As in problem 6.8 $\hat{A} = \dfrac{\sum_n x(n) r^n}{\sum_n r^{2n}}$

Also, $\hat{\sigma}_0^2 = 1/N \sum_n x^2(n)$

$$\hat{\sigma}_1^2 = \frac{1}{N} \sum_n (x(n) - \hat{A} r^n)^2$$

$$L_G(\underline{x}) = \frac{\dfrac{1}{(2\pi \hat{\sigma}_1^2)^{N/2}} e^{-\frac{1}{2\hat{\sigma}_1^2} \sum_n (x(n) - \hat{A} r(n))^2}}{\dfrac{1}{(2\pi\hat{\sigma}_0^2)^{N/2}} e^{-\frac{1}{2\hat{\sigma}_0^2} \sum x^2(n)}}$$

$$= \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{N/2} \frac{e^{-N/2}}{e^{-N/2}}$$

$$2 \ln L_G(\underline{x}) = N \ln \hat{\sigma}_0^2 / \hat{\sigma}_1^2$$

$$= N \ln \frac{\sum_n x^2(n)}{\sum_n (x(n) - \hat{A} r^n)^2}$$

Asymptotic PDF follows by noting that

$\mathcal{H}_0 : A = 0, \sigma^2 > 0$

$\mathcal{H}_1 : A \neq 0, \sigma^2 > 0$

$\Rightarrow$ same as in Ex 6.7

$$2 \ln L_G(\underline{x}) \overset{a}{\sim} \begin{cases} \chi_1^2 & \mathcal{H}_0 \\ \chi_1'^2(\lambda) & \mathcal{H}_1 \end{cases}$$

where $d = A^2 I_{AA}(0, \sigma^2) = \dfrac{A^2 \sum_n r^{2n}}{\sigma^2}$

since $I_{A\sigma^2} = 0$

6.10) $$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{\sigma}_1^2, \mathcal{H}_1)}{p(\underline{x}; \hat{\sigma}_0^2, \mathcal{H}_0)}$$

$$= \frac{\left(\frac{1}{\sqrt{2\hat{\sigma}_1^2}}\right)^N e^{-\sqrt{2/\hat{\sigma}_1^2}\sum_n |x(n)|}}{\frac{1}{(2\pi\hat{\sigma}_0^2)^{N/2}} e^{-\frac{1}{2\hat{\sigma}_0^2}\sum_n x^2(n)}}$$

But $\hat{\sigma}_0^2 = 1/N \sum_n x^2(n)$

$$\hat{\sigma}_1^2 = \left(\frac{\sqrt{2}}{N}\sum_n |x(n)|\right)^2$$

$$\Rightarrow \quad L_G(\underline{x}) = \frac{(2\pi\hat{\sigma}_0^2)^{N/2}}{(2\hat{\sigma}_1^2)^{N/2}} \frac{e^{-N}}{e^{-N/2}}$$

$$= (\pi/e)^{N/2} \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{N/2}$$

$$= \left(\frac{\pi}{e}\right)^{N/2} \left[\frac{\frac{1}{N}\sum x^2(n)}{\left(\frac{\sqrt{2}}{N}\sum |x(n)|\right)^2}\right]^{N/2}$$

6.11) $$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{\theta})}{p(\underline{x}; \theta_0)}$$

But to find MLE of θ we maximize

$$p(\underline{x};\theta) = g(T(\underline{x}),\theta)\, h(\underline{x})$$

or equivalently since $h(\underline{x}) \geq 0$ we maximize $g(T(\underline{x}),\theta)$ over $\theta \Rightarrow$

$\hat{\theta}$ = some function of $T(\underline{x})$, say $\hat{\theta} = \ell(T(\underline{x}))$

$$\Rightarrow \quad L_G(\underline{x}) = \frac{g(T(\underline{x}),\hat{\theta})\, h(\underline{x})}{g(T(\underline{x}),\theta_0)\, h(\underline{x})}$$

$$= \frac{g(T(\underline{x}),\ell(T(\underline{x})))}{g(T(\underline{x}),\theta_0)}$$

= function of $T(\underline{x})$.

