统计信号处理

第九章

线性贝叶斯估计

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一、引言

- MMSE估计： $\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}|\boldsymbol{x}) = \begin{bmatrix} \int \theta_1 p(\boldsymbol{\theta}|\boldsymbol{x}) d\boldsymbol{\theta} \\ \int \theta_2 p(\boldsymbol{\theta}|\boldsymbol{x}) d\boldsymbol{\theta} \\ \vdots \\ \int \theta_p p(\boldsymbol{\theta}|\boldsymbol{x}) d\boldsymbol{\theta} \end{bmatrix}$

多重积分

1.不一定好求解！

2.无pdf?

- MAP估计： $\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \{ p(\boldsymbol{\theta}|\boldsymbol{x}) \}$

多维最大值

- 线性MMSE (LMMSE)估计

二、LMMSE估计

- 观察数据 $\{x[0], x[1], x[2], \ldots, x[N-1]\}$
- LMMSE意味着：

$$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] + a_N$$

即限定估计量与观察数据间呈**线性关系**，然后使

$$\mathrm{Bmse}(\hat{\theta}) = E\left[\left(\theta - \hat{\theta}\right)^2\right]$$ **最小化**

即，LMMSE:

$$\begin{cases} \min\left\{E\left[\left(\theta - \hat{\theta}\right)^2\right]\right\} \\ s.t.\quad \hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] + a_N \end{cases}$$

LMMSE:

$$\begin{cases} \min\left\{E\left[\left(\theta-\hat{\theta}\right)^2\right]\right\} \\ s.t.\quad \hat{\theta}=\sum_{n=0}^{N-1}a_n x[n]+a_N \end{cases}$$

$$\left.\begin{aligned} &\mathrm{Bmse}\left(\hat{\theta}\right)=E\left[\left(\theta-\hat{\theta}\right)^2\right] \\ &\hat{\theta}=\sum_{n=0}^{N-1}a_n x[n]+a_N \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &\mathrm{Bmse}\left(\hat{\theta}\right)=E\left[\left(\theta-\sum_{n=0}^{N-1}a_n x[n]-a_N\right)^2\right] \\ &\frac{\partial \mathrm{Bmse}\left(\hat{\theta}\right)}{\partial a_N}=0 \end{aligned}\right\} \Rightarrow$$

$$E\left[\left(\theta-\sum_{n=0}^{N-1}a_n x[n]-a_N\right)\right]=0 \Rightarrow a_N=E(\theta)-\sum_{n=0}^{N-1}a_n E\left(x[n]\right)$$

$$a_N = E(\theta) - \sum_{n=0}^{N-1} a_n E(x[n])$$

$$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] + a_N$$

$$\hat{\theta} = E(\theta) + \sum_{n=0}^{N-1} a_n \left(x[n] - E(x[n])\right)$$

$$\mathrm{Bmse}(\hat{\theta}) = E\left[(\theta - \hat{\theta})^2\right]$$

$$\mathrm{Bmse}(\hat{\theta}) = E\left[\left(\theta - E(\theta) - \sum_{n=0}^{N-1} a_n \left(x[n] - E(x[n])\right)\right)^2\right]$$

$$= E\left[\left(\underline{\sum_{n=0}^{N-1} a_n \left(x[n] - E(x[n])\right)} - (\theta - E(\theta))\right)^2\right]$$

$$\boldsymbol{a} = [a_0, a_1, a_2, \ldots a_{N-1}]^T$$

$$= E\left[\left(\boldsymbol{a}^T (\boldsymbol{x} - E(\boldsymbol{x})) - (\theta - E(\theta))\right)^2\right]$$

$$= E\left[\left(\boldsymbol{a}^T (\boldsymbol{x} - E(\boldsymbol{x})) - (\theta - E(\theta))\right)\left(\boldsymbol{a}^T (\boldsymbol{x} - E(\boldsymbol{x})) - (\theta - E(\theta))\right)^T\right]$$

$$\mathrm{Bmse}\left(\hat{\theta}\right)=E\left[\left(\boldsymbol{a}^T\left(\boldsymbol{x}-E(\boldsymbol{x})\right)-\left(\theta-E(\theta)\right)\right)\left(\boldsymbol{a}^T\left(\boldsymbol{x}-E(\boldsymbol{x})\right)-\left(\theta-E(\theta)\right)\right)^T\right]$$

$$=E\begin{bmatrix}\boldsymbol{a}^T\left(\boldsymbol{x}-E(\boldsymbol{x})\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^T\boldsymbol{a}-\boldsymbol{a}^T\left(\boldsymbol{x}-E(\boldsymbol{x})\right)\left(\theta-E(\theta)\right)\\-\left(\theta-E(\theta)\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^T\boldsymbol{a}+\left(\theta-E(\theta)\right)^2\end{bmatrix}$$

$$\boxed{\mathbf{C}_{\theta x}^T=\mathbf{C}_{x\theta}}$$

$$\left.\begin{aligned}&=\boldsymbol{a}^T\mathbf{C}_{xx}\boldsymbol{a}-\boldsymbol{a}^T\mathbf{C}_{x\theta}-\mathbf{C}_{\theta x}\boldsymbol{a}+\mathbf{C}_{\theta\theta}\\&\frac{\partial\mathrm{Bmse}\left(\hat{\theta}\right)}{\partial\boldsymbol{a}}=0\end{aligned}\right\}\Rightarrow 2\mathbf{C}_{xx}\boldsymbol{a}-2\mathbf{C}_{x\theta}=0\Rightarrow\boldsymbol{a}=\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

$$\left.\begin{aligned}&\hat{\theta}=\sum_{n=0}^{N-1}a_n x[n]+a_N=\boldsymbol{a}^T\boldsymbol{x}+a_N\\&a_N=E(\theta)-\sum_{n=0}^{N-1}a_n E\left(x[n]\right)=E(\theta)-\boldsymbol{a}^T E(\boldsymbol{x})\end{aligned}\right\}\Rightarrow\hat{\theta}=E(\theta)+\boldsymbol{a}^T\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$$

$$\Rightarrow\left.\begin{aligned}&\underline{\hat{\theta}=E(\theta)+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)}\\&\mathrm{Bmse}\left(\hat{\theta}\right)=E\left[\left(\theta-\hat{\theta}\right)^2\right]\end{aligned}\right\}\Rightarrow\underline{\mathrm{Bmse}\left(\hat{\theta}\right)=\mathbf{C}_{\theta\theta}-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}}$$

