无研2/3 刘子源 2022310709.

7.3、8、9、17、18、19

7.3 两者均属于指数族函数，MLE正确性无需二阶求导验证

a. $p(x;\mu)=\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}(x-\mu)^2\}$

$p(\mathbf{x};\mu)=\frac{1}{(2\pi)^{N/2}}\exp\{-\frac{1}{2}\sum_{n=0}^{N-1}(x[n]-\mu)^2\}$

$\frac{d\ln p(\mathbf{x};\mu)}{d\mu}=\sum_{n=0}^{N-1}(x[n]-\mu)=0 \Rightarrow \hat{\mu}=\frac{1}{N}\sum_n x[n]=\bar{x}$ 可看作WGN模型中的直流分量

b. $p(x;\lambda)=\lambda\exp\{-\lambda x\}u(x)$.

$p(\mathbf{x};\lambda)=\begin{cases}\lambda^N\exp\{-\lambda\sum_n x[n]\}, & x[n]>0 \text{ for all } n\\ 0, & \text{otherwise}.\end{cases}$

$\frac{d\ln p(\mathbf{x};\lambda)}{d\lambda}=\frac{\partial}{\partial\lambda}(N\ln\lambda-\lambda\sum_n x[n])=\frac{N}{\lambda}-\sum_n x[n]=0 \Rightarrow \hat{\lambda}=\frac{1}{\bar{x}}$ 是对联立指数分布参数λ的估计.

指数分布中$EX=\frac{1}{\lambda}$，因此该估计很合理

7.8. $p(x;p)=\prod_{n=0}^{N-1}p^{x[n]}(1-p)^{1-x[n]}=p^{\sum_n x[n]}(1-p)^{N-\sum_n x[n]}$

$\ln p(x;p)=\sum_n x[n]\ln p+(N-\sum_n x[n])\ln(1-p)$.

$\frac{d\ln p(x;p)}{dp}=\frac{\sum_n x[n]}{p}-\frac{N-\sum_n x[n]}{1-p}=0 \Rightarrow \hat{p}=\bar{x}$

7.9. $p(x;\theta)=\begin{cases}\frac{1}{\theta}, & 0\le x\le\theta\\ 0, & \text{其他}\end{cases}$

$p(\mathbf{x};\theta)=\begin{cases}\frac{1}{\theta^N}, & x[n]\in[0,\theta] \text{ for all } n\\ 0, & \text{其他}\end{cases}$

当$p(\mathbf{x};\theta)\neq 0$时，$\theta\downarrow$ $p(\mathbf{x};\theta)\uparrow$，但是$\theta\ge x[n]$，$n=0,\dots,N-1$

所以θ能取到的min为x_{max}，即$\hat{\theta}=\max\{x[n]\}$

7.17 $x[n]\overset{iid}{\sim}N(0,\frac{1}{\theta})$

令$\alpha=\frac{1}{\theta}$，则$\hat{\theta}=\frac{1}{\hat{\alpha}}$

$p(\mathbf{x};\alpha)=\frac{1}{(2\pi\alpha)^{N/2}}\exp\{-\frac{1}{2\alpha}\sum_n x^2[n]\}$

$0=\frac{d\ln p(\mathbf{x};\alpha)}{d\alpha}=-\frac{N}{2}\cdot\frac{1}{\alpha}+\frac{1}{2\alpha^2}\sum_n x^2[n] \Rightarrow \hat{\alpha}=\frac{1}{N}\sum_n x^2[n]$

则 $\hat{\theta} = \frac{N}{\sum_n x^2[n]}$

$I(\alpha) = \frac{N}{2\alpha^2}$

$I^{-1}(\alpha) = I^{-1}(\theta)\left(\frac{\partial\alpha}{\partial\theta}\right)^2$

$$I^{-1}(\theta) = \frac{2\alpha^2}{N}\left(\frac{\partial\theta}{\partial\alpha}\right)^2 = \frac{2\alpha^2}{N}\left(-\frac{1}{\alpha^2}\right)^2 = \frac{2\theta^2}{N}$$

