统计信号处理

第十四章

复合假设检验 II
(基本方法II)

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内容概要

- 一、大数据量时GLRT等效方法
- 二、大数据量时GLRT的性能
- 三、局部最大势检测
- 四、最小错误概率检测——广义ML检测
- 五、应用案例
- 六、小结

一、大数据量时GLRT等效方法

假设检验模型

$$H_0:\boldsymbol{\theta}=\boldsymbol{\theta}_0$$

$$H_1:\boldsymbol{\theta}\neq\boldsymbol{\theta}_0 \quad (\boldsymbol{\theta}_1\approx\boldsymbol{\theta}_0) \quad \text{弱信号}$$

采用NP准则，若广义似然比

$$L_G(\mathbf{x})=\frac{p(\mathbf{x};\hat{\boldsymbol{\theta}}_1)}{p(\mathbf{x};\boldsymbol{\theta}_0)}>\gamma \quad \text{，则判 } H_1$$

隐含前提：不同假设下，观察数据具有相同的pdf形式

$N\to\infty$时，MLE是**渐近有效**的，

$$\frac{\partial\ln p(\mathbf{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}=\mathbf{I}(\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}) \approx\mathbf{I}(\hat{\boldsymbol{\theta}}_1)(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}) \Rightarrow$$

$$\ln p(\mathbf{x};\boldsymbol{\theta})=-\frac{1}{2}(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta})^T\mathbf{I}(\hat{\boldsymbol{\theta}}_1)(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta})+\ln p(\mathbf{x};\hat{\boldsymbol{\theta}}_1)$$

$$\ln p(\mathbf{x};\boldsymbol{\theta}) = -\frac{1}{2}\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}\right) + \ln p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1\right)$$

$$p(\mathbf{x};\boldsymbol{\theta}) = p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1\right)\exp\left\{-\frac{1}{2}\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}\right)\right\}$$

$$L_G(\mathbf{x}) = \frac{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1\right)}{p\left(\mathbf{x};\boldsymbol{\theta}_0\right)}$$

$$L_G(\mathbf{x}) = \frac{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1\right)}{p\left(\mathbf{x};\hat{\boldsymbol{\theta}}_1\right)\exp\left\{-\frac{1}{2}\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)\right\}}$$

$$2\ln L_G(\mathbf{x}) = \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)$$

$$\boxed{T_W(\mathbf{x}) = \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)}$$

前提条件：
- 双边检测
- 弱信号
- 大数据量
- 相同PDF形式

- 无需比较PDF
- 但仍需求MLE

Wald检验

$N \to \infty$ 时

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}(\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}) \Rightarrow \hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta} = \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

$$\Rightarrow \hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0 = \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$$

Wald检验: $2\ln L_G(\mathbf{x}) = (\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0)^T \mathbf{I}(\hat{\boldsymbol{\theta}}_1)(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0)$

$$2\ln L_G(\mathbf{x}) = \left(\left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)^T \underline{\mathbf{I}^{-1}(\boldsymbol{\theta}_0)\mathbf{I}(\hat{\boldsymbol{\theta}}_1)\mathbf{I}^{-1}(\boldsymbol{\theta}_0)} \left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$$

$$\Rightarrow 2\ln L_G(\mathbf{x}) = \left(\left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)^T \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$$

前提条件：

- 双边检测
- 弱信号
- 大数据量
- 相同PDF形式

$$\boxed{T_R(\mathbf{x}) = \left(\left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)^T \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}}$$

•无需比较PDF
•无需求MLE

Rao检验

例：WGN中未知信号检测——Wald与Rao检验

$$H_0: x[n] = w[n]$$

$$H_1: x[n] = A + w[n]$$

其中信号电平$A(-\infty < A < +\infty)$是未知的。噪声$w[n]$是方差为σ^2的WGN。如何检测是否存在信号？

Wald检验： $T_W(\mathbf{x}) = \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)$

$$p(\mathbf{x}; A, H_1) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n] - A\right)^2\right\}$$

$$\frac{\partial \ln p(\mathbf{x}; A, H_1)}{\partial A} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left(x[n] - A\right) \Rightarrow \hat{\boldsymbol{\theta}}_1 = \hat{A} = \bar{x}$$

$$\boldsymbol{\theta}_0 = 0$$

$$\frac{\partial^2 \ln p(\mathbf{x}; A, H_1)}{\partial A^2} = -\frac{N}{\sigma^2} \Rightarrow \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right) = \mathbf{I}\left(\hat{A}\right) = \frac{N}{\sigma^2}$$

$$\Rightarrow T_W(\mathbf{x}) = \frac{N\bar{x}^2}{\sigma^2} \Rightarrow T_W'(\mathbf{x}) = \bar{x}^2$$

例：WGN中未知信号检测——Wald与Rao检验

$$H_0: x[n] = w[n]$$
$$H_1: x[n] = A + w[n]$$

其中信号电平$A(-\infty < A < +\infty)$是未知的。噪声$w[n]$是方差为σ^2的WGN。如何检测是否存在信号？

Rao检验： $$T_R(\mathbf{x}) = \left(\left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)^T \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \left.\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$$

$$p(\mathbf{x};A) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\}$$

$$\frac{\partial \ln p(\mathbf{x};A)}{\partial A} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A) \Rightarrow \left.\frac{\partial \ln p(\mathbf{x};A)}{\partial A}\right|_{A=0} = \frac{N\bar{x}}{\sigma^2}$$

$$\frac{\partial^2 \ln p(\mathbf{x};A)}{\partial A^2} = -\frac{N}{\sigma^2} \Rightarrow \mathbf{I}(\boldsymbol{\theta}_0) = \left.\left\{E\left(-\frac{\partial^2 \ln p(\mathbf{x};A)}{\partial A^2}\right)\right\}\right|_{A=0} = \frac{N}{\sigma^2}$$