✓ 仅需一阶矩、二阶矩

✓ 无需PDF

● LMMSE Vs MMSE

LMMSE

LMMSE优化目标:
$$\begin{cases} \min E\left[\left(\theta-\hat{\theta}\right)^2\right] \\ s.t.\quad \hat{\theta}=\sum_{n=0}^{N-1} a_n x[n]+a_N \end{cases}$$

相应的LMMSE估计量及Bmse:
$$\begin{cases} \hat{\theta}=E(\theta)+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right) \\ \mathrm{Bmse}\left(\hat{\theta}\right)=\mathbf{C}_{\theta\theta}-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta} \end{cases}$$

- ✓ 附加了线性约束
- ✓ 仅需一阶矩和二阶矩
- ✓ 可得显示解——好求
- ✓ 仅在“线性”中最优

MMSE

MMSE优化目标：$\min\left\{E\left(\left(\theta-\hat{\theta}\right)^2\right)\right\}$

相应的MMSE估计量及Bmse:
$$\begin{cases} \hat{\theta}=E(\theta\,|\,\boldsymbol{x}) \\ \mathrm{Bmse}\left(\hat{\theta}\right)=E\left(\left(\theta-\hat{\theta}\right)^2\right) \end{cases}$$

- ✓ 无附加约束
- ✓ 需PDF
- ✓ 可能难以求得显示解
- ✓ 全局最优

贝叶斯一般线性模型的MMSE

若观测数据 $\mathbf{x}$ 满足如下数学模型：

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

其中，$\mathbf{H}$ 为观测矩阵，$\boldsymbol{\theta}$ 为待估计参数，具有先验概率密度函数 $N(\boldsymbol{\mu}_\theta, \mathbf{C}_\theta)$，噪声矢量 $\mathbf{w}$ 服从 $N(\mathbf{0}, \mathbf{C}_w)$，且与 $\boldsymbol{\theta}$ 无关，则MMSE为:

$$\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}|\mathbf{x}) = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}(\mathbf{x} - E(\mathbf{x}))$$

$$\mathbf{C}_{\theta|x} = \mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

Vs

LMMSE

$$\hat{\theta} = E(\theta) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}(\mathbf{x} - E(\mathbf{x}))$$

$$\mathrm{Bmse}(\hat{\theta}) = \mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

对贝叶斯一般线性模型，LMMSE与MMSE具有相同形式

例：白噪声中电平估计问题：

$$x[n]=A+w[n],\ n=0,1,\ldots,N-1$$

待估计参数为信号幅度 A，且 $A\sim U\left[-A_0,A_0\right]$，$w[n]$ 为高斯白噪声 $w[n]\sim N\left(0,\sigma^2\right)$，且和 A 相互独立。试比较其MMSE与LMMSE。

1. MMSE估计量

$$\hat{A}=\int A\underline{p\left(A\,|\,\mathbf{x}\right)}dA$$

$$p\left(A\,|\,\mathbf{x}\right)=\frac{p\left(\mathbf{x}\,|\,A\right)p\left(A\right)}{\int p\left(\mathbf{x}\,|\,A\right)p\left(A\right)dA}$$

$$p_x\left(x[n]\,\|\,A\right)=p_w\left(x[n]-A\,|\,A\right)$$

$$=p_w\left(x[n]-A\right)$$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}\left(x[n]-A\right)^2\right\}$$

$$p(\mathbf{x}\mid A)=\frac{1}{\left(2\pi\sigma^2\right)^{N/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A\right)^2\right\}$$

$$p(A)=\frac{1}{2A_0},\quad |A|\le A_0$$

$$p(A\mid\mathbf{x})=\frac{p(\mathbf{x}\mid A)\,p(A)}{\int p(\mathbf{x}\mid A)\,p(A)\,dA}$$

$$p(A\mid\mathbf{x})=\begin{cases}\dfrac{\dfrac{1}{2A_0\left(2\pi\sigma^2\right)^{N/2}}\exp\left\{-\dfrac{1}{2\sigma^2}\displaystyle\sum_{n=0}^{N-1}\left(x[n]-A\right)^2\right\}}{\displaystyle\int_{-A_0}^{A_0}\dfrac{1}{2A_0\left(2\pi\sigma^2\right)^{N/2}}\exp\left\{-\dfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A\right)^2\right\}dA}, & |A|\le A_0\\ 0, & |A|>A_0\end{cases}$$

$$=\begin{cases}\dfrac{\exp\left\{-\dfrac{1}{2\sigma^2}N\left(A-\bar{x}\right)^2\right\}}{\displaystyle\int_{-A_0}^{A_0}\exp\left\{-\dfrac{1}{2\sigma^2}N\left(A-\bar{x}\right)^2\right\}dA}, & |A|\le A_0\\ 0, & |A|>A_0\end{cases}$$

$$p(A|\mathbf{x})=\begin{cases}\dfrac{\exp\left\{-\dfrac{1}{2\dfrac{\sigma^2}{N}}(A-\bar{x})^2\right\}}{\int_{-A_0}^{A_0}\exp\left\{-\dfrac{1}{2\dfrac{\sigma^2}{N}}(A-\bar{x})^2\right\}dA}, & |A|\le A_0\\ 0, & |A|>A_0\end{cases}=\begin{cases}\dfrac{\dfrac{1}{\sqrt{2\pi\dfrac{\sigma^2}{N}}}\exp\left\{-\dfrac{1}{2\dfrac{\sigma^2}{N}}(A-\bar{x})^2\right\}}{\int_{-A_0}^{A_0}\dfrac{1}{\sqrt{2\pi\dfrac{\sigma^2}{N}}}\exp\left\{-\dfrac{1}{2\dfrac{\sigma^2}{N}}(A-\bar{x})^2\right\}dA}, & |A|\le A_0\\ 0, & |A|>A_0\end{cases}$$

MMSE: $\hat{A}=E(A|\mathbf{x})=\int Ap(A|\mathbf{x})dA$

$$=\frac{\int_{-A_0}^{A_0}A\dfrac{1}{\sqrt{2\pi\dfrac{\sigma^2}{N}}}\exp\left\{-\dfrac{1}{2\dfrac{\sigma^2}{N}}\sum_{n=0}^{N-1}(A-\bar{x})^2\right\}dA}{\int_{-A_0}^{A_0}\dfrac{1}{\sqrt{2\pi\dfrac{\sigma^2}{N}}}\exp\left\{-\dfrac{1}{2\dfrac{\sigma^2}{N}}\sum_{n=0}^{N-1}(A-\bar{x})^2\right\}dA}$$

——化简困难!

