Linear Conic Optimization Part II

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Preliminaries

Content

- Vectors, Matrices, and Spaces
- Inner Products and Norms
- Open, Closed, Interior, and Boundary Sets
- Functions
- Linear Systems
- Convex Sets and Functions

Vectors, Matrices and Spaces

- Real numbers: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$
- Euclidean space: $\mathbb{R}^n$
- First orthant: $\mathbb{R}^n_+$
- n-dimensional (column) vector:

$$x = (x_1, x_2, \ldots, x_n)^T$$

- Matrices space: $\mathbb{R}^{m \times n}$
- Matrix: $M \in \mathbb{R}^{m \times n}$, ith row $M_{i\bullet}$, jth column $M_{\bullet j}$, ijth entry M_{ij}
- Symmetric square matrices space ($n(n+1)/2$-dimensional space):

$$\mathcal{S}^n = \{M \in \mathbb{R}^{n \times n} \mid M = M^T\}.$$

Vectors, Matrices and Spaces

Given $M \in \mathbb{R}^{m\times n}$, $N \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{n\times n}$

- Determinant: $\det(S)$
- Trace: $\mathrm{tr}(S)$=$\sum_{i=1}^{n} s_{ii}$

$$\mathrm{tr}(MN) = \mathrm{tr}(NM)$$

- Null space: $\mathcal{N}(M)$= $\{x \in \mathbb{R}^n | Mx = 0\}$.
- Range space: $\mathcal{R}(M)$= $\{y \in \mathbb{R}^m | y = Mx \text{ for some } x \in \mathbb{R}^n\}$.
- Positive semidefinite matrix:

$$S \succeq 0 \iff z^T S z \geq 0, \; \forall\; z \in \mathbb{R}^n$$

- Positive definite matrix:

$$S \succ 0 \iff z^T S z > 0, \; \forall\; z \in \mathbb{R}^n, \; z \neq 0$$

Properties of Trace

Let $A, X, X_1, X_2 \in \mathcal{M}(m, n)$, $k_1, k_2 \in \mathbb{R}$. Define

$$A \bullet X = \mathrm{tr}(AX^T) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_{ij}.$$

- Linearity. $A \bullet (k_1 X_1 + k_2 X_2) = k_1 A \bullet X_1 + k_2 A \bullet X_2$.
- Symmetry. $A \bullet X = X \bullet A$.
- Nonnegativity. $X \bullet X \geq 0$ and $X \bullet X = 0$ if and only if $X = 0$.
- $x^T Q x = \mathrm{tr}(Q x x^T) = Q \bullet (x x^T)$, where $x \in \mathbb{R}^n$ and $Q \in \mathcal{S}^n$.

Vectors, Matrices and Spaces

Theorem: (Schur complementary theorem)

$$A \succ 0, X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, S = C - B^T A^{-1} B$$

Then

$$X \succeq (\succ) 0 \Leftrightarrow S \succeq (\succ) 0$$

An Example: QCQP

Quadratically constrained quadratic programming problem

$$
\begin{array}{ll}
\min & \frac{1}{2}x^TQ_0x + q_0^Tx + c_0 \\
s.t. & \frac{1}{2}x^TQ_ix + q_i^Tx + c_i \leq 0, i = 1, 2, \ldots, m \\
& x \in \mathbb{R}^n
\end{array}
$$

where $Q_i \in \mathcal{S}^n$, $q_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, m$ are given coefficients, x is a decision variable.

$$
\begin{array}{ll}
\min & \frac{1}{2}Q_0 \bullet X + q_0^Tx + c_0 \\
s.t. & \frac{1}{2}Q_i \bullet X + q_i^Tx + c_i \leq 0, i = 1, 2, \ldots, m \\
& X = xx^T \\
& x \in \mathbb{R}^n
\end{array}
$$

An Example: SDP Relaxation

Formulation 1

$$\begin{array}{ll}\min & \frac{1}{2}Q_0 \bullet X + q_0^T x + c_0 \\ s.t. & \frac{1}{2}Q_i \bullet X + q_i^T x + c_i \le 0, i = 1, 2, \ldots, m \\ & x \in \mathbb{R}^n, X \in \mathcal{S}_+^n. \end{array}$$