6.12) $$p(\underline{x};A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (x[n]-A)^2}$$

$$= \frac{e^{-\frac{1}{2\sigma^2}\left(\sum x^2[n] - 2A\sum x[n] + NA^2\right)}}{(2\pi\sigma^2)^{N/2}}$$

$$= \underbrace{e^{A/\sigma^2 \sum x[n] - \frac{NA^2}{2\sigma^2}}}_{g(T(\underline{x}),A)} \quad \underbrace{\frac{e^{-\frac{1}{2\sigma^2}\sum x^2[n]}}{(2\pi\sigma^2)^{N/2}}}_{h(\underline{x})}$$

$\Rightarrow \quad T(\underline{x}) = \sum_n x[n]$ is a sufficient statistic for A

We know that $\hat{A} = \frac{1}{N} T(\underline{x})$ so that from Problem 6.11 with

$$T(\underline{x}) = N\bar{x} \quad , \quad \theta_0 = 0$$
$$\ell(T(\underline{x})) = \hat{A} = \bar{x} \quad \text{we have}$$

$$L_G(\underline{x}) = \frac{g(N\bar{x}, \bar{x})}{g(N\bar{x}, 0)}$$
$$= \frac{e^{\frac{\bar{x}}{\sigma^2}N\bar{x} - \frac{N\bar{x}^2}{2\sigma^2}}}{e^0}$$
$$= e^{\frac{N\bar{x}^2}{2\sigma^2}}$$

6.13) Clairvoyant detector $\Rightarrow \sigma^2$ known

From (6.4) we decide $\mathcal{H}_1$ if

$$\bar{x} > \sqrt{\sigma^2/N}\, Q^{-1}(P_{FA}) \text{ or}$$

$$\frac{\bar{x}}{\sqrt{\sigma^2}} > \frac{1}{\sqrt{N}} Q^{-1}(P_{FA})$$

Let σ^2 be $\hat{\sigma}_0^2 = \frac{1}{N}\sum_n x^2[n]$ so that we decide $\mathcal{H}_1$ if

$$\frac{\bar{x}}{\sqrt{\frac{1}{N}\sum_n x^2[n]}} > \frac{1}{\sqrt{N}} Q^{-1}(P_{FA})$$

The threshold is incorrect since

$$\frac{\bar{x}}{\sqrt{\frac{1}{N}\sum x^2[n]}} \text{ is not Gaussian}$$

6.14) $L_G(\underline{x}) = \max_{\underline{\theta}_1} L(\underline{x}; \underline{\theta}_1)$

$$p(\underline{x}; A) = \frac{1}{(2\sigma^2)^{N/2}} e^{-\sqrt{2/\sigma^2}\sum_n |x[n]-A|}$$

Let $\underline{\theta}_1 = A$

$$p(\underline{x}; A=0) = \frac{1}{(2\sigma^2)^{N/2}} e^{-\sqrt{2/\sigma^2}\sum_n |x[n]|}$$

$$L(\underline{x}; A) = \frac{p(\underline{x}; A)}{p(\underline{x}; A=0)} = e^{-\sqrt{2/\sigma^2}\sum_n (|x[n]-A|-|x[n]|)}$$

$$L_G(\underline{x}) = \max_A e^{-\sqrt{2/\sigma^2}\sum_n (|x[n]-A|-|x[n]|)}$$

6.15) $p(\underline{x}; \underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}(\underline{x}-\underline{H}\underline{\theta})^T(\underline{x}-\underline{H}\underline{\theta})}$

$$L_G(\underline{x}) = \frac{p(\underline{x}; \hat{\underline{\theta}}_1)}{p(\underline{x}; \underline{\theta}=\underline{0})}$$

$$2\ln L_G(\underline{x}) = -\frac{1}{\sigma^2}\left((\underline{x}-\underline{H}\hat{\underline{\theta}}_1)^T(\underline{x}-\underline{H}\hat{\underline{\theta}}_1) - \underline{x}^T\underline{x}\right)$$

$$= -\frac{1}{\sigma^2}\left(-2\underline{x}^T\underline{H}\hat{\underline{\theta}}_1 + \hat{\underline{\theta}}_1^T\underline{H}^T\underline{H}\hat{\underline{\theta}}_1\right)$$

But $\underline{x}^T \underline{H}\hat{\underline{\theta}}_1 = \underbrace{\underline{x}^T \underline{H} (\underline{H}^T\underline{H})^{-1}}_{\hat{\underline{\theta}}_1^T} \underline{H}^T\underline{H}\,\underline{H}\hat{\underline{\theta}}_1$

$$2 \ln L_G(\underline{x}) = -\frac{1}{\sigma^2}\left(-2\hat{\underline{\theta}}_1^T \underline{H}^T\underline{H}\hat{\underline{\theta}}_1 + \hat{\underline{\theta}}_1^T \underline{H}^T\underline{H}\hat{\underline{\theta}}_1\right)$$

$$= \frac{\hat{\underline{\theta}}_1^T \underline{H}^T\underline{H}\hat{\underline{\theta}}_1}{\sigma^2}$$

$$T_W(\underline{x}) = (\hat{\underline{\theta}}_1 - \underline{\theta}_0)^T \underline{I}(\hat{\underline{\theta}}_1)(\hat{\underline{\theta}}_1 - \underline{\theta}_0)$$

