Let $\underline{x} = (\, x[-M] \;\ldots\; x[M]\,)^T$

$\underline{s} = (\, s[-M] \ldots s[M])^T$ so that

$$[C_s]_{ij} = E(\, s[i]\, s[j]\,) \qquad i,j = -M, \ldots, 0, \ldots, M$$
$$= r_{ss}[i-j]$$

$$\Rightarrow \sum_{j=-M}^{M} r_{xx}[i-j]\, \hat{s}[j] = \sum_{j=-M}^{M} r_{ss}[i-j]\, x[j]$$

for $i = -M, \ldots, M$

As $M \to \infty \Rightarrow$

$$\sum_{j=-\infty}^{\infty} r_{xx}[i-j]\, \hat{s}[j] = \sum_{j=-\infty}^{\infty} r_{ss}[i-j]\, x[j]$$

$$-\infty < i < \infty$$

or $r_{xx}[n] \star \hat{s}[n] = r_{ss}[n] \star x[n]$

Taking Fourier transforms

$$P_{xx}(f)\, \hat{S}(f) = P_{ss}(f)\, X(f)$$

$$\Rightarrow H(f) = \frac{P_{ss}(f)}{P_{xx}(f)} = \frac{P_{ss}(f)}{P_{ss}(f) + \sigma^2}$$

Wiener filter emphasizes high SNR regions and attenuates low SNR regions since

$$H(f) = \frac{\eta(f)}{\eta(f)+1} \qquad \text{where } \eta(f) = \frac{P_{ss}(f)}{\sigma^2}$$

If $\eta(f) \gg 1$ (high SNR), $H(f) \simeq 1$

If $\eta(f) \ll 1$ (low SNR), $H(f) \simeq 0$.

Chapter 12

1) $Bmse(\hat{\theta}) = E[(\theta - a x^2[0] - b x[0] - c)^2]$

$$\frac{\partial Bmse}{\partial a} = -2E[(\theta - a x^2[0] - b x[0] - c) x^2[0]] = 0$$

$$\frac{\partial Bmse}{\partial b} = -2E[(\quad '' \quad) x[0]] = 0$$

$$\frac{\partial Bmse}{\partial c} = -2E[(\quad '' \quad)] = 0$$

$$E(\theta x^2) = aE(x^4) + bE(x^3) + cE(x^2)$$
$$E(\theta x) = aE(x^3) + bE(x^2) + cE(x)$$
$$E(\theta) = aE(x^2) + bE(x) + c$$

$$\begin{bmatrix} E(x^4) & E(x^3) & E(x^2) \\ E(x^3) & E(x^2) & E(x) \\ E(x^2) & E(x) & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} E(\theta x^2) \\ E(\theta x) \\ E(\theta) \end{bmatrix}$$

The minimum MSE is

$$Bmse(\hat{\theta}) = E[\theta(\theta - \hat{a}x^2[0] - \hat{b}x[0] - \hat{c})]$$

$$= E(\theta^2) - \hat{a}E(\theta x^2) - \hat{b}E(\theta x) - \hat{c}E(\theta)$$

Now if $x[0] \sim U[-1/2, 1/2]$ and $\theta = \cos 2\pi x[0]$

$E(x) = 0, \quad E(\theta) = 0$

$$E(\theta x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x \cos 2\pi x \, dx = 0$$

$$E(\theta x^2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cos 2\pi x \, dx = -\frac{1}{2\pi^2}$$

$$E(x^2) = \int_{-1/2}^{1/2} x^2 dx = \frac{1}{12}$$

$$E(x^3) = 0$$

$$E(x^4) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^4 dx = \frac{1}{80}$$

$$\begin{bmatrix} \frac{1}{80} & 0 & \frac{1}{12} \\ 0 & \frac{1}{12} & 0 \\ \frac{1}{12} & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} -\frac{1}{2\pi^2} \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \hat{a} = -90/\pi^2$$
$$\hat{b} = 0$$
$$\hat{c} = 15/2\pi^2$$

$$Bmse(\hat{\theta}) = \frac{1}{2} - (-90/\pi^2)(-1/2\pi^2) - 0 - 0 = 0.04$$

Using a linear estimator we have

$$\hat{\theta} = b\,x(0) + c$$

$$\begin{bmatrix} E(x^2) & E(x) \\ E(x) & 1 \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} E(\theta x) \\ E(\theta) \end{pmatrix}$$

But $E(\theta x) = 0$, $E(\theta) = 0 \Rightarrow \hat{b} = \hat{c} = 0$ and $\hat{\theta} = 0$. The minimum MSE is $E(\theta^2) = 1/2$.

2) From (12.27)

$$\hat{A} = \mu_A + \left(1/\sigma_A^2 + \frac{\underline{h}^T\underline{h}}{\sigma^2} \right)^{-1} \frac{\underline{h}^T}{\sigma^2} (\underline{x} - \underline{h}\,\mu_A)$$

where $\underline{h} = [1\ r \ldots r^{N-1}]^T$

$$= \mu_A + \frac{\sum_{n=0}^{N-1} r^n (x[n] - r^n \mu_A)}{\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}}$$

$$Bmse(\hat{A}) = \frac{1}{1/\sigma_A^2 + \frac{1}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}}$$

from (12.29), (12.30).

3) $\hat{x}_2 = a_1 x_1 + b$

Let $\theta = x_2$. Then from (12.6)

$$\hat{\theta} = E(\theta) + C_{\theta x_1} C_{x_1 x_1}^{-1} (x_1 - E(x_1))$$

$$= E(x_2) + \frac{E(x_1 x_2)}{E(x_1^2)} (x_1 - E(x_1))$$

$$Bmse(\hat{\theta}) = C_{\theta\theta} - C_{\theta x_1} C_{x_1 x_1}^{-1} C_{x_1 \theta} \quad \text{from (12.8)}$$

$$= E(x_2^2) - \frac{E^2(x_1 x_2)}{E(x_1^2)}$$

But $\underline{C}_{xx} = \begin{bmatrix} E(x_1^2) & E(x_1 x_2) \\ E(x_2 x_1) & E(x_2^2) \end{bmatrix}$

This is singular if and only if

$$E(x_1^2) E(x_2^2) - E^2(x_1 x_2) = 0$$

and is equivalent to $Bmse(\hat{\theta}) = 0$.

In general, if

$$\hat{x}_1 = \sum_{i=2}^{N} a_i x_i$$

$$Bmse(\hat{x}_1) = E((x_1 - \hat{x}_1)^2)$$

$$= E\left[\left(x_1 - \sum_{i=2}^{N} a_i x_i\right)^2\right]$$

$$= E\left[(\underline{b}^T \underline{x})^2\right] \quad \text{where} \quad b_1 = 1$$

$$b_i = -a_i$$

$$i = 2, \ldots, N$$

$$= \underline{b}^T \underline{C}_{xx} \underline{b}$$

But $\underline{C}_{xx}$ is positive semidefinite and for the Bmse to be zero we require $\underline{b}^T \underline{C}_{xx} \underline{b} = 0$ for some $\underline{b} \neq \underline{0}$. Hence, $\underline{C}_{xx}$ must be singular.

4) $(x, x) = E(x^2) \geq 0$ and $= 0$ if and only if $x = 0$

$(x, y) = (y, x)$ obvious

$$(c_1 x_1 + c_2 x_2, y) = E((c_1 x_1 + c_2 x_2) y)$$

$$= E(c_1 x_1 y + c_2 x_2 y)$$

$$= c_1 E(x_1 y) + c_2 E(x_2 y)$$

$$= c_1 (x_1, y) + c_2 (x_2, y)$$

5) $(x, x) = 0 \Rightarrow \operatorname{cov}(x, x) = 0 \Rightarrow \operatorname{var}(x) = 0$

$\not\Rightarrow$ $x = 0$ but that x is a constant

6) Using (12.20)

$$\hat{\underline{s}} = \underline{C}_{sx} \underline{C}_{xx}^{-1} \underline{x}$$

$$\underline{C}_{xx} = E(\underline{x}\underline{x}^T) = \underline{R}_{ss} + \underline{R}_{ww} = (\sigma_s^2 + \sigma^2) \underline{I}$$

$$\underline{C}_{sx} = E(\underline{s}\underline{x}^T) = E(\underline{s}(\underline{s} + \underline{w})^T)$$

$$= E(\underline{s}\underline{s}^T) = \sigma_s^2 \underline{I}$$

$$\hat{\underline{s}} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} \underline{x} \quad \text{or} \quad \hat{s}[n] = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} x[n]$$

From (12.21)

$$\underline{M}_{\hat{s}} = \underline{C}_{ss} - \underline{C}_{sx}\underline{C}_{xx}^{-1}\underline{C}_{xs}$$

$$= \sigma_s^2 \underline{I} - \frac{(\sigma_s^2)^2}{\sigma_s^2+\sigma^2}\underline{I}$$

$$= \sigma_s^2\left[1 - \frac{\sigma_s^2}{\sigma_s^2+\sigma^2}\right]\underline{I}$$

$$= \frac{\sigma_s^2\sigma^2}{\sigma_s^2+\sigma^2}\underline{I}$$

7) $$\hat{\underline{\theta}} = E(\underline{\theta}) + \underline{C}_{\theta x}\underline{C}_{xx}^{-1}(\underline{x}-E(\underline{x}))$$

$$\underline{M}_{\hat{\theta}} = E\left[(\underline{\theta}-\hat{\underline{\theta}})(\underline{\theta}-\hat{\underline{\theta}})^T\right]$$

$$= E\left[\left(\underline{\theta}-E(\underline{\theta})-\underline{C}_{\theta x}\underline{C}_{xx}^{-1}(\underline{x}-E(\underline{x}))\right)\left(\underline{\theta}-E(\underline{\theta})-\underline{C}_{\theta x}\underline{C}_{xx}^{-1}(\underline{x}-E(\underline{x})\right)^T\right]$$

$$= \underline{C}_{\theta\theta} - E\left[(\underline{\theta}-E(\underline{\theta}))(\underline{x}-E(\underline{x}))^T\right]\underline{C}_{xx}^{-1}\underline{C}_{x\theta}$$
$$- \underline{C}_{\theta x}\underline{C}_{xx}^{-1}E\left[(\underline{x}-E(\underline{x}))(\underline{\theta}-E(\underline{\theta}))^T\right]$$
$$+ \underline{C}_{\theta x}\underline{C}_{xx}^{-1}\underline{C}_{xx}\underline{C}_{xx}^{-1}\underline{C}_{x\theta}$$

$$= \underline{C}_{\theta\theta} - \underline{C}_{\theta x}\underline{C}_{xx}^{-1}\underline{C}_{x\theta} - \underline{C}_{\theta x}\underline{C}_{xx}^{-1}\underline{C}_{x\theta}$$
$$+ \underline{C}_{\theta x}\underline{C}_{xx}^{-1}\underline{C}_{x\theta}$$

$$= \underline{C}_{\theta\theta} - \underline{C}_{\theta x}\underline{C}_{xx}^{-1}\underline{C}_{x\theta}$$

$$\text{Bmse}(\hat{\theta}_i) = E\left[(\theta_i-\hat{\theta}_i)^2\right]$$

where the expectation is with respect to $p(\underline{x}, \theta_i)$

But $$\text{Bmse}(\hat{\theta}_i) = \int (\theta_i-\hat{\theta}_i)^2 p(\underline{x},\theta_i)\,d\theta_i\,d\underline{x}$$

$$= \int\cdots\int (\theta_i-\hat{\theta}_i)^2 p(\underline{x},\underline{\theta})\,d\underline{\theta}\,d\underline{x}$$

since $\hat{\theta}_i$ depends on $\underline{x}$ only

$$= \int \cdots \int \left[(\underline{\theta} - \hat{\underline{\theta}})(\underline{\theta} - \hat{\underline{\theta}})^T \right]_{ii} \, p(\underline{x}, \underline{\theta}) \, d\underline{\theta} \, d\underline{x}$$

$$= \left[E\left((\underline{\theta} - \hat{\underline{\theta}})(\underline{\theta} - \hat{\underline{\theta}})^T \right) \right]_{ii} = [\underline{M}_{\hat{\theta}}]_{ii}$$

8)
$$\hat{\underline{\alpha}} = E(\underline{\alpha}) + C_{\alpha x} C_{xx}^{-1} (\underline{x} - E(\underline{x}))$$

But $E(\underline{\alpha}) = \underline{A} E(\underline{\theta}) + \underline{b}$

$$\begin{aligned} \underline{C}_{\alpha x} &= E\left[(\underline{\alpha} - E(\underline{\alpha}))(\underline{x} - E(\underline{x}))^T \right] \\ &= E\left[\underline{A}(\underline{\theta} - E(\underline{\theta}))(\underline{x} - E(\underline{x}))^T \right] \\ &= \underline{A} \, \underline{C}_{\theta x} \end{aligned}$$

$$\Rightarrow \hat{\underline{\alpha}} = \underline{A} E(\underline{\theta}) + \underline{b} + \underline{A} \underline{C}_{\theta x} \underline{C}_{xx}^{-1} (\underline{x} - E(\underline{x}))$$

$$= \underline{A} \hat{\underline{\theta}} + \underline{b}$$

$$\hat{\underline{\alpha}} = E(\underline{\alpha}) + C_{\alpha x} C_{xx}^{-1} (\underline{x} - E(\underline{x}))$$

If $\underline{\alpha} = \underline{\theta}_1 + \underline{\theta}_2$,

$$E(\underline{\alpha}) = E(\underline{\theta}_1) + E(\underline{\theta}_2)$$

$$\begin{aligned} \underline{C}_{\alpha x} &= E\left[(\underline{\theta}_1 + \underline{\theta}_2 - E(\underline{\theta}_1) - E(\underline{\theta}_2))(\underline{x} - E(\underline{x}))^T \right] \\ &= E\left[(\underline{\theta}_1 - E(\underline{\theta}_1))(\underline{x} - E(\underline{x}))^T \right] \\ &\quad + E\left[(\underline{\theta}_2 - E(\underline{\theta}_2))(\underline{x} - E(\underline{x}))^T \right] \\ &= \underline{C}_{\theta_1 x} + \underline{C}_{\theta_2 x} \end{aligned}$$

$$\Rightarrow \hat{\underline{\alpha}} = E(\underline{\theta}_1) + E(\underline{\theta}_2) + (\underline{C}_{\theta_1 x} + \underline{C}_{\theta_2 x}) \underline{C}_{xx}^{-1} (\underline{x} - E(\underline{x}))$$

$$= \hat{\underline{\theta}}_1 + \hat{\underline{\theta}}_2$$

9)
$$\hat{A}[N-1] = \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2/N} \bar{x} + \frac{\sigma^2/N}{\sigma_A^2 + \sigma^2/N} \mu_A$$

$$\hat{A}[N] = \frac{N\sigma_A^2}{(N+1)\sigma_A^2 + \sigma^2} \frac{1}{N} \left(\sum_0^{N-1} x[n] + x[N] \right)$$

$$+ \frac{\sigma^2}{(N+1)\sigma_A^2+\sigma^2}\,\mu_A$$

$$= \frac{N\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\;\frac{\sigma_A^2+\sigma^2/N}{\sigma_A^2}\;\underbrace{\frac{\sigma_A^2}{\sigma_A^2+\sigma^2/N}\;\frac{1}{N}\sum_{0}^{N-1} x(n)}\;\leftarrow \hat{A}(N-1)$$

$$+ \frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\,x[N] + \frac{N\sigma_A^2+\sigma^2}{(N+1)\sigma_A^2+\sigma^2}\;\underbrace{\frac{\sigma^2}{N\sigma_A^2+\sigma^2}\,\mu_A}$$

$$= \frac{N\sigma_A^2+\sigma^2}{(N+1)\sigma_A^2+\sigma^2}\,\hat{A}(N) + \frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\,x(N)$$

$$= \hat{A}(N-1) + \left(\frac{N\sigma_A^2+\sigma^2}{(N+1)\sigma_A^2+\sigma^2} - 1\right)\hat{A}(N-1)$$

$$+ \frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\,x(N)$$

$$= \hat{A}[N-1] - \frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\,\hat{A}[N-1]$$

$$+ \frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\,x[N]$$

$$= \hat{A}(N-1) + \frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\left(x(N) - \hat{A}(N-1)\right)$$

Since $\mathrm{Bmse}(\hat{A}[N-1])$ for the case $\mu_A \neq 0$ is also given by (12.31), the rest of the derivation is identical. Thus, we have the identical set of equations.