Formulation 2

$$\begin{array}{ll}\min & \frac{1}{2}Q_0 \bullet X + q_0^T x + c_0 \\ s.t. & \frac{1}{2}Q_i \bullet X + q_i^T x + c_i \le 0, i = 1, 2, \ldots, m \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}_+^{n+1}. \end{array}$$

Inner Products and Norms

- Inner products:

$$x \bullet y = x^T y = \sum_i x_i y_i$$

$$X \bullet Y = \text{tr}(X^T Y) = \sum_{i,j} X_{ij} Y_{ij}$$

- Norms:
  - Euclidean norm: $\|x\|_2 = \sqrt{x \bullet x}$
  - p-norm: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $p \geq 1$.
  - Infinity-norm: $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$
  - Frobenius norm:

$$\|X\|_F = \sqrt{X \bullet X} = \sqrt{\text{tr}(X^T X)}$$

- Note that: $x^T A x = A \bullet x x^T$

Open, Closed, Interior and Boundary Sets

- Neighborhood: $N(x^0;\epsilon) = \{x \in \mathbb{R}^n \mid \|x - x^0\| < \epsilon\}$.
- Open: $\mathcal{X} \subset \mathbb{R}^n$ is open if for any $x \in \mathcal{X}$, there exists $\epsilon > 0$ such that $N(x;\epsilon) \subset \mathcal{X}$.
- Closed: $\mathcal{X} \subset \mathbb{R}^n$ is closed, if $\mathbb{R}^n \backslash \mathcal{X} = \{x \in \mathbb{R}^n | x \notin \mathcal{X}\}$ is open.
- Closed: An equivalent statement: any accumulation point of $\mathcal{X}$ is in $\mathcal{X}$.
- Closure of a set $\mathcal{X} \subset \mathbb{R}^n$ is the smallest closed set containing $\mathcal{X}$ and is denoted as $\mathrm{cl}(\mathcal{X})$.

Open, Closed, Interior and Boundary Sets

- Interior: the interior of a given set $\mathcal{X} \subset \mathbb{R}^n$ is

$$\text{int}(\mathcal{X}) = \{x \in \mathcal{X} | \exists\ \epsilon_x > 0 \text{ such that } N(x; \epsilon_x) \subset \mathcal{X}\}$$

- Boundary of a set $\mathcal{X} \subset \mathbb{R}^n$:

$$\text{bdry}(\mathcal{X}) = \text{cl}(\mathcal{X}) \backslash \text{int}(\mathcal{X}) = \{x \in \text{cl}(\mathcal{X}) | x \notin \text{int}(\mathcal{X})\}$$

- Bounded: a set $\mathcal{X} \subset \mathbb{R}^n$ is bounded if there exists an $r > 0$ such that

$$\|x\| < r, \forall x \in \mathcal{X}$$

Functions

- Continuous: $f : \mathcal{X} \subset \mathbb{R}^n$ is continuous at x^0
  (i) $x^0 \in \mathcal{X}$. (ii) $\lim_{x \to x^0} f(x) = f(x^0)$.
- Continuous function: $f \in C^0(\mathcal{X})$ means f is continuous at all points in $\mathcal{X} \subset \mathbb{R}^n$.
- Gradient: For $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(x) = [\frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_n}]_{1 \times n}$$

- Hessian: For $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$

$$F(x) = [\frac{\partial^2 f(x)}{\partial x_i \partial x_j}]_{n \times n}$$

- Continuously differentiable function: $f \in C^p(\mathcal{X})$ $(p = 1, 2, \cdots)$ means f is p-th continuously differentiable over $\mathcal{X} \subset \mathbb{R}^n$.

Functions

Theorem (Taylor theorem)

Let $\mathcal{X}$ be open, $f \in C^p(\mathcal{X})$, $x^1, x^2 \in \mathcal{X}$, $x^1 \neq x^2$ and

$$x(\theta) = \theta x^1 + (1-\theta)x^2 \in \mathcal{X}, \ \forall\, 0 \leq \theta \leq 1.$$

Then $\exists\ \bar{x} = \bar{\theta}x^1 + (1-\bar{\theta})x^2 \in \mathcal{X}$, $0 < \bar{\theta} < 1$, s.t.

$$f(x^2) = f(x^1) + \sum_{k=1}^{p-1} \frac{1}{k!} d^k f(x^1; x^2 - x^1) + \frac{1}{p!} d^p f(\bar{x}; x^2 - x^1)$$

where $d^k f(x;h)$ is the k-th order differential of function f along h.

