since $\sum_{n=n_0}^{N-1} \cos^2 2\pi \hat{f}_0 n \simeq \frac{N-n_0}{2}$

$$\sum_{n=n_0}^{N-1} \cos^2 2\pi f_0 n \simeq \frac{N-n_0}{2}$$

or we decide $\mathcal{H}_1$ if

$$\max_f \sum_{n=n_0}^{N-1} x[n] \left[\cos 2\pi f n - \cos 2\pi f_0 n \right] > \gamma'$$

12.7 $$L_G(\underline{x}) = \frac{p(\underline{x}; A_1 = A, A_2 = \hat{A})}{p(\underline{x}; A_1 = A, A_2 = A)}$$

Since the noise samples are IID, the data before the jump is irrelevant (also need A_1 known).

$$L_G(\underline{x}) = \frac{\prod_{n=n_0}^{N-1} p(x[n] - \hat{A})}{\prod_{n=n_0}^{N-1} p(x[n] - A)}$$

$$= \frac{\max_{A_2} \prod_{n=n_0}^{N-1} p(x[n] - A_2)}{\prod_{n=n_0}^{N-1} p(x[n] - A)}$$

$$= \max_{A_1} \prod_{n=n_0}^{N-1} \frac{p(x[n]-A_1)}{p(x[n]-A)}$$

or equivalently

$$L_G(\underline{x}) = \max_{\Delta A} \prod_{n=n_0}^{N-1} \frac{p(x[n]-A-\Delta A)}{p(x[n]-A)}$$

12.81 $\mathcal{H}_0$: link OK

$\mathcal{H}_1$: link gone down

$$\text{or } \begin{aligned} \mathcal{H}_0 &: x[n] = s[n] + \overset{\swarrow \text{WGN}}{w[n]} \\ \mathcal{H}_1 &: x[n] = w[n] \end{aligned} \Bigg\} \; n = \quad \cdots, N-1$$

Data prior to n_0 is irrelevant

$$L(\underline{x}) = \frac{p(\underline{x};\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$= \frac{\dfrac{1}{(2\pi\sigma^2)^{\frac{N-n_0}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{n=n_0}^{N-1} x^2[n]}}{\dfrac{1}{(2\pi\sigma^2)^{\frac{N-n_0}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{n=n_0}^{N-1} (x[n]-s[n])^2}}$$

$$2\ln L(\underline{x}) = \frac{1}{\sigma^2}\sum\left(x^2[n] - 2x[n]s[n] + s^2[n] - x^2[n]\right)$$

$$= -\frac{2}{\sigma^2}\sum x[n]s[n] + \frac{1}{\sigma^2}\sum s^2[n]$$

Decide H_1 if

$$-\sum_{n=n_0}^{N-1} x[n]s[n] > \gamma'$$

or $$T(\underline{x}) = \sum_{n=n_0}^{N-1} x[n]s[n] < \gamma''$$

Note: we are detecting a failure based on a lack of correlation.

12.9) Need only maximize λ.

$$\lambda = \frac{(A_1-A_2)^2}{\sigma^2\left(\frac{1}{n_0}+\frac{1}{N-n_0}\right)}$$

But $$\frac{1}{\frac{1}{n_0}+\frac{1}{N-n_0}} = \frac{(N-n_0)n_0}{N} = N\left(1-\frac{n_0}{N}\right)\frac{n_0}{N}$$

Let $g(x) = x(1-x) \quad 0<x<1$

$$g'(x) = 1-2x = 0 \Rightarrow x = 1/2$$

$\Rightarrow \lambda$ maximized for $n_0 = N/2$ N even

$\frac{N-1}{2}$ or $\frac{N+1}{2}$ N odd

A_1-A_2 is most accurately estimated if n_0 is at the midpoint of data record

12.10) From Example 12.3

$$2 \ln L_G(\underline{x}) = 2 \ln \frac{p(\underline{x}; A_1 = \hat{A}_1, A_2 = \hat{A}_2)}{p(\underline{x}; A_1 = \hat{A}, A_2 = \hat{A})}$$

$$= \frac{(\hat{A}_1 - \hat{A}_2)^2}{\sigma^2\left(\frac{1}{n_0} + \frac{1}{N-n_0}\right)}$$

For our problem with n_0 unknown

$$L_G(\underline{x}) = \frac{p(\underline{x}; A_1 = \hat{A}_1, A_2 = \hat{A}_2, \hat{n}_0)}{p(\underline{x}; A_1 = \hat{A}, A_2 = \hat{A})}$$

$$= \max_{n_0} \frac{p(\underline{x}; A_1 = \hat{A}_1, A_2 = \hat{A}_2, n_0)}{p(\underline{x}; A_1 = \hat{A}, A_2 = \hat{A})}$$

$\Rightarrow$

$$2 \ln L_G(\underline{x}) = 2 \ln \max_{n_0} \quad ''$$

$$= \max_{n_0} 2 \ln \quad ''$$ See Prob. 7.21

$$= \max_{n_0} \frac{(\hat{A}_1 - \hat{A}_2)^2}{\sigma^2\left(\frac{1}{n_0} + \frac{1}{N-n_0}\right)}$$

12.11) $\mathcal{H}_0: \sigma^2 = \sigma_0^2 \quad n = 0, 1, \ldots$

$\mathcal{H}_1: \sigma^2 = \sigma_0^2 \quad n = 0, 1, \ldots, n_0 - 1$

$\sigma_0^2 + \Delta\sigma^2 \quad n = n_0, \ldots, N-1$

$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{\Delta\sigma^2}, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)}$$

Since σ^2 is known before n_0 the data before n_0 is irrelevant. To find $\hat{\Delta\sigma^2}$

$$p(\underline{x}; \Delta\sigma^2, \mathcal{H}_1) = \frac{1}{(2\pi\sigma_0^2)^{n_0/2}} e^{-\frac{1}{2\sigma_0^2}\sum_{n=0}^{n_0-1} x^2(n)} \cdot \frac{1}{(2\pi(\sigma_0^2 + \Delta\sigma^2))^{\frac{N-n_0}{2}}} e^{-\frac{1}{2(\sigma_0^2+\Delta\sigma^2)}\sum_{n_0}^{N-1} x^2(n)}$$

