Chapter 1

1.1) $P_0 = Pr\{ x(0) > \frac{1}{2}; \mathcal{H}_0 \}$

$$= Pr\{ w(0) > \tfrac{1}{2} \} = \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}t^2} dt$$

Let $u = t/\sigma \Rightarrow$

$$P_0 = \int_{\frac{1}{2\sigma}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}u^2} \sigma \, du$$

$$= \int_{\frac{1}{2\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= \Phi(\infty) - \Phi\left(\frac{1}{2\sigma}\right)$$

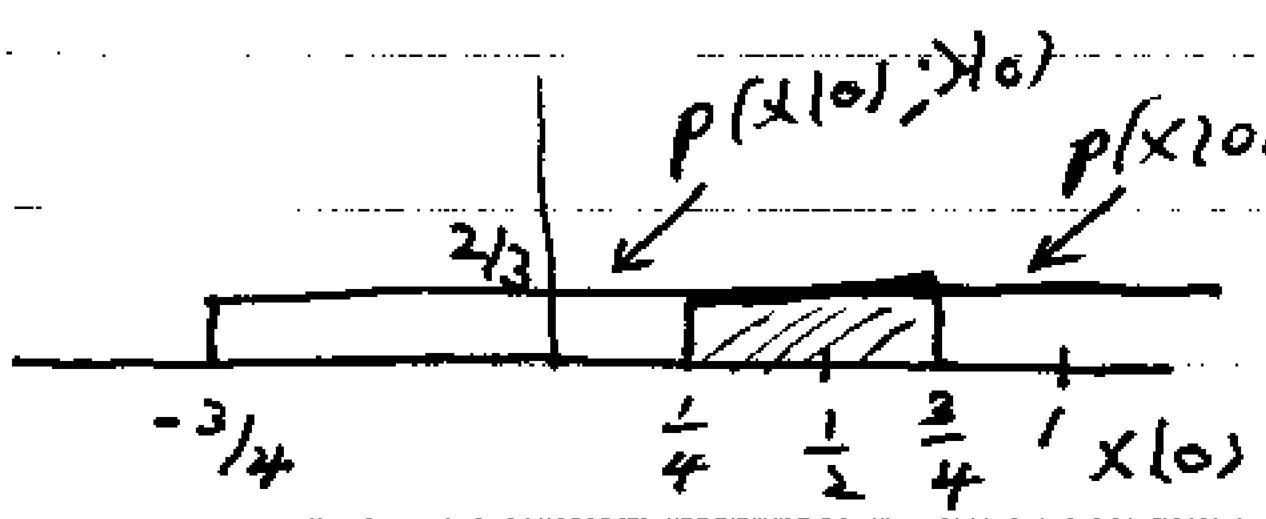
when $\Phi(x)$ is the CDF for a Gaussian random variable with mean zero and variance one.

$$P_0 = 1 - \Phi\left(\frac{1}{2\sigma}\right) = 0.001$$

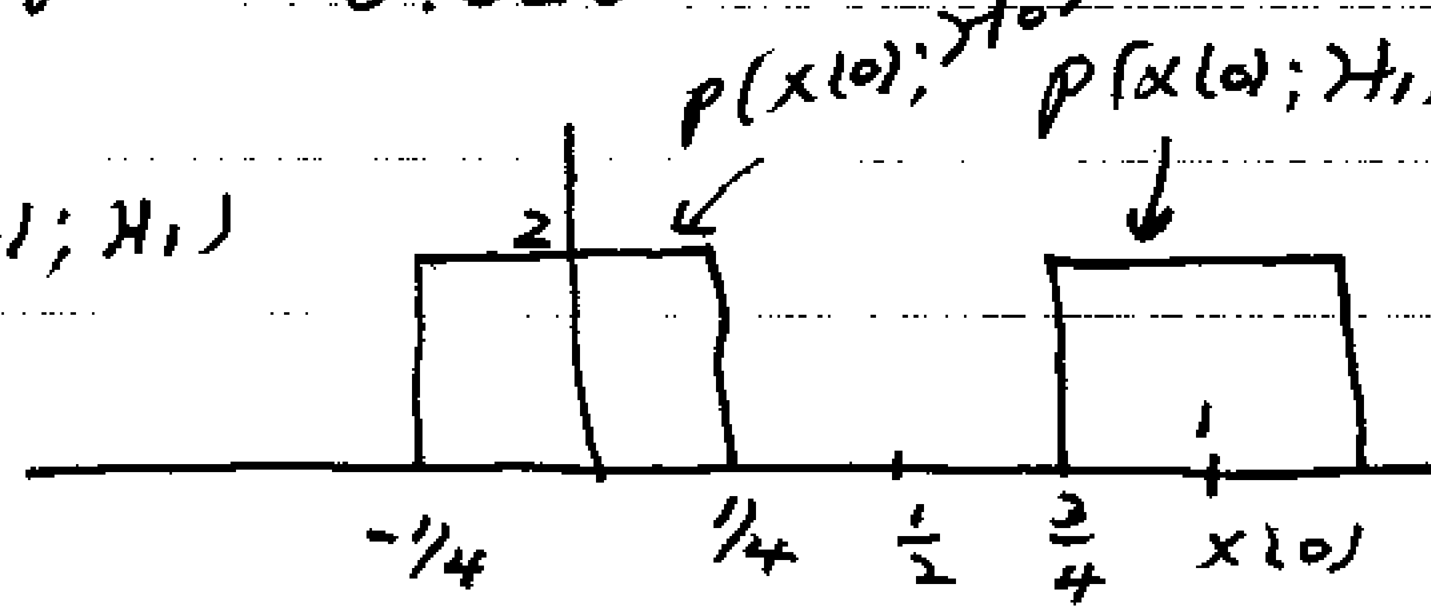
From Tables or using MATLAB program Qinv.m in Appendix 2C (y = Qinv(0.001))

$$\Rightarrow \frac{1}{2\sigma} = 3.09 \Rightarrow \sigma^2 = 0.026$$

1.2)



$a = 3/4$



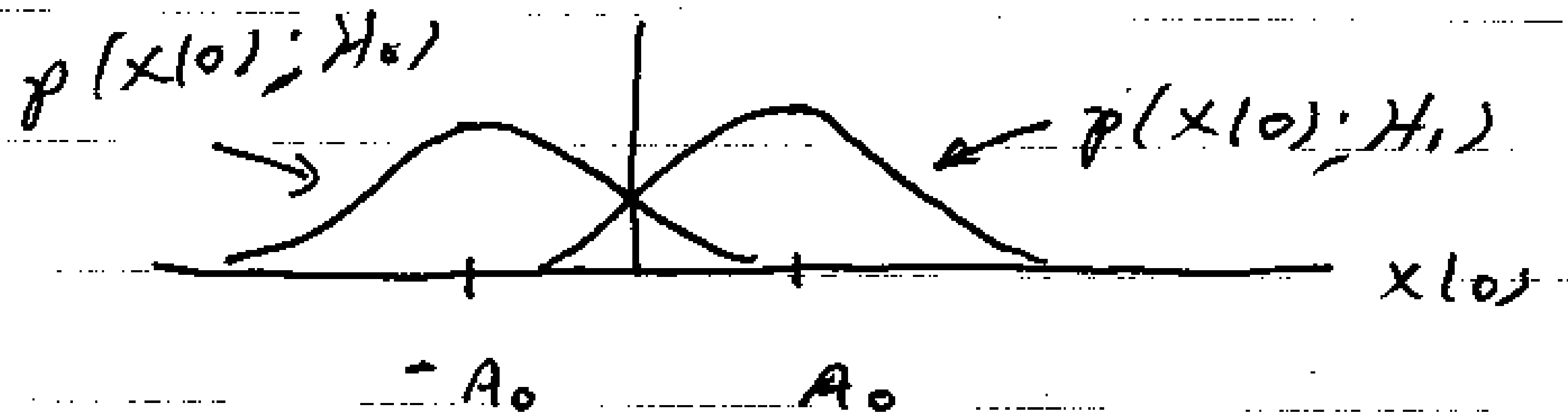
$a = 1/4$

If $a < \frac{1}{2}$, there is no overlap and thus we have perfect discrimination. Otherwise, not.

1.3) $\mathcal{H}_0 : A = -A_0$

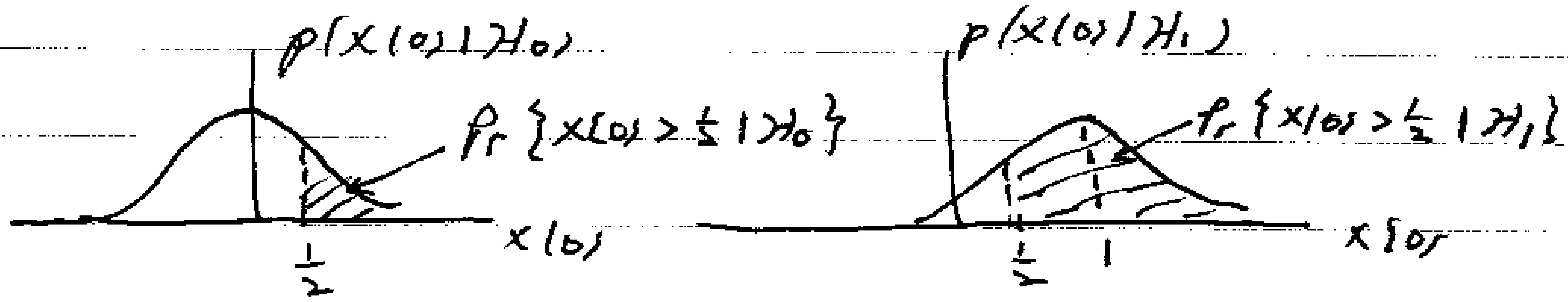
$\mathcal{H}_1 : A = A_0$ for $A_0 > 0$

$$p(x[0]) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x[0]-A)^2}$$



Makes sense to decide $\mathcal{H}_1$ if $x[0] > 0$. As A_0 increases, the performance will improve since the PDFs will overlap less.

1.4) $P(\mathcal{H}_0) = P(\mathcal{H}_1) = \frac{1}{2}$



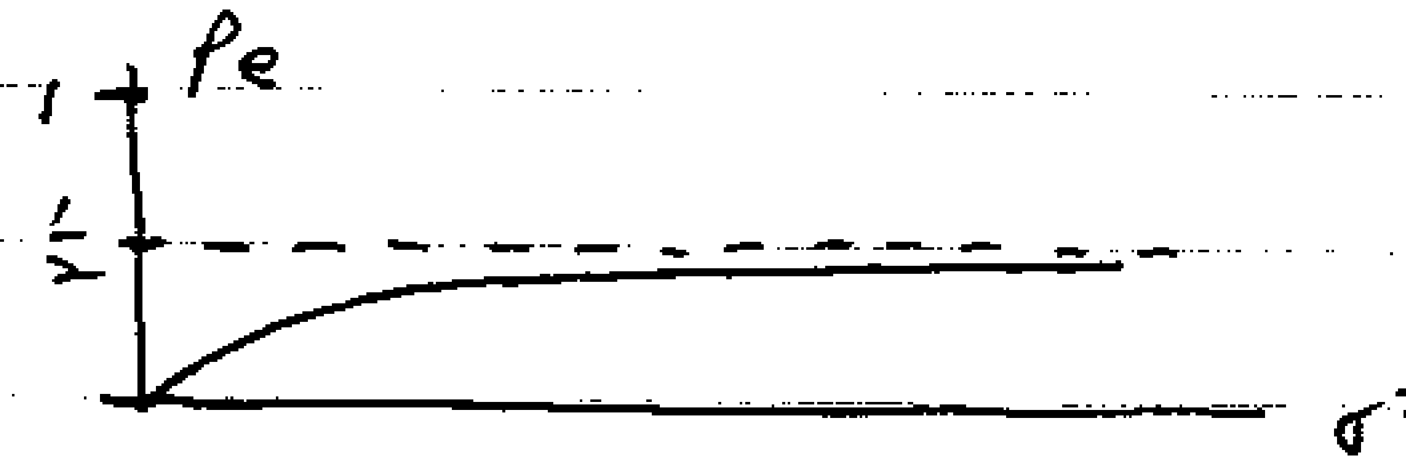
$$P_e = P_r\{x[0] > \tfrac{1}{2} | \mathcal{H}_0\} P(\mathcal{H}_0) + P_r\{x[0] < \tfrac{1}{2} | \mathcal{H}_1\} P(\mathcal{H}_1)$$

But $Pr\{x(0) > \frac{1}{2} \mid \mathcal{H}_0\} = 1 - \Phi(\frac{1}{2\sigma})$
from Problem 1.1. Also,

$$Pr\{x(0) < \tfrac{1}{2} \mid \mathcal{H}_1\} = Pr\{x(0) < -\tfrac{1}{2} \mid \mathcal{H}_0\}$$
$$= \Phi(-\tfrac{1}{2\sigma})$$
$$= 1 - \Phi(\tfrac{1}{2\sigma})$$

since $\Phi(-x) = 1 - \Phi(x)$

$$\Rightarrow P_e = 1 - \Phi(\tfrac{1}{2\sigma}) = 1 - \Phi(\tfrac{1}{2\sqrt{\sigma^2}})$$



As $\sigma^2 \to \infty$ $P_e \to \frac{1}{2}$ or we should discard $x(0)$ since it has no information and just flip a coin.

Decide $\mathcal{H}_1$ if heads
$\mathcal{H}_0$ if tails

$$\Rightarrow P_e = \tfrac{1}{2} Pr\{\text{heads} \mid \mathcal{H}_0\} + \tfrac{1}{2} Pr\{\text{tails} \mid \mathcal{H}_1\}$$
$$= \tfrac{1}{2} Pr\{\text{heads}\} + \tfrac{1}{2} Pr\{\text{tails}\}$$
$$= \tfrac{1}{2}$$

or an equivalent decision is
Decide $\mathcal{H}_1$ always!