Functions: Big O and Small o

Let $g(\cdot)$ be a real-valued function on $\mathbb{R}$.

- $g(x) = O(x)$

  $\exists c \geq 0$ such that
  $$\left|\frac{g(x)}{x}\right| \leq c \text{ as } x \to \ +\infty$$

- $g(x) = o(x)$
  $$\left|\frac{g(x)}{x}\right| \to 0 \text{ as } x \to 0$$

Functions

Taylor theorem in small o formulation:

- $p = 1$

$$f(x+h) = f(x) + \nabla f(x)h + o(\|h\|)$$

- $p = 2$

$$f(x+h) = f(x) + \nabla f(x)h + \frac{1}{2}h^T F(x)h + o(\|h\|^2)$$

Linear Systems

Given $x^1, \cdots, x^m \in \mathbb{R}^n$

- Linear combination:
$$\sum_{i=1}^{m} \lambda_i x^i,$$
where $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, m$.
- Linearly independent
$$\sum_{i=1}^{m} \lambda_i x^i = 0 \Rightarrow \lambda_1 = \cdots = \lambda_m = 0$$
- Affine combination: a linear combination with
$$\sum_{i=1}^{m} \lambda_i = 1$$
- Affinely independent: if $x^2 - x^1, \cdots, x^m - x^1$ are linearly independent.

Linear Systems

- Convex combination: a linear combination with
$$\sum_{i=1}^{m} \lambda_i = 1 \text{ and } \lambda_i \geq 0, i = 1, \dots, m$$
- Hyperplane:
$$\mathcal{X} = \{x \in \mathbb{R}^n | a^T x = \sum_{i=1}^{n} a_i x_i = b\}$$
- Affine space: affine combination of any two points in the space is still in the space. (An intersection of finitely many hyperplanes.)
- Linear subspace: an affine space containing the origin.

  We can always transform an affine space $\mathcal{Y} \subset \mathbb{R}^n$ into a linear subspace $\mathcal{X} \subset \mathbb{R}^n$ by choosing $x^0 \in \mathcal{Y}$ such that
$$\mathcal{X} = \{x - x^0 | x \in \mathcal{Y}\}$$

Linear Systems

- Half space:

$$\mathcal{X} = \{x \in \mathbb{R}^n | a^T x = \sum_{i=1}^{n} a_i x_i \leq b\}$$

- Polyhedron (Polytope): an intersection of finitely many half spaces.
- Dimension of a linear subspace: the maximum number of linearly independent vectors in the subspace.
- Dimension of an affine space: the dimension of the transformed linear subspace.
- Dimension of a polyhedron: the dimension of the smallest affine space containing it.

Linear Systems

- Linear equations

$$\left.\begin{array}{ccc} a^1 \bullet x & = & b_1 \\ a^2 \bullet x & = & b_2 \\ \cdots & \cdots & \cdots \\ a^m \bullet x & = & b_m \end{array}\right. \Rightarrow Ax = b,$$

where $a^1, \cdots, a^m$ and x are all in $\mathbb{R}^n$.

$$\left.\begin{array}{ccc} A_1 \bullet X & = & b_1 \\ A_2 \bullet X & = & b_2 \\ \cdots & \cdots & \cdots \\ A_m \bullet X & = & b_m \end{array}\right. \Rightarrow \mathcal{A}X = b,$$

where $A_1, \cdots, A_m$ and X are all in $\mathcal{S}^n$.

- For convenience, $\mathcal{A}^* y = \sum_{i=1}^m y_i A_i$.

