$$\underline{s}^T \underline{C}^{-1} \underline{x} = \underline{x}^T \underline{C}^{-1} \underline{s} > \gamma'$$

4.12) $$\underline{x}^T \underline{C}^{-1} \underline{s} = \underline{x}^T \sum_{i=0}^{N-1} 1/\lambda_i \underline{v}_i \underline{v}_i^H \underline{s}$$

$$= \sum_{i=0}^{N-1} \frac{(\underline{v}_i^T \underline{x})(\underline{v}_i^H \underline{s})}{\lambda_i}$$

$$\approx \sum_{i=0}^{N-1} \frac{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{j 2\pi/N \, i n} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(n) e^{-j\frac{2\pi}{N} i n}}{P_{ww}(i/N)}$$

$$= \sum_{i=0}^{N-1} \frac{X^*(f_i) S(f_i)}{P_{ww}(f_i)} \frac{1}{N}$$

where $f_i = i/N$

$$= \sum_{i=0}^{N-1} \frac{X(f_i) S^*(f_i)}{P_{ww}(f_i)} \frac{1}{N}$$ since $\underline{x}^T \underline{C}^{-1} \underline{s}$ is real

$$\rightarrow \int_0^1 \frac{X(f) S^*(f)}{P_{ww}(f)} df = \int^{\frac{1}{2}} \frac{X(f) S^*(f)}{P_{ww}(f)} df$$

4.13) $$d^2 = \underline{s}^T \underline{C}^{-1} \underline{s}$$

$$= \underline{s}^T \sum_i \frac{1}{\lambda_i} \underline{v}_i \underline{v}_i^H \underline{s}$$

$$= \sum_i \frac{|\underline{v}_i^H \underline{s}|^2}{\lambda_i}$$

$$\approx \sum_{i=0}^{N-1} \frac{\frac{1}{N} \left| \sum_{n=0}^{N-1} s(n) e^{-j 2\pi f_i n} \right|^2}{P_{ww}(f_i)}$$

$$= \sum_{i=0}^{N-1} \frac{|S(f_i)|^2}{P_{ww}(f_i)} \frac{1}{N}$$

$$\to \int_0^1 \frac{|S(f)|^2}{P_{ww}(f)} \, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|S(f)|^2}{P_{ww}(f)} \, df$$

4.14) $y[n] = \sum_{k=-\infty}^{\infty} h(k) s[n-k]$

$$\Rightarrow Y(f) = H(f)S(f)$$

$$y(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)S(f) e^{j2\pi fn} \, df$$

$$y(N-1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)S(f) e^{j2\pi f(N-1)} \, df$$

$$w(n) \longrightarrow \boxed{H(f)} \longrightarrow y(n) = \sum_{k=-\infty}^{\infty} h[k] w[n-k]$$

A standard result is that if $w(n)$ is WSS then so is $y(n)$ and

$$P_{yy}(f) = |H(f)|^2 P_{ww}(f)$$

$$\Rightarrow E(y^2[N-1]) = r_{yy}(0)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{yy}(f) \, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 P_{ww}(f) \, df$$

To maximize η let

$$g(f) = H(f)\sqrt{P_{WW}(f)}$$
$$h(f) = \frac{S(f)\, e^{j2\pi f(N-1)}}{\sqrt{P_{WW}(f)}}$$

$$\eta \le \int_{-\frac{1}{2}}^{\frac{1}{2}} |h(f)|^2 \, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|S(f)|^2}{P_{WW}(f)} \, df = \eta_{MAX}$$

and the maximum is attained for

$$g(f) = c\, h^*(f)$$
$$H(f)\sqrt{P_{WW}(f)} = \frac{c\, S^*(f)\, e^{-j2\pi f(N-1)}}{\sqrt{P_{WW}(f)}}$$

or letting $c = 1$

$$H(f) = \frac{S^*(f)\, e^{-j2\pi f(N-1)}}{P_{WW}(f)}$$

4.15) $T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s}$

$$= \sum_{n=0}^{N-1} \frac{x(n)s(n)}{\sigma_n^2}$$

$$= A \sum_{n=0}^{N-1} \frac{x(n)}{\sigma^2 r^n}$$

or we decide $\mathcal{H}_1$ if

$$\sum_{n=0}^{N-1} \frac{x(n)}{r^n} > \gamma'$$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\underline{s}^T \underline{C}^{-1} \underline{s}}\right)$$

$$\underline{s}^T \underline{C}^{-1} \underline{s} = A^2 \sum_{n=0}^{N-1} \frac{1}{\sigma^2 r^n}$$

$$= \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} 1/r^n$$

As $N \to \infty$ if $0 < r \leq 1$, $\underline{s}^T \underline{C}^{-1} \underline{s} \to \infty$
$\Rightarrow P_D \to 1$

if $r > 1$, $\underline{s}^T \underline{C}^{-1} \underline{s} \to \frac{A^2}{\sigma^2} \frac{1}{1 - 1/r}$

$$= \frac{A^2}{\sigma^2} \frac{r}{r-1}$$

In first case the noise "dies out" if $0 < r < 1$ and if $r = 1$ we have a DC level in WGN so that averaging causes $P_D \to 1$.

4.16) $d^2 = \underline{s}^T \underline{C}^{-1} \underline{s}$

Using Woodbury's identity

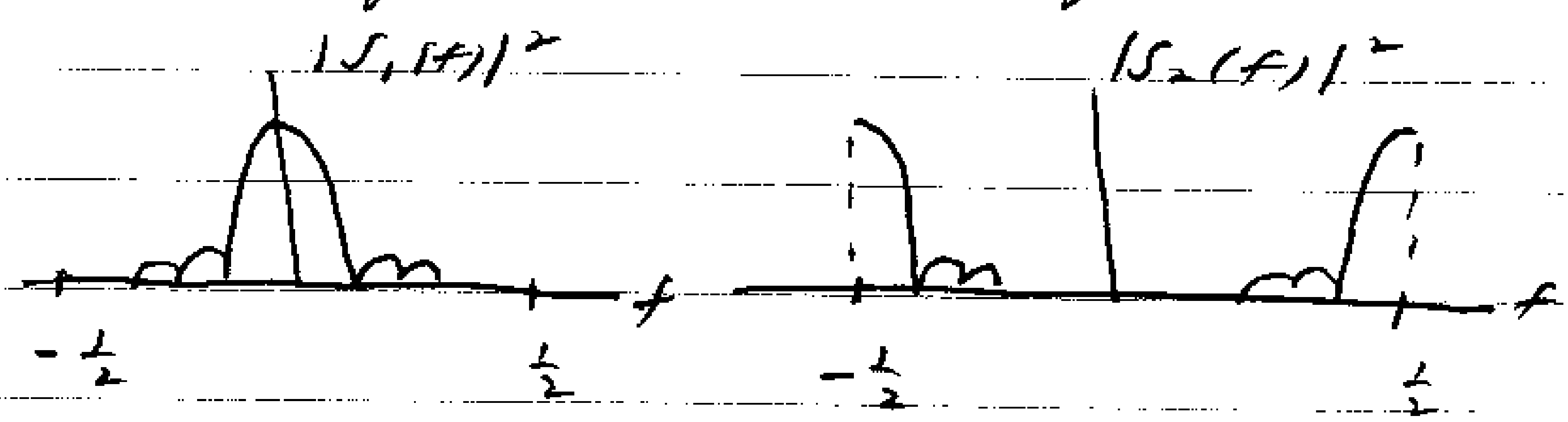
$$\underline{C}^{-1} = \frac{1}{\sigma^2}\underline{I} - \frac{1}{\sigma^4}\frac{P\underline{1}\underline{1}^T}{1+NP/\sigma^2}$$

$$d^2 = \frac{\underline{s}^T\underline{s}}{\sigma^2} - \frac{P/\sigma^4(\underline{s}^T\underline{1})^2}{1+NP/\sigma^2}$$

$$= \frac{\mathcal{E}}{\sigma^2} - \frac{1}{\sigma^2}\frac{P}{NP+\sigma^2}(\underline{s}^T\underline{1})^2$$

The better signal is the one that minimizes $\underline{s}^T\underline{1} = \sum_{n=0}^{N-1} s[n]$ which is $s_2[n]$.

This is because the noise PSD is $P_{WW}(f) = \sigma^2 + P\,\delta(f)$ or there is an impulse at DC. The FT magnitudes of the signals are



$s_2[n]$ has better overall SNR.

4.17) $T(\underline{x}) = \int \frac{X(f)S^*(f)}{P_{WW}(f)}df$

$$= \int \frac{1}{\sigma^2} X(f)S^*(f)\left|1+ae^{-j2\pi f}\right|^2 df$$

$$= \frac{1}{\sigma^2} \int [A(f)\, X(f)][A(f)\, S(f)]^* \, df$$

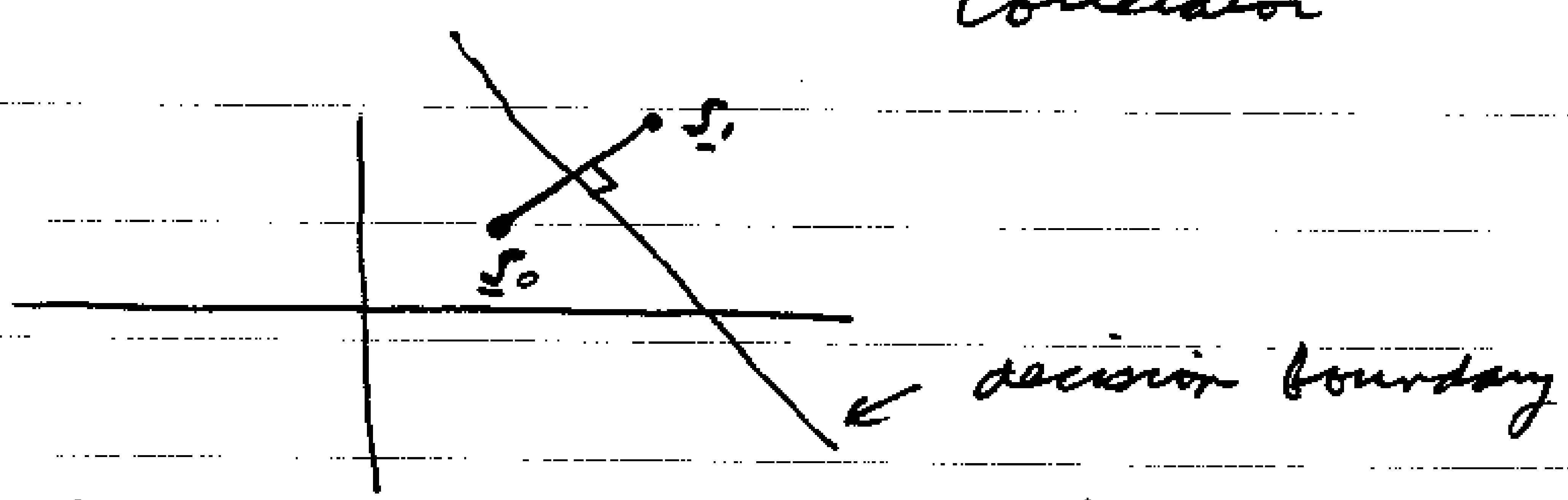
$$= \frac{1}{\sigma^2} \sum_{n=-\infty}^{\infty} (x(n) + a\,x(n-1))(s(n) + a\,s(n-1))$$

by Parseval's theorem

$$= \frac{1}{\sigma^2} \sum_{n=0}^{N} (x(n) + a\,x(n-1))(s(n) + a\,s(n-1))$$

$$\approx \underbrace{\frac{1}{\sigma^2} \sum_{n=1}^{N-1} \underbrace{(x(n) + a\,x(n-1))}_{\text{prewhitener}} \underbrace{(s(n) + a\,s(n-1))}_{\text{modified signal}}}_{\text{correlator}}$$

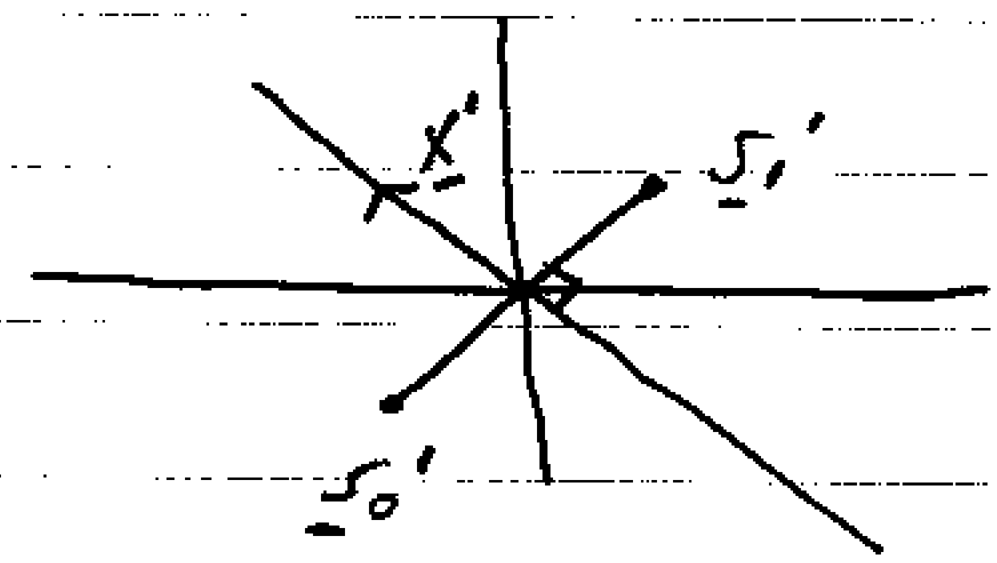
4.18)



Decision boundary is a hyperplane whose points satisfy $\| \underline{x} - \underline{s}_0 \| = \| \underline{x} - \underline{s}_1 \|$. The midpoint of the line segment is $\frac{\underline{s}_0 + \underline{s}_1}{2}$ so that $\| \frac{\underline{s}_0 + \underline{s}_1}{2} - \underline{s}_0 \| =$