Differentiating and setting equal to zero $\Rightarrow$

$$\sigma_0^2 + \hat{\Delta\sigma^2} = \frac{1}{N-n_0}\sum_{n=n_0}^{N-1} x^2(n)$$

$$\ln L_G(\underline{x}) = -\frac{N-n_0}{2}\ln(\sigma_0^2 + \hat{\Delta\sigma^2}) + \frac{N-n_0}{2}\ln\sigma_0^2 - \frac{1}{2(\sigma_0^2+\hat{\Delta\sigma^2})}\sum_{n_0}^{N-1} x^2(n) + \frac{1}{2\sigma_0^2}\sum_{n_0}^{N-1} x^2(n)$$

$$2\ln L_G(\underline{x}) = -(N-n_0)\ln\frac{\sigma_0^2 + \hat{\Delta\sigma^2}}{\sigma_0^2}$$

$$+\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_0^2+\hat{\Delta\sigma^2}}\right)\underbrace{\sum_{n=n_0}^{N-1}x^2[n]}_{(N-n_0)(\sigma_0^2+\hat{\Delta\sigma^2})}$$

$$=-(N-n_0)\ln\left(\frac{\sigma_0^2+\hat{\Delta\sigma^2}}{\sigma_0^2}\right)$$

$$+(N-n_0)\frac{\hat{\Delta\sigma^2}(\sigma_0^2+\hat{\Delta\sigma^2})}{\sigma_0^2(\sigma_0^2+\hat{\Delta\sigma^2})}$$

$$=(N-n_0)\left(\frac{\sigma_0^2+\hat{\Delta\sigma^2}}{\sigma_0^2}-\ln\frac{\sigma_0^2+\hat{\Delta\sigma^2}}{\sigma_0^2}-1\right)$$

Let $x=\dfrac{\sigma_0^2+\hat{\Delta\sigma^2}}{\sigma_0^2}$ and for $\hat{\Delta\sigma^2}>0$

$x>1$. Hence, an equivalent test is to decide $\mathcal{H}_1$ if $x>\gamma''$ or

$$\frac{\sigma_0^2+\hat{\Delta\sigma^2}}{\sigma_0^2}>\gamma''$$

or $$\underbrace{\frac{1}{N-n_0}\sum_{n=n_0}^{N-1}x^2[n]}_{\text{variance estimate after jump}}>\sigma_0^2\gamma''$$

12.12) $\mathcal{H}_0:\sigma^2=\sigma_1^2=\sigma_2^2$

$\mathcal{H}_1:\quad \sigma^2=\sigma_1^2 \qquad n=0,1,\dots,n_0-1$

$\qquad\qquad \sigma_2^2 \qquad n=n_0,\dots,N-1$

with $\sigma_1^2\neq\sigma_2^2$

or $\mathcal{H}_0 : \sigma_1^2 = \sigma_2^2$

$\mathcal{H}_1 : \sigma_1^2 \neq \sigma_2^2$

$$L_G(\underline{x}) = \frac{p(\underline{x};\ \sigma_1^2 = \hat{\sigma}_1^2,\ \sigma_2^2 = \hat{\sigma}_2^2)}{p(\underline{x};\ \sigma_1^2 = \hat{\sigma}^2,\ \sigma_2^2 = \hat{\sigma}^2)}$$

Under $\mathcal{H}_0$ $\quad \hat{\sigma}^2 = \frac{1}{N} \sum_{n=0}^{N-1} x^2(n)$

Under $\mathcal{H}_1$ $\quad \hat{\sigma}_1^2 = \frac{1}{n_0} \sum_{n=0}^{n_0-1} x^2(n)$

$$\hat{\sigma}_2^2 = \frac{1}{N-n_0} \sum_{n=n_0}^{N-1} x^2(n)$$

$$L_G(\underline{x}) = \frac{\dfrac{1}{(2\pi\hat{\sigma}_1^2)^{n_0/2}} e^{-\frac{1}{2\hat{\sigma}_1^2} \underbrace{\sum_{0}^{n_0-1} x^2(n)}_{n_0\hat{\sigma}_1^2}} \cdot \dfrac{1}{(2\pi\hat{\sigma}_2^2)^{\frac{N-n_0}{2}}} e^{-\frac{1}{2\hat{\sigma}_2^2} \underbrace{\sum_{n_0}^{N-1} x^2(n)}_{(N-n_0)\hat{\sigma}_2^2}}}{\dfrac{1}{(2\pi\hat{\sigma}^2)^{N/2}} e^{-\frac{1}{2\hat{\sigma}^2} \underbrace{\sum_{0}^{N-1} x^2(n)}_{N\hat{\sigma}^2}}}$$

$$= \frac{(\hat{\sigma}^2)^{N/2}}{(\hat{\sigma}_1^2)^{n_0/2} (\hat{\sigma}_2^2)^{\frac{N-n_0}{2}}}$$

$$2 \ln L_G(\underline{x}) = N \ln \frac{\hat{\sigma}^2}{(\hat{\sigma}_1^2)^{n_0/N} (\hat{\sigma}_2^2)^{\frac{N-n_0}{N}}}$$

For $n_0 = N/2$

$$2 \ln L_G(\underline{x}) = N \ln \frac{\hat{\sigma}^2}{\sqrt{\hat{\sigma}_1^2 \hat{\sigma}_2^2}}$$

$$= N \ln \frac{\frac{1}{2}(\hat{\sigma}_1^2 + \hat{\sigma}_2^2)}{\sqrt{\hat{\sigma}_1^2 \hat{\sigma}_2^2}}$$

For $\hat{\sigma}_1^2 \approx \hat{\sigma}_2^2 \Rightarrow 2 \ln L_G(\underline{x}) \approx 0$ and we decide $\mathcal{H}_0$. For $\hat{\sigma}_1^2 \gg \hat{\sigma}_2^2$ or vice-versa $2 \ln L_G(\underline{x})$ will be large and we decide $\mathcal{H}_1$.

(12.13) $$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{\sigma}_1^2, \hat{n}_0, \hat{n}_1, \hat{n}_2)}{p(\underline{x}; \hat{\sigma}_0^2)}$$

Since $p(\underline{x}; \hat{\sigma}_0^2)$ does not depend on n_0, n_1, n_2

$$L_G(\underline{x}) = \max_{n_0, n_1, n_2} \frac{p(\underline{x}; \hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{\sigma}_1^2; n_0, n_1, n_2)}{p(\underline{x}; \hat{\sigma}_0^2)}$$

But $\hat{\sigma}_0^2 = 1/N \sum_{n=0}^{N-1} x^2(n)$

$$\Rightarrow p(\underline{x}; \hat{\sigma}_0^2) = \frac{1}{(2\pi \hat{\sigma}_0^2)^{N/2}} e^{-N/2}$$

$$p(\underline{x}; A_0, A_1, A_2, \sigma^2, n_0, n_1, n_2) =$$

$$\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} J}$$

$$\text{where } J = \sum_{n=0}^{n_0-1} (x[n] - A_0)^2 + \sum_{n=n_0}^{n_1-1} (x[n] - A_1)^2 + \sum_{n=n_1}^{N-1} (x[n] - A_2)^2$$