$\underline{\theta}_0 = \underline{0}$ and $\underline{I}(\theta) = \frac{1}{\sigma^2}\underline{H}^T\underline{H}$

$$\Rightarrow T_W(\underline{x}) = \frac{\hat{\underline{\theta}}_1^T \underline{H}^T\underline{H}\hat{\underline{\theta}}_1}{\sigma^2}$$

$$T_R(\underline{x}) = \left.\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}}\right|^T_{\underline{\theta}=\underline{\theta}_0} \underline{I}^{-1}(\underline{\theta}_0)$$

But

$$\frac{\partial \ln p}{\partial \underline{\theta}} = \frac{\partial}{\partial \theta}\left[-\frac{1}{2\sigma^2}(\underline{x}-\underline{H}\underline{\theta})^T(\underline{x}-\underline{H}\underline{\theta})\right]$$

$$= -\frac{1}{2\sigma^2}\frac{\partial}{\partial \underline{\theta}}\left(\underline{x}^T\underline{x} - 2\underline{x}^T\underline{H}\underline{\theta} + \underline{\theta}^T\underline{H}^T\underline{H}\underline{\theta}\right)$$

$$= -\frac{1}{2\sigma^2}\left(-2\underline{H}^T\underline{x} + 2\underline{H}^T\underline{H}\underline{\theta}\right)$$

$$= \frac{1}{\sigma^2}\left(\underline{H}^T\underline{x} - \underline{H}^T\underline{H}\underline{\theta}\right)$$

$$= \frac{1}{\sigma^2}\underline{H}^T\underline{H}\left(\underbrace{(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}_{\hat{\underline{\theta}}_1} - \underline{\theta}\right)$$

$$\left.\frac{\partial \ln p}{\partial \underline{\theta}}\right|_{\underline{\theta}=\underline{\theta}_0=\underline{0}} = \frac{1}{\sigma^2} \underline{H}^T \underline{H} \, \underline{\hat{\theta}}_1$$

$$T_R(\underline{x}) = \frac{1}{\sigma^2}(\underline{H}^T\underline{H}\,\underline{\hat{\theta}}_1)^T\left(\frac{\underline{H}^T\underline{H}}{\sigma^2}\right)^{-1}\frac{1}{\sigma^2}\underline{H}^T\underline{H}\,\underline{\hat{\theta}}_1$$

$$= \frac{1}{\sigma^2}\,\underline{\hat{\theta}}_1^T \underline{H}^T\underline{H}\,\underline{\hat{\theta}}_1$$

6.16) $\mathcal{H}_0 : A = 0$

$\mathcal{H}_1 : A \neq 0$

$\Rightarrow \underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$ with $\underline{\theta} = A$, $\underline{\theta}_0 = 0$,

$\underline{H} = [1\ 1\ \dots\ 1]^T \quad N \times 1$

$\underline{\hat{\theta}}_1 = \hat{A} = \bar{x}$

$$2 \ln L_G(\underline{x}) = \frac{\underline{\hat{\theta}}_1^T \underline{H}^T \underline{H}\,\underline{\hat{\theta}}_1}{\sigma^2}$$

$$= \frac{\bar{x} N \bar{x}}{\sigma^2} = \frac{N\bar{x}^2}{\sigma^2}$$

Same result

6.17) $$T(\underline{x}) = \frac{\underline{\hat{\theta}}_1^T \underline{H}^T \underline{H}\,\underline{\hat{\theta}}_1}{\sigma^2}$$

$$\underline{\hat{\theta}}_1 = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}$$

$\hat{\underline{\theta}}_1$ is Gaussian since it is a linear transformation of $\underline{x}$.

$$E(\hat{\underline{\theta}}_1) = (\underline{H}^T\underline{H})^{-1}\underline{H}^T E(\underline{x})$$

$$= \underline{0} \quad \text{under } \mathcal{H}_0$$

$$= (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{H}\underline{\theta}_1 = \underline{\theta}_1 \quad \text{under } \mathcal{H}_1$$