Convex Sets and Properties

- A set $\mathcal{X} \subset \mathbb{R}^n$ is convex if for any $x^1 \in \mathcal{X}$ and $x^2 \in \mathcal{X}$, we have $\lambda x^1 + (1-\lambda)x^2 \in \mathcal{X}$, for all $0 \leq \lambda \leq 1$.
- Convex hull: the smallest convex set containing a given set

$$\mathrm{conv}(\mathcal{X}) = \{x \in \mathbb{R}^n | x = \textstyle\sum_{i=1}^m \lambda_i y^i \text{ for some } m \in \mathbb{N}_+, \\ \lambda_i \geq 0, \textstyle\sum_{i=1}^m \lambda_i = 1, \text{ and } y^i \in \mathcal{X}, i = 1, \ldots, m\}$$

- Dimension of a convex set: the dimension of the smallest affine space containing it.
- Relative interior of a convex set $\mathcal{X} \subset \mathbb{R}^n$: suppose $\mathcal{H}$ is the smallest affine space containing $\mathcal{X}$,

$$\mathrm{ri}(\mathcal{X}) = \{x \in \mathbb{R}^n | \exists \text{ open set } \mathcal{Y} \subseteq \mathbb{R}^n \text{ such that } x \in \mathcal{Y} \cap \mathcal{H} \subset \mathcal{X}\}$$

- Supporting hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n | a^T x = b\}$ of a convex set $\mathcal{X}$:

$$a^T y \geq b, \forall\, y \in \mathcal{X} \text{ and } \mathrm{cl}(\mathcal{X}) \cap \mathcal{H} \neq \emptyset.$$

Relative Interior–An Example

$$\mathcal{X} = \{x_1 \in \mathbb{R} | 0 \le x_1 \le 2\}.$$

A linear programming standard reformulation

$$\mathcal{Y} = \{(x_1, x_2) \in \mathbb{R}^2_+ | x_1 + x_2 = 2\}.$$

Relative interior

$$\mathrm{ri}(\mathcal{X}) = \mathrm{int}(\mathcal{X}) = \{x_1 \in \mathbb{R} | 0 < x_1 < 2\}.$$

$$\mathrm{ri}(\mathcal{Y}) = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 = 2, x_1 > 0, x_2 > 0\},$$

where the small affine space is $\mathcal{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 = 2\}$, and the open set is defined as in $\mathbb{R}^2$.

Supporting hyperplane–Examples

- Supporting hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n | a^T x = b\}$ of a convex set $\mathcal{X}$:

$$a^T y \geq b, \forall\ y \in \mathcal{X} \text{ and } \mathrm{cl}(\mathcal{X}) \cap \mathcal{H} \neq \emptyset.$$

- Example 1 (Y): $\mathcal{X} = \{x \in \mathbb{R}^2 | x_1^2 \leq x_2\}$, $\mathcal{H} = \{x \in \mathbb{R}^2 | x_2 = 0\}$.
- Example 2 (Y): $\mathcal{X} = \{x \in \mathbb{R}^2 | x_1^2 < x_2\}$, $\mathcal{H} = \{x \in \mathbb{R}^2 | x_2 = 0\}$.
- Example 3 (N): $\mathcal{X} = \{x \in \mathbb{R}^2 | x_1^2 \leq x_2\}$, $\mathcal{H} = \{x \in \mathbb{R}^2 | x_2 = -1\}$.

Convex Functions and Properties

- Epigraph of a function $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$

$$\operatorname{epi} f = \{(x, \lambda) \in \mathbb{R}^{n+1} | \lambda \geq f(x), x \in \mathcal{X}\}$$

- Closed function: if $\operatorname{epi} f$ is a closed set.
- Convex function: if $\operatorname{epi} f$ is a convex set.
- Concave function: if $-f$ is a convex function.
- Convex hull function $\operatorname{conv}(f)$ of a function $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ is a function on $\mathcal{X}$ such that $\operatorname{epi}(\operatorname{conv}(f)) = \operatorname{conv}(\operatorname{epi}(f))$.

Lemma

$f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ is a convex function if and only if for any x^1, $x^2 \in \mathcal{X}$, and $0 \leq \lambda \leq 1$, we have

$$f(\lambda x^1 + (1 - \lambda) x^2) \leq \lambda f(x^1) + (1 - \lambda) f(x^2).$$

Convex Functions and Properties

- Subgradient d of a convex function $f : \mathcal{X} \subset \mathbb{R}^n$ at $x \in \mathcal{X}$:

  if for any $y \in \mathcal{X}$,
$$f(y) \geq f(x) + d^T(y - x)$$

- The set $\{(y, \lambda) \in \mathbb{R}^{n+1} | \lambda - d^T y = f(x) - d^T x\}$ is a *supporting hyperplane* of $\mathrm{epi} f$ at x.