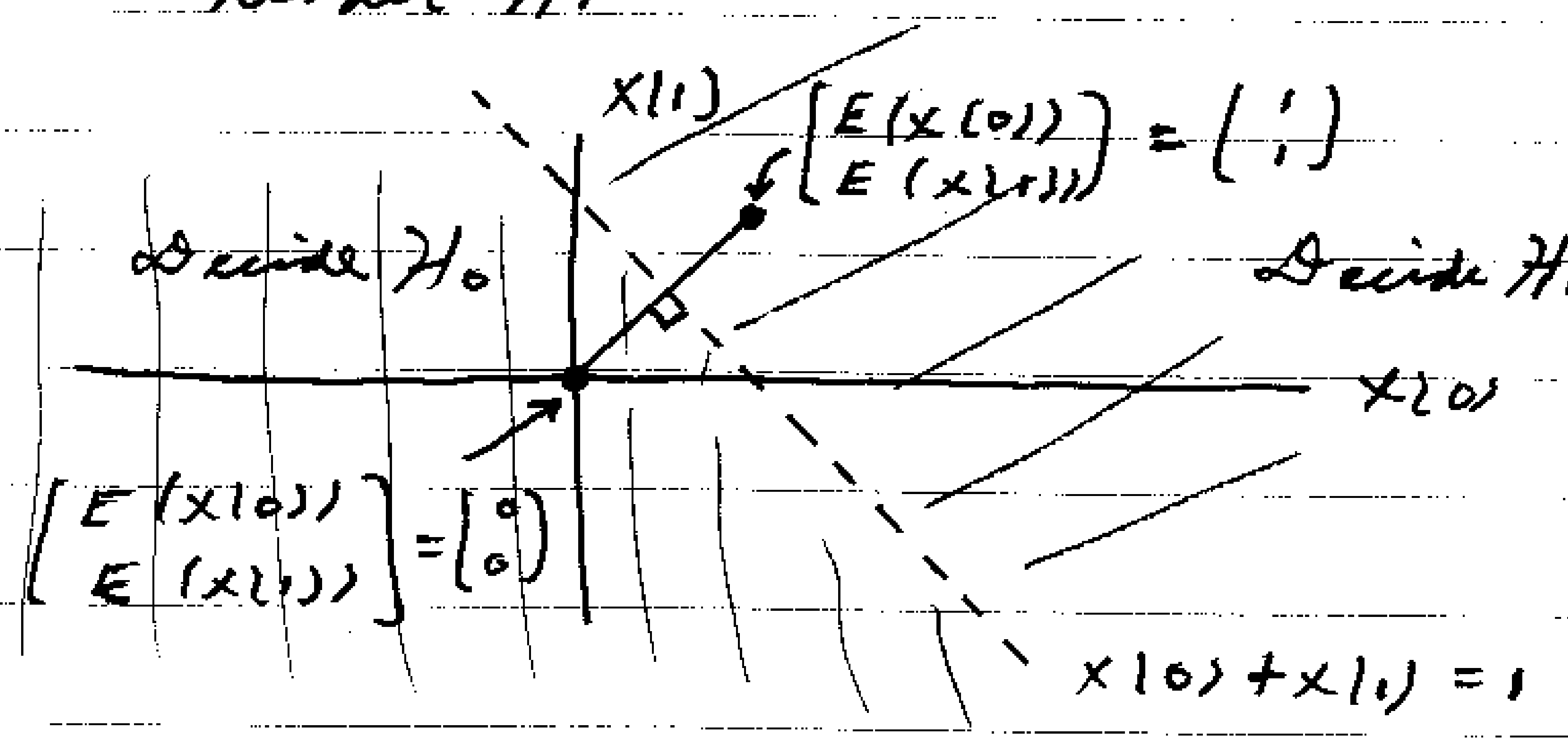
1.5) Decide signal present if
$\frac{1}{2}(x[0]+x[1]) > \frac{1}{2}$ or

$$x[0]+x[1] > 1$$

Also $E(x[0]) = E(x[1]) = 0$
under $\mathcal{H}_0$ and

$$E(x[0]) = E(x[1]) = 1$$

under $\mathcal{H}_1$



Decision boundary (dashed line)
is perpendicular bisector of line
segment shown. Says to choose $\mathcal{H}_1$
if $\begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$ is closer to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and
vice-versa. See also Example 4.6.

1.6) $T = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$

$$E(T; \mathcal{H}_0) = E\left(\frac{1}{N} \sum_n w[n]\right)$$
$$= \frac{1}{N} \sum_n E(w[n]) = 0$$
$$E(T; \mathcal{H}_1) = E\left(\frac{1}{N} \sum_n (A + w[n])\right)$$
$$= \frac{1}{N} \sum_n (A + E(w[n]))$$
$$= \frac{1}{N} \sum_n A = A$$

$\mathrm{var}(T; \mathcal{H}_0) = E(T^2; \mathcal{H}_0)$ since
$E(T; \mathcal{H}_0) = 0$

$$\mathrm{var}(T; \mathcal{H}_0) = E\left[\left(\frac{1}{N} \sum_n w[n]\right)^2\right]$$
$$= E\left[\frac{1}{N^2} \sum_m \sum_n w[m] w[n]\right]$$
$$= \frac{1}{N^2} \sum_m \sum_n \underbrace{E(w[m] w[n])}_{\substack{= \sigma^2 \text{ if } m=n \\ 0 \text{ if } m \neq n}}$$
$$= \frac{1}{N^2} \sum_n \underbrace{E(w^2[n])}_{\sigma^2}$$
$$= \sigma^2 / N$$

$$\mathrm{var}(T; \mathcal{H}_1) = E\left[\left(\frac{1}{N} \sum_n x[n] - A\right)^2; \mathcal{H}_1\right]$$

$$= E\left[\left(\frac{1}{N}\sum_n (x[n]-A)\right)^2 ; \mathcal{H}_1\right]$$

$$= E\left[\left(\frac{1}{N}\sum_n w[n]\right)^2\right] = \mathrm{var}(T;\mathcal{H}_0)$$

1.7) $d^2 = \frac{NA^2}{\sigma^2} = 100$

$SNR = A^2/\sigma^2 = 0.01$

$\Rightarrow N = 10^4$

Chapter 2

2.1) $$\Pr\{T > \gamma\} = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt$$

Let $t' = \frac{t-\mu}{\sigma}$ $\quad dt' = \frac{1}{\sigma} dt$

$$\Pr\{T > \gamma\} = \int_{\frac{\gamma-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}t'^2} \sigma\, dt'$$

$$= \int_{\frac{\gamma-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t'^2} dt'$$

$$= Q\left(\frac{\gamma-\mu}{\sigma}\right)$$

2.2) $$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= \int_x^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\, t}}_{u(t)} \underbrace{t e^{-\frac{1}{2}t^2} dt}_{dv(t)}$$

$$\Rightarrow \quad v(t) = -e^{-\frac{1}{2}t^2} \quad du(t) = -\frac{1}{\sqrt{2\pi}\, t^2} dt$$

$$Q(x) = -\frac{1}{\sqrt{2\pi}\, t} e^{-\frac{1}{2}t^2}\Big|_x^{\infty} - \int_x^{\infty} \frac{1}{\sqrt{2\pi}\, t^2} e^{-\frac{1}{2}t^2} dt$$

$$= \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{1}{2}x^2} - \int_x^{\infty} \frac{1}{\sqrt{2\pi}\, t^2} e^{-\frac{1}{2}t^2} dt$$

$$\Rightarrow \quad Q(x) \le \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{1}{2}x^2}$$ and

as $x \to \infty$ $\quad Q(x) \to \frac{1}{\sqrt{2\pi}\,x} e^{-\frac{1}{2}x^2}$

2.3) $\quad F_{\nu_1,\nu_2} = \dfrac{x_1/\nu_1}{x_2/\nu_2} \qquad x_1 \sim \chi^2_{\nu_1}$

$\qquad\qquad\qquad\qquad\qquad\qquad x_2 \sim \chi^2_{\nu_2}$

and x_1, x_2 are independent

Since

$$x_2 = \sum_{i=1}^{\nu_2} u_i^2 \qquad u_i \sim N(0,1)$$

and IID

$$\frac{x_2}{\nu_2} = \frac{1}{\nu_2} \sum_{i=1}^{\nu_2} u_i^2 \to E(u_i^2) = 1$$

by law of large numbers

$$\Rightarrow F_{\nu_1,\nu_2} \to \frac{x_1}{\nu_1} \sim \frac{\chi^2_{\nu_1}}{\nu_1}$$

2.4) $\quad \underline{C}^{-1} = \underline{D}^T\underline{D}$ where $\underline{x}' = \underline{D}\,\underline{x}$

has covariance matrix

$$E(\underline{x}'\underline{x}'^T) = E(\underline{D}\underline{x}\underline{x}^T\underline{D}^T)$$

$$= \underline{D}\,\underline{C}\,\underline{D}^T = \underline{D}\,\underline{D}^{-1}\underline{D}^{T^{-1}}\underline{D}^T = \underline{I}$$

Now $\underline{x}^T\underline{C}^{-1}\underline{x} = \underline{x}^T\underline{D}^T\underline{D}\underline{x}$

$$= \underline{y}^T\underline{y} = \sum_{i=1}^{n} y_i^2$$

where $\underline{y} = \underline{D}\underline{x} \sim N(\underline{0}, \underline{I})$

$\Rightarrow$ y_i's are IID and

$y_i \sim N(0, 1)$

$$\sum_{i=1}^{n} y_i^2 \sim \chi^2_n$$

2.5) As in Problem 2.4

$$\underline{x}^T \underline{C}^{-1} \underline{x} = \sum_{i=1}^{n} y_i^2$$

where $\underline{y} = \underline{D}\,\underline{x}$

Now, however, $\underline{y} \sim N(\underline{D}\,\underline{\mu}, \underline{I})$

or

y_i's are independent and

$y_i \sim N((\underline{D}\,\underline{\mu})_i, 1)$

$$\Rightarrow \sum_{i=1}^{n} y_i^2 \sim \chi'^2_n(\lambda)$$

$$\text{where } \lambda = \sum_{i=1}^{n} (\underline{D}\,\underline{\mu})_i^2$$

$$= (\underline{D}\,\underline{\mu})^T \underline{D}\,\underline{\mu} = \underline{\mu}^T \underline{D}^T \underline{D}\,\underline{\mu}$$

$$= \underline{\mu}^T \underline{C}^{-1} \underline{\mu}$$

2.6) Using the eigendecomposition

$$\underline{V}^T \underline{A} \underline{V} = \underline{\Lambda} \quad \text{where} \quad \underline{V}^T = \underline{V}^{-1}$$

and $\underline{\Lambda} = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{0, 0, \ldots 0}_{N-r})$

$$\Rightarrow \underline{x}^T \underline{A} \underline{x} = \underline{x}^T \underline{V} \underline{\Lambda} \underline{V}^T \underline{x} = \underline{y}^T \underline{\Lambda} \underline{y}$$

where $\underline{y} = \underline{V}^T \underline{x} \sim \mathcal{N}(\underline{0}, \underbrace{\underline{V}^T \underline{I} \underline{V}}_{\underline{I}})$

$$\underline{y}^T \underline{\Lambda} \underline{y} = \sum_{i=1}^{r} \lambda_i y_i^2$$

$$= \sum_{i=1}^{r} y_i^2$$

and $y_i \sim \mathcal{N}(0, 1)$ and are independent

$$\Rightarrow \sum_{i=1}^{r} y_i^2 \sim \chi_r^2$$

2.7) $$\underline{u}^T \underline{R} \underline{u} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} u \quad u(n)\, r_{xx}(m-n)$$

$$= \sum_m \sum_n u(m) u(n) \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(f) e^{j2\pi f(m-n)} df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_m \sum_n u(m) u(n) e^{j2\pi f m} e^{-j2\pi f n} P_{xx}(f)\, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} |\underbrace{\sum_n u(n) e^{-j2\pi f n}}_{U(f)}|^2 P_{xx}(f)\,df$$

Also, if $r_{xx}(m-n) = \delta(m-n) \Rightarrow$
$\underline{u}^T \underline{R} \underline{u} = \underline{u}^T \underline{u}$ and $P_{xx}(f) = 1$ so that

$$\underline{u}^T \underline{u} = \int_{-\frac{1}{2}}^{\frac{1}{2}} |U(f)|^2 df$$

(or use Parseval's theorem)

$$\lambda_{MAX} = \max_{\underline{u}} \frac{\int |U(f)|^2 P_{xx}(f)\,df}{\int |U(f)|^2 df}$$

But for any $\underline{u}$

$$\int |U(f)|^2 P_{xx}(f)\,df < \int |U(f)|^2 df \; P_{xx}(f)_{MAX}$$

$\Rightarrow \lambda_{MAX} < P_{xx}(f)_{MAX}$ (assuming $P_{xx}(f) \neq \sigma^2$)

Similarly, we have

$$\lambda_{MIN} > P_{xx}(f)_{MIN}$$

2.8) $P_{xx}(f) = r_{xx}(0) + 2 r_{xx}(1) \cos 2\pi f$

$\Rightarrow P_{xx}(f)_{MIN} = r_{xx}(0) - 2|r_{xx}(1)|$

$P_{xx}(f)_{MAX} = r_{xx}(0) + 2|r_{xx}(1)|$

If $r_{xx}(1) > 0$, $P_{xx}(f)_{MIN}$ is attained at $f = 0$ and otherwise at $f = 1/2$. Similarly, for $P_{xx}(f)_{MAX}$.

Now assume $r_{xx}(1) > 0$

$$c = \frac{r_{xx}(0) - \lambda}{r_{xx}(1)} < \frac{r_{xx}(0) - P_{xx}(f)_{MIN}}{r_{xx}(1)}$$

since $P_{xx}(f)_{MIN} < \lambda < P_{xx}(f)_{MAX}$. But

$$r_{xx}(0) - P_{xx}(f)_{MIN} = 2\,|r_{xx}(1)| = 2\,r_{xx}(1)$$

$\Rightarrow \quad c < 2$. Also,

$$c = \frac{r_{xx}(0) - \lambda}{r_{xx}(1)} > \frac{r_{xx}(0) - P_{xx}(f)_{MAX}}{r_{xx}(1)}$$

$$= \frac{-2|r_{xx}(1)|}{r_{xx}(1)}$$

$$= -2\,\frac{r_{xx}(1)}{r_{xx}(1)} = -2$$

Similarly, we can show that $-2 < c < 2$ for $r_{xx}(1) < 0$.

To solve $v_n + c v_{n-1} + v_{n-2} = 0$

let $v_n = z^n$ so that

$$z^n + c z^{n-1} + z^{n-2} = 0 \quad \text{or}$$

$$z^2 + cz + 1 = 0$$

$$\Rightarrow \quad z = \frac{-c \pm \sqrt{c^2-4}}{2} \quad \text{which are complex}$$

Also, $|z| = 1$ and thus $z = e^{\pm j\theta}$.

As a result $v_n = A e^{j\theta n} + A^* e^{-j\theta n}$

$$= B\cos(n\theta + \phi)$$

Noting that $c = -2\cos\theta$, we have

$$v_1 + c v_0 = 0 \Rightarrow$$

$$B\cos(\theta+\phi) - 2\cos\theta\, B\cos\phi = 0$$

$$\cos\theta\cos\phi - \sin\theta\sin\phi - 2\cos\theta\cos\phi = 0$$

$$\cos(\theta-\phi) = 0 \Rightarrow \theta = (2k+1)\frac{\pi}{2} + \phi$$

k an integer

$$v_{N-2} + c v_{N-1} = 0 \Rightarrow$$

$$B\cos((N-2)\theta+\phi) - 2\cos\theta B\cos((N-1)\theta+\phi) = 0$$

$$\cos((N-2)\theta+\phi) - \cos(N\theta+\phi) - \cos((N-2)\theta+\phi) = 0$$

$$\Rightarrow \cos(N\theta+\phi) = 0$$

$$N\theta + \phi = (2l+1)\pi/2 \qquad l \text{ an integer}$$

$$N\theta = (2l+1)\pi/2 + (2k+1)\frac{\pi}{2} - \theta$$

$$\theta = \frac{(k+l)\pi + \pi}{N+1} = \frac{m\pi}{N+1} = \frac{2m\pi}{2(N+1)}$$

for m an integer. Now

$$\lambda = r_{xx}[0] - c\, r_{xx}[1]$$
$$= r_{xx}[0] + 2\cos\theta\, r_{xx}[1]$$
$$= r_{xx}[0] + 2 r_{xx}[1] \cos \frac{2\pi m}{2(N+1)}$$

2.9) $r_{xx}[M] = 0.9^M < 0.001$
$\Rightarrow M > 66$

$E(x[n]) = E(A) = 0$ for all n
$E(x[n]x[n+k]) = E(A^2) = \sigma_A^2$ for all k
doesn't depend on n. Thus,
$r_{xx}[k] = \sigma_A^2$
$\Rightarrow$ WSS

correlation time is infinite
Can't have the asymptotic eigendecomposition since $P_{xx}(f) = \sigma_A^2\, \delta(f)$ doesn't exist.