10) This procedure is illustrated in Figure 12.5.

$$\underline{e}_1 = \frac{\begin{bmatrix}1\\1\\2\end{bmatrix}}{\sqrt{1+1+4}} = \begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}$$

$$\underline{z}_2 = \underline{x}_2 - (\underline{x}_2, \underline{e}_1)\underline{e}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} - [1\ 0\ 1]\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{3}{\sqrt{6}}\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}$$

$$\underline{e}_2 = \frac{\begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}}{\sqrt{1/4+1/4}} = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}$$

$$\underline{z}_3 = \underline{x}_3 - (\underline{x}_3, \underline{e}_1)\underline{e}_1 - (\underline{x}_3, \underline{e}_2)\underline{e}_2$$

$$= \begin{bmatrix}1/3\\1/3\\-1/3\end{bmatrix}$$

$$\underline{e}_3 = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}$$

(1) $$y_1 = \frac{x_1}{\sqrt{\mathrm{var}(x_1)}} = x_1$$

$$z_2 = x_2 - (x_2, y_1)y_1 = x_2 - E(x_2 y_1)y_1$$
$$= x_2 - E(x_1 x_2)x_1 = x_2 - \rho x_1$$

$$y_2 = \frac{x_2 - \rho x_1}{\sqrt{E((x_2-\rho x_1)^2)}} = \frac{x_2 - \rho x_1}{\sqrt{1-2\rho^2+\rho^2}} = \frac{x_2-\rho x_1}{\sqrt{1-\rho^2}}$$

$$z_3 = x_3 - (x_3, y_1)y_1 - (x_3, y_2)y_2$$

$$= x_3 - E(x_3 x_1)x_1 - E\left(x_3 \frac{x_2-\rho x_1}{\sqrt{1-\rho^2}}\right)\frac{x_2-\rho x_1}{\sqrt{1-\rho^2}}$$

$$= x_3 - \rho^2 x_1 - \frac{\rho-\rho^3}{\sqrt{1-\rho^2}}\,\frac{x_2-\rho x_1}{\sqrt{1-\rho^2}}$$

$$= x_3 - \rho^2 x_1 - \rho(x_2-\rho x_1) = x_3 - \rho x_2$$

$$y_3 = \frac{x_3-\rho x_2}{\sqrt{E((x_3-\rho x_2)^2)}} = \frac{x_3-\rho x_2}{\sqrt{1-\rho^2}}$$

$$\underline{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{-\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} & 0 \\ 0 & \frac{-\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix}}_{\underline{A}} \underline{x}$$

$\underline{A}$ is lower triangular and will be in general.

Since $\underline{C}_{yy} = \underline{I} \Rightarrow \underline{C}_{yy} = \underline{A}\,\underline{C}_{xx}\underline{A}^T$

$$C_{\underline{x}x} = \underline{A}^{-1}\underline{A}^{T^{-1}} \Rightarrow \underline{C}_{xx}^{-1} = \underline{A}^T\underline{A}$$

Thus, $\underline{A}$ is a whitening transformation and $\underline{A}^T\underline{A}$ provides a Cholesky decomposition of $\underline{C}_{xx}^{-1}$.

12) From (12.47) for a scalar parameter so that $M[n-1]$ and $b[n]$ are scalars, as $\sigma_n^2 \to 0$

$$K[n] \to \frac{M[n-1]\,h[n]}{M[n-1]\,h^2(n)} = \frac{1}{h[n]}$$

$$\Rightarrow \hat{\theta}[n] = \hat{\theta}[n-1] + \frac{1}{h[n]}\left(x[n] - h[n]\,\hat{\theta}[n-1]\right)$$
$$= x(n)/h[n]$$

We discard all previous data since $x[n]$ is a perfect measurement (no noise). Also,

$$M[n] = (1 - K[n]\,h[n])\,M[n-1] = 0$$

In vector case we do not obtain analogous result since we require p noiseless measurements to determine $\underline{\theta}$.

13) Here $h[n] = 1$, $\sigma_n^2 = \sigma^2$

$$\Rightarrow \hat{A}(n) = \hat{A}(n-1) + K(n)\left(x[n] - \hat{A}[n-1]\right)$$

$$K[n] = \frac{M[n-1]}{\sigma^2 + M[n-1]}$$

$$M[n] = (1 - K[n])\,M[n-1]$$

This is the same as the introductory example of Sect 12.6 except for the initialization since

$$\hat{A}[-1] = E(A) = 0$$

$$M[-1] = E\left[(A - \hat{A}[-1])^2\right] = E(A^2) = A_0^2/3$$

Solving for $K(n)$:

$$M(n) = \left(1 - \frac{M[n-1]}{\sigma^2 + M[n-1]}\right) M(n-1)$$

$$= \frac{\sigma^2 M[n-1]}{\sigma^2 + M[n-1]} = K[n]\,\sigma^2$$

$$\Rightarrow \quad K[n] = \frac{K[n-1]\,\sigma^2}{\sigma^2 + K[n-1]\,\sigma^2} = \frac{K[n-1]}{1+K[n-1]}$$

$$\frac{1}{K[n]} = \frac{1}{K[n-1]} + 1 \qquad n \geq 1$$

$$K[0] = \frac{M[-1]}{\sigma^2 + M[-1]} = \frac{A_0^2/3}{\sigma^2 + A_0^2/3}$$

$$\frac{1}{K[0]} = 1 + \frac{\sigma^2}{A_0^2/3}$$

$$\frac{1}{K[n]} = \frac{1}{K[0]} + n = (n+1) + \frac{\sigma^2}{A_0^2/3}$$

$$K[n] = \frac{A_0^2/3}{(n+1)A_0^2/3 + \sigma^2}$$

$$1 - K[n] = \frac{n A_0^2/3 + \sigma^2}{(n+1)A_0^2/3 + \sigma^2}$$

$$\hat{A}[n] = (1 - K[n])\,\hat{A}[n-1] + K[n]\,x[n]$$

$$= \frac{n A_0^2/3 + \sigma^2}{(n+1)A_0^2/3 + \sigma^2}\,\hat{A}[n-1] + \frac{A_0^2/3}{(n+1)A_0^2/3 + \sigma^2}\,x[n]$$

where $\hat{A}[-1] = 0$

$$\hat{A}[0] = \frac{A_0^2/3}{A_0^2/3 + \sigma^2}\,x[0]$$

$$\hat{A}[1] = \frac{A_0^2/3 + \sigma^2}{2A_0^2/3 + \sigma^2}\;\frac{A_0^2/3}{A_0^2/3 + \sigma^2}\,x[0]$$

$$+ \quad \frac{A_0^2/3}{2A_0^2/3+\sigma^2}\; x[1]$$

$$= \frac{A_0^2/3}{2A_0^2/3+\sigma^2}\;(x[0]+x[1])$$

$$= \frac{A_0^2/3}{A_0^2/3+\sigma^2/2}\;\underbrace{\tfrac{1}{2}(x[0]+x[1])}_{\bar{x}}$$

etc.

14) To minimize $E[(x[n]-\hat{x}[n])^2]$ we use the orthogonality principle or

$$E[(x[n]-\hat{x}[n])x[n-l]] = 0 \qquad l=-M,\dots,M \quad l\neq 0$$

$$r_{xx}[l] = E\left[\sum_k a_k x[n-k]x[n-l]\right] = \sum_k a_k r_{xx}[l-k]$$

To show that $a_{-k}=a_k$:

Let $k'=-k$

$$r_{xx}[l] = \sum_{\substack{k'=-M\\k'\neq 0}}^{M} a_{-k'} r_{xx}[l+k'] \qquad l=-M,\dots,M \quad l\neq 0$$

Let $l'=-l$

$$r_{xx}[-l'] = \sum_{\substack{k'=-M\\k'\neq 0}}^{M} a_{-k'} r_{xx}[-l'+k'] \qquad l'=-M,\dots,M \quad l'\neq 0$$

$$r_{xx}[l'] = \sum_{\substack{k'=-M\\k'\neq 0}}^{M} a_{-k'} r_{xx}[l'-k']$$

Since $r_{xx}[-k]=r_{xx}[k]$

But these are the same set of equations for which there is a unique solution. Hence $a_{-k} = a_k$. This must be true since the correlation of $x(n)$ with $x(n+k)$ is the same as that with $x(n-k)$, due to the even symmetry of the ACF.

15) $$W = \frac{r_{ss}[0]}{r_{ss}[0] + r_{ww}[0]} = \eta/_{\eta+1}$$

$$M_{\hat{s}} = \left(1 - \frac{\eta}{\eta+1}\right) r_{ss}[0]$$

But $$\rho = \frac{E(s[0](s[0] + w[0]))}{\sqrt{r_{ss}[0](r_{ss}[0] + r_{ww}[0])}}$$

$$= \frac{r_{ss}[0]}{\sqrt{r_{ss}[0](r_{ss}[0] + r_{ww}[0])}}$$

$$= \frac{1}{\sqrt{1 + 1/\eta}} = \sqrt{\eta/_{\eta+1}}$$

$\Rightarrow$ $W = \rho^2$ or $\hat{s}[0] = \rho^2 x[0]$

$M_{\hat{s}} = (1-\rho^2)\, r_{ss}[0]$

The higher the SNR, η, the larger is ρ and hence the better is the estimator.

16) $$\left[B(z^{-1}) \left(G(z) - \frac{P_{ss}(z)}{B(z^{-1})} \right) \right]_+ = 0$$

But $B(z^{-1})$ is the z-transform of an anticausal sequence and thus if $G(z) - \dfrac{P_{ss}(z)}{B(z^{-1})}$ is also anticausal with a sequence value of zero at $n = 0$ or

$$\mathcal{Z}^{-1}\{ G(z) - P_{ss}(z)/B(z^{-1}) \}$$

the convolution will be zero for $n \geq 0$ as required. Now, since $P_{ss}(z)/B(z^{-1})$ is a two-sided sequence we let

$$\mathcal{Z}^{-1}\{ G(z) \} = \mathcal{Z}^{-1}\left\{ \frac{P_{ss}(z)}{B(z^{-1})} \right\} \quad \text{for } n \geq 0$$

or $$G(z) = \left[\frac{P_{ss}(z)}{B(z^{-1})} \right]_+$$

or $$H(z) = \frac{1}{B(z)} \left[P_{ss}(z)/B(z^{-1}) \right]_+$$

17) $$E\{ (s[n] - \hat{s}[n]) \, x[n-\ell] \} = 0 \qquad -\infty < \ell < \infty$$

$$E\left\{ \left(s[n] - \sum_k h[k] \, x[n-k] \right) x[n-\ell] \right\} = 0$$

$$E\{ s[n] \, x[n-\ell] \} = \sum_k h[k] \, E\{ x[n-k] \, x[n-\ell] \}$$

$$r_{sx}[\ell] = \sum_{k=-\infty}^{\infty} h[k] \, r_{xx}[\ell-k]$$

18) $M_{\hat{s}} = E[(s[n]-\hat{s}[n])^2]$

$$= E[(s[n]-\hat{s}[n])(s[n]-\sum_k h[k]x[n-k])]$$

$$= E[(s[n]-\hat{s}[n])s[n]]$$

$$- \underbrace{E[(s[n]-\hat{s}[n])\sum_k h[k]x[n-k]]}_{=0 \text{ by orthogonality principle}}$$

$$= r_{ss}[0] - \sum_k h[k]\underbrace{E[x[n-k]s[n]]}_{r_{ss}[k]}$$

Now by Parseval's theorem

$$\sum_k h[k]r_{ss}[k] = \int P_{ss}(f)H(f)\,df$$

$$\Rightarrow \quad M_{\hat{s}} = \int P_{ss}(f)\,df - \int P_{ss}(f)H(f)\,df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{ss}(f)(1-H(f))\,df$$

For the example given

$$H(f) = \begin{cases} \dfrac{P_0}{P_0+\sigma^2} & |f| \le 1/4 \\ 0 & \frac{1}{4} < |f| \le \frac{1}{2} \end{cases}$$

$$M_{\hat{s}} = \int_{-1/4}^{1/4} P_0\left(1-\frac{P_0}{P_0+\sigma^2}\right)df = \frac{1}{2}\,\frac{P_0\sigma^2}{P_0+\sigma^2}$$

Since there is no signal power above $f = 1/4$, $H(f) = 0$. For $f \le 1/4$ we weight all frequencies (since signal and noise have flat PSDs) by an SNR weighting or

$$H(f) = \frac{P_0}{P_0 + \sigma^2} = \frac{\eta}{\eta + 1}$$

19) $$\hat{x}[n] = \sum_{k=1}^{N} h[k] x[n-k]$$

$$E[(x[n] - \hat{x}[n]) x[n-l]] = 0 \qquad l = 1, 2, \ldots, N$$

$$r_{xx}[l] = \sum_{k=1}^{N} h[k] E(x[n-k] x[n-l])$$

$$= \sum_{k=1}^{N} h[k] r_{xx}[l-k]$$

The equations are independent of n (in deriving (12.65) we assumed $n = N$ was the index of the sample to be predicted) since the ACF does not depend on n.

$$M_Q = E[(x[n] - \hat{x}[n]) x[n]] - \underbrace{E[(x[n] - \hat{x}[n]) \hat{x}[n]]}_{= 0 \text{ by orthogonality principle}}$$

$$= E[x^2[n]] - \sum_{k=1}^{N} h[k] E[x[n-k] x[n]]$$

$$= r_{xx}[0] - \sum_{k=1}^{N} h[k] r_{xx}[k]$$

20) From the previous problem we have

$$r_{xx}[l] = \sum_{k=1}^{N} h[k] r_{xx}[l-k] \qquad l = 1, 2, \ldots, N$$

must be solved for the optimal one-step predictor. But for an AR(N) process we know that (see Appendix 1)

$$r_{xx}[l] = -\sum_{k=1}^{N} a[k]\, r_{xx}[l-k] \qquad l \geq 1$$

which are the Yule-Walker equations. Since the solution for the $h[k]$'s is unique,

$$h[k] = -a[k]$$

so that $\hat{x}[n] = -\sum_{k=1}^{N} a(k)\, x[n-k]$ and the MMSE is

$$M_{\hat{x}} = r_{xx}[0] - \sum_{k=1}^{N} h[k]\, r_{xx}[k]$$

$$= r_{xx}[0] + \sum_{k=1}^{N} a[k]\, r_{xx}[k]$$

$$= \sigma_u^2 \quad \text{(see Appendix 1)}$$

Chapter 13

1) $s[n] = a^{n+1} s[-1] + \sum_{k=0}^{n} a^k u[n-k]$

Let $\underline{s} = [s[n_1]\ s[n_2]\ \ldots\ s[n_k]]^T$

Assume $n_k > n_{k-1} > \ldots > n_1$

$$\underline{s} = \begin{bmatrix} a^{n_1+1} & a^{n_1} & \ldots & 1 & 0 & 0 & \ldots & 0 \\ a^{n_2+1} & a^{n_2} & \ldots & & 1 & 0 & \ldots & 0 \\ \vdots & & & & & & & \\ a^{n_k+1} & a^{n_k} & \ldots & & & & & 1 \end{bmatrix} \begin{bmatrix} s[-1] \\ u[0] \\ \vdots \\ u[n_k] \end{bmatrix}$$

Since $s[-1], u[0], \ldots, u[n_k]$ are independent, they are jointly Gaussian and thus $\underline{s}$ has a multivariate Gaussian PDF, being a linear transformation.