Under $\mathcal{H}_0$

$$\underline{C}_{\hat{\theta}_1} = E(\hat{\underline{\theta}}_1\hat{\underline{\theta}}_1^T) = E((\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}\,\underline{x}^T\underline{H}(\underline{H}^T\underline{H})^{-1})$$

$$= (\underline{H}^T\underline{H})^{-1}\underline{H}^T\sigma^2\underline{I}\,\underline{H}(\underline{H}^T\underline{H})^{-1}$$

$$= \sigma^2(\underline{H}^T\underline{H})^{-1}$$

and also under $\mathcal{H}_1$ we have $\underline{C}_{\hat{\theta}_1} = \sigma^2(\underline{H}^T\underline{H})^{-1}$

$$\Rightarrow \quad \hat{\underline{\theta}}_1 \sim \begin{cases} \mathcal{N}(\underline{0}, \sigma^2(\underline{H}^T\underline{H})^{-1}) & \mathcal{H}_0 \\ \mathcal{N}(\underline{\theta}_1, \sigma^2(\underline{H}^T\underline{H})^{-1}) & \mathcal{H}_1 \end{cases}$$

so that $T(\underline{x}) = \hat{\underline{\theta}}_1^T\underline{C}_{\hat{\theta}_1}^{-1}\hat{\underline{\theta}}_1$

$$\sim \chi_p^2 \quad \mathcal{H}_0$$

$$\sim \chi_p'^2(\lambda) \quad \mathcal{H}_1$$

where $\lambda = \underline{\mu}^T\underline{C}^{-1}\underline{\mu} = \dfrac{\underline{\theta}_1^T\underline{H}^T\underline{H}\underline{\theta}_1}{\sigma^2}$

using the hint

6.18) See MATLAB code below.

```
%   fig65new.m                                       σ²
%
    randn('seed',0)
    lambda=5;sig2=1;N=10;A=sqrt(lambda*sig2/N);
    nreal=5000;
    for i=1:nreal
    x0=sqrt(sig2)*randn(N,1);x1=x0+A;        <-- data under H0, H1
    y0=mean(x0)^2/(x0'*x0/N-mean(x0)^2);  }
    y1=mean(x1)^2/(x1'*x1/N-mean(x1)^2);  }  SEE (6.18), (6.19)
    T0(i,1)=N*log(1+y0);   <-- 2 ln LG(x) under H0, see (6.18)
    T1(i,1)=N*log(1+y1);   <-- 2 ln LG(x) under H1        (6.19)
    end
    for i=1:51
    Pfaa(i,1)=(i-1)/50;
    u=Qinv(Pfaa(i)/2);
    Pda(i,1)=Q(u-sqrt(lambda))+Q(u+sqrt(lambda));  <-- THEORETICAL
    end
    [Pfa,Pd]=rocurve(T0,T1,51);  <-- ACTUAL
    plot(Pfa,Pd,'-',Pfaa,Pda,'--')
    xlabel('Probability of false alarm (Pfa)')
    ylabel('Probability of detection (Pd)')
    grid
%   print
```

6.19) $\underline{H} = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$ $\quad \underline{\theta} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$2 \ln L_G(\underline{x}) = \frac{\underline{\hat{\theta}}_1^T \underline{H}^T \underline{H} \underline{\hat{\theta}}_1}{\sigma^2}$$

$$\underline{\hat{\theta}}_1 = (\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{x}$$

$\underline{H}^T \underline{H} = N/2 \, \underline{I}$ (see Kay-I, 1993, Example 4.2)

$$\Rightarrow \quad \hat{\underline{\theta}}_1 = \frac{2}{N} \underline{H}^T \underline{x}$$

$$2 \ln L_G(\underline{x}) = \frac{\left(\frac{2}{N} \underline{H}^T \underline{x}\right)^T \frac{N}{2} \left(\frac{2}{N} \underline{H}^T \underline{x}\right)}{\sigma^2}$$

$$= \frac{N}{2\sigma^2} (\hat{a}^2 + \hat{b}^2)$$

where
$$\left.\begin{aligned} \hat{a} &= \frac{2}{N} \sum_n x(n) \cos 2\pi f_0 n \\ \hat{b} &= \frac{2}{N} \sum_n x(n) \sin 2\pi f_0 n \end{aligned}\right\} \text{MLE of } a, b$$

same statistic for Wald and Rao tests

$$2 \ln L_G(\underline{x}) \sim \begin{cases} \chi_2^2 & \mathcal{H}_0 \\ \chi_2'^2(\lambda) & \mathcal{H}_1 \end{cases}$$

where $\lambda = \dfrac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{\sigma^2}$

$$= \frac{\underline{\theta}_1^T \, N/2 \, \underline{\theta}_1}{\sigma^2} = \frac{N}{2\sigma^2}(a^2 + b^2)$$