2.10) $\underline{v}_m^H \underline{v}_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi f_n k} e^{-j2\pi f_m k}$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi\theta k} \qquad \theta = f_n - f_m = \frac{n-m}{N}$$

$$= \frac{1}{N} \frac{1 - e^{j2\pi\theta N}}{1 - e^{j2\pi\theta}}$$

$$= \frac{1}{N} \frac{e^{j\pi\theta N}}{e^{j\pi\theta}} \frac{2j \sin N\pi\theta}{2j \sin \pi\theta}$$

$$= \frac{1}{N} e^{j\pi(N-1)\frac{(n-m)}{N}} \frac{\sin(n-m)\pi}{\sin(n-m)\pi/N}$$

$= 0$ for $m \neq n$

$= 1$ for $m = n$ since for $m = n$

$$\underline{v}_m^H \underline{v}_n = \frac{1}{N} \sum_{k=0}^{N-1} 1 = 1$$

2.11) $$\underline{R} = \sum_{i=0}^{N-1} P_{xx}(f_i) \underline{v}_i \underline{v}_i^H$$

$$= P_{xx}(f_0) \underline{v}_0 \underline{v}_0^H + \sum_{i=1}^{\frac{N}{2}-1} P_{xx}(f_i) \underline{v}_i \underline{v}_i^H$$

$$+ P_{xx}(f_{N/2}) \underline{v}_{N/2} \underline{v}_{N/2}^H + \underbrace{\sum_{i=\frac{N}{2}}^{N-1} P_{xx}(f_i) \underline{v}_i \underline{v}_i^H}_{\sum_{i=1}^{\frac{N}{2}-1} P_{xx}(f_{N-i}) \underline{v}_{N-i} \underline{v}_{N-i}^H}$$

But $\underline{v}_0 = \underline{c}_0$, $\underline{v}_{N/2} = \underline{c}_{N/2}$

$P_{xx}(f_{N-i}) = P_{xx}(f_i)$ and

$$\underline{v}_{N-i} = \frac{1}{\sqrt{N}} (\underline{c}_{N-i} + j \underline{s}_{N-i})$$

$$= \frac{1}{\sqrt{N}} (\underline{c}_i - j \underline{s}_i) = \underline{v}_i^*$$

$$\Rightarrow \underline{R} = P_{xx}(f_0) \frac{\underline{c}_0 \underline{c}_0^T}{N} + \sum_{i=1}^{\frac{N}{2}-1} P_{xx}(f_i) (\underline{v}_i \underline{v}_i^H + \underline{v}_i^* \underline{v}_i^T)$$

$$+ P_{xx}(f_{N/2}) \underline{c}_{N/2} \underline{c}_{N/2}^T / N$$

and $\underline{v}_i \underline{v}_i^H + \underline{v}_i^* \underline{v}_i^T = 2\, Re(\underline{v}_i \underline{v}_i^H)$

$$= \frac{2}{N} Re\left((\underline{c}_i + j\underline{s}_i)(\underline{c}_i^T - j\underline{s}_i^T) \right)$$

$$= \frac{2}{N}(\underline{c}_i \underline{c}_i^T + \underline{s}_i \underline{s}_i^T)$$

To prove orthogonality consider as an example

$$\underline{s}_m^T \underline{c}_n = \sum_{i=0}^{N-1} \sin 2\pi f_m i \cos 2\pi f_n i$$

$$= \frac{1}{2} \sum_{i=0}^{N-1} \sin(2\pi(f_m - f_n) i) + \sin(2\pi(f_m + f_n) i)$$

$$= \frac{1}{2} Im \sum_{i=0}^{N-1} \left[e^{j2\pi(f_m - f_n) i} + e^{j2\pi(f_m + f_n) i} \right]$$

$$= \frac{1}{2} Im \left[N\delta(m-n) + 0 \right] = 0$$

since $\displaystyle\sum_{i=0}^{N-1} e^{j2\pi f_n i} = \begin{cases} 0 & i = 1, 2, \ldots, N-1 \\ N & n = 0 \end{cases}$

Similarly, for the others.

Eigenvalues are $\{ P_{xx}(f_0), P_{xx}(f_1), P_{xx}(f_1), \ldots, P_{xx}(f_{\frac{N}{2}-1}), P_{xx}(f_{\frac{N}{2}-1}), P_{xx}(f_{N/2}) \}$ and occur in pairs except for $i = 0, N/2$.

2.12) $[\underline{R}]_{mn} = r_{xx}(m-n)$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(f)\, e^{j2\pi f(m-n)}\, df$$

$$= \int_{0}^{1} P_{xx}(f)\, e^{j2\pi f(m-n)}\, df$$

$$\approx \frac{1}{N} \sum_{i=0}^{N-1} P_{xx}(f_i)\, e^{j2\pi f_i(m-n)}$$

$$= \sum_{i=0}^{N-1} P_{xx}(f_i)\, \frac{1}{\sqrt{N}} e^{j2\pi f_i m} \cdot \frac{1}{\sqrt{N}} e^{-j2\pi f_i n}$$

$$= \sum_{i=0}^{N-1} P_{xx}(f_i)\, (\underline{v}_i)_m\, [\underline{v}_i]_n^*$$

or $\underline{R} = \sum_{i=0}^{N-1} P_{xx}(f_i)\, \underline{v}_i \underline{v}_i^H$

Eigenvectors are $\underline{v}_i$'s, eigenvalues are $P_{xx}(f_i)$.

2.13) $\underline{R} = \sum_{i=0}^{N-1} \lambda_i \underline{v}_i \underline{v}_i^H = \underline{V}\, \underline{\Lambda}\, \underline{V}^H$

$$\det(\underline{R}) = \det(\underline{V}) \det(\underline{\Lambda}) \underbrace{\det(\underline{V}^H)}_{\underbrace{\det(\underline{V}^{-1})}_{\frac{1}{\det(\underline{V})}}}$$

$$= \det(\underline{\Lambda}) = \prod_{i=0}^{N-1} \lambda_i$$

$$\underline{R}^{-1} = (\underline{V}^H]^{-1} \underline{\Lambda}^{-1} \underline{V}^{-1} = \underline{V}\,\underline{\Lambda}^{-1}\underline{V}^H$$

$$= \sum_{i=0}^{N-1} \frac{1}{\lambda_i} \underline{v}_i \underline{v}_i^H$$

2.14) Using CLT we have

$$T(\underline{x}) = \frac{1}{N} \sum x^2[n] \overset{a}{\sim} N\left(E(x^2[n]), \frac{\mathrm{var}(x^2[n])}{N}\right)$$

But $x[n] \sim N(0, 5) \Rightarrow E(x^2[n]) = 5$

Also $\mathrm{var}(x^2[n]) = E(x^4[n]) - E^2(x^2[n])$

$$= 3E^2(x^2[n]) - E^2(x^2[n])$$

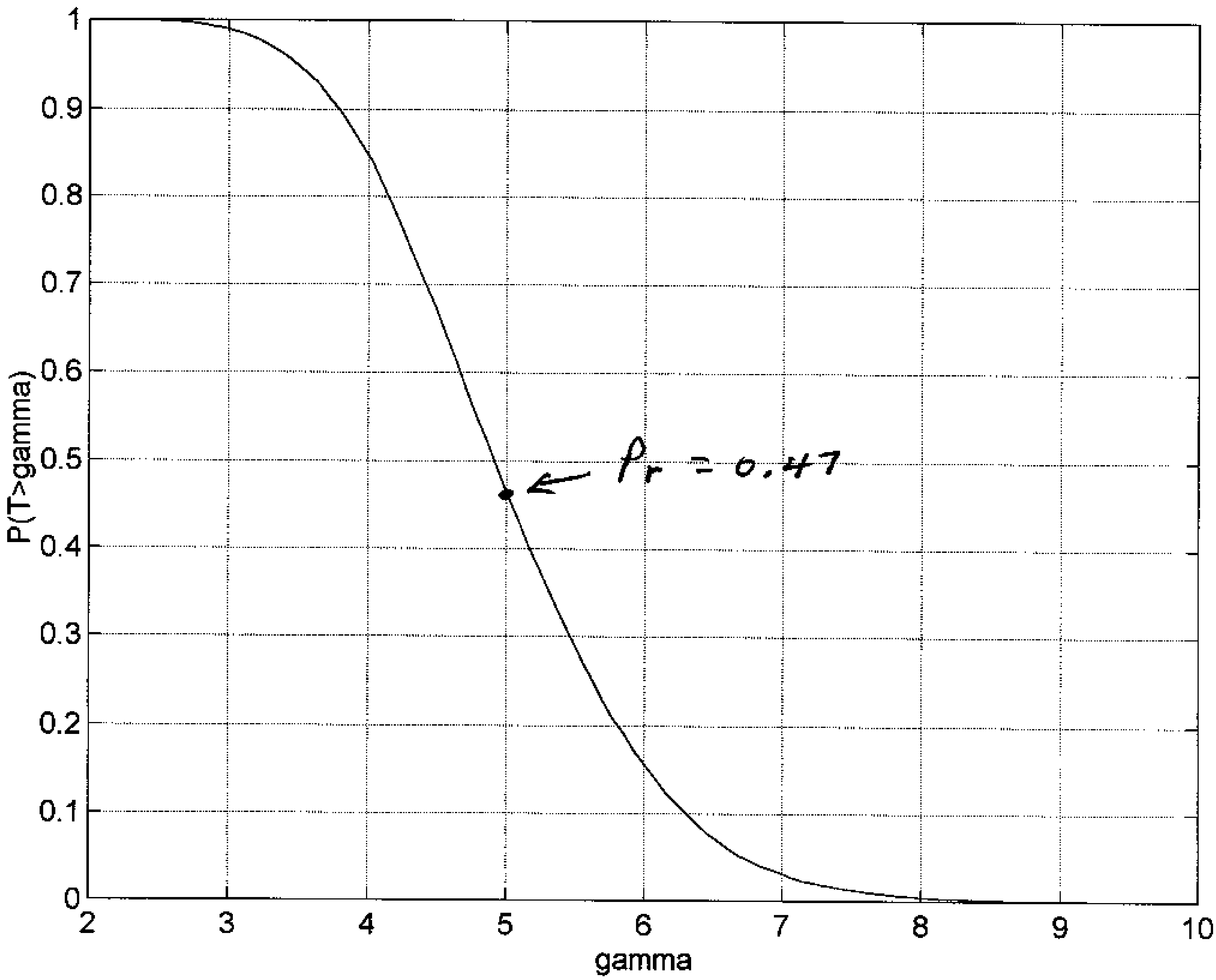
$$= 2(5)^2 = 50$$

$$T(\underline{x}) \overset{a}{\sim} N(5, 1)$$

$$Pr\{T(\underline{x}) > 5\} \approx \frac{1}{2}$$

See next two pages for Montecarlo results.

```
% prob214.m
%
% This program is a Monte Carlo computer simulation that solves
% Problem 2.14.
%
% Set seed of random number generator to initial value.
randn('seed',0);
% Set up values of variance, data record length, and number
% of realizations.
%  var=10;
var=5;  %MODIFY THE VARIANCE
%  N=10;
N=50;    %MODIFY THE DATA RECORD LENGTH
%  M=1000;
M=10000; %MODIFY THE NUMBER OF REALIZATIONS
% Dimension array of realizations.
T=zeros(M,1);
% Compute realizations of the sample mean.
for i=1:M
  x=sqrt(var)*randn(N,1);
%    T(i)=mean(x);
T(i)=x'*x/N; %MODIFY THE TEST STATISTIC
end
% Set number of values of gamma.
ngam=100;
% Set up gamma array.
gammamin=min(T);
gammamax=max(T);
gamdel=(gammamax-gammamin)/ngam;
gamma=[gammamin:gamdel:gammamax]';
% Dimension P (the Monte Carlo estimate) and Ptrue
% (the theoretical or true probability).
P=zeros(length(gamma),1);Ptrue=P;
% Determine for each gamma how many realizations exceeded
% gamma (Mgam) and use this to estimate the probability.
for i=1:length(gamma)
  clear Mgam;
  Mgam=find(T>gamma(i));
  P(i)=length(Mgam)/M;
end
% Compute the true probability.
Ptrue=Q(gamma/(sqrt(var/N)));
%  plot(gamma,P,'-',gamma,Ptrue,'--')
plot(gamma,P)  %MODIFY PLOT
xlabel('gamma')
ylabel('P(T>gamma)')
grid
```



PROB. 2.14

2.151 $95\% \Rightarrow \alpha = 0.05$

$\epsilon = 0.01 \qquad P_D \geq 0.8$

$$M \geq \frac{\left[Q^{-1}(\alpha/2)\right]^2 (1-P_D)}{\epsilon^2 P_D}$$

Since $\frac{1-P_D}{P_D}$ has a maximum at

$P_D = 0.8$ for $P_D \geq 0.8$, we use this value.

$$M \geq \frac{\left(Q^{-1}(0.025)\right)^2 \, 0.2}{(0.01)^2 \, 0.8} = 9604$$

Chapter 3

3.1)
$$L(x[0]) = \frac{p(x[0];\mathcal{H}_1)}{p(x[0];\mathcal{H}_0)}$$

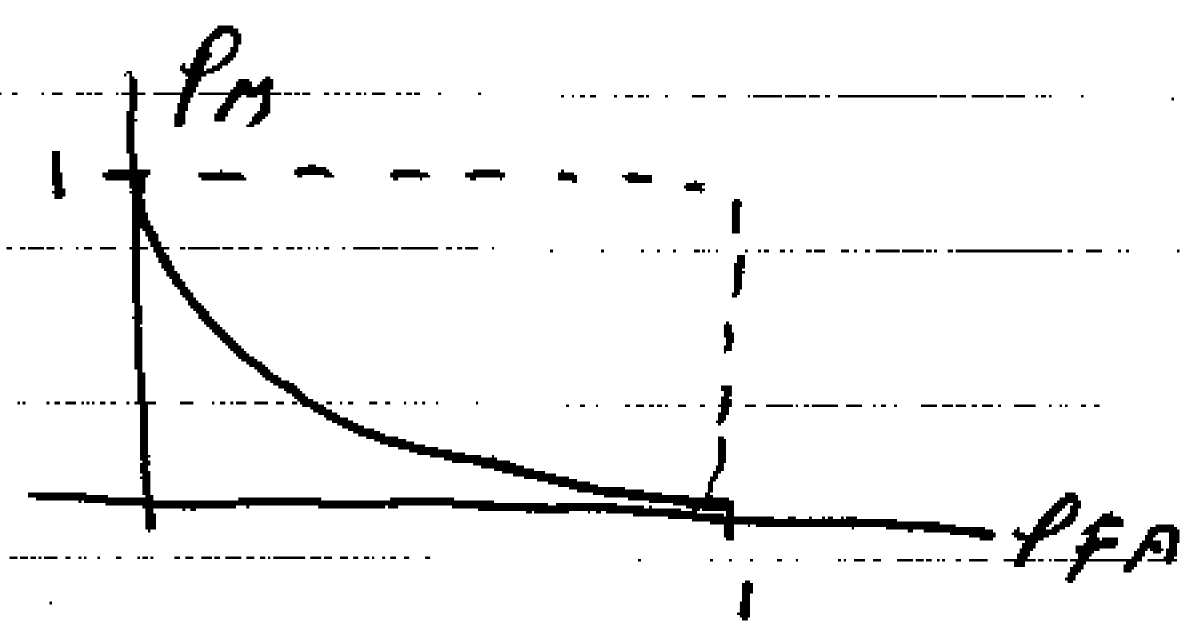
$$= \frac{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x[0]-1)^2}}{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2[0]}} > \gamma$$

