decide $\mathcal{H}_1$ if $\ln \dfrac{p(\underline{x}; \hat{P}_0, \mathcal{H}_1)}{p(\underline{x}; P_0 = 0, \mathcal{H}_1)} > \ln \gamma$

(see Prob 8.3)

Thus, if $\hat{P}_0 = 0$ decide $\mathcal{H}_0$ and if $\hat{P}_0 = \hat{P}_0^+$ decide $\mathcal{H}_1$ if

$$-\frac{1}{2}\ln\left(\frac{2\hat{P}_0^+}{\sigma^2}+1\right) + \frac{2\hat{P}_0^+}{2\hat{P}_0^+ + \sigma^2}\,\frac{3}{\sigma^2} > \gamma'$$

$$-\frac{1}{2}\ln\left(2 3/\sigma^2\right) + \frac{2 3 - \sigma^2}{2 3}\,\frac{3}{\sigma^2} > \gamma'$$

$$-\frac{1}{2}\ln\left(\frac{3/\sigma^2}{2}\right) + 3/\sigma^2 - \frac{1}{2} > \gamma'$$

or $\quad 3/\sigma^2 - \frac{1}{2}\ln 2 3/\sigma^2 - \frac{1}{2} > \gamma'$

$$2 3/\sigma^2 - \ln 2 3/\sigma^2 - 1 > \gamma'$$

$g(2 3/\sigma^2) > \gamma'$ where $g(x) = x - \ln x - 1$

$\Rightarrow \quad 2 3/\sigma^2 > g^{-1}(\gamma')$ or

$$3 = \int_{-\frac{1}{4}}^{\frac{1}{4}} I(f)\, df > \gamma''$$

GLRT compares estimated power over signal band to threshold.

8.12) $\quad T(\underline{x}) = \underline{x}^T \underline{C}\, \underline{x} \overset{a}{\sim}$ Gaussian

$\underline{x} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I})\quad \mathcal{H}_0$

$\mathcal{N}(\underline{0}, P_0 \underline{C} + \sigma^2 \underline{I})\quad \mathcal{H}_1$

$$E(T;\mathcal{H}_0) = \operatorname{tr}(\underline{C}(\sigma^2 \underline{I})) = \sigma^2 \operatorname{tr}(\underline{C})$$

$$E(T;\mathcal{H}_1) = \operatorname{tr}(\underline{C}(P_0\underline{C} + \sigma^2\underline{I}))$$

$$= P_0 \operatorname{tr}(\underline{C}^2) + \sigma^2 \operatorname{tr}(\underline{C})$$

$$\operatorname{var}(T;\mathcal{H}_0) = 2\operatorname{tr}\left[(\underline{C}(\sigma^2\underline{I}))^2\right]$$

$$= 2\sigma^4 \operatorname{tr}(\underline{C}^2)$$

$$\operatorname{var}(T;\mathcal{H}_1) = 2\operatorname{tr}\left[(\underline{C}(P_0\underline{C} + \sigma^2\underline{I}))^2\right]$$

$$P_{FA} = Q\left(\frac{\gamma' - \sigma^2 \operatorname{tr}(\underline{C})}{\sqrt{2\sigma^4 \operatorname{tr}(\underline{C}^2)}}\right)$$

$$P_D = Q\left(\frac{\gamma' - P_0 \operatorname{tr}(\underline{C}^2) - \sigma^2 \operatorname{tr}(\underline{C})}{\sqrt{2\operatorname{tr}\left[(\underline{C}(P_0\underline{C} + \sigma^2\underline{I}))^2\right]}}\right)$$

8.13) $r_{ss}[k] = E[s[n]s[n+k]]$

$$= \sum_{i=1}^{L} E[s_i[n]s_i[n+k]]$$

since the random variables are independent and furthermore $E(s_i[n]) = 0$

$$E[s_i[n]s_i[n+k]] = E[A_i\cos(2\pi f_i n + \phi_i)\, A_i\cos(2\pi f_i(n+k) + \phi_i)]$$

$$= E(A_i^2)\, E\left[\tfrac{1}{2}\cos 2\pi f_i k + \tfrac{1}{2}\cos[4\pi f_i n + 2\pi f_i k + 2\phi_i]\right]$$

$$= E(A_i^2)\tfrac{1}{2}\cos 2\pi f_i k$$

since $\phi_i \sim U[0, 2\pi] \Rightarrow E[\cos[4\pi f_i n$

$$= P_i \cos 2\pi f_i k$$

8.14) $$J(P_k) = \ln\left(\frac{NP_k/2}{\sigma^2}+1\right) - \frac{NP_k/2}{NP_k/2+\sigma^2}\,\frac{I(f_k)}{\sigma^2}$$

$$\frac{\partial J}{\partial P_k} = \frac{N/2\sigma^2}{\frac{NP_k}{2\sigma^2}+1} - \frac{I(f_k)/\sigma^2\left[(NP_k/2+\sigma^2)\frac{N}{2} - NP_k/2\; N/2\right]}{(NP_k/2+\sigma^2)^2}$$

$$= \frac{N/2}{N\frac{P_k}{2}+\sigma^2} - \frac{I(f_k)}{\sigma^2}\,\frac{\frac{N}{2}\sigma^2}{(NP_k/2+\sigma^2)^2} = 0$$

$$\frac{N}{2}\left(N\frac{P_k}{2}+\sigma^2\right) - \frac{N}{2} I(f_k) = 0$$

$$\hat{P}_k^{+} = \frac{2}{N}(I(f_k)-\sigma^2)$$

and from Problem 8.2

$$\hat{P}_k = \max\left(0,\ \frac{2}{N}(I(f_k)-\sigma^2)\right)$$

8.15) We wish to show that

$$\sum_{n=0}^{N-1} \bar{x}\, e^{-j2\pi f_i n} = 0 \qquad i=1,2,\ldots,\frac{M}{2}-1$$

But $$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{M} i n} = N = KM \quad \text{for } i=0$$

Otherwise, using

$$\sum_{n=0}^{N-1} a_n = \sum_{m=0}^{K-1} \sum_{r=0}^{M-1} a_{r+mM}$$

$$\sum_{n=0}^{N-1} e^{-j2\pi \frac{i}{M} n} = \sum_{m=0}^{K-1} \sum_{r=0}^{M-1} e^{-j2\pi \frac{i}{M}(r+mM)}$$

$$= \sum_{m=0}^{K-1} \sum_{r=0}^{M-1} e^{-j2\pi \frac{i}{M} r} \underbrace{e^{-j2\pi m i}}_{=1}$$

$$= K \sum_{r=0}^{M-1} e^{-j2\pi i/M\, r}$$

$$= 0 \quad \text{for } i = 1, 2, \dots, M/2 - 1$$

8.16) $$z[N-1] = \sum_{k=0}^{N-1} h(k)\, y(N-1-k)$$

$$= \sum_{k=0}^{N-1} \frac{1}{K} \sum_{r=0}^{K-1} \delta[k-rM]\, y[N-1-k]$$

$$= \frac{1}{K} \sum_{r=0}^{K-1} \underbrace{\sum_{k=0}^{N-1} \delta[k-rM]\, y[N-1-k]}_{=0 \text{ unless } k = rM}$$

$$= \frac{1}{K} \sum_{r=0}^{K-1} y[N-1-rM]$$

$$= \frac{1}{K} \sum_{r=0}^{K-1} y[(K-r)M-1]$$

Let $s = K-1-r$

$$= \frac{1}{K}\sum_{s=0}^{K-1} y[(s+1)M-1]$$

$$= \frac{1}{K}\sum_{s=0}^{K-1} y[(M-1)+sM]$$

To find the frequency response:

$$H(f) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j2\pi f n}$$

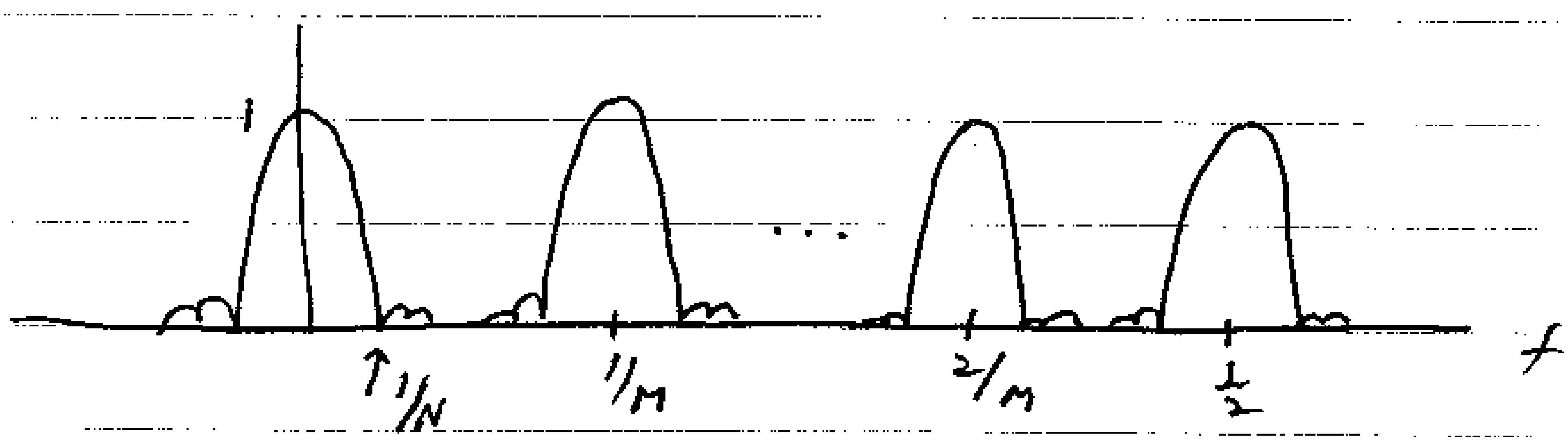
$$= \sum_{n} \frac{1}{K}\sum_{r=0}^{K-1} \delta(n-rM)\, e^{-j2\pi f n}$$

$$= \frac{1}{K}\sum_{r=0}^{K-1} e^{-j2\pi f r M}$$

$$|H(f)| = \frac{1}{K}\left|\sum_{r=0}^{K-1} e^{-j2\pi f r M}\right|$$

Let $\alpha = -2\pi f M$ and use hint $\Rightarrow$

$$|H(f)| = \left|\frac{\sin \pi f M K}{K \sin \pi f M}\right|$$



Bank of narrowband filters centered at harmonic frequencies. Looks like a "comb".