To find MLE of A_i just minimize appropriate sum in $J \Rightarrow \hat{A}_0, \hat{A}_1, \hat{A}_2$ as given. Also,

$$\hat{\sigma}_1^2 = 1/N \; J_{MIN}$$ J with A_i replaced by $\hat{A}_i$

$$p(\underline{x}; \hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{\sigma}_1^2, n_0, n_1, n_2) =$$

$$\frac{1}{(2\pi\hat{\sigma}_1^2)^{N/2}} e^{-\underbrace{\frac{1}{2\hat{\sigma}_1^2} J_{MIN}}_{N/2}}$$

$$L_G(\underline{x}) = \max_{n_0, n_1, n_2} \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} \right)^{N/2}$$

12.14) We would have to minimize over

$$\underline{A} = [A_0\ A_1\ A_2\ A_3]^T$$

$$\underline{B} = [B_0\ B_1\ B_2\ B_3]^T$$

$$\underline{n} = [n_0\ n_1\ n_2]^T$$

$$J(\underline{A}, \underline{B}, \underline{n}) = \sum_{n=0}^{n_0-1} (x[n] - A_0 - B_0 n)^2 + \sum_{n=n_0}^{n_1-1} (x[n] - A_1 - B_1 n)^2 + \sum_{n=n_1}^{n_2-1} (x[n] - A_2 - B_2 n)^2 + \sum_{n=n_2}^{N-1} (x[n] - A_3 - B_3 n)^2$$

Each minimization for a given $\underline{n}$ is just a simple least squares solution for A_i, B_i using $[n_{i-1}, n_i - 1]$ data set.

Also, we define

$$\Delta_i[n_{i-1}, n_i - 1] = \sum_{n=n_{i-1}}^{n_i-1} (x[n] - \hat{A}_i - \hat{B}_i n)^2$$

See [Kay-I 1993, pp 83-84] for computation of $\hat{A}_i, \hat{B}_i$.

12.15) From (12.18) we must minimize

$$J = \sum_{n=n_0+1}^{N-1} (x[n] + a[1]x[n-1])^2$$

$$\frac{\partial J}{\partial a[1]} = 2 \sum (x[n] + a[1]x[n-1])\, x[n-1] = 0$$

$$\Rightarrow \hat{a}[1] = -\frac{\sum_{n=n_0+1}^{N-1} x[n]x[n-1]}{\sum_{n=n_0+1}^{N-1} x^2[n-1]}$$

To find $\hat{\sigma}_z^2$:

$$\ln p(\underline{x}_2; \hat{a}[1], \sigma_z^2) = -\frac{(N-n_0)}{2} \ln 2\pi\sigma_z^2$$

$$- \frac{1}{2\sigma_z^2} \underbrace{\sum_{n_0+1}^{N-1} (x[n] + \hat{a}[1]x[n-1])^2}_{J_{MIN}}$$

$$\frac{\partial \ln p}{\partial \sigma_z^2} = -\frac{(N-n_0)}{2\sigma_z^2} + \frac{1}{2\sigma_z^4} J_{MIN} = 0$$

$$\Rightarrow \hat{\sigma}_z^2 = \frac{1}{N-n_0} J_{MIN}$$

12.16) The first term is

$$\mathfrak{Z} = \frac{\left(\frac{1}{N}\sum_{0}^{N-1} x^2(n)\right)^{N/2}}{\left(\frac{1}{n_0}\sum_{0}^{n_0} x^2(n)\right)^{n_0/2}\left(\frac{1}{N-n_0}\sum_{n_0}^{N-1} x^2(n)\right)^{\frac{N-n_0}{2}}}$$

$$= \frac{\left[\frac{1}{N}\left(n_0\hat{r}_1(0) + (N-n_0)\hat{r}_2(0)\right)\right]^{N/2}}{\left(\hat{r}_1(0)\right)^{n_0/2}\left(\hat{r}_2(0)\right)^{\frac{N-n_0}{2}}}$$

$$\mathfrak{Z}^{2/N} = \frac{\alpha\hat{r}_1(0) + (1-\alpha)\hat{r}_2(0)}{\hat{r}_1(0)^{\alpha}\ \hat{r}_2(0)^{1-\alpha}}$$

where $\alpha = n_0/N$

Now use given inequality $\Rightarrow \mathfrak{Z}^{2/N} \geq 1$

$\mathfrak{Z} \geq 1$ with equality if and only if $\hat{r}_1(0) = \hat{r}_2(0)$.

Chapter 13

13.1) From (13.3) we decide $\mathcal{H}_1$ if

$$T(\tilde{\underline{x}}) = \mathrm{Re}\left(\sum_{n=0}^{N-1} \tilde{x}[n]\tilde{A}^* e^{-j2\pi f_0 n}\right) > \gamma'$$

Performance is given by (12.8) with

$$d^2 = 2\mathcal{E}/\sigma^2 = \frac{2}{\sigma^2}\sum_{n=0}^{N-1}|\tilde{s}[n]|^2$$

$$= 2N|\tilde{A}|^2/\sigma^2$$

Also, $\gamma' = \sqrt{\frac{\sigma^2}{2}N|\tilde{A}|^2}\, Q^{-1}(P_{FA})$.

13.2) $$T(\underline{x}) = \mathrm{Re}\left[\sum_{n=0}^{N-1} \tilde{x}(n)\tilde{s}^*(n)\right) = \mathrm{Re}\left[\sum_{k=0}^{N-1} \tilde{s}^*(k)\tilde{x}(k)\right)$$

$$= \mathrm{Re}\left(\sum_{k=0}^{n} \tilde{h}(n-k)\tilde{x}(k)\right)\Big|_{n=N-1}$$

$$\Rightarrow \tilde{s}^*(k) = \begin{cases} \tilde{h}(N-1-k) & k=0,1,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{or } \tilde{h}(k) = \begin{cases} \tilde{s}^*(N-1-k) & k=0,1,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