- Subdifferential of a convex function $f : \mathcal{X} \subset \mathbb{R}^n$ at $x \in \mathcal{X}$:

$$\partial f(x) = \{d | d \text{ is a subgradient of } f \text{ at } x\}$$

Convex Functions and Properties

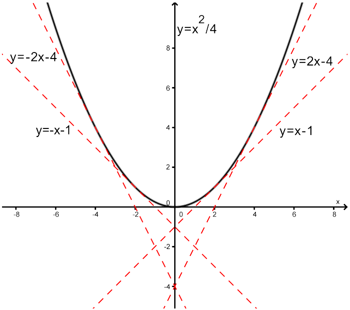


Figure: $(x,\ f(x)) \leftrightarrow (y,\ \lambda)$

- $(x, f(x))$: The curve of $f(x)$.
- $(y,\ \lambda)$: The supporting plane of $\{(x, f(x))|x \in \mathbb{R}^n\}$ at x.

Conjugate Functions

- The negative of λ-intercept: $\lambda - d^T y = f(x) - d^T x, y \in \mathbb{R}^n$.

$$f^*(d) = \sup_{x \in \mathcal{X}} \{d \bullet x - f(x)\}$$

- Conjugate of $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$:

$$f^*(y) = \sup_{x \in \mathcal{X}} \{y \bullet x - f(x)\}$$

with f^* being defined on $\mathcal{Y} = \{y \in \mathbb{R}^n | f^*(y) < +\infty\}$.

Conjugate Functions

Lemma

If $f^* : \mathcal{Y}$ exists then $\mathcal{Y}$ is a convex set and $f^* : \mathcal{Y}$ is a convex function.

Lemma (Fenchel's inequality)

Given $f : \mathcal{X}$ and its conjugate $f^* : \mathcal{Y}$, then

$$x \bullet y \leq f(x) + f^*(y), \; \forall \; x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$

Moreover,

$$x \bullet y = f(x) + f^*(y) \iff y \in \partial f(x)$$

Conjugate Functions–Examples

Example 1: $f(x) = x^2$.

$$f^*(y) = \sup_{x \in \mathbb{R}} (xy - x^2) = \frac{y^2}{4}, \quad \mathcal{Y} = \mathbb{R}.$$

Example 2: $f(x) = x^3$.

$$f^*(y) = \sup_{x \in \mathbb{R}} (xy - x^3) = +\infty, \quad \mathcal{Y} = \emptyset.$$

Example 3: $f(x) = 2x^2, x \geq 1$.

$$f^*(y) = \sup_{x \geq 1} (xy - 2x^2) = \begin{cases} \frac{y^2}{8}, & y \geq 4 \\ y - 2, & y < 4, \end{cases} \qquad \mathcal{Y} = \mathbb{R}.$$

Conjugate Functions and Properties

Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be a function with its conjugate transform $f^* : \mathcal{Y}$.

- For $\alpha \in \mathbb{R}$, the conjugate of $f + \alpha$ is $f^* - \alpha$.
- For $a \in \mathbb{R}^n$, the conjugate of $\tilde{f}(x) = f(x) + x \bullet a$ on $\mathcal{X}$ is $\tilde{f}^*(y) = f^*(y - a)$, $\forall\, y \in \mathcal{Y}$.
- For $a \in \mathbb{R}^n$, the conjugate of $\bar{f}(x) = f(x - a)$ on $\mathcal{X}$ is $\bar{f}^*(y) = f^*(y) + y \bullet a$, $\forall\, y \in \mathcal{Y}$.
- For $\lambda > 0$, the conjugate of $f_1(x) = \lambda f(x)$ on $\mathcal{X}$ is $f_1^*(y) = \lambda f^*(\frac{y}{\lambda})$, $\forall\, y \in \lambda\mathcal{Y}$.
- For $\lambda > 0$, the conjugate of $f_2(x) = f(\frac{x}{\lambda})$ on $\lambda\mathcal{X}$ is $f_2^*(y) = f^*(\lambda y)$, $\forall\, y \in \mathcal{Y}/\lambda$.

Theorem

Assume that $f_1 : \mathcal{X}$ and $f_2 : \mathcal{X}$ have the same convex hull function. Then they have the same conjugate transform $f^* : \mathcal{Y}$ when it exists.