Chapter 9

9.1) $x = \sqrt{N}\ \bar{x}/\sigma$

$\bar{x} \sim \mathcal{N}(0, \sigma^2/N) \Rightarrow x = \sqrt{N}\dfrac{\bar{x}}{\sigma} \sim \mathcal{N}(0,1)$

$$y = \frac{\sum_{n=0}^{N-1} w_R^2(n)}{\sigma^2} \sim \chi^2_N \text{ since } w_R(n) \sim \mathcal{N}(0, \sigma^2)$$

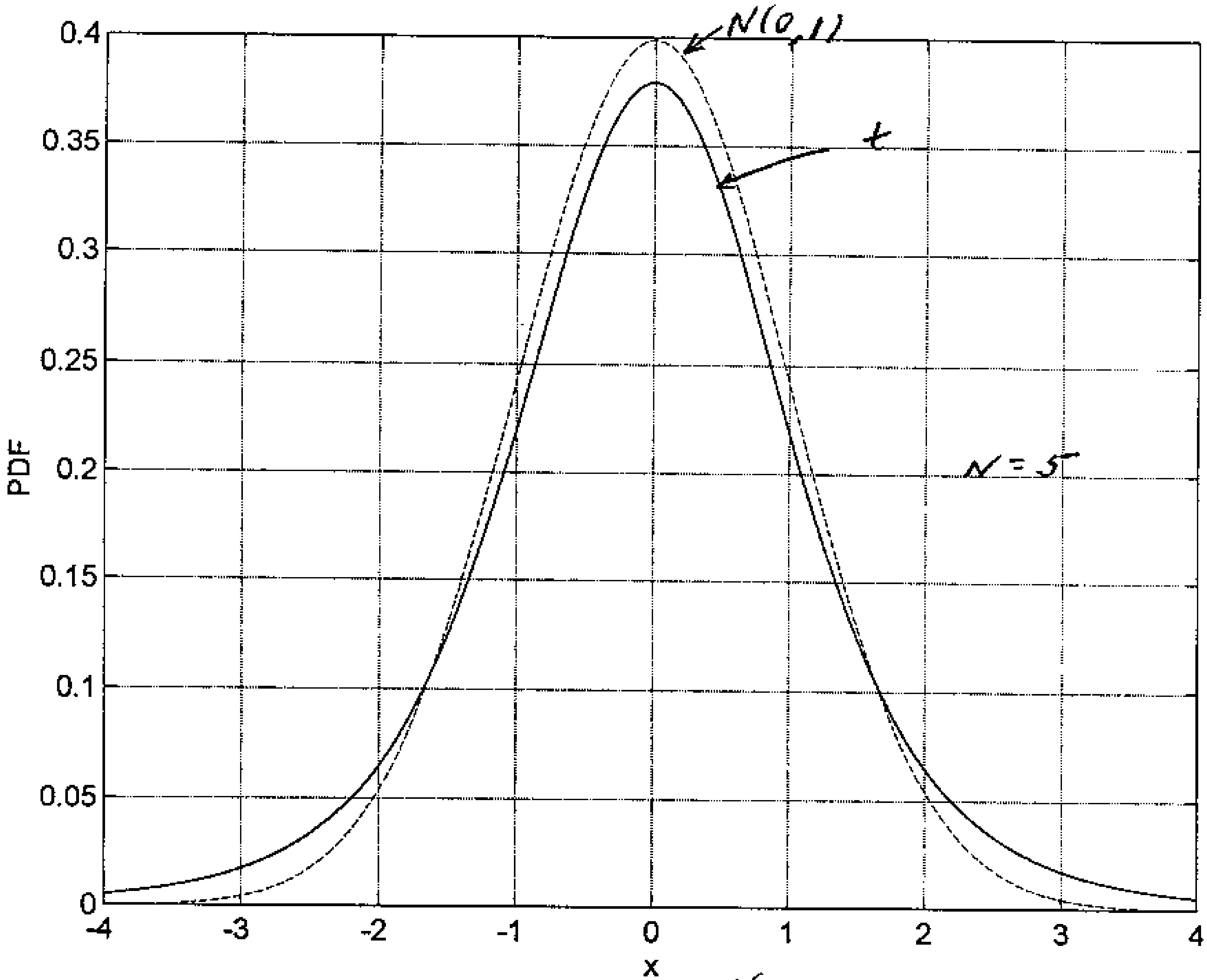
Also, x, y are independent since $x(n)$ and $w_R(n)$ are independent processes.

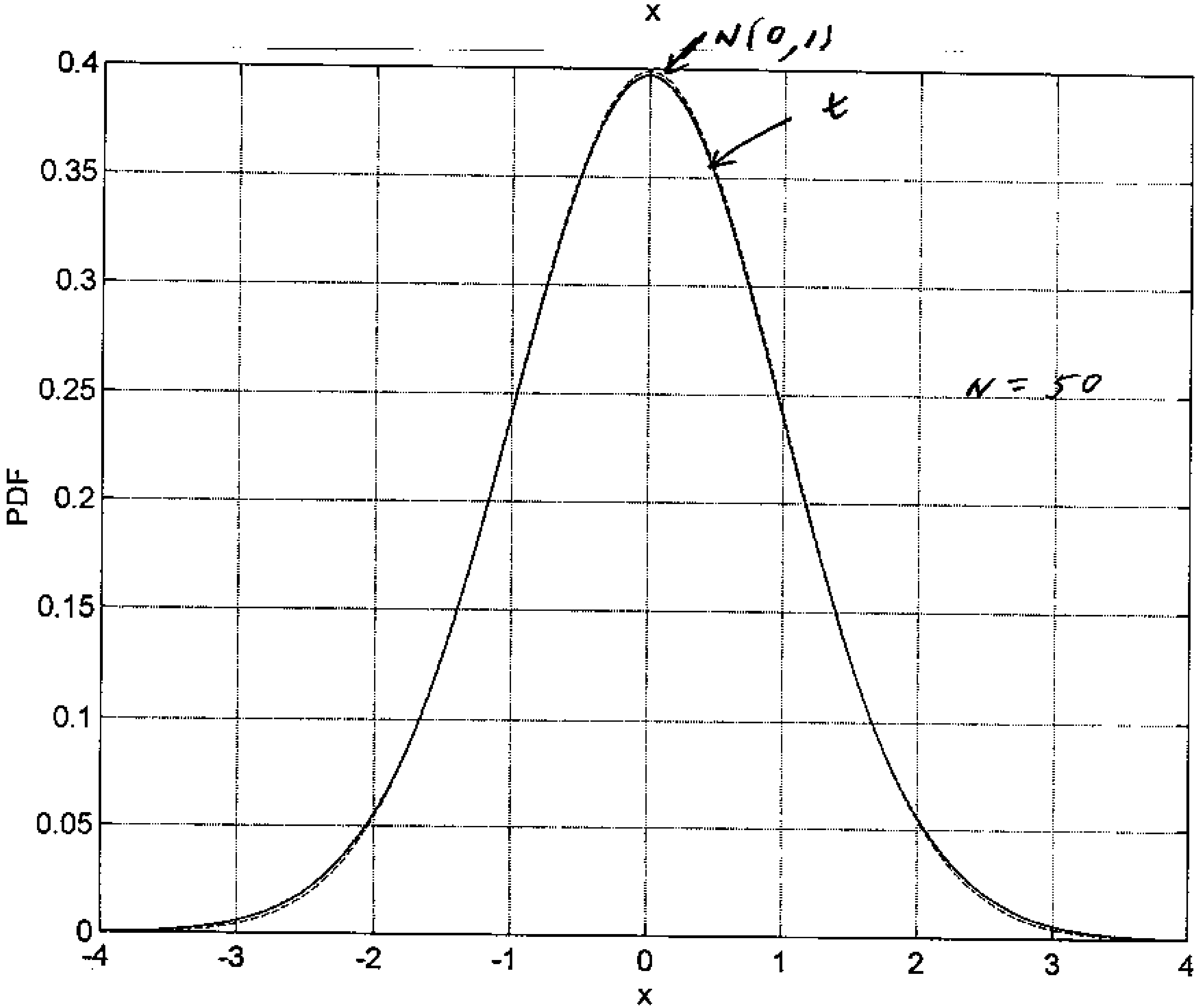
$$\frac{x}{\sqrt{y/N}} = \frac{\sqrt{N}\bar{x}/\sigma}{\sqrt{\frac{1}{N}\sum w_R^2(n)/\sigma^2}}$$

$$= \frac{\sqrt{N}\,\bar{x}}{\sqrt{\frac{1}{N}\sum w_R^2(n)}} = T(\underline{x}, \underline{w_R}) \sim t_N$$

9.2) See MATLAB code below

```
% prob92.m
%
x=[-4:0.01:4]';
N=5;
pg=(1/sqrt(2*pi))*exp(-0.5*x .^2);
c=gamma((N+1)/2)/(sqrt(N*pi)*gamma(N/2));
pt=c*(  (1+(x .^2)/N)  .^(-(N+1)/2)  );
plot(x,pg,'--',x,pt,'-')
xlabel('x')
ylabel('PDF')
grid
```


N(0,1)
t
N = 5
PDF
x
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-4
-3
-2
-1
0
1
2
3
4


N(0,1)
t
N = 50
PDF
x
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-4
-3
-2
-1
0
1
2
3
4

9.3) $P_{FA} = \Pr\{T(\underline{x}) > \gamma'; \mathcal{H}_0\}$

$$= \int_{\{\underline{x}: T(\underline{x}) > \gamma'\}} p(\underline{x}; \mathcal{H}_0)\, d\underline{x}$$

$$= \int_{\{\underline{x}: T(\underline{x}) > \gamma'\}} \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n x^2(n)}\, d\underline{x}$$

Let $u(n) = x(n)/\sigma$

$du(n) = dx(n)/\sigma$

$$P_{FA} = \int_{\{\underline{u}: T(\sigma\underline{u}) > \gamma'\}} \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}\sum_n u^2(n)}\, d\underline{u}$$

$$= \int_{\{\underline{u}: T(\underline{u}) > \gamma'\}} \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}\sum_n u^2(n)}\, d\underline{u}$$

does not depend on σ^2

9.4) $$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}, \hat{\sigma}_1^2, \mathcal{H}_1)}{p(\underline{x}; \hat{\sigma}_0^2, \mathcal{H}_0)}$$

From Section 9.4 $\hat{\sigma}_0^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2(n)$

$$\hat{\sigma}_1^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x(n) - \hat{A})^2$$

By the hint $\hat{A} = \mathrm{sgn}(\bar{x})$

so that

$$L_G(\underline{x}) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{N/2}$$

$$= \left[\frac{1/N \sum x^2(n)}{\frac{1}{N}\sum (x(n) - \mathrm{sgn}(\bar{x}))^2}\right]^{N/2}$$

Let $x(n) = \sigma u(n)$

$$L_G(\underline{x}) = \left[\frac{1/N \sum (\sigma u(n))^2}{1/N \sum (\sigma u(n) - \mathrm{sgn}(\sigma \bar{u}))^2}\right]^{N/2}$$

For $\sigma > 0$

$$L_G(\underline{x}) = \left[\frac{\sigma^2 \, 1/N \sum u^2(n)}{1/N \sum (\sigma u(n) - \mathrm{sgn}(\bar{u}))^2}\right]^{N/2}$$

depends on $\sigma \Rightarrow$ not CFAR
since not scale invariant

9.5) $$T(\underline{x}) = \frac{\bar{x} - A/2}{\frac{1}{N}\sum (x(n) - A)^2}$$

$$\frac{dT}{dA} = \frac{\frac{1}{N}\sum (x(n) - A)^2 \left(-\frac{1}{2}\right) - (\bar{x} - A/2) \cdot \frac{2}{N}\sum (x(n) - A)(-1)}{\left[\frac{1}{N}\sum (x(n) - A)^2\right]^2}$$

$$dT/dA\Big|_{A=0} = \frac{\frac{1}{N}\sum x^2(n)(-\frac{1}{2}) + \bar{x}\frac{2}{N}N\bar{x}}{\left(\frac{1}{N}\sum x^2(n)\right)^2}$$

$$= \frac{2\bar{x}^2 - \frac{1}{2}\frac{1}{N}\sum x^2(n)}{\left(\frac{1}{N}\sum x^2(n)\right)^2}$$

$$T' \approx T\Big|_{A=0} + \frac{dT}{dA}\Big|_{A=0} A$$

$$= \frac{\bar{x}}{\frac{1}{N}\sum x^2(n)} + \frac{2A\bar{x}^2 - \frac{A}{2}\frac{1}{N}\sum x^2(n)}{\left(1/N\sum x^2(n)\right)^2}$$

$$= \frac{\bar{x}\hat{\sigma}_0^2 + 2A\bar{x}^2 - A/2\,\hat{\sigma}_0^2}{(\hat{\sigma}_0^2)^2}$$

$$\approx \frac{(\bar{x} - A/2)\hat{\sigma}_0^2}{(\hat{\sigma}_0^2)^2} = \frac{\bar{x} - A/2}{\hat{\sigma}_0^2}$$

since $A\bar{x}^2 \approx A^3 << A$

Using Slutsky's Theorem the PDF as $N \to \infty$ is equivalent to

$$T'(\underline{x}) = \frac{\bar{x} - A/2}{\sigma^2} \quad \text{under } \mathcal{H}_0$$