$\| \frac{\underline{s}_1}{2} - \frac{\underline{s}_0}{2} \| = \| \frac{\underline{s}_0 + \underline{s}_1}{2} - \underline{s}_1 \|$ and it lies on hyperplane. To show that line segment is perpendicular let

$\underline{x}' = \underline{x} - \frac{\underline{s}_0 + \underline{s}_1}{2}$ so that



$$\underline{x}'^T \frac{(\underline{s}_1' - \underline{s}_0')}{2} = \left(\underline{x} + \frac{\underline{s}_0 + \underline{s}_1}{2}\right)^T \left(\frac{\underline{s}_1 - \underline{s}_0}{2}\right)$$

$$= \frac{1}{2}\underline{x}^T(\underline{s}_1 - \underline{s}_0) + \frac{1}{4}\underline{s}_0^T\underline{s}_0 - \frac{1}{4}\underline{s}_1^T\underline{s}_1$$

$$= \frac{1}{4}\left[(\underline{x} - \underline{s}_0)^T(\underline{x} - \underline{s}_0) - (\underline{x} - \underline{s}_1)^T(\underline{x} - \underline{s}_1)\right]$$

$$= \frac{1}{4}(\|\underline{x} - \underline{s}_0\|^2 - \|\underline{x} - \underline{s}_1\|^2) = 0$$

4.19) Using the MAP rule we decide $\mathcal{H}_1$ if

$$\mathfrak{Z}_1 = \ln p(\underline{x}|\mathcal{H}_1) + \ln P(\mathcal{H}_1)$$
$$> \ln p(\underline{x}|\mathcal{H}_0) + \ln P(\mathcal{H}_0) = \mathfrak{Z}_0$$

But $p(\underline{x}|\mathcal{H}_i) = \frac{1}{2\pi} e^{-\frac{1}{2}(\underline{x} - \underline{s}_i)^T(\underline{x} - \underline{s}_i)}$

for $N = 2$ and $\sigma^2 = 1$

$$\mathfrak{Z}_i = -\ln 2\pi - \frac{1}{2}(\underline{x} - \underline{s}_i)^T(\underline{x} - \underline{s}_i) + \ln P(\mathcal{H}_i)$$

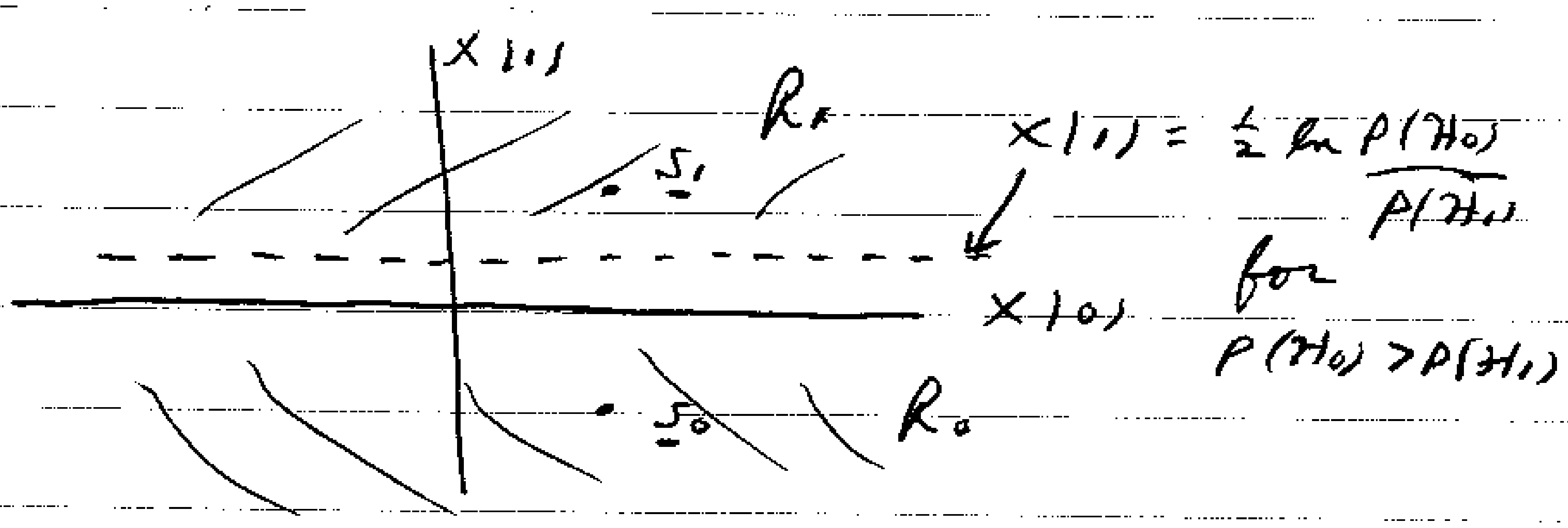
The decision boundary is obtained by letting $\mathfrak{J}_1 = \mathfrak{J}_0$ or

$$-\tfrac{1}{2}(\underline{x}-\underline{s}_1)^T(\underline{x}-\underline{s}_1) + \ln P(\mathcal{H}_1) =$$
$$-\tfrac{1}{2}(\underline{x}-\underline{s}_0)^T(\underline{x}-\underline{s}_0) + \ln P(\mathcal{H}_0)$$

$$\underline{x}^T\underline{s}_1 - \tfrac{1}{2}\mathcal{E}_1 + \ln P(\mathcal{H}_1) = \underline{x}^T\underline{s}_0 - \tfrac{1}{2}\mathcal{E}_0 + \ln P(\mathcal{H}_0)$$
$$\underline{x}^T(\underline{s}_1 - \underline{s}_0) = \tfrac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0) + \ln \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)}$$

For given $\underline{s}_0, \underline{s}_1$ we have

$$\underline{x}^T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \ln P(\mathcal{H}_0)/P(\mathcal{H}_1)$$
$$x(1) = \tfrac{1}{2}\ln P(\mathcal{H}_0)/P(\mathcal{H}_1)$$



If $P(\mathcal{H}_0) = P(\mathcal{H}_1)$, we decide $\mathcal{H}_1$ if $x(1) > 0$ (minimum distance) and if $P(\mathcal{H}_0) > P(\mathcal{H}_1)$ we move decision boundary to include more of $\underline{x}$ in R_0.

4.20) $(\underline{s}_1 \pm \underline{s}_0)^T (\underline{s}_1 \pm \underline{s}_0) \geq 0$

$$\underline{s}_1^T \underline{s}_1 \pm 2\,\underline{s}_1^T \underline{s}_0 + \underline{s}_0^T \underline{s}_0 \geq 0$$

$$\frac{1}{2}\left(\underline{s}_1^T \underline{s}_1 + \underline{s}_0^T \underline{s}_0\right) \geq \mp \underline{s}_1^T \underline{s}_0$$

$$1 \geq \mp \rho_s \Rightarrow \rho_s \leq 1, \; -\rho_s \leq 1$$

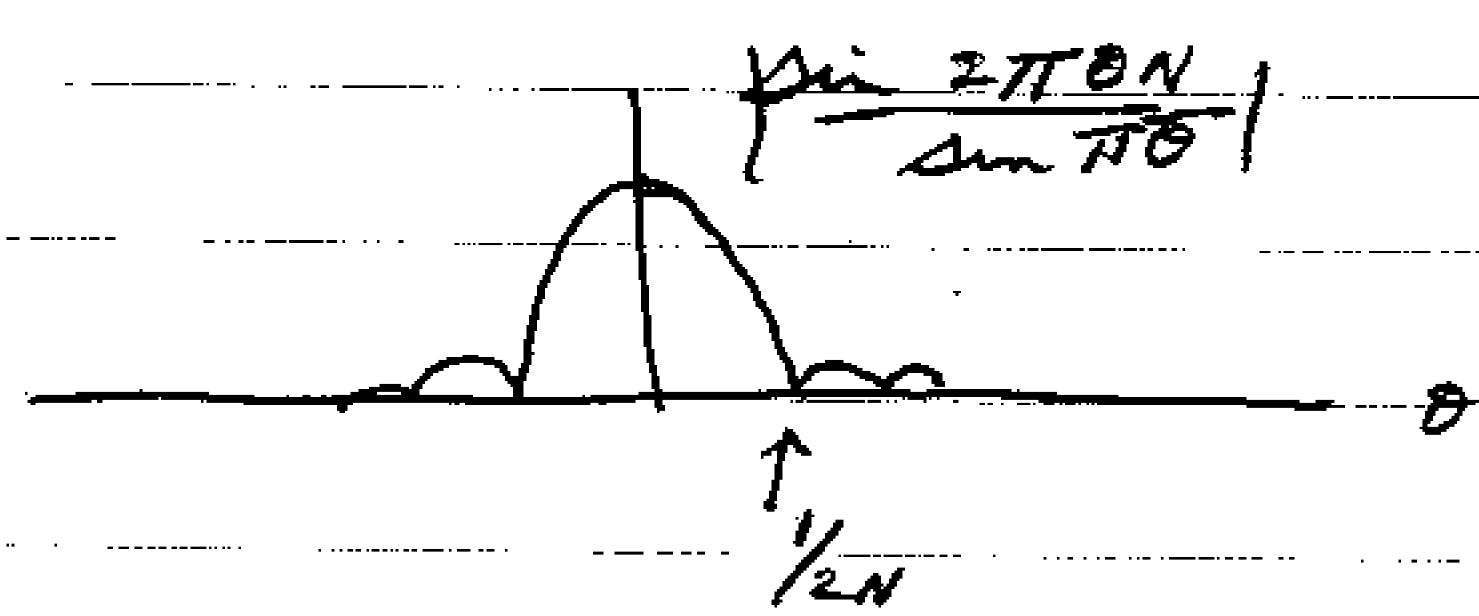
$$\text{or } \rho_s \geq -1 \Rightarrow |\rho_s| \leq 1$$

4.21) $\displaystyle\sum_{n=0}^{N-1} s_0[n] s_1[n] = A^2 \sum_n \cos 2\pi f_0 n \cdot \cos 2\pi f_1 n$

$$= \frac{A^2}{2} \sum_n \left[\cos 2\pi(f_0+f_1)n + \cos 2\pi(f_1-f_0)n\right]$$

$$\approx \frac{A^2}{2}\left[\frac{1}{2}\frac{\sin 2\pi(f_0+f_1)N}{\sin \pi(f_0+f_1)} + \frac{1}{2}\frac{\sin 2\pi(f_1-f_0)N}{\sin \pi(f_1-f_0)}\right]$$

For $(f_1 - f_0) \gg 1/2N$ and f_0, f_1 not near 0 or $\frac{1}{2}$, this is ≈ 0



4.22) Must minimize ρ_s.
But from Problem 4.21

$$\rho_s = \frac{\underline{s}_1^T \underline{s}_0}{\frac{1}{2}(\underline{s}_1^T\underline{s}_1 + \underline{s}_0^T\underline{s}_0)}$$

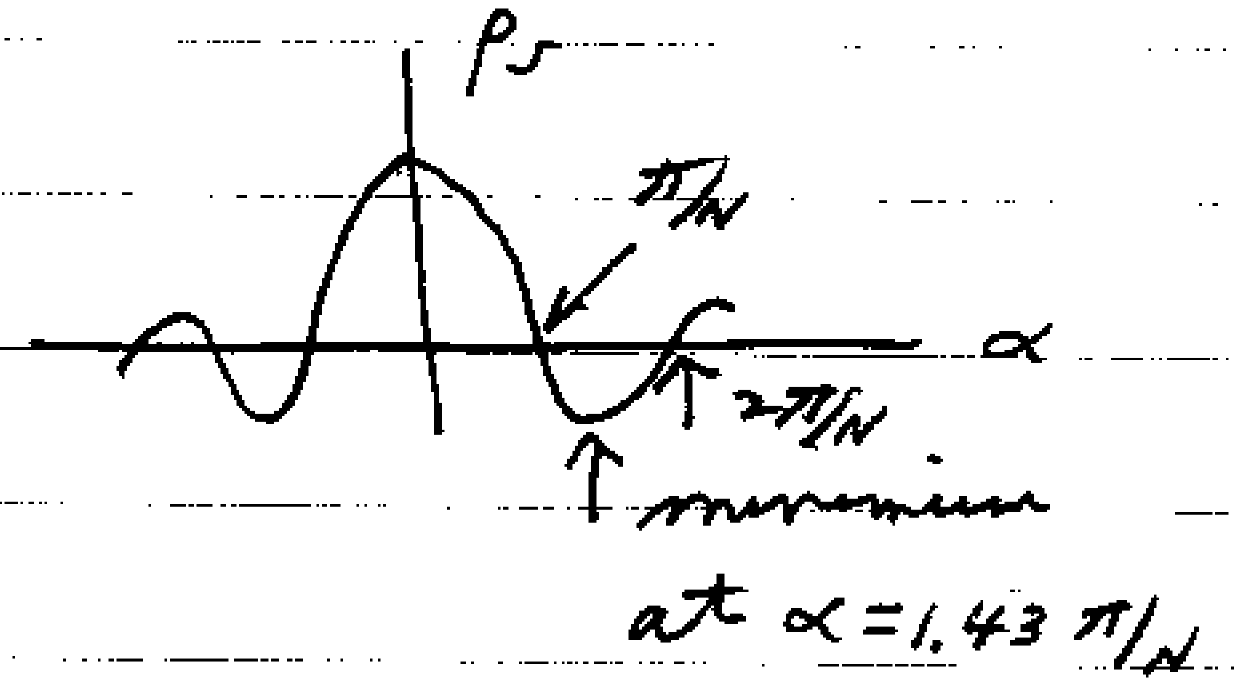
$$\approx \frac{\sin 2\pi(f_1-f_0)N}{2N \sin \pi(f_1-f_0)}$$

since $\underline{s}_1^T\underline{s}_1 \simeq \underline{s}_0^T\underline{s}_0 \simeq NA^2/2$.