Taking the logarithm of both sides

$$-\frac{1}{2}\left(x^2[0]-2x[0]+1\right)+\frac{1}{2}x^2[0] > \ln\gamma$$

or $x[0] > \frac{1}{2} + \ln\gamma = \gamma'$

$$P_{FA} = \Pr\{x[0] > \gamma';\mathcal{H}_0\}$$
$$= Q(\gamma')$$
$$P_D = \Pr\{x[0] > \gamma';\mathcal{H}_1\}$$
$$= Q(\gamma'-1)$$
$$\Rightarrow P_M = 1 - Q(\gamma'-1)$$
$$= 1 - Q\left(Q^{-1}(P_{FA}) - 1\right)$$

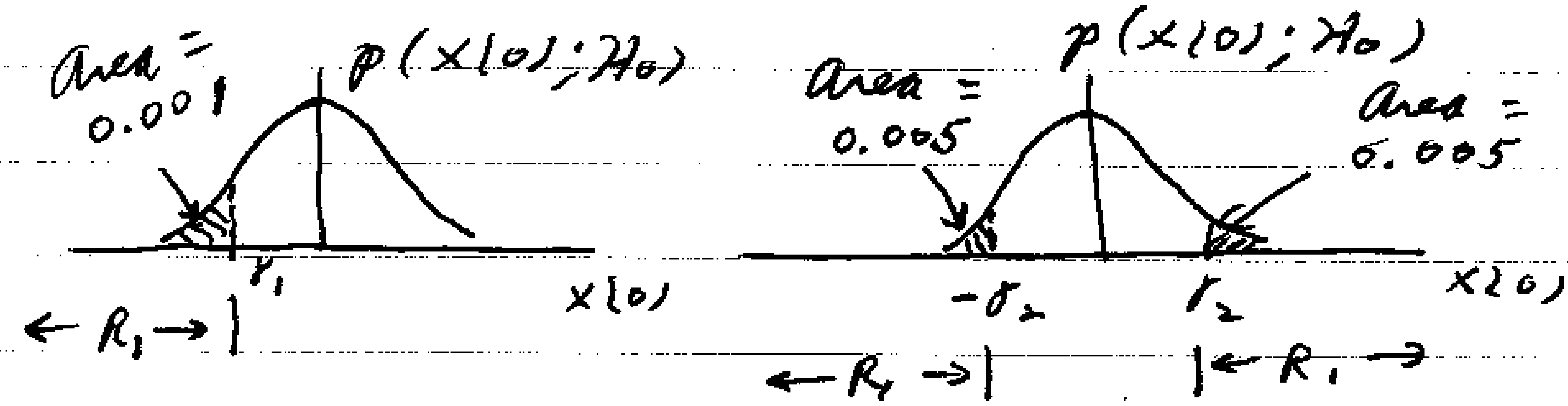


3.2) Let $R_1 = \{\underline{x} : \text{decide } \mathcal{H}_1\}$

$$P_{FA} = \int_{R_1} p(\underline{x}; \mathcal{H}_0)\, d\underline{x} = 10^{-3}$$

But $p(\underline{x}; \mathcal{H}_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2(0)}$

$$\int_{R_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt = 10^{-3}$$



where $\gamma_1 = -3$
$\gamma_2 = 3.3$

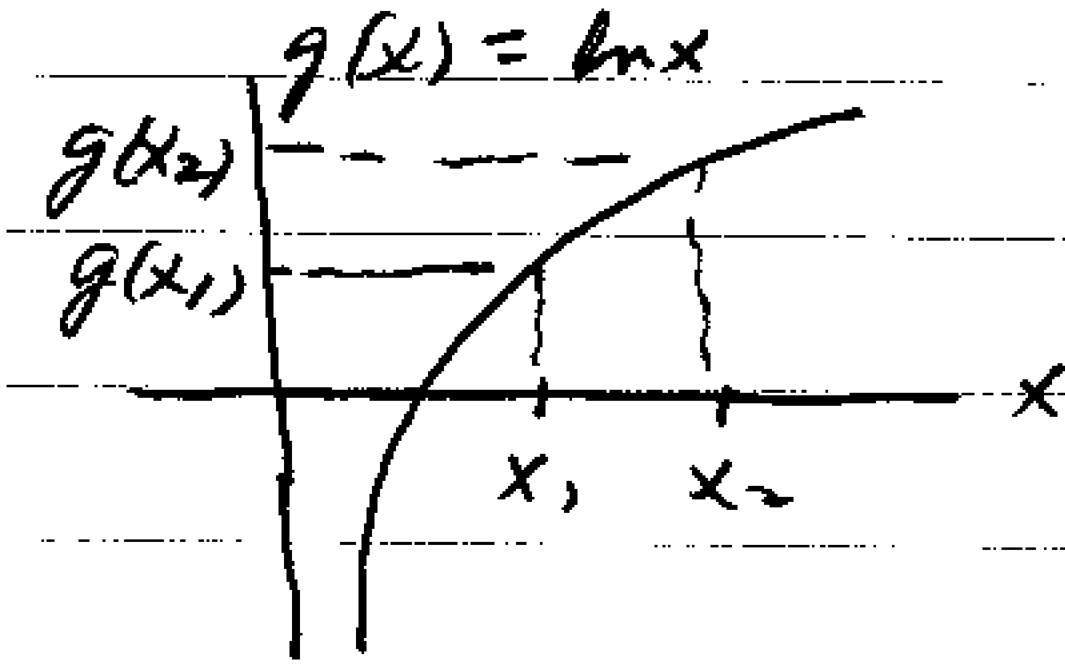
$R_1 = \{x[0] : x[0] < -3\}$
$R_1 = \{x[0] : |x[0]| > 3.3\}$

3.3) Since $g(x_2) > g(x_1)$ if and only if $x_2 > x_1$, we have

$g(L(\underline{x})) > g(\gamma)$ if and only if $L(\underline{x}) > \gamma$, where we have let

$x_1 = \gamma,\ x_2 = L(\underline{x})$

An example of $g(x)$ is



3.4) From (3.8)

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

where $d^2 = NA^2/\sigma^2$

$$Q^{-1}(P_D) = Q^{-1}(P_{FA}) - \sqrt{d^2}$$

$$d^2 = \left[Q^{-1}(P_{FA}) - Q^{-1}(P_D)\right]^2$$

or

$$N = \frac{\left[Q^{-1}(P_{FA}) - Q^{-1}(P_D)\right]^2}{A^2/\sigma^2}$$

$$= \frac{\left[Q^{-1}(10^{-4}) - Q^{-1}(0.99)\right]^2}{0.001}$$

Using Qinv.m in 2C, we have

$$N = 36,546$$

3.5) $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

As shown in Chapter 1, $E(\hat{A}) = A$ and

$$\mathrm{var}(\hat{A}) = \sigma^2/N$$

$$\frac{E^2(\hat{A})}{\mathrm{var}(\hat{A})} = \frac{A^2}{\sigma^2/N} = \frac{NA^2}{\sigma^2} = d^2$$

Thus this quantity, which measures estimation accuracy is actually the ENR.

3.6) For $A<0$ we have that we decide $\mathcal{H}_1$ if

$$\frac{A}{\sigma^2}\sum_n x[n] > \ln\gamma + \frac{NA^2}{2\sigma^2} \quad \text{as before}$$

but now $A<0$ so that we decide $\mathcal{H}_1$ if

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n] < \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma'$$

as before $\frac{1}{N}\sum_n x[n] \sim \mathcal{N}(0, \sigma^2/N)$ $\mathcal{H}_0$
$\mathcal{N}(A, \sigma^2/N)$ $\mathcal{H}_1$

so that

$$\begin{aligned} P_{FA} &= \Pr\{T(\underline{x}) < \gamma'; \mathcal{H}_0\} \\ &= 1 - \Pr\{T(\underline{x}) > \gamma'; \mathcal{H}_0\} \\ &= 1 - Q(\gamma'/\sqrt{\sigma^2/N}) \end{aligned}$$

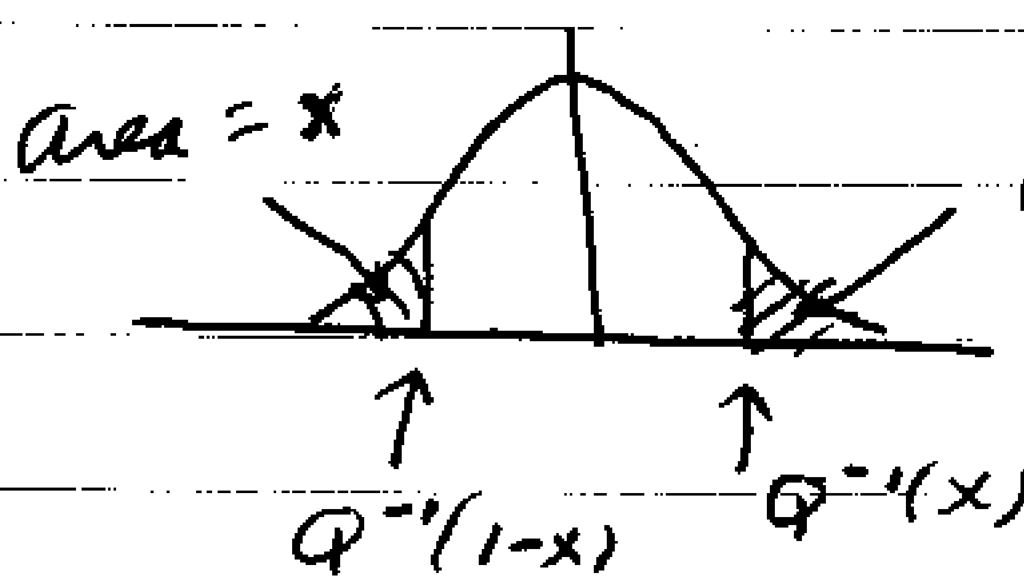
$$P_D = P_r\{T(\underline{x}) < \gamma' ; \mathcal{H}_1\}$$
$$= 1 - P_r\{T(\underline{x}) > \gamma' ; \mathcal{H}_1\}$$
$$= 1 - Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right)$$

$$\gamma' = \sqrt{\sigma^2/N}\, Q^{-1}(1 - P_{FA})$$

$$\Rightarrow P_D = 1 - Q\left[Q^{-1}(1-P_{FA}) - A/\sqrt{\sigma^2/N}\right]$$

But $Q(-x) = 1 - Q(x)$

$$P_D = Q\left[-Q^{-1}(1-P_{FA}) + A/\sqrt{\sigma^2/N}\right]$$



$$\Rightarrow Q^{-1}(x) = -Q^{-1}(1-x)$$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \frac{|A|}{\sqrt{\sigma^2/N}}\right) \quad \text{since } A < 0$$

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$

3.7) Decide $\mathcal{H}_1$ if $L(\underline{x}) > \gamma$ or

$$\frac{\prod_{n=0}^{N-1} \frac{x[n]}{\sigma_1^2} e^{-\frac{1}{2}x^2[n]/\sigma_1^2}}{\prod_{n=0}^{N-1} \frac{x[n]}{\sigma_0^2} e^{-\frac{1}{2}x^2[n]/\sigma_0^2}} > \gamma$$

$$\frac{\sigma_0^{2N}}{\sigma_1^{2N}} e^{\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_n x^2(n)} > \gamma$$

Taking logarithms

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_n x^2(n) > \ln\left(\frac{\sigma_1^{2N}}{\sigma_0^{2N}}\gamma\right)$$

or since $\sigma_1^2 > \sigma_0^2$

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2(n) > \frac{\ln\left(\frac{\sigma_1^{2N}}{\sigma_0^{2N}}\gamma\right)}{\frac{N}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)} = \gamma'$$

Test statistic is estimator of second moment. For Rayleigh random variable $E(x^2) = 2\sigma^2$.

3.8) $$L(\underline{x}) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2(0)}}{\frac{1}{2} e^{-|x(0)|}} > \gamma$$

Taking logarithms

$$-\frac{1}{2}x^2(0) + |x(0)| > \ln\left(\sqrt{\tfrac{\pi}{2}}\,\gamma\right)$$

$$x^2(0) - 2|x(0)| < -2\ln\left(\sqrt{\tfrac{\pi}{2}}\,\gamma\right)$$

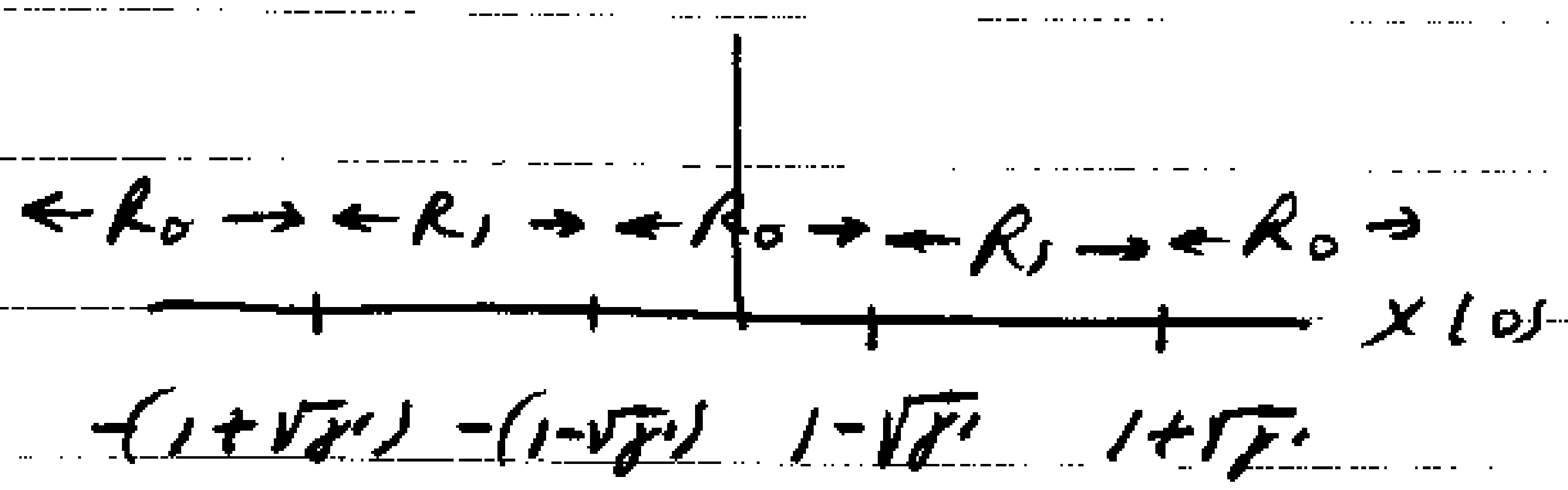
$$x^2(0) - 2|x(0)| + 1 < 1 - 2\ln\left(\sqrt{\tfrac{\pi}{2}}\,\gamma\right)$$

$$(|x(0)|-1)^2 < \gamma'$$

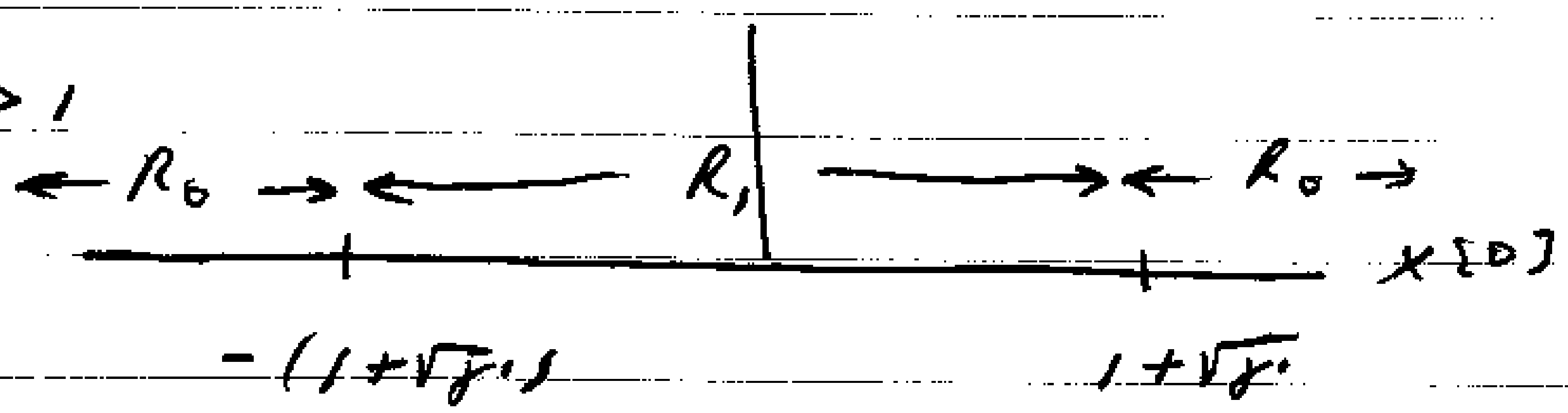
Note that $\gamma \in (0, \infty) \Rightarrow \gamma' \in (-\infty, \infty)$. If $\gamma' < 0$, the inequality is never satisfied and we always choose $\mathcal{H}_0 \Rightarrow P_{FA} = 1$. To avoid this let $\gamma' > 0$. Then, decide $\mathcal{H}_1$ if

$$-\sqrt{\gamma'} < |x[0]| - 1 < \sqrt{\gamma'}$$
$$1 - \sqrt{\gamma'} < |x[0]| < 1 + \sqrt{\gamma'}$$