2) If $\mu_s = 0$, from (13.4), $E(s[n]) = 0$

$$C_s[m,n] = a^{m+n+2}\sigma_s^2 + \sigma_u^2 a^{m-n} \cdot \sum_{k=0}^{n} a^{2k}$$

$$= \frac{a^{m+n+2}\sigma_u^2}{1-a^2} + \frac{\sigma_u^2 a^{m-n}(1-a^{2(n+1)})}{1-a^2}$$

$$= \frac{\sigma_u^2}{1-a^2} a^{m-n}$$

$\Rightarrow$ WSS. By setting the initial conditions as given the process is in steady-state for $n = -1$.

3) $E(s[n]) = a^{n+1}\mu_s = 5(0.98)^{n+1}$

$$\mathrm{var}(s[n]) = a^{2n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=0}^{n} a^{2k}$$

$$= (0.98)^{2n+2} + 0.1 \sum_{k=0}^{n} (0.98)^{2k}$$

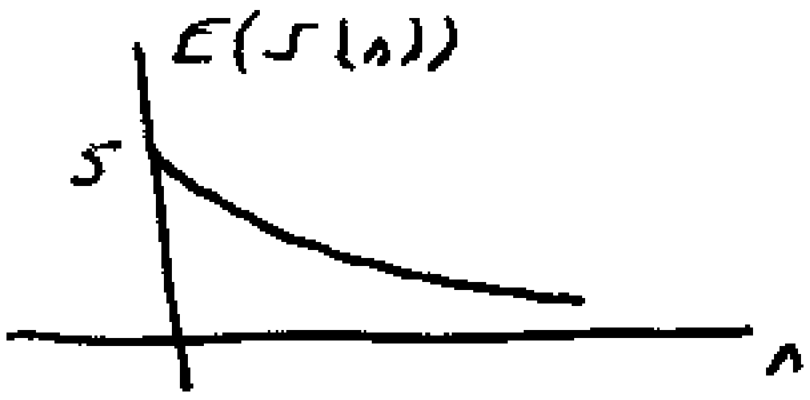
$$c_s[m,n] \rightarrow \frac{\sigma_u^2}{1-a^2}\, a^{m-n}$$

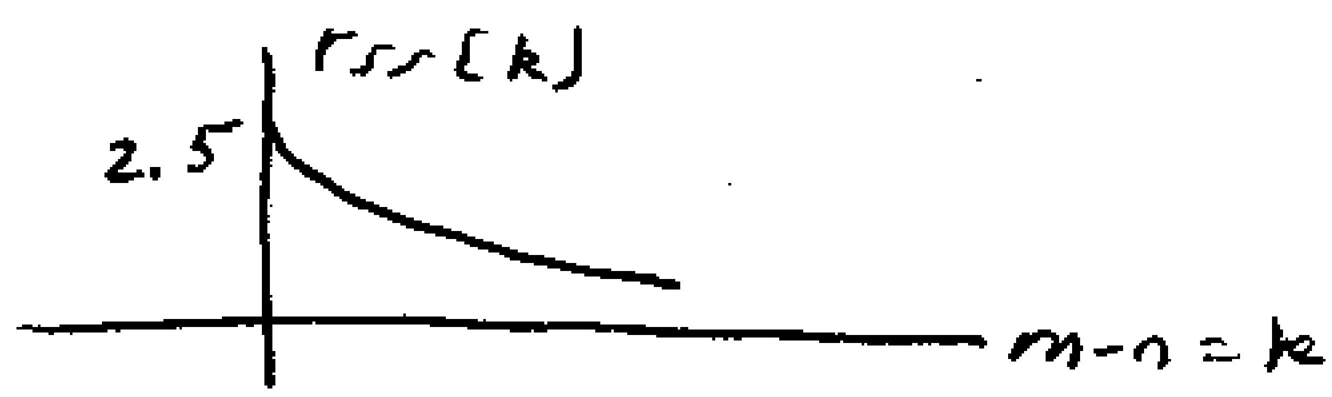
$$= \frac{0.1}{1-0.98^2}\; 0.98^{m-n}$$

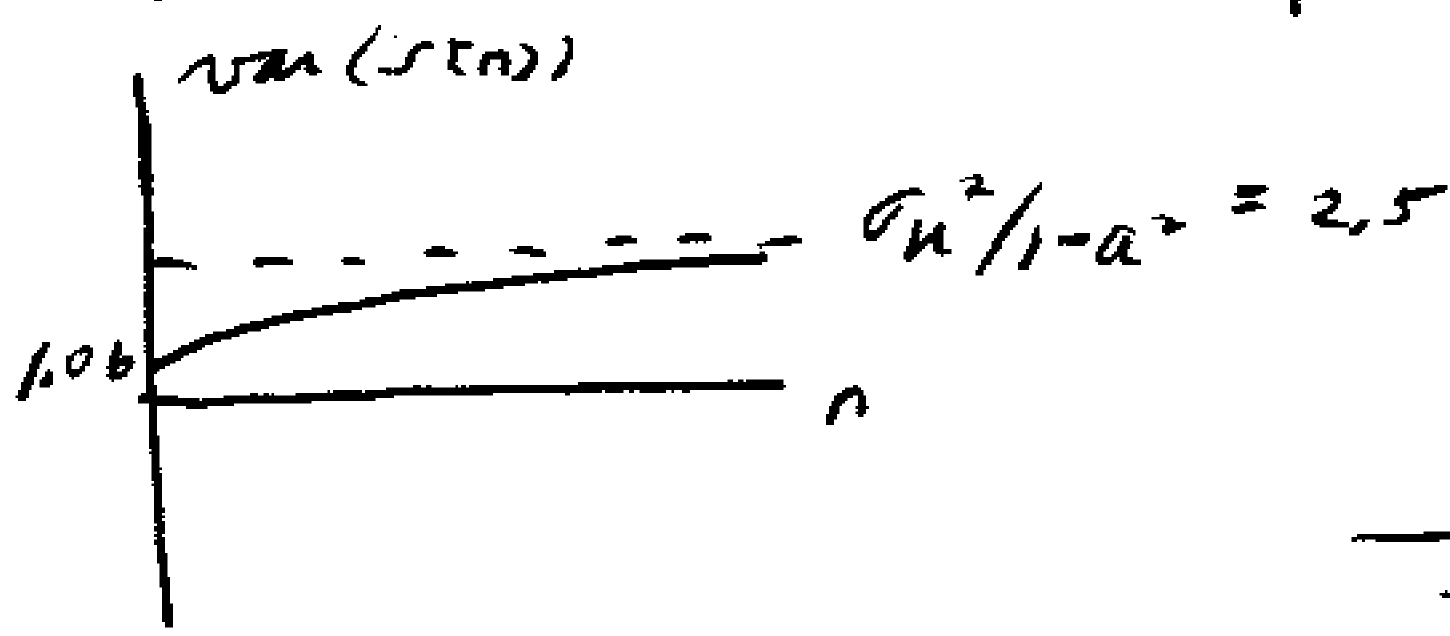
PSD is just that of an AR(1) process or

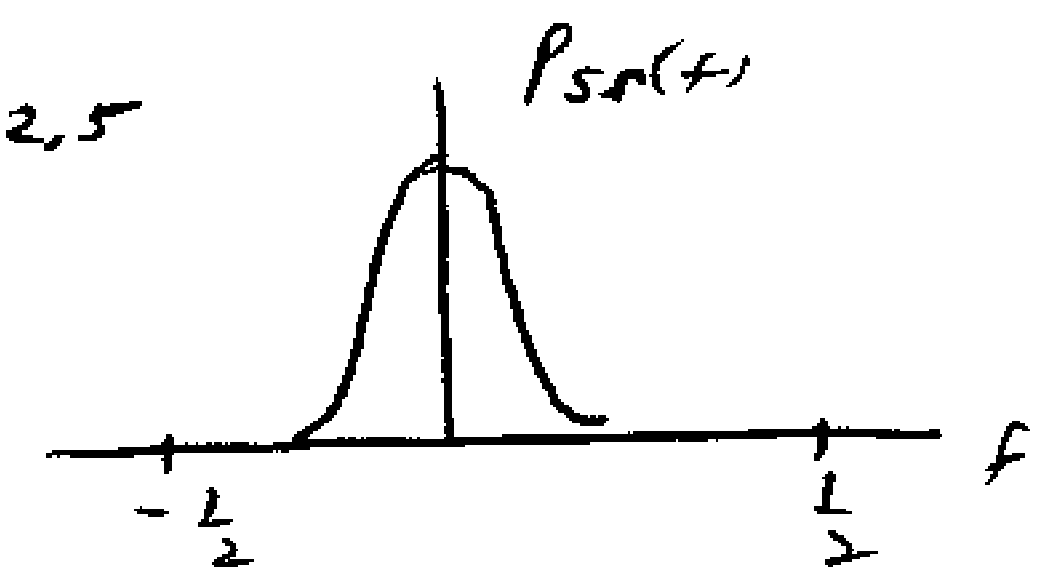
$$P_{ss}(f) = \frac{\sigma_u^2}{|1-a e^{-j2\pi f}|^2}$$

$$= \frac{0.1}{|1-0.98 e^{-j2\pi f}|^2}$$









4) From (13.5)

$$c_s[m,n] = a^{m-n}\underbrace{\left[a^{2n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=0}^{n} a^{2k}\right]}_{\mathrm{var}(s[n])}$$

$$C_s[m,n) = a^{m-n} C_s[n,n]$$

5) $\underline{s}[n] = \underline{A}\,\underline{s}[n-1] + \underline{B}\,\underline{u}[n)$

$$E(\underline{s}[n]) = \underline{A}\,E(\underline{s}[n-1]) + \underline{B}\,E(\underline{u}[n])$$
$$= \underline{A}\,E(\underline{s}[n-1])$$

$$\underline{C}[n) = E[(\underline{s}[n) - E(\underline{s}[n)))(\underline{s}[n) - E(\underline{s}[n))^T]$$
$$= E[(\underline{A}\,\underline{s}[n-1) + \underline{B}\,\underline{u}[n) - \underline{A}\,E(\underline{s}[n-1)))(\quad '' \quad)^T]$$

$$= E[(\underline{A}(\underline{s}[n-1] - E(\underline{s}[n-1])) + \underline{B}\,\underline{u}[n])(\quad '' \quad)^T)$$

$$= \underline{A}\,\underline{C}[n-1)\underline{A}^T + \underline{B}\,\underline{Q}\,\underline{B}^T$$ since
$$E(\underline{s}[n-1)\underline{u}[n)^T) = \underline{0}$$
($\underline{s}[n-1)$ depends on $\underline{u}[k)$ for $k \le n-1$).

6) From (3.14) $E(\underline{s}[n)) = \underline{A}^{n+1}\mu_s$
Let $\underline{V}$ be the modal matrix for $\underline{A}$
and $\underline{\Lambda}$ the diagonal matrix of eigenvalues.
Thus, $\underline{V}^T\underline{A}\,\underline{V} = \underline{\Lambda}$ or $\underline{A} = \underline{V}\,\underline{\Lambda}\,\underline{V}^T$

$$\Rightarrow \underline{A}^{n+1} = \underline{V}\,\underline{\Lambda}^{n+1}\underline{V}^T$$
$$E(\underline{s}[n)) = \underline{V}\,\underline{\Lambda}^{n+1}\underline{V}^T\underline{\mu}_s$$
$$= \sum_{i=1}^{p} a_i \lambda_i^{n+1}\underline{v}_i$$

where $a_i = (\underline{V}^T \underline{u}_s)_i$

$\underline{V} = (\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_p)$

If any $|\lambda_i| > 1$ $\quad E(\underline{s}(n)) \to \infty$

If all $|\lambda_i| < 1$ $\quad E(\underline{s}(n)) \to \underline{0}$

7) $\underline{A}^n = (\underline{V}\,\underline{\Lambda}\,\underline{V}^T)^n = \underline{V}\,\underline{\Lambda}^n\,\underline{V}^T$

$\underline{A}^n \underline{C}_s \underline{A}^{nT} = \underline{V}\,\underline{\Lambda}^n \underline{V}^T \underline{C}_s \underline{V}\,\underline{\Lambda}^n \underline{V}^T$

$$\underline{e}_i^T \underline{A}^n \underline{C}_s \underline{A}^{nT} \underline{e}_j = \underbrace{\underline{e}_i^T \underline{V}\,\underline{\Lambda}^n}_{\underline{b}^T} (\underline{V}^T \underline{C}_s \underline{V}) \underbrace{\underline{\Lambda}^n \underline{V}^T \underline{e}_j}_{\underline{a}}$$

But $\underline{b} = \underline{\Lambda}^n \underline{V}\,\underline{e}_i \to \underline{0}$ if $|\lambda_i| < 1$

$\underline{a} = \underline{\Lambda}^n \underline{V}\,\underline{e}_j \to \underline{0}$ if $|\lambda_i| < 1$

$$\Rightarrow \quad \underline{b}^T \underline{V}^T \underline{C}_s \underline{V}\,\underline{a} = [\underline{A}^n \underline{C}_s \underline{A}^{nT}]_{ij} \to 0$$

for all i, j

8)

$$\underbrace{\begin{bmatrix} r(n-p+1) \\ r(n-p+2) \\ \vdots \\ r(n) \end{bmatrix}}_{\underline{s}(n)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & & & \\ -a(p) & -a(p-1) & \ldots & & -a(1) \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} r(n-p) \\ r(n-p+1) \\ \vdots \\ r(n-1) \end{bmatrix}}_{\underline{s}(n-1)}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ & & & \\ b(q) & b(q-1) & \ldots & 1 \end{bmatrix}}_{\underline{B}} \underbrace{\begin{bmatrix} u(n-q) \\ u(n-q+1) \\ \vdots \\ u(n) \end{bmatrix}}_{\underline{u}(n)}$$

Not a Gauss-Markov process since $\underline{u}[n]$ is not vector WGN. This is because $\underline{u}[n]$ is correlated in time. If $q=1$ for example

$$E(\underline{u}[n]\,\underline{u}[n+1]^T) = E\begin{bmatrix} u[n-1] \\ u[n] \end{bmatrix}[u[n]\; u[n+1]]$$

$$= \begin{bmatrix} E[u[n-1]u[n]] & E[u[n-1]u[n+1]] \\ E[u^2[n]] & E[u[n]\,u[n+1]] \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \sigma_u^2 & 0 \end{bmatrix} \neq \underline{0}$$

9) $$r[n] = s[n] + \sum_{l=1}^{q} b[l]s[n-l]$$

$$\sum_{k=0}^{p} a[k]r[n-k] = \sum_{k=0}^{p} a[k]s[n-k] + \sum_{l=1}^{q} b[l] \sum_{k=0}^{p} a[k]s[n-k+l]$$

$$= u[n] + \sum_{l=1}^{q} b[l]\,u[n-l]$$

Now let the state be given by (13.10)

$\Rightarrow \quad \underline{s}[n] = \underline{A}\,\underline{s}[n-1] + \underline{B}\,u[n]$ (see development leading to (13.11))

Since $\underline{s}[n] = \begin{bmatrix} s[n-p+1] \\ s[n-p+2] \\ \vdots \\ s[n] \end{bmatrix}$, if $q \le p-1$

$$r[n] = \underbrace{(0 \ldots 0\; b[q] \ldots b[1]\; 1)}_{h^T[n] = h^T}\,\underline{s}[n]$$

$\Rightarrow x(n) = \underline{h}^T \underline{s}[n] + \underline{w}[n]$

Now use (13.51) - (13.56) with

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \\ -a(p) & -a(p-1) & \cdots & & -a[1] \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{Q} = \sigma_u^2$$

10) From (13.38) - (13.42) with $a = 1$, $\sigma_u^2 = 0$ so that $s[n] = s[n-1] = A$. We can skip the prediction stage since

$$\hat{A}[n|n-1] = \hat{A}[n-1|n-1]$$
$$M[n|n-1] = M[n-1|n-1]$$

$$\Rightarrow K[n] = \frac{M[n-1|n-1]}{\sigma^2 + M[n-1|n-1]}$$

$$\hat{A}[n|n] = \hat{A}[n-1|n-1] + K[n](x[n] - \hat{A}[n-1|n-1])$$
$$M[n|n] = (1-K[n])M[n-1|n-1]$$

or changing the notation we have

$$K[n] = \frac{M[n-1]}{\sigma^2 + M[n-1]}$$

$$\hat{A}[n] = \hat{A}[n-1] + K[n](x[n] - \hat{A}[n-1])$$
$$M[n] = (1-K[n])M[n-1]$$

These equations are just (12.34) - (12.36) with obvious changes in notation.