Let $\alpha = 2\pi(f_1 - f_0)$

$$\rho_s \simeq \frac{\sin N\alpha}{2N \sin \alpha/2}$$

$$\simeq \frac{\sin N\alpha}{N\alpha}$$



or $2\pi|f_1 - f_0| = 1.4\,\pi/N$

$\Rightarrow \quad |f_1 - f_0| = 0.7/N$

4.23) $\quad P_e = Q\left(\sqrt{\dfrac{\bar{\mathcal{E}}(1-\rho_s)}{2\sigma^2}}\right)$

$$\bar{\mathcal{E}} \approx \frac{1}{2}\left(NA^2/2 + 0\right) = NA^2/4$$

$\rho_s = 0$

$\Rightarrow \quad P_e = Q\left(\sqrt{\dfrac{NA^2}{8\sigma^2}}\right)$

For PSK $\quad P_e = Q\left(\sqrt{\mathcal{E}/\sigma^2}\right)$

$$= Q\left(\sqrt{\frac{NA^2}{2\sigma^2}}\right)$$

For FSK $\quad P_e = Q\left(\sqrt{\frac{\mathcal{E}}{2\sigma^2}}\right)$

$$= Q\left(\sqrt{\frac{NA^2}{4\sigma^2}}\right)$$

OOK is 6 dB poorer than PSK and 3 dB poorer than FSK, assuming a peak power constraint or the sinusoidal amplitude A is the same for all three systems.

4.24) Use ML rule $\Rightarrow$ use (4.26)

$$T_i(\underline{x}) = \sum_n x[n]\, s_i[n] - \frac{1}{2}\mathcal{E}_i$$

with $s_i[n] = A_i$

$$T_i(\underline{x}) = NA_i\bar{x} - \frac{1}{2}NA_i^2$$

Decide H_k for which $T_i(\underline{x})$ is maximum.

For $M=2$ we have

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}(1-\rho_s)}{2\sigma^2}}\right)$$

To minimize P_e must minimize ρ_s

But

$$\rho_s = \frac{\sum_n s_0[n]s_1[n]}{\frac{1}{2}\left(\sum_n s_0^2[n] + \sum_n s_1^2[n]\right)}$$

$$= \frac{N A_0 A_1}{\frac{1}{2}(N A_0^2 + N A_1^2)}$$

$$= \frac{A_0 A_1}{\frac{1}{2}(A_0^2 + A_1^2)}$$

Choose $A_1 = -A_0 \Rightarrow$ antipodal signals

Then, $\rho_s = -1$.

4.25) $$P_e = 1 - \int_{-\infty}^{\infty} \Phi(u) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(u-\sqrt{\mathcal{E}/\sigma^2}\right)^2} du$$

$$= 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(u-\sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)^2} du$$

$$= 1 - \int_{-\infty}^{\infty}\int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(t^2 + \left(u-\sqrt{\mathcal{E}/\sigma^2}\right)^2\right)} dt\, du$$

Let $v = u - \sqrt{\varepsilon/\sigma^2}$

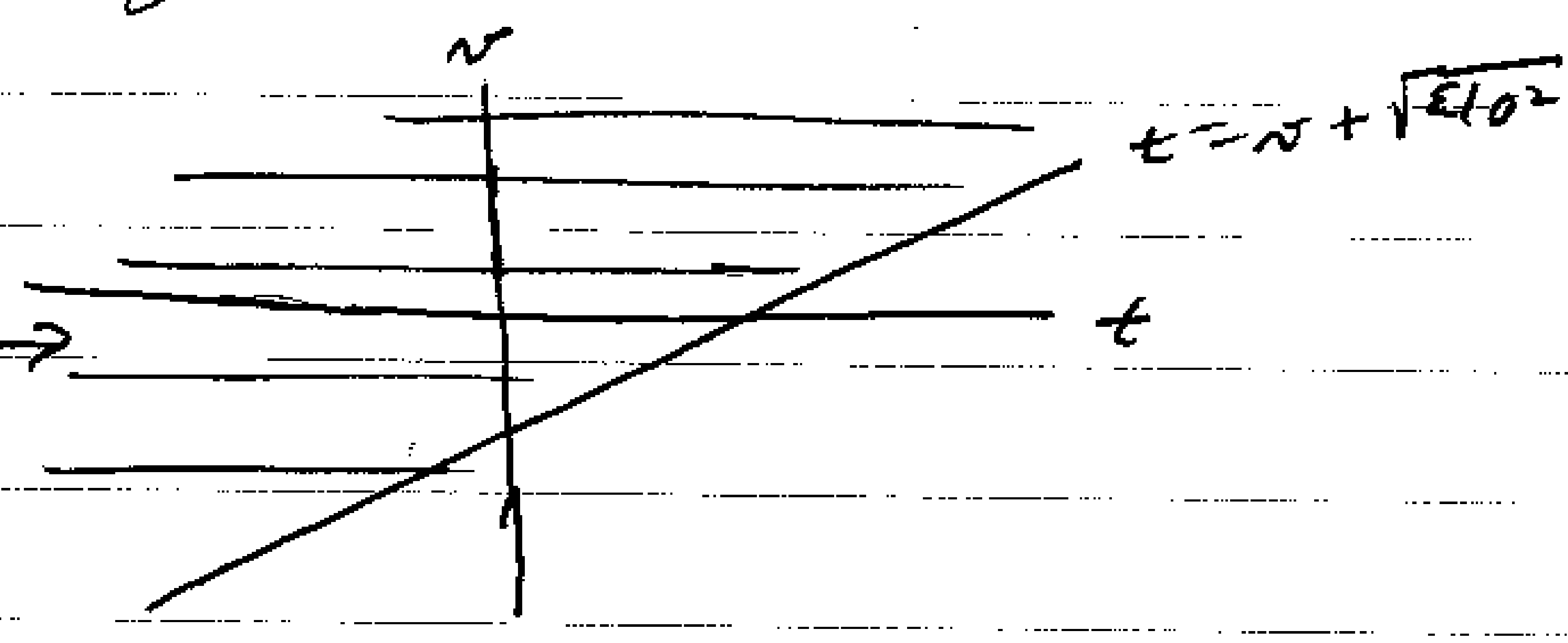
$$P_e = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{v + \sqrt{\varepsilon/\sigma^2}} \frac{1}{2\pi} e^{-\frac{1}{2}(t^2 + v^2)} \, dt \, dv$$

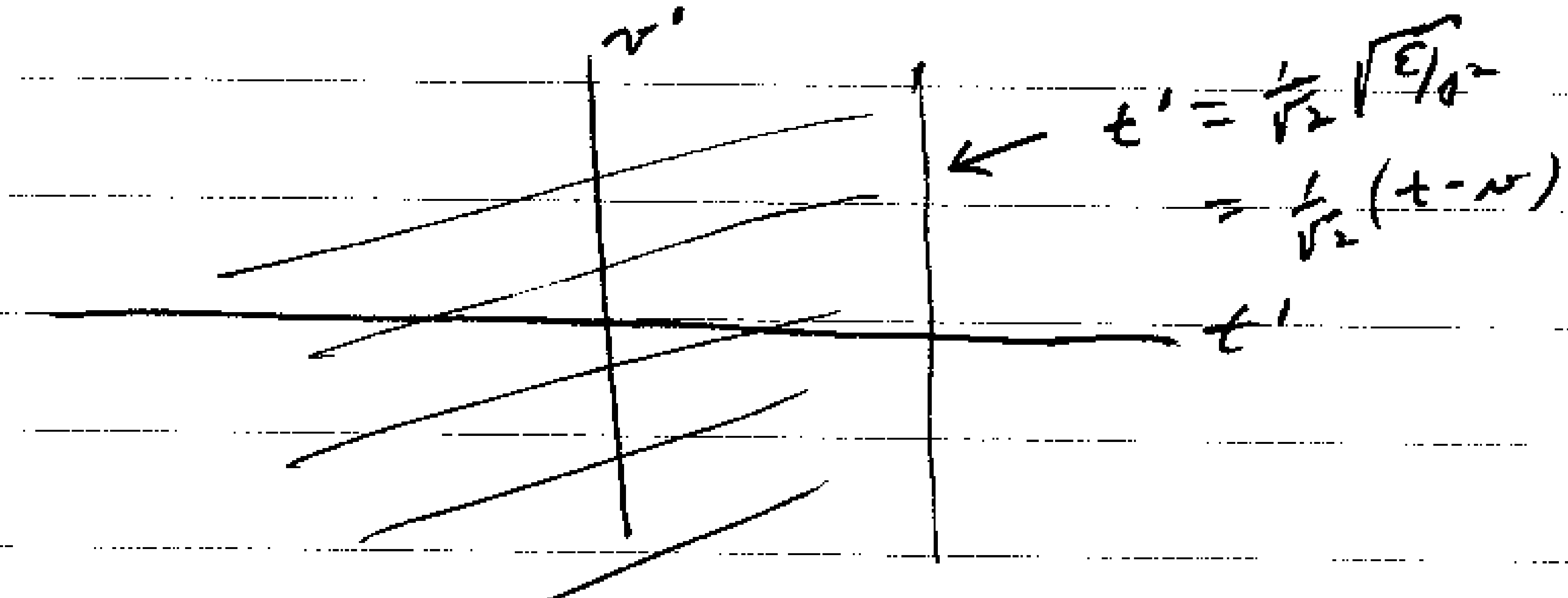
Let $\begin{bmatrix} t' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\underline{A}} \begin{bmatrix} t \\ v \end{bmatrix}$

$\Rightarrow \quad t^2 + v^2 = t'^2 + v'^2$ since $\underline{A}^T = \underline{A}^{-1}$

and the Jacobian has determinant $= 1$

since the Jacobian is $\underline{A}$





$$P_e = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\sqrt{\varepsilon/2\sigma^2}} \frac{1}{2\pi} e^{-\frac{1}{2}(t'^2 + v'^2)} \, dt' \, dv'$$

$$= 1 - \int_{-\infty}^{\sqrt{\varepsilon/2\sigma^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t'^2} \, dt'$$

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}}{2\sigma^2}}\right)$$

4.26) $$\underline{H} = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$$

To show it has rank 2 we need only show that there is a 2×2 submatrix that has a nonzero determinant.

$$\begin{vmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \end{vmatrix} = \sin 2\pi f_0 \neq 0$$

for $0 < f_0 < \frac{1}{2}$

4.27) $$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{bmatrix} \frac{2}{N} \sum_n x[n] \cos 2\pi f_0 n \\ \frac{2}{N} \sum_n x[n] \sin 2\pi f_0 n \end{bmatrix}$$

$$E(\hat{a}) = \frac{2}{N} \sum_n (a \cos^2 2\pi f_0 n + b \sin 2\pi f_0 n \cos 2\pi f_0 n)$$

$$= \frac{2}{N} \sum_n \left(\frac{a}{2} + \frac{a}{2} \cos 4\pi f_0 n + b/2 \sin 4\pi f_0 n\right)$$

$$\approx \frac{2}{N} a \frac{N}{2} = a$$

$$E(\hat{b}) = \frac{2}{N} \sum_n (a \cos 2\pi f_0 n \sin 2\pi f_0 n + b \sin^2 2\pi f_0 n)$$

$$= \frac{2}{N} \sum_n \frac{a}{2} \sin 4\pi f_0 n + b/2 - b/2 \cos 4\pi f_0 n)$$

$$\approx \frac{2}{N} N \frac{b}{2} = b$$

4.28) $\underline{x} = \underline{H}\underline{\theta} + \underline{w}$

where $\underline{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix}$ $\quad \underline{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$

NP detector is to decide $\mathcal{H}_1$ if

$$T(\underline{x}) = \underline{x}^T \underline{C}^{-1} \underline{s} > \gamma'$$

$$T(\underline{x}) = \frac{1}{\sigma^2} \underline{x}^T \underline{H} \underline{\theta}_1$$

$$= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} x[n](A + Bn)$$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\underline{s}^T \underline{C}^{-1} \underline{s}}\right)$$

$$\underline{s}^T \underline{C}^{-1} \underline{s} = \frac{1}{\sigma^2} \underline{s}^T \underline{s} = \mathcal{E}/\sigma^2 = \frac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{\sigma^2}$$

or

$$\underline{s}^T \underline{C}^{-1} \underline{s} = \frac{\sum_{n=0}^{N-1} (A + Bn)^2}{\sigma^2}$$

4.29) Since $L = \log_2 M$ we have $L = \ln M / \ln 2$ and the first step follows by letting $v = u - \sqrt{\frac{LPT}{\Delta\sigma^2}}$

Now let $\alpha = \frac{PT}{\Delta\sigma^2}$ and define

$$g(x) = \ln\left[\Phi\left(v + \sqrt{\frac{\alpha \ln x}{\ln 2}}\right)^{x-1}\right]$$

want $\lim_{x\to 0} g(x)$, or $\lim_{y\to 0} g(1/y)$

$$\lim_{y\to 0} g(1/y) = \lim_{y\to 0} \frac{\ln \Phi\left(v + \sqrt{\frac{\alpha \ln 1/y}{\ln 2}}\right)}{\frac{1}{\frac{1}{y}-1}}$$

$$= \lim_{y\to 0} \frac{\ln \Phi\left(\sqrt{-\frac{\alpha}{\ln 2}\ln y}\right)}{y}$$

