统计信号处理

第十三章

复合假设检验 I

(基本方法I)

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一、引言

简单假设检验

- 确定信号检测
- 随机信号检测

PDF已知

$$\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)} > \gamma$$

$$\frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} > \gamma$$

? → 复合假设检验

二、未知参数对检测的影响

WGN中未知信号检测

$$H_0 : x[n] = w[n]$$

$$H_1 : x[n] = A + w[n]$$

其中信号电平A是未知的。噪声 $w[n]$ 是方差为σ^2 的WGN。如何检测是否存在信号？

采用NP准则，若似然比

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; A, H_1)}{p(\mathbf{x}; H_0)} > \gamma$$

则判 H_1

$$L(\boldsymbol{x})=\frac{p(\boldsymbol{x};A,H_1)}{p(\boldsymbol{x};H_0)}=\frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\}}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}}>\gamma$$

$$\Rightarrow \quad -\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2+\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]>\ln\gamma$$

$$-\frac{1}{2\sigma^2}\left\{-2A\sum_{n=0}^{N-1}x[n]+NA^2\right\}>\ln\gamma$$

$$\boxed{A\sum_{n=0}^{N-1}x[n]>\sigma^2\ln\gamma+\frac{NA^2}{2}}$$

检测三要素：

- ✓ 检测统计量
- ✓ 判决方法
- ✓ 门限

1 若 $A>0$

$$\underbrace{\frac{1}{N}\sum_{n=0}^{N-1}x[n]}_{T(\boldsymbol{x})}>\frac{\sigma^2}{NA}\ln\gamma+\frac{A}{2}=\gamma'$$

$$\boxed{T(\boldsymbol{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] > \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma'}$$

是否可实现?

$$T(\boldsymbol{x}) \sim \begin{cases} N\left(0, \dfrac{\sigma^2}{N}\right), & H_0 \\ N\left(A, \dfrac{\sigma^2}{N}\right), & H_1 \end{cases}$$

$$\Rightarrow P_{FA} = \Pr\left(T(\boldsymbol{x}) > \gamma'; H_0\right) = Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right)$$

$$\Downarrow$$

$$\boxed{\gamma' = \sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})}$$

与信号幅度无关

——可实现!

$$\Downarrow$$

$$P_D = \Pr\left(T(\boldsymbol{x}) > \gamma'; H_1\right) = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right)$$

$$\Rightarrow P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{N}{\sigma^2}}A\right)$$

$$\Rightarrow \boxed{T(\boldsymbol{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] > \sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})}$$

一致最大势检验（uniformly most powerful test, UMP）

$$A\sum_{n=0}^{N-1} x[n] > \sigma^2 \ln\gamma + \frac{NA^2}{2}$$

2 若 $A<0$

$$\Rightarrow \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} x[n]}_{T(\mathbf{x})} < \frac{\sigma^2}{NA}\ln\gamma + \frac{A}{2} = \gamma^{'}$$

$$T(\mathbf{x}) \sim N\left(0, \frac{\sigma^2}{N}\right), \quad H_0 \Rightarrow P_{FA} = \Pr\left(T(\mathbf{x}) < \gamma^{'}; H_0\right) = Q\left(-\frac{\gamma^{'}}{\sqrt{\sigma^2/N}}\right)$$

$$\Rightarrow \gamma^{'} = -\sqrt{\frac{\sigma^2}{N}} Q^{-1}(P_{FA})$$

单边检验

即，若 $T(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] < -\sqrt{\frac{\sigma^2}{N}} Q^{-1}(P_{FA})$ ，判 H_1

双边检验

$A>0$ 时，若 $T(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] > \sqrt{\frac{\sigma^2}{N}} Q^{-1}(P_{FA})$ ，判 H_1

$-\infty < A < +\infty$?

WGN中未知信号检测

$$H_0 : x[n] = w[n]$$
$$H_1 : x[n] = A + w[n]$$

其中信号电平A是未知的。噪声 $w[n]$ 是方差为σ^2 的WGN。如何检测是否存在信号？

单边检验:

$$\begin{cases} H_0 : A = 0 \\ H_1 : A > 0 \end{cases} \text{或} \begin{cases} H_0 : A = 0 \\ H_1 : A < 0 \end{cases}$$

存在UMP

双边检验:

$$\begin{cases} H_0 : A = 0 \\ H_1 : A \neq 0 \end{cases}$$

不存在UMP

推广至一般情况

单边检验:

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta > \theta_0 \end{cases} \text{或} \begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta < \theta_0 \end{cases}$$