$$= NP/\sigma^2$$

where P = sinusoidal power

$\Rightarrow \lambda$ is an energy-to-noise ratio

6.20) $$i(A) = \int_{-\infty}^{\infty} \frac{\left(\frac{dp(u)}{du}\right)^2}{p(u)} du$$

$$= \int \left(\frac{d \ln p(u)}{du}\right)^2 p(u) du$$

$$\geq \frac{\left(\int \frac{d \ln p(u)}{du} u p(u) du\right)^2}{\int u^2 p(u) du}$$

with equality iff $\frac{d \ln p(u)}{du} = cu$

Integrating we have

$$\ln p(u) = \tfrac{1}{2} c u^2 + d \leftarrow \text{constant}$$

$$p(u) = e^d e^{\frac{1}{2} c u^2}$$

Must have $c < 0$ for $p(u)$ to be a valid PDF $\Rightarrow$ let $c = -1/\sigma^2$ for $\sigma^2 > 0$

$$p(u) = e^d e^{-\frac{1}{2\sigma^2} u^2}$$

To integrate to one must have

$$e^d = \frac{1}{\sqrt{2\pi\sigma^2}}$$

and a variance of $1 \Rightarrow \sigma^2 = 1$. Thus

$$p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \qquad -\infty < u < \infty$$

minimizes $i(A)$.

6.21) $$T(\underline{x}) = \frac{\left.\frac{\partial \ln p}{\partial \sigma^2}\right|_{\sigma^2=\sigma_0^2}}{\sqrt{I(\sigma_0^2)}}$$

$$p(\underline{x};\sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n x^2(n)}$$

$$\frac{\partial \ln p}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_n x^2(n)$$

also, $I(\sigma^2) = \frac{N}{2\sigma^4}$ see Kay-I 1993 Eq 3.10

$$T(\underline{x}) = \frac{-\frac{N}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum x^2(n)}{\sqrt{N/2\sigma_0^4}}$$

$$= \frac{\frac{N}{2\sigma_0^4}\left(-\sigma_0^2 + \frac{1}{N}\sum x^2(n)\right)}{\sqrt{N/2\sigma_0^4}}$$

$$= \sqrt{\frac{N}{2\sigma_0^4}}\left(\underbrace{\frac{1}{N}\sum_n x^2(n)}_{\hat{\sigma}^2} - \sigma_0^2\right)$$

To see if UMP test exists:

$$L(\underline{x}) = \frac{\frac{1}{(2\pi\sigma_1^2)^{N/2}} e^{-\frac{1}{2\sigma_1^2}\sum x^2(n)}}{\frac{1}{(2\pi\sigma_0^2)^{N/2}} e^{-\frac{1}{2\sigma_0^2}\sum x^2(n)}} > \gamma$$

$$\Rightarrow \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{N/2} e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum x^2[n]} > \gamma$$

$$e^{-\frac{1}{2}(1/\sigma_1^2 - 1/\sigma_0^2)\sum_n x^2[n]} > \gamma\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{N/2}$$

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_n x^2[n] > \ln \gamma\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{N/2}$$

Since $\sigma_1^2 > \sigma_0^2$, we decide $\mathcal{H}_1$ if

$$\sum_n x^2[n] > \gamma'$$

Also, γ' will be independent of σ_1^2 since under $\mathcal{H}_0$ ~~the~~ PDF only depends on σ_0^2, which is known. Thus, UMP test exists.

Note: UMP and LMP tests are the same for this example.

6.22) $T_R(\underline{x}) = \left.\frac{\partial \ln p(\underline{x};\rho)}{\partial \rho}\right|_{\rho=0}^2 I^{-1}(\rho=0)$

$$= T_{LMP}(\underline{x})^2 = N\hat{\rho}^2$$

From (6.37)