For $\sqrt{\gamma'} < 1$



For $\sqrt{\gamma'} > 1$



3.9) Decide $\mathcal{H}_1$ if $\frac{1}{2}(x^2[0] + x^2[1]) > \gamma'$

or $x^2[0] + x^2[1] > 2\gamma'$

Under $\mathcal{H}_0$ $\sigma^2 = \sigma_0^2$ so that

$$P_{FA} = \Pr\left\{ \frac{x^2[0] + x^2[1]}{\sigma_0^2} > \frac{2\gamma'}{\sigma_0^2} ; \mathcal{H}_0 \right\}$$

$$= \Pr\{ \chi_2^2 > 2\gamma'/\sigma_0^2 \}$$

$$= \int_{2\gamma'/\sigma_0^2}^{\infty} \tfrac{1}{2} e^{-\frac{1}{2}t}\, dt$$

$$= -e^{-\frac{1}{2}t}\Big|_{2\gamma'/\sigma_0^2}^{\infty} = e^{-\gamma'/\sigma_0^2}$$

Likewise

$$P_D = \Pr\left\{ \frac{x^2(0)+x^2(1)}{\sigma_1^2} > \frac{2\gamma'}{\sigma_1^2}; \mathcal{H}_1 \right\}$$

$$= \Pr\{ \chi_2^2 > 2\gamma'/\sigma_1^2 \} = e^{-\gamma'/\sigma_1^2}$$

$\Rightarrow$

$$\sigma_0^2 \ln P_{FA} = \sigma_1^2 \ln P_D \quad \text{or} \quad P_D = P_{FA}^{\sigma_0^2/\sigma_1^2}$$

3.10) $p(\underline{x}; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2}$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\left[\sum_n x^2[n] - 2AN\bar{x} + NA^2\right]}$$

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{\frac{AN}{\sigma^2}\bar{x} - NA^2/2\sigma^2}}_{g(\bar{x}, A)} \underbrace{e^{-\frac{1}{2\sigma^2}\sum_n x^2[n]}}_{h(\underline{x})}$$

$\Rightarrow$ $\bar{x}$ is a sufficient statistic for A (assumes σ^2 is known)

3.11) $p(x; \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2 + \mu/\sigma^2 x - \mu^2/2\sigma^2}$

$$= e^{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2 - \frac{\mu^2}{2\sigma^2} + \ln\frac{1}{\sqrt{2\pi\sigma^2}}}$$

$$\Rightarrow \quad A(\mu) = \mu$$
$$B(x) = x/\sigma^2$$
$$C(x) = -x^2/2\sigma^2$$
$$D(\mu) = -\frac{\mu^2}{2\sigma^2} + \ln\frac{1}{\sqrt{2\pi\sigma^2}}$$

$$p(\underline{x};\theta) = \prod_{n=0}^{N-1} e^{A(\theta)B(x[n]) + C(x[n]) + D(\theta)}$$

$$= \underbrace{e^{A(\theta)\sum_n B(x[n]) + ND(\theta)}}_{g(T(\underline{x}),\theta)} \; \underbrace{e^{\sum_n C(x[n])}}_{h(\underline{x})}$$

$\Rightarrow \quad T(\underline{x}) = \sum_{n=0}^{N-1} B(x[n])$ is a sufficient statistic for θ

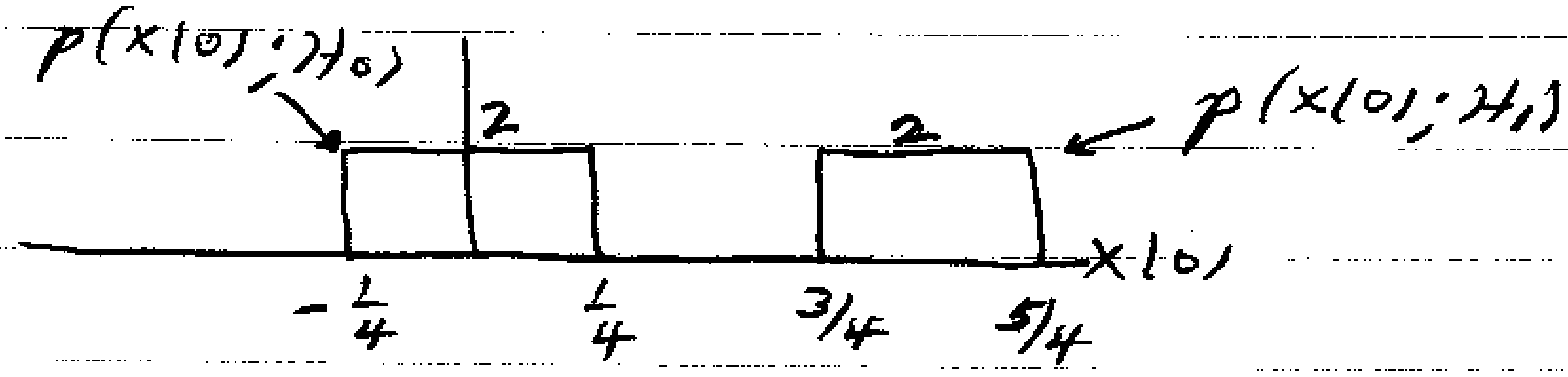
In Gaussian case we have

$$B(x) = x/\sigma^2 \text{ or}$$
$$T(\underline{x}) = \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]$$

or $T'(\underline{x}) = \sigma^2/N \, T(\underline{x}) = \bar{x}$ since sufficient statistics are not unique and 1-1 transformations are also sufficient statistics

3.12) For perfect detection PDFs cannot overlap. Hence, if $1-c > c$ or

$c < 1/2$. As an example, for $c = 1/4$



We decide $\mathcal{H}_1$ if $x[0] > 1/2$.

3.13) Clearly, the performance of the randomized detector must be poorer than NP detector. Randomized test decides $\mathcal{H}_1$ if

heads and $T(\underline{x}) > \gamma_1$

or if tails and $T(\underline{x}) > \gamma_2$

where test 1 corresponds to deciding $\mathcal{H}_1$ if $T(\underline{x}) > \gamma_1$ and test 2 corresponds to deciding $\mathcal{H}_1$ if $T(\underline{x}) > \gamma_2$. Now

$$P_{FA}(\text{randomized}) = \Pr\{\text{heads}, T(\underline{x}) > \gamma_1; \mathcal{H}_0\} + \Pr\{\text{tails}, T(\underline{x}) > \gamma_2; \mathcal{H}_0\}$$

$$= \alpha \Pr\{T(\underline{x}) > \gamma_1; \mathcal{H}_0\} + (1-\alpha)\Pr\{T(\underline{x}) > \gamma_2; \mathcal{H}_0\}$$
$$= \alpha p_1 + (1-\alpha) p_2$$

and similarly

$$P_D(\text{randomized}) = \alpha P_D(p_1) + (1-\alpha) P_D(p_2)$$

Now assume a NP detector whose P_{FA} is

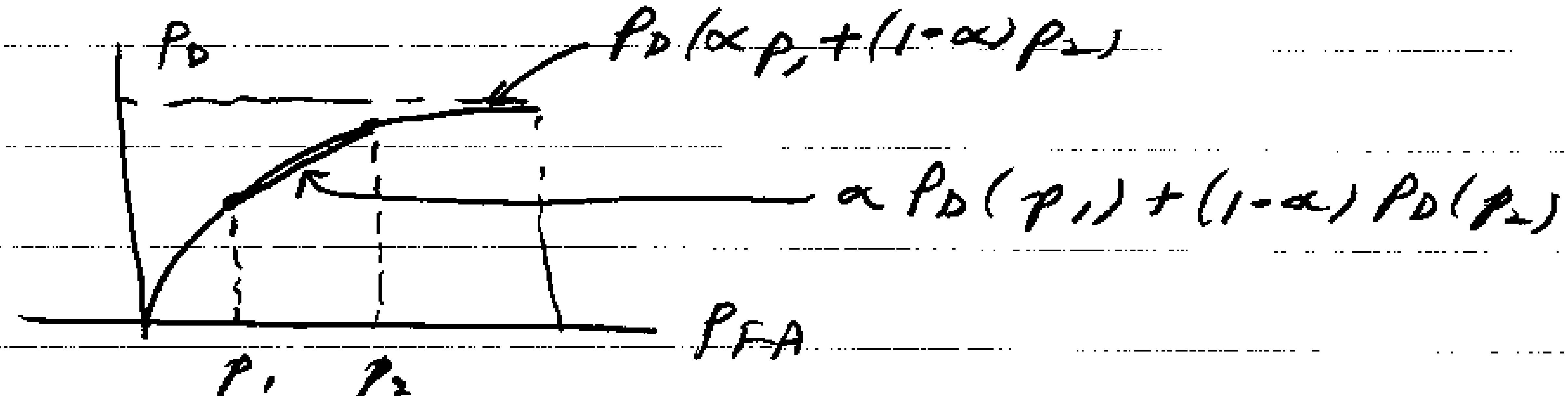
$$P_{FA} = \alpha p_1 + (1-\alpha) p_2 .$$

Then we must have (due to optimality of NP)

$$P_D \geq \alpha P_D(p_1) + (1-\alpha) P_D(p_2)$$

or

$$\alpha P_D(p_1) + (1-\alpha) P_D(p_2) \leq P_D(\alpha p_1 + (1-\alpha) p_2)$$



3.14)
$$L(\underline{x}) = \frac{\dfrac{1}{2\pi \det^{1/2}(\underline{C})} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_1)^T \underline{C}^{-1}(\underline{x}-\underline{\mu}_1)}}{\dfrac{1}{2\pi \det^{1/2}(\underline{C})} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_0)^T \underline{C}^{-1}(\underline{x}-\underline{\mu}_0)}}$$

Taking logarithms we have

$$\ln L(\underline{x}) = -\frac{1}{2}\Big[\underline{x}^T \underline{C}^{-1} \underline{x} - 2\underline{x}^T \underline{C}^{-1} \underline{\mu}_1 + \underline{\mu}_1^T \underline{C}^{-1} \underline{\mu}_1 - \underline{x}^T \underline{C}^{-1} \underline{x} + 2\underline{x}^T \underline{C}^{-1} \underline{\mu}_0 - \underline{\mu}_0^T \underline{C}^{-1} \underline{\mu}_0 \Big]$$

or we decide H_1 if

$$\underline{x}^T \underline{C}^{-1} (\underline{\mu}_1 - \underline{\mu}_0) > \underbrace{\ln \gamma + \frac{1}{2}(\underline{\mu}_1^T \underline{C}^{-1} \underline{\mu}_1 - \underline{\mu}_0^T \underline{C}^{-1} \underline{\mu}_0)}_{\gamma'}$$

Now $\underset{\sim}{C}^{-1} = \dfrac{\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}}{1-\rho^2}$

$$\underline{\mu}_1 - \underline{\mu}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{C}^{-1}(\underline{\mu}_1 - \underline{\mu}_0) = \frac{1}{1-\rho^2}\begin{bmatrix} 1 \\ -\rho \end{bmatrix}$$

$$\underline{x}^T \underset{\sim}{C}^{-1}(\underline{\mu}_1 - \underline{\mu}_0) = \frac{x[0] - \rho x[1]}{1-\rho^2}$$

or we decide $\mathcal{H}_1$ if $x[0] - \rho x[1] > \gamma''$

If $\rho = 0$ we decide $\mathcal{H}_1$ if $x[0] > \gamma''$, since ~~then~~ $x[0]$ and $x[1]$ are independent and the PDF of $x[1]$ under $\mathcal{H}_0$ is the same as under $\mathcal{H}_1$. Thus $x[1]$ is irrelevant.

3.15) $$p(\underline{x};\mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2} \cdot \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=N}^{2N-1}x^2[n]}$$

$$p(\underline{x};\mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2} \cdot \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=N}^{2N-1}(x[n]-2A)^2}$$

$$\ln L(\underline{x}) = -\frac{1}{2\sigma^2}\left[\sum_{n=N}^{2N-1}(x[n]-2A)^2 - \sum_{n=N}^{2N-1} x^2[n]\right]$$

$$= -\frac{1}{2\sigma^2}\left(-2A\sum_{n=N}^{2N-1} x[n] + 4NA^2\right)$$

$$= \frac{A}{\sigma^2}\sum_{n=N}^{2N-1} x[n] - 2NA^2/\sigma^2$$

or since $A > 0$ we decide $\mathcal{H}_1$ if

$$T(\underline{x}) = \frac{1}{N}\sum_{n=N}^{2N-1} x[n] > \gamma'$$

The observed samples $\{x[0], x[1], \ldots, x[N-1]\}$ are irrelevant since they provide no discrimination between $\mathcal{H}_0$ and $\mathcal{H}_1$.

$$T(\underline{x}) \sim \begin{cases} \mathcal{N}(0, \sigma^2/N) & \mathcal{H}_0 \\ \mathcal{N}(2A, \sigma^2/N) & \mathcal{H}_1 \end{cases}$$

which is the mean-shifted Gauss-Gauss problem $\Rightarrow$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

where $d^2 = 4NA^2/\sigma^2$

3.1 b) Decide $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x} \mid \mathcal{H}_1)}{p(\mathbf{x} \mid \mathcal{H}_0)} > \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)}$$

$$e^{-\frac{1}{2\sigma^2}\left(-2A\sum x(n) + NA^2\right)} > \frac{1-P(\mathcal{H}_1)}{P(\mathcal{H}_1)}$$

$$\frac{A}{\sigma^2}\sum x(n) - \frac{NA^2}{2\sigma^2} > \ln\left[\frac{1-P(\mathcal{H}_1)}{P(\mathcal{H}_1)}\right]$$

$$\frac{1}{N}\sum_{n=0}^{N-1} x(n) > \underbrace{\frac{A}{2} + \frac{\sigma^2}{NA}\ln\left[\frac{1-P(\mathcal{H}_1)}{P(\mathcal{H}_1)}\right]}_{\gamma'}$$

For $N=1$, $A=1$, $\sigma^2=1$ decide $\mathcal{H}_1$ if

$$x(0) > \frac{1}{2} + \ln\frac{1-P(\mathcal{H}_1)}{P(\mathcal{H}_1)}$$

If $P(\mathcal{H}_0) = P(\mathcal{H}_1) \Rightarrow \gamma' = 1/2$

If $P(\mathcal{H}_0) = 1/4$, $P(\mathcal{H}_1) = 3/4 \Rightarrow \gamma' = -0.6$

Since $\mathcal{H}_1$ is more likely, the detector "biases" its decision toward $\mathcal{H}_1$ by lowering the threshold.