Hence, from Section 12.6

$$\hat{A}[n] = \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2/n+1} \frac{1}{n+1} \sum_{k=0}^{n} x[k]$$

$$M[n-1] = \frac{\sigma_A^2 \sigma^2}{n\sigma_A^2 + \sigma^2}$$

$$K[n] = \frac{\sigma_A^2}{(n+1)\sigma_A^2 + \sigma^2}$$

11) From (13.39), (13.40), (13.42)

$$M[n|n-1] = 0.81\, M[n-1|n-1] + 1$$

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

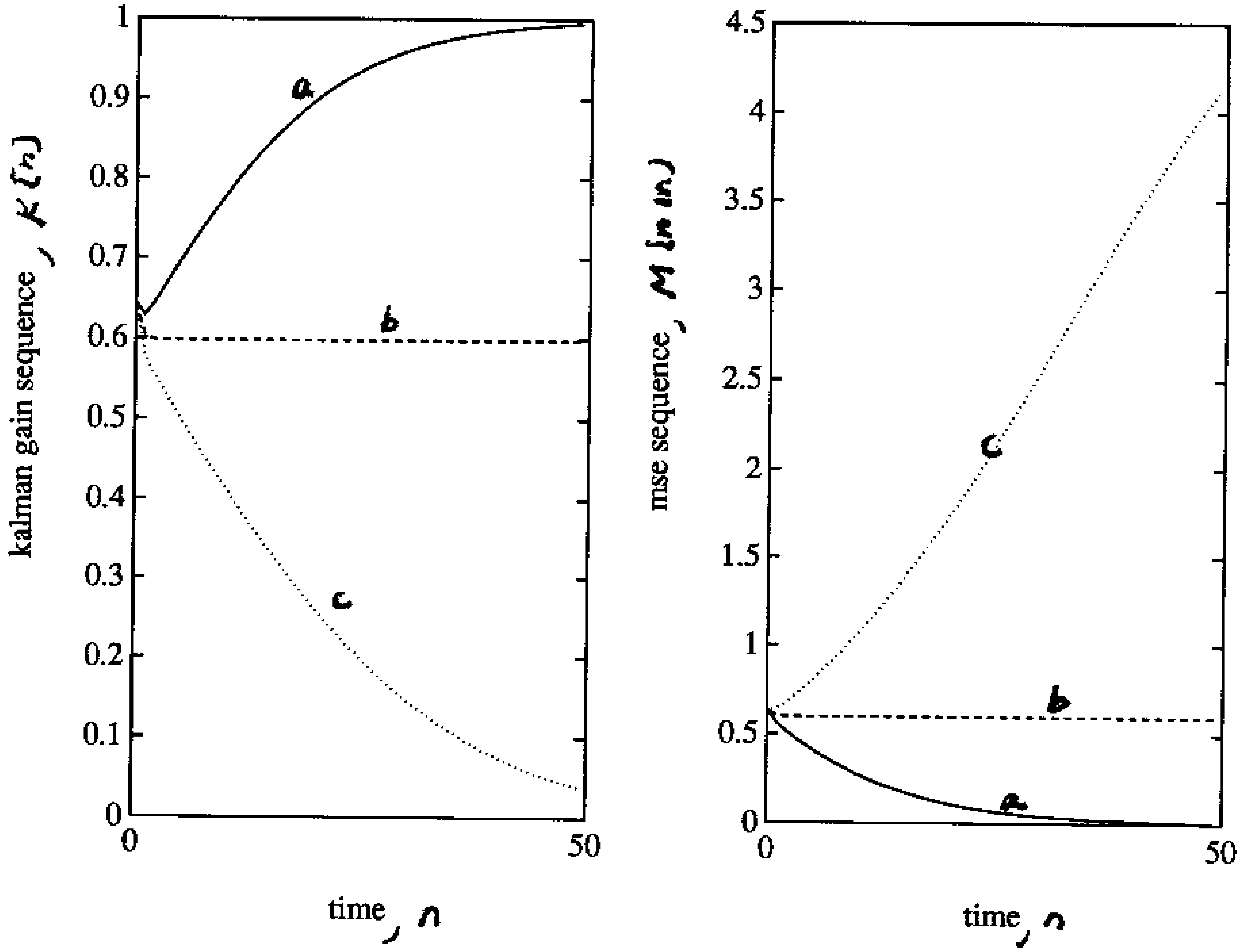
$$M[n|n] = (1 - K[n])\, M[n|n-1]$$

where $M[-1|-1] = \sigma_s^2 = 1$

See next page for plots. For (a) the gain $\to 1$ since the observations become less noisy and thus $\hat{s}[n|n] \to x[n]$. Also, the signal estimate improves with time since $M[n|n] \to 0$. For (c) the gain $\to 0$ since the observations become noisier with time, and also $\hat{s}[n|n] \to \hat{s}[n|n-1]$.

$$= 0.9\,\hat{s}[n-1|n-1]$$

The signal estimate will decay to zero so that the MMSE will be $E[s^2[n]] = \sigma_u^2/1-a^2$ $= 5.26$, which is the steady-state variance of $s[n]$. For (b) the gain and MMSE sequences attain a steady state after a few iterations. This is the steady-state Kalman filter or Wiener filter (see Section 13.5).



A Monte Carlo simulation produces the plots on the following page.

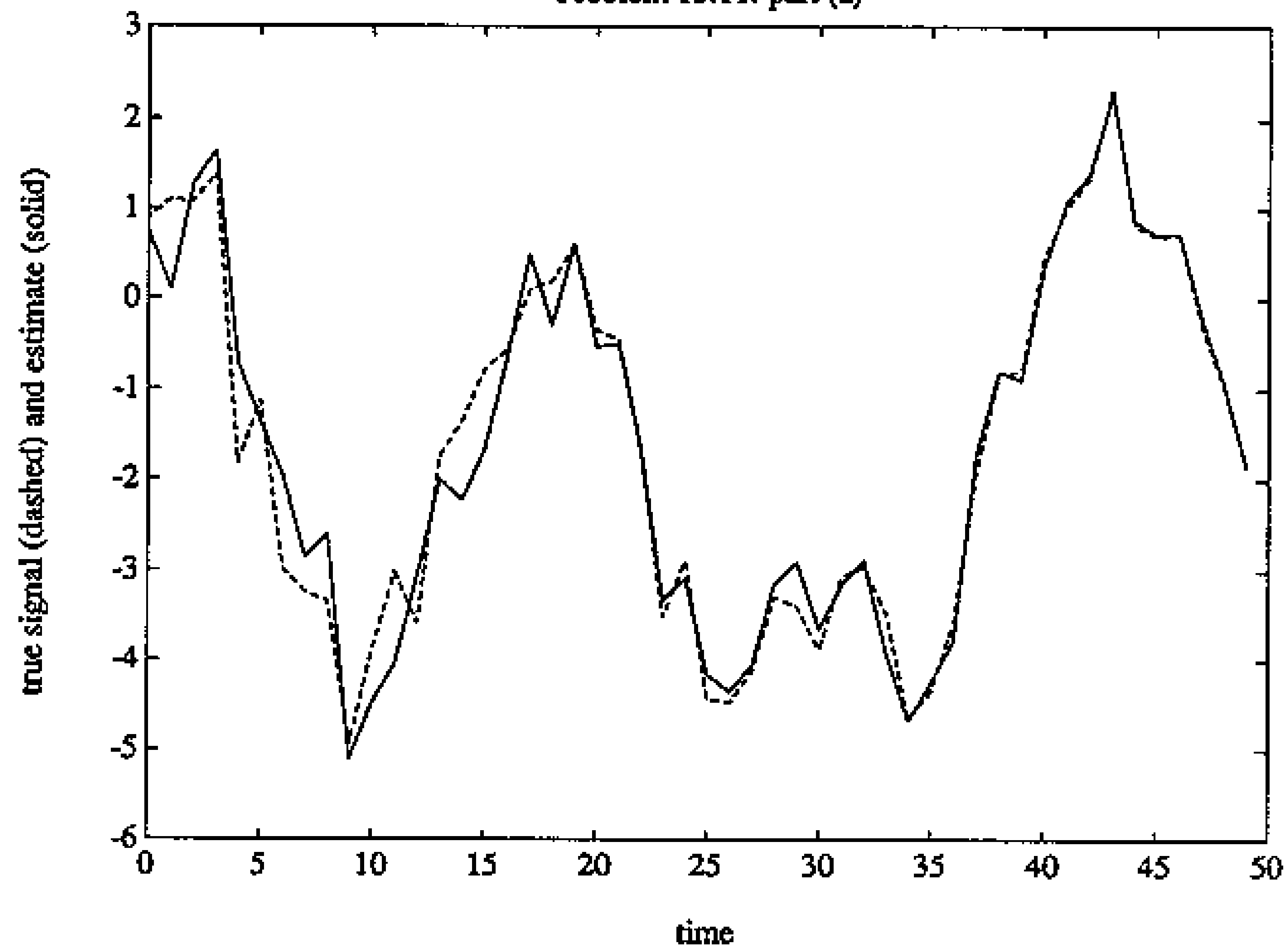
12) If $\sigma_n^2 = 0$, $K[n] = 1$ and $\hat{s}[n|n] = x[n]$

Also, $\hat{s}[n|n-1] = a\,\hat{s}[n-1|n-1]$

$= a\,x[n-1] = a\,s[n-1]$

Thus, $\tilde{x}[n] = x[n] - \hat{s}[n|n-1]$

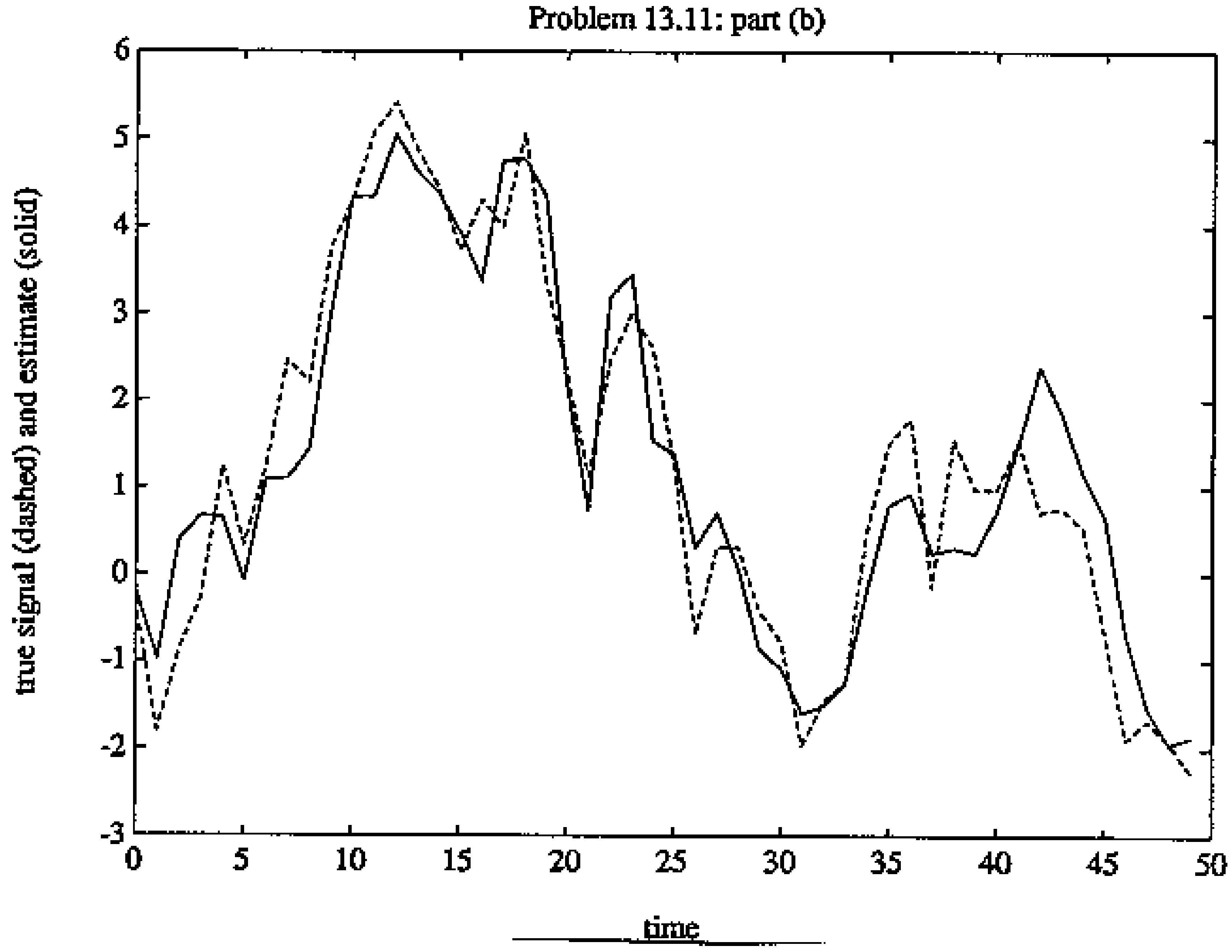
Problem 13.11: part (a)



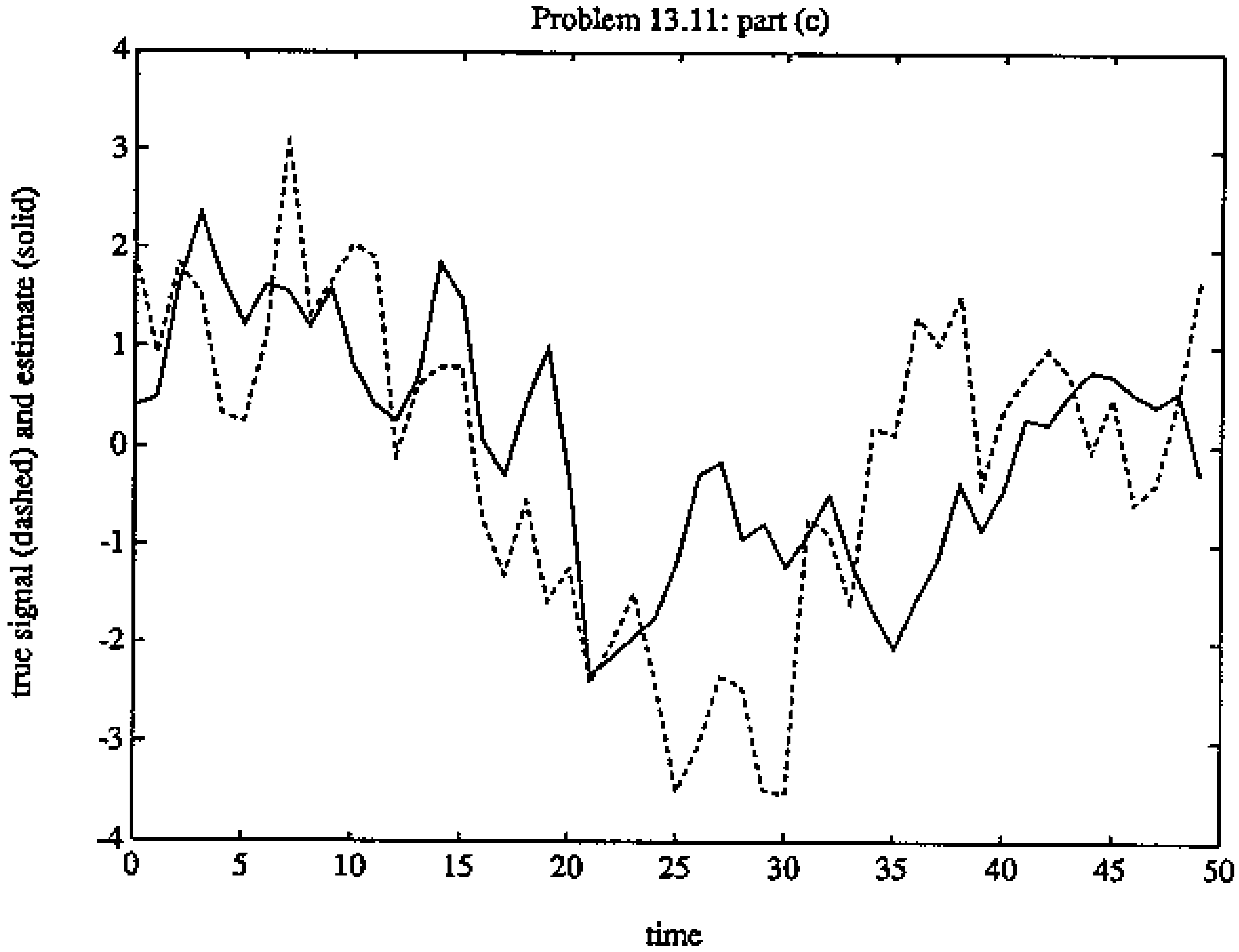
$s[n]$ = DASHED

$\hat{s}[n|n]$ = SOLID

Problem 13.11: part (b)



Problem 13.11: part (c)



$$= s[n] - a s[n-1] = u[n]$$

Yes.