可能存在UMP

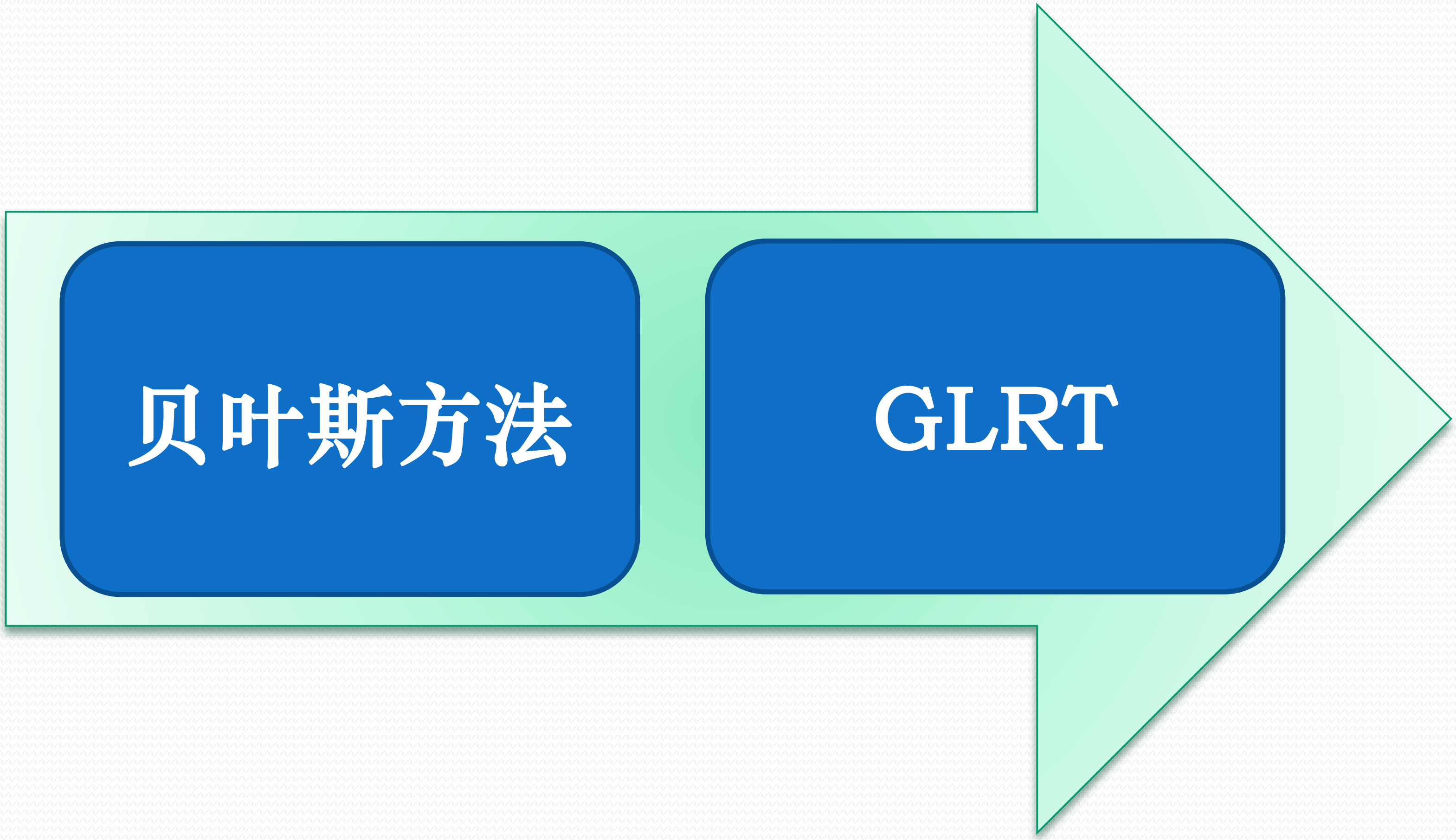
双边检验:

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0 \end{cases}$$

不可能存在UMP

——需**复合假设检验**方法

复合假设检验的基本方法



三、方法一：贝叶斯方法

未知参数：

$$H_0 : \boldsymbol{\theta}_0$$

$$H_1 : \boldsymbol{\theta}_1$$

数据PDF：

$$p(\mathbf{x}; H_0) = \int p(\mathbf{x} | \boldsymbol{\theta}_0; H_0) p(\boldsymbol{\theta}_0) d\boldsymbol{\theta}_0$$

$$p(\mathbf{x}; H_1) = \int p(\mathbf{x} | \boldsymbol{\theta}_1; H_1) p(\boldsymbol{\theta}_1) d\boldsymbol{\theta}_1$$

采用NP准则，若似然比

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; H_1)}{p(\mathbf{x}; H_0)} > \gamma$$