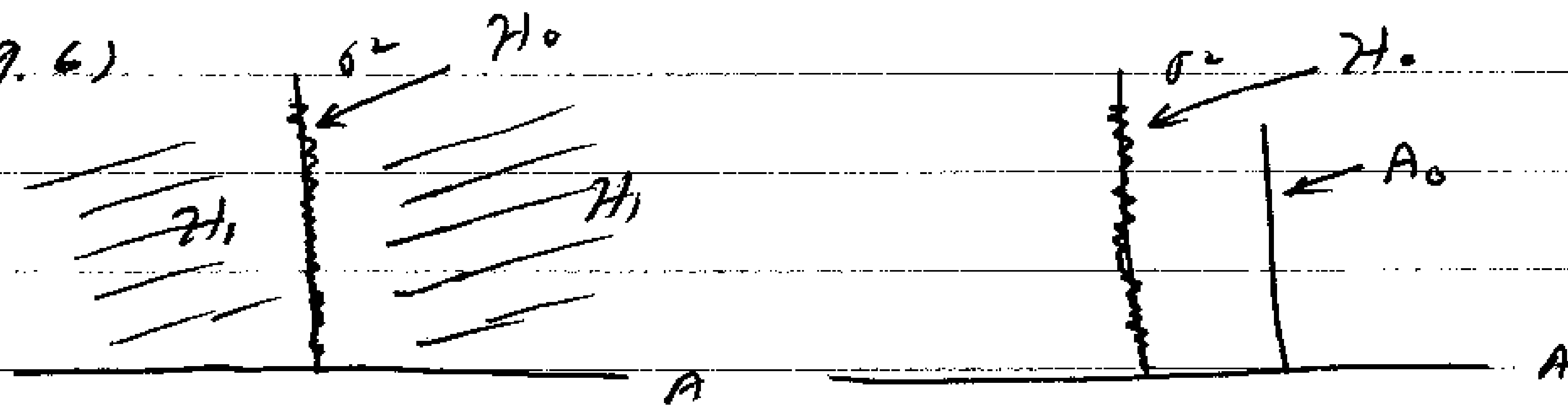
since $\hat{\sigma}_0^2 \to \sigma^2$

and $T(\underline{x}) = \dfrac{\bar{x} - A/2}{\sigma^2}$ under $\mathcal{H}_1$

since $\frac{1}{N} \sum_n (x[n]-A)^2 \to \sigma^2$

$$E(T'; \mathcal{H}_0) = -A/2\sigma^2$$
$$E(T; \mathcal{H}_1) = A/2\sigma^2$$
$$\mathrm{var}(T'; \mathcal{H}_0) = 1/\sigma^4 \, \mathrm{var}(\bar{x}) = \frac{\sigma^2/N}{\sigma^4} = \frac{1}{N\sigma^2}$$
$$= \mathrm{var}(T; \mathcal{H}_1)$$

9.6)



$A=0$ vs $A \neq 0$

$A=0$ vs $A=A_0$

Subspaces under $\mathcal{H}_0$ and $\mathcal{H}_1$ must abut.

9.7) MLE of A can be shown to be
$$\hat{A} = \frac{\sum_n x[n](-1)^n}{N}$$

$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}, \hat{\sigma}_1^2, \mathcal{H}_1)}{p(\underline{x}; \hat{\sigma}_0^2, \mathcal{H}_0)}$$

Also, it can be shown that

$$\hat{\sigma}_0^2 = 1/N \sum_n x^2(n)$$

$$\hat{\sigma}_1^2 = 1/N \sum_n (x(n) - \hat{A}(-1)^n)^2$$

$$LG(\underline{x}) = \frac{\frac{1}{(2\pi\hat{\sigma}_1^2)^{N/2}} e^{-N/2}}{\frac{1}{(2\pi\hat{\sigma}_0^2)^{N/2}} e^{-N/2}}$$

$$2 \ln LG(\underline{x}) = N \ln \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}$$

$$= N \ln \frac{\sum_n x^2(n)}{\sum_n (x(n) - \hat{A}(-1)^n)^2}$$

Since we have $\mathcal{H}_0 : A = 0, \sigma^2 > 0$

$\mathcal{H}_1 : A \neq 0, \sigma^2 > 0$

$\theta_r = A$, $\theta_{r_0} = 0$, $r = 1$, $\theta_s = \sigma^2$

and from (6.23)

$$2 \ln LG(\bar{x}) \overset{a}{\sim} \begin{cases} \chi_1^2 & \mathcal{H}_0 \\ \chi_1'^2(\lambda) & \mathcal{H}_1 \end{cases}$$

where $\lambda = A^2 (I_{AA} - I_{A\sigma^2} I_{\sigma^2\sigma^2}^{-1} I_{\sigma^2 A})$

from (6.24)

and using the hint

$$d = A^2 I_{AA} = NA^2/\sigma^2 = ENR$$

The larger the ENR the better the detection performance since P_D is monotone with λ

9.8) Using (6.35) with $\underline{\theta} = [A\ \sigma^2]^T$, $\theta_r = A$
$\theta_s = \sigma^2$

$$p(\underline{x}; A, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (x(n) - A(-1)^n)^2}$$

$$\frac{\partial \ln p}{\partial A} = -\frac{1}{\sigma^2} \sum_n (x(n) - A(-1)^n)(-1)(-1)^n$$

$$\left.\frac{\partial \ln p}{\partial A}\right|_{\substack{A=0 \\ \sigma^2=\hat{\sigma}_0^2}} = \frac{1}{\hat{\sigma}_0^2} \sum_n x(n)(-1)^n$$

since $\tilde{\underline{\theta}} = (\theta_{r_0}\ \hat{\theta}_{s_0})^T$
$= (A=0\ \ \sigma^2=\hat{\sigma}_0^2)^T$

$[\underline{I}^{-1}(\tilde{\theta})]_{\theta_r\theta_r} = I_{\theta_r\theta_r}(\tilde{\theta})^{-1}$
due to the decoupling of the Fisher info. matrix

$$[\underline{I}^{-1}(\tilde{\theta})]_{rr} = I_{AA}(A=0, \sigma^2=\hat{\sigma}_0^2)^{-1}$$
$$= (N/\hat{\sigma}_0^2)^{-1}$$

$$T_R(\underline{x}) = \frac{\left(\frac{1}{\hat{\sigma}_0^2} \sum x[n](-1)^n\right)^2}{N/\hat{\sigma}_0^2}$$

$$= \frac{\left(\frac{1}{N} \sum_n (-1)^n x[n]\right)^2}{\hat{\sigma}_0^2/N}$$

Same asymptotic performance as GLRT – see Prob 9.7.

9.9) Let $a > 0$ and

$$T(a\underline{x}) = \frac{\sum a x[n] s[n]}{\sqrt{\frac{1}{N} \sum a^2 x^2[n]}}$$

$$= \frac{a \sum x[n] s[n]}{a \sqrt{\frac{1}{N} \sum x^2[n]}} = T(\underline{x})$$

By Problem 9.3 the PDF cannot depend on σ^2.

9.10) $\sum_{i=1}^{N} \lambda_i = \mathrm{tr}(\underline{C}_s)$ well known result

or from
$$\mathrm{tr}(\underline{C}_s) = \mathrm{tr}(\underline{V}\underline{\Lambda}_s\underline{V}^T)$$
$$= \mathrm{tr}(\underline{V}\underline{\Lambda}_s\underline{V}^{-1})$$
$$= \mathrm{tr}(\underline{V}^{-1}\underline{V}\underline{\Lambda}_s)$$
$$= \mathrm{tr}(\underline{\Lambda}_s)$$

$$\sum_{1}^{N} (\underline{v}_i^T\underline{x})^2 = \sum_{1}^{N} \underline{x}^T\underline{v}_i\underline{v}_i^T\underline{x}$$
$$= \underline{x}^T \underbrace{\sum_{1}^{N} \underline{v}_i\underline{v}_i^T}_{\underline{V}\underline{V} = \underline{I}} \underline{x}$$

9.11) Using Thm 9.1 with $\underline{A} = \underline{I}$, $\underline{b} = \underline{0}$,
$\underline{\theta} = A$, $\underline{H} = \underline{s}$ and $\underline{H}^T\underline{H} = \underline{s}^T\underline{s} = 1$
$r = 1$, $p = 1$

$$\hat{A} = \underline{s}^T\underline{x}$$

$$T(\underline{x}) = \frac{(N-1)\,\hat{A}^2}{\underline{x}^T(\underline{I} - \underline{s}\underline{s}^T)\underline{x}}$$

$$= \frac{(N-1)\,(\underline{s}^T\underline{x})^2}{\underline{x}^T\underline{x} - (\underline{s}^T\underline{x})^2}$$

$$P_{FA} = Q_{F_{1,N-1}}(\gamma') \qquad P_D = Q_{F'_{1,N-1}(\lambda)}(\gamma')$$

where $\lambda = A^2/\sigma^2$

9.12) From Theorem 9.1 with (see Eq 7.2)

$$\underline{\theta} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} A\cos\phi \\ -A\sin\phi \end{bmatrix}$$

$$\underline{H} = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$$

$$\underline{A} = \underline{I}, \quad \underline{b} = \underline{0} \quad r = 2, \ p = 2$$

Since $f_0 = k/N$ for $k = 1, 2, \ldots, N/2 - 1$

$\Rightarrow \quad \underline{H}^T\underline{H} = N/2\ \underline{I}$ (this is exact)

$$\Rightarrow \quad T(\underline{x}) = \frac{N-2}{2} \frac{\hat{\underline{\theta}}_1^T \underline{H}^T\underline{H}\, \hat{\underline{\theta}}_1}{\underline{x}^T(\underline{I} - \underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T)\underline{x}}$$

$$= \frac{N-2}{2} \frac{\hat{\underline{\theta}}_1^T \hat{\underline{\theta}}_1\, N/2}{\underline{x}^T\underline{x} - 2/N\ \underline{x}^T\underline{H}\underline{H}^T\underline{x}}$$

$$= \frac{N-2}{2} \frac{\frac{N}{2}\left(\frac{2}{N}\underline{H}^T\underline{x}\right)^T\left(\frac{2}{N}\underline{H}^T\underline{x}\right)}{\underline{x}^T\underline{x} - 2/N\ \underline{x}^T\underline{H}\underline{H}^T\underline{x}}$$

$$= \frac{N-2}{2} \frac{\frac{N}{2}(2/N)^2 \|\underline{H}^T\underline{x}\|^2}{\underline{x}^T\underline{x} - 2/N \|\underline{H}^T\underline{x}\|^2}$$

But $\underline{H}^T \underline{x} = \begin{bmatrix} \sum x(n) \cos 2\pi f_0 n \\ \sum x(n) \sin 2\pi f_0 n \end{bmatrix}$

$$\|\underline{H}^T \underline{x}\|^2 = \left| \sum x(n) e^{-j2\pi f_0 n} \right|^2 = N\, I(f_0)$$

$$\Rightarrow T(\underline{x}) = \frac{N-2}{2} \; \frac{\frac{2}{N} N\, I(f_0)}{\underline{x}^T \underline{x} - 2 I(f_0)}$$

$$= \frac{N-2}{2} \; \frac{2 I(f_0)}{\sum x^2(n) - 2 I(f_0)}$$

$$= \frac{N-2}{2} \; \frac{\frac{2}{N} I(f_0)}{\frac{1}{N} \sum x^2(n) - \frac{2}{N} I(f_0)}$$

$\frac{2}{N} I(f_0) = 2 \times$ bandwidth $\times$ PSD

$=$ signal + noise power in band

$\simeq$ signal power

$\frac{1}{N} \sum x^2(n) - \frac{2}{N} I(f_0) \approx$ total power $-$ signal power

$=$ noise power

$T(\underline{x})$ to within a scale factor is determining the SNR.

$$P_{FA} = Q_{F_{2,N-2}}(\gamma')$$

$$P_D = Q_{F'_{2,N-2}(\lambda)}(\gamma')$$

$$\lambda = \frac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{\sigma^2} = \frac{\frac{N}{2} \underline{\theta}_1^T \underline{\theta}_1}{\sigma^2}$$

$$= \frac{NA^2}{2\sigma^2}$$

9.13) From Theorem 9.1 we have

$$\underline{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} A \\ B \end{bmatrix}}_{\underline{\theta}} + \underline{w}$$

where $\underline{A} = \underline{I}$, $\underline{b} = \underline{0}$, $p = 2$, $r = 2$

$$\hat{\underline{\theta}}_1 = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x} = \begin{bmatrix} N/2 & 0 \\ 0 & N/2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{0}^{\frac{N}{2}-1} x(n) \\ \sum_{N/2}^{N-1} x(n) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{N/2} \sum_{n=0}^{\frac{N}{2}-1} x(n) \\ \frac{1}{N/2} \sum_{n=N/2}^{N-1} x(n) \end{bmatrix}$$

and $(\underline{H}^T\underline{H})^{-1} = \frac{2}{N}\underline{I}$

$$T(\underline{x}) = \frac{N-2}{2} \frac{\hat{\underline{\theta}}_1^T \underline{H}^T \underline{H} \hat{\underline{\theta}}_1}{\underline{x}^T(\underline{I} - \underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T)\underline{x}}$$

$$= \frac{N-2}{2} \frac{\frac{N}{2}(\hat{A}^2 + \hat{B}^2)}{\underline{x}^T\underline{x} - 2/N\,(N/2)^2(\hat{A}^2 + \hat{B}^2)}$$

$$= \frac{N-2}{2} \frac{\hat{A}^2 + \hat{B}^2}{\frac{2}{N}\underline{x}^T\underline{x} - (\hat{A}^2 + \hat{B}^2)}$$

$$= \frac{N-2}{2} \frac{\frac{1}{2}(\hat{A}^2 + \hat{B}^2)}{\frac{1}{N}\sum_0^{N-1} x^2[n] - \frac{1}{2}(\hat{A}^2 + \hat{B}^2)}$$

$$P_{FA} = Q_{F_{2,N-2}}(\gamma')$$

$$P_D = Q_{F'_{2,N-2}(\lambda)}(\gamma')$$

$$\lambda = \frac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{\sigma^2} = \frac{N/2\,(A^2 + B^2)}{\sigma^2}$$

$$= \frac{N\,\frac{1}{2}(A^2 + B^2)}{\sigma^2}$$

Almost the same as Ex 9.2 if we replace $\bar{x}$ by $\frac{1}{2}(\hat{A}^2 + \hat{B}^2)$. Note if $A = B$ the performance of this detector is slightly worse due to $F_{2,N-2}$ vs $F_{1,N-1}$ statistics.

