12.1
12.2
12.6
无研203 刘子源 2022310709.

12.1 $\hat{\theta} = a x^2[0] + b x[0] + c$

$Bmse(\hat{\theta}) = E(\theta - a x^2[0] - b x[0] - c)^2$.

$\frac{\partial Bmse(\hat{\theta})}{\partial a} = -2E[(\theta - a x^2[0] - b x[0] - c) x^2[0]] = 0$.

$\frac{\partial Bmse(\hat{\theta})}{\partial b} = -2E[(\theta - a x^2[0] - b x[0] - c) x[0]] = 0$.

$\frac{\partial Bmse(\hat{\theta})}{\partial c} = -2E[(\theta - a x^2[0] - b x[0] - c)] = 0$.

得 $E(\theta x^2[0]) = aE x^4[0] + bE x^3[0] + cE x^2[0]$

$E(\theta x[0]) = aE x^3[0] + bE x^2[0] + cE x[0]$

$E(\theta) = aE x^2[0] + bE x[0] + c$

得Bmse最小时、$Bmse(\hat{\theta}) = E[\theta(\theta - \hat{a} x^2[0] - \hat{b} x[0] - \hat{c})]$

其中 $\hat{a}, \hat{b}, \hat{c}$ 为 $= E(\theta^2) - \hat{a}E(\theta x^2[0]) - \hat{b}E(\theta x[0]) - \hat{c}E(\theta)$

$$\begin{bmatrix} Ex^4[0] & Ex^3[0] & Ex^2[0] \\ Ex^3[0] & Ex^2[0] & Ex[0] \\ Ex^2[0] & Ex[0] & 1 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = \begin{bmatrix} E(\theta x^2[0]) \\ E(\theta x) \\ E(\theta) \end{bmatrix}$$ 的解.

若 $x[0] \sim U[-\frac{1}{2}, \frac{1}{2}]$, $\theta = \cos 2\pi x[0]$

x[0]都换成x.

$Ex[0] = 0$, $E\theta = 0$

$E\theta x[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} x[0] \cos 2\pi x[0]\, dx = 0$.

$E\theta x^2[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2[0] \cos 2\pi x[0]\, dx = -\frac{1}{2\pi^2}$

$Ex^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{1}{12}$

$Ex^3 = 0$

$Ex^4 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^4 dx = \frac{1}{80}$

$$\begin{bmatrix} \frac{1}{80} & 0 & \frac{1}{12} \\ 0 & \frac{1}{12} & 0 \\ \frac{1}{12} & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\pi^2} \\ 0 \\ 0 \end{bmatrix} \Rightarrow \cancel{\hat{a} = -\frac{1}{2\pi^2}}, \cancel{\hat{b} = 0}, \cancel{\hat{c} = 0}$$

$$\begin{cases} \hat{a} = -\frac{90}{\pi^2} \\ \hat{b} = 0 \\ \hat{c} = \frac{15}{2\pi^2} \end{cases}$$

$Bmse(\hat{\theta}) = \frac{1}{2} - (\frac{90}{\pi^2})(-\frac{1}{2\pi^2}) - 0 - 0 = 0.04$

若线性估计、$\hat{\theta} = bx + c$.

$$\begin{bmatrix} Ex^2 & Ex \\ Ex & 1 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} E\theta x \\ E\theta \end{bmatrix} \Rightarrow \begin{matrix} E\theta x = 0, E\theta = 0 \\ \hat{b} = \hat{c} = 0, \hat{\theta} = 0 \end{matrix}$$

LMMSE最小时为 $E\theta^2 = \frac{1}{2}$

2、$x[n] = Ar^n + w[n]$, $n = 0, 1, \ldots, N-1$

$\hat{A} = \mu_A + (\frac{1}{\sigma_A^2} + \frac{h^T h}{\sigma^2})^{-1} \frac{h^T}{\sigma^2}(x - h\mu_A)$.

$h = [1, r, \ldots, r^{N-1}]^T$

$$\hat{A} = \mu_A + \frac{\sum_{n=0}^{N-1} r^n (x[n] - r^n \mu_A)}{\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}}$$

$$Bmse(\hat{A}) = \frac{1}{\frac{1}{\sigma_A^2} + \frac{1}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}$$

12.6.

$\hat{s} = C_{sx} C_{xx}^{-1} x$

$C_{xx} = E(xx^T) = R_{ss} + R_{ww} = (\sigma_s^2 + \sigma^2) I$

$C_{sx} = E(sx^T) = E(s(s+w)^T) = E(ss^T) = \sigma_s^2 I$

$\hat{s} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} x$

相应的最小MSE矩阵为

$M_{\hat{s}} = [C_{ss} - C_{sx} C_{xx}^{-1} C_{xs}]$

$= \sigma_s^2 I - \frac{(\sigma_s^2)^2}{\sigma_s^2 + \sigma^2} I$

$= \sigma_s^2 [1 - \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2}] I$

$= \frac{\sigma_s^2 \sigma^2}{\sigma_s^2 + \sigma^2} I$.