则判 H_1

方法特点：

- 该方法要求多重积分，积分维数等于未知参数维数 **——有时难求解**
- 要求先验知识 **——有时难把握**
- 当具备条件时，可以使用

例：WGN中未知信号检测——贝叶斯方法

$$H_0 : x[n] = w[n]$$
$$H_1 : x[n] = A + w[n]$$

其中信号电平$A(-\infty < A < +\infty)$是未知的，其先验分布为$N(0,\sigma_A^2)$。噪声$w[n]$是方差为σ^2的WGN，且与信号统计独立。如何检测是否存在信号？

$$\begin{cases} H_0 : A = 0 \\ H_1 : A \neq 0 \end{cases}$$

不存在UMP

采用贝叶斯方法:

$$p(\mathbf{x} \mid A; H_1) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\}$$

$$p(A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left\{-\frac{1}{2\sigma_A^2}A^2\right\}$$

$$p(\mathbf{x}; H_1) = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\} \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left\{-\frac{1}{2\sigma_A^2}A^2\right\} dA$$

$$p\left(\boldsymbol{x};H_0\right)=\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}$$

$$L\left(\boldsymbol{x}\right)=\frac{p\left(\boldsymbol{x};H_1\right)}{p\left(\boldsymbol{x};H_0\right)}=\frac{\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A\right)^2\right\}\frac{1}{\sqrt{2\pi\sigma_A^2}}\exp\left\{-\frac{1}{2\sigma_A^2}A^2\right\}dA}{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}}$$

$$=\frac{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A\right)^2\right\}\frac{1}{\sqrt{2\pi\sigma_A^2}}\exp\left\{-\frac{1}{2\sigma_A^2}A^2\right\}dA}{\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}}$$

$$=\frac{1}{\sqrt{2\pi\sigma_A^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\underbrace{\left(\frac{N}{\sigma^2}+\frac{1}{\sigma_A^2}\right)}_{1/\sigma_{A|x}^2}A^2+\frac{N}{\sigma^2}\bar{x}A\right\}dA$$

$$=\frac{1}{\sqrt{2\pi\sigma_A^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_{A|x}^2}\left(A-\frac{N\bar{x}\sigma_{A|x}^2}{\sigma^2}\right)^2-\frac{N^2\bar{x}^2\sigma_{A|x}^2}{\sigma^4}\right)\right\}dA$$

$$L(\mathbf{x})=\frac{p(\mathbf{x};H_1)}{p(\mathbf{x};H_0)}=\frac{1}{\sqrt{2\pi\sigma_A^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_{A|x}^2}\left(A-\frac{N\overline{x}\sigma_{A|x}^2}{\sigma^2}\right)^2-\frac{N^2\overline{x}^2\sigma_{A|x}^2}{\sigma^4}\right)\right\}dA$$

$$=\frac{1}{\sqrt{2\pi\sigma_A^2}}\exp\left\{\frac{N^2\overline{x}^2\sigma_{A|x}^2}{2\sigma^4}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\sigma_{A|x}^2}\left(A-\frac{N\overline{x}\sigma_{A|x}^2}{\sigma^2}\right)^2\right\}dA$$

$$=\frac{\sqrt{2\pi\sigma_{A|x}^2}}{\sqrt{2\pi\sigma_A^2}}\exp\left\{\frac{N^2\overline{x}^2\sigma_{A|x}^2}{2\sigma^4}\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_{A|x}^2}}\exp\left\{-\frac{1}{2\sigma_{A|x}^2}\left(A-\frac{N\overline{x}\sigma_{A|x}^2}{\sigma^2}\right)^2\right\}dA$$

$$=\sqrt{\frac{\sigma_{A|x}^2}{\sigma_A^2}}\exp\left\{\frac{N^2\overline{x}^2\sigma_{A|x}^2}{2\sigma^4}\right\}$$

当 $\sqrt{\frac{\sigma_{A|x}^2}{\sigma_A^2}}\exp\left\{\frac{N^2\overline{x}^2\sigma_{A|x}^2}{2\sigma^4}\right\}>\gamma$ 时，判 H_1

仅保留与数据有关的项

$\overline{x}^2>\gamma'$ 时，判 H_1 ，即 $|\overline{x}|>\sqrt{\gamma'}=\gamma''$ 时，判 H_1

——未知参数不再影响判决！

● 未知参数对检测性能的影响？

检测器： $|\bar{x}| > \sqrt{\gamma'} = \gamma''$

$$\bar{x} \sim \begin{cases} N\left(0, \dfrac{\sigma^2}{N}\right), & H_0 \\ N\left(A, \dfrac{\sigma^2}{N}\right), & H_1 \end{cases}$$

$$P_{FA} = \Pr\left(|\bar{x}| > \gamma''; H_0\right)$$
$$= \Pr\left(\bar{x} > \gamma''; H_0\right) + \Pr\left(\bar{x} < -\gamma''; H_0\right)$$
$$= 2Q\left(\frac{\gamma''}{\sqrt{\sigma^2 / N}}\right)$$

$$\gamma'' = \sqrt{\frac{\sigma^2}{N}} Q^{-1}\left(\frac{P_{FA}}{2}\right)$$

$$P_D = \Pr\left(|\bar{x}| > \gamma''; H_1\right)$$
$$= \Pr\left(\bar{x} > \gamma''; H_1\right) + \Pr\left(\bar{x} < -\gamma''; H_1\right)$$
$$= Q\left(\frac{\gamma'' - A}{\sqrt{\sigma^2 / N}}\right) + 1 - Q\left(\frac{-\gamma'' - A}{\sqrt{\sigma^2 / N}}\right)$$
$$= Q\left(\frac{\gamma'' - A}{\sqrt{\sigma^2 / N}}\right) + Q\left(\frac{\gamma'' + A}{\sqrt{\sigma^2 / N}}\right)$$

$$P_D = Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - A\sqrt{\frac{N}{\sigma^2}}\right) + Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + A\sqrt{\frac{N}{\sigma^2}}\right)$$