9.14) $$\frac{\chi^2_{N-p}}{N-p} = \frac{1}{N-p}\sum_{n=0}^{N-p-1} x^2(n)$$

where $x(n) \sim N(0,1)$ and IID

By law of large numbers

$$\frac{1}{N-p}\sum x^2(n) \to E(x^2(n)) = 1 \text{ as } N \to \infty$$

9.15) $$P_H \underline{x} = \underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}$$
$$= \frac{\underline{1}\,\underline{1}^T\underline{x}}{N} = \bar{x}\,\underline{1}$$

$$P_H^{\perp}\underline{x} = \underline{x} - \bar{x}\,\underline{1}$$

$$T(\underline{x}) = (N-1)\frac{\|\bar{x}\,\underline{1}\|^2}{\|\underline{x}-\bar{x}\,\underline{1}\|^2} = \frac{\|\bar{x}\,\underline{1}\|^2}{\frac{1}{N-1}\|\underline{x}-\bar{x}\,\underline{1}\|^2}$$

$$\approx \frac{\text{estimated signal energy}}{\text{estimated noise power}}$$

9.16) $$\underline{\theta} = [A\ f_0\ \sigma^2]^T$$

$$x(n) = A\cos 2\pi f_0 n + w(n)$$

$$p(\underline{x};\underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (x(n) - A\cos 2\pi f_0 n)^2}$$

$$\frac{\partial \ln p}{\partial A} = \frac{1}{\sigma^2}\sum_n (x(n) - A\cos 2\pi f_0 n)\cos 2\pi f_0 n$$

$$\frac{\partial^2 \ln p}{\partial A^2} = -\frac{1}{\sigma^2}\sum_n \cos^2 2\pi f_0 n$$

$$\approx -N/2\sigma^2$$

$$I_{AA} = N/2\sigma^2$$

$$\frac{\partial \ln p}{\partial A \partial f_0} = \frac{1}{\sigma^2} \sum_n (x(n) - A\cos 2\pi f_0 n)\left(\frac{\partial \cos 2\pi f_0 n}{\partial f_0}\right)$$

$$+ \frac{1}{\sigma^2} \sum_n A 2\pi n \sin 2\pi f_0 n \cos 2\pi f_0 n$$

$$E\left[\frac{\partial \ln p}{\partial A \partial f_0}\right] = 0 + \frac{A\pi}{\sigma^2} \sum_{n=0}^{N-1} n \sin 4\pi f_0 n \approx 0$$

$$\frac{\partial \ln p}{\partial f_0} = -\frac{1}{\sigma^2} \sum_n (x(n) - A\cos 2\pi f_0 n)(A 2\pi n \sin 2\pi f_0 n)$$

$$\approx -\frac{1}{\sigma^2} \sum_n x(n) A 2\pi n \sin 2\pi f_0 n$$

$$\frac{\partial^2 \ln p}{\partial f_0^2} = -\frac{A}{\sigma^2} \sum_n x(n)(2\pi n)^2 \cos 2\pi f_0 n$$

$$E\left[\frac{\partial^2 \ln p}{\partial f_0^2}\right] = -\frac{A 4\pi^2}{\sigma^2} \sum_n A(\cos^2 2\pi f_0 n)\, n^2$$

$$= \frac{-A^2 4\pi^2}{\sigma^2} \sum_n \left(\frac{1}{2} + \frac{1}{2}\cos 4\pi f_0 n\right) n^2$$

$$\approx -\frac{A^2 2\pi^2}{\sigma^2} \sum_n n^2$$

$$I_{f_0 f_0} = \frac{2A^2\pi^2}{\sigma^2} \sum_0^{N-1} n^2$$

$$\frac{\partial^2 \ln p}{\partial f_0 \partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\left(-\frac{1}{\sigma^2}\sum_n x[n] A 2\pi n \sin 2\pi f_0 n\right)$$

$$= \frac{1}{\sigma^4}\sum_n x[n] A 2\pi n \sin 2\pi f_0 n$$

$$E\left(\frac{\partial^2 \ln p}{\partial f_0 \partial \sigma^2}\right) = \frac{1}{\sigma^4} 2\pi A^2 \sum_n \cos 2\pi f_0 n \sin 2\pi f_0 n$$

$$\approx 0$$

Similarly $E\left(\frac{\partial^2 \ln p}{\partial A \partial \sigma^2}\right) \approx 0$

$$\frac{\partial \ln p}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\left(-N/2 \ln 2\pi - \frac{N}{2}\ln \sigma^2 - \frac{1}{2\sigma^2} S\right) \qquad S = \sum_n (x[n] - A \cos 2\pi f_0 n)^2$$

$$= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4} S$$

$$\frac{\partial^2 \ln p}{\partial \sigma^{2^2}} = \frac{N}{2\sigma^4} - \frac{1}{\sigma^6} S$$

$$E\left(\frac{\partial^2 \ln p}{\partial \sigma^{2^2}}\right) = \frac{N}{2\sigma^4} - \frac{1}{\sigma^6} N\sigma^2 = -\frac{N}{2\sigma^4}$$

$$I_{\sigma^2\sigma^2} = \frac{N}{2\sigma^4}$$

9.17) $$w[n] = \sum_{\ell=0}^{\infty} h[\ell] u[n-\ell]$$

$$E(w[n] w[n+k]) = E\left(\sum_\ell h[\ell] u[n-\ell] \sum_m h[m] u[n+k-m]\right)$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} h(\ell)h(m)\underbrace{E(u(n-\ell)u(n+k-m))}_{\sigma_u^2\,\delta(\ell+k-m)}$$

assume $k \geq 0 \Rightarrow m = \ell + k \geq 0$ for contribution to sum

$$= \sum_{\ell=0}^{\infty} h(\ell)\, h(\ell+k)\, \sigma_u^2$$

or $r_{ww}(k) = \sigma_u^2 \sum_{n=0}^{\infty} h(n)h(n+|k|)$ all k

For AR(1) $h(n) = (-a(1))^n$ $n \geq 0$

0 $n < 0$

For $k \geq 0$

$$r_{ww}(k) = \sigma_u^2 \sum_n (-a(1))^n(-a(1))^{n+k}$$

$$= \sigma_u^2(-a(1))^k \underbrace{\sum_{n=0}^{\infty}(-a(1))^{2n}}_{\sum_{n=0}^{\infty}(a^2(1))^n = \frac{1}{1-a^2(1)}}$$

for $|a(1)| < 1$

$$P_{ww}(f) = \frac{\sigma_u^2}{1-a^2(1)} \mathcal{F}\{(-a(1))^{|k|}\}$$

But $\mathcal{F}\{a^k, k \geq 0\} = \dfrac{1}{1-a e^{-j2\pi f}}$

$$\mathcal{F}\{(-a[1])^{|k|}\} = \mathcal{F}\{(-a[1])^k \quad k \geq 0\}$$
$$+ \mathcal{F}\{(-a[1])^k \quad k < 0\}$$
$$- 1$$

$$= \frac{1}{1+a[1]e^{-j2\pi f}} + \frac{1}{1+a[1]e^{j2\pi f}} - 1$$

$$= \frac{2+2a[1]\cos 2\pi f - |1+a[1]e^{-j2\pi f}|^2}{|1+a[1]e^{-j2\pi f}|^2}$$

$$= \frac{1-a^2[1]}{|1+a[1]e^{-j2\pi f}|^2}$$

9.18) $I = \int_{-\frac{1}{2}}^{\frac{1}{2}} |A(f)X(f)|^2 \, df = \sum_{n=-\infty}^{\infty} y^2[n]$

where $y[n] = \mathcal{F}^{-1}\{A(f)X(f)\}$ by Parseval's theorem

But $y[n] = \sum_{k=0}^{1} a[k]x[n-k]$

$$= x[n] + a[1]x[n-1]$$

$$I = \sum_{n=-\infty}^{\infty} (x[n] + a[1]x[n-1])^2$$

$$= \sum_{n=1}^{N-1} (x[n] + a[1]x[n-1])^2$$
$$+ x^2[0] + a^2[1]x^2[N-1]$$

since $x[n] = 0$ for $n < 0$ and $n > N-1$

$$I \approx \sum_{n=1}^{N-1} (x[n]+a[1]x[n-1])^2 \quad \text{for } N \text{ large}$$

9.19) For σ^2 known we have

$\mathcal{H}_0 : A = 0$

$\mathcal{H}_1 : A \neq 0$

or $\theta_r = A$, $\theta_{r_0} = 0$ and no θ_s (nuisance parameters)

From Sect 6.5 with $r=1$

$$2 \ln L_G(\underline{x}) \overset{a}{\sim} \begin{cases} \chi_1^2 & \mathcal{H}_0 \\ \chi_1'^2(\lambda) & \mathcal{H}_1 \end{cases}$$

where $\lambda = A^2 I(A=0)$ ((6.27))

For σ^2 unknown we have

$\mathcal{H}_0 : A = 0, \sigma^2 > 0$

$\mathcal{H}_1 : A \neq 0, \sigma^2 > 0$

or $\theta_r = A$, $\theta_{r_0} = 0$, $\theta_s = \sigma^2$

From Sect 6.5 PDF is the same except from (6.24)

$$d = A^2 \left[I_{AA} - I_{A\sigma^2} I^{-1}_{\sigma^2\sigma^2} I_{\sigma^2 A} \right] \Big|_{\substack{A=0 \\ \sigma^2}}$$

But from Example 6.7 $I_{A\sigma^2} = 0$

$\Rightarrow \quad d = A^2 I_{AA}(A=0)$

same asymptotic performance.
as $N \to \infty$ we can estimate A as accurately when σ^2 is known as when it is not due to diagonal Fisher info. matrix. Hence detection performance is the same.