Using L'Hospital's rule

$$= \lim_{y\to 0} \frac{\Phi'\left(\sqrt{-\frac{\alpha}{\ln 2}\ln y}\right) \frac{1}{2}\left(-\frac{\alpha}{\ln 2}\ln y\right)^{-1/2}\left(-\frac{\alpha}{y\ln 2}\right)}{\Phi\left(\sqrt{-\frac{\alpha}{\ln 2}\ln y}\right)}$$

$$= \lim_{y\to 0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{-\alpha}{\ln 2}\ln y\right)} \frac{\frac{-\alpha}{2y\ln 2}}{\sqrt{\frac{-\alpha}{\ln 2}\ln y}}$$

since $\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

and $\Phi(\infty) = 1$

$$= \lim_{y \to 0} \frac{1}{\sqrt{2\pi}} e^{\ln y^{\frac{\alpha}{2\ln 2}}} \quad \frac{\frac{-\alpha}{2\ln 2}}{\sqrt{\frac{\alpha}{\ln 2}}} \quad \frac{1}{y\sqrt{\ln 1/y}}$$

$$= \lim_{y \to 0} c \, \frac{y^{\frac{\alpha}{2\ln 2}}}{y\sqrt{\ln 1/y}} \qquad \text{where } c < 0$$

$$= c \lim_{y \to 0} \frac{y^{\frac{\alpha}{2\ln 2} - 1}}{\sqrt{\ln 1/y}}$$

Now if $y \to 0$ and $\beta = \frac{\alpha}{2\ln 2} - 1 > 0$, we have

$$c \lim_{y \to 0} \frac{y^{\beta}}{\sqrt{\ln 1/y}} = 0$$

or if $\frac{\alpha}{2\ln 2} - 1 < 0$, we have

$$c \lim_{y \to 0} \frac{1}{y^{\beta}\sqrt{\ln 1/y}} = -\infty \qquad \text{since } c < 0.$$

$$\Rightarrow \lim = \begin{cases} -\infty & \text{if } \alpha < 2\ln 2 \\ 0 & \text{if } \alpha > 2\ln 2 \end{cases}$$

4.30) $\det\left(\underline{I}_{N-1} - 1/N \, \underline{1}_{N-1} \underline{1}_{N-1}^T\right) =$

$\det\left(1 - 1/N \, \underline{1}_{N-1}^T \underline{1}_{N-1}\right)$ since $M = 1$

$$= 1 - \frac{1}{N}(N-1) = 1/N \neq 0$$

4.31) $(\underline{x}-\underline{H}\underline{\theta})^T \underline{C}^{-1}(\underline{x}-\underline{H}\underline{\theta}) =$

$$[\underline{x}-\underline{H}\hat{\underline{\theta}}-(\underline{H}\underline{\theta}-\underline{H}\hat{\underline{\theta}})]^T \underline{C}^{-1}[\underline{x}-\underline{H}\hat{\underline{\theta}}-(\underline{H}\underline{\theta}-\underline{H}\hat{\underline{\theta}})]$$

$$= [(\underline{x}-\underline{H}\hat{\underline{\theta}})-\underline{H}(\underline{\theta}-\hat{\underline{\theta}})]^T \underline{C}^{-1}[(\underline{x}-\underline{H}\hat{\underline{\theta}})-\underline{H}(\underline{\theta}-\hat{\underline{\theta}})]$$

$$= (\underline{x}-\underline{H}\hat{\underline{\theta}})^T\underline{C}^{-1}(\underline{x}-\underline{H}\hat{\underline{\theta}}) + (\underline{\theta}-\hat{\underline{\theta}})^T\underline{H}^T\underline{C}^{-1}\underline{H}(\underline{\theta}-\hat{\underline{\theta}})$$

$$-2(\underline{\theta}-\hat{\underline{\theta}})^T\underline{H}^T\underline{C}^{-1}(\underline{x}-\underline{H}\hat{\underline{\theta}})$$

But $\underline{H}^T\underline{C}^{-1}(\underline{x}-\underline{H}\hat{\underline{\theta}}) = \underline{H}^T\underline{C}^{-1}\underline{x}-(\underline{H}^T\underline{C}^{-1}\underline{H})\hat{\underline{\theta}} = \underline{0}$

Now

$$p(\underline{x};\underline{\theta}) = \underbrace{\frac{1}{(2\pi)^{N/2}\det^{1/2}(\underline{C})} \cdot e^{-\frac{1}{2}(\underline{x}-\underline{H}\hat{\underline{\theta}})^T\underline{C}^{-1}(\underline{x}-\underline{H}\hat{\underline{\theta}})}}_{h(\underline{x})} \underbrace{e^{-\frac{1}{2}(\underline{\theta}-\hat{\underline{\theta}})^T\underline{H}^T\underline{C}^{-1}\underline{H}(\underline{\theta}-\hat{\underline{\theta}})}}_{g(T(\underline{x}),\underline{\theta})}$$

Chapter 5

5.1) For $N = 6$ and $P_{FA} = 10^{-3}$

$$\gamma''_{k+1} = -\ln 10^{-3} + \ln\left[1 + \gamma''_k + \tfrac{1}{2}\gamma_k''^{2}\right)$$

k	γ_k''
0	1
1	7.8240
2	10.5823
3	11.1210
4	11.2113
5	11.2260
6	11.2284
7	11.2289
8	11.2289

$$\Rightarrow \gamma' = 2\sigma^2(11.2289) = 22.4578\,\sigma^2$$

5.2) Let $x = \sum_{i=1}^{N} x_i^2$

$$x \overset{a}{\sim} N\left(N E(x_i^2),\ N \operatorname{var}(x_i^2)\right)$$

But $E(x_i^2) = E(\chi_1^2) = 1$

$$\operatorname{var}(x_i^2) = \operatorname{var}(\chi_1^2) = 2$$

(see Chapter 2) $\Rightarrow$ $x \overset{a}{\sim} N(N, 2N)$

or

$$\frac{x-N}{\sqrt{2N}} \overset{a}{\sim} N(0,1)$$

$$\begin{aligned} Q_{\chi^2_N}(x) &= \Pr\{\chi_N^2 > x\} \\ &= \Pr\left\{\frac{\chi_N^2 - N}{\sqrt{2N}} > \frac{x-N}{\sqrt{2N}}\right\} \\ &\approx \Pr\left\{N(0,1) > \frac{x-N}{\sqrt{2N}}\right\} \\ &= Q\left(\frac{x-N}{\sqrt{2N}}\right) \end{aligned}$$

Now

$$\begin{aligned} P_D &= Q_{\chi^2_N}\left(\frac{\gamma'/\sigma^2}{\sigma_s^2/\sigma^2+1}\right) \\ &\approx Q\left[\frac{\frac{\gamma'/\sigma^2}{\sigma_s^2/\sigma^2+1} - N}{\sqrt{2N}}\right] \end{aligned}$$

$$\begin{aligned} P_{FA} &= Q_{\chi^2_N}\left(\frac{\gamma'}{\sigma^2}\right) \\ &\approx Q\left(\frac{\gamma'/\sigma^2 - N}{\sqrt{2N}}\right) \end{aligned}$$

$$\Rightarrow \quad \gamma'/\sigma^2 = N + \sqrt{2N}\, Q^{-1}(P_{FA})$$

$$P_D = Q\left[\frac{\gamma'/\sigma^2 - N\sigma_s^2/\sigma^2 - N}{(\sigma_s^2/\sigma^2+1)\sqrt{2N}}\right]$$

$$= Q\left(\frac{N + \sqrt{2N}\,Q^{-1}(P_{FA}) - N\sigma_s^2/\sigma^2 - N}{(\sigma_s^2/\sigma^2 + 1)\sqrt{2N}}\right)$$

$$= Q\left(\frac{Q^{-1}(P_{FA}) - \sqrt{N/2}\,\sigma_s^2/\sigma^2}{\sigma_s^2/\sigma^2 + 1}\right)$$

5.3) From (5.5) decide $\mathcal{H}_1$ if

$$T(\underline{x}) = \underline{x}^T \underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x} > \gamma''$$

$$\hat{\underline{s}} = \underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x}$$

$$= \begin{bmatrix} \frac{\sigma_{s_0}^2}{\sigma_{s_0}^2+\sigma^2} & & \underline{0} \\ & \ddots & \\ \underline{0} & & \frac{\sigma_{s_{N-1}}^2}{\sigma_{s_{N-1}}^2+\sigma^2} \end{bmatrix} \underline{x}$$

$$T(\underline{x}) = \sum_{n=0}^{N-1} \frac{\sigma_{s_n}^2}{\sigma_{s_n}^2+\sigma^2}\, x^2[n]$$

5.4) $\hat{\underline{s}} = \underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x}$

But $\underline{s} = A\underline{1}$

$$\underline{C}_s = E(\underline{s}\underline{s}^T) = E(A^2\underline{1}\underline{1}^T)$$

$$= \sigma_A^2\underline{1}\underline{1}^T$$

By Woodbury's identity

$$(\underline{C}_s + \sigma^2\underline{I})^{-1} = (\sigma^2\underline{I} + \sigma_A^2\underline{1}\underline{1}^T)^{-1}$$

$$= \frac{1}{\sigma^2}\underline{I} - \frac{1}{\sigma^4}\,\frac{\sigma_A^2\underline{1}\underline{1}^T}{1 + \frac{\sigma_A^2}{\sigma^2}\underline{1}^T\underline{1}}$$

$$\hat{\underline{s}} = \sigma_A^2 \underline{1}\underline{1}^T \left(\frac{1}{\sigma^2} I - \frac{\sigma_A^2}{\sigma^2} \frac{\underline{1}\underline{1}^T}{\sigma^2 + N\sigma_A^2} \right) \underline{x}$$

$$= \frac{\sigma_A^2}{\sigma^2} N \bar{x} \underline{1} - \frac{N\sigma_A^4/\sigma^2}{\sigma^2 + N\sigma_A^2} N \bar{x} \underline{1}$$

$$= \frac{N\sigma_A^2(\sigma^2 + N\sigma_A^2) - N^2\sigma_A^4}{\sigma^2(\sigma^2 + N\sigma_A^2)} \bar{x} \underline{1}$$

$$= \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2/N} \bar{x} \underline{1}$$

$$T(\underline{x}) = \underline{x}^T \hat{\underline{s}} = \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2/N} \bar{x} N \bar{x}$$

$$= \frac{N\sigma_A^2}{\sigma_A^2 + \sigma^2/N} (\bar{x})^2$$

5.5)

$$J = E\left[(\underline{\alpha}^T(\underline{s} - \hat{\underline{s}}))^2 \right]$$

$$= E\left[(\alpha^T(\underline{s} - \underline{W}\underline{x}))^2 \right]$$

$$= E\left[(\alpha^T s - \alpha^T(\underline{W}_{opt} + \epsilon\,\delta\underline{W})\underline{x})^2 \right]$$

$$\frac{\partial J}{\partial \epsilon} = E\left[2(\underline{\alpha}^T \underline{s} - \underline{\alpha}^T(W_{opt} + \epsilon\,\delta\underline{W})\underline{x}) \cdot (-\underline{\alpha}^T \delta\underline{W}\,\underline{x}) \right]$$

$$\left.\frac{\partial J}{\partial \epsilon}\right|_{\epsilon = 0} = 0 \Rightarrow$$

$$E\left[(\underline{\alpha}^T\underline{s} - \underline{\alpha}^T\underline{W}_{opt}\underline{x})(-\underline{\alpha}^T\delta\underline{W}\,\underline{x}) \right] = 0$$

$$E\left[\underline{\alpha}^T(\underline{s} - \underline{W}_{opt}\underline{x})(\underline{x}^T\delta\underline{W}^T\underline{\alpha}) \right] = 0$$

$$\underline{\alpha}^T E\left[(\underline{s} - \underline{W}_{opt}\underline{x})\underline{x}^T \right] \delta\underline{W}^T\underline{\alpha} = 0$$

Now since we must have

$$E\left[(\underline{s} - \underline{W}_{opt}\underline{x})\,\underline{x}^T \right] = \underline{0}$$

$$E(\underline{s}\,\underline{x}^T) - \underline{W}_{opt} E(\underline{x}\,\underline{x}^T) = \underline{0}$$

or

$$\begin{aligned}\underline{W}_{opt} &= E(\underline{s}\,\underline{x}^T)\, E(\underline{x}\,\underline{x}^T)^{-1} \\ &= E(\underline{s}(\underline{s}+\underline{w})^T)\, E((\underline{s}+\underline{w})(\underline{s}+\underline{w})^T)^{-1} \\ &= \underline{C}_s(\underline{C}_s + \sigma^2 \underline{I})^{-1}\end{aligned}$$

5.6) $\underline{V}_0 = \frac{1}{\sqrt{N}}\underline{1}$, $\underline{V}_1 = \frac{1}{\sqrt{N}}\underline{1}^{*}$

$\lambda_{s_0} = N\sigma_s^2/2$ $\quad \lambda_{s_1} = N\sigma_s^2/2$

From (5.10), (5.11) with

$$\alpha_0 = \frac{N\sigma_s^2/2\;\sigma^2}{N\sigma_s^2/2 + \sigma^2} = \alpha_1 = \alpha$$

$$P_{FA} = \int_{\gamma''}^{\infty} \underbrace{\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{1 - 2j\alpha\omega}} \right)^2 e^{-j\omega t} \frac{d\omega}{2\pi}}_{\frac{1}{2\alpha} e^{-t/2\alpha} \quad t > 0 \text{ from } (5.13)} dt$$

$$= \int_{\gamma''}^{\infty} \frac{1}{2\alpha} e^{-t/2\alpha} dt = e^{-\gamma''/2\alpha}$$

also,

$$P_D = e^{-\gamma''/2\lambda_s}$$

Now $\gamma'' = 2\alpha \ln 1/P_{FA}$

$$\ln P_D = -\frac{\sigma''}{2\lambda_s} = -\frac{2\alpha \ln 1/P_{FA}}{2\lambda_s}$$