- 假定未知参数已知时——透视检测器——其性能可作为上界

$A>0$ 时

$$T(\boldsymbol{x})=\bar{x}=\frac{1}{N}\sum_{n=0}^{N-1}x[n]>\sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})=\gamma_+^{'}$$

$A<0$ 时

$$T(\boldsymbol{x})=\bar{x}=\frac{1}{N}\sum_{n=0}^{N-1}x[n]<-\sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})=\gamma_-^{'}$$

✓ 当 $A>0$ 时，

$$\bar{x}\sim\begin{cases}N\left(0,\dfrac{\sigma^2}{N}\right), & H_0\\ N\left(A,\dfrac{\sigma^2}{N}\right), & H_1\end{cases}$$

$$P_{FA}=\Pr\left(\bar{x}>\gamma_+^{'};H_0\right)=Q\left(\frac{\gamma_+^{'}}{\sqrt{\sigma^2/N}}\right)$$

$$\gamma_+^{'}=\sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})$$

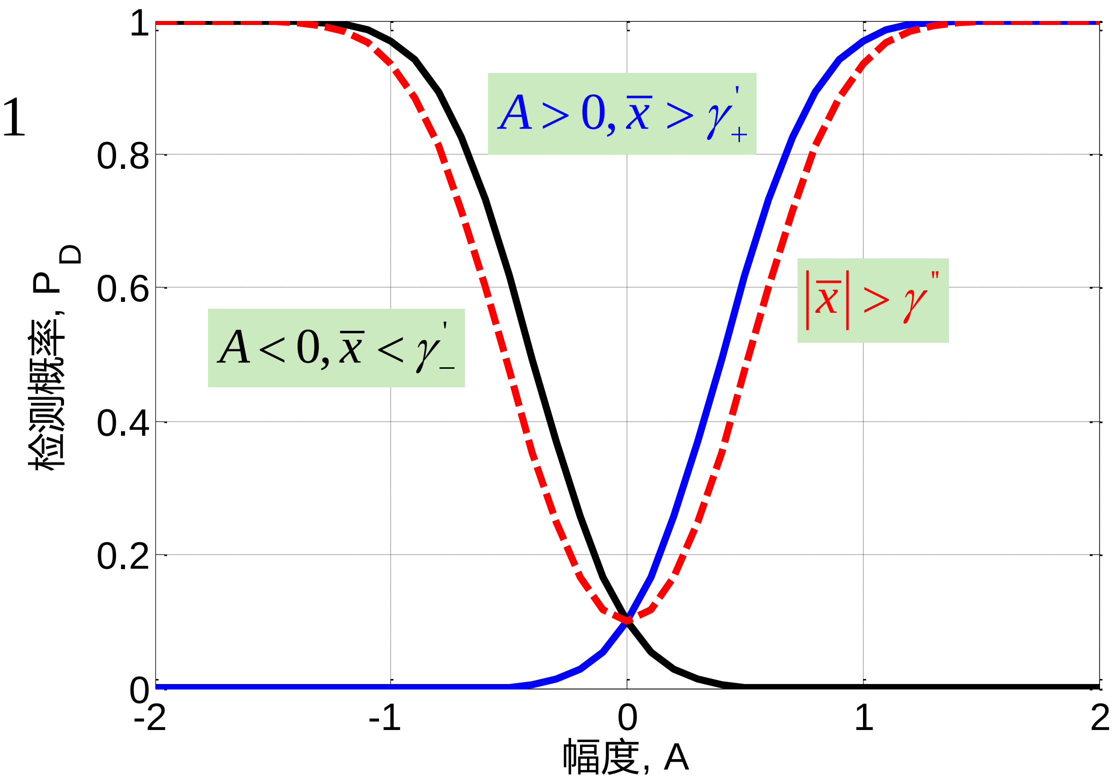
$$P_D=\Pr\left(\bar{x}>\gamma_+^{'};H_1\right)=Q\left(\frac{\gamma_+^{'}-A}{\sqrt{\sigma^2/N}}\right)\longrightarrow P_D=Q\left(Q^{-1}(P_{FA})-A\sqrt{\frac{N}{\sigma^2}}\right)$$

- ✓ 同理，当 $A<0$ 时，$P_D = Q\left(Q^{-1}(P_{FA}) + A\sqrt{\frac{N}{\sigma^2}}\right)$
- ✓ 当 $A>0$ 时，$P_D = Q\left(Q^{-1}(P_{FA}) - A\sqrt{\frac{N}{\sigma^2}}\right)$
- ✓ 当 A 未知时，$P_D = Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - A\sqrt{\frac{N}{\sigma^2}}\right) + Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + A\sqrt{\frac{N}{\sigma^2}}\right)$