$$\Rightarrow T_R(\mathbf{x}) = \frac{N\bar{x}^2}{\sigma^2}$$

$$\Rightarrow T_R'(\mathbf{x}) = \bar{x}^2$$

✓ *对线性模型，GLRT、Wald、Rao的检验统计量是相同的*

例：非高斯噪声中未知信号检测——Rao检验

$$H_0: x[n]=w[n]$$
$$H_1: x[n]=A+w[n]$$

其中信号电平$A(-\infty<A<+\infty)$是未知的。噪声$w[n]$是独立同分布的，服从均值为零、方差为 σ^2 的广义高斯分布，其PDF为：

$$p(w[n])=\frac{1}{a\sigma\Gamma\left(\frac{5}{4}\right)2^{\frac{5}{4}}}\exp\left\{-\frac{1}{2}\left(\frac{w[n]}{a\sigma}\right)^4\right\},-\infty<w[n]<+\infty$$

其中常数为

$$a=\left(\frac{\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\Gamma\left(\frac{3}{4}\right)}\right)^{\frac{1}{2}}=1.4464$$

$$p(w[n])=\frac{1}{a\sigma\Gamma\left(\frac{5}{4}\right)2^{\frac{5}{4}}}\exp\left\{-\frac{1}{2}\left(\frac{w[n]}{a\sigma}\right)^4\right\},-\infty<w[n]<+\infty$$

$$\Rightarrow\quad p(\mathbf{x};A)=\left(\frac{1}{a\sigma\Gamma\left(\frac{5}{4}\right)2^{\frac{5}{4}}}\right)^N\exp\left\{-\frac{1}{2}\sum_{n=0}^{N-1}\left(\frac{x[n]-A}{a\sigma}\right)^4\right\}$$

$$\Rightarrow\quad \ln p(\mathbf{x};A)=-N\ln\left(a\sigma\Gamma\left(\frac{5}{4}\right)2^{\frac{5}{4}}\right)-\frac{1}{2}\sum_{n=0}^{N-1}\left(\frac{x[n]-A}{a\sigma}\right)^4$$

$$\Rightarrow\quad \frac{\partial\ln p(\mathbf{x};A)}{\partial A}=\frac{2}{a\sigma}\sum_{n=0}^{N-1}\left(\frac{x[n]-A}{a\sigma}\right)^3$$

MLE: $\frac{\partial\ln p(\mathbf{x};A)}{\partial A}=0$ $\quad\hat{A}=?$ 难以求解！

$$\text{Rao检验：} T_R(\mathbf{x}) = \left(\left. \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right)^T \mathbf{I}^{-1}(\boldsymbol{\theta}_0) \left. \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$$

$$\frac{\partial \ln p(\mathbf{x};A)}{\partial A} = \frac{2}{a\sigma} \sum_{n=0}^{N-1} \left(\frac{x[n]-A}{a\sigma} \right)^3 \Rightarrow$$

$$\left. \frac{\partial \ln p(\mathbf{x};A)}{\partial A} \right|_{A=0} = \left. \frac{2}{a\sigma} \sum_{n=0}^{N-1} \left(\frac{x[n]-A}{a\sigma} \right)^3 \right|_{A=0} = \frac{2}{a^4\sigma^4} \sum_{n=0}^{N-1} x^3[n]$$

$$\frac{\partial^2 \ln p(\mathbf{x};A)}{\partial A^2} = -\frac{6}{a^2\sigma^2} \sum_{n=0}^{N-1} \left(\frac{x[n]-A}{a\sigma} \right)^2 \Rightarrow$$

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{I}(A) = E\left(-\frac{\partial^2 \ln p(\mathbf{x};A)}{\partial A^2} \right) = \frac{6}{a^2\sigma^2} \sum_{n=0}^{N-1} E\left(\left(\frac{x[n]-A}{a\sigma} \right)^2 \right) = \frac{6N}{a^4\sigma^2}$$

$$\Rightarrow T_R(\mathbf{x}) = \frac{2N}{3a^4\sigma^6} \left(\frac{1}{N} \sum_{n=0}^{N-1} x^3[n] \right)^2$$

存在多余参数时Wald检验与Rao检验

假设检验模型（存在多余参数）

$$H_0: \boldsymbol{\theta}_r = \boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s$$
$$H_1: \boldsymbol{\theta}_r \neq \boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s \qquad \left(\boldsymbol{\theta}_{r_1} \approx \boldsymbol{\theta}_{r_0}\right)$$

$\boldsymbol{\theta}_s$ 是多余且未知的参数

Wald检验：

$$T_W(\mathbf{x}) = \left(\hat{\boldsymbol{\theta}}_{r_1} - \boldsymbol{\theta}_{r_0}\right)^T \left(\left[\mathbf{I}^{-1}\left(\hat{\boldsymbol{\theta}}_1\right)\right]_{\theta_r\theta_r}\right)^{-1} \left(\hat{\boldsymbol{\theta}}_{r_1} - \boldsymbol{\theta}_{r_0}\right)$$

其中 $\hat{\boldsymbol{\theta}}_1 = \left[\hat{\boldsymbol{\theta}}_{r_1}^T, \hat{\boldsymbol{\theta}}_{s_1}^T\right]^T$ 是 $\boldsymbol{\theta}$ 在 H_1 下的MLE，$\left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{\theta_r\theta_r}$ 是 $\mathbf{I}^{-1}(\boldsymbol{\theta})$ 的左上分块矩阵，

$$\left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{\theta_r\theta_r} = \left(\mathbf{I}_{\theta_r\theta_r}(\boldsymbol{\theta}) - \mathbf{I}_{\theta_r\theta_s}(\boldsymbol{\theta})\mathbf{I}_{\theta_s\theta_s}^{-1}(\boldsymbol{\theta})\mathbf{I}_{\theta_s\theta_r}(\boldsymbol{\theta})\right)^{-1}$$