13.3) $\mathcal{H}_0$: $x_R[n] = u(n)$ where $\tilde{A} = A_R + jA_I$

$x_I(n) = v(n)$ $\quad \tilde{x}(n) = x_R(n) + jx_I(n)$

$\mathcal{H}_1$: $x_R(n) = A_R + u(n)$

$x_I(n) = A_I + v(n)$

Since $u[n], v[n]$ are independent processes and each is WGN with variance $\sigma^2/2$,

$$L(\underline{x}) = \frac{\dfrac{1}{(2\pi\sigma^2/2)^{N/2}} e^{-\frac{1}{2(\sigma^2/2)}\sum(u[n]-A_R)^2} \cdot \dfrac{1}{(2\pi\frac{\sigma^2}{2})^{N/2}} e^{-\frac{1}{2(\sigma^2/2)}\sum(v[n]-A_I)^2}}{\dfrac{1}{(2\pi\sigma^2/2)^{N/2}} e^{-\frac{1}{2(\sigma^2/2)}\sum u^2[n]} \cdot \dfrac{1}{(2\pi\sigma^2/2)^{N/2}} e^{-\frac{1}{2(\sigma^2/2)}\sum v^2[n]}}$$

$$\ln L(\underline{x}) = -\frac{1}{\sigma^2}\Big[\sum\left[(u[n]-A_R)^2 - u^2[n]\right) + \sum\left[(v[n]-A_I)^2 - v^2[n]\right)\Big]$$

$$= -\frac{1}{\sigma^2}\left(-2A_R\sum u[n] + NA_R^2 - 2A_I\sum v[n] + NA_I^2\right) > \ln \gamma$$

or $$\underbrace{A_R\sum u[n] + A_I\sum v[n]}_{Re\left(\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{A}^*\right)} > \frac{\sigma^2}{2}\ln\gamma + \frac{N}{\sigma^2}(A_R^2 + A_I^2)$$

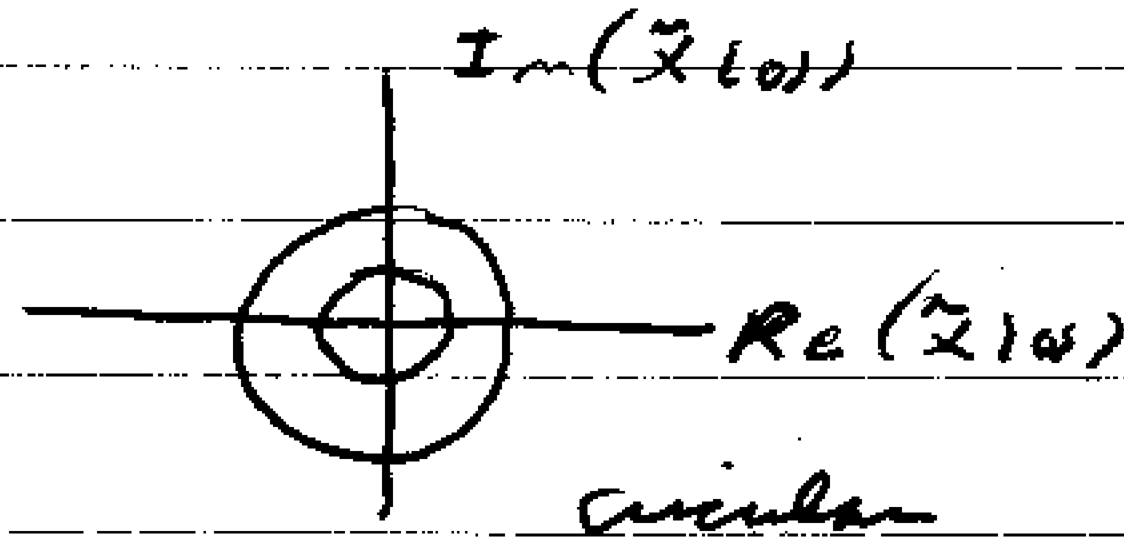
13.4) Under $\mathcal{H}_0$ $\tilde{x}[0] = \tilde{w}[0]$

$$= \underbrace{w_R[0]}_{\mathcal{N}(0,\sigma^2/2)} + j\underbrace{w_I[0]}_{\mathcal{N}(0,\sigma^2/2)}$$