2. LMMSE估计量

$$\hat{A} = E(A) + \mathbf{C}_{Ax}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x} - E(\boldsymbol{x})\right)$$

其中， $E(A) = 0$

$$\mathbf{C}_{Ax} = E\left(\left(A - E(A)\right)\left(\boldsymbol{x} - E(\boldsymbol{x})\right)^T\right) = E\left(A\left(A\mathbf{1} + \boldsymbol{w}\right)^T\right) = E\left(A^2\right)\mathbf{1}^T = \sigma_A^2\mathbf{1}^T$$

$$\mathbf{C}_{xx} = E\left(\left(\boldsymbol{x} - E(\boldsymbol{x})\right)\left(\boldsymbol{x} - E(\boldsymbol{x})\right)^T\right) = E\left(\left(A\mathbf{1} + \boldsymbol{w}\right)\left(A\mathbf{1} + \boldsymbol{w}\right)^T\right) = \sigma_A^2\mathbf{1}\mathbf{1}^T + \sigma^2\mathbf{I}$$

$$\hat{A} = \sigma_A^2\mathbf{1}^T\left(\sigma_A^2\mathbf{1}\mathbf{1}^T + \sigma^2\mathbf{I}\right)^{-1}\boldsymbol{x}$$

$$= \frac{\sigma_A^2}{\sigma^2}\mathbf{1}^T\left(\frac{\sigma_A^2}{\sigma^2}\mathbf{1}\mathbf{1}^T + \mathbf{I}\right)^{-1}\boldsymbol{x}$$

Woodbury恒等式：

$$\left(\mathbf{B} + \boldsymbol{u}\boldsymbol{u}^T\right)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\boldsymbol{u}\boldsymbol{u}^T\mathbf{B}^{-1}}{1 + \boldsymbol{u}^T\mathbf{B}^{-1}\boldsymbol{u}}$$

$$\hat{A} = \frac{\sigma_A^2}{\sigma^2}\mathbf{1}^T\left(\mathbf{I} - \frac{\frac{\sigma_A^2}{\sigma^2}\mathbf{1}\mathbf{1}^T}{1 + \frac{\sigma_A^2}{\sigma^2}N}\right)\boldsymbol{x}$$

$$= \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}}\bar{x}$$

——显示解

- ✓ 比MMSE更好求解！
- ✓ 但，是准最佳的！

几何解释

假定θ和$x[n]$是零均值的

矢量"长度"：$\|x\|=\sqrt{E\left(x^2\right)}$

线性约束：$\hat{\theta}=\sum_{n=0}^{N-1}a_n x[n]$

$$\mathrm{Bmse}\left(\hat{\theta}\right)=E\left(\left(\theta-\hat{\theta}\right)^2\right)$$

$$\mathrm{Bmse}\left(\hat{\theta}\right)=E\left(\left(\theta-\sum_{n=0}^{N-1}a_n x[n]\right)^2\right)$$

$$\mathrm{Bmse}\left(\hat{\theta}\right)=\left\|\theta-\sum_{n=0}^{N-1}a_n x[n]\right\|^2$$

——误差矢量长度的平方

误差：$\varepsilon=\theta-\hat{\theta}$

优化目标：$\min\left\{\mathrm{Bmse}\left(\hat{\theta}\right)\right\}$ $\Leftrightarrow$ 误差矢量长度的平方最小化

$\varepsilon\perp\left\{x[0],x[1],x[2],\ldots,x[N-1]\right\}$ $\Rightarrow$ $E\left(\left(\theta-\hat{\theta}\right)x[m]\right)=0$ **正交原理**

$$\left.\begin{aligned} E\left(\left(\theta-\hat{\theta}\right)x[m]\right)=0 \\ \hat{\theta}=\sum_{n=0}^{N-1}a_n x[n] \end{aligned}\right\} \Rightarrow E\left(\left(\theta-\sum_{n=0}^{N-1}a_n x[n]\right)x[m]\right)=0 \Rightarrow$$

$$E\left(\theta x[m]\right)=\sum_{n=0}^{N-1}a_n E\left(x[m]x[n]\right) \Rightarrow$$

$$\underbrace{\begin{bmatrix} E\left(\theta x[0]\right) \\ E\left(\theta x[1]\right) \\ \vdots \\ E\left(\theta x[N-1]\right) \end{bmatrix}}_{\mathbf{C}_{x\theta}} = \underbrace{\begin{bmatrix} E\left(x[0]x[0]\right) & E\left(x[0]x[1]\right) & \cdots & E\left(x[0]x[N-1]\right) \\ E\left(x[1]x[0]\right) & E\left(x[1]x[1]\right) & \cdots & E\left(x[1]x[N-1]\right) \\ \vdots & \vdots & \ddots & \vdots \\ E\left(x[N-1]x[0]\right) & E\left(x[N-1]x[1]\right) & \cdots & E\left(x[N-1]x[N-1]\right) \end{bmatrix}}_{\mathbf{C}_{xx}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}}_{\boldsymbol{a}}$$

$$\Rightarrow \mathbf{C}_{x\theta}=\mathbf{C}_{xx}\boldsymbol{a} \Rightarrow \boldsymbol{a}=\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

$$\left.\begin{aligned} \boldsymbol{a} &= \mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta} \\ \hat{\theta} &= \sum_{n=0}^{N-1} a_n x[n] \;= \boldsymbol{a}^T \boldsymbol{x} \end{aligned}\right\} \Rightarrow \hat{\theta} = \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\boldsymbol{x}$$