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{I}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) = \begin{bmatrix} \mathbf{I}_{\theta_r\theta_r}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) & \mathbf{I}_{\theta_r\theta_s}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) \\ \mathbf{I}_{\theta_s\theta_r}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) & \mathbf{I}_{\theta_s\theta_s}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) \end{bmatrix}$$

前提条件：

- 双边检测
- 弱信号
- 大数据量
- 相同pdf形式

（推导见附录6B）

假设检验模型（存在多余参数）

$$H_0:\boldsymbol{\theta}_r=\boldsymbol{\theta}_{r_0},\boldsymbol{\theta}_s$$
$$H_1:\boldsymbol{\theta}_r\neq\boldsymbol{\theta}_{r_0},\boldsymbol{\theta}_s \qquad \left(\boldsymbol{\theta}_{r_1}\approx\boldsymbol{\theta}_{r_0}\right)$$

$\boldsymbol{\theta}_s$ 是多余且未知的参数

Rao检验： $$T_R(\mathbf{x})=\left(\left.\frac{\partial\ln p(\mathbf{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_r}\right|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}}\right)^T\left[\mathbf{I}^{-1}(\tilde{\boldsymbol{\theta}})\right]_{\theta_r\theta_r}\left.\frac{\partial\ln p(\mathbf{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_r}\right|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}}$$

其中 $\tilde{\boldsymbol{\theta}}=\left[\boldsymbol{\theta}_{r_0}^T,\hat{\boldsymbol{\theta}}_{s_0}^T\right]^T$ 是 $\boldsymbol{\theta}$ 在 H_0下的MLE，$\left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{\theta_r\theta_r}$ 是 $\mathbf{I}^{-1}(\boldsymbol{\theta})$ 的左上分块矩阵，

$$\left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{\theta_r\theta_r}=\left(\mathbf{I}_{\theta_r\theta_r}(\boldsymbol{\theta})-\mathbf{I}_{\theta_r\theta_s}(\boldsymbol{\theta})\mathbf{I}_{\theta_s\theta_s}^{-1}(\boldsymbol{\theta})\mathbf{I}_{\theta_s\theta_r}(\boldsymbol{\theta})\right)^{-1}$$

$$\mathbf{I}(\boldsymbol{\theta})=\mathbf{I}(\boldsymbol{\theta}_r,\boldsymbol{\theta}_s)=\begin{bmatrix}\mathbf{I}_{\theta_r\theta_r}(\boldsymbol{\theta}_r,\boldsymbol{\theta}_s) & \mathbf{I}_{\theta_r\theta_s}(\boldsymbol{\theta}_r,\boldsymbol{\theta}_s)\\ \mathbf{I}_{\theta_s\theta_r}(\boldsymbol{\theta}_r,\boldsymbol{\theta}_s) & \mathbf{I}_{\theta_s\theta_s}(\boldsymbol{\theta}_r,\boldsymbol{\theta}_s)\end{bmatrix}$$

前提条件：
- 双边检测
- 弱信号
- 大数据量
- 相同pdf形式

（推导见附录6B）

二、大数据量时GLRT的性能

GLRT检验统计量：

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x};\hat{\boldsymbol{\theta}}_1)}{p(\mathbf{x};\boldsymbol{\theta}_0)}$$

ROC ?

$$P_{FA} = \Pr\left(L_G(\mathbf{x}) > \gamma; H_0\right)$$
$$P_D = \Pr\left(L_G(\mathbf{x}) > \gamma; H_1\right)$$

1. 无多余参数时

$$H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$$
$$H_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0 \quad (\boldsymbol{\theta}_1 \approx \boldsymbol{\theta}_0)$$

GLRT检验统计量

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x};\hat{\boldsymbol{\theta}}_1)}{p(\mathbf{x};\boldsymbol{\theta}_0)}$$

大数据量时 →

Wald检验统计量

$$2\ln L_G(\mathbf{x}) = \left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)^T \mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_0\right)$$

$$2\ln L_G(\mathbf{x})=\left(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\right)^T\mathbf{I}\left(\hat{\boldsymbol{\theta}}_1\right)\left(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\right)$$

当 $N\to\infty$ 时

$$\hat{\boldsymbol{\theta}}_1\overset{a}{\sim}\begin{cases}N\left(\boldsymbol{\theta}_0,\mathbf{I}^{-1}(\boldsymbol{\theta}_0)\right), & H_0\\ N\left(\boldsymbol{\theta}_1,\mathbf{I}^{-1}(\boldsymbol{\theta}_1)\right), & H_1\end{cases}\quad\Rightarrow\quad\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\overset{a}{\sim}\begin{cases}N\left(\mathbf{0},\mathbf{I}^{-1}(\boldsymbol{\theta}_0)\right), & H_0\\ N\left(\boldsymbol{\theta}_1-\boldsymbol{\theta}_0,\mathbf{I}^{-1}(\boldsymbol{\theta}_1)\right), & H_1\end{cases}$$

若 $\mathbf{x}\sim N(\boldsymbol{\mu},\mathbf{C})$，那么 $y=\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}$ 服从如下非中心chi方分布

$$y\sim\chi_n^2(\lambda)$$

其中非中心参量 $\lambda=\boldsymbol{\mu}^T\mathbf{C}^{-1}\boldsymbol{\mu}$ 。

H_1 时， 当 $N\to\infty$ ，$\hat{\boldsymbol{\theta}}_1\to\boldsymbol{\theta}_1\approx\boldsymbol{\theta}_0$