Conjugate Functions and Properties

We know the dual problem of LD is LP again. When will the conjugate transform of $f^* : \mathcal{Y}$ become $f : \mathcal{X}$?

Proper function

A convex function f is proper if its epigraph is non-empty and contains no vertical lines, i.e. if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x.

Theorem

Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be a proper closed convex function with conjugate transform $f^* : \mathcal{Y}$. Then the conjugate transform of $f^* : \mathcal{Y}$ is $f : \mathcal{X}$. Moreover, $y \in \partial f(x)$ if and only if $x \in \partial f^*(y)$. In this case,

$$x \bullet y = f(x) + f^*(y) \quad \Longleftrightarrow \quad y \in \partial f(x) \text{ or } x \in \partial f^*(y)$$

Convex Cone Structure

Content

- Convex Cones and Properties
- Dual Cones
- Partial Order and Ordered Vector Space
- Some Examples

Convex Cones and Properties

- A set $K \subset \mathbb{R}^n$ is a cone if
$$\forall x \in K \text{ and } \lambda > 0 \Rightarrow \lambda x \in K;$$
- A cone $K \subset \mathbb{R}^n$ is pointed if
$$K \cap -K = \{0\};$$
- A cone $K \subset \mathbb{R}^n$ is solid if
$$\text{int}K \neq \emptyset;$$
- A cone $K \subset \mathbb{R}^n$ is proper if it is pointed, solid, closed and convex.

Dual Cones

- Conic combination: a linear combination $\sum_{i=1}^{m} \lambda_i x^i$ with $\lambda_i \geq 0$, $x^i \in \mathbb{R}^n$ for all $i = 1, \ldots, m$.
- The conic hull of a set $\mathcal{X} \subset \mathbb{R}^n$ is

$$\text{cone}(\mathcal{X}) = \{x \in \mathbb{R}^n | x = \textstyle\sum_{i=1}^{m} \lambda_i x^i, \text{ for some } m \in \mathbb{N}_+ \\ \text{and } x^i \in \mathcal{X}, \lambda_i \geq 0, i = 1, \ldots, m.\}$$

- The dual cone $K^* \subset \mathbb{R}^n$ of a cone $K \subset \mathbb{R}^n$ is

$$K^* = \{y \in \mathbb{R}^n | y \bullet x \geq 0, \forall\, x \in K\}$$

  K^* is a *closed, convex* cone.
- If $K^* = K$, then K is a self-dual cone.

Properties

K, K_1, K_2 are convex cones in $\mathbb{R}^n$.

- $(K^*)^* = \text{cl}(K)$
- $K_1 \cap K_2$, $K_1 \cup K_2$, $K_1 + K_2$ are all cones
- $(K_1 + K_2)^* = K_1^* \cap K_2^*$
- K_1 and K_2 are both closed $\Rightarrow$ $K_1 + K_2$ is closed.
- $\text{ri}(K_1 + K_2) = \text{ri}(K_1) + \text{ri}(K_2)$
- The supporting hyperplane of K always contains the origin
- If K is solid (pointed), then K^* is pointed (solid).

Partial Order and Ordered Vector Space

- A relation "$\geq$" is a partial order on a set $\mathcal{X}$ if it has:

  1. *reflexivity*: $a \geq a$ for all $a \in \mathcal{X}$;
  2. *antisymmetry*: $a \geq b$ and $b \geq a$ imply $a = b$;
  3. *transitivity*: $a \geq b$ and $b \geq c$ imply $a \geq c$.

- An ordered vector space $\mathcal{X}$ is equipped with a partial order "$\geq$" which also satisfies:

  - *homogeneity*: $a \geq b$ and $\lambda \in \mathbb{R}_+$ imply $\lambda a \geq \lambda b$;
  - *additivity*: $a \geq b$ and $c \geq d$ imply $a + c \geq b + d$.