➢ 相应的贝叶斯均方误差：

$$\mathrm{Bmse}\left(\hat{\theta}\right) = E\left(\left(\theta-\hat{\theta}\right)^2\right) = E\left(\left(\theta-\sum_{n=0}^{N-1}a_n x[n]\right)^2\right)$$

$$= E\left(\left(\theta-\sum_{n=0}^{N-1}a_n x[n]\right)\left(\theta-\sum_{n=0}^{N-1}a_n x[n]\right)\right)$$

$$= E\left(\underbrace{\theta^2}_{\mathbf{C}_{\theta\theta}} - \sum_{n=0}^{N-1}a_n \underbrace{x[n]\theta}_{\mathbf{C}_{x\theta}} - \underbrace{\sum_{n=0}^{N-1}a_n x[n]\left(\theta-\sum_{n=0}^{N-1}a_n x[n]\right)}_{0}\right)$$

$$= \mathbf{C}_{\theta\theta} - \boldsymbol{a}^T\mathbf{C}_{x\theta}$$

$$= \mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

推广至矢量参数情况

待估计参数$\boldsymbol{\theta}=\left[\theta_1,\theta_2,\ldots,\theta_p\right]^T$，其每个参数的LMMSE定义为：

$$\begin{cases}\min E\left[\left(\theta_i-\hat{\theta}_i\right)^2\right]\\ s.t.\ \ \hat{\theta}_i=\sum_{n=0}^{N-1}a_{in}x[n]+a_{iN}\end{cases}$$

标量参数LMMSE：

$$\hat{\theta}=E(\theta)+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$$

$$\text{Bmse}\left(\hat{\theta}\right)=C_{\theta\theta}-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

$$\Rightarrow\quad \hat{\boldsymbol{\theta}}=\begin{bmatrix}E(\theta_1)\\E(\theta_2)\\\vdots\\E(\theta_p)\end{bmatrix}+\begin{bmatrix}\mathbf{C}_{\theta_1 x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)\\\mathbf{C}_{\theta_2 x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)\\\vdots\\\mathbf{C}_{\theta_p x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)\end{bmatrix}$$

$$=E(\boldsymbol{\theta})+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$$

三、LMMSE性能评估及性质

1. LMMSE性能评估

LMMSE估计量: $\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x} - E(\mathbf{x})\right)$

估计量误差: $\boldsymbol{\varepsilon} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$

——利用误差来进行性能评估

$$\boldsymbol{\varepsilon} = \boldsymbol{\theta} - E(\boldsymbol{\theta}) - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x} - E(\mathbf{x})\right)$$

- 误差均值

$$\begin{aligned}
E(\boldsymbol{\varepsilon}) &= E\left\{\boldsymbol{\theta} - E(\boldsymbol{\theta}) - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x} - E(\mathbf{x})\right)\right\} \\
&= E_{x,\theta}\left\{\boldsymbol{\theta} - E(\boldsymbol{\theta}) - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x} - E(\mathbf{x})\right)\right\} \\
&= E_{x,\theta}\left(\boldsymbol{\theta} - E(\boldsymbol{\theta})\right) - E_{x,\theta}\left(\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x} - E(\mathbf{x})\right)\right) \\
&= \mathbf{0}
\end{aligned}$$

即LMMSE估计量的误差是零均值的!

- 误差协方差矩阵：

其对角线元素为: $\left[E_{x,\theta}\left(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right)\right]_{ii}=E_{x,\theta}\left(\left(\theta_i-\hat{\theta}_i\right)\left(\theta_i-\hat{\theta}_i\right)^T\right)$

$$\left.\begin{aligned}&E_{x,\theta}\left(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right)=E_{x,\theta}\left(\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)^T\right)\\&\hat{\boldsymbol{\theta}}=E(\boldsymbol{\theta})+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\end{aligned}\right\}\Rightarrow$$

$$E_{x,\theta}\left(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right)=E\left(\left(\underline{\boldsymbol{\theta}-E(\boldsymbol{\theta})}-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\right)\left(\underline{\boldsymbol{\theta}-E(\boldsymbol{\theta})}-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\right)^T\right)$$

$$=E\begin{pmatrix}\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)^T-\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)\left(\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\right)^T\\-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)^T+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\left(\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\mathbf{x}-E(\mathbf{x})\right)\right)^T\end{pmatrix}$$

$$=\mathbf{C}_{\theta\theta}-E\left\{\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)\left(\mathbf{x}-E(\mathbf{x})\right)^T\right\}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

$$-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}E\left(\left(\mathbf{x}-E(\mathbf{x})\right)\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)^T\right)+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}E\left(\left(\mathbf{x}-E(\mathbf{x})\right)\left(\mathbf{x}-E(\mathbf{x})\right)^T\right)\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

$$=\mathbf{C}_{\theta\theta}-\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta}$$

- ✓ 其对角线元素表示相应待估计参数的**最小贝叶斯均方误差**
- ✓ 也称为**贝叶斯均方误差矩阵**，并常记: $\mathbf{M}_{\hat{\theta}}$

2. LMMSE的性质

（1）线性变换不变性

若 $\boldsymbol{\alpha}=\mathbf{A}\boldsymbol{\theta}+\boldsymbol{b}$，$\boldsymbol{\theta}$ 的LMMSE估计是 $\hat{\boldsymbol{\theta}}$，则 $\boldsymbol{\alpha}$ 的LMMSE估计为：

$$\hat{\boldsymbol{\alpha}}=\mathbf{A}\hat{\boldsymbol{\theta}}+\boldsymbol{b}$$

证明：

按定义，有 $\hat{\boldsymbol{\alpha}}=E(\boldsymbol{\alpha})+\mathbf{C}_{\alpha x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$

$$E(\boldsymbol{\alpha})=E(\mathbf{A}\boldsymbol{\theta}+\boldsymbol{b})=\mathbf{A}E(\boldsymbol{\theta})+\boldsymbol{b}$$

