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4.1 $x[n]=\sum_{i=1}^{p}A_i r_i^n + w[n]$

$x = H\theta + w$

$\theta = [A_1 \ A_2 \ \dots \ A_p]^T$

$$H=\begin{bmatrix}1 & \cdots & 1\\ r_1 & \cdots & r_p\\ \vdots & & \vdots\\ r_1^{N-1} & \cdots & r_p^{N-1}\end{bmatrix}$$

$\hat{\theta}=(H^TH)^{-1}H^Tx$ ， $C_{\hat{\theta}}=\sigma^2(H^TH)^{-1}$

$p=2$ ， $r_1=1$ ， $r_2=-1$ ，N偶数

$$H=\begin{bmatrix}1 & 1\\ 1 & -1\\ \vdots & \vdots\\ 1 & -1\end{bmatrix} \quad H^TH=\begin{bmatrix}N & 0\\ 0 & N\end{bmatrix}$$

$$\hat{\theta}=\frac{1}{N}H^Tx=\begin{bmatrix}\frac{1}{N}\sum_{n=0}^{N-1}x[n]\\ \frac{1}{N}\sum_{n=0}^{N-1}(-1)^n x[n]\end{bmatrix}$$

$$C_{\hat{\theta}}=\frac{\sigma^2}{N}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{\sigma^2}{N} & 0\\ 0 & \frac{\sigma^2}{N}\end{bmatrix}$$

4.14 $x=H\theta+w$

$$H=\begin{cases}[1,\dots,1]^T, & p=1-\varepsilon\\ [\underbrace{1,\dots,1}_{M},\underbrace{0,\dots,0}_{N-M}], & p=\varepsilon\end{cases}$$

$$\hat{\theta}=\hat{A}=(H^TH)^{-1}H^Tx=\begin{cases}\frac{1}{N}\sum_{n=0}^{N-1}x[n] & \text{不衰落}\\ \frac{1}{M}\sum_{n=0}^{M-1}x[n] & \text{衰落}\end{cases}$$

$C_{\hat{A}}=\sigma^2 E_H[(H^TH)^{-1}]$

$=\sigma^2\left((1-\varepsilon)\frac{1}{N}+\varepsilon\frac{1}{M}\right)$

$=\frac{\sigma^2}{N}\left(1+(\frac{N}{M}-1)\varepsilon\right)\geq\frac{\sigma^2}{N}$ 当且仅当 $M=N$ 或 $\varepsilon=0$ 时取"="

即在衰落情况下方差会变大

6.6. $E\,x[n]=\theta s[n]+\beta$

$\hat{\theta}=\sum_{n=0}^{N-1}a_n x[n]+b$

$E\hat{\theta}=E\left[\sum_{n=0}^{N-1}a_n x[n]+b\right]=\sum_{n=0}^{N-1}a_n(\theta s[n]+\beta)+b=\theta$

即 $\sum_{n=0}^{N-1}a_n s[n]=1$ ， $\beta\sum_n a_n=-b$

所以 $\hat{\theta}=\sum_{n=0}^{N-1}a_n x[n]-\beta\sum_n a_n=\sum_n a_n(x[n]-\beta)$

且 $C_{x'}=C_x=C$

令 $x'[n]=x[n]-\beta$ 则回到 BLUE问题

$$\hat{\theta}=\frac{s^TC^{-1}x'}{s^TC^{-1}s}=\frac{s^TC^{-1}(x-\beta 1)}{s^TC^{-1}s}$$

$$\mathrm{var}\,\hat{\theta}=\frac{1}{s^TC^{-1}s}$$

6.9、

$x[n]=A\cos 2\pi f_1 n+w[n]$

$E\,x[n]=\theta s[n]=A\cos 2\pi f_1 n$

$s=[1,\cos 2\pi f_1,\dots,\cos 2\pi f_1(N-1)]^T$

$$\hat{A}=\frac{s^TC^{-1}x}{s^TC^{-1}s}=\frac{s^Tx}{s^Ts}=\frac{\sum_n x[n]\cos 2\pi f_1 n}{\sum_n \cos^2 2\pi f_1 n}$$

$$=\frac{N/2}{\sum_n\cos^2 2\pi f_1 n}\cdot\underbrace{\frac{2}{N}\sum_n x[n]\cos\frac{2\pi(Nf_1)n}{N}}_{\text{DFT系数}}$$

$$\mathrm{var}\,\hat{A}=\frac{1}{s^TC^{-1}s}=\frac{\sigma^2}{s^Ts}=\frac{\sigma^2}{\sum_n\cos^2 2\pi f_1 n}\geq\frac{\sigma^2}{N}$$

当 $f_1=0$ 时， $\hat{A}$ 方差最小为 $\frac{\sigma^2}{N}$ ，此时 $x[n]=A+w[n]$

注：$w_b[n]$, $w_I[n]$ 被简化写成 w_b, w_I.

6.14 $p(w[n]) = \cancel{(1-\varepsilon)N(0,\sigma_B^2) + \varepsilon N(0,\sigma_I^2) = (1-\varepsilon)p(w_b) + \varepsilon p(w_I)}$ ~~其中~~ $w_b \sim N(0,\sigma_b^2)$

$w_I \sim N(0,\sigma_I^2)$.

$Ew[n] = \cancel{(1-\varepsilon)\cdot 0 + \varepsilon 0} = 0$　　则 $w[n] \cancel{= (1-\varepsilon)w_b + \varepsilon w_I}$ ~~两者独立~~

$\mathrm{var}\,w[n] = \cancel{Ew^2[n] = E\left[((1-\varepsilon)w_b + \varepsilon w_I)^2\right]}$

$= E\left[\cancel{(1-\varepsilon)^2 w_b^2 + \varepsilon^2 w_I^2 + 2\varepsilon(1-\varepsilon) w_b w_I}\right]$

$p(w[n]) = (1-\varepsilon)N(0,\sigma_b^2) + \varepsilon N(0,\sigma_I^2)$

$Ew[n] = 0$

$\mathrm{var}\,w[n] = Ew^2[n] = \int_{-\infty}^{+\infty} w^2(1-\varepsilon)\frac{1}{\sqrt{2\pi\sigma_b^2}} e^{-\frac{w^2}{2\sigma_b^2}}\,dw + \int_{-\infty}^{+\infty} w^2 \varepsilon \frac{1}{\sqrt{2\pi\sigma_I^2}} e^{-\frac{w^2}{2\sigma_I^2}}\,dw$

$= (1-\varepsilon)\sigma_b^2 + \varepsilon\sigma_I^2$

即 $\sigma^2 = (1-\varepsilon)\sigma_B^2 + \varepsilon\sigma_I^2$

利用数据变量 $y[n] = w^2[n]$

$\hat{\sigma}^2 = \sum_{n=0}^{N-1} a_n w^2[n]$, $E\hat{\sigma}^2 = \sum_n a_n \sigma^2 = \sigma^2$

$Ew^2[n] = 1\cdot\sigma^2$ 即 $s = [1, \ldots, 1]^T$,

$\hat{\sigma}^2 = \frac{s^T C^{-1} w^2[n]}{s^T C^{-1} s} = \frac{s^T w^2[n]}{s^T s} = \frac{1}{N}\sum_{n=0}^{N} w^2[n]$

由6.12知，$\sigma_I^2 = \frac{\sigma^2 - (1-\varepsilon)\sigma_B^2}{\varepsilon}$

即 $\hat{\sigma}_I^2 = \frac{\frac{1}{N}\sum_n w^2[n] - (1-\varepsilon)\sigma_B^2}{\varepsilon}$