性能对比？

参数：

$N=10$，$\sigma^2=1$，$P_{FA}=0.1$



- ✓ 与透视检测器（假定未知参数已知时）相比，性能有所下降
- ✓ 次最佳检测器

四、方法二：广义似然比检验

未知参数：

$H_0: \boldsymbol{\theta}_0$

$H_1: \boldsymbol{\theta}_1$

$p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1, H_1\right)$

$p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_0, H_0\right)$

$\mathrm{MLE}: \hat{\boldsymbol{\theta}}_0,\ \hat{\boldsymbol{\theta}}_1$

采用NP准则，若似然比

$$L_G(\mathbf{x}) = \frac{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1, H_1\right)}{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_0, H_0\right)} > \gamma \quad \Leftrightarrow \quad L_G(\mathbf{x}) = \frac{\max_{\boldsymbol{\theta}_1} p\left(\mathbf{x};\boldsymbol{\theta}_1, H_1\right)}{\max_{\boldsymbol{\theta}_0} p\left(\mathbf{x};\boldsymbol{\theta}_0, H_0\right)} > \gamma$$

则判 H_1

——Generalized Likelihood Ratio Test（GLRT)

- **内在含义？**
- **与贝叶斯方法思想差异**

例：WGN中未知信号检测——GLRT

$$H_0 : x[n] = w[n]$$
$$H_1 : x[n] = A + w[n]$$

其中信号电平 $A(-\infty < A < +\infty)$ 是未知的。噪声 $w[n]$ 是方差为 σ^2 的WGN。如何检测是否存在信号？

$$\begin{cases} H_0 : A = 0 \\ H_1 : A \neq 0 \end{cases}$$

采用NP准则，若似然比

$$L_G(\mathbf{x}) = \frac{p\left(\mathbf{x}; \hat{A}, H_1\right)}{p\left(\mathbf{x}; H_0\right)} > \gamma$$

则判 H_1

$$p(\mathbf{x};A,H_1)=\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\}$$

$$\frac{\partial \ln p(\mathbf{x};A,H_1)}{\partial A}=\frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)$$

$$\frac{\partial \ln p(\mathbf{x};A,H_1)}{\partial A}=0$$

$$\Rightarrow \quad \hat{A}=\overline{x}$$

$$\Rightarrow \quad p(\mathbf{x};\hat{A},H_1)=\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\overline{x})^2\right\}$$

$$p(\mathbf{x};H_0)=\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}$$

$$\Rightarrow \quad L_G(\mathbf{x})=\frac{p(\mathbf{x};\hat{A},H_1)}{p(\mathbf{x};H_0)}=\frac{\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\overline{x})^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}}$$

$$L_G(\boldsymbol{x}) = \frac{p(\boldsymbol{x};\hat{A},H_1)}{p(\boldsymbol{x};H_0)} = \frac{\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-\overline{x}\right)^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}}$$

$$\ln L_G(\boldsymbol{x}) = -\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}x^2[n]-2\overline{x}\sum_{n=0}^{N-1}x[n]+N\overline{x}^2-\sum_{n=0}^{N-1}x^2[n]\right)$$

$$= -\frac{1}{2\sigma^2}\left(-2N\overline{x}^2+N\overline{x}^2\right)$$

$$= \frac{N\overline{x}^2}{2\sigma^2}$$

仅保留与数据有关的项

$\overline{x}^2 > \gamma'$ 时，判 H_1

即 $|\overline{x}| > \sqrt{\gamma'} = \gamma''$ 时，判 H_1

Vs 贝叶斯方法

- 无需先验知识
- 运算简单
- 因此应用更广

例：WGN中未知信号检测——GLRT

$$H_0 : x[n] = w[n]$$
$$H_1 : x[n] = A + w[n]$$

其中信号电平$A(-\infty < A < +\infty)$是未知的。噪声$w[n]$是方差未知的WGN。如何检测是否存在信号？

$$\begin{cases} H_0 : A = 0,\ \sigma^2 \\ H_1 : A \neq 0,\ \sigma^2 \end{cases}$$

多余参数

采用NP准则，若广义似然比

$$L_G(\mathbf{x}) = \frac{p\left(\mathbf{x}; \hat{A}, \hat{\sigma}_1^2, H_1\right)}{p\left(\mathbf{x}; \hat{\sigma}_0^2, H_0\right)} > \gamma$$

则判 H_1

➢ H_1 时

$$p\left(\mathbf{x};A,\sigma^2,H_1\right)=\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A\right)^2\right\}$$

$$\frac{\partial\ln p\left(\mathbf{x};A,\sigma^2,H_1\right)}{\partial A}=\frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A\right)$$

$$\frac{\partial\ln p\left(\mathbf{x};A,\sigma^2,H_1\right)}{\partial A}=0$$

$$\frac{\partial\ln p\left(\mathbf{x};A,\sigma^2,H_1\right)}{\partial\sigma^2}=-\frac{N}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{n=0}^{N-1}\left(x[n]-A\right)^2$$

$$\frac{\partial\ln p\left(\mathbf{x};A,\sigma^2,H_1\right)}{\partial\sigma^2}=0$$

$$\Rightarrow\begin{cases}\hat{A}=\bar{x}\\ \hat{\sigma}_1^2=\dfrac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-\bar{x}\right)^2\end{cases}$$

$$\Rightarrow\quad p\left(\mathbf{x};\hat{A},\hat{\sigma}_1^2,H_1\right)=\frac{1}{\left(2\pi\hat{\sigma}_1^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{N}{2}\right\}$$