and $w_R(0)$, $w_I(0)$ are independent

Under $\mathcal{H}_1$ $\quad \tilde{x}(0) = A e^{j\phi} + \tilde{w}(0)$

$$= (A\cos\phi + w_R(0)) + j(A\sin\phi + w_I(0))$$

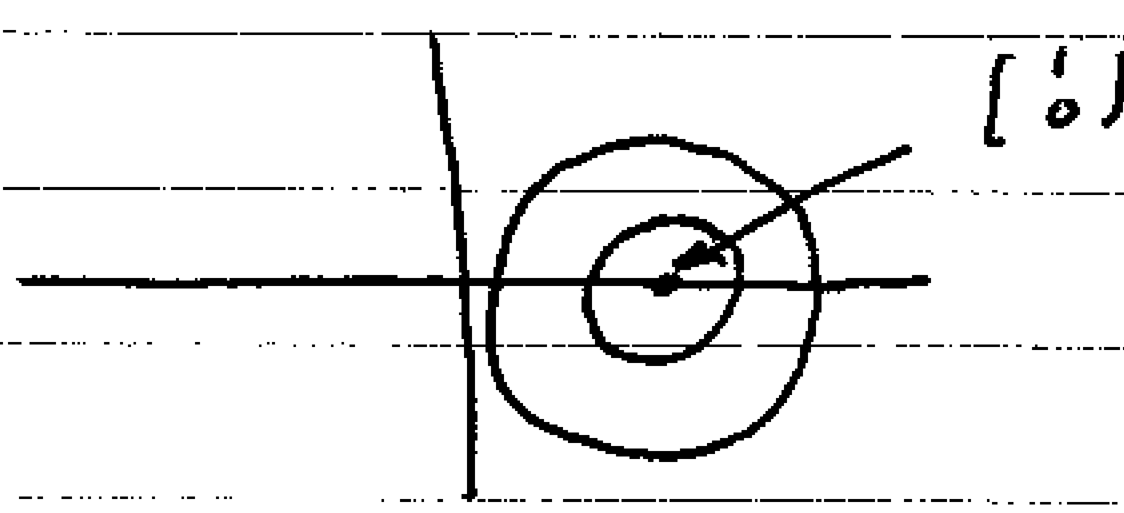


circular contours

Under $\mathcal{H}_1$

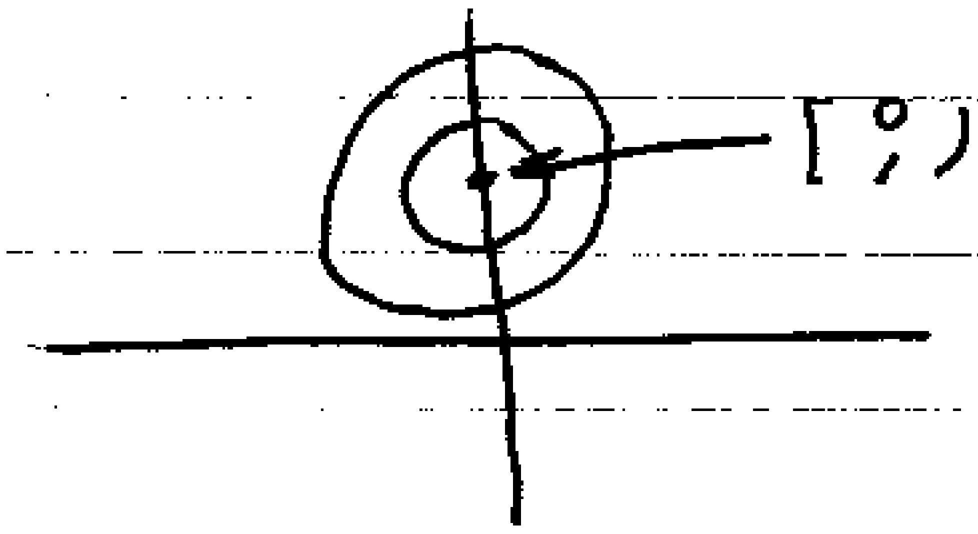
For $A=1$, $\phi=0$

$\tilde{x}(0) = (1 + w_R(0)) + j\,w_I(0)$



For $A=1$, $\phi=\pi/2$

$\tilde{x}(0) = w_R(0) + j(1 + w_I(0))$



The discrimination between the PDFs

under $\mathcal{H}_0$ and $\mathcal{H}_1$ is the same for all ϕ. In fact we could always rotate the axes by applying a $e^{-j\phi}$ transformation to $\tilde{x}[0]$ without changing the problem. Consider the data

$$\tilde{y}[0] = \tilde{x}[0]\, e^{-j\phi}$$

$$\Rightarrow \tilde{y}[0] \sim \begin{cases} \mathcal{CN}(0, \sigma^2) & \mathcal{H}_0 \\ \mathcal{CN}(A, \sigma^2) & \mathcal{H}_1 \end{cases}$$

13.5) $\tilde{z} = \tilde{\underline{s}}^H \underline{C}^{-1} \tilde{\underline{x}}$

Under $\mathcal{H}_0$

$$E(\tilde{z}) = \tilde{\underline{s}}^H \underline{C}^{-1} E(\tilde{\underline{x}}) = 0$$

$$\begin{aligned} \mathrm{var}(\tilde{z}) = E(|\tilde{z}|^2) &= E(\tilde{\underline{s}}^H \underline{C}^{-1} \tilde{\underline{x}}\, \tilde{\underline{x}}^H \underline{C}^{-1} \tilde{\underline{s}}) \\ &= \tilde{\underline{s}}^H \underline{C}^{-1} \underline{C}\, \underline{C}^{-1} \tilde{\underline{s}} \\ &= \tilde{\underline{s}}^H \underline{C}^{-1} \tilde{\underline{s}} \end{aligned}$$

Under $\mathcal{H}_1$

$$E(\tilde{z}) = \tilde{\underline{s}}\, \underline{C}^{-1} E(\tilde{\underline{x}}) = \tilde{\underline{s}}^H \underline{C}^{-1} \tilde{\underline{s}}$$

$$\begin{aligned} \mathrm{var}(\tilde{z}) &= E(|\tilde{z} - E(\tilde{z})|^2) \\ &= E(|\tilde{z} - \tilde{\underline{s}}^H \underline{C}^{-1} \tilde{\underline{s}}|^2) \\ &= E(|\tilde{w}|^2) = \mathrm{var}(\tilde{w}) \\ &= \tilde{\underline{s}}^H \underline{C}^{-1} \tilde{\underline{s}} \end{aligned}$$

13.6) From (13.14), (13.5)

$$\hat{\tilde{s}} = \sigma_{\tilde{s}}^2 (\sigma_{\tilde{s}}^2 + \sigma^2)^{-1}\, \tilde{\underline{x}}$$

$$T(\tilde{\underline{x}}) = \tilde{\underline{x}}^H \hat{\tilde{\underline{s}}} = \frac{\sigma_{\tilde{s}}^2}{\sigma_{\tilde{s}}^2 + \sigma^2} \sum_{n=0}^{N-1} |\tilde{x}[n]|^2$$

which is an energy detector.

(3.7) a) $(\underline{A}\underline{B})^H = (\underline{A}\underline{B})^{T*} = (\underline{B}^T\underline{A}^T)^*$
$= \underline{B}^H\underline{A}^H$

b) $(\underline{A}+\underline{B})^H = (\underline{A}+\underline{B})^{T*}$
$= (\underline{A}^T+\underline{B}^T)^*$
$= \underline{A}^H+\underline{B}^H$

c) $(\underline{A}^{-1})^H = (\underline{A}^{-1})^{T*}$
$= (\underline{A}^{T^{-1}})^*$
$= ((\underline{A}^*)^{-1})^* = \underline{A}^{-1}$

$$\left[\underline{C}_{\tilde{s}}(\underline{C}_{\tilde{s}} + \sigma^2\underline{I})^{-1}\right]^H = \left[(\underline{C}_{\tilde{s}} + \sigma^2\underline{I})^{-1}\right]^H \underline{C}_{\tilde{s}}^H$$

using a)

But $\underline{C}_{\tilde{s}}$ is Hermitian $\Rightarrow$ $\underline{C}_{\tilde{s}} + \sigma^2\underline{I}$ is Hermitian by b) $\Rightarrow$ $(\underline{C}_{\tilde{s}} + \sigma^2\underline{I})^{-1}$ is Hermitian by c)

$$= (\underline{C}_{\tilde{s}} + \sigma^2\underline{I})^{-1}\underline{C}_{\tilde{s}}$$
$$= (\underline{C}_{\tilde{s}} + \sigma^2\underline{C}_{\tilde{s}}\underline{C}_{\tilde{s}}^{-1})^{-1}\underline{C}_{\tilde{s}}$$
$$= [(\underline{C}_{\tilde{s}}^2 + \sigma^2\underline{C}_{\tilde{s}})\underline{C}_{\tilde{s}}^{-1}]^{-1}\underline{C}_{\tilde{s}}$$

$$= \tilde{C_s}\left(\tilde{C_s}^2 + \sigma^2 \tilde{C_s}\right)^{-1} \tilde{C_s}$$

$$= \tilde{C_s}\left[\tilde{C_s}^{-1}\left(\tilde{C_s}^2 + \sigma^2 \tilde{C_s}\right)\right]^{-1}$$

$$= \tilde{C_s}\left(\tilde{C_s} + \sigma^2 I\right)^{-1}$$

13.8) $T(\tilde{x}) = \tilde{x}^H \tilde{C_s}\left(\tilde{C_s} + \sigma^2 I\right)^{-1} \tilde{x}$