13) Since $E(s[n])$ is known, the MMSE estimator of $s'[n] = s[n] - E(s[n])$ is $\hat{s}'[n] = \hat{s}[n] - E(s[n])$ and the minimum MSE is the same as for $\hat{s}[n]$. Thus, $M[n|n]$ and $M[n|n-1]$ do not change. Also, then $K[n]$ does not change.

Prediction: $\hat{s}'[n|n-1] = a\,\hat{s}'[n-1|n-1]$

$$\hat{s}[n|n-1] - E(s[n]) = a\left(\hat{s}[n-1|n-1] - E(s[n-1])\right)$$

or $\hat{s}[n|n-1] = a\,\hat{s}[n-1|n-1]$

since $E(s[n]) = a\,E(s[n-1])$

Correction: $\hat{s}'[n|n] = \hat{s}'[n|n-1]$

$$+ K[n]\left(x'[n] - \hat{s}'[n|n-1]\right)$$

$$\hat{s}[n|n] - E(s[n]) = \hat{s}[n|n-1] - E(s[n])$$

$$+ K[n]\left(x[n] - E(x[n]) - \hat{s}[n|n-1] + E(s[n])\right)$$

But $E(x[n]) = E(s[n]) + E(w[n]) = E(s[n])$

Thus, we have the same equations as before. The only difference arises in the initialization since $\mu_s \neq 0$.

14) In steady state we have $M[n|n] = M[\infty]$, $M[n|n-1] = M_p[\infty]$ and from (13.42)

$$M[\infty] = (1-K[\infty])M_p[\infty]$$
$$< M_p[\infty]$$

since $K[\infty] < 1$. Thus, for large n $M[n|n-1] > M[n-1|n-1]$. This is reasonable since $s[n]$ is harder to estimate than $s[n-1]$ based on $\{x[0], x[1], \ldots, x[n-1]\}$ due to the added variability of the $u[n]$ noise term.

15) From (13.38) $\hat{s}[n+1|n] = a\hat{s}[n|n]$
Now if $\sigma_{n+1}^2 \to \infty$, the future measurements will be useless so that the corrected estimates will be predictions or will be based on only $\{x[0], \ldots, x[n]\}$.

$$\Rightarrow \hat{s}[n+1|n+1] \to \hat{s}[n+1|n] = a\hat{s}[n|n]$$
$$\hat{s}[n+2|n+1] = a\hat{s}[n+1|n+1] = a^2\hat{s}[n|n]$$
etc.

16) From (13.47)
$$H_\infty(z) = \frac{K[\infty]}{1-a(1-K[\infty])z^{-1}}$$

From (13.46)
$$M[\infty] = \frac{0.64M[\infty]+1}{0.64M[\infty]+2}$$

Let $x = M[\infty]$

$$0.64x^2 + 2x = 0.64x + 1$$

$$x^2 + 2.125x - 1.5625 = 0$$

$$\Rightarrow \quad x = 0.5781, \; -2.703$$

$$M[\infty] = 0.5781$$

$$M_p[\infty] = a^2 M[\infty] + \sigma_u^2$$

$$= 0.64 M[\infty] + 1 = 1.37$$

$$K[\infty] = \frac{M_p[\infty]}{1 + M_p[\infty]} = 0.5781$$

$$H_\infty(z) = \frac{0.5781}{1 - 0.3375 z^{-1}}$$

$$\hat{s}[n|n] = 0.3375\,\hat{s}[n-1|n-1] + 0.5781\,x[n]$$

17) See plot on next page.

18) Here $h[n] = r^n$. Using (13.51) – (13.54)

$$x[n] = h[n]A + w[n]$$

$$\underline{Q} = \underline{0}$$

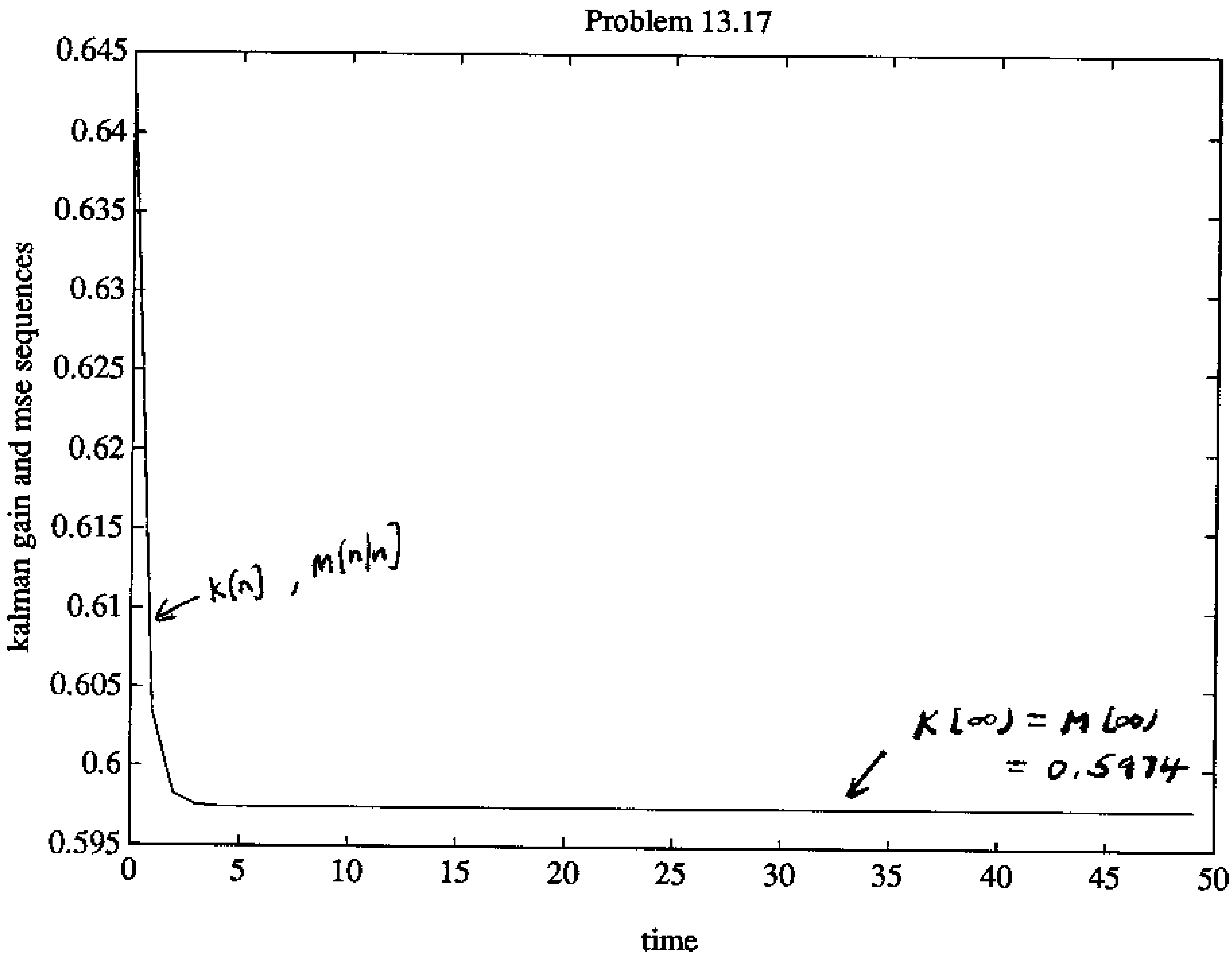
$$\underline{A} = \underline{I}$$

Can omit prediction stage.

$$K[n] = \frac{M[n|n-1]\, r^n}{\sigma^2 + r^{2n} M[n|n-1]}$$

$$M[n|n] = (1 - K[n] r^n) M[n|n-1]$$

$$\hat{A}[n|n] = \hat{A}[n|n-1] + K[n](x[n] - r^n \hat{A}[n|n-1])$$



where $\hat{A}[-1|-1] = \mu_A$, $M[-1|-1] = \sigma_A^2$.

19) If $\mathbf{C}[n] = \mathbf{0}$, from (13.60)

$$\mathbf{K}[n] = \mathbf{M}[n|n-1]\mathbf{H}^T[n]\left(\mathbf{H}[n]\mathbf{M}[n|n-1]\mathbf{H}^T[n]\right)^{-1}$$

$$= \mathbf{M}[n|n-1]\mathbf{H}^T[n]\mathbf{H}^{T^{-1}}[n]\mathbf{M}^{-1}[n|n-1]\mathbf{H}^{-1}[n]$$

$$= \mathbf{H}^{-1}[n]$$

$$\hat{\mathbf{s}}[n|n] = \hat{\mathbf{s}}[n|n-1] + \mathbf{H}^{-1}[n]\left(\mathbf{x}[n] - \mathbf{H}[n]\hat{\mathbf{s}}[n|n-1]\right)$$

$$= \mathbf{H}^{-1}[n]\mathbf{x}[n]$$

$\Rightarrow$ discard all previous data since

$\mathbf{x}[n] = \mathbf{H}[n]\mathbf{s}[n]$ and thus $\mathbf{s}[n] = \mathbf{H}^{-1}[n]\mathbf{x}[n]$

$\Rightarrow$ no error in $\hat{\mathbf{s}}[n|n]$.

If $\underline{C}[n] \to \infty$, $\underline{K}[n] \to \underline{0} \Rightarrow \hat{\underline{s}}[n|n] \to \hat{\underline{s}}[n|n-1]$
or we ignore the data sample $\underline{x}[n]$.

20) Same approach as in Problem 13.15.

21) Note that the state equation is linear but the observation equation is not.
From (13.67) - (13.71)

$$\hat{f}_0[n|n-1] = a\hat{f}_0[n-1|n-1]$$

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2$$

$$K[n] = \frac{M[n|n-1]\,H[n]}{\sigma^2 + H^2[n]\,M[n|n-1]}$$

where $H[n] = \left.\frac{\partial h}{\partial f_0[n]}\right|_{f_0[n]=\hat{f}_0[n|n-1]}$

But $h(f_0[n]) = \cos 2\pi f_0[n]$

$$H[n] = -2\pi \sin 2\pi \hat{f}_0[n|n-1]$$

$$\hat{f}_0[n|n] = \hat{f}_0[n|n-1] + K[n]\left(x[n] - \cos 2\pi \hat{f}_0[n|n-1]\right)$$

$$M[n|n] = \left(1 - K[n]H[n]\right)M[n|n-1]$$

22) $v_x[n] = v_x[n-1] + u_x[n]$

$$\Rightarrow v_x[n] = \sum_{k=0}^{n} u_x[k] + v_x[-1]$$

$$\mathrm{var}(v_x[n]) = \sum_{k=0}^{n} \mathrm{var}(u_x[k]) + \mathrm{var}(v_x[-1])$$

since $u_x[n]$ is WGN and is independent of $v_x[-1]$

$$\mathrm{var}(v_x[n]) = (n+1)\,\mathrm{var}(u_x[k]) + \sigma_v^2$$
$$= (n+1)\sigma_u^2 + \sigma_v^2$$

A better model would be

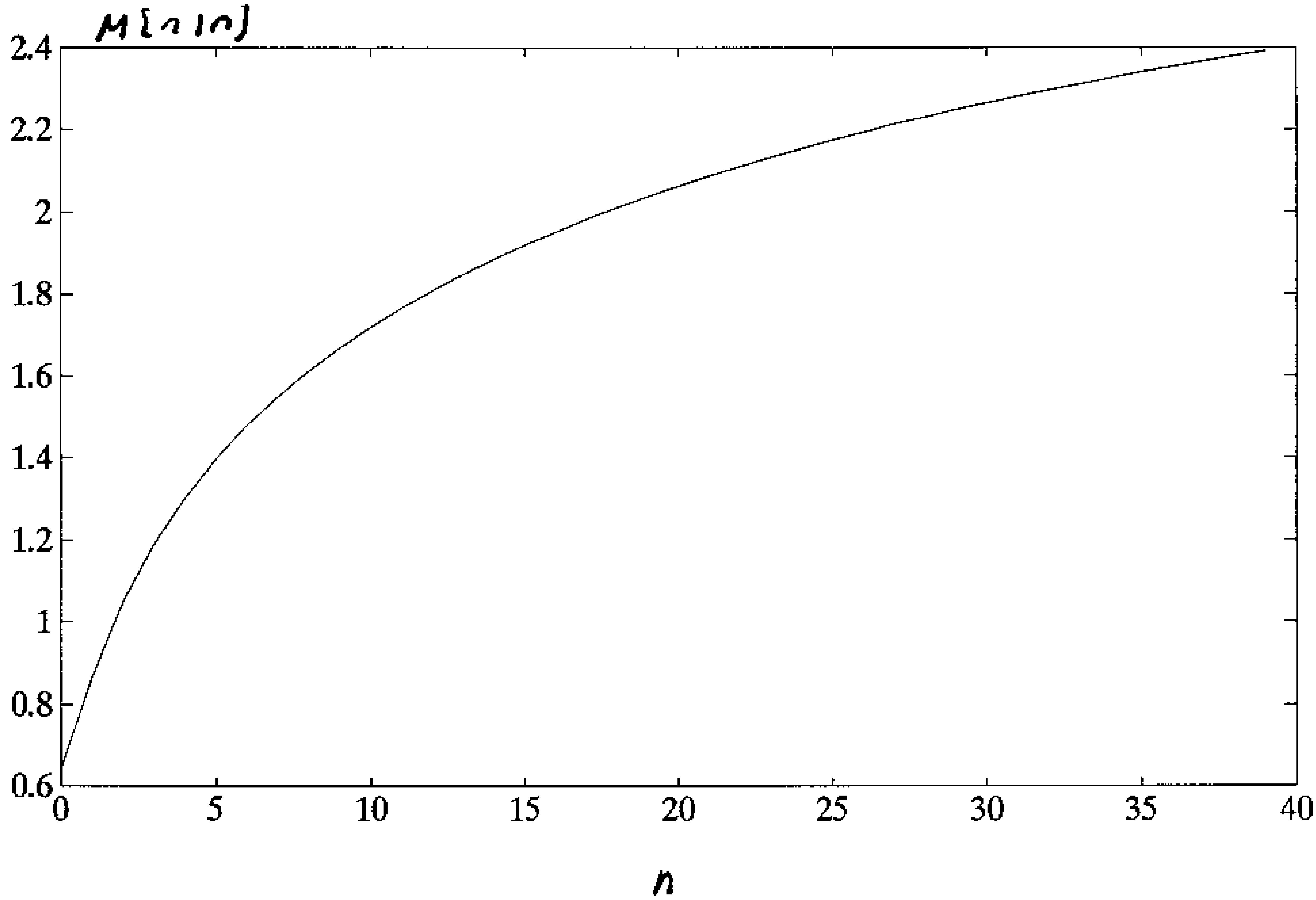
$$v_x[n] = a\,v_x[n-1] + u_x[n]$$

for $0<a<1$ and a should be near one. Then, in steady-state the variance would be a constant.