➢ H_0 时

$$p\left(\mathbf{x};\sigma^2,H_0\right)=\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right\}$$

$$\frac{\partial\ln p\left(\mathbf{x};\sigma^2,H_0\right)}{\partial\sigma^2}=-\frac{N}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{n=0}^{N-1}x^2[n]$$

$$\frac{\partial\ln p\left(\mathbf{x};\sigma^2,H_0\right)}{\partial\sigma^2}=0$$

$$\hat{\sigma}_0^2=\frac{1}{N}\sum_{n=0}^{N-1}x^2[n]$$

$$p\left(\mathbf{x};\hat{\sigma}_0^2,H_0\right)=\frac{1}{\left(2\pi\hat{\sigma}_0^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{N}{2}\right\}$$

$$p\left(\mathbf{x};\hat{A},\hat{\sigma}_1^2,H_1\right)=\frac{1}{\left(2\pi\hat{\sigma}_1^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{N}{2}\right\}$$

$$L_G(\mathbf{x})=\frac{p\left(\mathbf{x};\hat{A},\hat{\sigma}_1^2,H_1\right)}{p\left(\mathbf{x};\hat{\sigma}_0^2,H_0\right)}=\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{\frac{N}{2}}$$，即 $2\ln L_G(\mathbf{x})=N\ln\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)$

$$2\ln L_G(\mathbf{x}) = N\ln\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)$$

$$\left.\begin{aligned}\hat{\sigma}_0^2 &= \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] \\ \hat{\sigma}_1^2 &= \frac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-\bar{x}\right)^2\end{aligned}\right\} \Rightarrow \hat{\sigma}_1^2 = \frac{1}{N}\left(\sum_{n=0}^{N-1} x^2[n] + N\bar{x}^2 - 2\sum_{n=0}^{N-1} x[n]\bar{x}\right) = \hat{\sigma}_0^2 - \bar{x}^2 \Rightarrow \hat{\sigma}_0^2 = \hat{\sigma}_1^2 + \bar{x}^2$$

$$\Rightarrow 2\ln L_G(\mathbf{x}) = N\ln\left(1+\frac{\bar{x}^2}{\hat{\sigma}_1^2}\right) \Rightarrow \boxed{T'(\mathbf{x}) = \frac{\bar{x}^2}{\hat{\sigma}_1^2} > \gamma'}$$

门限:

$$\left.\begin{aligned} & w[n] \sim N\left(0,\sigma^2\right) \\ & u[n] \sim N(0,1) \\ & w[n] = \sigma u[n]\end{aligned}\right\} \Rightarrow T'(\mathbf{x};H_0) = \frac{\left(\frac{1}{N}\sum_{n=0}^{N-1} w[n]\right)^2}{\frac{1}{N}\sum_{n=0}^{N-1}\left(w[n]-\bar{w}\right)^2} = \frac{\left(\frac{1}{N}\sigma\sum_{n=0}^{N-1} u[n]\right)^2}{\frac{1}{N}\sum_{n=0}^{N-1}\left(\sigma u[n]-\sigma\bar{u}\right)^2} = \frac{\left(\frac{1}{N}\sum_{n=0}^{N-1} u[n]\right)^2}{\frac{1}{N}\sum_{n=0}^{N-1}\left(u[n]-\bar{u}\right)^2}$$

五、经典线性模型下的GLRT

假定数据满足线性模型：$\mathbf{x}=\mathbf{H}\boldsymbol{\theta}+\mathbf{w}$，其中 $\mathbf{H}$ 是 $N\times p$ $(N>p)$ 秩为 p 的观测矩阵，$\boldsymbol{\theta}$ 是 $p\times 1$ 的参数矢量，$\mathbf{w}$ 是$N\times 1$ 的噪声矢量，PDF为 $N\left(\mathbf{0},\sigma^2\mathbf{I}\right)$。对两类假设检验：

$$H_0:\mathbf{A}\boldsymbol{\theta}=\mathbf{b}$$

$$H_1:\mathbf{A}\boldsymbol{\theta}\neq\mathbf{b}$$

其中 $\mathbf{A}$ 是 $r\times p\,(r\le p)$ 秩为 r 的矩阵，$\mathbf{b}$ 是 $r\times 1$ 的矢量。如何检验是哪类？

采用NP准则，若广义似然比

$$L_G(\mathbf{x})=\frac{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1,H_1\right)}{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_0,H_0\right)}>\gamma$$