Partial Order and Ordered Vector Space

- A *proper* cone K in a vector space can induce a partial order "$\geq_K$"

$$a \geq_K b \Leftrightarrow a - b \in K$$

  which leads to an ordered vector space.
- Similarly, we can define "$\leq_K$"

$$a \leq_K b \Leftrightarrow b \geq_K a,$$

- *Closeness* of K allows passing limits in $\geq_K$:

$$a^i \geq_K b^i,\ a^i \to a,\ b^i \to b \text{ as } i \to \infty \ \Rightarrow \ a \geq_K b.$$

- *Solidness* of K allows us to define a strict inequality:

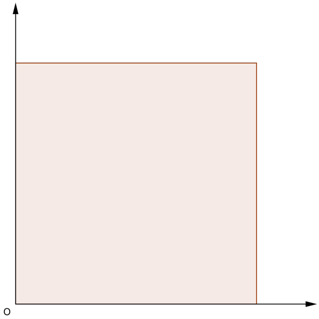
$$a >_K b \Leftrightarrow a - b \in \mathrm{int} K,$$

  and

$$a <_K b \Leftrightarrow b >_K a.$$

Examples: $\mathbb{R}^n_+$

- $\mathbb{R}^n_+$ is a proper cone;
- Inner product: $x \bullet y = x^T y$;
- $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ (self-dual);
- Partial order: "$\geq_{\mathbb{R}^n_+}$"

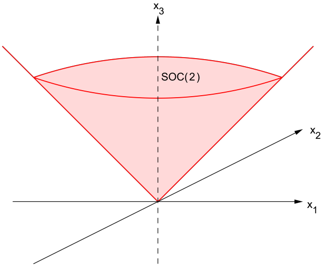


Examples: $\mathcal{L}^n$

- $\mathcal{L}^n$ / $SOC(n-1)$Lorentz cone (second order cone)

$$\mathcal{L}^n = \{x \in \mathbb{R}^n | x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\}$$

- $\mathcal{L}^n$ is a proper cone;
- Inner product: $x \bullet y = x^T y$;
- $(\mathcal{L}^n)^* = \mathcal{L}^n$ (self-dual);
- Partial order: "$\geq_{\mathcal{L}^n}$"



Examples: $\mathcal{S}_+^n$

- $\mathcal{S}_+^n \subset \mathcal{S}^n$: the set of symmetric positive semidefinite matrices
- $\mathcal{S}_+^n$ is a proper cone;
- Inner product:

$$X \bullet Y = \mathrm{tr}(X^T Y)$$

- *Another view*:

$$\mathrm{vec}(X) = [X_{11}, \sqrt{2}X_{12}, X_{22}, \sqrt{2}X_{13}, \sqrt{2}X_{23}, X_{33}, \cdots, X_{nn}]^T \in \mathbb{R}^{\frac{n(n+1)}{2}}$$

Then

$$X \bullet Y = \mathrm{vec}(X) \bullet \mathrm{vec}(Y) = \sum_{i,j} X_{ij} Y_{ij}$$

- Partial order: "$\geq_{\mathcal{S}_+^n}$" or "$\succeq$"

Examples: $\mathcal{S}_+^n$

Lemma

$(\mathcal{S}_+^n)^* = \mathcal{S}_+^n$ (self-dual)

Proof.

"$\subseteq$": If $X \in (\mathcal{S}_+^n)^*$, then $z^T X z = X \bullet zz^T \geq 0$, for all $z \in \mathbb{R}^n$. Therefore, $X \in \mathcal{S}_+^n$. "$\supseteq$": For any $Y \in \mathcal{S}_+^n$,

$$Y = \textstyle\sum_{i=1}^n \lambda_i z^i (z^i)^T,$$

with $\lambda_i \geq 0$.

If $X \in \mathcal{S}_+^n$, then

$$X \bullet Y = \sum_{i=1}^n \lambda_i X \bullet z^i (z^i)^T = \sum_{i=1}^n \lambda_i (z^i)^T X z^i \geq 0.$$

Therefore, $X \in (\mathcal{S}_+^n)^*$. $\square$

Examples: $\mathcal{C}_n$ and $\mathcal{C}_n^*$

- Copositive cone:

$$\mathcal{C}_n = \{X \in \mathcal{S}^n | z^T X z \geq 0, \forall\ z \geq_{\mathbb{R}^n_+} 0\}$$

- Completely positive(nonnegative) cone:

$$\mathcal{C}_n^* = \left\{ X \in \mathcal{S}^n \middle| \begin{array}{l} X = \sum_{i=1}^m z^i (z^i)^T, \text{ for some } m \in \mathbb{N}_+ \\ \text{and } z^i \geq_{\mathbb{R}^n_+} 0, i = 1, \ldots, m \end{array} \right\}$$