3.17) Since $P(\mathcal{H}_0) = P(\mathcal{H}_1)$, we use ML rule or we decide $\mathcal{H}_1$ if

$$p(\underline{x} \mid \mathcal{H}_1) > p(\underline{x} \mid \mathcal{H}_0)$$

$$\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}(\underline{x}-\underline{\mu})^T(\underline{x}-\underline{\mu})} > \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\underline{x}^T\underline{x}}$$

Taking logarithms

$$-\frac{1}{2\sigma^2}(\underline{x}^T\underline{x} - 2\underline{\mu}^T\underline{x} + \underline{\mu}^T\underline{\mu} - \underline{x}^T\underline{x}) > 0$$

$$\frac{1}{\sigma^2}(\underline{\mu}^T\underline{x} - \frac{1}{2}\underline{\mu}^T\underline{\mu}) > 0$$

or

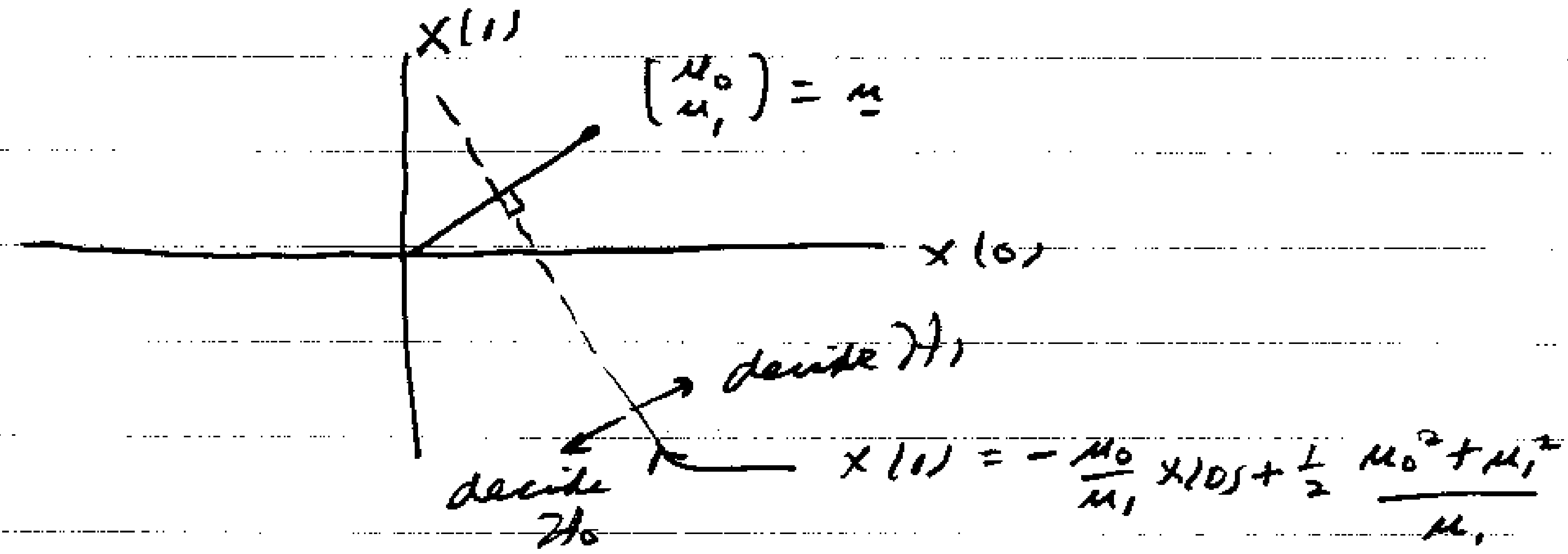
$$\underline{\mu}^T\underline{x} > \frac{1}{2}\underline{\mu}^T\underline{\mu}$$

For $N=2$ we decide $\mathcal{H}_1$ if

$$\mu_0 x[0] + \mu_1 x[1] > \frac{1}{2}(\mu_0^2 + \mu_1^2)$$

or if $$x[1] > -\frac{\mu_0}{\mu_1} x[0] + \frac{\frac{1}{2}(\mu_0^2+\mu_1^2)}{\mu_1}$$

(assuming $\mu_1 > 0$)



slope of line segment from origin to $\underline{\mu}$ is $\mu_1/\mu_0 \Rightarrow$ decision boundary line is perpendicular and also it intersects at $(\mu_0/2, \mu_1/2)$ or midpoint.

3.18) Decide $\mathcal{H}_1$ if $p(\mathcal{H}_1 | \underline{x}) > p(\mathcal{H}_0 | \underline{x})$

or if

$$p(\underline{x} | \mathcal{H}_1) P(\mathcal{H}_1) > p(\underline{x} | \mathcal{H}_0) P(\mathcal{H}_0)$$

$$\frac{p(\underline{x} | \mathcal{H}_1)}{p(\underline{x} | \mathcal{H}_0)} > \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma$$

$$\frac{\frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2 \cdot 2} x^2(0)}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2(0)}} > \gamma$$

$$\ln 1/\sqrt{2} - \frac{1}{4} x^2(0) + \frac{1}{2} x^2(0) > \ln \gamma$$

$$1/4\, x^2(0) > \ln \sqrt{2}\gamma$$

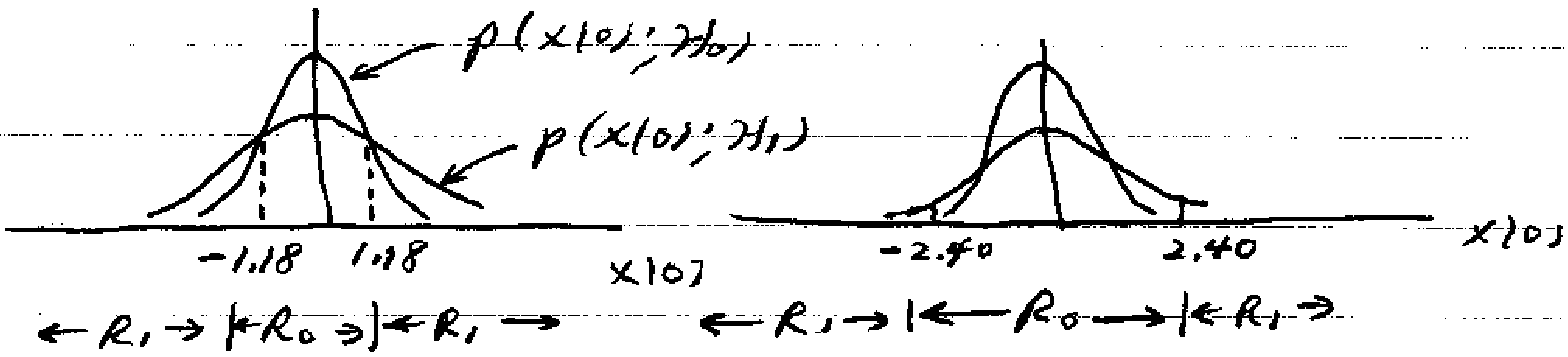
$$|x(0)| > 2 \sqrt{\ln \sqrt{2}\gamma}$$

For $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2 \Rightarrow \gamma = 1$

$|x[0]| > 1.18$

For $P(\mathcal{H}_0) = 3/4,\ P(\mathcal{H}_1) = 1/4 \Rightarrow \gamma = 3$

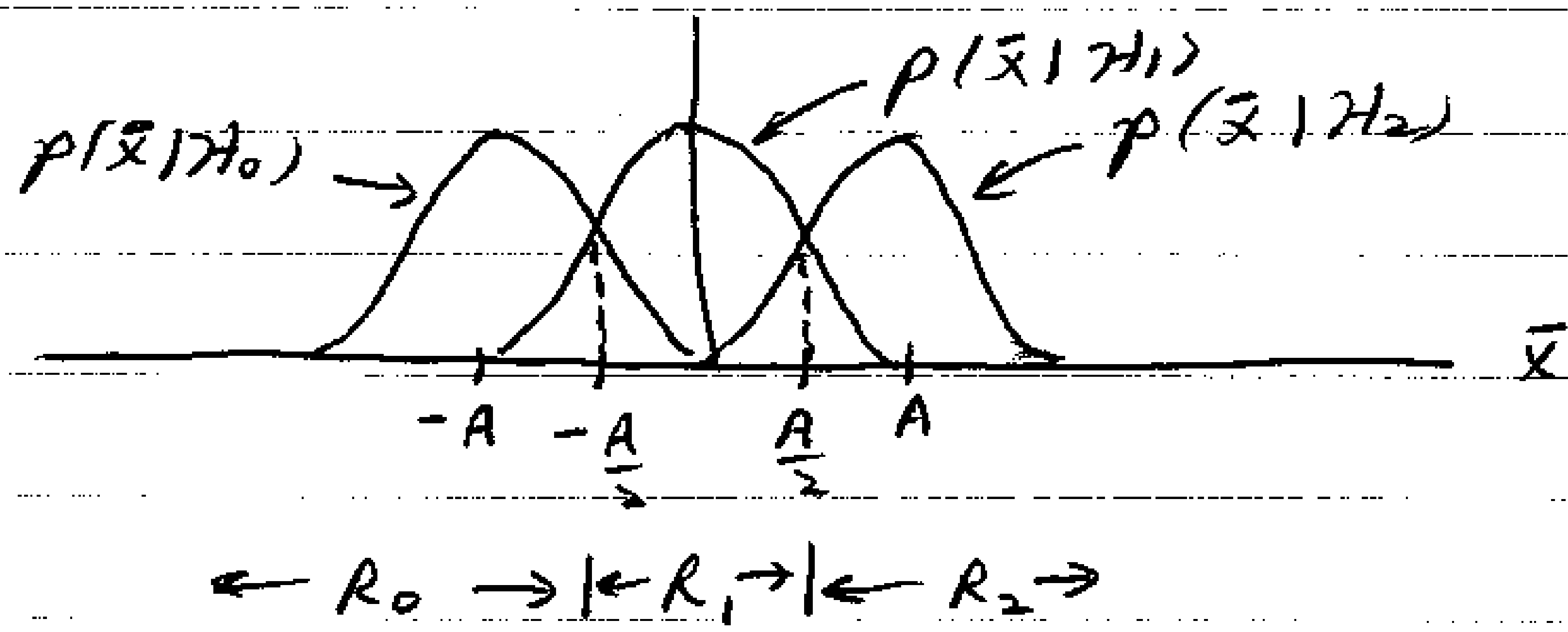
$|x[0]| > 2.40$



When $P(\mathcal{H}_0) > 1/2$, the threshold becomes larger than for $P(\mathcal{H}_0) = 1/2$ since $\mathcal{H}_0$ is more likely. Note that for $P(\mathcal{H}_0) = \frac{1}{2}$ we have the ML rule so that we decide $\mathcal{H}_1$ if $p(x[0]; \mathcal{H}_1) > p(x[0]; \mathcal{H}_0)$.

3.19) ML detector chooses $\mathcal{H}_i$ for which $p(\underline{x} | \mathcal{H}_i)$ is maximum. Since $\bar{x}$ is a sufficient statistic, we can equivalently decide $\mathcal{H}_i$ for which $p(\bar{x} | \mathcal{H}_i)$ is maximum. But

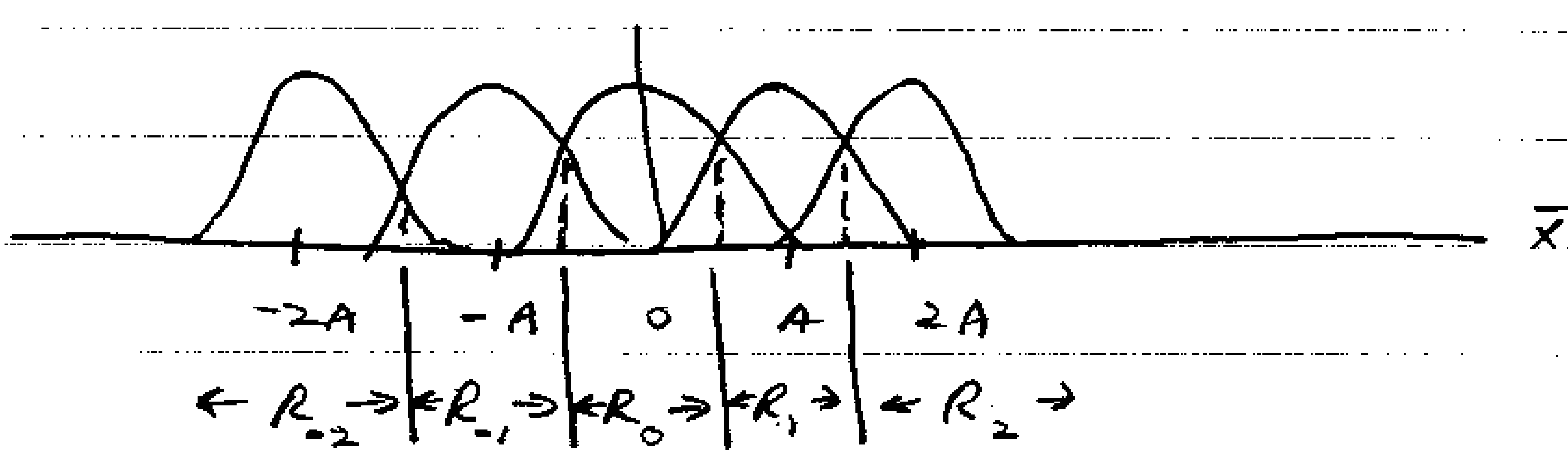
$$\bar{x} \sim \begin{array}{ll} \mathcal{N}(-A, \sigma^2/N) & \mathcal{H}_0 \\ \mathcal{N}(0, \sigma^2/N) & \mathcal{H}_1 \\ \mathcal{N}(A, \sigma^2/N) & \mathcal{H}_2 \end{array}$$



Same result as in Example 3.6.