23)
$$M[n|n] = (1-K[n])\,M[n|n-1]$$
$$= \left(1 - \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}\right) M[n|n-1]$$
$$= \frac{\sigma_n^2 M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$
$$= \frac{\sigma_n^2 (a^2 M[n-1|n-1] + \sigma_u^2)}{\sigma_n^2 + a^2 M[n-1|n-1] + \sigma_u^2}$$
$$= \frac{(n+1)(0.81\,M[n-1|n-1]+1)}{n+1+0.81\,M[n-1|n-1]+1}$$

where $M[-1|-1] = 1$

See plot on next page. Since the observations are progressively noisier, the MMSE increases.


M[η|n]
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30
35
40
n

Chapter 15

1) $\text{Cov}(\tilde{x}_1, \tilde{x}_2) = E\left[(\tilde{x}_1 - E(\tilde{x}_1))^* (\tilde{x}_2 - E(\tilde{x}_2))\right]$

The expectation is with respect to $p(u_1, v_1, u_2, v_2)$

But $p(u_1, v_1, u_2, v_2) = p(u_1 v_1)\, p(u_2 v_2)$

Thus,

$$\text{Cov}(\tilde{x}_1, \tilde{x}_2) = E_{u_1,v_1}(\tilde{x}_1 - E(\tilde{x}_1)) E_{u_2,v_2}(\tilde{x}_2 - E(\tilde{x}_2))$$
$$= 0$$

2) Consider $\tilde{y} = \underline{a}^H \tilde{\underline{x}}$

$E(|\tilde{y}|^2) = E(\underline{a}^H \tilde{\underline{x}} \tilde{\underline{x}}^H \underline{a}) = \underline{a}^H \underline{C}_{\tilde{x}} \underline{a} \geq 0$ for all $\underline{a}$. It will be positive definite if and only if $E(|\tilde{y}|^2) > 0$ or $\tilde{y} \neq 0$ for any $\underline{a}$. Thus, if any random variable (element of $\tilde{\underline{x}}$) can be expressed as a linear combination of the others, $\underline{C}_{\tilde{x}}$ will only be positive semidefinite.

3) $$\underline{C}_x = \frac{1}{2}\begin{bmatrix} \underline{A} & -\underline{B} \\ \underline{B} & \underline{A} \end{bmatrix} \quad \text{where } \underline{A} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$
$$\underline{B} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Note that $\underline{A}^T = \underline{A}$, $\underline{B}^T = -\underline{B}$ as required.

$$\Rightarrow \underline{C}_{\tilde{x}} = \underline{A} + j\underline{B} = \begin{bmatrix} 4 & 2+2j \\ 2-2j & 4 \end{bmatrix}$$

The complex Gaussian vector $\tilde{\underline{x}} = \begin{bmatrix} u_1 + jv_1 \\ u_2 + jv_2 \end{bmatrix}$ will have the covariance matrix $\underline{C}_{\tilde{x}}$.

4) a) $$\underline{C}_x^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

$$\underline{C}_{\tilde{x}}^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1-j \\ -1+j & 2 \end{bmatrix}$$

Let $\underline{x} = [\underline{u}^T \underline{v}^T]^T$ when $\underline{u} = [u_1 u_2]$, $\underline{v} = [v_1 v_2]^T$

$$2\underline{x}^T\underline{C}_x^{-1}\underline{x} = [\underline{u}^T \underline{v}^T]\left[\begin{array}{cc|cc} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ \hline 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{array}\right]\begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix}$$

$$= [\underline{u}^T \underline{v}^T]\begin{bmatrix} \underline{C} & \underline{D}^T \\ \underline{D} & \underline{C} \end{bmatrix}\begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix}$$

$$= \underline{u}^T\underline{C}\underline{u} + \underline{v}^T\underline{C}\underline{v} + \underline{u}^T\underline{D}^T\underline{v} + \underline{v}^T\underline{D}\underline{u}$$

$$= \underline{u}^T\underline{C}\underline{u} + \underline{v}^T\underline{C}\underline{v} + \underline{v}^T\underline{D}\underline{u} + \underline{v}^T\underline{D}\underline{u}$$

$$= \underline{u}^T\underline{C}\underline{u} + \underline{v}^T\underline{D}\underline{v} + 2\underline{v}^T\underline{D}\underline{u}$$

$$\tilde{\underline{x}}^H\underline{C}_{\tilde{x}}^{-1}\tilde{\underline{x}} = (\underline{u}-j\underline{v})^T\frac{1}{4}(\underline{C}+j\underline{D})(\underline{u}+j\underline{v})$$

$$= \frac{1}{4}(\underline{u}-j\underline{v})^T(\underline{C}\underline{u} + j\underline{C}\underline{v} + j\underline{D}\underline{u} - \underline{D}\underline{v})$$

$$= \frac{1}{4}(\underline{u}^T\underline{C}\underline{u} + j\underline{u}^T\underline{C}\underline{v} + j\underline{u}^T\underline{D}\underline{u} - \underline{u}^T\underline{D}\underline{v}$$

$$- j\underline{v}^T\underline{C}\underline{u} + \underline{v}^T\underline{C}\underline{v} + \underline{v}^T\underline{D}\underline{u} + j\underline{v}^T\underline{D}\underline{v})$$

But $\underline{u}^T\underline{D}\underline{u} = (\underline{u}^T\underline{D}\underline{u})^T = \underline{u}^T\underline{D}^T\underline{u} = -\underline{u}^T\underline{D}\underline{u} = 0$

since $\underline{D}^T = -\underline{D}$ and similarly for $\underline{v}^T\underline{D}\underline{v} = 0$

Also, $\underline{v}^T\underline{C}\underline{u} = \underline{u}^T\underline{C}\underline{v}$ since $\underline{C}^T = \underline{C}$

$$\tilde{\underline{x}}^H \underline{C}_{\tilde{x}}^{-1} \tilde{\underline{x}} = 1/4(\underline{u}^T \underline{C}\underline{u} - \underline{u}^T \underline{D}\underline{v} + \underline{v}^T \underline{C}\underline{v} + \underline{v}^T \underline{D}\underline{u})$$

$$= 1/4(\underline{u}^T \underline{C}\underline{u} - \underline{v}^T \underline{D}^T \underline{u} + \underline{v}^T \underline{C}\underline{v} + \underline{v}^T \underline{D}\underline{u})$$

$$= 1/4(\underbrace{\underline{u}^T \underline{C}\underline{u} + \underline{v}^T \underline{C}\underline{v} + 2\underline{v}^T \underline{D}\underline{u}}_{2\underline{x}^T \underline{C}_x^{-1} \underline{x}})$$

$$= \frac{1}{2} \underline{x}^T \underline{C}_x^{-1} \underline{x}$$

b) $\det(C_{\underline{x}}) = 4$

$$\det(\underline{C}_{\tilde{x}}) = \det \begin{pmatrix} 4 & 2+2j \\ 2-2j & 4 \end{pmatrix}$$

$$= 16 - |2+2j|^2 = 16 - 8 = 8$$

$$\frac{\det^2(C_{\tilde{x}})}{16} = \frac{64}{16} = 4$$

5) $M_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad M_2 = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

$M_1 \to a + jb \qquad M_2 \to c + jd$

a) $\alpha(a+jb) = \alpha a + j\alpha b \to \begin{bmatrix} \alpha a & -\alpha b \\ \alpha b & \alpha a \end{bmatrix}$

$= \alpha M_1$

b) $(a+jb) + (c+jd) = (a+c) + j(b+d)$

$$\to \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} = M_1 + M_2$$

c) $(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$

$$\to \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} = M_1 M_2$$

6) Transform 2×2 blocks or let

$$\underline{A} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ a_3 + jb_3 \end{bmatrix}$$

But $\underline{A}^T\underline{A} = m_1^T m_1 + m_2^T m_2 + m_3^T m_3$

(Note that if $m \rightarrow a + jb$, $m^T \rightarrow a - jb$)

$$\Rightarrow \quad \underline{A}^T\underline{A} \rightarrow (a_1 - jb_1)(a_1 + jb_1) + (a_2 - jb_2)(a_2 + jb_2) + (a_3 - jb_3)(a_3 + jb_3)$$

$$= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 + j0$$

Transforming back we have

$$\underline{A}^T\underline{A} = \begin{bmatrix} a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 & & 0 \\ & \ddots & \\ 0 & & \end{bmatrix}$$

7) Since $E(\tilde{x}^{*3}) = E(\tilde{x}^3)^*$

$$E(\tilde{x}^* \tilde{x}^2) = E(\tilde{x}\tilde{x}^{*2})^*$$

we need only show that $E(\tilde{x}^3) = 0$, $E(\tilde{x}^* \tilde{x}^2) = 0$.

But $\dfrac{\partial^3 \phi_{\tilde{x}}(\tilde{\omega})}{\partial \tilde{\omega}^{*3}} = (j/2)^3 E(\tilde{x}^3)$

(see development of (15B.3) in App 15B)

$$\phi_{\tilde{x}}(\tilde{\omega}) = e^{-1/4\,\sigma^2 \tilde{\omega}\tilde{\omega}^*}$$

$$\frac{\partial \phi}{\partial \tilde{\omega}^*} = e^{-1/4\,\sigma^2 |\tilde{\omega}|^2} \left(-1/4\,\sigma^2 \tilde{\omega}\right)$$

$$\frac{\partial^2 \phi}{\partial \tilde{\omega}^{*2}} = e^{-1/4\,\sigma^2 |\tilde{\omega}|^2} \left(-1/4\,\sigma^2 \tilde{\omega}\right)^2$$

$$\frac{\partial^3 \phi}{\partial \tilde{\omega}^{*3}} = e^{-1/4\,\sigma^2 |\tilde{\omega}|^2} \left(-1/4\,\sigma^2 \tilde{\omega}\right)^3$$

$$\Rightarrow \quad \left.\frac{\partial^3 \phi}{\partial \tilde{\omega}^{*3}}\right|_{\tilde{\omega}=0} = 0 \quad \Rightarrow \quad E(\tilde{x}^3) = 0$$

Similarly,

$$E(\tilde{x}^* \tilde{x}^2) = (j/2)^3 \left.\frac{\partial^3 \phi_{\tilde{x}}(\tilde{\omega})}{\partial \tilde{\omega}\, \partial \tilde{\omega}^{*2}}\right|_{\tilde{\omega}=0}$$

$$\frac{\partial^2 \phi}{\partial \tilde{\omega}^{*2}} = e^{-1/4\,\sigma^2 \tilde{\omega}\tilde{\omega}^*} \left(-1/4\,\sigma^2 \tilde{\omega}\right)^2$$

$$\frac{\partial^3 \phi}{\partial \tilde{\omega}\, \partial \tilde{\omega}^{*2}} = e^{-1/4\,\sigma^2 |\tilde{\omega}|^2} \left(\tfrac{1}{16}\,\sigma^4\, 2\tilde{\omega}\right) + \left(-1/4\,\sigma^2 \tilde{\omega}\right)^2 e^{-1/4\,\sigma^2 |\tilde{\omega}|^2} \left(-1/4\,\sigma^2 \tilde{\omega}^*\right)$$

Which when evaluated at $\tilde{\omega} = 0$ is zero.

8) $E(\tilde{x}^2) = 0$

$$\Rightarrow E\left((u+jv)(u+jv)\right) = 0$$

$$E(u^2) - E(v^2) + j\,E(uv) + j\,E(uv) = 0$$

$$\Rightarrow E(u^2) = E(v^2), \quad E(uv) = 0$$

or $\mathrm{var}(u) = \mathrm{var}(v)$, $\mathrm{cov}(u,v) = 0$

This is the usual complex Gaussian random variable assumption.

9) $E\left[(\tilde{\underline{x}}-\tilde{\underline{\mu}})(\tilde{\underline{x}}-\tilde{\underline{\mu}})^T\right] =$

$$E\left[\left((\underline{u}-\underline{\mu}_u)+j(\underline{v}-\underline{\mu}_v)\right)(\quad)^T\right]$$

$$= E\left[(\underline{u}-\underline{\mu}_u)(\underline{u}-\underline{\mu}_u)^T\right] - E\left[(\underline{v}-\underline{\mu}_v)(\underline{v}-\underline{\mu}_v)^T\right]$$

$$+ j\left(E\left[(\underline{u}-\underline{\mu}_u)(\underline{v}-\underline{\mu}_v)^T\right] + E\left[(\underline{v}-\underline{\mu}_v)(\underline{u}-\underline{\mu}_u)^T\right]\right)$$

$$= \underline{0}$$

$$\Rightarrow \quad \underline{C}_{uu} = \underline{C}_{vv}, \quad \underline{C}_{uv} = -\underline{C}_{vu}$$

$$= \underline{A}/2 \qquad\qquad = -\underline{B}/2$$

or $\underline{C}_x = E\left[\left(\begin{bmatrix}\underline{u}\\ \underline{v}\end{bmatrix} - E\begin{bmatrix}\underline{u}\\ \underline{v}\end{bmatrix}\right)\left(\begin{bmatrix}\underline{u}\\ \underline{v}\end{bmatrix} - E\begin{bmatrix}\underline{u}\\ \underline{v}\end{bmatrix}\right)^T\right]$

$$= \begin{bmatrix} \underline{A}/2 & -\underline{B}/2 \\ \underline{B}/2 & \underline{A}/2 \end{bmatrix}$$

10) $E(\hat{\sigma}^2) = E(\tilde{\underline{x}}^H \underline{A}\tilde{\underline{x}}) = \mathrm{tr}(\underline{A}\,\sigma^2\underline{B})$ (see (15.29))

$$= \sigma^2\,\mathrm{tr}(\underline{A}\underline{B})$$

$$\Rightarrow \mathrm{tr}(\underline{A}\underline{B}) = 1$$

$\mathrm{var}(\hat{\sigma}^2) = \mathrm{tr}(\underline{A}\,\sigma^2\underline{B}\,\underline{A}\,\sigma^2\underline{B})$ (see (15.30))

$$= \sigma^4\,\mathrm{tr}((\underline{A}\underline{B})^2)$$

But $\mathrm{tr}(\underline{A}\underline{B}) = \sum_{i=1}^{N} \lambda_i = 1$

$$\mathrm{tr}((\underline{A}\underline{B})^2) = \sum_{i=1}^{N} \lambda_i^2$$

Using the constraint we have to minimize

$$tr((\underline{A}\underline{B})^2) = \sum_{i=2}^{N} \lambda_i^2 + \left(1 - \sum_{i=2}^{N} \lambda_i\right)^2$$

$$\frac{\partial\, tr((\underline{A}\underline{B})^2}{\partial \lambda_k} = 2\lambda_k + 2\left(1 - \sum_2^N \lambda_i\right)(-1) = 0$$

$$k = 2, 3, \ldots, N$$

$$\Rightarrow \quad \lambda_k = 1 - \sum_2^N \lambda_i$$

Since $\sum_2^N \lambda_i$ is a constant, λ_k must be the same for all k. Thus,

$$\lambda_k = 1/N \text{ for } tr(\underline{A}\underline{B}) = 1.$$

Since

$$\underline{V}^T \underline{A}\,\underline{B}\,\underline{V} = \underline{\Lambda} \qquad \underline{V} = \text{modal matrix}$$

$$= 1/N\ \underline{I} \qquad \underline{\Lambda} = \text{eigenvalue matrix}$$

$$\Rightarrow \quad \underline{A}\underline{B} = 1/N\ \underline{V}^{T^{-1}} \underline{V}^{-1} = 1/N\ \underline{I}$$

$$\Rightarrow \quad \underline{A} = 1/N\ \underline{B}^{-1} \quad \text{or} \quad \hat{\sigma}^2 = 1/N\ \tilde{\underline{x}}^H \underline{B}^{-1} \tilde{\underline{x}}$$

If $\underline{B} = \underline{I}$, $\hat{\sigma}^2 = 1/N\ \tilde{\underline{x}}^H \tilde{\underline{x}}$.