9.20) $\underline{\theta} = \begin{bmatrix} \underline{\alpha}_s \\ \underline{\alpha}_w \end{bmatrix} = \begin{bmatrix} r\times 1 \\ s\times 1 \end{bmatrix}$

Noting that $\dfrac{\partial \underline{\mu}(\underline{\theta})}{\partial \alpha_{w_i}} = \underline{0}$

$$\frac{\partial \underline{C}(\underline{\theta})}{\partial \alpha_{s_i}} = \underline{0}$$

we have

$$\underline{I}(\underline{\theta}) = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{B}^T & \underline{C} \end{bmatrix} = \begin{bmatrix} r\times r & r\times s \\ s\times r & s\times s \end{bmatrix}$$

$$(\underline{A})_{ij} = -E\left[\frac{\partial^2 \ln p}{\partial \alpha_{s_i} \partial \alpha_{s_j}} \right]$$

$$= \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \alpha_{s_i}} \right]^T \underline{C}^{-1}(\underline{\theta}) \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \alpha_{s_j}} \right]$$

$$= \left[\frac{\partial \underline{\mu}(\underline{\alpha}_s)}{\partial \alpha_{s_i}}\right]^T \underline{C}^{-1}(\underline{\alpha}_w) \left[\frac{\partial \underline{\mu}(\underline{\alpha}_s)}{\partial \alpha_{s_j}}\right]$$

$\underline{B} = \underline{0}$ since $\left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \alpha_{s_i}}\right]^T \underline{C}^{-1}(\underline{\theta}) \underbrace{\left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \alpha_{w_j}}\right]}_{= \underline{0}} = 0$

and $\mathrm{tr}\left[\underline{C}^{-1}(\underline{\theta}) \underbrace{\frac{\partial \underline{C}(\underline{\theta})}{\partial \alpha_{s_i}}}_{= \underline{0}} \underline{C}^{-1}(\underline{\theta}) \frac{\partial \underline{C}(\underline{\theta})}{\partial \alpha_{w_j}}\right] = 0$

and similarly for $\underline{C}$.

9.21) As an example for $N = 5$

$$\underline{A}^T\underline{A} = \begin{bmatrix} 1 & a[1] & 0 & 0 & 0 \\ 0 & 1 & a[1] & 0 & 0 \\ 0 & 0 & 1 & a[1] & 0 \\ 0 & 0 & 0 & 1 & a[1] \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a[1] & 1 & 0 & 0 & 0 \\ 0 & a[1] & 1 & 0 & 0 \\ 0 & 0 & a[1] & 1 & 0 \\ 0 & 0 & 0 & a[1] & 1 \end{bmatrix}$$

only error

$$= \begin{bmatrix} 1+a^2[1] & a[1] & 0 & 0 & 0 \\ a[1] & 1+a^2[1] & a[1] & 0 & 0 \\ 0 & a[1] & 1+a^2[1] & a[1] & 0 \\ 0 & 0 & a[1] & 1+a^2[1] & a[1] \\ 0 & 0 & 0 & a[1] & 1 \end{bmatrix}$$

Noting that

$$\underline{A}\underline{x} = \begin{bmatrix} x[0] \\ x[1] + a[1]x[0] \\ x[2] + a[1]x[1] \\ \vdots \\ x[N-1] + a[1]x[N-2] \end{bmatrix}$$

the result follows if we omit the $n=0$ term from the sum. Note that $\underline{A}$ is an approximate whitener.

9.22) From Theorem 9.2 with

$$\underline{H} = \underline{s}, \quad \underline{\theta} = A$$

$$T_R(\underline{x}) = \underline{x}^T \underline{C}^{-1}(\underline{\theta}_w)\underline{s}\left(\underline{s}^T\underline{C}^{-1}(\underline{\theta}_w)\underline{s}\right)^{-1}\underline{s}^T\underline{C}^{-1}(\underline{\theta}_w)\underline{x}$$

$$= \frac{\left(\underline{s}^T\underline{C}^{-1}(\underline{\theta}_w)\underline{x}\right)^2}{\underline{s}^T\underline{C}^{-1}(\underline{\theta}_w)\underline{s}}$$

9.23) Since $\underline{H} = \underline{s}$ we have

$$\hat{A} = (\underline{s}^T\underline{C}^{-1}\underline{s})^{-1}\underline{s}^T\underline{C}^{-1}\underline{x} = \frac{\underline{s}^T\underline{C}^{-1}\underline{x}}{\underline{s}^T\underline{C}^{-1}\underline{s}}$$

and $\underline{C}_{\hat{A}} = \mathrm{var}(\hat{A}) = (\underline{s}^T\underline{C}^{-1}\underline{s})^{-1}$

$$\Rightarrow \quad \frac{\hat{A}^2}{\mathrm{var}(\hat{A})} = \frac{(\underline{s}^T\underline{C}^{-1}\underline{x})^2/(\underline{s}^T\underline{C}^{-1}\underline{s})^2}{(\underline{s}^T\underline{C}^{-1}\underline{s})^{-1}} = T_R(\underline{x})$$

9.24) From Theorem 9.2

$$\underline{C}_w = P_0 \qquad \underline{\theta} = A \qquad \underline{H} = \underline{1}$$

Need the MLE of P_0

$$p(\underline{x}; P_0, \mathcal{H}_0) = \frac{1}{(2\pi)^{N/2} \underbrace{\det^{1/2}(P_0 \underline{Q})}_{P_0^{N/2} \det^{1/2}(\underline{Q})}} e^{-\frac{1}{2}\underline{x}^T \underbrace{(P_0 \underline{Q})^{-1}}_{\frac{1}{P_0}\underline{Q}^{-1}} \underline{x}}$$

$$\frac{\partial \ln p}{\partial P_0} = \frac{\partial}{\partial P_0}\left(-\frac{N}{2}\ln P_0 - \frac{1}{2P_0}\underline{x}^T \underline{Q}^{-1}\underline{x}\right)$$

$$= -\frac{N}{2P_0} + \frac{1}{2P_0^2}\underline{x}^T \underline{Q}^{-1}\underline{x} = 0$$

$$\Rightarrow \quad \hat{P}_0 = \frac{1}{N}\underline{x}^T \underline{Q}^{-1}\underline{x}$$

$$T_R(\underline{x}) = \underline{x}^T \underline{C}^{-1}\underline{1}(\underline{1}^T \underline{C}^{-1}\underline{1})^{-1}\underline{1}^T \underline{C}^{-1}\underline{x}$$

$$= \frac{(\underline{1}^T \underline{C}^{-1}\underline{x})^2}{\underline{1}^T \underline{C}^{-1}\underline{1}} = \frac{\left(\frac{1}{\hat{P}_0}\underline{1}^T \underline{Q}^{-1}\underline{x}\right)^2}{\frac{1}{\hat{P}_0}\underline{1}^T \underline{Q}^{-1}\underline{1}}$$

$$= \frac{(\underline{1}^T \underline{Q}^{-1}\underline{x})^2}{\frac{1}{N}\underline{x}^T \underline{Q}^{-1}\underline{x}\;\underline{1}^T \underline{Q}^{-1}\underline{1}}$$

If $\underline{Q} = \underline{I}$ (WGN), $T_R(\underline{x}) = \dfrac{N\bar{x}^2}{\frac{1}{N}\sum x^2[n]}$

See Ex. 6.10

Chapter 10

10.1) $Pr\{w(n) > 3\sigma\} = Q(3) = 0.0013$

for Gaussian

$$Pr\{w(n) > 3\sigma\} = \int_{3\sigma}^{\infty} \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}\,w}\, dw$$

$$= -\left.\frac{e^{-\sqrt{2/\sigma^2}\,w}}{2}\right|_{3\sigma}^{\infty}$$

$$= \frac{e^{-\sqrt{2/\sigma^2}\,3\sigma}}{2} = \frac{e^{-3\sqrt{2}}}{2}$$

$$= 0.0072$$

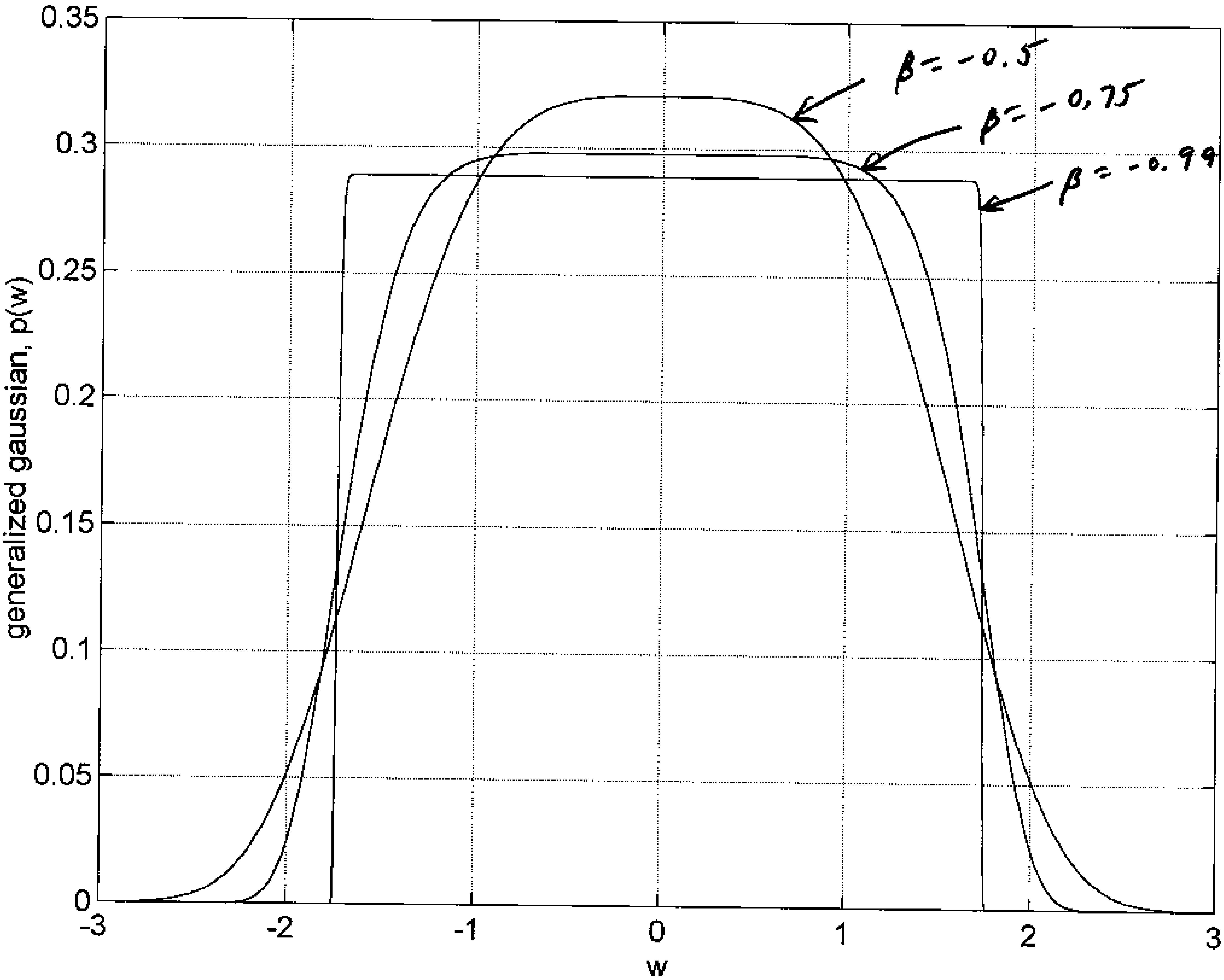
Much more probable that a high level event will occur for Laplacian PDF.