3.20) As before $\bar{x}$ is a sufficient statistic. Hence, the ML detector decides $\mathcal{H}_i$ for which $p(\bar{x}|\mathcal{H}_i)$ is maximum. Consider $M=5$. Then we have levels $\{-2A, -A, 0, A, 2A\}$ so that the PDFs are (we relabel hypotheses)

$$
\begin{aligned}
\bar{x} \sim\ & \mathcal{N}(-2A, \sigma^2/N) && \mathcal{H}_{-2}\\
& \mathcal{N}(-A, \sigma^2/N) && \mathcal{H}_{-1}\\
& \mathcal{N}(0, \sigma^2/N) && \mathcal{H}_0\\
& \mathcal{N}(A, \sigma^2/N) && \mathcal{H}_1\\
& \mathcal{N}(2A, \sigma^2/N) && \mathcal{H}_2
\end{aligned}
$$



ML detector chooses the level which is closest to $\bar{x}$ since

$$p(\bar{x} \mid \mathcal{H}_i) = \frac{1}{\sqrt{2\pi\sigma^2/N}} e^{-\frac{1}{2\sigma^2}(\bar{x}-A_i)^2}$$

where $A_i = iA$

To maximize $p(\bar{x} \mid \mathcal{H}_i) \Rightarrow$ minimize $|\bar{x}-A_i|$

To find P_e we note that except for the $\pm\left(\frac{M-1}{2}\right)A$ levels we make an error if $|\bar{x}-A_i| > A/2$. There are $(M-2)$ of these types of errors. For the $\pm\left(\frac{M-1}{2}\right)A$ levels we make an error if

$$\bar{x} - A_{\frac{M-1}{2}} < A/2$$

$$\bar{x} - A_{-\left(\frac{M-1}{2}\right)} > A/2$$

or if

$$\bar{x} - \frac{M-1}{2}A < A/2$$

$$\bar{x} + \frac{M-1}{2}A > A/2$$

Hence, since $P(\mathcal{H}_i) = 1/M$ we have

$$P_e = 1/M \sum_{i=-(M-1)/2}^{(M-1)/2} P_e(\mathcal{H}_i)$$

$$= \frac{1}{M}\Big[(M-2)\,P_r\{|\bar{x}-A_i| > A/2 \mid \mathcal{H}_i,\ i \neq \pm\tfrac{M-1}{2}\}$$

$$+ P_r\{\bar{x} - \tfrac{M-1}{2}A < \tfrac{A}{2} \mid \mathcal{H}_{\frac{M-1}{2}}\}$$

$$+ P_r\{\bar{x} + \tfrac{M-1}{2}A > \tfrac{A}{2} \mid \mathcal{H}_{-\frac{M-1}{2}}\}\Big]$$

$$= \frac{1}{M}\left[(M-2)\Pr\{|\bar{w}| > A/2\} + \Pr\{\bar{w} < A/2\} + \Pr\{\bar{w} > A/2\}\right]$$

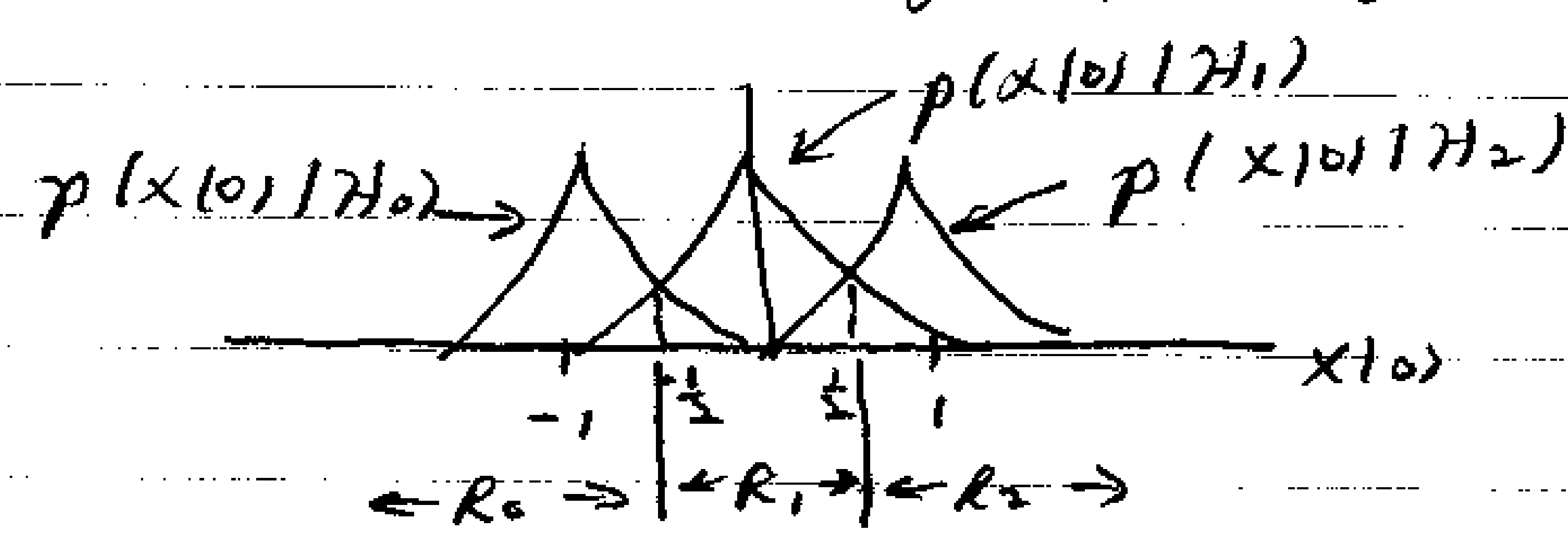
where $\bar{w} = \bar{x} - E(\bar{x}) \sim \mathcal{N}(0, \sigma^2/N)$

$$P_e = \frac{1}{M}\left[(M-2)\, 2Q\left(A/2\sqrt{\sigma^2/N}\right) + Q\left(\frac{A}{2\sqrt{\sigma^2/N}}\right) + Q\left(\frac{A}{2\sqrt{\sigma^2/N}}\right)\right]$$

$$= \frac{2M-2}{M}\, Q\left(\frac{A}{2\sqrt{\sigma^2/N}}\right)$$

$$= \frac{2M-2}{M}\, Q\left(\sqrt{\frac{NA^2}{4\sigma^2}}\right)$$

3.21) Decide $\mathcal{H}_i$ for which $p(x[0]|\mathcal{H}_i)$ is maximum. By symmetry we have



Decide $\mathcal{H}_0$ if $x[0] < -1/2$

$\mathcal{H}_1$ if $|x[0]| < 1/2$

$\mathcal{H}_2$ if $x[0] > 1/2$

$$P_e = 1 - P_c$$
$$= 1 - \frac{1}{3}\Big[P_r\{x(0) < -\tfrac{1}{2} \mid H_0\} + P_r\{-\tfrac{1}{2} < x(0) < \tfrac{1}{2} \mid H_1\} + P_r\{x(0) > \tfrac{1}{2} \mid H_2\}\Big]$$

$$= 1 - \frac{1}{3}\Big[P_r\{-\tfrac{1}{2} < x(0) < \tfrac{1}{2} \mid H_1\} + 2\, P_r\{x(0) > \tfrac{1}{2} \mid H_2\}\Big]$$

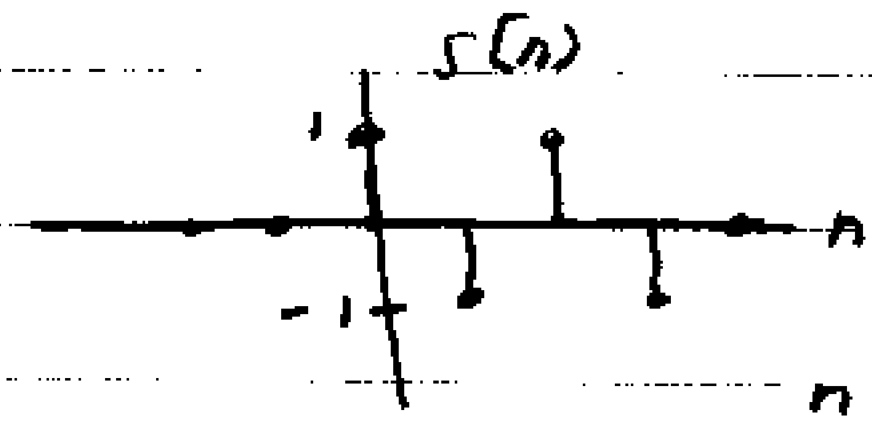
$$P_r\{-\tfrac{1}{2} < x(0) < \tfrac{1}{2} \mid H_1\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} e^{-|t|}\, dt$$
$$= 2\int_0^{\frac{1}{2}} \frac{1}{2} e^{-t}\, dt = -e^{-t}\Big|_0^{\frac{1}{2}}$$
$$= 1 - e^{-1/2}$$

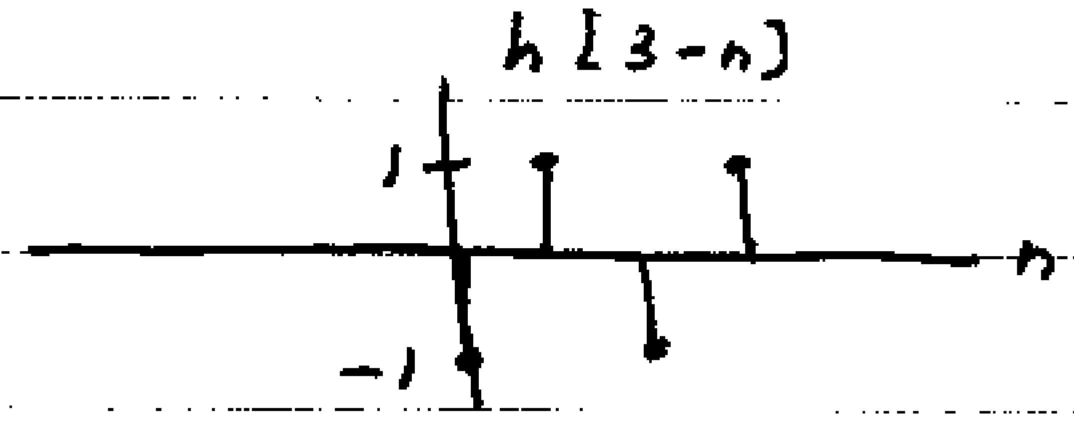
$$P_r\{x(0) > \tfrac{1}{2} \mid H_2\} = \int_{\frac{1}{2}}^{\infty} \frac{1}{2} e^{-|t-1|}\, dt$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{2} e^{t-1}\, dt + \int_1^{\infty} \frac{1}{2} e^{-t+1}\, dt$$
$$= \frac{1}{2} e^{-1}(e^{t})\Big|_{\frac{1}{2}}^{1} - \frac{1}{2} e\, e^{-t}\Big|_1^{\infty}$$
$$= \frac{1}{2} e^{-1}(e - e^{\frac{1}{2}}) + \frac{1}{2} = 1 - \frac{1}{2} e^{-\frac{1}{2}}$$

$$\Rightarrow P_e = 1 - \frac{1}{3}\left[1 - e^{-1/2} + 2 - e^{-1/2}\right] = \frac{2}{3} e^{-1/2}$$

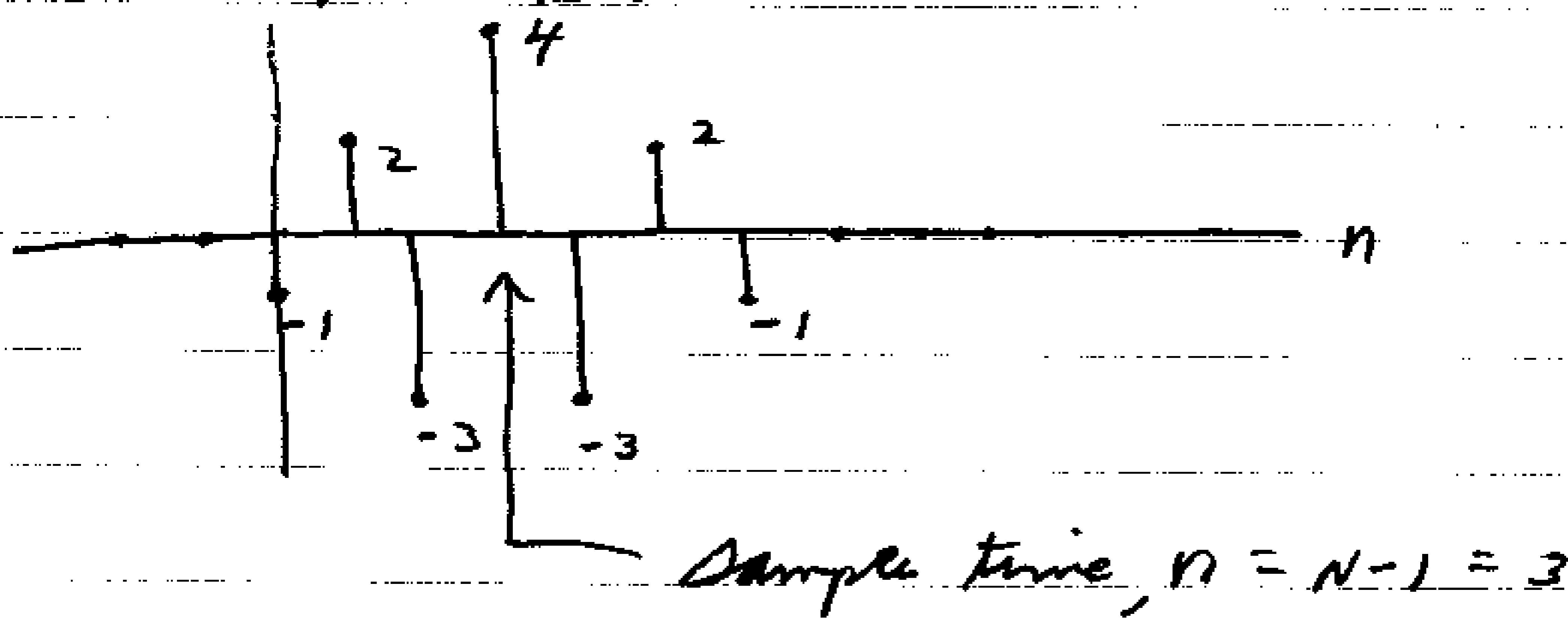
Chapter 4

4.1)





$$y[n] = \sum_{k=0}^{n} h[n-k]\, s[k]$$



4.2) Signal output is

$$y[n] = \int S^*(f)\, S(f)\, e^{j2\pi f(n-(N-1))}\, df$$
$$= \int |S(f)|^2 e^{j2\pi f(n-(N-1))}\, df$$
$$\le \int |S(f)|^2\, df$$

with equality if and only if $n = N-1$.

