SDP}) = b^T y^* = v(\mathrm{SDD})$.

(iii) If there exists a feasible solution $(\bar{y}, \bar{S})$ such that $\bar{S} \succ 0$, and $v(\mathrm{SDD})$ is finite, then there exist $X^* \in \mathrm{feas}(\mathrm{SDP})$ such that $v(\mathrm{SDP}) = C \bullet X^* = v(\mathrm{SDD})$.

Positive Semidefinite Program (SDP)

Theorem (SDP duality theorem)

(iv) If both SDP and SDD are feasible, and there exists a feasible solution $\bar{X}$ such that $\bar{X} \succ 0$, then X^* is optimal for SDP if and only if the following conditions hold:

(a) $\mathcal{A}X^* = b$, $X^* \succeq 0$;

(b) there exists (y^*, S^*) satisfying $\mathcal{A}^* y^* + S^* = C$, $S^* \succeq 0$;

(c) $X^* \bullet S^* = C \bullet X^* - b^T y^* = 0$.

Positive Semidefinite Program (SDP)

Interior feasible solution

Infinite duality gap:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b = 0$$

$X^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and SDD is infeasible.

Zero duality gap with non-attainable value:

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad b = 1$$

$v(SDP) = 0$ but is not attainable. $y^* = 0$ and $S^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Positive Semidefinite Program (SDP)

Finite nonzero duality gap:

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, y^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, S^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v(SDP) = 0 \neq -1 = v(SDD)$$