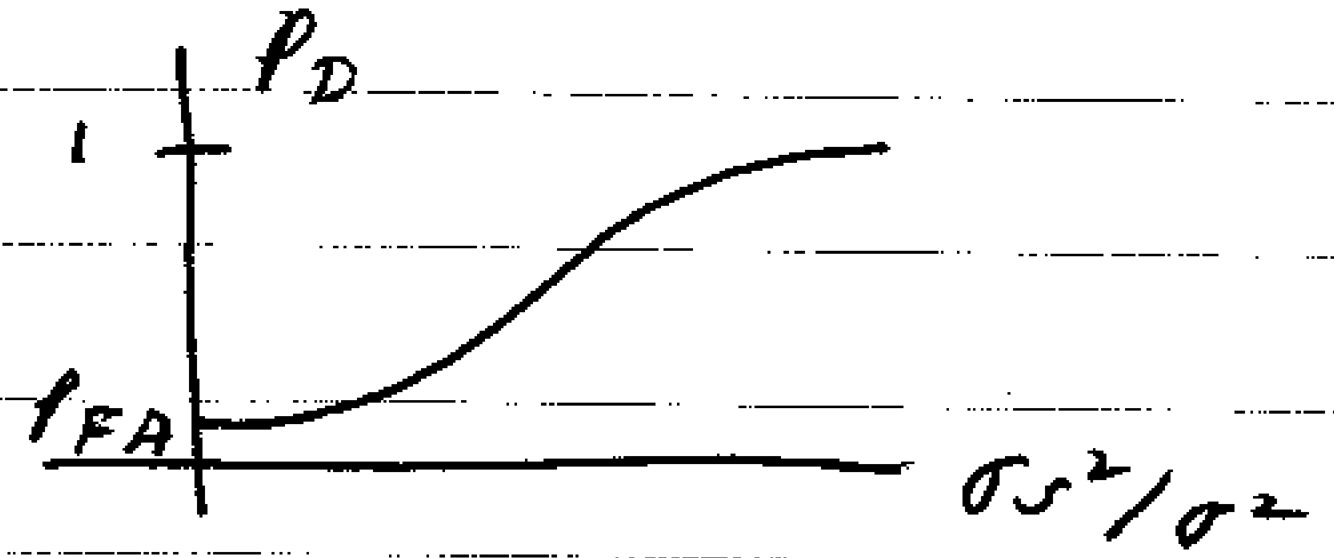
$$= \alpha/\lambda_s \ln P_{FA}$$

$$P_D = P_{FA}^{\ \alpha/\lambda_s}$$

$$\alpha/\lambda_s = \frac{\sigma^2}{N\sigma_s^2/2 + \sigma^2} = \frac{1}{1 + \frac{N}{2}\sigma_s^2/\sigma^2}$$

$$P_D = P_{FA}^{\ \frac{1}{1+N/2\,\sigma_s^2/\sigma^2}}$$

$$= (0.01)^{\frac{1}{1+5\sigma_s^2/\sigma^2}}$$



5.7) as $\rho \to 1$ $\quad T(\underline{x}) \to \dfrac{2\sigma_s^2}{2\sigma_s^2+\sigma^2}\, y^2(0)$

$\Rightarrow$ $y(1)$ is discarded

But $y(1) = (\underline{V}^T\underline{x})_1 = \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \right)_1$

$$= \frac{1}{\sqrt{2}}\left(x(0) - x(1) \right)$$

If $\rho \to 1$ then $s[0] \to s[1]$ since the PDF of $[s[0]\ s[1]]^T$ is concentrated along $s[1] = s[0]$. Thus $y[1] =$
$1/\sqrt{2}\,(s[0] + w[0] - s[1] - w[1]) \to$
$1/\sqrt{2}\,(w[0] - w[1])$ under $\mathcal{H}_1$.
Thus $y[1]$ provides no discrimination

5.81 $\quad \underline{x}' = \underline{W}\,\underline{x} = \underline{C}_s(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x}$

But $\underline{V}^T\underline{C}_s\underline{V} = \underline{\Lambda}_s \Rightarrow \underline{C}_s = \underline{V}\,\underline{\Lambda}_s\underline{V}^T$

$$
\begin{aligned}
\underline{x}' &= \underline{V}\,\underline{\Lambda}_s\underline{V}^T(\underline{V}\,\underline{\Lambda}_s\underline{V}^T + \sigma^2\underline{I})^{-1}\underline{x} \\
&= \underline{V}\,\underline{\Lambda}_s\,\underline{V}^T[\underline{V}(\underline{\Lambda}_s + \sigma^2\underline{I})\underline{V}^T]^{-1}\underline{V}\,\underline{V}^T\underline{x} \\
&= \underline{V}\,\underline{\Lambda}_s\,\underline{V}^T\,\underline{V}^{T^{-1}}(\underline{\Lambda}_s + \sigma^2\underline{I})^{-1}\underline{V}^{-1}\underline{V}\,\underline{V}^T\underline{x} \\
&= \underline{V}\,\underline{\Lambda}_s(\underline{\Lambda}_s + \sigma^2\underline{I})^{-1}\underbrace{\underline{V}^T\underline{x}}_{\underline{y}} \\
&= \underline{V}\begin{bmatrix} \frac{\lambda_{s_0}}{\lambda_{s_0}+\sigma^2} & & \\ & \ddots & \\ & & \frac{\lambda_{s_{N-1}}}{\lambda_{s_{N-1}}+\sigma^2} \end{bmatrix}\underline{y} \\
&= \underline{V}\underbrace{\begin{bmatrix} \frac{\lambda_{s_0}}{\lambda_{s_0}+\sigma^2}\,y[0] \\ \vdots \\ \frac{\lambda_{s_{N-1}}}{\lambda_{s_{N-1}}+\sigma^2}\,y[N-1] \end{bmatrix}}_{\underline{y}'}
\end{aligned}
$$

5.9) $N = 4$ Let $M = 2$ in Ex 5.4

$$\Rightarrow \quad P_{FA} = A_0 e^{-\gamma''/2\alpha_0} + A_1 e^{-\gamma''/2\alpha_1}$$

$$A_0 = \frac{1}{1 - \alpha_1/\alpha_0} \qquad A_1 = \frac{1}{1-\alpha_0/\alpha_1}$$
$$= 1 - A_0$$

$$\alpha_0 = \frac{\lambda_{s_0}\sigma^2}{\lambda_{s_0} + \sigma^2} \qquad \alpha_1 = \frac{\lambda_{s_1}\sigma^2}{\lambda_{s_1} + \sigma^2}$$

$\lambda_{s_0} = 2, \quad \lambda_{s_1} = 1, \quad \sigma^2 = 1$

$$\Rightarrow \quad \alpha_0 = \frac{2}{3} \qquad \alpha_1 = \frac{1}{2}$$

$$A_0 = \frac{1}{1 - \frac{1/2}{2/3}} = 4$$

$$A_1 = -3$$

$$P_{FA} = 4 e^{-\frac{3}{4}\gamma''} - 3 e^{-\gamma''}$$

$$P_D = B_0 e^{-\gamma''/2\lambda_{s_0}} + B_1 e^{-\gamma''/2\lambda_{s_1}}$$

$$B_0 = \frac{1}{1 - \lambda_{s_1}/\lambda_{s_0}} \qquad B_1 = \frac{1}{1-\lambda_{s_0}/\lambda_{s_1}}$$
$$= 1 - B_0$$

$$B_0 = \frac{1}{1 - 1/2} = 2 \qquad B_1 = -1$$

$$P_D = 2 e^{-1/4\gamma''} - e^{-\frac{1}{2}\gamma''}$$

5.10) Decide $\mathcal{H}_1$ if

$$L(\underline{x}) = \frac{p(\underline{x};\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

$$L(\underline{x}) = \frac{\dfrac{1}{(2\pi)^{N/2}\det^{1/2}(\underline{C}_s+\underline{C}_w)} e^{-\frac{1}{2}\underline{x}^T(\underline{C}_s+\underline{C}_w)^{-1}\underline{x}}}{\dfrac{1}{(2\pi)^{N/2}\det^{\frac{1}{2}}(\underline{C}_w)} e^{-\frac{1}{2}\underline{x}^T\underline{C}_w^{-1}\underline{x}}}$$

or decide $\mathcal{H}_1$ if

$$-\tfrac{1}{2}\underline{x}^T(\underline{C}_s+\underline{C}_w)^{-1}\underline{x} + \tfrac{1}{2}\underline{x}^T\underline{C}_w^{-1}\underline{x} > \gamma'$$

$$T(\underline{x}) = \underline{x}^T\left(\underline{C}_w^{-1} - (\underline{C}_s+\underline{C}_w)^{-1}\right)\underline{x} > 2\gamma'$$

From matrix inversion lemma

$$(\underline{C}_s+\underline{C}_w)^{-1} = \underline{C}_w^{-1} - \underline{C}_w^{-1}\underline{C}_s(\underline{C}_w^{-1}\underline{C}_s+\underline{I})^{-1}\underline{C}_w^{-1}$$

$$T(\underline{x}) = \underline{x}^T\underline{C}_w^{-1}\underbrace{\left[\underline{C}_s(\underline{C}_w^{-1}\underline{C}_s+\underline{I})^{-1}\underline{C}_w^{-1}\right]\underline{x}}_{\hat{\underline{s}}}$$

$$\begin{aligned}\hat{\underline{s}} &= \left[\underline{C}_w(\underline{C}_w^{-1}\underline{C}_s+\underline{I})\underline{C}_s^{-1}\right]^{-1}\underline{x}\\ &= \left[(\underline{C}_s+\underline{C}_w)\underline{C}_s^{-1}\right]^{-1}\underline{x}\\ &= \underline{C}_s(\underline{C}_s+\underline{C}_w)^{-1}\underline{x}\end{aligned}$$

5.11) $\underline{V}_W^T \underline{C}_W \underline{V}_W = \underline{\Lambda}_W$

$$= \sqrt{\underline{\Lambda}_W}\sqrt{\underline{\Lambda}_W}$$

$$\underbrace{\sqrt{\underline{\Lambda}_W}^{-1}\underline{V}_W^T}_{\underline{A}^T} \underline{C}_W \underbrace{\underline{V}_W\sqrt{\underline{\Lambda}_W}^{-1}}_{\underline{A}} = \underline{I}$$

$$\Rightarrow \underline{C}_W = \underline{A}^{T^{-1}}\underline{A}^{-1}$$

$$T(\underline{x}) = \underline{x}^T \underline{C}_W^{-1} \underline{C}_S (\underline{C}_S + \underline{C}_N)^{-1}\underline{x}$$

$$= \underline{x}^T \underline{A}\,\underline{A}^T \underline{C}_S (\underline{C}_S + \underline{A}^{T^{-1}}\underline{A}^{-1})^{-1}\underline{x}$$

$$= \underline{x}^T \underline{A}\,\underline{A}^T \underline{C}_S \underline{A}\,\underline{A}^{-1} (\underline{A}^{T^{-1}}(\underline{A}^T\underline{C}_S\underline{A} + \underline{I})\underline{A}^{-1})^{-1}\underline{x}$$

$$= \underline{x}^T \underline{A}\,\underline{B}\,\underline{A}^{-1}\underline{A}(\underline{B}+\underline{I})^{-1}\underline{A}^T\underline{x}$$

$$= \underline{x}^T \underline{A}\,\underline{B}(\underline{B}+\underline{I})^{-1}\underline{A}^T\underline{x}$$

Let $\underline{y} = \underline{V}_B^T \underline{A}^T \underline{x} \Rightarrow \underline{A}^T\underline{x} = \underline{V}_B \underline{y}$

$$T(\underline{x}) = \underline{y}^T \underline{V}_B^T \underline{B}\,\underline{V}_B \underline{V}_B^{-1} (\underline{B}+\underline{I})^{-1}\underline{V}_B \underline{y}$$

$$= \underline{y}^T \underline{\Lambda}_B (\underline{V}_B^{-1}(\underline{B}+\underline{I})\underline{V}_B)^{-1}\underline{y}$$

$$= \underline{y}^T \underline{\Lambda}_B (\underline{\Lambda}_B + \underline{I})^{-1}\underline{y}$$

5.12) $T(\underline{x}) = \underline{x}^T \underline{C}_s(\underline{C}_s+\sigma^2\underline{I})^{-1}\underline{x}$

$$\underline{s} = A\,\underline{h} \qquad \underline{h} = \begin{bmatrix} 1 \\ r \\ \vdots \\ r^{N-1} \end{bmatrix}$$

$$\underline{C}_s = E(A\underline{h}\,A\underline{h}^T) = \sigma_A^2\,\underline{h}\,\underline{h}^T$$

$$(\underline{C}_s + \sigma^2\underline{I})^{-1} = (\sigma^2\underline{I} + \sigma_A^2\,\underline{h}\,\underline{h}^T)^{-1}$$

Using Woodbury's identity

$$= \frac{1}{\sigma^2}\underline{I} - \frac{1}{\sigma^4}\,\frac{\sigma_A^2/\sigma^2\,\underline{h}\,\underline{h}^T}{1+\underline{h}^T\underline{h}\,\sigma_A^2/\sigma^2}$$

$$T(\underline{x}) = \underline{x}^T\sigma_A^2\,\underline{h}\,\underline{h}^T\left[\frac{1}{\sigma^2}\underline{I} - \frac{\sigma_A^2/\sigma^4\,\underline{h}\,\underline{h}^T}{1+\underline{h}^T\underline{h}\,\frac{\sigma_A^2}{\sigma^2}}\right]\underline{x}$$

$$= (\underline{h}^T\underline{x})^2\underbrace{\left[\frac{\sigma_A^2}{\sigma^2} - \frac{\sigma_A^4/\sigma^4\,\underline{h}^T\underline{h}}{1+\underline{h}^T\underline{h}\,\sigma_A^2/\sigma^2}\right]}_{\frac{\sigma_A^2}{\sigma^2} - \frac{\sigma_A^4\,\underline{h}^T\underline{h}}{\sigma^2(\sigma^2+\sigma_A^2\underline{h}^T\underline{h})}}$$

$$= \frac{\sigma_A^2(\sigma^2+\sigma_A^2\underline{h}^T\underline{h}) - \sigma_A^4\,\underline{h}^T\underline{h}}{\sigma^2(\sigma^2+\sigma_A^2\underline{h}^T\underline{h})}$$

$$= \frac{\sigma_A^2}{\sigma_A^2\underline{h}^T\underline{h}+\sigma^2}$$

or we decide $\mathcal{H}_1$ if

$$T'(\underline{x}) = (\underline{h}^T \underline{x})^2$$

$$= \left(\sum_{n=0}^{N-1} x(n) r^n \right)^2 > \gamma''$$

5.13) $s[n] = A h(n)$

$$\Rightarrow \underline{s} = A \underline{h}$$

$$C_s = E(\underline{s}\,\underline{s}^T) = E(A^2 \underline{h}\,\underline{h}^T)$$

$$= \sigma_A^2 \underline{h}\,\underline{h}^T$$

$$\Rightarrow \underline{u} = \sigma_A \underline{h}$$

5.14) See Problem 5.12. To find P_{FA}, P_D

$$T'(\underline{x}) = (\underline{h}^T \underline{x})^2$$

Under $\mathcal{H}_0$ $\underline{x} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I})$

$$\Rightarrow \underline{h}^T \underline{x} \sim \mathcal{N}(0, \sigma^2 \underline{h}^T \underline{h})$$

$$\frac{(\underline{h}^T \underline{x})^2}{\sigma^2 \underline{h}^T \underline{h}} \sim \chi_1^2$$

$$P_{FA} = P_r\{T'(\underline{x}) > \gamma''; \mathcal{H}_0\}$$

$$= P_r\left\{ \underbrace{\frac{T'(\underline{x})}{\sigma^2 \underline{h}^T \underline{h}}}_{\chi_1^2} > \frac{\gamma''}{\sigma^2 \underline{h}^T \underline{h}} ; \mathcal{H}_0 \right\}$$