H_1 时，$2\ln L_G(\mathbf{x})\to\left(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\right)^T\mathbf{I}(\boldsymbol{\theta}_1)\left(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\right)\overset{a}{\sim}\chi_p^2(\lambda)$,$\lambda=(\boldsymbol{\theta}_1-\boldsymbol{\theta}_0)^T\mathbf{I}(\boldsymbol{\theta}_1)(\boldsymbol{\theta}_1-\boldsymbol{\theta}_0)$

$=(\boldsymbol{\theta}_1-\boldsymbol{\theta}_0)^T\mathbf{I}(\boldsymbol{\theta}_0)(\boldsymbol{\theta}_1-\boldsymbol{\theta}_0)$

H_0 时, $2\ln L_G(\mathbf{x})\to\left(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\right)^T\mathbf{I}(\boldsymbol{\theta}_0)\left(\hat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_0\right)\overset{a}{\sim}\chi_p^2(0)$

例：WGN中未知信号检测

$$H_0: x[n] = w[n]$$
$$H_1: x[n] = A + w[n]$$

其中信号电平$A(-\infty < A < +\infty)$是未知的。噪声$w[n]$是方差为σ^2的WGN。GLRT检测统计量的性能？

GLRT检测统计量：$2\ln L_G(\mathbf{x}) = \dfrac{N\bar{x}^2}{\sigma^2}$ （见第十三章P21）

H_1时，$2\ln L_G(\mathbf{x}) \to (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0)^T \mathbf{I}(\boldsymbol{\theta}_1)(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0) \overset{a}{\sim} \chi_p'^2(\lambda)$ ，$\lambda = (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0)^T \mathbf{I}(\boldsymbol{\theta}_0)(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0)$

H_0时，$2\ln L_G(\mathbf{x}) \to (\boldsymbol{\theta}_0 - \boldsymbol{\theta}_0)^T \mathbf{I}(\boldsymbol{\theta}_0)(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_0) \overset{a}{\sim} \chi_p'^2(0)$

$\boldsymbol{\theta}_1 = A,\quad \boldsymbol{\theta}_0 = 0,\quad \mathbf{I}(\boldsymbol{\theta}_0) = \dfrac{N}{\sigma^2}$

✓ *对线性模型，渐近统计特性对有限数据记录是有效、且精确的*

$$2\ln L_G(\mathbf{x}) \overset{a}{\sim} \begin{cases} \chi_1'^2(0), & H_0 \\ \chi_1'^2\left(\dfrac{NA^2}{\sigma^2}\right), & H_1 \end{cases}$$

，且进一步有 $2\ln L_G(\mathbf{x}) \sim \begin{cases} \chi_1'^2(0), & H_0 \\ \chi_1'^2\left(\dfrac{NA^2}{\sigma^2}\right), & H_1 \end{cases}$

2. 存在多余参数时

$$H_0: \boldsymbol{\theta}_r = \boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s$$

$$H_1: \boldsymbol{\theta}_r \neq \boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s \qquad \left(\boldsymbol{\theta}_{r_1} \approx \boldsymbol{\theta}_{r_0}\right)$$

$\boldsymbol{\theta}_s$ 是多余且未知的参数

$$2\ln L_G(\boldsymbol{x}) \overset{a}{\sim} \begin{cases} \chi_r^2(0), & H_0 \\ \chi_r^2(\lambda), & H_1 \end{cases}$$

其中，非中心参量

$$\lambda = \left(\boldsymbol{\theta}_{r_1} - \boldsymbol{\theta}_{r_0}\right)^T \left[\mathbf{I}_{\theta_r\theta_r}\left(\boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s\right) - \mathbf{I}_{\theta_r\theta_s}\left(\boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s\right)\mathbf{I}_{\theta_s\theta_s}^{-1}\left(\boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s\right)\mathbf{I}_{\theta_s\theta_r}\left(\boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s\right)\right]\left(\boldsymbol{\theta}_{r_1} - \boldsymbol{\theta}_{r_0}\right)$$

Fisher信息矩阵为

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{I}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) = \begin{bmatrix} \mathbf{I}_{\theta_r\theta_r}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) & \mathbf{I}_{\theta_r\theta_s}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) \\ \mathbf{I}_{\theta_s\theta_r}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) & \mathbf{I}_{\theta_s\theta_s}(\boldsymbol{\theta}_r, \boldsymbol{\theta}_s) \end{bmatrix}$$

（推导见附录6C）

三、局部最大势检测

单边检验:

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta > \theta_0 \end{cases} \quad (\theta_1 \approx \theta_0)$$ **弱信号**

采用NP准则，若似然比

$$L(\mathbf{x}) = \frac{p(\mathbf{x};\theta,H_1)}{p(\mathbf{x};\theta_0,H_0)} > \gamma$$

简记为：$L(\mathbf{x}) = \frac{p(\mathbf{x};\theta)}{p(\mathbf{x};\theta_0)} > \gamma$

则判 H_1

$$L(\mathbf{x})=\frac{p(\mathbf{x};\theta)}{p(\mathbf{x};\theta_0)}>\gamma \iff \ln p(\mathbf{x};\theta)-\ln p(\mathbf{x};\theta_0)>\ln\gamma$$

$$\ln p(\mathbf{x};\theta)=\ln p(\mathbf{x};\theta_0)+\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}(\theta-\theta_0)$$

$$\Rightarrow \left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}(\theta-\theta_0)>\ln\gamma$$