11) $$\sum_{n=0}^{N-1} |\tilde{x}[n]|^2 = \tilde{\underline{x}}^H \tilde{\underline{x}}$$

$$E(\tilde{\underline{x}}^H \tilde{\underline{x}}) = N\sigma^2$$

$$\text{var}(\tilde{\underline{x}}^H \tilde{\underline{x}}) = tr(\underline{C}_{\tilde{x}} \underline{C}_{\tilde{x}}) \qquad \text{(see (15.30))}$$

$$= tr(\sigma^4 \underline{I})$$

$$= N\sigma^4$$

$$\Rightarrow \hat{\sigma}^2 = 1/N\ \tilde{\underline{x}}^H \tilde{\underline{x}}$$

where $\text{var}(\hat{\sigma}^2) = \sigma^4/N$

For real WGN, $\hat{\sigma}^2 = 1/N\ \underline{x}^T\underline{x}$ and $\text{var}(\hat{\sigma}^2) = 2\sigma^4/N$. Now, since $\tilde{x}(n) \sim CN(0,\sigma^2)$, $\tilde{x}[n] = u[n] + j v[n]$, where $u[n] \sim N(0, \sigma^2/2)$, $v[n] \sim N(0, \sigma^2/2)$. Hence, for the complex case $\hat{\sigma}^2 = 1/N \sum_{n=0}^{N-1} (u^2[n] + v^2[n])$ and we are averaging $2N$ independent samples as opposed to only N for the real case. Thus, $\text{var}(\hat{\sigma}^2)|_{\text{COMPLEX}} = 1/2\ \text{var}(\hat{\sigma}^2)|_{\text{REAL}}$.

(2) $$v[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

$$E(v[n]) = \sum_k h[k]\, E(u[n-k]) = 0$$

$$E(v[n]v[n+m]) = E\Big[\sum_k h[k]\, u[n-k] \cdot \sum_l h[l]\, u[n+m-l]\Big]$$

$$= \sum_k \sum_l h[k] h[l] \underbrace{E(u[n-k]\, u[n+m-l])}_{r_{uu}[m+k-l]}$$

doesn't depend on $n \Rightarrow$ WSS

Also, since $u[n]$ is Gaussian and $v[n]$ is the result of a linear transformation, $v[n]$ is Gaussian (u, v are jointly Gaussian)

To verify the ACF relationships

$$P_{vv}(f) = |H(f)|^2 P_{uu}(f)$$
$$= 1 \cdot P_{uu}(f)$$
$$\Rightarrow \quad r_{uu}[k] = r_{vv}[k]$$

also, $P_{uv}(f) = H(f) P_{uu}(f)$

$$= \begin{cases} -j P_{uu}(f) & f \geq 0 \\ j P_{uu}(f) & f < 0 \end{cases}$$

But $P_{vu}(f) = P_{uv}^*(f)$,

$$\Rightarrow \quad P_{vu}(f) = \begin{cases} j P_{uu}(f) & f \geq 0 \\ -j P_{uu}(f) & f < 0 \end{cases}$$

$$= -P_{uv}(f)$$

and thus, $r_{uv}[k] = -r_{vu}[k]$. The PSD of $\tilde{x}[n]$ is from (15.3

$$P_{\tilde{x}\tilde{x}}(f) = 2(P_{uu}(f) + j P_{uv}(f))$$
$$= \begin{cases} 2(P_{uu}(f) + j(-j P_{uu}(f))) & f \geq 0 \\ 2(P_{uu}(f) + j(j P_{uu}(f))) & f < 0 \end{cases}$$

$$= \begin{cases} 4 P_{uu}(f) & f \geq 0 \\ 0 & f < 0 \end{cases}$$

B) $$\frac{\partial \theta^*}{\partial \theta} = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - j\frac{\partial}{\partial \beta}\right)(\alpha - j\beta)$$

$$= \frac{1}{2}\left(\frac{\partial \alpha}{\partial \alpha} - j\frac{\partial \beta}{\partial \alpha} - j\frac{\partial \alpha}{\partial \beta} - \frac{\partial \beta}{\partial \beta}\right)$$

$$= \frac{1}{2}\left(1 - 0 - 0 - 1\right) = 0$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \alpha} + j\frac{\partial}{\partial \beta}$$

$$\frac{\partial \theta}{\partial \theta} = \left(\frac{\partial}{\partial \alpha} + j\frac{\partial}{\partial \beta}\right)(\alpha + j\beta)$$

$$= 1 - 1 = 0$$

$$\frac{\partial \theta^*}{\partial \theta} = \left(\frac{\partial}{\partial \alpha} + j\frac{\partial}{\partial \beta}\right)(\alpha - j\beta)$$

$$= 1 + 1 = 2$$

14) $$\frac{\partial \underline{\theta}^H \underline{b}}{\partial \theta_k} = \frac{\partial}{\partial \theta_k} \sum_l b_l \theta_l^*$$

$$= \sum_l b_l \frac{\partial \theta_l^*}{\partial \theta_k} = 0 \quad \text{all } k$$

$$\Rightarrow \frac{\partial \underline{\theta}^H \underline{b}}{\partial \underline{\theta}} = \underline{0}$$

15) Same proof as in deriving (15.46)

$$\frac{\partial \underline{\theta}^H \underline{A} \underline{\theta}}{\partial \underline{\theta}} = \underline{A}^T \underline{\theta}^*$$

Since $\underline{A}^H \neq \underline{A}$, $\frac{\partial \underline{\theta}^H \underline{A} \underline{\theta}}{\partial \underline{\theta}} = (\underline{A}^H \underline{\theta})^*$

16) To find LSE

$$J = \sum_{n=0}^{N-1} |\tilde{x}[n] - \tilde{A}\, r^n|^2$$

From Example 15.2

$$\hat{\tilde{A}} = \frac{\sum_{n=0}^{N-1} \tilde{x}[n] \tilde{s}^*[n]}{\sum_{n=0}^{N-1} |\tilde{s}[n]|^2}$$

$$= \frac{\sum_{n=0}^{N-1} \tilde{x}(n) \gamma^{*n}}{\sum_{n=0}^{N-1} |\gamma|^{2n}}$$

MLE will be the same.

$$E(\hat{\tilde{A}}) = \frac{\sum_n E(\tilde{x}[n]) \gamma^{*n}}{\sum_n |\gamma|^{2n}}$$

$$= \frac{\tilde{A} \sum_n \gamma^n \gamma^{*n}}{\sum_n |\gamma|^{2n}} = \tilde{A}$$

$$\mathrm{var}(\hat{\tilde{A}}) = \frac{\sum_n \mathrm{var}(\tilde{x}(n)) |\gamma|^{2n}}{(\sum_n |\gamma|^{2n})^2}$$

$$= \frac{\sigma^2}{\sum_n |\gamma|^{2n}}$$

If $|\gamma| < 1$, $\mathrm{var}(\hat{\tilde{A}}) \to \frac{\sigma^2}{(1-|\gamma|^2)^{-1}}$

$$= \sigma^2 (1 - |\gamma|^2)$$

If $|\gamma| \geq 1$, $\mathrm{var}(\hat{\tilde{A}}) \to 0$

17) Equivalently, we have

$$\tilde{x}[n] = A + \tilde{w}[n] \quad \text{or}$$

$$u[n] = A + w_R[n]$$

$$v[n] = w_I[n]$$

Now let $\underline{x} = \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} \underline{w}_R \\ \underline{w}_I \end{bmatrix}$ so that

$$\underline{x} = A \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix} + \underline{w}$$

where $E(\underline{w}) = \underline{0}$, $\underline{C}_w = \sigma^2/2\, \underline{I} = \underline{C}_x$

The real BLUE can now be used.

$$\hat{A} = \frac{[\underline{1}^T\ \underline{0}^T]\ \underline{C}_w^{-1}\ \underline{x}}{[\underline{1}^T\ \underline{0}^T]\ \underline{C}_w^{-1} \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} u[n] = \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{Re}(\tilde{x}[n])$$

In Example 15.7 $\tilde{A}$ is complex and if $\underline{C} = \sigma^2 \underline{I}$, which conforms to the assumptions for this problem,

$$\hat{\tilde{A}} = \frac{\underline{1}^T \tilde{\underline{x}}}{\underline{1}^T \underline{1}} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n]$$

18) From (15.52)

$$I(\theta) = 2\,\mathrm{Re}\left[\frac{\partial \tilde{\underline{u}}^H(\theta)}{\partial \theta}\ \underline{C}_{\tilde{x}}^{-1}\ \frac{\partial \tilde{\underline{u}}(\theta)}{\partial \theta} \right]$$

$$= \frac{2}{\sigma^2} \mathrm{Re}\left[\sum_{n=0}^{N-1} \left| \frac{\partial [\tilde{\underline{u}}(\theta)]_n}{\partial \theta} \right|^2 \right]$$

$$= \frac{2}{\sigma^2} \sum_{n=0}^{N-1} \left| \frac{\partial \tilde{s}[n;\theta]}{\partial \theta} \right|^2$$

$$\operatorname{var}(\hat{\theta}) \geq \frac{\sigma^2/2}{\sum_{n=0}^{N-1} \left(\frac{\partial s_R[n;\theta]}{\partial \theta}\right)^2 + \left(\frac{\partial s_I[n;\theta]}{\partial \theta}\right)^2}$$

For real data (see (3.14))

$$\operatorname{var}(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2}$$

In complex case there is information in both the real and imaginary parts of the signal. Also, the $\sigma^2/2$ factor accounts for the variance of each <u>real</u> noise sample.

19) From (15.52) with $\underline{C}_{\tilde{x}} = \sigma^2 \underline{I}$

$$I(\sigma^2) = \operatorname{tr}\left[\underline{C}_{\tilde{x}}^{-1} \frac{\partial \underline{C}_{\tilde{x}}}{\partial \sigma^2} \underline{C}_{\tilde{x}}^{-1} \frac{\partial \underline{C}_{\tilde{x}}}{\partial \sigma^2} \right]$$

$$= \operatorname{tr}\left[\left(\frac{1}{\sigma^2}\underline{I}\right)(\underline{I})\left(\frac{1}{\sigma^2}\underline{I}\right)(\underline{I}) \right]$$

$$= N/\sigma^4$$

$$\operatorname{var}(\hat{\sigma^2}) \geq \sigma^4/N$$

$$p(\tilde{\underline{x}};\sigma^2) = \frac{1}{\pi^N \det(\sigma^2 \underline{I})} e^{-\frac{1}{\sigma^2}\tilde{\underline{x}}^H \tilde{\underline{x}}}$$

$$\frac{\partial \ln p}{\partial \sigma^2} = -\frac{\partial \ln \det(\sigma^2 \underline{I})}{\partial \sigma^2} - \frac{\partial}{\partial \sigma^2}\left(\frac{1}{\sigma^2} \underline{\tilde{x}}^H \underline{\tilde{x}}\right)$$

$$= -\frac{\partial \ln \sigma^{2N}}{\partial \sigma^2} - \frac{\partial}{\partial \sigma^2}\left(\frac{1}{\sigma^2} \underline{\tilde{x}}^H \underline{\tilde{x}}\right)$$

$$= -N/\sigma^2 + 1/\sigma^4 \, \underline{\tilde{x}}^H \underline{\tilde{x}}$$

$$= N/\sigma^4 \Big(\underbrace{\frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}(n)|^2}_{\text{efficient estimator}} - \sigma^2 \Big)$$

20) $\hat{\theta}_i = \underline{a}_i^H \underline{\tilde{x}} \Rightarrow \underline{\hat{\theta}} = \underline{A}\,\underline{\tilde{x}} \qquad \underline{A} = \begin{bmatrix} \underline{a}_1^H \\ \vdots \\ \underline{a}_p^H \end{bmatrix}$

$E(\underline{\hat{\theta}}) = \underline{\theta} \Rightarrow E(\underline{A}(\underline{H}\underline{\theta} + \underline{\tilde{w}}))$

$= \underline{A}\,\underline{H}\, E(\underline{\theta}) = \underline{\theta}$

or $\underline{A}\,\underline{H} = \underline{I}$

$\text{var}(\hat{\theta}_i) = E\left(|\theta_i - \hat{\theta}_i|^2 \right)$

$= E\left(|\underline{a}_i^H \underline{\tilde{x}} - \underline{a}_i^H E(\underline{\tilde{x}})|^2 \right)$

$= E\left[\underline{a}_i^H (\underline{x} - E(\underline{x}))(\underline{x} - E(\underline{x}))^H \underline{a}_i \right]$

$= \underline{a}_i^H \underline{C}\, \underline{a}_i$

$\underline{A}\,\underline{H} = \underline{I} \Rightarrow \begin{bmatrix} \underline{a}_1^H \\ \vdots \\ \underline{a}_p^H \end{bmatrix} \underline{H} = \underline{I}$

or $\underline{H}^H [\underline{a}_1 \ldots \underline{a}_p] = \underline{I}$

$\underline{H}^H \underline{a}_i = \underline{e}_i$ where $\underline{e}_i = [0\,0 \ldots 0\,1\,0 \ldots 0]^T$
↑ i^{th} place

Using (15.51) we have

$$\underline{a}_{i_{opt}} = \underline{C}^{-1}\underline{H}(\underline{H}^H \underline{C}^{-1}\underline{H})^{-1}\underline{e}_i$$

$$\hat{\theta}_i = \underline{a}_{i_{opt}}{}^H \tilde{\underline{x}} = \underline{e}_i^H (\underline{H}^H \underline{C}^{-1} \underline{H})^{-1} \underline{H}^H \underline{C}^{-1} \tilde{\underline{x}}$$

$$\Rightarrow \quad \hat{\underline{\theta}} = (\underline{H}^H \underline{C}^{-1} \underline{H})^{-1} \underline{H}^H \underline{C}^{-1} \tilde{\underline{x}}$$

21) $\tilde{\underline{x}} = \tilde{A}_1 \underline{e}_1 + \tilde{A}_2 \underline{e}_2 + \tilde{\underline{w}} = \underline{E}\tilde{\underline{A}} + \tilde{\underline{w}}$

where $\underline{E} = (\underline{e}_1 \ \underline{e}_2)$, $\tilde{\underline{A}} = (\tilde{A}_1 \ \tilde{A}_2)^T$

The MLE will minimize

$$J(\tilde{\underline{A}}, f) = (\tilde{\underline{x}} - \underline{E}\tilde{\underline{A}})^H (\tilde{\underline{x}} - \underline{E}\tilde{\underline{A}})$$

$$= \tilde{\underline{x}}^H \tilde{\underline{x}} - \tilde{\underline{x}}^H \underline{E}\tilde{\underline{A}} - \tilde{\underline{A}}^H \underline{E}^H \tilde{\underline{x}} + \tilde{\underline{A}}^H \underline{E}^H \underline{E}\tilde{\underline{A}}$$

Taking the complex gradient we have

$$\frac{\partial J}{\partial \tilde{\underline{A}}} = \underline{0} - \frac{\partial}{\partial \tilde{\underline{A}}} \tilde{\underline{x}}^H \underline{E}\tilde{\underline{A}} - \underline{0} + \frac{\partial}{\partial \tilde{\underline{A}}} \tilde{\underline{A}}^H \underline{E}^H \underline{E}\tilde{\underline{A}}$$