Let $\tilde{x} = P\tilde{y}$

$$T(\tilde{x}) = \tilde{y}^H P^H \tilde{C_s} P P^H \left(P \Lambda_{\tilde{s}} P^H + \sigma^2 P P^H\right)^{-1} P \tilde{y}$$

$$= \tilde{y}^H \Lambda_{\tilde{s}} P^H \left[P\left(\Lambda_{\tilde{s}} + \sigma^2 I\right) P^H\right]^{-1} P \tilde{y}$$

$$= \tilde{y}^H \Lambda_{\tilde{s}} P^H \, P^{H^{-1}} \left(\Lambda_{\tilde{s}} + \sigma^2 I\right)^{-1} \cdot P^{-1} P \tilde{y}$$

$$= \tilde{y}^H \Lambda_{\tilde{s}} \left(\Lambda_{\tilde{s}} + \sigma^2 I\right)^{-1} \tilde{y}$$

$$= \sum_{n=0}^{N-1} \frac{\lambda_{\tilde{s}_n}}{\lambda_{\tilde{s}_n} + \sigma^2} \left|\tilde{y}[n]\right|^2$$

Under $\mathcal{H}_0$ $\quad \tilde{y} = P^H \tilde{x} \sim \mathcal{CN}(0, \underbrace{P^H \sigma^2 I P}_{\sigma^2 I})$

$$T(\tilde{x}) = \sum_n \frac{\lambda_{\tilde{s}_n} \sigma^2}{2(\lambda_{\tilde{s}_n} + \sigma^2)} \underbrace{\left|\frac{\tilde{y}[n]}{\sigma/\sqrt{2}}\right|^2}_{\chi^2_2}$$

Under $\mathcal{H}_1$ $\quad \tilde{y} = P^H \tilde{x} \sim \mathcal{CN}(0, \underbrace{P^H (\tilde{C_s} + \sigma^2 I) P}_{\Lambda_{\tilde{s}} + \sigma^2 I})$

$$T(\tilde{x}) = \sum_n \frac{\lambda \tilde{s}_n}{2} \underbrace{\frac{|\tilde{y}[n]|^2}{(\lambda \tilde{s}_n + \sigma^2)/2}}_{\chi_2^2}$$

Let $\alpha_n = \dfrac{\lambda \tilde{s}_n \sigma^2}{2(\lambda \tilde{s}_n + \sigma^2)} > 0 \qquad \mathcal{H}_0$

$$\frac{\lambda \tilde{s}_n}{2} > 0 \qquad \mathcal{H}_1$$

PDFs follow from given results

$$\begin{matrix} P_{FA} \\ P_D \end{matrix} = \int_{\gamma'}^{\infty} p(y)\, dy$$

$$= \sum_{n=0}^{N-1} A_n \underbrace{\int_{\gamma'}^{\infty} \frac{e^{-y/2\alpha_n}}{2\alpha_n}\, dy}_{e^{-\gamma'/2\alpha_n}}$$

$$\Rightarrow P_{FA} = \sum_{n=0}^{N-1} A_n e^{-\gamma'/2\alpha_n}$$

where $\alpha_n = \dfrac{\lambda \tilde{s}_n \sigma^2}{2(\lambda \tilde{s}_n + \sigma^2)}$

$$A_n = \prod_{\substack{i=0 \\ i \neq n}}^{N-1} \frac{1}{1 - \alpha_i/\alpha_n}$$

$$P_D = \sum_{n=0}^{N-1} B_n e^{-\gamma'/\lambda \tilde{s}_n}$$

where $B_n = \displaystyle\prod_{\substack{i=0 \\ i \neq n}}^{N-1} \frac{1}{1 - \lambda \tilde{s}_i / \lambda \tilde{s}_n}$

13.9) From (13.17) with $\tilde{h}[n]=1$

$$T'(\tilde{x}) = \left| \sum_{n=0}^{N-1} \tilde{x}[n] \right|^2 = N^2 |\bar{\tilde{x}}|^2$$

This is usual sample mean but we take $||^2$ since effect of $\tilde{A}$ is to rotate $\bar{\tilde{x}}$. For no signal $E(\bar{\tilde{x}}) = 0$ while for a signal $E(\bar{\tilde{x}} | \tilde{A}) = \tilde{A} = A_R + jA_I$.

$$P_D = P_{FA}^{\frac{1}{1+\bar{\eta}}}$$

where $\bar{\eta} = \bar{\mathcal{E}}/\sigma^2 = \sigma_A^2 \tilde{h}^H \tilde{h} / \sigma^2$
$= N\sigma_A^2/\sigma^2$.

13.10) $T'(\tilde{x}) = \left| \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi f_0 n} \right|^2$

$$P_D = P_{FA}^{\frac{1}{1+\bar{\eta}}}$$

$$\bar{\eta} = \bar{\mathcal{E}}/\sigma^2 = \sigma_A^2 \tilde{h}^H \tilde{h} / \sigma^2$$
$$= N\sigma_A^2/\sigma^2$$

Same performance as for complex random DC level. Only depends on $\tilde{h}^H \tilde{h}$ or energy since noise is CWGN.

13.11) From (13.19) with $\underline{H} = \underline{1}$ and $\hat{\underline{\theta}}_1 = \hat{A} = \bar{\tilde{x}}$

$$T(\tilde{\underline{x}}) = N|\bar{\tilde{x}}|^2/\sigma^2/2 > \gamma'$$

$$P_{FA} = Q_{\chi_2^2}(\gamma') \quad \text{since } p=1$$
$$= e^{-\frac{1}{2}\gamma'}$$

$$\Rightarrow \gamma' = 2\ln 1/P_{FA}$$

13.12) From (13.19) with $\underline{H} = [1 \; e^{j2\pi f_0} \ldots e^{j2\pi f_0(N-1)}]^T$ $\hat{\underline{\theta}}_1 = \hat{A} = \frac{1}{N}\underline{H}^H\tilde{\underline{x}}$

or

$$\hat{\tilde{A}} = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi f_0 n}$$

$$T(\tilde{\underline{x}}) = \frac{N|\hat{\tilde{A}}|^2}{\sigma^2/2} = \frac{I(f_0)}{\sigma^2/2} > \gamma'$$

where $I(f_0) = \frac{1}{N}\left|\sum_{n=0}^{N-1}x[n]e^{-j2\pi f_0 n}\right|^2$

is the periodogram for $f = f_0$.