则GLRT判 H_1

$$L_G(\boldsymbol{x}) = \frac{p\left(\boldsymbol{x};\hat{\boldsymbol{\theta}}_1, H_1\right)}{p\left(\boldsymbol{x};\hat{\boldsymbol{\theta}}_0, H_0\right)} > \gamma \quad \Leftrightarrow \quad L_G(\boldsymbol{x}) = \frac{\max\limits_{\boldsymbol{\theta}_1} p\left(\boldsymbol{x};\boldsymbol{\theta}_1, H_1\right)}{\max\limits_{\boldsymbol{\theta}_0} p\left(\boldsymbol{x};\boldsymbol{\theta}_0, H_0\right)} > \gamma$$

$$p(\boldsymbol{x};\boldsymbol{\theta}) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\underline{(\boldsymbol{x}-\mathbf{H}\boldsymbol{\theta})^T(\boldsymbol{x}-\mathbf{H}\boldsymbol{\theta})}\right\}$$

$$\text{MLE:} \quad \min\left\{(\boldsymbol{x}-\mathbf{H}\boldsymbol{\theta})^T(\boldsymbol{x}-\mathbf{H}\boldsymbol{\theta})\right\}$$

1 $H_1: \mathbf{A}\boldsymbol{\theta} \neq \boldsymbol{b}$

$$J(\boldsymbol{\theta}) = (\boldsymbol{x}-\mathbf{H}\boldsymbol{\theta})^T(\boldsymbol{x}-\mathbf{H}\boldsymbol{\theta})$$

$$= \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\mathbf{H}\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{H}^T\mathbf{H}\boldsymbol{\theta}$$

$$\frac{\partial J(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -2\mathbf{H}^T\boldsymbol{x} + 2\mathbf{H}^T\mathbf{H}\boldsymbol{\theta} \quad \Rightarrow \quad \hat{\boldsymbol{\theta}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\boldsymbol{x}$$

即 $\hat{\boldsymbol{\theta}}_1 = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\boldsymbol{x}$

2

$$H_0 : \mathbf{A}\boldsymbol{\theta} = \boldsymbol{b}$$

$$J_c(\boldsymbol{\theta}) = (\boldsymbol{x} - \mathbf{H}\boldsymbol{\theta})^T (\boldsymbol{x} - \mathbf{H}\boldsymbol{\theta}) + \boldsymbol{\lambda}^T (\mathbf{A}\boldsymbol{\theta} - \boldsymbol{b})$$

$$= \boldsymbol{x}^T \boldsymbol{x} - 2\boldsymbol{x}^T \mathbf{H}\boldsymbol{\theta} + \boldsymbol{\theta}^T \mathbf{H}^T \mathbf{H}\boldsymbol{\theta} + \boldsymbol{\lambda}^T \mathbf{A}\boldsymbol{\theta} - \boldsymbol{\lambda}^T \boldsymbol{b}$$

$$\frac{\partial J_c(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -2\mathbf{H}^T \boldsymbol{x} + 2\mathbf{H}^T \mathbf{H}\boldsymbol{\theta} + \mathbf{A}^T \boldsymbol{\lambda}$$

$$\Rightarrow \quad \left.\begin{aligned} \hat{\boldsymbol{\theta}}_0 &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\boldsymbol{x} - \frac{1}{2}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\boldsymbol{\lambda} \\ \mathbf{A}\boldsymbol{\theta}_0 &= \boldsymbol{b} \end{aligned}\right\} \Rightarrow$$

$$\frac{\boldsymbol{\lambda}}{2} = \left(\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\boldsymbol{x} - \boldsymbol{b}\right) \Rightarrow$$

$$\hat{\boldsymbol{\theta}}_0 = \underline{(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\boldsymbol{x}} - (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\left(\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\underline{(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\boldsymbol{x}} - \boldsymbol{b}\right)$$

$$\Rightarrow \quad \hat{\boldsymbol{\theta}}_0 = \hat{\boldsymbol{\theta}}_1 - \underbrace{(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\left(\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \boldsymbol{b}\right)}_{\boldsymbol{d}}$$

$$= \hat{\boldsymbol{\theta}}_1 - \boldsymbol{d}$$

$$L_G(\boldsymbol{x}) = \frac{p\left(\boldsymbol{x};\hat{\boldsymbol{\theta}}_1, H_1\right)}{p\left(\boldsymbol{x};\hat{\boldsymbol{\theta}}_0, H_0\right)}$$

$$= \frac{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_1\right)^T\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_1\right)\right\}}{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_0\right)^T\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_0\right)\right\}}$$

$$\ln L_G(\boldsymbol{x}) = -\frac{1}{2\sigma^2}\left\{\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_1\right)^T\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_1\right)-\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_0\right)^T\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_0\right)\right\}$$

$$\hat{\boldsymbol{\theta}}_0 = \hat{\boldsymbol{\theta}}_1 - \boldsymbol{d}$$

$$\ln L_G(\boldsymbol{x}) = -\frac{1}{2\sigma^2}\left\{-\underbrace{\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_1\right)^T\mathbf{H}\boldsymbol{d}}_{0} - \underbrace{\boldsymbol{d}^T\mathbf{H}^T\left(\boldsymbol{x}-\mathbf{H}\hat{\boldsymbol{\theta}}_1\right)}_{0} - \boldsymbol{d}^T\mathbf{H}^T\mathbf{H}\boldsymbol{d}\right\}$$

$$= \frac{1}{2\sigma^2}\boldsymbol{d}^T\mathbf{H}^T\mathbf{H}\boldsymbol{d}$$

$$\boldsymbol{d} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \boldsymbol{b}\right)$$

$$2\ln L_G(\mathbf{x}) = \frac{\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b}\right)^T \left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b}\right)}{\sigma^2}$$