由于 $\theta>\theta_0$

$$\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}>\frac{\ln\gamma}{\theta-\theta_0}=\gamma'$$

$$T_{LMP}(\mathbf{x})=\frac{\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}}{\sqrt{I(\theta_0)}}>\frac{\ln\gamma}{\sqrt{I(\theta_0)}(\theta-\theta_0)}=\gamma''$$

局部最大势（Locally Most Powerful, LMP）检验

● 大数据量时LMP的性能

检测统计量：$T_{LMP}(\mathbf{x}) = \dfrac{\left.\dfrac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}}{\sqrt{I(\theta_0)}}$

若观测数据是IID的，

$$\ln p(\mathbf{x};\theta) = \sum_{n=0}^{N-1} \ln p(x[n];\theta)$$

$$\Rightarrow \left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0} = \sum_{n=0}^{N-1} \left.\frac{\partial \ln p(x[n];\theta)}{\partial \theta}\right|_{\theta=\theta_0} \stackrel{a}{\sim} N\left(\mu_{LMP}, \sigma_{LMP}^2\right)$$ （中心极限定理）

- 在 H_0条件下

$$E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}\right) = 0$$

$$\operatorname{var}\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}\right)=E\left\{\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}-\underline{E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}\right)}\right)^2\right\}=I(\theta_0)$$

$$\Rightarrow \quad \left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}\overset{a}{\sim}N\left(0,I(\theta_0)\right)$$

$$\Rightarrow \quad \frac{\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}}{\sqrt{I(\theta_0)}}\overset{a}{\sim}N(0,1)$$

- 在 H_1条件下 $E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}\right)$? $=0$?

$$\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_0}=\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta=\theta_1}+\left.\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right|_{\theta=\theta_1}(\theta_0-\theta_1)$$

$$E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}\right)=\underline{E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_1}\right)}+E\left(\left.\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial\theta^2}\right|_{\theta=\theta_1}\right)(\theta_0-\theta_1)$$

$$\Rightarrow\quad E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}\right)=-I(\theta_1)(\theta_0-\theta_1)$$

$$\operatorname{var}\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}\right)=E\left\{\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}-E\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}\right)\right)^2\right\}$$

$$\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}=\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_1}+\left.\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial\theta^2}\right|_{\theta=\theta_1}(\theta_0-\theta_1)$$

$$\Rightarrow$$

$$\operatorname{var}\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}\right)=E\left\{\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_1}+\left(\left.\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial\theta^2}\right|_{\theta=\theta_1}+I(\theta_1)\right)(\theta_0-\theta_1)\right)^2\right\}$$

$$\operatorname{var}\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}\right) = E\left\{\begin{aligned}&\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_1}\right)^2 + \left(\left.\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial\theta^2}\right|_{\theta=\theta_1} + I(\theta_1)\right)^2 (\theta_0-\theta_1)^2 \\ &+2\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_1}\left(\left.\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial\theta^2}\right|_{\theta=\theta_1} + I(\theta_1)\right)(\theta_0-\theta_1)\end{aligned}\right\}$$

$$\approx E\left\{\left(\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_1}\right)^2\right\} = I(\theta_1) \quad \approx I(\theta_0)$$

$$\Rightarrow \left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0} \overset{a}{\sim} N\left(-I(\theta_1)(\theta_0-\theta_1), I(\theta_0)\right)$$

$$\Rightarrow \left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0} \overset{a}{\sim} N\left(I(\theta_0)(\theta_1-\theta_0), I(\theta_0)\right) \Rightarrow \frac{\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}}{\sqrt{I(\theta_0)}} \overset{a}{\sim} N\left(\sqrt{I(\theta_0)}(\theta_1-\theta_0), 1\right)$$

因此，对大数据量：

$$T_{LMP}(\mathbf{x}) = \frac{\left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right|_{\theta=\theta_0}}{\sqrt{I(\theta_0)}} \overset{a}{\sim} \begin{cases} N(0,1), & H_0 \\ N\left(\sqrt{I(\theta_0)}(\theta_1-\theta_0), 1\right), & H_1 \end{cases}$$

四、最小错误概率检测——广义ML准则

例：WGN中电平检测——多元假设检验

$$H_0 : x[n] = w[n]$$
$$H_1 : x[n] = A + w[n]$$
$$H_2 : x[n] = A + Bn + w[n]$$

噪声 $w[n]$是方差为σ^2的WGN。A, B, σ^2 是未知参数。采用最小错误概率准则时，应该如何检测？

最小错误概率准则/最大后验概率准则

$$\max_i p(H_i \mid \mathbf{x}) \xrightarrow{\text{先验相同时}} \max_i p(\mathbf{x} \mid H_i)$$

GLRT?

NP准则下的GLRT：

若 $L_G(\mathbf{x}) = \dfrac{\max\limits_{\theta_1} p(\mathbf{x};\boldsymbol{\theta}_1, H_1)}{\max\limits_{\theta_0} p(\mathbf{x};\boldsymbol{\theta}_0, H_0)} > \gamma$