For f_0 unknown we would need to maximize $I(f)$ over f (note that γ' will change - see Section 7.6.3)

For f_0 known

$$P_{FA} = Q_{\chi^2_2}(\gamma') = e^{-\gamma'/2}$$

$$\Rightarrow \gamma' = 2 \ln 1/P_{FA}$$

13.13)
$$\underline{s} = \underbrace{\begin{bmatrix} \tilde{\phi}_1(0) & \tilde{\phi}_2(0) \\ \vdots & \vdots \\ \tilde{\phi}_1(N-1) & \tilde{\phi}_2(N-1) \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}}_{\underline{\theta}}$$

Due to orthonormality the columns of $\underline{H}$ are orthonormal $\Rightarrow \underline{H}^H \underline{H} = \underline{I}$

$$\hat{\underline{\theta}}_1 = (\underline{H}^H \underline{H})^{-1} \underline{H}^H \tilde{\underline{x}} = \underline{H}^H \tilde{\underline{x}}$$

$$= \begin{bmatrix} \sum_{n=0}^{N-1} \tilde{\phi}_1^*(n) \tilde{x}(n) \\ \sum_{n=0}^{N-1} \tilde{\phi}_2^*(n) \tilde{x}(n) \end{bmatrix}$$

$$T(\tilde{\underline{x}}) = \|\hat{\underline{\theta}}_1\|^2 / \sigma^2/2$$

$$= \frac{\left| \sum_{n=0}^{N-1} \tilde{x}(n) \tilde{\phi}_1^*(n) \right|^2 + \left| \sum_{n=0}^{N-1} \tilde{x}(n) \tilde{\phi}_2(n) \right|^2}{\sigma^2/2}$$

For $\tilde{\phi}_1(n) = \frac{1}{\sqrt{N}} e^{j2\pi f_1 n}$

$\tilde{\phi}_2(n) = \frac{1}{\sqrt{N}} e^{j2\pi f_2 n}$

$$T(\underline{\tilde{x}}) = \frac{I(f_1) + I(f_2)}{\sigma^2/2}$$

where $I(f)$ is the periodogram.

(13.14) $\tilde{\underline{x}}[n] = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad n = 0$

$\begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad n = 1$

$\begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad n = 2$

$\tilde{\underline{x}}_m = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad m = 0$

$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad m = 1$.

$$\tilde{\underline{x}} = \begin{bmatrix} \tilde{\underline{x}}[0] \\ \tilde{\underline{x}}[1] \\ \tilde{\underline{x}}[2] \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{bmatrix}$$

$$\tilde{\underline{\underline{x}}} = \begin{bmatrix} \tilde{\underline{x}}_1 \\ \tilde{\underline{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

(13.15) $(\underline{A}, \underline{B}) = \mathrm{tr}(\underline{A}^H \underline{B})$

$$\underline{A}^H \underline{B} = \begin{bmatrix} \underline{a}_0^H \\ \vdots \\ \underline{a}_{N-1}^H \end{bmatrix} [\underline{b}_0 \ldots \underline{b}_{N-1}]$$

$$(\underline{A}, \underline{B}) = \sum_{i=0}^{N-1} \underbrace{\underline{a}_i^H \underline{b}_i}_{\text{correlation over column } i}$$

$$(\underline{A}, \underline{B}) = tr(\underline{A}^H \underline{B}) = tr(\underline{B}\underline{A}^H)$$

$$\underline{B}\,\underline{A}^H = \begin{bmatrix} \underline{d}_0^T \\ \vdots \\ \underline{d}_{N-1}^T \end{bmatrix} [\underline{c}_0^* \ldots \underline{c}_{N-1}^*]$$

$$tr(\underline{B}\underline{A}^H) = \sum_{i=0}^{N-1} \underline{d}_i^T \underline{c}_i^* = \sum_{i=0}^{N-1} \underbrace{\underline{c}_i^H \underline{d}_i}_{\text{correlation over row } i}$$

13.16)
$$r_{mm'}[k] = E[\tilde{x}_m^*[n]\tilde{x}_{m'}[n+k]]$$
$$= E\{\tilde{A}^* e^{-j[2\pi(f_0 m + f_1 n) + \phi]} \cdot \tilde{A} e^{j[2\pi(f_0 m' + f_1(n+k)) + \phi]}\}$$

$$= E[|\tilde{A}|^2 e^{j2\pi(f_0(m'-m) + f_1 k)}]$$

$$= |\tilde{A}|^2 e^{j2\pi(f_0(m'-m) + f_1 k)}$$

13.17)
$$\underline{R}_{\tilde{x}\tilde{x}}[k] = \begin{bmatrix} r_{00}[k] & r_{01}[k] \\ r_{10}[k] & r_{11}[k] \end{bmatrix}$$

$$= |\tilde{A}|^2 \begin{bmatrix} e^{j2\pi f_1 k} & e^{j2\pi(f_0 + f_1 k)} \\ e^{j2\pi(-f_0 + f_1 k)} & e^{j2\pi f_1 k} \end{bmatrix}$$

$$= |\tilde{A}|^2 e^{j2\pi f_1 k} \begin{bmatrix} 1 & e^{j2\pi f_0} \\ e^{-j2\pi f_0} & 1 \end{bmatrix}$$

$$\underline{P}_{\tilde{x}\tilde{x}}(f) = \mathcal{F}\{\underline{R}_{\tilde{x}\tilde{x}}(k)\}$$

$$= |\tilde{A}|^2 \, 2\pi \, \delta(f-f_1) \begin{bmatrix} 1 & e^{j2\pi f_0} \\ e^{-j2\pi f_0} & 1 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} \underline{R}_{\tilde{x}\tilde{x}}(0) & \underline{R}_{\tilde{x}\tilde{x}}^T(-1) & \underline{R}_{\tilde{x}\tilde{x}}^T(-2) \\ \underline{R}_{\tilde{x}\tilde{x}}^T(1) & \underline{R}_{\tilde{x}\tilde{x}}^T(0) & \underline{R}_{\tilde{x}\tilde{x}}^T(-1) \\ \underline{R}_{\tilde{x}\tilde{x}}^T(2) & \underline{R}_{\tilde{x}\tilde{x}}^T(1) & \underline{R}_{\tilde{x}\tilde{x}}^T(0) \end{bmatrix}$$

13.18) The process is uncorrelated between sensors.

$$\underline{R}_{\tilde{x}\tilde{x}}(k) = \begin{bmatrix} r_{00}(k) & \cancel{r_{01}(k)}\ 0 \\ \cancel{r_{10}(k)}\ 0 & r_{11}(k) \end{bmatrix}$$

$$\underline{P}_{\tilde{x}\tilde{x}}(f) = \mathcal{F}\{\underline{R}_{\tilde{x}\tilde{x}}(k)\}$$

$$= \begin{bmatrix} P_{00}(f) & 0 \\ 0 & P_{11}(f) \end{bmatrix}$$

Each $\underline{R}_{\tilde{x}\tilde{x}}(k)$ for $k = 0, 1, 2$ is diagonal but not $\underline{C}$.

For spatial ordering $\underline{C}$ will be diagonal.

