刘子源 2022310704

① $\hat{\mu}_x = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$

$$\mathrm{var}(\hat{\mu}_x) = E\left\{\left[\frac{1}{N}\sum_{n=0}^{N-1} x(n) - E\left(\frac{1}{N}\sum_{n=0}^{N-1} x(n)\right)\right]^2\right\}$$

$$= \frac{1}{N^2} E\left\{\left(\sum_{n=0}^{N-1}(x(n) - Ex(n))\right)^2\right\}$$

$$= \frac{1}{N^2} E\left\{\cancel{\sum_{n=0}^{N-1}(x(n) - Ex(n))^2} + \sum_{\substack{0\le n\le N-1 \\ 0\le m\le N-1}} (x(n) - Ex(n))(x(m) - Ex(m))\right\}$$

包括

$$= \frac{1}{N^2}\left\{\cancel{\sum_{n=0}^{N-1} E(x(n)-Ex(n))^2} + \sum_{0\le n,m\le N-1} E(x(n)-Ex(n))(x(m)-Ex(m))\right\}$$

$$= \frac{1}{N^2}\{N c_x(0) + (N-1)(c_x(1)+c_x(-1)) + (N-2)(c_x(2)+c_x(-2)) + \cdots + c_x(N-1) + c_x(-(N-1))\}$$

$$= \frac{1}{N}\sum_{l=-N-1}^{N-1}\left(1-\frac{|l|}{N}\right)c_x(l)$$

$$= \frac{1}{N}\sum_{l=-N}^{N}\left(1-\frac{|l|}{N}\right)c_x(l)$$

（$c_x(N)$不存在，但 $1-\frac{|l|}{N}\ (l=N) = 0$）

② 由 $w(n)$ 为 Gauss 白噪声可得

$$p(\vec{x};\theta) = \frac{1}{(2\pi\delta^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2}\sum_{k=1}^{n}(x(k)-s(k,\theta))^2/\delta^2\right\}$$

$$\ln p(\vec{x};\theta) = -\frac{N}{2}\ln 2\pi\delta^2 - \frac{1}{2}\sum_{k=1}^{n}[x(k)-s(k,\theta)]^2/\delta^2$$

$$\cancel{\frac{\partial^2 \ln p(\vec{x};\theta)}{\partial\theta^2} = -\frac{1}{\delta^2}\sum_{k=1}^{n}\frac{\partial^2 s(k,\theta)}{\partial\theta^2}}$$

~~由CRLB可得，对θ无偏估计满足~~

$$\cancel{\mathrm{var}(\hat{\theta}) \ge -\frac{1}{E\left[\frac{\partial^2 \ln p(\vec{x};\theta)}{\partial\theta^2}\right]} = \frac{\delta^2}{\sum_{k=1}^{n}\frac{\partial^2 s(k,\theta)}{\partial\theta^2}}}$$

$$\frac{\partial \ln p(\vec{x};\theta)}{\partial\theta} = \frac{1}{\delta^2}\sum_{n=0}^{N-1}(x(n) - s(n,\theta))\frac{\partial s(n,\theta)}{\partial\theta}$$

$$\frac{\partial^2 \ln p(\vec{x};\theta)}{\partial\theta^2} = \frac{1}{\delta^2}\sum_{n=0}^{N-1}\left(x(n)\frac{\partial^2 s(n,\theta)}{\partial\theta^2} - \left(\frac{\partial s(n,\theta)}{\partial\theta}\right)^2 - s(n,\theta)\frac{\partial^2 s(n,\theta)}{\partial\theta^2}\right)$$

$$E\left[\frac{\partial^2 \ln p(\vec{x};\theta)}{\partial\theta^2}\right] = \frac{1}{\delta^2}\sum_{n=0}^{N-1}\left\{\cancel{s(n,\theta)\frac{\partial^2 s(n,\theta)}{\partial\theta^2}} - \left(\frac{\partial s(n,\theta)}{\partial\theta}\right)^2 - \cancel{s(n,\theta)\frac{\partial^2 s(n,\theta)}{\partial\theta^2}}\right\}$$

$$= \frac{-1}{\delta^2}\sum_{n=0}^{N-1}\left(\frac{\partial s(n,\theta)}{\partial\theta}\right)^2$$

$$\therefore \mathrm{var}(\hat{\theta}) \ge \frac{\delta^2}{\sum_{n=0}^{N-1}\left(\frac{\partial s(n,\theta)}{\partial\theta}\right)^2}.$$