则判 H_1

推广至

$$\max_{\theta_i} p(\mathbf{x};\boldsymbol{\theta}_i \mid H_i)$$

$$\Updownarrow$$

$$\max_{\theta_i} \ln p(\mathbf{x};\boldsymbol{\theta}_i \mid H_i)$$

$$p(\mathbf{x};A,B,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-A-Bn\right)^2\right\}$$

$$\max_{\theta_i} p(\mathbf{x};\boldsymbol{\theta}_i \mid H_i) \Rightarrow \left\{\begin{array}{l} H_0 : \max\limits_{\sigma^2} p(\mathbf{x};A=0,B=0,\sigma^2) \\ H_1 : \max\limits_{A,\sigma^2} p(\mathbf{x};A,B=0,\sigma^2) \\ H_2 : \max\limits_{A,B,\sigma^2} p(\mathbf{x};A,B,\sigma^2) \end{array}\right\} \Rightarrow \text{总是判 } H_2$$

$$\max_{\theta_i} \ln p\left(\boldsymbol{x};\boldsymbol{\theta}_i \mid H_i\right) \Rightarrow \max_i \left\{ \ln p\left(\boldsymbol{x};\hat{\boldsymbol{\theta}}_i \mid H_i\right) - \underline{\frac{1}{2}\ln\left(\det\left(\mathbf{I}\left(\hat{\boldsymbol{\theta}}_i\right)\right)\right)} \right\}$$

修正项

或惩罚项(penalty term)

（推导见附录6F）

最小错误概率准则下（假定先验概率相同）检测：

$$\max_i \left\{ \ln p\left(\boldsymbol{x};\hat{\boldsymbol{\theta}}_i \mid H_i\right) - \frac{1}{2}\ln\left(\det\left(\mathbf{I}\left(\hat{\boldsymbol{\theta}}_i\right)\right)\right) \right\}$$

广义ML准则

五、应用案例

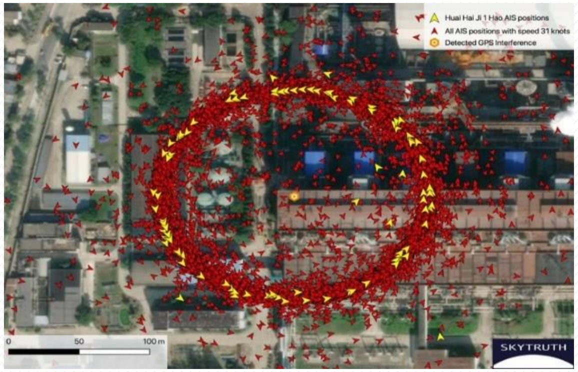
- 卫星导航系统脆弱性

2011年

伊朗宣称通过GPS欺骗干扰在内的电子对抗手段，成功截获美中央情报局派出执行监控任务的哨兵无人侦查机

2013年

美德州大学奥斯汀分校Humphreys教授团队，成功对游艇开展了GPS欺骗攻击，使游艇偏出预定航线一公里



2019年

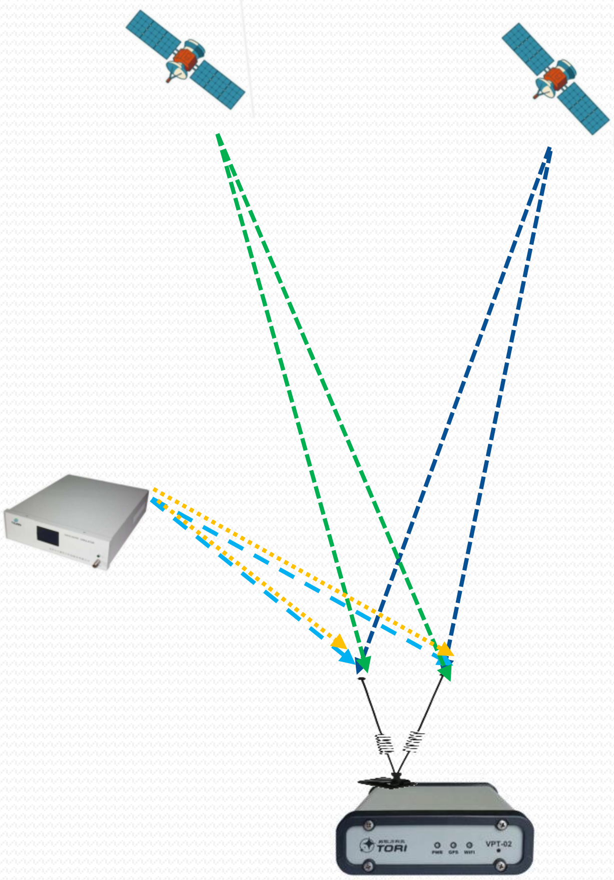
GPS“怪圈：一旦进入某区域，定位结果呈现“圆圈”形状

如何有效防御欺骗干扰?