But $Q_{\chi_1^2}(x) = 2Q(\sqrt{x})$ (see Chapter 2)

$$\Rightarrow P_{FA} = 2Q\left(\sqrt{\frac{\gamma''}{\sigma^2 \underline{h}^T \underline{h}}}\right)$$

Under $\mathcal{H}_1$ $\underline{x} \sim N(\underline{0}, \sigma_A^2 \underline{h}\underline{h}^T + \sigma^2 \underline{I})$

$$\Rightarrow \underline{h}^T \underline{x} \sim N(0, \sigma_A^2(\underline{h}^T\underline{h})^2 + \sigma^2 \underline{h}^T \underline{h})$$

$$\frac{(\underline{h}^T \underline{x})^2}{\sigma_A^2(\underline{h}^T\underline{h})^2 + \sigma^2 \underline{h}^T\underline{h}} \sim \chi_1^2$$

As before we have

$$P_D = 2Q\left(\sqrt{\frac{\gamma''}{\sigma_A^2(\underline{h}^T\underline{h})^2 + \sigma^2 \underline{h}^T\underline{h}}}\right)$$

5.15) $\hat{\underline{\theta}} = \underline{C}_{\theta x} \underline{C}_{xx}^{-1} \underline{x}$

where $\underline{x} = \underline{H}\underline{\theta} + \underline{w}$

$$\underline{C}_{xx} = E(\underline{x}\underline{x}^T) = E((\underline{H}\underline{\theta} + \underline{w})(\underline{H}\underline{\theta} + \underline{w})^T)$$

$$= E(\underline{H}\underline{\theta}\underline{\theta}^T\underline{H}^T) + E(\underline{w}\underline{w}^T)$$

since $\underline{\theta}$ and $\underline{w}$ are independent and zero mean

$$\underline{C}_{xx} = \underline{H}\underline{C}_\theta \underline{H}^T + \sigma^2 \underline{I}$$

$$\underline{C}_{\theta x} = E(\underline{\theta}(\underline{H}\underline{\theta} + \underline{w})^T)$$

$$= E(\underline{\theta}\underline{\theta}^T\underline{H}^T) = \underline{C}_\theta \underline{H}^T$$

$$\Rightarrow \hat{\underline{\theta}} = \underline{C}_\theta \underline{H}^T(\underline{H}\underline{C}_\theta\underline{H}^T + \sigma^2\underline{I})^{-1}\underline{x}$$

5.16) $T(\underline{x}) = \frac{c}{N} \underline{x}^T \underline{H}\underline{H}^T \underline{x}$

$= \underline{x}^T \hat{\underline{s}}$ where $\hat{\underline{s}} = \frac{c}{N} \underline{H}\underline{H}^T \underline{x}$

$$\underline{H}^T \underline{x} = \begin{bmatrix} \sum_n x(n) \cos 2\pi f_0 n \\ \sum_n x(n) \sin 2\pi f_0 n \end{bmatrix}$$

$$\hat{\underline{s}} = \frac{c}{N} \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix} \begin{bmatrix} \sum x(n) \cos 2\pi f_0 n \\ \sum x(n) \sin 2\pi f_0 n \end{bmatrix}$$

$$\Rightarrow \hat{s}(n) = \frac{c}{N} \Big\{ \cos 2\pi f_0 n \left(\sum x(n) \cos 2\pi f_0 n \right) + \sin 2\pi f_0 n \left(\sum x(n) \sin 2\pi f_0 n \right) \Big\}$$

$$= \hat{a} \cos 2\pi f_0 n + \hat{b} \sin 2\pi f_0 n$$

where $\hat{a} = \dfrac{\sigma_s^2}{\frac{N\sigma_s^2}{2} + \sigma^2} \displaystyle\sum_{n=0}^{N-1} x(n) \cos 2\pi f_0 n$

$\hat{b} = \dfrac{\sigma_s^2}{\frac{N\sigma_s^2}{2} + \sigma^2} \displaystyle\sum_{n=0}^{N-1} x(n) \sin 2\pi f_0 n$

or $\hat{a} = \dfrac{N\sigma_s^2/2}{N\sigma_s^2/2 + \sigma^2} \dfrac{2}{N} \displaystyle\sum_{n=0}^{N-1} x(n) \cos 2\pi f_0 n$

$\hat{b} = \dfrac{N\sigma_s^2/2}{N\sigma_s^2/2 + \sigma^2} \dfrac{2}{N} \displaystyle\sum_{n=0}^{N-1} x(n) \sin 2\pi f_0 n$

5.17) $E(T;\mathcal{H}_0) = 0$

$$E(T;\mathcal{H}_1) = \sum_n s[n] A \cos 2\pi f_0 n$$

$$= A^2 \sum_n \cos(2\pi f_0 n + \phi) \cos 2\pi f_0 n$$

$$= \frac{A^2}{2} \sum_n \cos\phi + \cos(4\pi f_0 n + \phi)$$

$$\approx \frac{NA^2}{2} \cos\phi$$

$$\operatorname{var}(T;\mathcal{H}_0) = E\left(\left(\sum_n w[n] A \cos 2\pi f_0 n\right)^2\right)$$

$$= A^2 \sum_m \sum_n \underbrace{E(w[m]w[n])}_{\sigma^2 \delta[m-n]} \cos 2\pi f_0 m \cos 2\pi f_0 n$$

$$= A^2 \sum_n \sigma^2 \cos^2 2\pi f_0 n$$

$$= \frac{A^2 \sigma^2}{2} \sum_n (1 + \cos 4\pi f_0 n)$$

$$\approx NA^2\sigma^2/2 = \operatorname{var}(T;\mathcal{H}_1)$$

$$d^2 = \frac{(NA^2/2 \cos\phi)^2}{NA^2\sigma^2/2} = \frac{NA^2}{2\sigma^2} \cos^2\phi$$

If $\phi = 0$ then $d^2 = NA^2/2\sigma^2$

If $\phi = 90°$ then $d^2 = 0 \Rightarrow$ poor performance ($P_D = P_{FA}$).

5.18) $T'(\underline{x}) = \frac{1}{N}(I^2 + Q^2)$

First note that I, Q are jointly Gaussian, being linear transformations of $\underline{x}$. Also, since $s[n]$ is zero mean, all the means are zero.

To find the variances:

Under $\mathcal{H}_0$

$$\begin{aligned}
\mathrm{var}(I) &= E(I^2) \\
&= E\left(\sum_m \sum_n w[m]w[n]\cos 2\pi f_0 m \cos 2\pi f_0 n\right) \\
&= \sum_m \sum_n E(w[m]w[n]) \cos 2\pi f_0 m \cos 2\pi f_0 n \\
&= \sigma^2 \sum_n \cos^2 2\pi f_0 n \\
&= \frac{\sigma^2}{2} \sum_n (1 + \cos 4\pi f_0 n) \\
&\approx N\sigma^2/2
\end{aligned}$$

and similarly for $\mathrm{var}(Q)$. Under $\mathcal{H}_1$

$$\begin{aligned}
I &= \sum_n (s[n] + w[n]) \cos 2\pi f_0 n \\
&= \sum_n (a\cos 2\pi f_0 n + b \sin 2\pi f_0 n + w[n]) \cdot \cos 2\pi f_0 n \\
&= a \sum_n \cos^2 2\pi f_0 n + b \sum_n \sin 2\pi f_0 n \cos 2\pi f_0 n \\
&\quad + \sum_n w[n] \cos 2\pi f_0 n
\end{aligned}$$

$$\simeq aN/2 + \sum_n w(n) \cos 2\pi f_0 n$$

since $\sum_n \sin 2\pi f_0 n \cos 2\pi f_0 n \simeq 0$

$$\text{var}(I) = (N/2)^2 \text{var}(a) + N\sigma^2/2$$
$$= \frac{N^2 \sigma_s^2}{4} + N\sigma^2/2$$

and similarly for Q. To show that I and Q are uncorrelated under $\mathcal{H}_1$

$$I \simeq aN/2 + \sum_n w[n] \cos 2\pi f_0 n$$
$$Q \simeq bN/2 + \sum_n w(n) \sin 2\pi f_0 n$$

$$E(IQ) = (N/2)^2 \underbrace{E(ab)}_{=0} + E\left(\sum_n w(n) \cos 2\pi f_0 n \sum_n w(n) \sin 2\pi f_0 n\right)$$

$$= \sum_m \sum_n \underbrace{E(w(m) w(n))}_{\sigma^2 \delta[m-n]} \cos 2\pi f_0 m \sin 2\pi f_0 n$$

$$= \sigma^2 \sum_n \underbrace{\cos 2\pi f_0 n \sin 2\pi f_0 n}_{\frac{1}{2}\sin 4\pi f_0 n} \simeq 0$$

and similarly under $\mathcal{H}_0$.

Now we decide $\mathcal{H}_1$ if $T'(\underline{x}) > \gamma'''$

where $T'(\underline{x}) = 1/N(I^2+Q^2)$

Under $\mathcal{H}_0$

$$\frac{I^2+Q^2}{N\sigma^2/2} \sim \chi_2^2$$

Under $\mathcal{H}_1$

$$\frac{I^2+Q^2}{\frac{N\sigma^2}{2}+\frac{N^2\sigma_s^2}{4}} \sim \chi_2^2$$

$$P_{FA} = P_r\left\{\frac{I^2+Q^2}{N} > \gamma''' ; \mathcal{H}_0\right\}$$

$$= P_r\left\{\chi_2^2 > \frac{\gamma'''}{\sigma^2/2}\right\} = e^{-1/2\left(\frac{\gamma'''}{\sigma^2/2}\right)} = e^{-\gamma'''/\sigma^2}$$

Similarly, $P_D = e^{-\frac{1}{2}\left(\frac{\gamma'''}{\sigma^2/2+N\sigma_s^2/4}\right)}$

$$= e^{-\frac{\gamma'''}{N\sigma_s^2/2+\sigma^2}}$$

5.19) Let $\underline{c}_i = [1 \ \cos 2\pi f_i \ldots \cos 2\pi f_i(N-1)]^T$

$\underline{s}_i = [0 \ \sin 2\pi f_i \ldots \sin 2\pi f_i(N-1)]^T$

$i = 0, 1$

$$\underline{W}(f_i) = \begin{bmatrix} \underline{c}_i^T \underline{w} \\ \underline{s}_i^T \underline{w} \end{bmatrix}$$

$$E\left(\underline{W}(f_0)\underline{W}^T(f_1)\right) = E\begin{bmatrix} \underline{c}_0^T \underline{w}\,\underline{w}^T \underline{c}_1 & \underline{c}_0^T \underline{w}\,\underline{w}^T \underline{s}_1 \\ \underline{s}_0^T \underline{w}\,\underline{w}^T \underline{c}_1 & \underline{s}_0^T \underline{w}\,\underline{w}^T \underline{s}_1 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} \underline{c}_0^T \underline{c}_1 & \underline{c}_0^T \underline{s}_1 \\ \underline{s}_0^T \underline{c}_1 & \underline{s}_0^T \underline{s}_1 \end{bmatrix} \approx \underline{0}$$

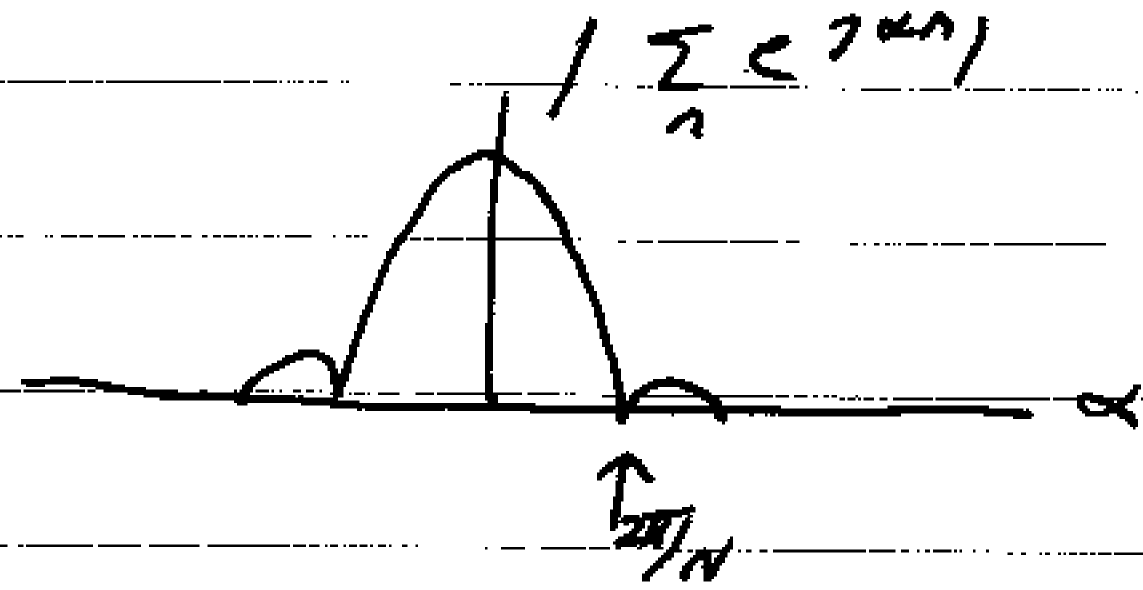
for $|f_1 - f_0| >> 1/N$ since as an example

$$\underline{c}_0^T \underline{c}_1 = \sum_n \cos 2\pi f_0 n \cos 2\pi f_1 n$$

$$= \sum_n \left(\tfrac{1}{2} \cos 2\pi (f_0 + f_1) n + \tfrac{1}{2} \cos 2\pi (f_0 - f_1) n \right)$$

$$= \tfrac{1}{2} \operatorname{Re} \left(\sum_n e^{j 2\pi (f_0 + f_1) n} + e^{j 2\pi (f_0 - f_1) n} \right)$$

and $\displaystyle\sum_{n=0}^{N-1} e^{j\alpha n} = e^{j(N-1)\alpha/2} \frac{\sin N\alpha/2}{\sin \alpha/2}$



For $|\alpha| >> 2\pi/N$ this is ≈ 0.