③ $x(n) = A\cos(2\pi f_0 n + \varphi) + w(n)$

$$p(\vec{x};f_0) = \frac{1}{(2\pi\delta^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2}\sum_{n=0}^{N-1}(x(n) - A\cos(2\pi f_0 n+\varphi))^2/\delta^2\right\}$$

$$\ln p(\vec{x};f_0) = -\frac{N}{2}\ln(2\pi\delta^2) - \frac{1}{2}\sum_{n=0}^{N-1}(x(n) - A\cos(2\pi f_0 n+\varphi))^2/\delta^2$$

$$\cancel{\frac{\partial \ln p(\vec{x};f_0)}{\partial f_0} = \frac{-1}{\delta^2}\sum_{n=0}^{N-1}(x(n) - A\cos(2\pi f_0 n+\varphi))\cdot A\sin(2\pi f_0 n+\varphi)\cdot 2\pi n}$$

$$\cancel{\frac{\partial^2 \ln p(\vec{x};f_0)}{\partial f_0^2} = \frac{-2\pi n A}{\delta^2}}$$

由上题结论可知

$$\mathrm{var}(\hat{f}_0) \ge \frac{\delta^2}{\sum_{n=0}^{N-1}\left(\frac{\partial(A\cos(2\pi f_0 n+\varphi))}{\partial f_0}\right)^2} = \frac{\delta^2}{A^2\sum_{n=0}^{N-1}[2\pi n\sin(2\pi f_0 n+\varphi)]^2}$$

⑪ $p(\theta|\vec{x}) = p(\theta,\vec{x})/p(\vec{x}) = \frac{p(\vec{x}|\theta)p(\theta)}{p(\vec{x})}$.

$$= \frac{p(\vec{x}|\theta)p(\theta)}{\int p(\vec{x}|\theta)p(\theta)d\theta}$$

$$p(\theta) = \frac{1}{\theta_2-\theta_1} I\{\theta_1 \le \theta \le \theta_2\}$$

$$p(\vec{x}|\theta) = \frac{1}{(2\pi\delta_\omega^2)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2\delta_\omega^2}\sum_{n=0}^{N-1}(x(n)-\theta)^2\right\}$$

则 $\ln(p(\vec{x}|\theta)p(\theta)) = -\ln(\theta_2-\theta_1) - \frac{N}{2}\ln(2\pi\delta_\omega^2) + \frac{1}{2\delta_\omega^2}\sum_{n=0}^{N-1}(x(n)-\theta)^2$, $\theta_1 \le \theta \le$

$$\frac{\partial \ln(p(\vec{x}|\theta)p(\theta))}{\partial\theta} = \frac{1}{\delta_\omega^2}\sum_{n=0}^{N-1}(x(n)-\theta) = 0,$$

$$\Rightarrow \hat{\theta}_{MLE} = \begin{cases}\theta_1, & \bar{x} < \theta \\ \bar{x}, & \theta_1 \le \bar{x} \le \theta_2 \\ \theta_2, & \bar{x} > \theta_2\end{cases}.$$

⑦ $$p(\vec{x};\alpha) = \begin{cases}\frac{1}{\alpha^{2N}}\prod_{n=0}^{N-1} x(n)\, e^{-\frac{x^2(n)}{2\alpha^2}} & \text{for all } x(n) \ge 0. \\ 0 & \text{otherwise.}\end{cases}$$

$$\ln p(\vec{x};\alpha) = -2N\ln\alpha + \sum_{n=0}^{N-1}\ln x(n) - \frac{1}{2\alpha^2}\sum_{n=0}^{N-1}x^2(n)$$

$$\frac{\partial \ln p(\vec{x};\alpha)}{\partial\alpha} = -\frac{2N}{\alpha} + \frac{1}{\alpha^3}\sum_{n=0}^{N-1}x^2(n) = 0 \Rightarrow \hat{\alpha}_{MLE} = \pm\sqrt{\frac{1}{2N}\sum_{n=0}^{N-1}x^2(n)}$$

其实题目需要估计 α^2，所以 α 可以有正负2个值，实际上只需 $\hat{\alpha}^2_{MLE} = \frac{1}{2N}\sum x^2(n)$