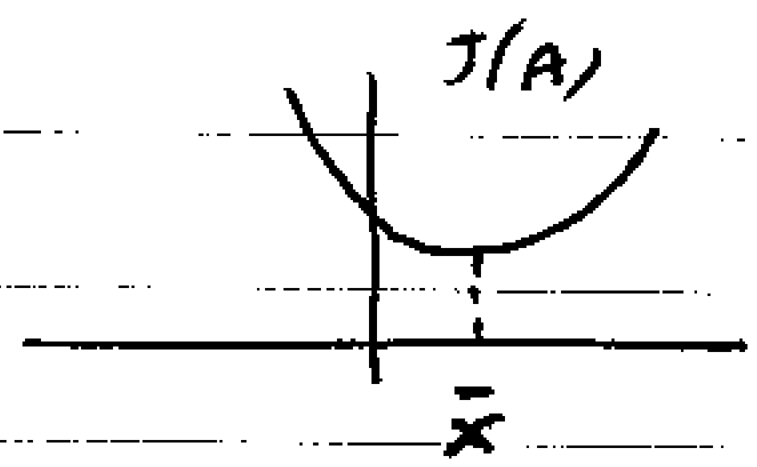
$$T_R(\underline{x}) \sim \chi_1^2 \quad \mathcal{H}_0$$
$$\chi_1'^2(\lambda) \quad \mathcal{H}_1$$
$$\text{where } \lambda = \left(\sqrt{I(\rho=0)}\,\rho\right)^2 = (\sqrt{N}\rho)^2 = N\rho^2$$

or use (6.23) directly.

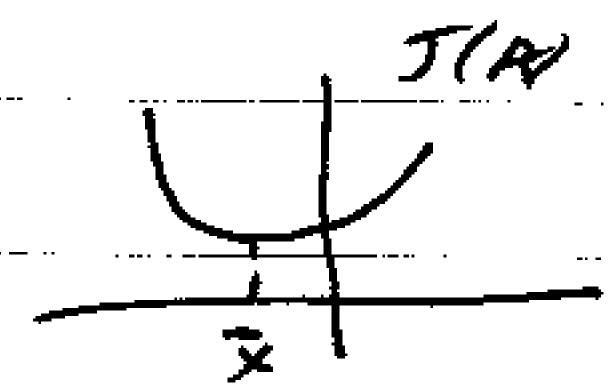
6.23) $$p(\underline{x}; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (x[n]-A)^2}$$

To find $\hat{A}$ we minimize

$$J(A) = \sum_n (x[n]-A)^2$$
$$= \sum x^2[n] - 2AN\bar{x} + NA^2$$
$$= N(A-\bar{x})^2 + \sum_n x^2[n] - N\bar{x}^2$$



$\bar{x} > 0$



$\bar{x} < 0$

$\Rightarrow \hat{A} = \bar{x}$ $\qquad\qquad \hat{A} = 0$

or $\hat{A} = \max(0, \bar{x})$

$$2\ln L_G(\underline{x}) = 2\ln \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum (x[n]-\hat{A})^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum x^2[n]}}$$

$$= \frac{1}{\sigma^2}\left[\sum x^2[n] - \sum (x[n]-\hat{A})^2\right]$$

$$= \frac{1}{\sigma^2}\left(2\hat{A}N\bar{x} - N\hat{A}^2\right) = \frac{N}{\sigma^2}\left(2\hat{A}\bar{x} - \hat{A}^2\right)$$

$$= \begin{cases} \frac{N}{\sigma^2}\bar{x}^2 & \text{if} \quad \bar{x} > 0 \\ 0 & \text{if} \quad \bar{x} \le 0 \end{cases}$$

Under $\mathcal{H}_0$ $\bar{x} \sim \mathcal{N}(0, \sigma^2/N)$. Let $y = 2\ln L_G(\underline{x})$

$$y = \begin{cases} \frac{N}{\sigma^2}\bar{x}^2 & \bar{x} > 0 \\ 0 & \bar{x} \le 0 \end{cases}$$

or $y \sim \begin{cases} \mathcal{N}(0,1)^2 & \text{if } \bar{x} > 0 \\ 0 & \text{if } \bar{x} \le 0 \end{cases}$

Since $\frac{\bar{x}}{\sqrt{\sigma^2/N}} \sim \mathcal{N}(0,1)$

or $y \sim \chi_1^2 \quad \text{if } \bar{x} > 0$

$\quad\quad 0 \quad \text{if } \bar{x} \le 0$

$$\Pr\{y \le \gamma\} = \begin{cases} 0 & \gamma < 0 \\ \Pr\{\bar{x} \le 0\} & \gamma = 0 \\ \Pr\{\bar{x} \le 0\} + \Pr\{\bar{x} > 0 \text{ and } \frac{N\bar{x}^2}{\sigma^2} \le \gamma\} & \gamma > 0 \end{cases}$$

$$= \begin{cases} 0 & z < 0 \\ \frac{1}{2} & z = 0 \\ \frac{1}{2} + \underbrace{Pr\left\{ \frac{N\bar{x}^2}{\sigma^2} \le z \,\middle|\, \bar{x} > 0 \right\}}_{Pr\{\chi_1^2 \le z\}} \underbrace{Pr\{\bar{x} > 0\}}_{\frac{1}{2}} & z > 0 \end{cases}$$