性能如何?

$$\hat{\boldsymbol{\theta}}_1 = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x} \quad \Rightarrow \quad \hat{\boldsymbol{\theta}}_1 \sim N\left(\boldsymbol{\theta}, \sigma^2\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\right)$$

MLE　MVU

$$\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b} \sim N\left(\mathbf{A}\boldsymbol{\theta} - \mathbf{b}, \sigma^2\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right) \sim \begin{cases} N\left(\mathbf{0}, \sigma^2\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right), & H_0 \\ N\left(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b}, \sigma^2\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right), & H_1 \end{cases}$$

若 $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{C})$，那么 $y = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}$ 服从如下非中心chi方分布

$$y \sim \chi_n^2(\lambda)$$

其中非中心参量 $\lambda = \boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}$ 。（见书P479~480，第2章2.3节）

因此，$2\ln L_G(\mathbf{x}) \sim \begin{cases} \chi_r^2(0), & H_0 \\ \chi_r^2(\lambda), & H_1 \end{cases}$ ，$\lambda = \dfrac{\left(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b}\right)^T\left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b}\right)}{\sigma^2}$

假定数据满足线性模型：$\mathbf{x}=\mathbf{H}\boldsymbol{\theta}+\mathbf{w}$，其中 $\mathbf{H}$ 是 $N\times p$ $(N>p)$ 秩为 p 的观测矩阵，$\boldsymbol{\theta}$ 是 $p\times 1$ 的参数矢量，$\mathbf{w}$ 是$N\times 1$ 的噪声矢量，PDF为 $N\left(\mathbf{0},\sigma^2\mathbf{I}\right)$。对两类假设检验：

$$H_0:\mathbf{A}\boldsymbol{\theta}=\boldsymbol{b}$$

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其中 $\mathbf{A}$ 是 $r\times p\,(r\le p)$ 秩为 r 的矩阵，$\boldsymbol{b}$ 是 $r\times 1$ 的矢量。如何检验是哪类？

采用NP准则，若广义似然比

$$2\ln L_G(\mathbf{x})=\frac{\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1-\boldsymbol{b}\right)^T\left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1-\boldsymbol{b}\right)}{\sigma^2}>\gamma'$$，则判 H_1

且

$$2\ln L_G(\mathbf{x})\sim\begin{cases}\chi_r^2(0), & H_0\\ \chi_r^2(\lambda), & H_1\end{cases},\quad \lambda=\frac{\left(\mathbf{A}\boldsymbol{\theta}_1-\boldsymbol{b}\right)^T\left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\boldsymbol{\theta}_1-\boldsymbol{b}\right)}{\sigma^2}$$

例：WGN中未知信号检测

$$H_0: x[n] = w[n]$$
$$H_1: x[n] = D + w[n]$$

其中信号电平 $A(-\infty < A < +\infty)$ 是未知的。噪声 $w[n]$ 是方差为 σ^2 的WGN。如何检测是否存在信号？

假设检验为：

$$H_0: D = 0$$
$$H_1: D \neq 0$$

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w} \qquad \begin{matrix} H_0: \mathbf{A}\boldsymbol{\theta} = \mathbf{b} \\ H_1: \mathbf{A}\boldsymbol{\theta} \neq \mathbf{b} \end{matrix}$$

对经典线性模型：$\mathbf{H} = \mathbf{1},\ \mathbf{A} = 1,\ \boldsymbol{\theta} = D,\ \mathbf{b} = 0\ ,\hat{\boldsymbol{\theta}}_1 = \bar{x}$

$$2\ln L_G(\mathbf{x}) = \frac{\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b}\right)^T \left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b}\right)}{\sigma^2} = \frac{N}{\sigma^2}\bar{x}^2$$

$$2\ln L_G(\mathbf{x}) \sim \begin{cases} \chi_r^2(0),\ H_0 \\ \chi_r^2(\lambda),\ H_1 \end{cases}, \quad \lambda = \frac{\left(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b}\right)^T \left(\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}\left(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b}\right)}{\sigma^2} = \frac{ND^2}{\sigma^2}$$

六、小结

- 复合假设检验：含未知参数的检验
- 一致最大势检测（UMP）
  - 在双边检测中**不可能**存在
  - 在单边检测中**可能**存在
- 贝叶斯方法
  - 需要未知参数的先验知识
  - 多重积分——运算量较大
- GLRT方法
  - 无需先验知识
  - 运算相对简单
  - 应用广泛
- 经典线性模型下的GLRT

相比透视检测器，性能有所恶化

——参数未知的代价!