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1.4 $E\hat{A}_1 = E\frac{1}{N}\sum_{n=0}^{N-1}x[n] = \frac{1}{N}E\sum_{n=0}^{N-1}(A+w[n])$

$= A + \frac{1}{N}\sum_{n=0}^{N-1}Ew[n]$

$= A$

$\mathrm{var}\hat{A}_1 = \frac{1}{N^2}\sum_{n=0}^{N-1}\mathrm{var}\, w[n] = \frac{1}{N}$

$E\hat{A}_2 = \frac{1}{N+2}\left(2Ex[0] + \sum_{n=1}^{N-2}Ex[n] + 2Ex[N-1]\right)$

$= \frac{1}{N+2}\cdot(N+2)A$

$= A$

$\mathrm{var}\hat{A}_2 = \frac{1}{(N+2)^2}\left(4\sigma^2 + (N-2)\sigma^2 + 4\sigma^2\right)$

$= \frac{(N-2)^2+8}{(N+2)^2} = \frac{N+6}{(N+2)^2}$

$\cancel{= \frac{1}{N} - \frac{N+6}{N^2+4N+4}}$

当 $\frac{1}{N} > \frac{N+6}{(N+2)^2}$ 时，$4N+4 > 6N \Rightarrow N<2$

综上，$\hat{A}_1$，$\hat{A}_2$均是对A的无偏估计，当

$0 \le N < 2$ 时，$\mathrm{var}\hat{A}_2 < \mathrm{var}\hat{A}_1$，$\hat{A}_2$更好

$N=2$ 时，$\mathrm{var}\hat{A}_2 = \mathrm{var}\hat{A}_1$，一样好

$N>2$ 时，$\mathrm{var}\hat{A}_2 > \mathrm{var}\hat{A}_1$，$\hat{A}_1$更好。

与A的值无关

1.5. $E\hat{A}_3 = A$ 显然

$A^2 \le 1000$ 时，$\hat{A}_3 = \hat{A}_1$

$A^2 > 1000$ 时，$\hat{A}_3 = x[0] = A + w[0]$

$\mathrm{var}\hat{A}_3 = \sigma^2 = 1$

$|E\hat{A}_3| = |A| > \sqrt{1000}$.

合理，对高SNR情况，噪声对信号影响很小，无需遍历所有检测，且将复杂度从O(N)降为O(1)。

2.2 记 $x_{max} = \max\{x[0], \dots, x[N-1]\}$

易知 x_{max} 是对θ的一个估计，即 $\hat{\theta} = x_{max}$

$F_{x_{max}}(x) = P(x_{max} \le x)$

$= P(x[0] \le x, \dots, x[N-1] \le x)$

$= P(x[0] \le x) \cdots P(x[N-1] \le x)$

$= P^N(x[0] \le x)$

$= (\frac{x}{\theta})^N$

则 $f_{x_{max}}(x) = N\frac{x^{N-1}}{\theta^N}$

$E\hat{\theta} = \int_0^\theta x \cdot N\frac{x^{N-1}}{\theta^N}dx$

$= \frac{N}{\theta^N}\cdot\frac{\theta^{N+1}}{N+1}$

$= \frac{N}{N+1}\theta$

所以 $\frac{N+1}{N}x_{max}$ 是θ的无偏估计量

2.4 $\alpha = 1$：

$E\hat{h} = E\frac{1}{10}\sum_{i=1}^{10}\hat{h}_i = \frac{1}{10}\sum_{i=1}^{10}E\hat{h}_i = h$

$\mathrm{var}\hat{h} = \frac{1}{100}\sum_{i=1}^{10}\mathrm{var}\hat{h}_i = \frac{1}{10}$

$\alpha = \frac{1}{2}$：

$E\hat{h} = E\hat{h}_i = \frac{1}{2}h$

$\mathrm{var}\hat{h} = \frac{1}{10}\mathrm{var}\hat{h}_i = \frac{1}{10}$

对 $\alpha=1$ 或 $\frac{1}{2}$，偏差没有改变，但方差~~有所~~减小，因此有改善

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$x[n] = A + w[n]$, $w[n] \sim N(0, \sigma^2)$.

$E x^2[0] = \text{var}\, x[0] + (E x[0])^2$
$= \sigma^2 + A^2$

$E\hat{\theta} = E\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right)^2$

$= E\left\{\frac{1}{N^2}\left(\sum_{n=0}^{N-1} x[n]\right)^2\right\}$

$= \frac{1}{N^2} E\left\{\sum_{n=0}^{N-1} x^2[n] + 2\sum_{0\le i<j\le N-1} x[i]x[j]\right\}$

$= \frac{1}{N^2}\left(N E x^2[0] + N(N-1)(E x[0])^2\right)$

$= \frac{1}{N^2}\left(N(A^2+\sigma^2) + (N^2-N)A^2\right)$

$= A^2 + \frac{1}{N}\sigma^2$

$\neq A^2$

$\hat{\theta}$是有偏的. 当$N \to \infty$时 $E\hat{\theta} = A^2 + \frac{1}{N}\sigma^2 \to A^2$

7.10.

$E\hat{A} = E\frac{1}{N}\sum_{n=0}^{N-1} x[n] = A$ ✓

$E\hat{\sigma}^2 = E\frac{1}{N-1}\sum_{n=0}^{N-1}(x[n]-\hat{A})^2$

$(N-1)E\hat{\sigma}^2 = E\sum_{n=0}^{N-1}(x[n] - A + A - \hat{A})^2$

$= E\sum_{n=0}^{N-1}\left[(x[n]-A)^2 + (\hat{A}-A)^2 + 2(x[n]-A)(A-\hat{A})\right]$

$= \sum_{n=0}^{N-1}\left[E w^2[n] + \text{var}\hat{A} + 2EA(\cancel{x[n]}-\hat{A}) - 2E\hat{A}w[n]\right]$

$= \sum_{n=0}^{N-1}\left[\sigma^2 + \frac{1}{N}\sigma^2 - \cancel{2\frac{1}{N}\sum_{n=0}^{N-1} EA+w}\ 2\frac{1}{N}\sum_{i=0}^{N-1} E(A+w[i])w[n]\right]$

$= \sum_{n=0}^{N-1}\left[\sigma^2 + \frac{1}{N}\sigma^2 - \frac{2}{N}\left((N-1)\underset{i\neq n}{\cancel{E w[i] w[n]}} + \cancel{NEAw[n]} + \sigma^2\right)\right]$

$= \sum_{n=0}^{N-1}\frac{N-1}{N}\sigma^2$

$= (N-1)\sigma^2$

则 $E\hat{\sigma}^2 = \sigma^2$

$\hat{\sigma}^2$是无偏的