$$Pr\{y \le z\} = \begin{cases} 0 & z < 0 \\ \frac{1}{2} & z = 0 \\ \frac{1}{2} + \frac{1}{2} Pr\{\chi_1^2 \le z\} & z > 0 \end{cases}$$

$$p(z) = \frac{d\,Pr\{y \le z\}}{dz} = \begin{cases} 0 & z < 0 \\ \frac{1}{2}\delta(z) & z = 0 \\ \frac{1}{2} p_{\chi_1^2}(z) & z > 0 \end{cases}$$

or

$$p(y) = \frac{1}{2}\delta(y) + \frac{1}{2} p_{\chi_1^2}(y)$$

6.24) $\quad z_i = \ln p(\underline{x}; \hat{\theta}_i \,|\, \mathcal{H}_i) - \frac{1}{2} \ln \det(I(\hat{\theta}_i))$

$$p(\underline{x}; \underline{\theta}_i \,|\, \mathcal{H}_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}(\underline{x} - \underline{H}_i\underline{\theta}_i)^T(\underline{x} - \underline{H}_i\underline{\theta}_i)}$$

$$(\underline{x} - \underline{H}_i\hat{\underline{\theta}}_i)^T(\underline{x} - \underline{H}_i\hat{\underline{\theta}}_i) = (\underline{x} - \underline{H}_i(\underline{H}_i^T\underline{H}_i)^{-1}\underline{H}_i^T\underline{x})^T(\quad)$$

$$= [(\underline{I} - \underline{H}_i(\underline{H}_i^T\underline{H}_i)^{-1}\underline{H}_i^T)\underline{x}]^T[\quad]$$

$$= \underline{x}^T(\underline{I} - \underline{H}_i(\underline{H}_i^T\underline{H}_i)^{-1}\underline{H}_i^T)(\underline{I} - \underline{H}_i(\underline{H}_i^T\underline{H}_i)^{-1}\underline{H}_i^T)\underline{x}$$

$$= \underline{x}^T (\underline{I} - \underline{H}_i (\underline{H}_i^T \underline{H}_i)^{-1} \underline{H}_i^T) \underline{x}$$

Also, $\det (I(\hat{\underline{\theta}}_i)) = \det \left(\frac{\underline{H}_i^T \underline{H}_i}{\sigma^2} \right)$

$$= \frac{1}{\sigma^{2i}} \det (\underline{H}_i^T \underline{H}_i)$$

$$S_i = -\frac{N}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} \underline{x}^T (\underline{I} - \underline{H}_i (\underline{H}_i^T \underline{H}_i)^{-1} \underline{H}_i^T) \underline{x}$$

$$- \frac{1}{2} \ln {}^1/\sigma^{2i} - \frac{1}{2} \ln \det (\underline{H}_i^T \underline{H}_i)$$

$$= -\frac{N}{2} \ln 2\pi\sigma^2 + \frac{i}{2} \ln \sigma^2 - \frac{1}{2} \ln \det (\underline{H}_i^T \underline{H}_i)$$

$$- \frac{1}{2\sigma^2} \underline{x}^T (\underline{I} - \underline{H}_i (\underline{H}_i^T \underline{H}_i)^{-1} \underline{H}_i^T) \underline{x}$$

If $\underline{H}_i^T \underline{H}_i = N/2\, \underline{I}_i$ (as in Fourier analysis)

$$\det (\underline{H}_i^T \underline{H}_i) = (N/2)^i$$

Also,

$$\underline{x}^T \underline{H}_i (\underline{H}_i^T \underline{H}_i)^{-1} \underline{H}_i^T \underline{x} =$$

$$\underline{x}^T \underline{H}_i (\underline{H}_i^T \underline{H}_i)^{-1} (\underline{H}_i^T \underline{H}_i)(\underline{H}_i^T \underline{H}_i)^{-1} \underline{H}_i^T \underline{x}$$

$$= \hat{\underline{\theta}}_i^T \underline{H}_i^T \underline{H}_i \hat{\underline{\theta}}_i = \frac{N}{2} \hat{\underline{\theta}}_i^T \hat{\underline{\theta}}_i$$

Dropping terms not dependent on i

$$\frac{J_i'}{2} = \frac{i}{2} \ln \sigma^2 - \frac{1}{2} \ln (N/2)^i + \frac{1}{2\sigma^2} \frac{N}{2} \hat{\underline{\theta}}_i^T \hat{\underline{\theta}}_i$$

$$= i/2 \ln 2\sigma^2/N + \frac{N}{4\sigma^2} \hat{\underline{\theta}}_i^T \hat{\underline{\theta}}_i$$

or

$$J_i' = \frac{\hat{\underline{\theta}}_i^T \hat{\underline{\theta}}_i}{2\sigma^2/N} + i \ln \frac{2\sigma^2}{N}$$

$$= \sum_{k=1}^{i} \frac{[\hat{\underline{\theta}}_i]_k^2}{2\sigma^2/N} - i \ln N/2\sigma^2$$

First term goes up with i and is a measure of fit while the second term goes down with i and relates to the increased variability of estimating more parameters.