Need $f_0 + f_1 >> 1/N$ and $|f_0 - f_1| >> 1/N$

Now $E(W^*(f_0) W(f_1)) =$

$$E\left((\underline{c}_0^T \underline{w} - j \underline{s}_0^T \underline{w})(\underline{c}_1^T \underline{w} + j \underline{s}_1^T \underline{w}) \right)$$

$$= \underline{c}_0^T E(\underline{w}\underline{w}^T) \underline{c}_1 + j E(\underline{c}_0^T \underline{w}\underline{w}^T \underline{s}_1)$$

$$- j \underline{s}_0^T E(\underline{w}\underline{w}^T) \underline{c}_1 + \underline{s}_0^T E(\underline{w}\underline{w}^T) \underline{s}_1$$

$$= \sigma^2 \left(\underline{c}_0^T \underline{c}_1 + j \underline{c}_0^T \underline{s}_1 - j \underline{s}_0^T \underline{c}_1 + \underline{s}_0^T \underline{s}_1 \right) \approx 0$$

5.20) $$T(\underline{x}) = \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n}+\sigma^2} y^2(n)$$

$$\underline{y} = \underline{V}^T \underline{x} \Rightarrow y(n) = \underline{v}_n^T \underline{x}$$

$$y(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x(k) e^{j2\pi f_n k} = \frac{1}{\sqrt{N}} X^*(f_n)$$

Also, $\lambda_{s_n} = P_{ss}(f_n)$. Using the hint, $y^2(n) + y^2(N-n) = 2|y(n)|^2$

$$= \frac{2}{N} |X(f_n)|^2$$

$$T(\underline{x}) = \sum_{n=0}^{\frac{N}{2}-1} + \sum_{n=N/2}^{N-1}$$

$$= \sum_{n=0}^{N/2-1} \frac{P_{ss}(f_n)}{P_{ss}(f_n)+\sigma^2} y^2(n)$$

$$+ \sum_{n=N/2}^{N-1} \frac{P_{ss}(f_n)}{P_{ss}(f_n)+\sigma^2} y^2(n)$$

$$= \sum_{n=0}^{N/2-1} \frac{P_{ss}(f_n)}{P_{ss}(f_n)+\sigma^2} y^2(n) + \sum_{n=1}^{N/2} \frac{P_{ss}(f_{N-n})}{P_{ss}(f_{N-n})+\sigma^2} y^2(N-n)$$

$$\approx \sum_{n=0}^{N/2} \frac{P_{ss}(f_n)}{P_{ss}(f_n)+\sigma^2} \left(y^2(n) + y^2(N-n)\right)$$

$$\rightarrow 2\int_0^{\frac{1}{2}} \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2} |X(f)|^2 \, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2} |X(f)|^2 \, df$$

$$= N \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2} I(f)\,df$$

5.21) $$T(\underline{x}) \simeq N \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_{ss}(f)}{\sigma^2} I(f)\,df$$

= correlation of PSD and estimated PSD

By Parseval's theorem

$$T(\underline{x}) = N/\sigma^2 \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}\{P_{ss}(f)\}\mathcal{F}^{-1}\{I(f)\}$$

$$= \frac{N}{\sigma^2} \sum_{k=-\infty}^{\infty} r_{ss}[k]\,\mathcal{F}^{-1}\{I(f)\}$$

$$= \frac{N}{\sigma^2} \sum_{k=-(N-1)}^{N-1} r_{ss}[k]\,\hat{r}_{xx}[k]$$

To find $\mathcal{F}^{-1}\{I(f)\}$:

$$\hat{r}_{xx}[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N} |X(f)|^2 e^{j2\pi f k}\,df$$

$$= \frac{1}{N} \int \sum_m x[m] e^{-j2\pi f m} \sum_n x[n] e^{j2\pi f n} e^{j2\pi f k}\,df$$

$$= \frac{1}{N} \sum_m \sum_n x[m]x[n] \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k+n-m)}\,df}_{\substack{1 \text{ for } m=n+k \\ 0 \text{ otherwise}}}$$

$$= \frac{1}{N} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m]x[n]\,\delta[m-n-k]$$

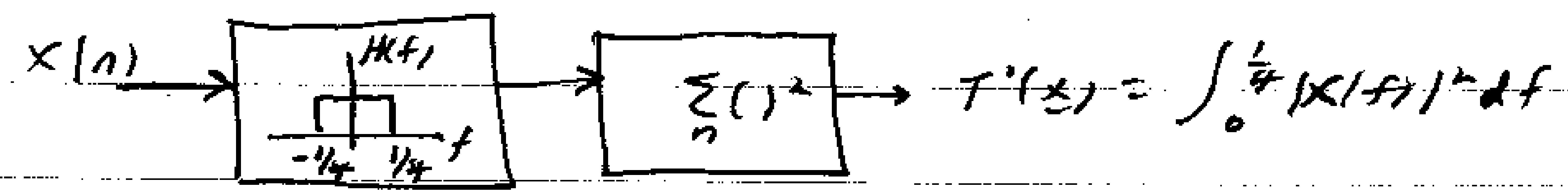
where $x[n] = 0$ for $n < 0$, $n > N-1$

$$\hat{r}_{xx}(k) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n]x[n+k]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1-|k|} x[n]x[n+|k|]$$

5.22) $$T(\underline{x}) = 2N \int_0^{1/4} \frac{P_0}{P_0 + \sigma^2} I(f)\,df$$

$$= \frac{2P_0}{P_0 + \sigma^2} \int_0^{1/4} |X(f)|^2 df$$



Filter Energy detector

5.23) From (5.30)

$$T'(\underline{x}) = \frac{1}{2} \underline{x}^T \underline{C}_w^{-1} \underline{C}_s (\underline{C}_s + \underline{C}_w)^{-1} \underline{x}$$

$$= \frac{1}{2} \underline{x}^T \underline{C}_w^{-1} \eta \underline{C}_w (\eta \underline{C}_w + \underline{C}_w)^{-1} \underline{x}$$

$$= \frac{1}{2} \eta \underline{x}^T \frac{\underline{C}_w^{-1}}{\eta + 1} \underline{x}$$

$$= \frac{1}{2} \frac{\eta}{\eta + 1} \underline{x}^T \underline{C}_w^{-1} \underline{x}$$

or $T''(\underline{x}) = \underline{x}^T \underline{C}_w^{-1} \underline{x}$

This is just a prewhitener + energy detector. No Wiener filtering is possible.

But under $\mathcal{H}_0$ $\quad \underline{x} \sim N(\underline{0}, \underline{C}_w)$

$\Rightarrow \quad T''(\underline{x}) \sim \chi^2_N$

$$P_{FA} = Q_{\chi^2_N}(\gamma'')$$

Under $\mathcal{H}_1$ $\quad \underline{x} \sim N(\underline{0}, \underbrace{\underline{C}_s + \underline{C}_w}_{(\eta+1)\underline{C}_w})$

$$\Rightarrow \quad \frac{T''(\underline{x})}{\eta+1} \sim \chi^2_N$$

$$\begin{aligned} P_D &= \Pr\{T''(\underline{x}) > \gamma''; \mathcal{H}_1\} \\ &= \Pr\left\{\frac{T''(\underline{x})}{\eta+1} > \frac{\gamma''}{\eta+1}; \mathcal{H}_1\right\} \\ &= Q_{\chi^2_N}(\gamma''/(\eta+1)) \end{aligned}$$

For $N=2$ $\quad P_{FA} = e^{-1/2\,\gamma''}$

$$P_D = e^{-\frac{1}{2}\gamma''/\eta+1}$$

$$P_D = (P_{FA})^{\frac{1}{\eta+1}}$$

5.24) $\quad s(n) = \sum_{k=0}^{p-1} h(k)\,u(n-k)$

$$= \sum_{k=0}^{p-1} h(k)\cos 2\pi f_0(n-k) \qquad n \geq p-1$$

$$= \sum_k h(k) \cos 2\pi f_0 n \cos 2\pi f_0 k + \sum_k h(k) \sin 2\pi f_0 n \sin 2\pi f_0 k$$

$$= a \cos 2\pi f_0 n + b \sin 2\pi f_0 n$$

Let $\underline{c} = [1 \ \cos 2\pi f_0 \ldots \cos 2\pi f_0(p-1)]^T$

$\underline{s} = [0 \ \sin 2\pi f_0 \ldots \sin 2\pi f_0(p-1)]^T$

$$\underline{\xi} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \underline{c}^T \underline{h} \\ \underline{s}^T \underline{h} \end{pmatrix} \qquad \text{where } \underline{h} \sim N(\underline{0}, C_{\underline{h}})$$

$\underline{\xi}$ is Gaussian since $\underline{h}$ is Gaussian. Also, $E(\underline{\xi}) = \underline{0}$

$$C_{\xi} = E(\underline{\xi}\underline{\xi}^T) = E\left(\begin{bmatrix} \underline{c}^T \underline{h}\underline{h}^T \underline{c} & \underline{c}^T \underline{h}\underline{h}^T \underline{s} \\ \underline{s}^T \underline{h}\underline{h}^T \underline{c} & \underline{s}^T \underline{h}\underline{h}^T \underline{s} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \underline{c}^T C_h \underline{c} & \underline{c}^T C_h \underline{s} \\ \underline{s}^T C_h \underline{c} & \underline{s}^T C_h \underline{s} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{p-1} \sigma_k^2 \cos^2 2\pi f_0 k & \sum_{k=0}^{p-1} \sigma_k^2 \cos 2\pi f_0 k \sin 2\pi f_0 k \\ \sum_{k=0}^{p-1} \sigma_k^2 \underbrace{\sin 2\pi f_0 k \cos 2\pi f_0 k}_{\frac{1}{2} \sin 4\pi f_0 k} & \sum_{k=0}^{p-1} \sigma_k^2 \sin^2 2\pi f_0 k \end{bmatrix}$$

By hint the cross-diagonal terms $\simeq 0$

$$\approx \begin{bmatrix} \sum_k \frac{\sigma_k^2}{2}(1+\cos 4\pi f_0 k) & 0 \\ 0 & \sum_k \frac{\sigma_k^2}{2}(1-\cos 4\pi f_0 k) \end{bmatrix}$$

Again by the hint

$$\approx \begin{bmatrix} \frac{1}{2}\sum_k \sigma_k^2 & 0 \\ 0 & \frac{1}{2}\sum_k \sigma_k^2 \end{bmatrix}$$

Chapter 6

6.1) From Example 6.1 we decide $\mathcal{H}_1$ if

$$A \sum_n x[n] > \sigma^2 \ln \gamma + NA^2/2$$

If $A < 0$, we decide $\mathcal{H}_1$ if

$$\bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] < A/2 + \frac{\sigma^2}{NA} \ln \gamma = \gamma'$$

But $\bar{x} \sim \mathcal{N}(0, \sigma^2/N)$ under $\mathcal{H}_0$

$$\Rightarrow \quad P_{FA} = Q\left(\gamma'/\sqrt{\sigma^2/N}\right)$$

$$\text{or} \quad \gamma' = \sqrt{\sigma^2/N}\, Q^{-1}(P_{FA})$$

which is (6.4)

6.2)
$$\frac{p(x[0], x[1]; \mathcal{H}_1)}{p(x[0], x[1]; \mathcal{H}_0)} = \frac{\lambda^2 e^{-\lambda(x[0]+x[1])}}{\lambda_0^2 e^{-\lambda_0(x[0]+x[1])}} > \gamma$$