$$\mathbf{C}_{\alpha x}=E\left(\left(\boldsymbol{\alpha}-E(\boldsymbol{\alpha})\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^{T}\right)=E\left(\mathbf{A}\left(\boldsymbol{\theta}-E(\boldsymbol{\theta})\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^{T}\right)=\mathbf{A}\mathbf{C}_{\theta x}$$

$$\Rightarrow\quad \hat{\boldsymbol{\alpha}}=\mathbf{A}E(\boldsymbol{\theta})+\boldsymbol{b}+\mathbf{A}\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$$

$$=\mathbf{A}\left(E(\boldsymbol{\theta})+\mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)\right)+\boldsymbol{b}$$

$$=\mathbf{A}\hat{\boldsymbol{\theta}}+\boldsymbol{b}$$ 得证！

(2) 可加性

若 $\boldsymbol{\alpha}=\boldsymbol{\theta}_1+\boldsymbol{\theta}_2$，$\boldsymbol{\theta}_1$和 $\boldsymbol{\theta}_2$的LMMSE估计分别是$\hat{\boldsymbol{\theta}}_1$、$\hat{\boldsymbol{\theta}}_2$，则$\boldsymbol{\alpha}$ 的LMMSE估计为：

$$\hat{\boldsymbol{\alpha}}=\hat{\boldsymbol{\theta}}_1+\hat{\boldsymbol{\theta}}_2$$

证明：

按定义，有 $\hat{\boldsymbol{\alpha}}=E(\boldsymbol{\alpha})+\mathbf{C}_{\alpha x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$

$\boldsymbol{\alpha}=\boldsymbol{\theta}_1+\boldsymbol{\theta}_2 \Rightarrow E(\boldsymbol{\alpha})=E(\boldsymbol{\theta}_1)+E(\boldsymbol{\theta}_2)$

$$\begin{aligned}\mathbf{C}_{\alpha x}&=E\left(\left(\boldsymbol{\alpha}-E(\boldsymbol{\alpha})\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^T\right)\\&=E\left(\left(\boldsymbol{\theta}_1+\boldsymbol{\theta}_2-E(\boldsymbol{\theta}_1+\boldsymbol{\theta}_2)\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^T\right)\\&=E\left(\left(\left(\boldsymbol{\theta}_1-E(\boldsymbol{\theta}_1)\right)+\left(\boldsymbol{\theta}_2-E(\boldsymbol{\theta}_2)\right)\right)\left(\boldsymbol{x}-E(\boldsymbol{x})\right)^T\right)\\&=\mathbf{C}_{\theta_1 x}+\mathbf{C}_{\theta_2 x}\end{aligned}$$

$$\hat{\boldsymbol{\alpha}}=E(\boldsymbol{\theta}_1)+E(\boldsymbol{\theta}_2)+\left(\mathbf{C}_{\theta_1 x}+\mathbf{C}_{\theta_2 x}\right)\mathrm{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)$$

$$=E(\boldsymbol{\theta}_1)+\mathbf{C}_{\theta_1 x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)+E(\boldsymbol{\theta}_2)+\mathbf{C}_{\theta_2 x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x}-E(\boldsymbol{x})\right)=\hat{\boldsymbol{\theta}}_1+\hat{\boldsymbol{\theta}}_2$$ 得证！

（3）贝叶斯高斯-马尔可夫定理

如果数据具有贝叶斯一般线性模型的**形式**，即

$$\boldsymbol{x} = \mathbf{H}\boldsymbol{\theta} + \boldsymbol{w}$$

其中 $\mathbf{H}$ 是已知的 $N\times p$ 观测矩阵；$\boldsymbol{\theta}$ 是 $p\times 1$ 的随机矢量，其均值为 $E(\boldsymbol{\theta})$，协方差矩阵为 $\mathbf{C}_{\theta\theta}$，其现实待估计；$\boldsymbol{w}$ 是 $N\times 1$ 的随机矢量，其均值为零、协方差为 $\mathbf{C}_w$，且与 $\boldsymbol{\theta}$ 不相关（联合PDF $p(\boldsymbol{x};\boldsymbol{\theta})$ 是**任意的**）。那么，$\boldsymbol{\theta}$ 的**LMMSE估计量**是

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x} - E(\boldsymbol{x})\right) \\ &= E(\boldsymbol{\theta}) + \mathbf{C}_{\theta\theta}\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_{\theta\theta}\mathbf{H}^T + \mathbf{C}_w\right)^{-1}\left(\boldsymbol{x} - \mathbf{H}E(\boldsymbol{\theta})\right) \\ &= E(\boldsymbol{\theta}) + \left(\mathbf{C}_{\theta\theta}^{-1} + \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_w^{-1}\left(\boldsymbol{x} - \mathbf{H}E(\boldsymbol{\theta})\right)\end{aligned}$$

误差 $\varepsilon = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ 的均值为零、协方差矩阵为

若为高斯分布，则LMMSE为MMSE

$$\begin{aligned}\mathbf{C}_\varepsilon &= E\left(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right) \\ &= \mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{C}_{x\theta} \\ &= \mathbf{C}_{\theta\theta} - \mathbf{C}_{\theta\theta}\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_{\theta\theta}\mathbf{H}^T + \mathbf{C}_w\right)^{-1}\mathbf{H}\mathbf{C}_{\theta\theta} \\ &= \left(\mathbf{C}_{\theta\theta}^{-1} + \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{H}\right)^{-1}\end{aligned}$$

$$\mathrm{Bmse}\left(\hat{\theta}_i\right) = \left[\mathbf{C}_\varepsilon\right]_{ii} = \left[\mathbf{M}_{\hat{\theta}}\right]_{ii}$$