- $(\mathcal{C}_n)^* = \mathcal{C}_n^*$ and $\mathcal{C}_n = (\mathcal{C}_n^*)^*$
- $\mathcal{C}_n^* \subset \mathcal{S}_+^n \subset \mathcal{C}_n$

Examples: Cones of Nonnegative Quadratic Functions — Homogeneous

- $\mathcal{F} \subset \mathbb{R}^n$
- Nonnegative homogeneous quadratic functions over $\mathcal{F}$

$$f(x) = x^T A x \geq 0, \forall x \in \mathcal{F}$$

$$f \Leftrightarrow A$$

- $\mathcal{HD}_{\mathcal{F}} = \{A \in \mathcal{S}^n | x^T A x \geq 0, \forall x \in \mathcal{F}\}$ is a closed, convex cone.
  (i) Closeness:

$$x^T A_i x \geq 0 \text{ and } A_i \to A \Rightarrow x^T A x \geq 0$$

  (ii) Convexity:

$$x^T A_i X \geq 0, i = 1, 2 \Rightarrow x^T(\lambda A_1 + (1 - \lambda) A_2) x \geq 0, \forall\ 0 \leq \lambda \leq 1$$

Examples: Cones of Nonnegative Quadratic Functions — Homogeneous

- $\mathcal{HD}^*_{\mathcal{F}} = \mathrm{cl}(\mathrm{cone}\{xx^T | x \in \mathcal{F}\})$
- $(\mathcal{HD}_{\mathcal{F}})^* = \mathcal{HD}^*_{\mathcal{F}}$ and $(\mathcal{HD}^*_{\mathcal{F}})^* = \mathcal{HD}_{\mathcal{F}}$
- Examples:
  - $\mathcal{F} = \mathbb{R}^n$
    $\mathcal{HD}_{\mathcal{F}} = \mathcal{HD}^*_{\mathcal{F}} = \mathcal{S}^n_+$
  - $\mathcal{F} = \mathbb{R}^n_+$
    $\mathcal{HD}_{\mathcal{F}} = \mathcal{C}_n$ and $\mathcal{HD}^*_{\mathcal{F}} = \mathcal{C}^*_n$
  - $\mathcal{F} = \{x | e^T x = 1, x \in \mathbb{R}^n_+\}$
    $\mathcal{HD}_{\mathcal{F}} = \mathcal{C}_n$ and $\mathcal{HD}^*_{\mathcal{F}} = \mathcal{C}^*_n$

Examples: Cones of Nonnegative Quadratic Functions — Nonhomogeneous

- Nonnegative quadratic functions over $\mathcal{F} \subset \mathbb{R}^n$

$$f(x) = x^T A x + 2b^T x + c \geq 0, \forall x \in \mathcal{F}$$

$$f \Leftrightarrow \begin{bmatrix} c & b^T \\ b & A \end{bmatrix}$$

- $\mathcal{D}_{\mathcal{F}} = \{ \begin{bmatrix} c & b^T \\ b & A \end{bmatrix} \in \mathcal{S}^{n+1} | \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c & b^T \\ b & A \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \forall x \in \mathcal{F} \}$ is a closed, convex cone.
- $\mathcal{D}_{\mathcal{F}}^* = \mathrm{cl}(\mathrm{cone}\{ \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} | x \in \mathcal{F} \})$
- $(\mathcal{D}_{\mathcal{F}}^*)^* = \mathcal{D}_{\mathcal{F}}$ and $(\mathcal{D}_{\mathcal{F}})^* = \mathcal{D}_{\mathcal{F}}^*$

Examples: Cones of Nonnegative Quadratic Functions — Nonhomogeneous

- Examples:
  - $\mathcal{F} = \mathbb{R}^n$
    $\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}^* = \mathcal{S}_+^{n+1}$
  - $\mathcal{F} = \mathbb{R}_+^n$
    $\mathcal{D}_{\mathcal{F}} = \mathcal{C}_{n+1}$ and $\mathcal{D}_{\mathcal{F}}^* = \mathcal{C}_{n+1}^*$