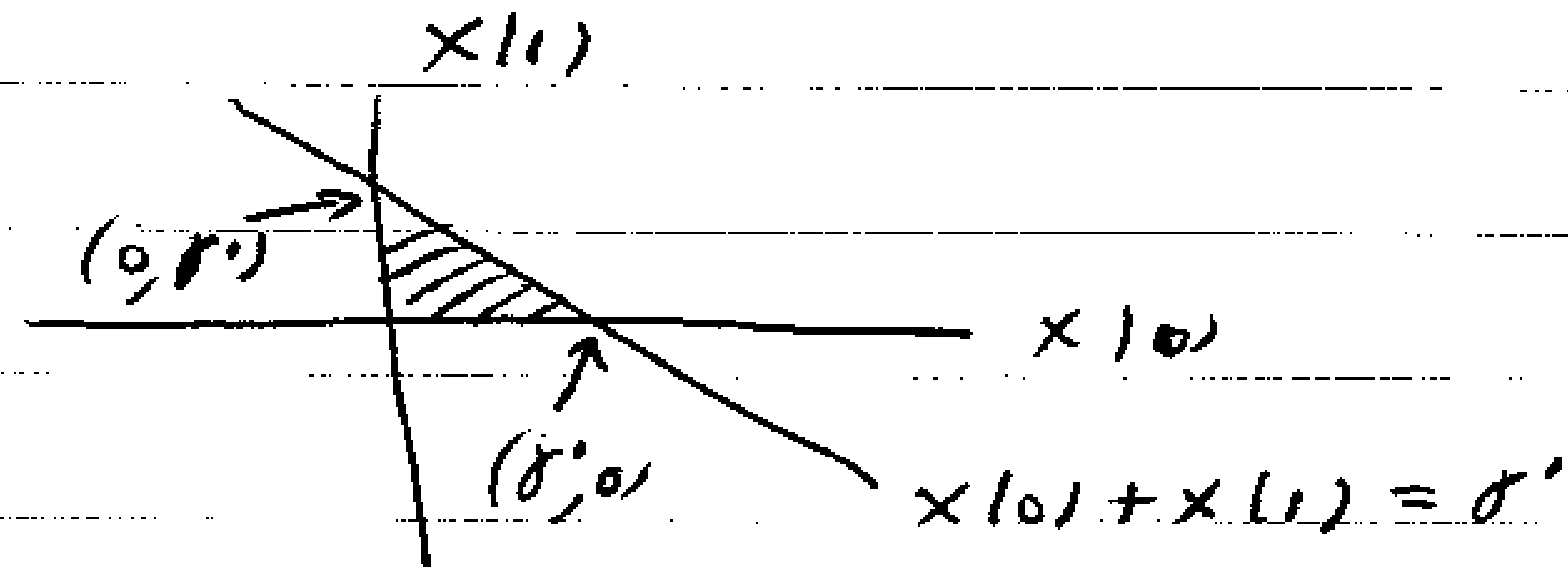
$$e^{-(\lambda-\lambda_0)(x[0]+x[1])} > \frac{\lambda_0^2 \gamma}{\lambda^2}$$

$$-(\lambda-\lambda_0)(x[0]+x[1]) > \ln \lambda_0^2 \gamma/\lambda^2$$

Since $\lambda > \lambda_0$ we decide $\mathcal{H}_1$ if

$$T = x[0] + x[1] < -\frac{\ln \lambda_0^2 \gamma/\lambda^2}{(\lambda-\lambda_0)} = \gamma'$$

$$P_{FA} = \Pr\{x(0)+x(1) < \gamma' ; \lambda_0\}$$



$$P_{FA} = \int_0^{\gamma'}\int_0^{\gamma'-x(1)} \lambda_0^2 e^{-\lambda_0(x(0)+x(1))}\,dx(0)\,dx(1)$$

$$= \int_0^{\gamma'} \lambda_0 e^{-\lambda_0 x(1)} \underbrace{-e^{-\lambda_0 x(0)}\Big|_0^{\gamma'-x(1)}}_{1-e^{-\lambda_0(\gamma'-x(1))}}\,dx(1)$$

$$= \int_0^{\gamma'} \left(\lambda_0 e^{-\lambda_0 x(1)} - \lambda_0 e^{-\lambda_0\gamma'}\right) dx(1)$$

$$= -e^{-\lambda_0 x(1)}\Big|_0^{\gamma'} - \gamma'\lambda_0 e^{-\lambda_0\gamma'}$$

$$= 1 - e^{-\lambda_0\gamma'} - \gamma'\lambda_0 e^{-\lambda_0\gamma'}$$

$\Rightarrow$ For given P_{FA} can find threshold without knowledge of λ under $\mathcal{H}_1$.

6.31 No UMP test since hypothesis is two-sided.

For $\lambda > \lambda_0$ should use $x(0)+x(1) < \gamma'$

For $\lambda < \lambda_0$ " " $x(0)+x(1) > \gamma'$

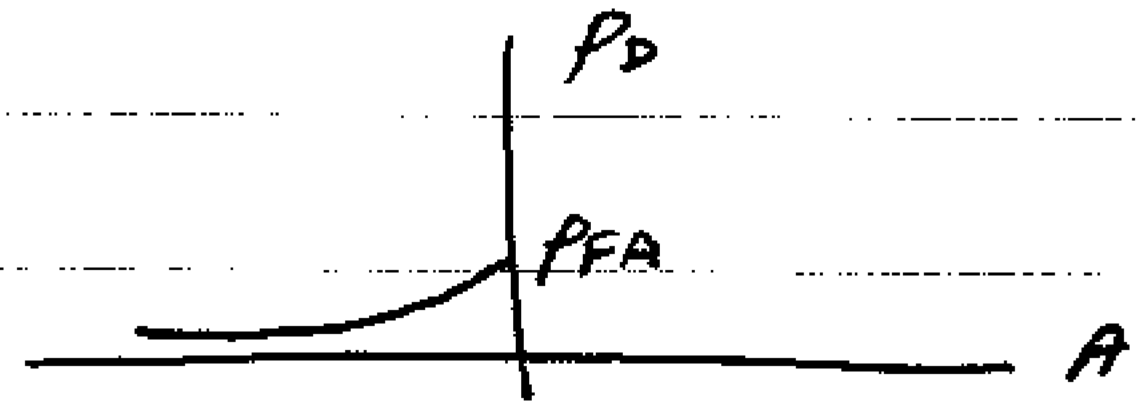
6.4) Decide $\mathcal{H}_1$ if $\bar{x} > \gamma' = \sqrt{\sigma^2/N}\, Q^{-1}(P_{FA})$

Under $\mathcal{H}_1$ $\bar{x} \sim \mathcal{N}(A, \sigma^2/N)$ $A < 0$

$$P_D = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right) = Q\left(Q^{-1}(P_{FA}) - \frac{A}{\sqrt{\sigma^2/N}}\right)$$

$$= Q\left(Q^{-1}(P_{FA}) + \sqrt{NA^2/\sigma^2}\right)$$

As $|A|$ increases P_D decreases



6.5) Decide $\mathcal{H}_1$ if $|\bar{x}| > \gamma''$

$$P_D = P_r\{|\bar{x}| > \gamma''; \mathcal{H}_1\}$$

$$= P_r\{\bar{x} > \gamma''; \mathcal{H}_1\} + P_r\{\bar{x} < -\gamma''; \mathcal{H}_1\}$$

$$= Q\left(\frac{\gamma'' - A}{\sqrt{\sigma^2/N}}\right) + 1 - Q\left(\frac{-\gamma'' - A}{\sqrt{\sigma^2/N}}\right)$$

$$= Q\left(\frac{\gamma'' - A}{\sqrt{\sigma^2/N}}\right) + Q\left(\frac{\gamma'' + A}{\sqrt{\sigma^2/N}}\right)$$

$$= Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - \frac{A}{\sqrt{\sigma^2/N}}\right)$$

$$+ Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + \frac{A}{\sqrt{\sigma^2/N}}\right)$$

since $P_{FA} = P_r\{|\bar{x}| > \gamma''; \mathcal{H}_0\}$

$$= 2P_r\{\bar{x} > \gamma''; \mathcal{H}_0\}$$

$$= 2Q\left(\gamma''/\sqrt{\sigma^2/N}\right)$$

$$P_D = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{NA^2/\sigma^2}\right) + Q\left(Q^{-1}(P_{FA}/2) + \sqrt{NA^2/\sigma^2}\right)$$

6.6) $p(\underline{x} \mid \sigma^2; \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \underbrace{\sum_n (x(n)-A)^2}_{Q}}$

$$p(\underline{x}; \mathcal{H}_1) = \int_0^\infty p(\underline{x} \mid \sigma^2; \mathcal{H}_1)\, p(\sigma^2)\, d\sigma^2$$

$$= \int_0^\infty \frac{1}{(2\pi)^{N/2} \sigma^N} e^{-\frac{1}{2\sigma^2} Q} \, \lambda \frac{e^{-\lambda/\sigma^2}}{\sigma^4}\, d\sigma^2$$

$$= \frac{\lambda}{(2\pi)^{N/2}} \int_0^\infty \frac{1}{(\sigma^2)^{N/2+2}} e^{-(\lambda + 1/2 Q) 1/\sigma^2}\, d\sigma^2$$

Let $u = 1/\sigma^2 \quad du = -1/\sigma^4\, d\sigma^2 = -u^2 d\sigma^2$

$$p(\underline{x}; \mathcal{H}_1) = \frac{\lambda}{(2\pi)^{N/2}} \int_0^\infty \frac{u^{N/2+2}}{u^2} e^{-(\lambda + \frac{1}{2} Q) u}\, du$$

$$= \frac{\lambda}{(2\pi)^{N/2}} (\lambda + 1/2 Q)^{\frac{N}{2}+1}\, \Gamma(\tfrac{N}{2}+1)$$

$$= \frac{\lambda}{(2\pi)^{N/2}} \left(\lambda + \frac{1}{2} \sum_n (x(n)-A)^2\right)^{\frac{N}{2}+1} \Gamma(\tfrac{N}{2}+1)$$

$$\Rightarrow p(\underline{x}; \mathcal{H}_0) = \frac{\lambda}{(2\pi)^{N/2}} \left(\lambda + \frac{1}{2} \sum x^2(n)\right)^{\frac{N}{2}+1} \Gamma(\tfrac{N}{2}+1)$$

Decide $\mathcal{H}_1$ if $\dfrac{p(\underline{x}; \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} > \gamma$

$$L(\underline{x}) = \left(\frac{\lambda + \frac{1}{2} \sum_n x^2(n)}{\lambda + \frac{1}{2} \sum (x(n)-A)^2}\right)^{\frac{N}{2}+1}$$

or we decide $\mathcal{H}_1$ if

$$L(\underline{x})^{-\frac{N}{2}-1} = \frac{\lambda + \frac{1}{2}\sum x^2(n)}{\lambda + \frac{1}{2}\sum(x(n)-A)^2} > \gamma'$$

If $\lambda \to 0$, we decide $\mathcal{H}_1$ if

$$\frac{\frac{1}{N}\sum_n x^2(n)}{\frac{1}{N}\sum_n (x(n)-A)^2} > \gamma'$$

or $$\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} > \gamma'$$

As $\lambda \to 0$ $p(\sigma^2) \to$ constant and thus we have no prior knowledge. Hence, test statistic is ratio of fitting errors (similar to GLRT).

6.7) $|\bar{x}| > \sqrt{\gamma'}$

$\bar{x} \sim \mathcal{N}(0, \sigma^2/N)$ under $\mathcal{H}_0$

$\mathcal{N}(0, \sigma_A^2 + \sigma^2/N)$ under $\mathcal{H}_1$

since under $\mathcal{H}_1$

$$\bar{x} = \underset{\uparrow \atop \mathcal{N}(0,\sigma_A^2)}{A} + \bar{w} \leftarrow \mathcal{N}(0, \sigma^2/N)$$

$$P_{FA} = P_r\{|\bar{x}| > \sqrt{\gamma'}; \mathcal{H}_0\}$$
$$= 2Q(\sqrt{\gamma'}/\sqrt{\sigma^2/N})$$

$$P_D = P_r\{|\bar{x}| > \sqrt{\gamma'}; \mathcal{H}_1\}$$
$$= 2P_r\{\bar{x} > \sqrt{\gamma'}; \mathcal{H}_1\}$$
$$= 2Q(\sqrt{\gamma'}/\sqrt{\sigma_A^2+\sigma^2/N})$$

or
$$P_D = 2Q\left(\frac{\sqrt{\sigma^2/N}\,Q^{-1}(P_{FA}/2)}{\sqrt{\sigma_A^2+\sigma^2/N}}\right)$$

$$= 2Q\left(\frac{Q^{-1}(P_{FA}/2)}{\sqrt{1+\sigma_A^2/\sigma^2/N}}\right)$$

As $\sigma_A^2 \to \infty$, the argument $\to 0 \Rightarrow P_D \to 1$. Signal power is much greater than noise power.

6.81
$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)}$$

$$p(\underline{x}; A, \mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (x[n]-Ar^n)^2}$$

To find $\hat{A}$ we minimize $J(A) = \sum_n (x[n]-Ar^n)^2$

$$\frac{\partial J}{\partial A} = 0 \Rightarrow \sum_n (x[n]-\hat{A}r^n)r^n = 0$$

$$\Rightarrow \hat{A} = \sum_n x[n]r^n \Big/ \sum_n r^{2n}$$

$$L_G(\underline{x}) = \frac{e^{-\frac{1}{2\sigma^2}\sum_n (x(n)-\hat{A}r^n)^2}}{e^{-\frac{1}{2\sigma^2}\sum_n x^2(n)}}$$

$$\ln L_G(\underline{x}) = -\frac{1}{2\sigma^2}\left[\sum_n(-2\hat{A}r^n x(n) + \hat{A}^2 r^{2n})\right]$$

$$= \frac{1}{\sigma^2}\sum_n\left(\hat{A}r^n x(n) - \frac{1}{2}\hat{A}^2 r^{2n}\right)$$

But $\sum_n r^n x(n) = \hat{A}\sum_n r^{2n}$

$$\ln L_G(\underline{x}) = \frac{\hat{A}}{\sigma^2}\sum_n r^n x(n) - \frac{1}{2}\frac{\hat{A}^2}{\sigma^2}\sum_n r^{2n}$$

$$= \frac{\hat{A}^2}{\sigma^2}\sum_n r^{2n} - \frac{1}{2}\frac{\hat{A}^2}{\sigma^2}\sum_n r^{2n}$$

$$= \frac{1}{2\sigma^2}\sum_n r^{2n}\hat{A}^2$$

Decide $\mathcal{H}_1$ if $L_G(\underline{x}) > \gamma$ or

$$\frac{1}{2\sigma^2}\sum_n r^{2n}\hat{A}^2 > \ln\gamma$$

$$\hat{A}^2 > \frac{2\sigma^2\ln\gamma}{\sum_n r^{2n}} = \gamma'$$

6.9) $L_G(\underline{x}) = \dfrac{p(\underline{x};\hat{A},\hat{\sigma}_1^2,\mathcal{H}_1)}{p(\underline{x};\hat{\sigma}_0^2,\mathcal{H}_0)}$