四、序贯LMMSE估计

例：白噪声中电平估计问题：

$$x[n] = A + w[n],\ n = 0,1,\dots,N-1$$

待估计参数为信号幅度 A，其服从 $N\left(0,\sigma_A^2\right)$，$w[n]$ 为高斯白噪声，且其方差为 σ^2，即 $w[n] \sim N\left(0,\sigma^2\right)$，且与幅度不相关。$A$ 的LMMSE估计量为？

$$\hat{\theta} = E(\theta) + \left(\mathbf{C}_{\theta\theta}^{-1} + \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_w^{-1}\left(\mathbf{x} - \mathbf{H}E(\theta)\right)$$

$$\mathrm{Bmse}\left(\hat{\theta}\right) = \left(\mathbf{C}_{\theta\theta}^{-1} + \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{H}\right)^{-1}$$

$$E(\theta) = 0 \qquad \mathbf{C}_{\theta\theta} = \sigma_A^2$$

$$\mathbf{H} = [1,1,1,\dots,1]^T \qquad \mathbf{C}_w = \sigma^2\mathbf{I}$$

$$\Rightarrow \begin{cases} \hat{A} = \dfrac{\sigma_A^2}{\sigma_A^2 + \dfrac{\sigma^2}{N}}\bar{x} \\[2ex] \mathrm{Bmse}\left(\hat{A}\right) = \dfrac{\sigma_A^2\sigma^2}{N\sigma_A^2 + \sigma^2} \end{cases}$$

LMMSE估计量：

$$\hat{A}=\frac{\sigma_A^2}{\sigma_A^2+\frac{\sigma^2}{N}}\bar{x}$$

重记为：

$$\hat{A}[N-1]=\frac{\sigma_A^2}{\sigma_A^2+\frac{\sigma^2}{N}}\frac{1}{N}\sum_{n=0}^{N-1}x[n]$$

$$\hat{A}[N]=\frac{\sigma_A^2}{\sigma_A^2+\frac{\sigma^2}{N+1}}\frac{1}{N+1}\sum_{n=0}^{N}x[n]$$

$$=\frac{N\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\frac{1}{N}\left(\sum_{n=0}^{N-1}x[n]+x[N]\right)$$

$$=\frac{N\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\frac{1}{N}\sum_{n=0}^{N-1}x[n]+\frac{N\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\frac{1}{N}x[N]$$

$$=\frac{N\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\frac{\sigma_A^2+\frac{\sigma^2}{N}}{\sigma_A^2}\hat{A}[N-1]+\frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}x[N]$$

$$=\frac{N\sigma_A^2+\sigma^2}{(N+1)\sigma_A^2+\sigma^2}\hat{A}[N-1]+\frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}x[N]$$

$$=\hat{A}[N-1]+\underbrace{\frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}}_{K[N]}\left(x[N]-\hat{A}[N-1]\right)$$

$K[N]$ 增益因子

$$\frac{\frac{1}{\sigma^2}}{\frac{1}{\mathrm{Bmse}\left(\hat{A}[N-1]\right)}+\frac{1}{\sigma^2}}=\frac{\mathrm{Bmse}\left(\hat{A}[N-1]\right)}{\mathrm{Bmse}\left(\hat{A}[N-1]\right)+\sigma^2}$$

$$=\frac{\frac{\sigma_A^2\sigma^2}{N\sigma_A^2+\sigma^2}}{\frac{\sigma_A^2\sigma^2}{N\sigma_A^2+\sigma^2}+\sigma^2}=\frac{\sigma_A^2}{\left(N\sigma_A^2+\sigma^2\right)+\sigma_A^2}=K[N]$$

最小贝叶斯MSE：

$$\mathrm{Bmse}\left(\hat{A}\right)=\frac{\sigma_A^2\sigma^2}{N\sigma_A^2+\sigma^2}$$

重记为：

$$\mathrm{Bmse}\left(\hat{A}[N-1]\right)=\frac{\sigma_A^2\sigma^2}{N\sigma_A^2+\sigma^2}$$

$$\mathrm{Bmse}\left(\hat{A}[N]\right)=\frac{\sigma_A^2\sigma^2}{(N+1)\sigma_A^2+\sigma^2}$$

$$=\frac{N\sigma_A^2+\sigma^2}{(N+1)\sigma_A^2+\sigma^2}\frac{\sigma_A^2\sigma^2}{N\sigma_A^2+\sigma^2}$$

$$=\left(1-\frac{\sigma_A^2}{(N+1)\sigma_A^2+\sigma^2}\right)\frac{\sigma_A^2\sigma^2}{N\sigma_A^2+\sigma^2}$$

$$=\left(1-K[N]\right)\mathrm{Bmse}\left(\hat{A}[N-1]\right)$$

序贯计算方法

初始化：

$$\hat{A}[-1]=E(A)$$

$$\mathrm{Bmse}\left(\hat{A}[-1]\right)=\mathrm{var}(A)$$

增益因子：

$$K[N]=\frac{\mathrm{Bmse}\left(\hat{A}[N-1]\right)}{\mathrm{Bmse}\left(\hat{A}[N-1]\right)+\sigma^2}$$

估计量更新：

$$\hat{A}[N]=\hat{A}[N-1]+K[N]\left(x[N]-\hat{A}[N-1]\right)$$

最小贝叶斯MSE更新：

$$\mathrm{Bmse}\left(\hat{A}[N]\right)==\left(1-K[N]\right)\mathrm{Bmse}\left(\hat{A}[N-1]\right)$$

一般化，且可推广至矢量参数情况

信号模型：

$$\boldsymbol{x} = \mathbf{H}\boldsymbol{\theta} + \boldsymbol{w}$$

待估计参数 $\boldsymbol{\theta}$ ，其均值为 $E(\boldsymbol{\theta})$，协方差矩阵为 $\mathbf{C}_{\theta\theta}$。$\boldsymbol{w}$为零均值不相关噪声，且 $\mathrm{var}(w[n]) = \sigma_n^2$，$\boldsymbol{w}$ 与 $\boldsymbol{\theta}$ 不相关。