Using (15.44) and (15.46)

$$\frac{\partial J}{\partial \tilde{\underline{A}}} = -(\underline{E}^H \tilde{\underline{x}})^* + (\underline{E}^H \underline{E}\tilde{\underline{A}})^* = \underline{0}$$

$$\Rightarrow \quad \hat{\tilde{\underline{A}}} = (\underline{E}^H \underline{E})^{-1} \underline{E}^H \tilde{\underline{x}}$$

$$J(\hat{\tilde{\underline{A}}}, f) = (\tilde{\underline{x}} - \underline{E}(\underline{E}^H \underline{E})^{-1} \underline{E}^H \tilde{\underline{x}})^H \cdot (\tilde{\underline{x}} - \underline{E}(\underline{E}^H \underline{E})^{-1} \underline{E}^H \tilde{\underline{x}})$$

$$= \tilde{\underline{x}}^H (\underline{I} - \underline{E}(\underline{E}^H \underline{E})^{-1} \underline{E}^H) \tilde{\underline{x}}$$

Since $\underline{I} - \underline{E}(\underline{E}^H \underline{E})^{-1} \underline{E}^H$ is idempotent

To minimize over f we need to maximize

$$J(\underline{f}) = \tilde{\underline{x}}^H \underline{E} (\underline{E}^H \underline{E})^{-1} \underline{E}^H \tilde{\underline{x}}$$

This will require a 2-D search. An approximate MLE for $|f_1 - f_2| >> 1/N$ is found as follows:

$$\underline{E}^H \underline{E} = \begin{bmatrix} \underline{e}_1^H \\ \underline{e}_2^H \end{bmatrix} [\underline{e}_1 \ \underline{e}_2] = \begin{bmatrix} N & \underline{e}_1^H \underline{e}_2 \\ \underline{e}_2^H \underline{e}_1 & N \end{bmatrix}$$

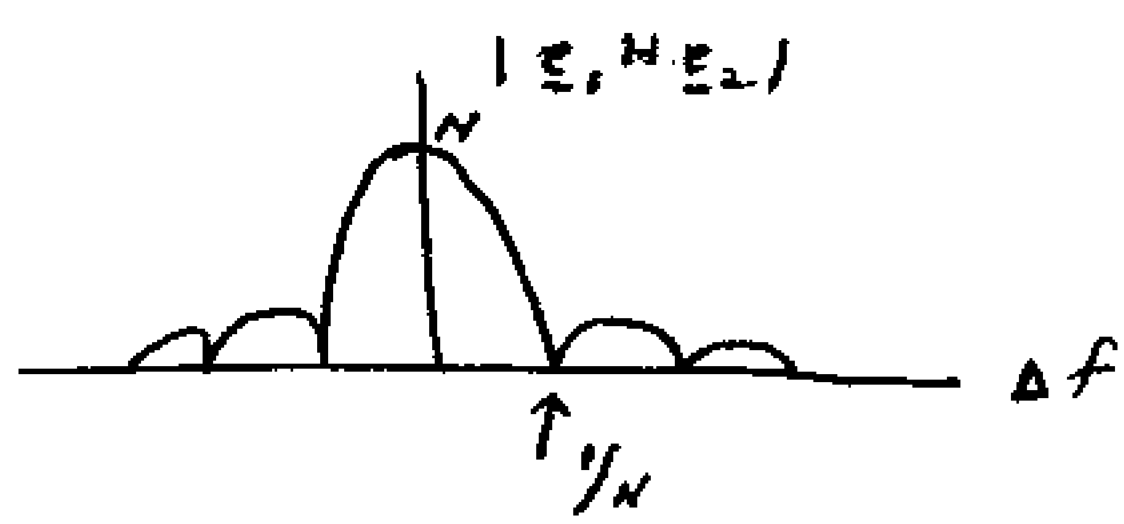
$$\underline{e}_1^H \underline{e}_2 = \sum_{n=0}^{N-1} e^{-j2\pi f_1 n} e^{j2\pi f_2 n}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi \Delta f n} \qquad \Delta f = f_2 - f_1$$

$$= \frac{1 - e^{j2\pi \Delta f N}}{1 - e^{j2\pi \Delta f}}$$

$$= \frac{e^{j\pi \Delta f N}}{e^{j\pi \Delta f}} \frac{e^{-j\pi \Delta f N} - e^{j\pi \Delta f N}}{e^{-j\pi \Delta f} - e^{j\pi \Delta f}}$$

$$= e^{j\pi \Delta f (N-1)} \frac{\sin \pi \Delta f N}{\sin \pi \Delta f}$$



For $|\Delta f| >> 1/N$, $\underline{e}_1^H \underline{e}_2 << N$ and thus

$$\underline{E}^H \underline{E} \simeq N \underline{I}$$

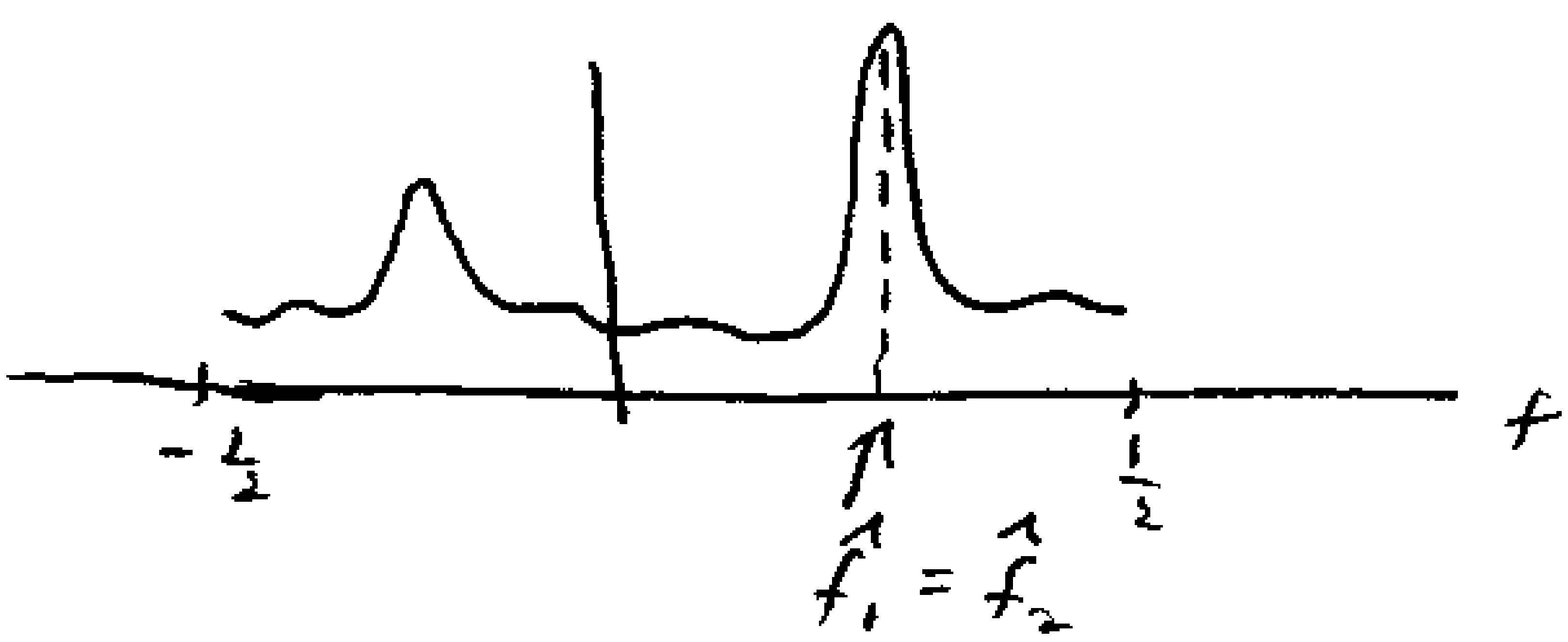
$$J(\underline{f}) \simeq 1/N \ \tilde{\underline{x}}^H \underline{E} \underline{E}^H \tilde{\underline{x}} = \frac{1}{N} \| \underline{E}^H \tilde{\underline{x}} \|^2$$

$$= \frac{1}{N}\left(|\underline{e}_1^H \tilde{\underline{x}}|^2 + |\underline{e}_2^H \tilde{\underline{x}}|^2\right)$$

$$= \frac{1}{N}\left|\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi f_1 n}\right|^2$$

$$+ \frac{1}{N}\left|\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi f_2 n}\right|^2$$

Approximate MLE finds peak locations of the two largest peaks of a single periodogram, with the constraint $|f_1 - f_2| >> 1/N$.
Without constraint we could have



22) An MLE will minimize
$$\sum_{n=0}^{N-1} |\tilde{x}[n] - \tilde{A}\tilde{s}[n]|^2$$

where $\tilde{s}[n] = e^{j2\pi(f_0 n + 1/2 \alpha n^2)}$
This is a linear LS problem with respect to $\tilde{A}$. Thus, from Example 15.2

$$\hat{\tilde{A}} = \frac{\sum_n \tilde{x}[n]\tilde{s}^*[n]}{\sum_n |\tilde{s}[n]|^2}$$

Substituting for $\hat{A}$

$$\sum_{n=0}^{N-1} (\tilde{x}[n] - \hat{\tilde{A}}\tilde{s}[n])^* (\tilde{x}[n] - \hat{\tilde{A}}\tilde{s}[n])$$

$$= \sum_{n=0}^{N-1} \tilde{x}^*[n] (\tilde{x}[n] - \hat{\tilde{A}}\tilde{s}[n])$$

$$- \underbrace{\hat{\tilde{A}} \sum_{n=0}^{N-1} \tilde{s}^*[n] (\tilde{x}[n] - \hat{\tilde{A}}\tilde{s}[n])}_{=0}$$

$$= \sum_n |\tilde{x}[n]|^2 - \hat{\tilde{A}} \sum_n \tilde{x}^*[n]\tilde{s}[n]$$

$$= \sum_n |\tilde{x}[n]|^2 - \frac{\left|\sum_n \tilde{x}[n] s^*[n]\right|^2}{\sum_n |s[n]|^2}$$

$$= \sum_n |\tilde{x}[n]|^2 - \frac{\left|\sum_n \tilde{x}[n] e^{-j2\pi(f_0 n + \frac{1}{2}\alpha n^2)}\right|^2}{N}$$

Hence, we need to maximize over f_0, α

$\left|\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi(f_0 n + 1/2\alpha n^2)}\right|^2$. To

compute this efficiently let $\tilde{y}[n] = \tilde{x}[n] e^{-j\pi\alpha n^2}$

$\Rightarrow \left|\sum_{n=0}^{N-1} \tilde{y}[n] e^{-j2\pi f_0 n}\right|^2$, can use

FFT on $\tilde{y}[n]$ for each α.

23) $\hat{\alpha} = E(\alpha \mid \underline{u}, \underline{v}) = \int \alpha \, p(\alpha \mid \underline{u}, \underline{v}) \, d\alpha$

$\hat{\beta} = E(\beta \mid \underline{u}, \underline{v}) = \int \beta \, p(\beta \mid \underline{u}, \underline{v}) \, d\beta$

or $\hat{\alpha} = \iint \alpha \, p(\alpha, \beta \mid \underline{u}, \underline{v}) \, d\alpha \, d\beta$

$\hat{\beta} = \iint \beta \, p(\alpha, \beta \mid \underline{u}, \underline{v}) \, d\alpha \, d\beta$

$$\begin{aligned}\hat{\theta} = \hat{\alpha} + j\hat{\beta} &= \iint (\alpha + j\beta) \, p(\alpha, \beta \mid \underline{u}, \underline{v}) \, d\alpha \, d\beta \\ &= \iint \theta \, p(\alpha, \beta \mid \underline{u}, \underline{v}) \, d\alpha \, d\beta \\ &= E(\theta \mid \underline{u}, \underline{v}) = E(\underline{\theta} \mid \tilde{\underline{x}})\end{aligned}$$

Note that $p(\theta \mid \tilde{\underline{x}})$ is just $p(\alpha, \beta \mid \underline{u}, \underline{v})$ in disguise.

24) From (15.52) with $\underline{C}_{\tilde{x}} = P_0 \underline{Q}$ where $\underline{Q}$ is the covariance matrix corresponding to a process with PSD $Q(f)$,

$$\begin{aligned}I(P_0) &= \operatorname{tr}\left(\underline{C}_{\tilde{x}}^{-1} \frac{\partial \underline{C}_{\tilde{x}}}{\partial P_0} \underline{C}_{\tilde{x}}^{-1} \frac{\partial \underline{C}_{\tilde{x}}}{\partial P_0} \right) \\ &= \operatorname{tr}\left[\frac{1}{P_0} \underline{Q}^{-1} \underline{Q} \, \frac{1}{P_0} \underline{Q}^{-1} \underline{Q} \right] \\ &= \frac{1}{P_0^2} \operatorname{tr}(\underline{I}) = N / P_0^2\end{aligned}$$

$\Rightarrow \operatorname{var}(\hat{P}_0) \geq P_0^2 / N$

From (15.68)

$$I(P_0) = N \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln P_{\tilde{x}\tilde{x}}(f)}{\partial P_0} \right)^2 df$$

$$= N \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln P_0 Q(f)}{\partial P_0} \right)^2 df$$

$$= N \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{P_0^2} \, df = N / P_0^2$$

same result (<u>not</u> true in general)

25) $$X(f_k) = \sum_n \tilde{x}(n) e^{-j2\pi f_k n} \qquad (f_k = k/N)$$

$$= \sum_n (\tilde{A} e^{j2\pi f_c n} + \tilde{w}(n)) e^{-j2\pi f_k n}$$

$$= \sum_n \tilde{w}(n) e^{-j2\pi f_k n} \qquad f_k \neq f_c$$

$$\sum_n (\tilde{A} + \tilde{w}(n) e^{-j2\pi f_k n}) \qquad f_k = f_c$$

$$= \begin{cases} W(f_k) & k \neq l \\ N\tilde{A} + W(f_k) & k = l \end{cases}$$

where $W(f_k)$ is the DFT of $\tilde{w}(n)$.
But $\underline{W} \sim CN(\underline{0}, N\sigma^2 \underline{I})$ from Sect 15.9

$$\Rightarrow \underline{X} \sim CN\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ N\tilde{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, N\sigma^2 \underline{I} \right)$$

where $\underline{X} = [\, X(f_0) \; X(f_1) \ldots X(f_{N-1})]^T$

Now, $$SNR(\text{input}) = |E(\tilde{x}[n])|^2 / \sigma^2$$
$$= |\tilde{A}|^2 / \sigma^2$$

$$SNR(\text{output}) = \frac{|E(X(f_c))|^2}{\text{var}(X(f_c))}$$

$$= \frac{|N\tilde{A}|^2}{N\sigma^2} = \frac{N|\tilde{A}|^2}{\sigma^2}$$

Processing gain $= 10 \log_{10} N$ dB

To detect a sinusoid of unknown frequency we could form

$$\frac{1}{N}|X(f_k)|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi f_k n}\right|^2$$

and choose the maximum over f_k for comparison to a threshold. If no signal is present,

$$E\left(\frac{1}{N}|X(f_k)|^2\right) = \frac{1}{N}\operatorname{var}(X(f_k))$$
$$= \sigma^2 \quad \text{for all } k$$

and for a signal present

$$E\left[\frac{1}{N}|X(f_k)|^2\right] = \frac{1}{N}\left[\operatorname{var}(X(f_k)) + |E(X(f_k))|^2\right]$$

$$= \frac{1}{N}\left(N\sigma^2 + N^2|\tilde{A}|^2\right)$$
$$= \sigma^2 + N|\tilde{A}|^2 \qquad \text{for } k = l$$
$$= \sigma^2 \qquad \text{for } k \neq l$$

This statistic is just a sampled in frequency periodogram!