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5.3 5.4 12 16 17

5.3 由(5.5)知 $T(x) = x^T C_s (C_s + \sigma^2 I)^{-1} x > \gamma''$ 判定 H_1

$$\hat{s} = C_s (C_s + \sigma^2 I)^{-1} x = \begin{bmatrix} \frac{\sigma_{s_0}^2}{\sigma_{s_0}^2 + \sigma^2} & & \\ & \ddots & \\ & & \frac{\sigma_{s_{N-1}}^2}{\sigma_{s_{N-1}}^2 + \sigma^2} \end{bmatrix} x.$$

$$T(x) = \sum_{n=0}^{N-1} \frac{\sigma_{s_n}^2}{\sigma_{s_n}^2 + \sigma^2} x^2[n]$$

$$\hat{s} = C_s (C_s + \sigma^2 I)^{-1} x$$

5.4 而 $s = A1$

$$C_s = E(ss^T) = E(A^2 1 1^T) = \sigma_A^2 1 1^T$$

由Woodbury恒等式有

$$(C_s + \sigma^2 I)^{-1} = (\sigma_A^2 1 1^T + \sigma^2 I)^{-1} = \frac{1}{\sigma^2} I - \frac{1}{\sigma^4} \frac{\sigma_A^2 1 1^T}{1 + \frac{\sigma_A^2}{\sigma^2} 1^T 1}$$

$$\hat{s} = \sigma_A^2 1 1^T \left(\frac{1}{\sigma^2} I - \frac{\sigma_A^2}{\sigma^2} \frac{1 1^T}{\sigma^2 + N\sigma_A^2} \right) x$$

$$= \frac{\sigma_A^2}{\sigma^2} N \bar{x} 1 - \frac{N\sigma_A^4/\sigma^2}{\frac{\sigma^2}{N} + N\sigma_A^2} N \bar{x} 1$$

$$= \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2/N} \bar{x} 1$$

则 $T(x) = x^T \hat{s} = \frac{N\sigma_A^2}{\sigma_A^2 + \sigma^2/N} (\bar{x})^2$

5.16

$$T(x) = \frac{c}{N} x^T H H^T x = x^T \left(\frac{c}{N} H H^T x \right) = x^T \hat{s}$$

$$H^T x = \begin{bmatrix} \sum_n x[n] \cos 2\pi f_0 n \\ \sum_n x[n] \sin 2\pi f_0 n \end{bmatrix}$$

$$\hat{s} = \frac{c}{N} \begin{bmatrix} \vdots & \vdots \\ \cos 2\pi f_0 n & \sin 2\pi f_0 n \end{bmatrix} \begin{bmatrix} \sum x[n] \cos 2\pi f_0 n \\ \sum x[n] \sin 2\pi f_0 n \end{bmatrix}$$

$$\hat{s}[n] = \frac{c}{N} \left[\cos 2\pi f_0 n \left(\sum x[n] \cos 2\pi f_0 n \right) + \sin 2\pi f_0 n \left(\sum x[n] \sin 2\pi f_0 n \right) \right]$$

$$= \hat{a} \cos 2\pi f_0 n + \hat{b} \sin 2\pi f_0 n.$$

其中

$$\hat{a} = \frac{\sigma_s^2}{\frac{N\sigma_s^2}{2} + \sigma^2} \sum_{n=0}^{N-1} x[n] \cos 2\pi f_0 n$$

$$\hat{b} = \frac{\sigma_s^2}{\frac{N\sigma_s^2}{2} + \sigma^2} \sum_{n=0}^{N-1} x[n] \sin 2\pi f_0 n.$$

5.12 $T(x) = x^T C_s (C_s + \sigma^2 I)^{-1} x$

$$s = Ah = A \begin{bmatrix} 1 \\ r \\ r^{N-1} \end{bmatrix}$$

$$C_s = E(A h A h^T) = \sigma_A^2 h h^T$$

$$(C_s + \sigma^2 I)^{-1} = (\sigma^2 I + \sigma_A^2 h h^T)^{-1} = \frac{I}{\sigma^2} - \frac{1}{\sigma^4} \frac{\sigma_A^2 h h^T}{1 + h^T h \frac{\sigma_A^2}{\sigma^2}}$$

$$T(x) = x^T \sigma_A^2 h h^T \left(\frac{1}{\sigma^2} I - \frac{\sigma_A^2/\sigma^4 \, h^T h}{1 + h^T h \frac{\sigma_A^2}{\sigma^2}} \right) x$$

$$= (h^T x)^2 \left[\frac{\sigma_A^2}{\sigma^2} - \frac{\sigma_A^2/\sigma^4 \, h^T h}{1 + h^T h \, \sigma_A^2/\sigma^2} \right]$$

$$= \frac{\sigma_A^2}{\sigma_A^2 h^T h + \sigma^2} (h^T x)^2 > \gamma'$$

或 $T(x) = (h^T x)^2 > \gamma''$ · decide H_1

5.17

则 $E(T; H_0) = 0$

$$E(T; H_1) = \sum_n s[n] A \cos 2\pi f_0 n$$

$$= A^2 \sum_n \cos(2\pi f_0 n + \phi) \cos 2\pi f_0 n.$$

$$= \frac{A^2}{2} \sum_n \left(\cos\phi + \cos(4\pi f_0 n + \phi) \right).$$

$$\approx \frac{NA^2}{2} \cos\phi$$

$$\mathrm{var}(T; H_0) = E\left[\left(\sum_n w[n] A \cos 2\pi f_0 n \right)^2 \right]$$

$$= A^2 \sum_{0 \le m, n \le N-1} E(w[m] w[n]) (\cos 2\pi f_0 m)(\cos 2\pi f_0 n)$$

$$= A^2 \sum_n \sigma^2 \cos^2 2\pi f_0 n$$

$$= \frac{A^2 \sigma^2}{2} \sum_n (1 + \cos 4\pi f_0 n)$$

$$\approx \frac{NA^2 \sigma^2}{2} = \mathrm{var}(T; H_1)$$

$$d^2 = \frac{(NA^2/2 \cos\phi)^2}{NA^2 \sigma^2/2} = \frac{NA^2}{2\sigma^2} \cos^2\phi$$

若 $\phi = 0$ 则 $d^2 = \frac{NA^2}{2\sigma^2}$

若 $\phi \to 90^\circ$, 则 $d^2 \to 0$ 性能变差.