对 $\boldsymbol{\theta}$ 的LMMSE估计可采用如下序贯方式进行：

初始化：$\hat{\boldsymbol{\theta}}[-1] = E(\boldsymbol{\theta})$ $\quad \mathbf{M}[-1] = \mathbf{C}_{\theta\theta}$

$$\mathbf{H}[n] = \begin{bmatrix} \mathbf{H}[n-1] \\ \boldsymbol{h}^T[n] \end{bmatrix}$$

$$x[n] = \boldsymbol{h}^T[n]\boldsymbol{\theta} + w[n]$$

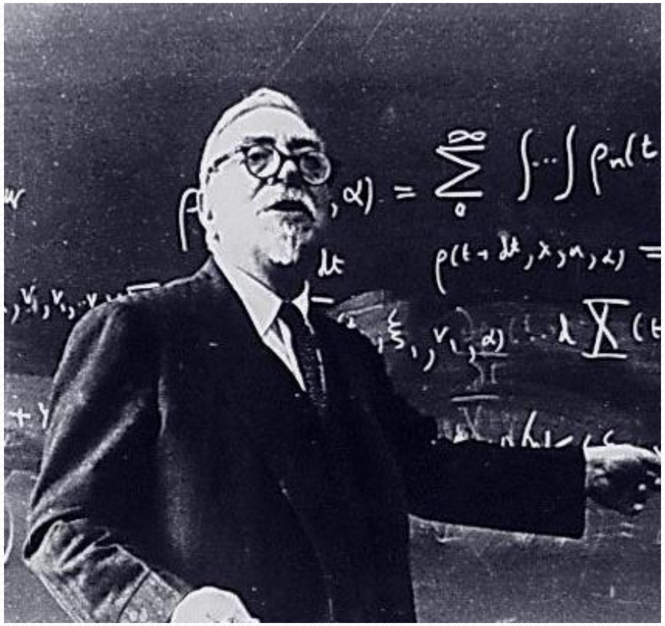
增益因子：$\mathbf{K}[n] = \dfrac{\mathbf{M}[n-1]\boldsymbol{h}[n]}{\sigma_n^2 + \boldsymbol{h}^T[n]\mathbf{M}[n-1]\boldsymbol{h}[n]}$

估计量更新：$\hat{\boldsymbol{\theta}}[n] = \hat{\boldsymbol{\theta}}[n-1] + \mathbf{K}[n]\left(x[n] - \boldsymbol{h}^T[n]\hat{\boldsymbol{\theta}}[n-1]\right)$

最小贝叶斯MSE更新：$\mathbf{M}[n] = \left(\mathbf{I} - \mathbf{K}[n]\boldsymbol{h}^T[n]\right)\mathbf{M}[n-1]$

五、LMMSE的应用

● 维纳滤波



- 诺伯特·维纳(Norbert Wiener，1894-1964)，美国数学家，控制论的创始人。1894年11月26日生于密苏里州的哥伦比亚，1964年3月18日卒于斯德哥尔摩
- 维纳在其50年的科学生涯中，先后涉足哲学、数学、物理学和工程学，最后转向生物学，在各个领域中都取得了丰硕成果，是一位多才多艺、学识渊博的科学巨人
- 他一生发表论文240多篇，著作14本。他的主要著作有《控制论》(1948)、《维纳选集》(1964)和《维纳数学论文集》(1980)等

$$x[n]=s[n]+w[n],\quad n=0,1,\ldots,N-1$$

假定观测数据是零均值的WSS，信号也是零均值的，信号与噪声不相关

- **滤波**

$\theta=s[n]$ 用 $x[0],x[1],x[2],\ldots,x[n]$ 来估计

- **平滑**

$\theta=s[n]$ 用 $x[0],x[1],x[2],\ldots,x[N-1]$ 来估计

- **预测**

$\theta=x[N-1+l]$ 用 $x[0],x[1],x[2],\ldots,x[N-1]$ 来估计

滤波

$\theta = s[n]$ 用 $x[0], x[1], x[2], \ldots, x[n]$ 来估计

LMMSE: $\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x} - E(\boldsymbol{x})\right)$

$$\hat{s}[n] = \hat{\theta} = \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\boldsymbol{x}$$

$$\begin{aligned}
\mathbf{C}_{\theta x} &= E\left(s[n]\left(\left[x[0], x[1], \ldots, x[n]\right]\right)\right) \\
&= E\left(s[n]\left(\left[s[0], s[1], \ldots, s[n]\right]\right)\right) \\
&\quad + E\left(s[n]\left(\left[w[0], w[1], \ldots, w[n]\right]\right)\right) \\
&= \left[r_{ss}[n], r_{ss}[n-1], \ldots, r_{ss}[0]\right] \\
&\triangleq \boldsymbol{r}_{ss}^{'T}
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}_{xx} &= E\left(\boldsymbol{x}\boldsymbol{x}^T\right) \\
&= E\left((\boldsymbol{s}+\boldsymbol{w})(\boldsymbol{s}+\boldsymbol{w})^T\right) \\
&= E\left(\boldsymbol{s}\boldsymbol{s}^T\right) + E\left(\boldsymbol{w}\boldsymbol{w}^T\right) \\
&= \mathbf{C}_{ss} + \mathbf{C}_{ww} \\
&= \mathbf{R}_{ss} + \mathbf{R}_{ww}
\end{aligned}$$

$$\Rightarrow \quad \hat{s}[n] = \boldsymbol{r}_{ss}^{'T}\left(\mathbf{R}_{ss} + \mathbf{R}_{ww}\right)^{-1}\boldsymbol{x}$$

滤波器：$\hat{s}[n]=\sum_{k=0}^{n}h^{(n)}[n-k]x[k]=\sum_{k=0}^{n}h^{(n)}[k]x[n-k]$

$$\left.\begin{aligned}&\hat{s}[n]=\boldsymbol{r}_{ss}^{'T}(\mathbf{R}_{ss}+\mathbf{R}_{ww})^{-1}\boldsymbol{x}\\&\hat{s}[n]=\boldsymbol{a}^{T}\boldsymbol{x}=\sum_{k=0}^{n}a_k x[k]\end{aligned}\right\}\Rightarrow \boldsymbol{a}=(\mathbf{R}_{ss}+\mathbf{R}_{ww})^{-1}\boldsymbol{r}_{ss}^{'}\Rightarrow(\mathbf{R}_{ss}+\mathbf{R}_{ww})\boldsymbol{a}=\boldsymbol{r}_{ss}^{'}$$

$$\left.\begin{aligned}&(\mathbf{R}_{ss}+\mathbf{R}_{ww})\boldsymbol{a}=\boldsymbol{r}_{ss}^{'}\\&\text{令 } h^{(n)}[k]=a_{n-k},\ k=0,1,\ldots,n\end{aligned}\right\}\Rightarrow$$

$$\underline{(\mathbf{R}_{ss}+\mathbf{R}_{ww})}\boldsymbol{h}=\boldsymbol{r}_{ss}\ \text{,其中}\ \boldsymbol{r}_{ss}=\left[r_{ss}[0],r_{ss}[1],\ldots,r_{ss}[n]\right]^{T}$$