10.2) $$E(w^4(n)) = \int_{-\infty}^{\infty} w^4 \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}\,|w|}\, dw$$

$$= \int_0^{\infty} w^4 \frac{1}{\sqrt{2/\sigma^2}} e^{-\sqrt{2/\sigma^2}\,w}\, dw$$

$$= \frac{1}{\sqrt{2/\sigma^2}} \frac{4!}{(\sqrt{2/\sigma^2})^5} = \frac{24}{(2/\sigma^2)^2} = 6\sigma^4$$

10.32



PDF converges to a uniform PDF as $\beta \to -1$

10.4) $$L(\underline{x}) = \frac{p(\underline{x};\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$= \prod_{n=0}^{N-1} \frac{p(x[n];\mathcal{H}_1)}{p(x[n];\mathcal{H}_0)}$$

$$= \prod_{n=0}^{N-1} \frac{\frac{1}{\pi}\frac{1}{1+(x[n]-A)^2}}{\frac{1}{\pi}\frac{1}{1+x^2[n]}}$$

$$= \prod_{n=0}^{N-1} \frac{1+x^2[n]}{1+(x[n]-A)^2}$$

or we decide $\mathcal{H}_1$ if

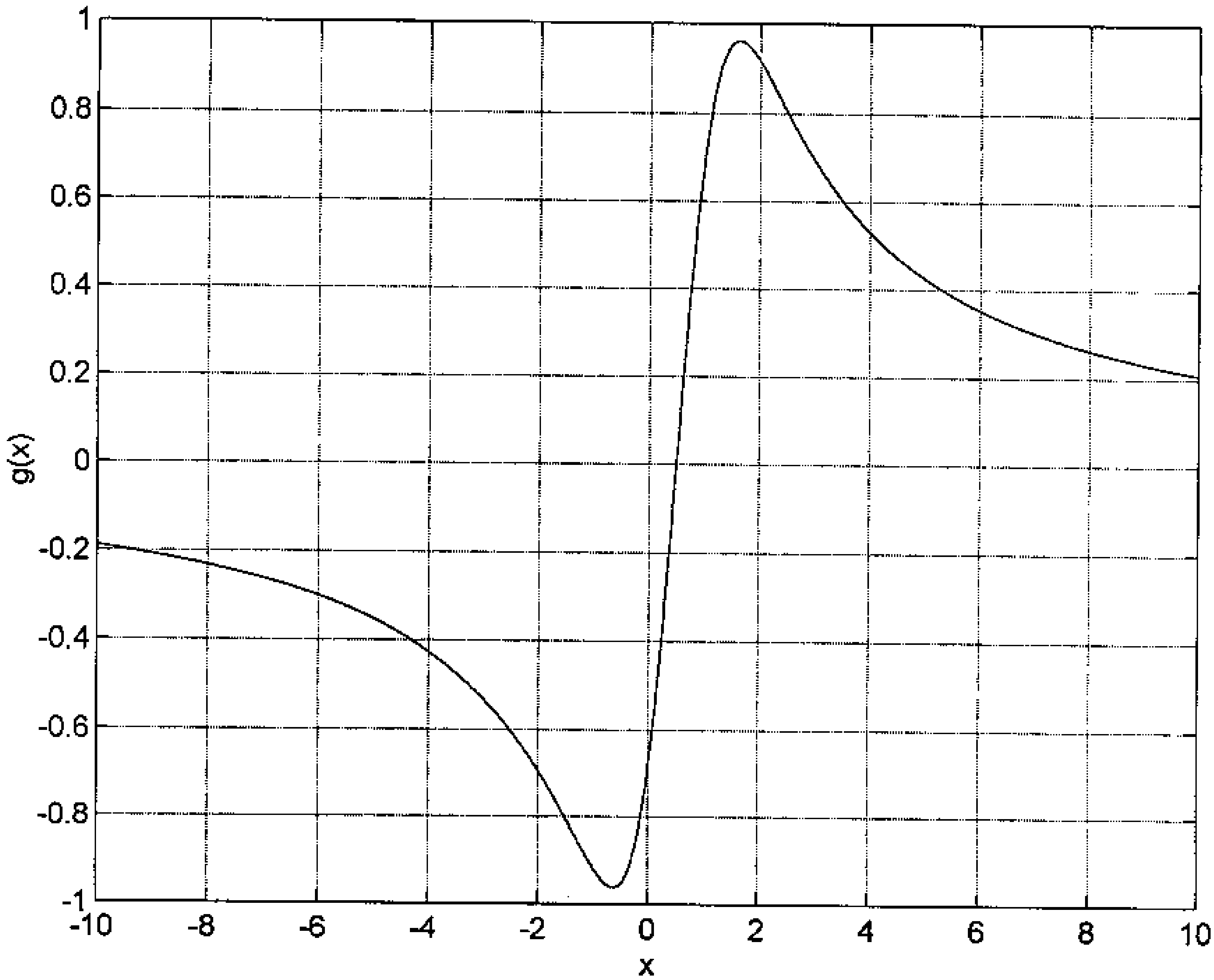
$$\ln L(\underline{x}) = \sum_{n=0}^{N-1} \ln \frac{1+x^2[n]}{1+(x[n]-A)^2} > \ln \gamma$$

$$g(x) = \ln \frac{p(x-A)}{p(x)}$$

$$= \ln \frac{\frac{1}{\pi}\frac{1}{1+(x-1)^2}}{\frac{1}{\pi}\frac{1}{1+x^2}}$$

$$= \ln \frac{1+x^2}{1+(x-1)^2}$$

See next page



10.5) From (6.36)

$$T_{LMP}(\underline{x}) = \frac{\left.\dfrac{\partial \ln p(\underline{x};A)}{\partial A}\right|_{A=0}}{\sqrt{I(A=0)}}$$

$$p(\underline{x};A) = \prod_{n=0}^{N-1} p(x[n]-As[n])$$

$$\frac{\partial \ln p}{\partial A} = \sum_{n=0}^{N-1} \frac{\partial}{\partial A} \ln p(x[n]-As[n])$$

$$= \sum_{n=0}^{N-1} \frac{\left.\dfrac{\partial p(w)}{\partial w}\right|_{w=x[n]-As[n]} (-s[n])}{p(x[n]-As[n])}$$

$$\left.\frac{\partial \ln p}{\partial A}\right|_{A=0} = \sum_{n=0}^{N-1} \frac{-\dfrac{dp(x[n])}{dx[n]}}{p(x[n])} s[n]$$

$$T_{LMP}(\underline{x}) = \frac{\displaystyle\sum_{n=0}^{N-1} \frac{-\frac{dp(x[n])}{dx[n]}}{p(x[n])} s[n]}{\sqrt{I(A=0)}}$$

10.6) $$g(w) \equiv \frac{-\frac{dp(w)}{dw}}{p(w)}$$

$$g(-w) = \frac{-\frac{dp(-w)}{d(-w)}}{p(-w)}$$

$$= \frac{-\frac{dp(w)}{dw}\frac{dw}{d(-w)}}{p(w)}$$ since $p(-w) = p(w)$

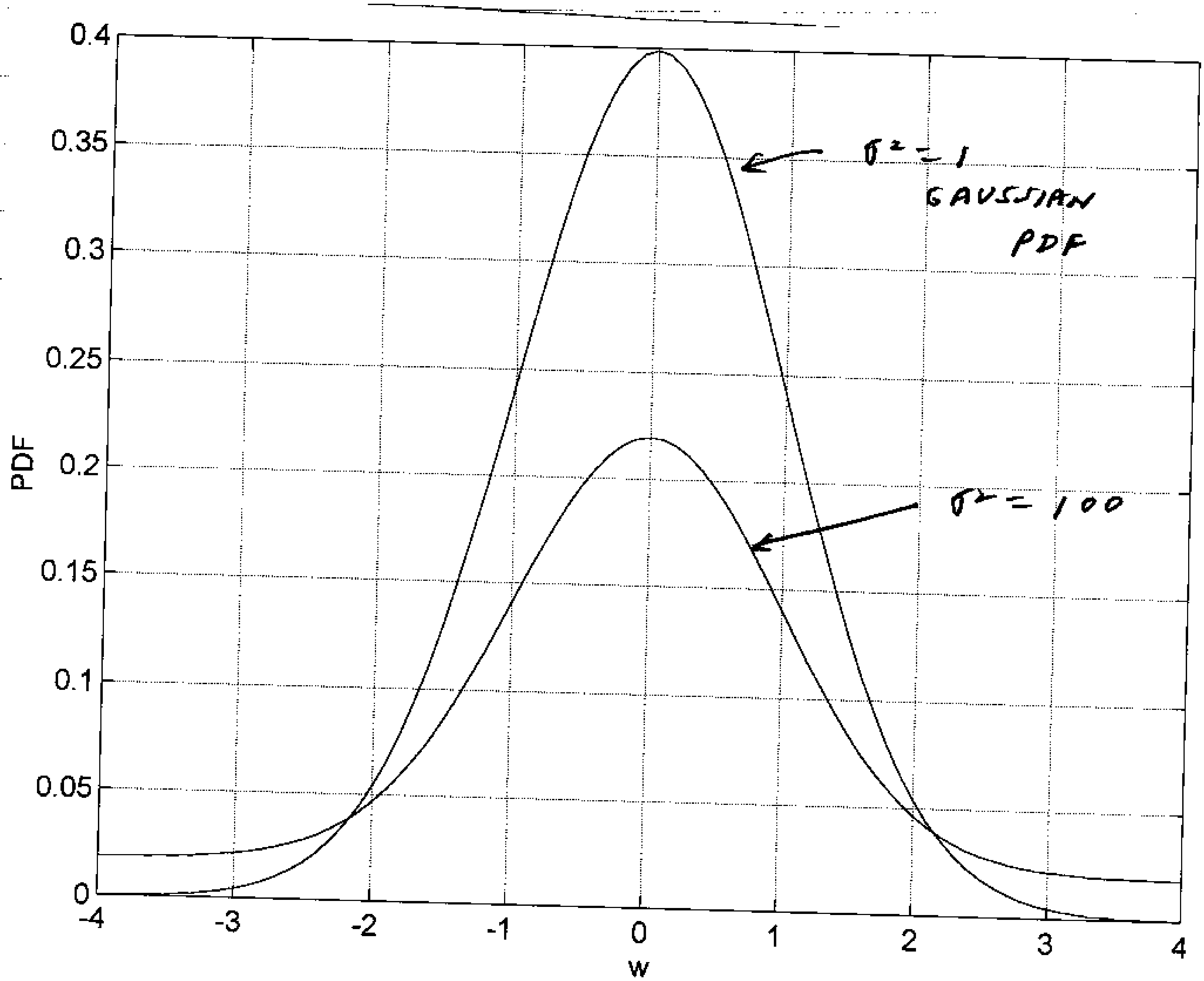
$$= \frac{\frac{dp(w)}{dw}}{p(w)} = -g(w)$$

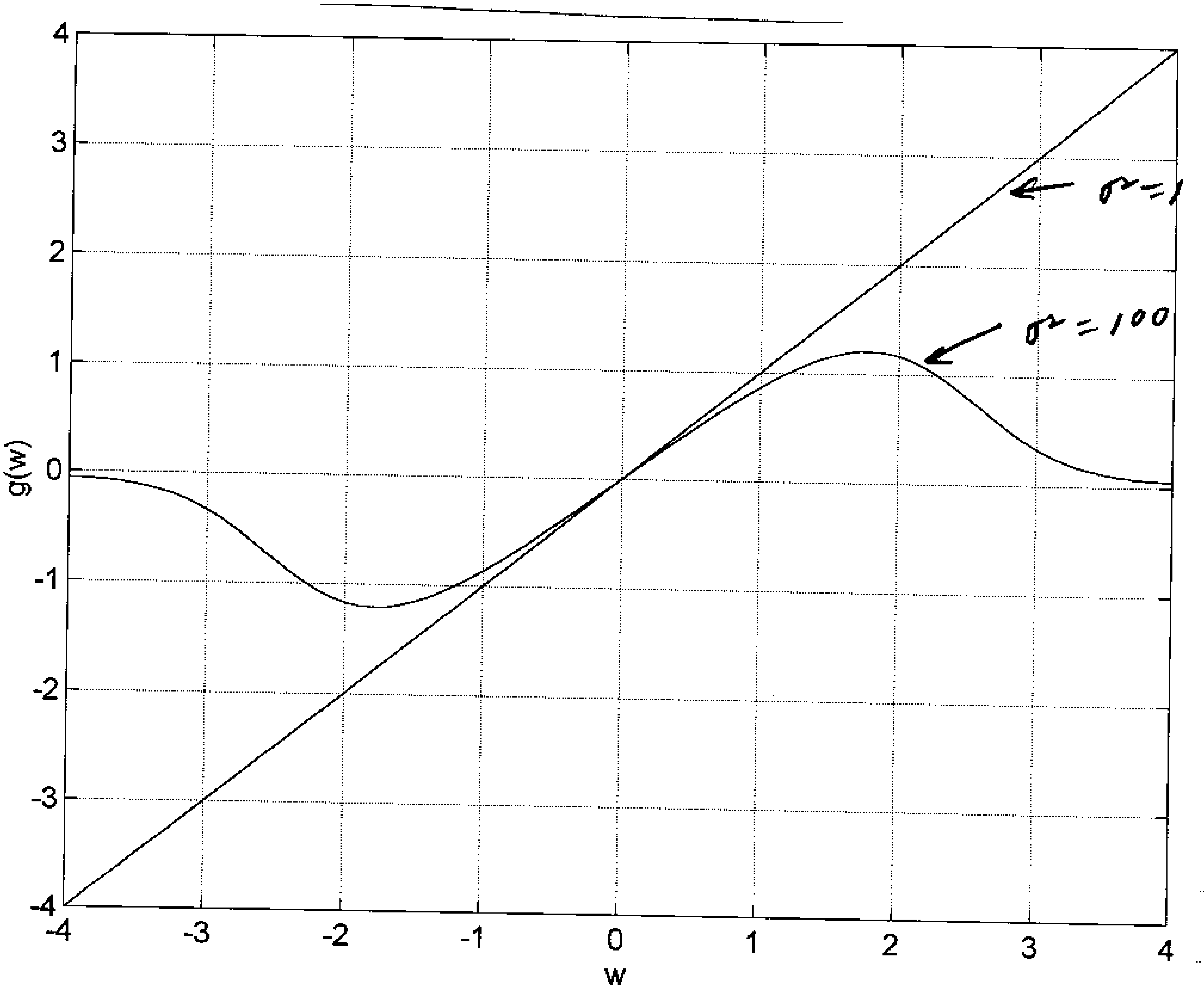
10.7) $$\frac{\frac{dp(w)}{dw}}{p(w)} = \frac{\frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}(-w) + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}w^2}(-w/\sigma^2)}{\frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}w^2}}$$

$$= -w \frac{e^{-\frac{1}{2}w^2} + \frac{1}{\sqrt{\sigma^2}} e^{-\frac{1}{2\sigma^2}w^2} \, 1/\sigma^2}{e^{-\frac{1}{2}w^2} + \frac{1}{\sqrt{\sigma^2}} e^{-\frac{1}{2\sigma^2}w^2}}$$

$$g(w) = w \frac{e^{-\frac{1}{2}w^2} + \frac{1}{(\sigma^2)^{3/2}} e^{-\frac{1}{2\sigma^2}w^2}}{e^{-\frac{1}{2}w^2} + \frac{1}{\sqrt{\sigma^2}} e^{-\frac{1}{2\sigma^2}w^2}}$$