➤ 双天线载波相位双差欺骗检测技术

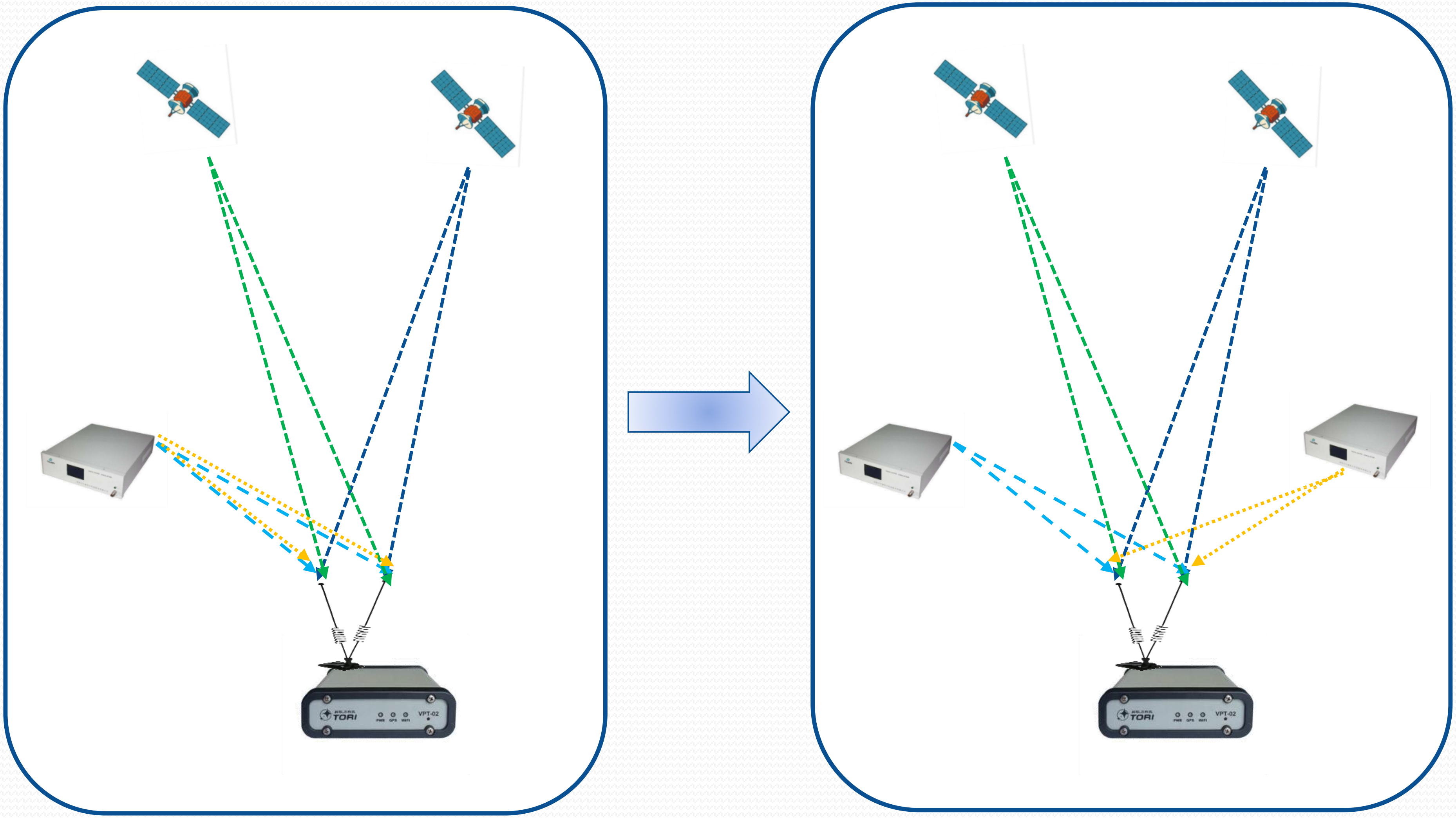
$$\begin{cases} H_0 : \nabla\Delta\varphi = w[n] \\ H_1 : \nabla\Delta\varphi = \theta + w[n] \end{cases}$$

——GLRT检测问题

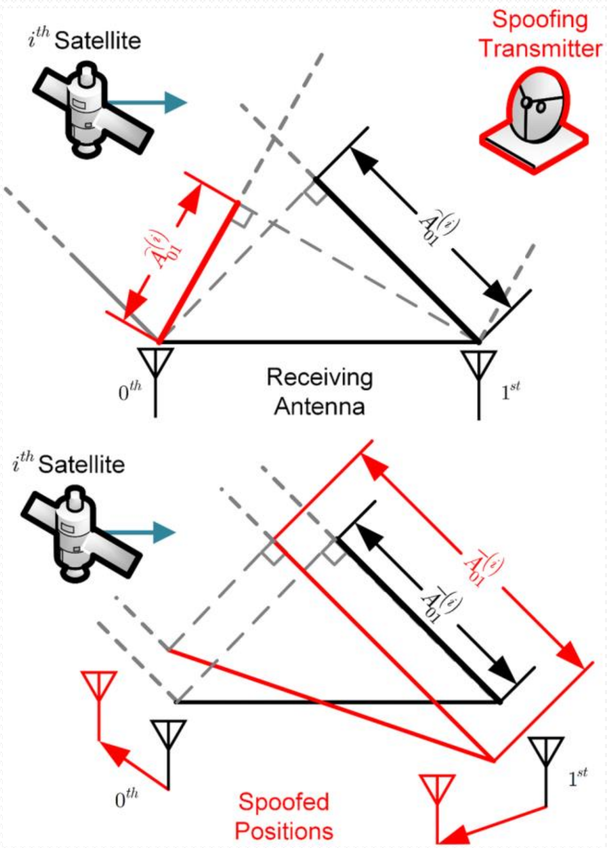


- Jafarnia A, etc, A double antenna approach toward detection, classification and mitigation of GNSS structural interference, Proceedings of NAVITEC 2014, ESA-ESTEC, Noordwijk, December 3–5
- Psiaki ML, et, GNSS spoofing detection using two-antenna differential carrier phase, Proceedings of the ION GNSS + 2014, September 8–12, pp 2776–2800
- Borio D, etc, A sum-of-squares approach to GNSS spoofing detection, IEEE Transactions on Aerospace and Electronics System, 2016, 52(4):1756–1768