13.19) $\underline{P}_{\tilde{x}\tilde{x}}(f) = \mathcal{F}\{\underline{R}_{\tilde{x}\tilde{x}}(k)\}$

$$\underline{P}_{\tilde{x}\tilde{x}}^{H}(f) = \sum_{k=-\infty}^{\infty} \underline{R}_{\tilde{x}\tilde{x}}^{H}(k)\, e^{j2\pi fk}$$

$$= \sum_{k=-\infty}^{\infty} \underline{R}_{\tilde{x}\tilde{x}}(-k)\, e^{j2\pi fk}$$

$$= \sum_{k=-\infty}^{\infty} \underline{R}_{\tilde{x}\tilde{x}}(k)\, e^{-j2\pi fk}$$

$$= \underline{P}_{\tilde{x}\tilde{x}}(f)$$

$$r_{\tilde{y}\tilde{y}}(k) = E(\tilde{y}^{*}(n)\tilde{y}(n+k))$$
$$= E(\underline{\tilde{\xi}}^{H}\underline{\tilde{x}}^{*}(n)\underline{\tilde{x}}^{T}(n+k)\underline{\tilde{\xi}})$$
$$= \underline{\tilde{\xi}}^{H}\underline{R}_{\tilde{x}\tilde{x}}(k)\underline{\tilde{\xi}}$$

$$P_{\tilde{y}\tilde{y}}(f) = \mathcal{F}\{R_{\tilde{y}\tilde{y}}(k)\}$$
$$= \underline{\tilde{\xi}}^{H}\underline{P}_{\tilde{x}\tilde{x}}(f)\underline{\tilde{\xi}} \geq 0 \quad \text{all } \underline{\tilde{\xi}}$$

since $P_{\tilde{y}\tilde{y}}(f) \geq 0$

13.20) $$T(\tilde{x}) = \sum_{m=0}^{M-1}\sum_{i=0}^{N-1} \frac{P_{mm}(f_i)}{P_{mm}(f_i)+\sigma^2} I_m(f_i)$$

$$= N \sum_{m=0}^{M-1} \underbrace{\sum_{i=0}^{N-1} \frac{P_{mm}(f_i)}{P_{mm}(f_i)+\sigma^2} I_m(f_i)\frac{1}{N}}$$

$$\to \int_0^1 \frac{P_{mm}(f)}{P_{mm}(f)+\sigma^2} I_m(f)\,df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_{mm}(f)}{P_{mm}(f) + \sigma^2} I_m(f)\, df$$

due to periodicity of 1.

This is just the sum of the large N-WSS estimator correlator outputs for each sensor (see Section 5.5)

13.21) For $M=3$

$$\underline{R}_{\tilde{x}\tilde{x}}[k] = \begin{bmatrix} r_{00}[k] & r_{01}[k] & r_{02}[k] \\ r_{10}[k] & r_{11}[k] & r_{12}[k] \\ r_{20}[k] & r_{21}[k] & r_{22}[k] \end{bmatrix}$$

$$= \begin{bmatrix} r[0,k] & r[1,k] & r[2,k] \\ r[-1,k] & r[0,k] & r[1,k] \\ r[-2,k] & r[-1,k] & r[0,k] \end{bmatrix}$$

Toeplitz but not Hermitian since

$$r[j-i,k] = r_{ij}[k] = r^*_{ji}[-k]$$

$$= r^*(i-j,-k)$$

13.22) $$\underline{C}_{\tilde{s}} = \begin{bmatrix} \underline{R}^T_{\tilde{x}\tilde{x}}[0] & \underline{R}^T_{\tilde{x}\tilde{x}}[-1] & \underline{R}^T_{\tilde{x}\tilde{x}}[-2] \\ \underline{R}^T_{\tilde{x}\tilde{x}}[1] & \underline{R}^T_{\tilde{x}\tilde{x}}[0] & \underline{R}^T_{\tilde{x}\tilde{x}}[-1] \\ \underline{R}^T_{\tilde{x}\tilde{x}}[2] & \underline{R}^T_{\tilde{x}\tilde{x}}[1] & \underline{R}^T_{\tilde{x}\tilde{x}}[0] \end{bmatrix}$$

Each $\underline{R}^T_{\tilde{x}\tilde{x}}[k]$ is 2×2.

$\underline{C}_{\tilde{\xi}}$ is seen to be block-Toeplitz and each $\underline{R}_{\tilde{x}\tilde{x}}[k]$ is Toeplitz from Problem 13.21. To see if it is Toeplitz

$\underline{C}_{\tilde{\xi}} =$

$$\left[\begin{array}{cc|cc|cc}
r[0,0] & r[-1,0] & r[0,-1] & r[-1,-1] & r[0,-2] & r[-1,-2] \\
r[1,0] & r[0,0] & r[1,-1] & r[0,-1] & r[1,-2] & r[0,-2] \\
\hline
r[0,1] & r[-1,1] & r[0,0] & r[-1,0] & r[0,-1] & r[-1,-1] \\
r[1,1] & r[0,1] & r[1,0] & r[0,0] & r[1,-1] & r[0,-1] \\
\hline
r[0,2] & r[-1,2] & r[0,1] & r[-1,1] & r[0,0] & r[-1,0] \\
r[1,2] & r[0,2] & r[1,1] & r[0,1] & r[1,0] & r[0,0]
\end{array}\right]$$

$\Rightarrow$ not Toeplitz

13.23) For $\beta = \pi/2$ $\quad \eta_m(\beta) = -\dfrac{\underline{r}_m^T \underline{u}}{c\Delta}$

But $\underline{u} = [\cos\beta \;\; \sin\beta]^T = [0 \;\; 1]^T$

$\Rightarrow \eta_m(\beta) = 0$

$$T(\tilde{\underline{x}}) = \frac{1}{MN\sigma^2/2}\left|\sum_m \sum_n \tilde{x}_m[n]\, e^{-j2\pi f_D n}\right|^2$$

$$= \frac{M}{N\sigma^2/2}\left|\frac{1}{M}\sum_{m=0}^{M-1} \underbrace{\sum_{n=0}^{N-1} \tilde{x}_m[n]\, e^{-j2\pi f_D n}}_{X_m(f_D)}\right|^2$$

All signals at sensors are in phase since there is no time delay. Hence, we just add up Fourier transforms in our beamforming.

13.24) As in Problem 13.23 $n_m(\beta) = 0$

$$T(\underset{\sim}{x}) = \frac{M}{N} \sum_{i=0}^{N-1} H(f_i) \underbrace{\left| \frac{1}{M} \sum_{m=0}^{M-1} X_m(f_i) \right|^2}_{\hat{S}_B(f_i)}$$

We first beamform by averaging Fourier transforms at sensors. Then, we take the magnitude-squared since the phase is random (the signal is a WSS random process). Finally, we use a Wiener filter to get rid of the CWGN.