For $\sigma^2 = 1$ we have a standard normal Gaussian and $g(w) = w$. Hence there is no limiting.





g(w)
w
4
3
2
1
0
-1
-2
-3
-4
σ² = 1
σ² = 100

10.8) From (10.10) with $s[n]=1$

$$T(\underline{x}) = \sum_{n=0}^{N-1} - \frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])}$$

$$p(x) = \frac{1}{\pi(1+x^2)}$$

$$\frac{dp}{dx} = -\frac{1}{\pi(1+x^2)^2}(2x)$$

$$= \frac{-2x}{\pi(1+x^2)^2}$$

$$g(x) = \frac{-dp(x)/dx}{p(x)} = \frac{2x}{1+x^2}$$

$$T(\underline{x}) = \sum_{n=0}^{N-1} \frac{2x[n]}{1+x^2[n]}$$

For $|x[n]| << 1$ we have

$$\frac{2x[n]}{1+x^2[n]} \approx 2x[n]$$

For $|x[n]| >> 1$

$$\frac{2x[n]}{1+x^2[n]} \approx \frac{2}{x[n]} \approx 0$$

Thus, we average the samples (to within a scale factor) if they are small and discard the large ones.

10.9) By CLT $T(\underline{x}) \sim N(E(x[n]), \mathrm{var}(x[n])/N)$
since $x[n]$'s are IID
Thus, if $E(x[n])$, $\mathrm{var}(x[n])$ are fixed, the performance (as $N \to \infty$) remains the same

For a Laplacian PDF
$\mathrm{var}(x[n]) = \sigma^2$

$$\Rightarrow \quad T(\underline{x}) \sim \begin{cases} N(0, \sigma^2/N) & \mathcal{H}_0 \\ N(A, \sigma^2/N) & \mathcal{H}_1 \end{cases}$$

$$d^2 = \frac{A^2}{\sigma^2/N} = NA^2/\sigma^2$$

From Example 10.2 we see that the loss in performance is 3 dB. However, its performance is robust or is the same for all $p(w[n])$.

10.10) $$T = \sum_{n=0}^{N-1} - \frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])} \underbrace{s[n]}_{=1}$$

$$E(T;A) = A\,i(A) \sum_n s^2(n) = N A\, i(A)$$
$$\mathrm{var}(T;A) = i(A) \sum_n s^2(n) = N\, i(A)$$

Since $i(A)$ does not depend on A,

$$\frac{dE(T;A)}{dA} = N\, i(A)$$

$$\Rightarrow \quad \mathfrak{Z}(T) = \lim_{N\to\infty} \frac{(N\, i(A))^2}{N\,(N\, i(A))} = i(A)$$

But $d^2 = N A^2 i(A)$

or $\mathfrak{Z}(T) = \dfrac{d^2}{NA^2}$

10.11) For $T_1(\underline{x})$ from (10.11)

$$T_1 \overset{a}{\sim} N(0,\ N i(A)) \qquad \mathcal{H}_0$$
$$N(NA\,i(A),\ N i(A)) \qquad \mathcal{H}_1$$

For $T_2(\underline{x})$

$$E(T_2) = 0 \qquad \mathcal{H}_0$$
$$A \qquad \mathcal{H}_1$$
$$\mathrm{var}(T_2) = \sigma^2/N$$

where $\sigma^2 = \mathrm{var}(w(n))$

or using CLT

$$T_2 \overset{a}{\sim} N(0, \sigma^2/N) \quad \mathcal{H}_0$$
$$N(A, \sigma^2/N) \quad \mathcal{H}_1$$

$$\Rightarrow d_1^2 = \frac{(NAi(A))^2}{Ni(A)} = NA^2 i(A)$$

$$d_2^2 = \frac{A^2}{\sigma^2/N} = \frac{NA^2}{\sigma^2}$$

For same performance we must have

$d_1^2 = d_2^2$ or $N_1 A^2 i(A) = \frac{N_2 A^2}{\sigma^2}$

$$N_1/N_2 = \frac{1/\sigma^2}{i(A)}$$

or $ARE_{2,1} = \lim_{N\to\infty} N_1/N_2 = \frac{1}{\sigma^2 i(A)}$

Since $\mathfrak{z}(T) = d^2/NA^2$

$$\frac{\mathfrak{z}(T_2)}{\mathfrak{z}(T_1)} = d_2^2/d_1^2$$

$$= \frac{NA^2/\sigma^2}{NA^2 i(A)} = \frac{1}{\sigma^2 i(A)} = ARE_{2,1}$$

10.12) $T_R = N\left(\frac{1}{N} \sum \operatorname{sgn} x(n)\right)^2$

Under $\mathcal{H}_0$

$$T_R = N\left(\frac{1}{N}\sum \mathrm{sgn}(w[n])\right)$$

where $w[n]$ is Laplacian noise

Let $u = \frac{1}{N}\sum \mathrm{sgn}\, w[n]$

$E(u) = E(\mathrm{sgn}\, w[n])$ due to identically distributed

$$E(\mathrm{sgn}\, w[n]) = 1 \Pr\{w[n] > 0\} + (-1)\Pr\{w[n] < 0\}$$
$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$\mathrm{var}(\mathrm{sgn}\, w[n]) = E((\mathrm{sgn}\, w[n])^2) = E(1) = 1$$

$$\mathrm{var}(u) = \mathrm{var}(\mathrm{sgn}(w[n]))/N = 1/N$$

due to IID

$\Rightarrow \quad u \overset{a}{\sim} \mathcal{N}(0, 1/N)$

or $\sqrt{N}u \overset{a}{\sim} \mathcal{N}(0, 1)$

$$T_R = (\sqrt{N}u)^2 \sim \chi_1^2$$

10.13) For WGN $\quad p(w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}w^2}$

$$g(w) = -\frac{dp(w)/dw}{p(w)} = -\frac{d\ln p(w)}{dw}$$

$$= -d/dw\left(-\frac{1}{2\sigma^2}w^2\right) = w/\sigma^2$$

$$y(n) = g(x(n)) = x(n)/\sigma^2 \Rightarrow \underline{y} = \frac{1}{\sigma^2}\underline{x}$$

$$T_R(\underline{x}) = \frac{1}{\sigma^4}\frac{\underline{x}^T\underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}{1/\sigma^2}$$

$$= \frac{\underline{x}^T\underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}{\sigma^2}$$

since

$$i(A) = \int_{-\infty}^{\infty}\frac{\left(\frac{dp(w)}{dw}\right)^2}{p(w)}dw$$

$$= \int\left(\frac{d\ln p(w)}{dw}\right)^2 p(w)\,dw$$

$$= E\left[\left(\frac{d\ln p(w)}{dw}\right)^2\right]$$

$$= E\left[w^2/\sigma^4\right] = 1/\sigma^2$$

For linear model Rao test and GLRT are identical (see Prob. 6.15)

10.14) $$i(A) = \int_{-\infty}^{\infty}\frac{\left(\frac{dp(w)}{dw}\right)^2}{p(w)}dw$$

$$= \int_{-\infty}^{\infty} \left(\frac{d \ln p(w)}{dw} \right)^2 p(w)\, dw$$

$$= \int_{-\infty}^{\infty} \left[\frac{d}{dw} \left(-c_2 \, |w/\sigma|^{2/1+\beta} \right) \right]^2 p(w)\, dw$$

$$= 2c_2^2 \int_0^{\infty} \left(\frac{d}{dw} (w/\sigma)^{2/1+\beta} \right)^2 p(w)\, dw$$

Let $u = w/\sigma$

$$= 2c_2^2 \int_0^{\infty} \left(\frac{d}{d(\sigma u)} u^{2/1+\beta} \right)^2 p(\sigma u)\, \sigma\, du$$

$$= 2c_2^2 \int_0^{\infty} \frac{4}{\sigma^2 (1+\beta)^2} \left(u^{\frac{1-\beta}{1+\beta}} \right)^2 \frac{c_1}{\sigma} e^{-c_2 u^{\frac{2}{1+\beta}}} \sigma\, du$$

$$= \frac{8 c_2^2 c_1}{(1+\beta)^2 \sigma^2} \int_0^{\infty} u^{\frac{2-2\beta}{1+\beta}} e^{-c_2 u^{2/1+\beta}}\, du$$

Let $v = u^{\frac{2}{1+\beta}}$ $\qquad dv = \frac{2}{1+\beta} u^{\frac{1-\beta}{1+\beta}}\, du$

$u = v^{\frac{1+\beta}{2}}$ $\qquad = \frac{2}{1+\beta} v^{\frac{1-\beta}{2}}\, du$

$$= \frac{4 c_2^2 c_1}{(1+\beta)\, \sigma^2} \int_0^{\infty} v^{1-\beta} e^{-c_2 v} v^{-\left(\frac{1-\beta}{2}\right)}\, dv$$

$$= \frac{4 c_2^2 c_1}{(1+\beta)\, \sigma^2} \int_0^{\infty} \underset{\nu}{v^{\frac{1-\beta}{2}}}\, \underset{a}{e^{-c_2 v}}\, dv$$

$$= \frac{4 c_2^2 c_1}{(1+\beta)\, \sigma^2} \; \frac{\Gamma\left(3/2 - \beta/2 \right)}{c_2^{3/2 - \beta/2}}$$

$$= \frac{4\, c_2^{\frac{1}{2}+\beta/2}\, c_1\, \Gamma(3/2-\beta/2)}{(1+\beta)\,\sigma^2}$$

$$= \left[\frac{\Gamma(3/2(1+\beta))}{\Gamma(\frac{1}{2}(1+\beta))}\right]^{1/2} \frac{(\Gamma(3/2(1+\beta)))^{\frac{1}{2}}}{(1+\beta)[\Gamma(\frac{1}{2}(1+\beta))]^{3/2}}$$

$$\cdot\ \frac{4\,\Gamma(3/2-\beta/2)}{(1+\beta)\,\sigma^2}$$

$$= \frac{4\,\Gamma(3/2(1+\beta))\,\Gamma(3/2-\beta/2)}{\sigma^2(1+\beta)^2\,\Gamma^2(\frac{1}{2}(1+\beta))}$$

10.15) From Section 7.6.2 we have the same means and variances of z_1, z_2 for the nonGaussian case since the noise samples are IID. By the CLT z_1, z_2 are jointly Gaussian. Thus the performance is identical to the Gaussian case.