失效场景

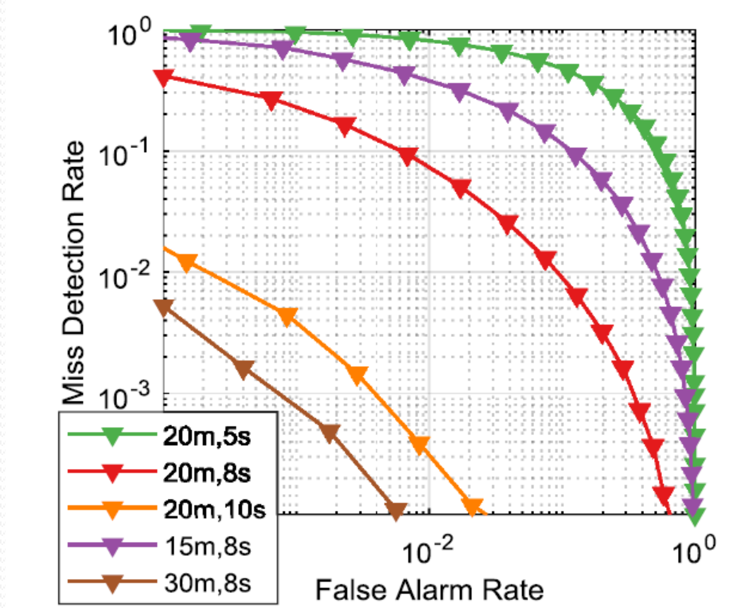


双天线到达频率差欺骗检测技术



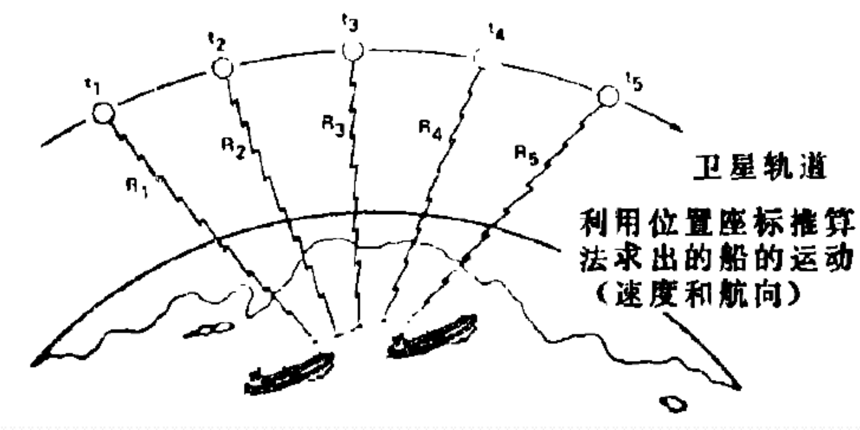
$$\begin{cases} H_0 : \bar{A}_{01}^{(i)} - \hat{A}_{01}^{(i)} = 0 \\ H_1 : \bar{A}_{01}^{(i)} - \hat{A}_{01}^{(i)} \neq 0 \end{cases}$$ ——GLRT检测问题

- 检测预测的单位时间内载波相位变化即多普勒，与实测值间是否一致



- Li He, Hong Li, Mingquan Lu, Dual-Antenna GNSS Spoofing Detection Method Based on Doppler Frequency Difference of Arrival, GPS Solutions, 2019 23:78, DOI: 10.1007/s10291-019-0868-5

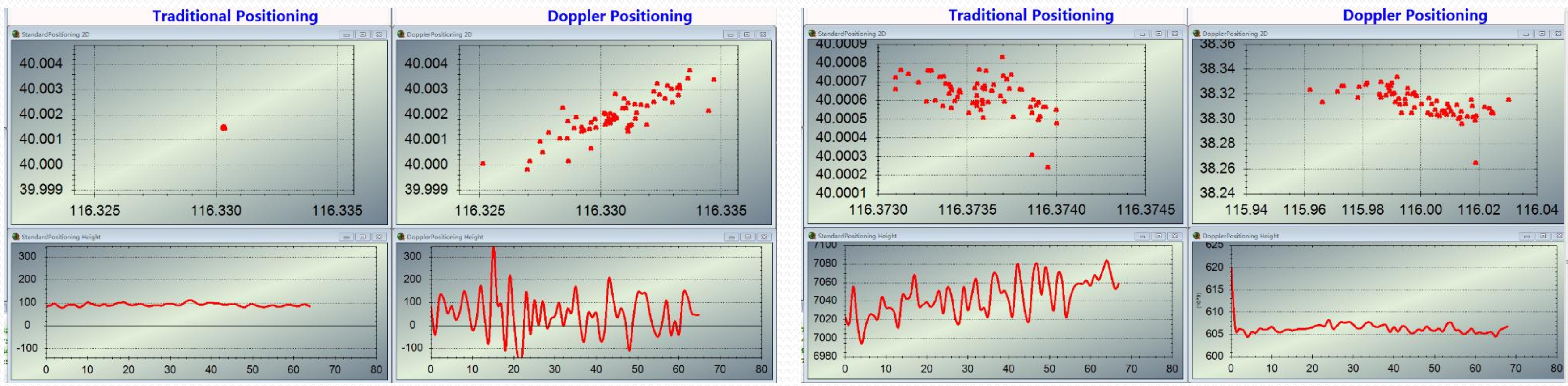
多普勒与伪距定位结果一致性欺骗检测技术



$$s[k]=\left\|\boldsymbol{P}^{d}[k]-\boldsymbol{P}^{t}[k]\right\|$$

$$\begin{cases} H_0: s[k]=n[k] \\ H_1: s[k]=d+n[k] \end{cases}$$

——GLRT检测问题



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六、小结

- GLRT等效方法
  - 条件：双边、弱信号、大数据量
  - Wald检验、Rao检验
- GLRT的性能
  - 条件：双边、弱信号、大数据量
  - 渐近服从chi方分布；对Wald检验、Rao检验同样适用
- 单边检测时UMP的渐近方法：LMP
  - 条件：单边、标量、弱信号、无多余参数
  - 大数据量时，渐近服从高斯分布
- 最小错误概率准则下信号检测
  - GLRT不再适用，需引入修正项
- GLRT方法的应用