无研773 刘子源 ~~2022~~ 2022310709,

11.1

$$p(x[n]|\mu)=\frac{1}{\sqrt{2\pi\delta^2}}\exp\left\{-\frac{1}{2\delta^2}(x[n]-\mu)^2\right\}$$

$\mu\sim N(\mu_0,\delta_0^2)$

MMSE: $\hat{\mu}_{MMSE}=\frac{\delta_0^2}{\delta_0^2+\frac{\delta^2}{N}}\bar{x}+\frac{\frac{\delta^2}{N}}{\delta_0^2+\frac{\delta^2}{N}}\mu_0$

MAP: 后验概率的PDF就是Gaussian的，则

$$\hat{\mu}_{MAP}=\hat{\mu}_{MMSE}$$

$\delta_0^2\to0$ 时 $\hat{\mu}\to\mu_0$ 先验主导

$\delta_0^2\to\infty$ 时，$\hat{\mu}\to\bar{x}$，观测主导

11.3 $p(\theta|x)=\begin{cases}\exp[-(\theta-x)], & \theta>x\\ 0, & \theta<x\end{cases}$

MMSE: $\hat{\theta}=\int_x^{\infty}\theta e^{-(\theta-x)}d\theta=x+1$

MAP: $\hat{\theta}=\arg\max_{\theta}p(\theta|x)=x$.

11.1, 3, 4, 9, 16

11.4. $p(A)=\begin{cases}\lambda e^{-\lambda A}. & A>0\\ 0 & 其他\end{cases}$ $x[n]=A+w[n]$ $w[n]\sim N(0,\delta^2)$

~~$p(A|x)=p(A)p(x|A)$~~

$\hat{A}=\arg\max_{A}p(A)p(x|A)$

$=\arg\max_{A}(\ln p(A)+\ln p(x|A))\triangleq\arg\max_{A}l(A)$

$$l(A)=\begin{cases}-\frac{N}{2}\ln(2\pi\delta^2)-\frac{1}{2\delta^2}\sum_n(x[n]-A)^2+\ln\lambda-\lambda A & (A>0)\\ 0 & (A<0)\end{cases}$$

$$\frac{\partial l(A)}{\partial A}=\frac{1}{\delta^2}\sum_n(x[n]-A)-\lambda=0$$

$$\Rightarrow\hat{A}=\begin{cases}\bar{x}-\frac{\delta^2}{N}\lambda, & \hat{A}>0\\ 0 & \hat{A}\le0\end{cases}$$

即 $\hat{A}=\max\{\bar{x}-\frac{\delta^2}{N}\lambda,\ 0\}$

11.9

$$\Theta[n] = \begin{bmatrix} x[n] \\ y[n] \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} x[n-1] + v_x \\ y[n-1] + v_y \\ v_x \\ v_y \end{bmatrix}$$

因为 $x[n] - x[n-1] = v_x$，$y[n] - y[n-1] = v_y$

则 $$\Theta[n] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x[n-1] \\ y[n-1] \\ v_x \\ v_y \end{bmatrix} = \cancel{A\Theta}\, A\Theta[n-1]$$

即 $$\hat{\Theta}[n] = A^n \hat{\Theta}[0], \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\hat{A}$ 得到的益处更多，因为 $\sum n^2$ 对 Bmse 造成的衰减比 N 更大。

11.16 $x[n] = A + Bn + w[n]$，$-M \le n \le M$

$$\begin{bmatrix} A \\ B \end{bmatrix} \sim N\left(\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix} \right), \quad w[n] \sim N(0, \sigma^2)$$

由书上 (11.33) 可知

$$\hat{\theta} = \mu_\theta + (C_\theta^{-1} + H^T C_w^{-1} H)^{-1} H^T C_w^{-1} (x - H\mu_\theta)$$

$$\mu_\theta = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \quad C_\theta = \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & -M \\ 1 & -M+1 \\ \vdots & \vdots \\ 1 & M \end{bmatrix} \quad C_w = \sigma^2 I$$

则 $$H^T C_w^{-1} H + C_\theta^{-1} = \begin{bmatrix} \frac{1}{\sigma_A^2} + \frac{N}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma_B^2} + \frac{\sum n^2}{\sigma^2} \end{bmatrix}$$

$$(H^T C_w^{-1} H + C_\theta^{-1})^{-1} = \begin{bmatrix} \frac{1}{\frac{1}{\sigma_A^2} + \frac{N}{\sigma^2}} & 0 \\ 0 & \frac{1}{\frac{1}{\sigma_B^2} + \frac{\sum n^2}{\sigma^2}} \end{bmatrix}$$

$$\hat{A} = A_0 + \frac{\frac{N}{\sigma^2}}{\frac{1}{\sigma_A^2} + \frac{N}{\sigma^2}} (\bar{x} - A_0)$$

$$\mathrm{Bmse}(\hat{A}) = 1 \Big/ \left(\frac{1}{\sigma_A^2} + \frac{N}{\sigma^2} \right)$$

$$\hat{B} = B_0 + \frac{\sum_{-M}^{M} n^2 / \sigma^2}{\frac{1}{\sigma_B^2} + \frac{\sum n^2}{\sigma^2}} \left[\frac{\sum n\, x[n]}{\sum n^2} - B_0 \right]$$

$$\mathrm{Bmse}(\hat{B}) = 1 \Big/ \left(\frac{1}{\sigma_B^2} + \frac{\sum n^2}{\sigma^2} \right)$$