$\mathbf{R}_{xx}$

$$\Rightarrow\begin{bmatrix}r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[n]\\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[n-1]\\ \vdots & \vdots & \ddots & \vdots\\ r_{xx}[n] & r_{xx}[n-1] & \cdots & r_{xx}[0]\end{bmatrix}\begin{bmatrix}h^{(n)}[0]\\ h^{(n)}[1]\\ \vdots\\ h^{(n)}[n]\end{bmatrix}=\begin{bmatrix}r_{ss}[0]\\ r_{ss}[1]\\ \vdots\\ r_{ss}[n]\end{bmatrix}$$

维纳-霍夫滤波方程

Levinson递推算法

平滑

$\theta = s[n]$ 用 $\ldots, x[-1], x[0], x[1], x[2], \ldots,$ 来估计

$$\text{LMMSE:}\quad \hat{s}[n] = \sum_{k=-\infty}^{\infty} a_k x[k]$$

令 $h[k] = a_{n-k}$

$$\hat{s}[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

正交原理

$$E\left(\left(s[n] - \hat{s}[n]\right) x[m]\right) = 0$$

$$E\left(s[n] x[m]\right) = E\left(\hat{s}[n] x[m]\right)$$

$$E\left(s[n]\left(s[m] + w[m]\right)\right) = E\left(\sum_{k=-\infty}^{\infty} h[k] x[n-k] x[m]\right)$$

$$E\left(s[n]\left(s[m]+w[m]\right)\right)=E\left(\sum_{k=-\infty}^{\infty}h[k]x[n-k]x[m]\right)$$

$$r_{ss}[n-m]=\sum_{k=-\infty}^{\infty}h[k]E\left(x[n-k]x[m]\right)$$

$$r_{ss}[n-m]=\sum_{k=-\infty}^{\infty}h[k]r_{xx}[n-m-k]$$

$$r_{ss}[l]=\sum_{k=-\infty}^{\infty}h[k]r_{xx}[l-k],\quad -\infty<l<\infty$$

$$r_{ss}[n]=h[n]*r_{xx}[n]$$

$$H(f)=\frac{P_{ss}(f)}{P_{xx}(f)}=\frac{P_{ss}(f)}{P_{ss}(f)+P_{ww}(f)}=\frac{\eta(f)}{\eta(f)+1}$$,其中 $\eta(f)=\frac{P_{ss}(f)}{P_{ww}(f)}$

——无限维纳平滑器的频率响应

- **预测**

$\theta = x[N-1+l]$ 用 $x[0], x[1], x[2], \ldots, x[N-1]$ 来估计

$$\text{LMMSE:} \quad \hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\left(\boldsymbol{x} - E(\boldsymbol{x})\right)$$

$$\hat{x}[N-1+l] = \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\boldsymbol{x}$$

$$\mathbf{C}_{\theta x} = E\left(x[N-1+l]\left[x[0], x[1], \ldots, x[N-1]\right]\right)$$

$$= \left[r_{xx}[N-1+l], r_{xx}[N-2+l], \ldots, r_{xx}[l]\right]$$

$$\triangleq \boldsymbol{r}_{xx}^{'T}$$

$$\mathbf{C}_{xx} = \mathbf{R}_{xx}$$

$$\hat{x}[N-1+l] = \boldsymbol{r}_{xx}^{'T}\mathbf{R}_{xx}^{-1}\boldsymbol{x}$$

“滤波”器：$\hat{x}[N-1+l]=\sum_{k=0}^{N-1}h[N-k]x[k]=\sum_{k=1}^{N}h[k]x[N-k]$

$$\left.\begin{aligned}\hat{x}[N-1+l]&=\boldsymbol{r}_{xx}^{'T}\mathbf{R}_{xx}^{-1}\boldsymbol{x}\\ \hat{x}[N-1+l]&=\boldsymbol{a}^{T}\boldsymbol{x}=\sum_{k=0}^{n}a_k x[k]\end{aligned}\right\} \Rightarrow \boldsymbol{a}=\mathbf{R}_{xx}^{-1}\boldsymbol{r}_{xx}^{'} \Rightarrow \left.\begin{aligned}&\mathbf{R}_{xx}\boldsymbol{a}=\boldsymbol{r}_{xx}^{'}\\ &\text{令 } h[k]=a_{N-k}\end{aligned}\right\}$$

$\Rightarrow$ $\mathbf{R}_{xx}\boldsymbol{h}=\boldsymbol{r}_{xx}$，其中 $\boldsymbol{r}_{xx}=\left[r_{xx}[l],r_{xx}[l+1],\ldots,r_{xx}[N-1+l]\right]^{T}$

$$\Rightarrow \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[N-1]\\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[N-2]\\ \vdots & \vdots & \ddots & \vdots\\ r_{xx}[N-1] & r_{xx}[N-2] & \cdots & r_{xx}[0]\end{bmatrix}\begin{bmatrix} h[1]\\ h[2]\\ \vdots\\ h[N]\end{bmatrix}=\begin{bmatrix} r_{xx}[l]\\ r_{xx}[l+1]\\ \vdots\\ r_{xx}[N-1+l]\end{bmatrix}$$

线性预测维纳-霍夫方程

Levinson递推算法

六、小结

- LMMSE估计
  - 限定估计量是线性的，然后再找贝叶斯MSE最小者
  - 无需PDF，仅需前两阶矩即可
  - 一般情况下并非最佳（在贝叶斯MSE准则下），但相比MMSE较易得到显示解
- LMMSE的性质
  - 线性变换不变性
  - 可加性
- 贝叶斯高斯马尔科夫定理
- 对贝叶斯一般线性模型，LMMSE与MMSE具有相同形式
- 序贯LMMSE
- LMMSE的“应用”
  - 维纳滤波
  - 卡尔曼滤波