4.3) Signal output = energy

$$y[N-1] = \sum_{n=0}^{N-1} s^2[n]$$

$$= \sum_{n=0}^{N-1} A^2 \cos^2 2\pi f_0 n$$

$$= A^2 \sum_{n=0}^{N-1} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi f_0 n\right) \simeq NA^2/2$$

$$y[N-1] = \sum_{n=0}^{N-1} x(n)s(n)$$

$$= \sum_{n=0}^{N-1} s(n-n_0)s(n)$$

$$= \sum_{n=n_0}^{N-1} s(n-n_0)s(n)$$

$$= \sum_{n=0}^{N-1-n_0} s(n)s(n+n_0)$$ signal correlation

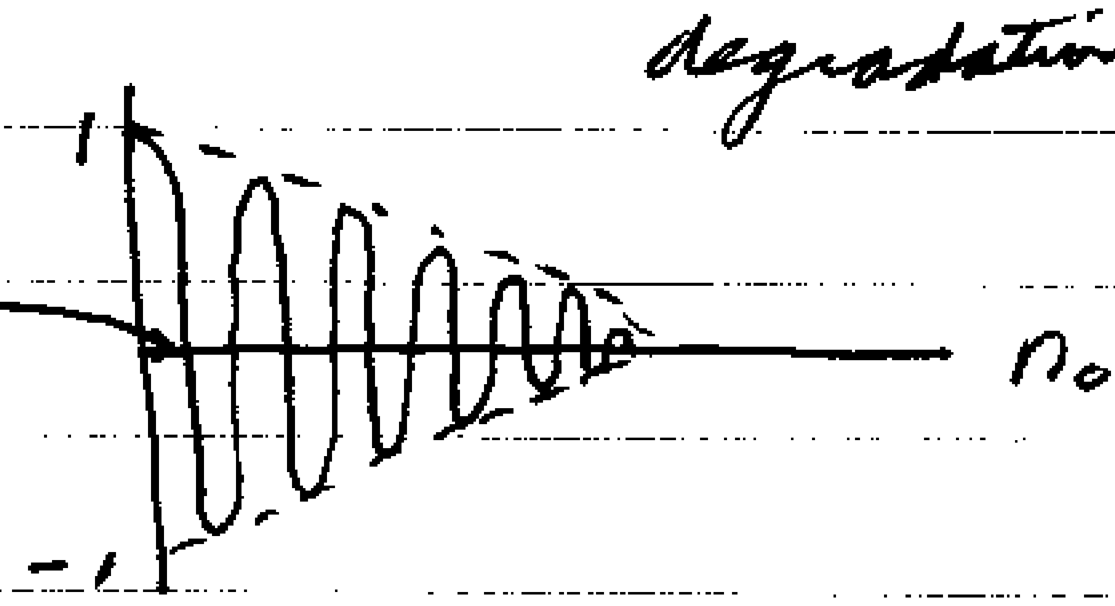
$$= A^2 \sum_{n=0}^{N-1-n_0} \cos 2\pi f_0 n \cos 2\pi f_0 (n+n_0)$$

$$= A^2 \sum_{n=0}^{N-1-n_0} \tfrac{1}{2}\cos 2\pi f_0 n_0 + \tfrac{1}{2}\cos(4\pi f_0 n + 2\pi f_0 n_0)$$

$$\simeq \frac{A^2}{2} \sum_{n=0}^{N-1-n_0} \cos 2\pi f_0 n_0$$

$$= \frac{NA^2}{2} \underbrace{\frac{N-n_0}{N} \cos 2\pi f_0 n_0}_{\text{degradation for } n_0 \neq 0}$$

$n_0 = \frac{1}{4f_0}$



4.4) $$\eta = \frac{\left(\sum_{k=0}^{N-1} h[N-1-k]\, s[k]\right)^2}{\sum_k \sum_l h[N-1-k]\, h[N-1-l] \cdot \underbrace{E(w[k]\, w[l])}_{\sigma^2 \delta[k-l]}}$$

$$= \frac{\left(\sum_{k=0}^{N-1} h[N-1-k]\, s[k]\right)^2}{\sigma^2 \sum_{k=-\infty}^{\infty} h^2[N-1-k]}$$

$$= \frac{\left(\sum_{l=0}^{N-1} h[l]\, s[N-1-l]\right)^2}{\sigma^2 \sum_{l=-\infty}^{\infty} h^2[l]}$$

Since the numerator does not depend on $h[l]$ for l outside the $[0, N-1]$ interval, we maximize η by minimizing the denominator. Hence, we set $h(l) = 0$ if l is outside $[0, N-1]$.

4.5) $$\eta = \frac{\left(\sum_{k=-\infty}^{\infty} h(k)\, s[N-1-k]\right)^2}{E\left(\sum_{k=-\infty}^{\infty} h[k]\, w[N-1-k]\right)^2}$$

$$= \frac{\left(\sum_{k=0}^{2N-1} h(k) s(N-1-k)\right)^2}{E\left[\left(\sum_{k=0}^{2N-1} h(k) w(N-1-k)\right)^2\right]}$$

But $\sum_{k=0}^{2N-1} h(k) w(N-1-k)$

$$= \sum_{k=0}^{N-1} 1 \cdot w(N-1-k) + \sum_{k=N}^{2N-1} (-1) w(N-1-k)$$

$$= \sum_{k=0}^{N-1} w(k) - \sum_{k=0}^{N-1} w(-k-1)$$

$$= \sum_{k=0}^{N-1} w(k) - \sum_{k=0}^{N-1} w(N-k-1) = 0$$

since $w(n)$ is periodic with period N. Noise is perfectly cancelled due to its periodic structure $\Rightarrow \eta \to \infty$ (as long as signal is not cancelled!)

4.6) From (4.3) decide H_1 if

$$T(\underline{x}) = \sum_{n=0}^{N-1} x(n) A r^n > \gamma'$$

From (4.14)

$$P_D = Q(Q^{-1}(P_{FA}) - \sqrt{\mathcal{E}/\sigma^2})$$

where $\mathcal{E}/\sigma^2 = \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}$

For $0 < r < 1$ $\quad \frac{\mathcal{E}}{\sigma^2} \to \frac{A^2}{\sigma^2} \frac{1}{1-r^2}$

$r = 1$ $\quad \mathcal{E}/\sigma^2 \to \infty \Rightarrow P_D \to 1$

$r > 1$ $\quad \mathcal{E}/\sigma^2 \to \infty \Rightarrow P_D \to 1$

4.7) From (4.14) a NP detector has

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\mathcal{E}/\sigma^2}\right)$$

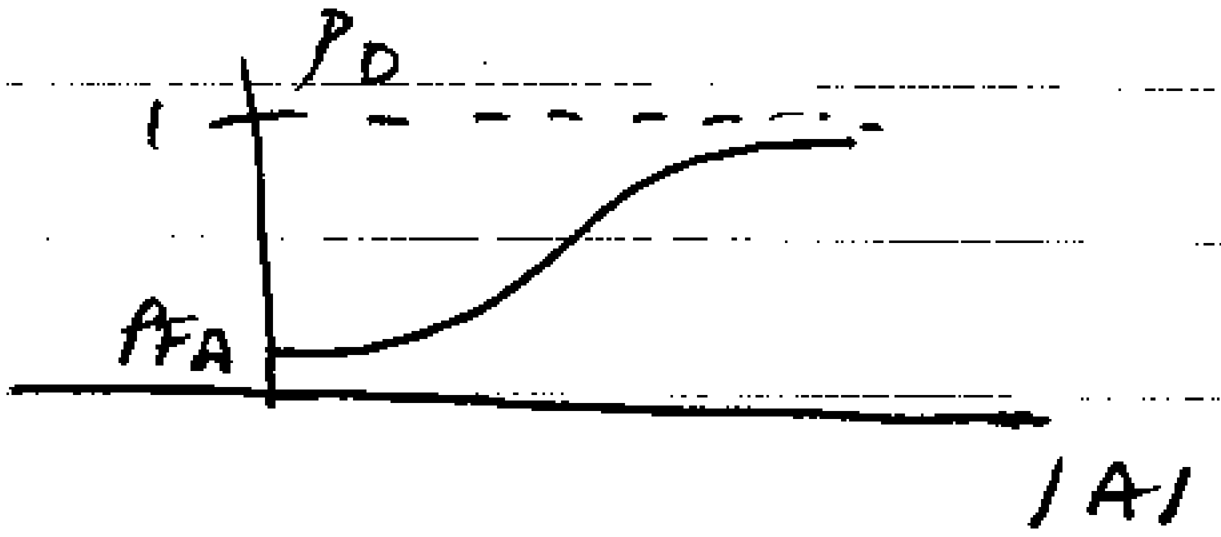
$$\mathcal{E}/\sigma^2 = \frac{\sum_{n=0}^{N-1} s^2[n]}{\sigma^2} = A^2 \sum_n \cos^2 2\pi f_0 n$$

$$= A^2 \sum_{n=0}^{24} \cos^2 \frac{\pi}{2} n$$

$$= A^2 (1 + 0 + 1 + 0 + \ldots + 1)$$

$$= 13 A^2$$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{13A^2}\right)$$



Same performance for $\pm A$

4.8) Both have same energy $\Rightarrow$ both have same detection performance

4.9) $\eta_{OUT} = \mathcal{E}/\sigma^2 = \dfrac{\sum_{n=0}^{N-1} A^2 \cos^2 2\pi f_0 n}{\sigma^2}$

$$\simeq NA^2/2\sigma^2$$

$\eta_{IN} = A^2/2\sigma^2$

$$PG = 10 \log_{10} \frac{NA^2/2\sigma^2}{A^2/2\sigma^2}$$

$$= 10 \log_{10} N \quad dB$$

$h[n] = s[N-1-n]$

$$= A \cos(2\pi f_0 (N-1-n)) \qquad 0 \le n \le N-1$$

$$H(f) = A \sum_{n=0}^{N-1} \cos 2\pi f_0 (N-1-n)\, e^{-j2\pi f n}$$

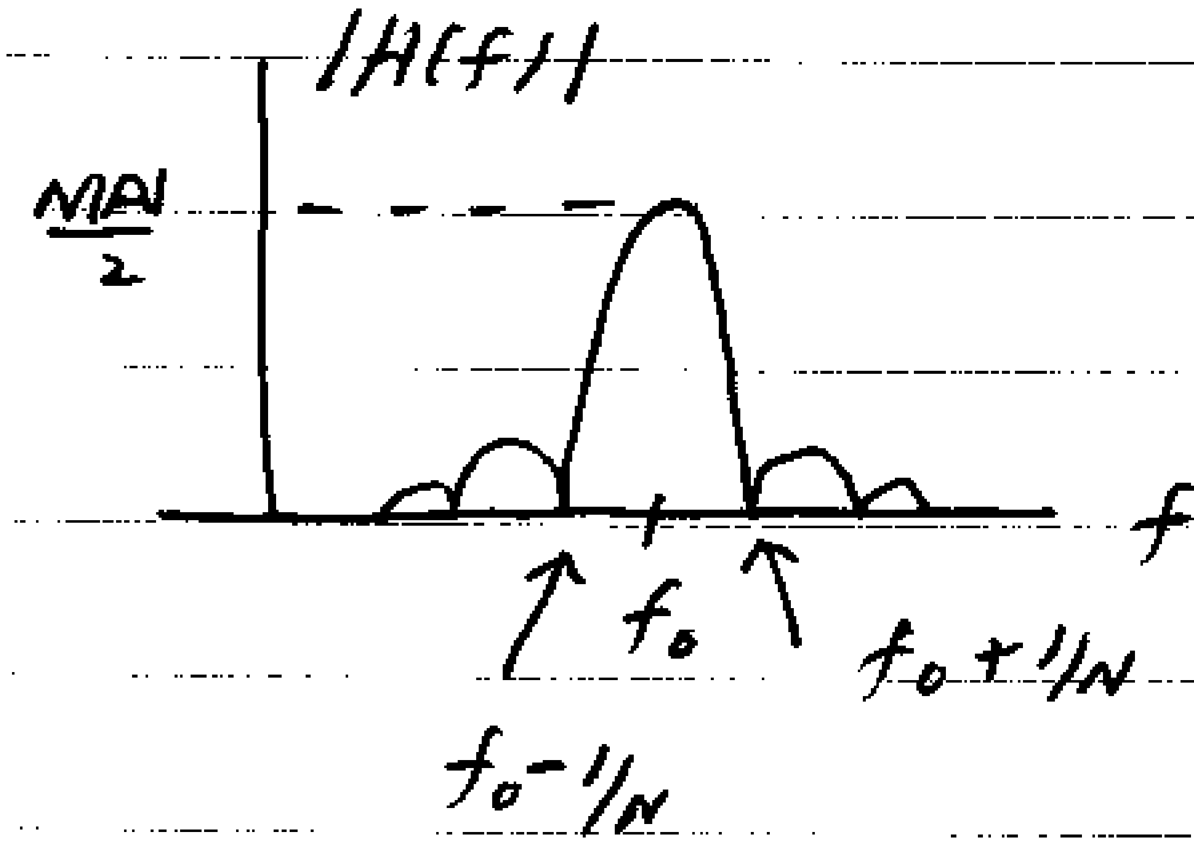
$$= A \sum_{n=0}^{N-1} \cos 2\pi f_0 n \; e^{-j2\pi f(N-1-n)}$$

$$= \frac{A}{2} e^{-j2\pi f(N-1)} \sum_{n=0}^{N-1} \left(e^{j2\pi(f+f_0)n} + e^{j2\pi(f-f_0)n} \right)$$

For N large

$$|H(f)| \approx \frac{|A|}{2} \left| \sum_{n=0}^{N-1} e^{j2\pi(f-f_0)n} \right|$$

$$= \frac{|A|}{2} \left| \frac{\sin N\pi(f-f_0)}{\sin \pi(f-f_0)} \right| \qquad f>0$$



As N increases, passband of filter decreases to reduce the noise but at the same time the signal is passed.

4.10) From Example 4.5

$$\underline{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underline{\Lambda} = \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}$$

$$\underline{V}^T \underline{C} \underline{V} = \underline{\Lambda} \quad \Rightarrow \quad \underline{C} = \underline{V}\,\underline{\Lambda}\,\underline{V}^T$$

$$\underline{C}^{-1} = \underline{V}\,\underline{\Lambda}^{-1}\underline{V}^T$$

$$= \underline{D}^T\underline{D}$$

$$\Rightarrow \quad \underline{D} = \sqrt{\underline{\Lambda}^{-1}}\,\underline{V}^T \qquad \text{where } \underline{\Lambda}^{-1} = \sqrt{\underline{\Lambda}^{-1}}\sqrt{\underline{\Lambda}^{-1}}$$

$$\underline{D} = \begin{bmatrix} \frac{1}{\sqrt{1+\rho}} & 0 \\ 0 & \frac{1}{\sqrt{1-\rho}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2(1+\rho)}} & \frac{1}{\sqrt{2(1+\rho)}} \\ \frac{1}{\sqrt{2(1-\rho)}} & -\frac{1}{\sqrt{2(1-\rho)}} \end{bmatrix}$$

4.11) $\mathcal{H}_0: \underline{x} = \underline{w}$

$\mathcal{H}_1: \underline{x} = \underline{s} + \underline{w}$

or equivalently

$\mathcal{H}_0: \underline{y} = \underline{D}\underline{w}$

$\mathcal{H}_1: \underline{y} = \underline{D}\underline{s} + \underline{D}\underline{w}$

Under $\mathcal{H}_0$ $\underline{y} \sim \mathcal{N}(\underline{0}, \underline{I})$

$\mathcal{H}_1$ $\underline{y} \sim \mathcal{N}(\underline{D}\underline{s}, \underline{I})$

$$L(\underline{y}) = \frac{p(\underline{y}; \mathcal{H}_1)}{p(\underline{y}; \mathcal{H}_0)}$$

$$= \frac{\frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}(\underline{y}-\underline{D}\underline{s})^T(\underline{y}-\underline{D}\underline{s})}}{\frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}\underline{y}^T\underline{y}}}$$

$$\ln L(\underline{y}) = -\frac{1}{2}(\underline{y}^T\underline{y} - 2\underline{s}^T\underline{D}^T\underline{y} + \underline{s}^T\underline{D}^T\underline{D}\underline{s} - \underline{y}^T\underline{y})$$

$$= \underline{s}^T\underline{D}^T\underline{y} - \frac{1}{2}\underline{s}^T\underline{C}^{-1}\underline{s}$$

or in terms of $\underline{x}$

$$\ln L(\underline{y}) = \underline{s}^T\underbrace{\underline{D}^T\underline{D}}_{\underline{C}^{-1}}\underline{x} - \frac{1}{2}\underline{s}^T\underline{C}^{-1}\underline{s}$$

and we decide $\mathcal{H}_1$ if