即 $\hat{\theta} \sim N\left(\theta, \frac{2\theta^2}{N}\right)$

记 $g(f_0) = \sum_n x[n]\cos 2\pi f_0 n$

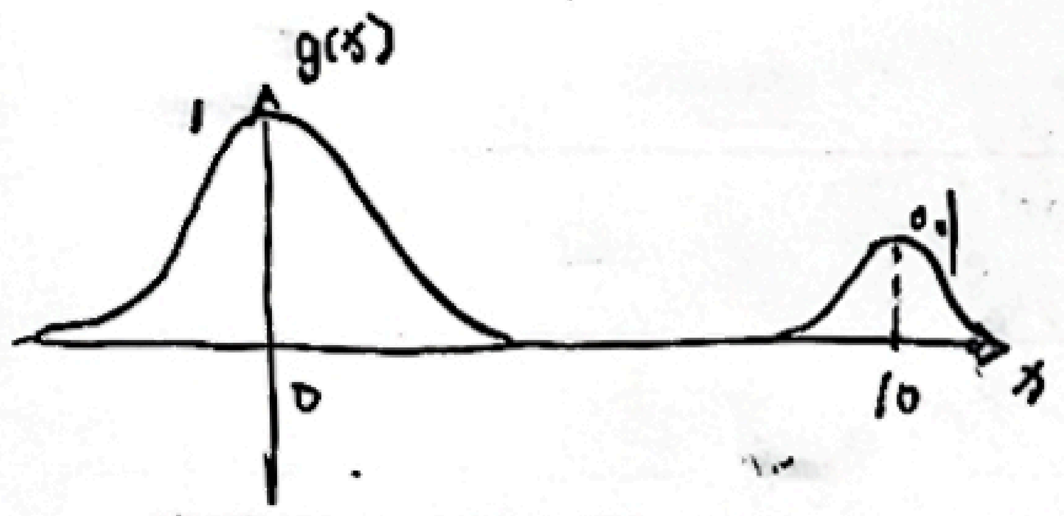
$g'(f_0) = -\sum_n 2\pi n x[n]\sin 2\pi f_0 n$

$g''(f_0) = -\sum_n (2\pi n)^2 x[n]\cos 2\pi f_0 n$

NR: $f_{0k+1} = f_{0k} - \frac{g'(f_{0k})}{g''(f_{0k})}$

数值求解略

7.18. $g(x) = e^{-\frac{x^2}{2}} + 0.1 e^{-\frac{1}{2}(x-10)^2}$



$g'(x) = -xe^{-\frac{1}{2}x^2} - 0.1(x-10)e^{-\frac{1}{2}(x-10)^2}$

$g''(x) = x^2 e^{-\frac{1}{2}x^2} + 0.1(x-10)^2 e^{-\frac{1}{2}(x-10)^2} - 0.1 e^{-\frac{1}{2}(x-10)^2} - e^{-\frac{1}{2}x^2}$

$x_{k+1} = x_k - \frac{g'(x_k)}{g''(x_k)}$

当 $x_0 = 0.5$ 时收敛到 $x=0$

当 $x_0 = 9.5$ 时收敛到 $x=10$

NR可迭代收敛至局部最优，不一定是全局最优，因此选择初值要尽可能接近真值

7.19. $x[n] = \cos 2\pi f_0 n + w[n]$

$p(x;f_0) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_n (x[n]-\cos 2\pi f_0 n)^2\right\}$

$\ln p(x;f_0) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_n (x[n]-\cos 2\pi f_0 n)^2$

只需 $\min \sum_n (x[n]-\cos 2\pi f_0 n)^2 = \sum_n x^2[n] - 2\sum_n x[n]\cos 2\pi f_0 n + \sum_n \cos^2 2\pi f_0 n$

$$\approx \sum_n x^2[n] - 2\sum_n x[n]\cos 2\pi f_0 n + \frac{N}{2} + \underbrace{\cancel{\frac{1}{2}\sum_n \cos 4\pi f_0 n}}_{\approx 0 \text{ 当 } 0<f_0<\frac{1}{2}}$$

$\Longrightarrow$ 只需 $\max \sum_n x[n]\cos 2\pi f_0 n$