When we use a limiter as in (10.28) the performance is given by (10.31). The only difference is in d_1 with the loss being

$$10 \log_{10} \frac{d - \text{nonGaussian detector}}{d - \text{"linear" detector}}$$

$$= 10 \log_{10} \frac{N A^2 i(A)/2}{N A^2 / 2\sigma^2}$$

$$= 10 \log_{10} \sigma^2 i(A) \qquad db$$

Chapter 12

12.1) $$T(N) = \frac{1}{N-n_0+1} \sum_{n=n_0}^{N} (x[n]-A_0)$$

$$= \frac{1}{N-n_0+1}\left[\sum_{n=n_0}^{N-1} (x[n]-A_0) + (x[N]-A_0)\right]$$

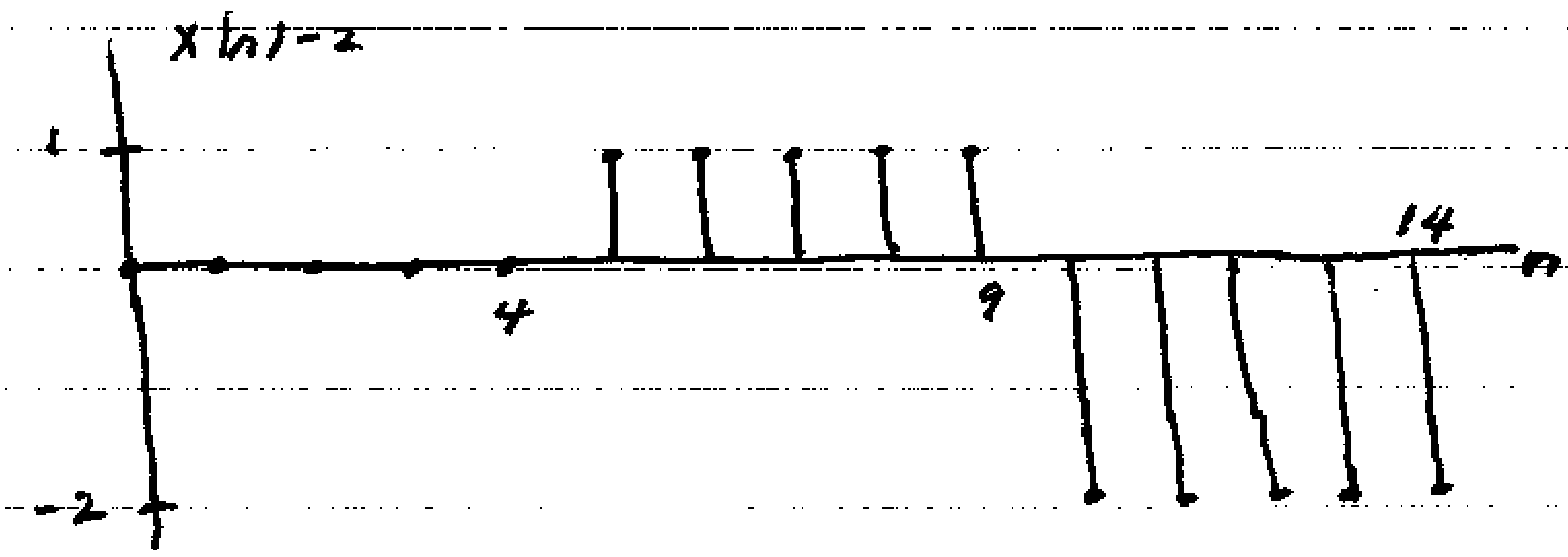
$$= \frac{1}{N-n_0+1} (N-n_0) T[N-1] + \frac{x[N]-A_0}{N-n_0+1}$$

$$= T[N-1] + \left[\frac{N-n_0}{N-n_0+1} - 1\right] T[N-1]$$

$$+ \frac{x[N]-A_0}{N-n_0+1}$$

$$= T[N-1] + \frac{1}{N-n_0+1}\left(x[N]-A_0 - T[N-1]\right)$$

12.2)



$T(5) = 1$ $\quad$ $T(10) = 3/6 = 1/2$

$T(6) = 1$ $\quad$ $T(11) = 1/7$

$\vdots$ $\quad$ $T(12) = -1/8$

$T(9) = 1$ $\quad$ $T(13) = -3/9 = -1/3$

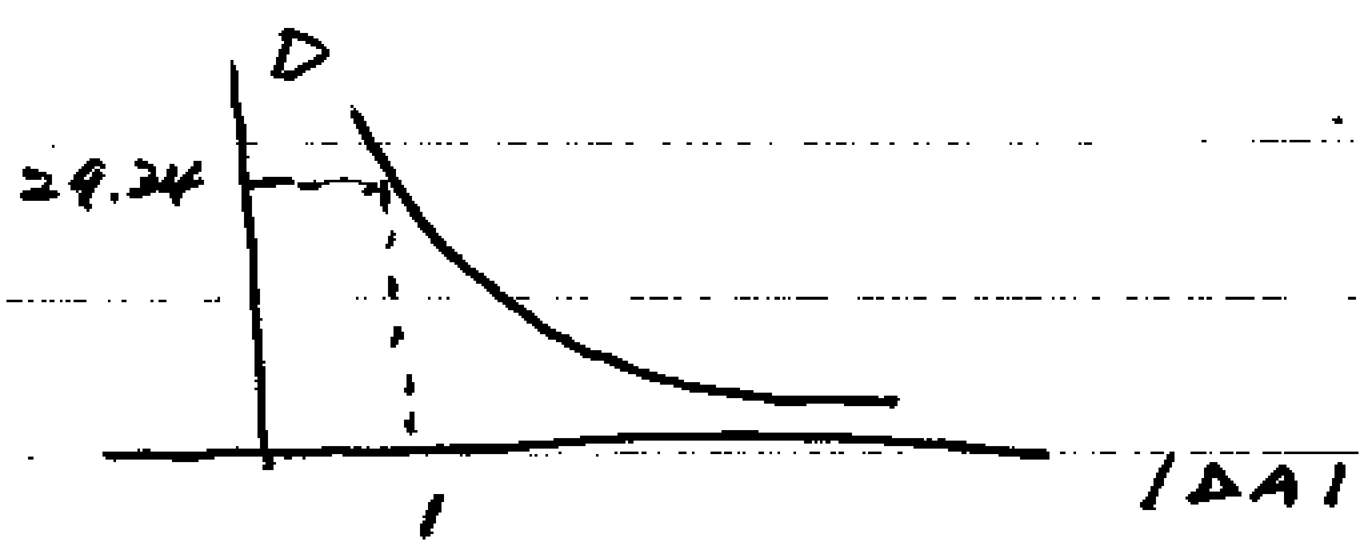
$T(14) = -5/10 = -1/2$

For $5 \le N \le 9$ $T(N) = 1$ and we would detect the change for a threshold $\gamma' < 1$. However, for $N > 9$ we may miss the jump if $\gamma > 1/2$ due to the non-stationarity of the signal, i.e., the unanticipated jump at $n = 10$.

12.31
$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$
$$0.99 = Q\left(Q^{-1}(10^{-3}) - \sqrt{d^2}\right)$$
$$\Rightarrow \quad d^2 = \left[Q^{-1}(10^{-3}) - Q^{-1}(0.99)\right]^2$$
$$= 29.34$$

But $d^2 = (N - n_0)\Delta A^2/\sigma^2$

The delay time is $N - n_0$. Thus
$$D = N - n_0 = d^2\sigma^2/\Delta A^2$$
$$= 29.34/|\Delta A|^2$$



12.4) From (12.4) we decide $\mathcal{H}_1$ if

$$\ln L(\underline{x}) = \frac{\Delta A}{\sigma^2} \sum (x[n]-A_0) - \frac{(N-n_0)\Delta A^2}{2\sigma^2} > \ln \gamma$$

For $\Delta A > 0$ we have for any ΔA

$$\sum(x[n]-A_0) > \frac{\sigma^2}{\Delta A} \ln \gamma + (N-n_0)\frac{\Delta A}{2}$$

or $T(\underline{x}) = \frac{1}{N-n_0} \sum (x[n]-A_0)$

$$> \frac{\Delta A}{2} + \frac{\sigma^2}{(N-n_0)\Delta A} \ln \gamma = \gamma'$$

Note that γ' does not depend on ΔA

since $T(\underline{x}) \sim N(0, \sigma^2/(N-n_0))$ under $\mathcal{H}_0$

12.5) If A_1 is known then

$$L_G(\underline{x}) = \frac{p(\underline{x}; A_1, \hat{A}_2)}{p(\underline{x}; A_1, A_1)}$$

where $\hat{A}_2$ = MLE under $\mathcal{H}_1$. As in Ex. 12.3 $\hat{A}_2 = \frac{1}{N-n_0} \sum_{n=n_0}^{N-1} x[n]$

Also the data before the jump is irrelevant so that

$$\ln L_G(\underline{x}) = -\frac{1}{2\sigma^2}\left[\sum_{n=n_0}^{N-1}\left((x(n)-\hat{A}_2)^2-(x(n)-A_1)^2\right)\right]$$

$$= -\frac{1}{2\sigma^2}\Big[\sum\big(x^2(n)-2\hat{A}_2 x(n)+\hat{A}_2^2$$

$$-x^2(n)+2A_1 x(n)-A_1^2\big)\Big]$$

$$= -\frac{1}{2\sigma^2}\Big(-2\hat{A}_2^2(N-n_0)+(N-n_0)\hat{A}_2^2$$

$$+2(N-n_0)A_1\hat{A}_2-(N-n_0)A_1^2\Big)$$

$$= -\frac{N-n_0}{2\sigma^2}\left(-\hat{A}_2^2+2A_1\hat{A}_2-A_1^2\right)$$

$$= \frac{(\hat{A}_2-A_1)^2}{2\sigma^2/N-n_0}$$

or we decide $\mathcal{H}_1$ if $(\hat{A}_2-A_1)^2 > \gamma'$

Note that

$$2\ln L_G(\underline{x}) = \frac{(\hat{A}_2-A_1)^2}{\sigma^2/N-n_0} \sim \begin{array}{ll}\chi_1^2 & \mathcal{H}_0\\ \chi_1'^2(\lambda) & \mathcal{H}_1\end{array}$$

$$\lambda = \frac{(A_2-A_1)^2}{\sigma^2/N-n_0}$$

(12.6) $$L_G(\underline{x}) = \frac{p(\underline{x};f_1=f_0,f_2=\hat{f}_0)}{p(\underline{x};f_1=f_0,f_2=f_0)}$$

since we have $\mathcal{H}_0:\ f_1=f_0,\ f_2=f_0$

$\mathcal{H}_1:\ f_1=f_0,\ f_2\neq f_0$

$$p(\underline{x}; f_1, f_2) = \frac{1}{(2\pi\sigma^2)^{N/2}}$$

$$\cdot\; e^{-\frac{1}{2\sigma^2}\left[\sum_{n=0}^{n_0-1}(x(n)-\cos 2\pi f_1 n)^2 + \sum_{n=n_0}^{N-1}(x(n)-\cos 2\pi f_2 n)^2\right]}$$

To find $\hat{f}_0$ we minimize

$$J(f_0) = \sum_{n=n_0}^{N-1}(x(n)-\cos 2\pi f_0 n)^2$$

$$= \sum (x^2(n) - 2x(n)\cos 2\pi f_0 n + \cos^2 2\pi f_0 n)$$

$$\approx \sum x^2(n) - 2\sum x(n)\cos 2\pi f_0 n + N/2 \quad \text{for } N \text{ large}$$

$\Rightarrow$ Must maximize $\sum_{n=n_0}^{N-1} x(n)\cos 2\pi f_0 n$ over f_0

$$L_G(\underline{x}) = e^{-\frac{1}{2\sigma^2}\sum_{n=n_0}^{N-1}\left[(x(n)-\cos 2\pi \hat{f}_0 n)^2 - (x(n)-\cos 2\pi f_0 n)^2\right]}$$

$$\ln L_G(\underline{x}) = -\frac{1}{2\sigma^2}\sum_{n=n_0}^{N-1}\left[-2x(n)\cos 2\pi\hat{f}_0 n + \cos^2 2\pi\hat{f}_0 n + 2x(n)\cos 2\pi f_0 n - \cos^2 2\pi f_0 n\right]$$

$$\approx -\frac{1}{2\sigma^2}\sum_{n=n_0}^{N-1} -2\left[x(n)\cos 2\pi\hat{f}_0 n - x(n)\cos 2\pi f_0 n\right]$$