Chapter 7

7.1) Two-sided hypothesis $\Rightarrow$ no UMP

$\mathcal{H}_0: A = 0$

$\mathcal{H}_1: A = \pm 1$

$$p(\underline{x}; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2}$$

To find MLE of A we minimize

$$J(A) = \sum_n (x[n]-A)^2$$

$$= \sum x^2[n] - 2AN\bar{x} + NA^2$$

$$= \sum x^2[n] - 2AN\bar{x} + N$$

since $A = \pm 1$. Must maximize $A\bar{x}$ over $A = \pm 1$ $\Rightarrow$

$$A = 1 \text{ if } \bar{x} > 0$$

$$-1 \text{ if } \bar{x} < 0$$

or $\hat{A} = \operatorname{sgn}(\bar{x})$.

$$L_G(\underline{x}) = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum(x[n]-\hat{A})^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum x^2[n]}}$$

$$\ln L_G(\underline{x}) = -\frac{1}{2\sigma^2}(-2\hat{A}N\bar{x} + N\hat{A}^2) > \ln\gamma$$

$$\frac{N\hat{A}\bar{x}}{\sigma^2} - \frac{N}{2\sigma^2} > \ln\gamma$$

$$\hat{A}\bar{x} > \gamma'$$

or $\operatorname{sgn}(\bar{x})\bar{x} > \gamma'$

$\Rightarrow \quad |\bar{x}| > \gamma'$

7.2) $$L(\underline{x}) = \frac{\int p(\underline{x}|A;\mathcal{H}_1)\,p(A)\,dA}{p(\underline{x};\mathcal{H}_0)}$$

$$= \frac{\frac{1}{2}p(\underline{x}|A=1;\mathcal{H}_1) + \frac{1}{2}p(\underline{x}|A=-1;\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$= \frac{\frac{1}{2}\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum(x(n)-1)^2} + \frac{1}{2}\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum(x(n)+1)^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum x^2(n)}}$$

$$= \frac{1}{2}e^{-\frac{1}{2\sigma^2}(-2N\bar{x}+N)} + \frac{1}{2}e^{-\frac{1}{2\sigma^2}(2N\bar{x}+N)}$$

$L(\underline{x}) > \gamma \quad \Rightarrow$

$$e^{\frac{N\bar{x}}{\sigma^2}} + e^{-\frac{N\bar{x}}{\sigma^2}} > 2\gamma e^{N/2\sigma^2}$$

Using the hint

$$\left|\frac{N\bar{x}}{\sigma^2}\right| > \gamma' \text{ or } |\bar{x}| > \gamma''$$

7.3) $$L_G(\underline{x}) = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum(x(n)-\hat{r}(n))^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum x^2(n)}}$$

$$\ln L_G(\mathbf{x}) = -\frac{1}{2\sigma^2}\sum_n \left(-2x[n]\hat{r}^n + \hat{r}^{2n}\right)$$

$$= \frac{1}{\sigma^2}\sum_n x[n]\hat{r}^n - \frac{1}{2\sigma^2}\sum_n \hat{r}^{2n}$$

or we decide $\mathcal{H}_1$ if

$$\sum_n x[n]\hat{r}^n - \frac{1}{2}\sum_n \hat{r}^{2n} > \gamma'$$

7.41) $$L_G(\mathbf{x}) = \frac{\dfrac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum (x[n]-\hat{A})^2}}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum x^2[n]}}$$

$$\ln L_G(\mathbf{x}) = -\frac{1}{2\sigma^2}\sum_n \left(-2\hat{A}x[n] + \hat{A}^2\right)$$

$$= \frac{N}{\sigma^2}\hat{A}\bar{x} - \frac{N\hat{A}^2}{2\sigma^2}$$

$$= \begin{cases} -\dfrac{N}{\sigma^2}A_0\bar{x} - \dfrac{NA_0^2}{2\sigma^2} & \text{if } \bar{x} < -A_0 \\ \dfrac{N}{2\sigma^2}\bar{x}^2 & -A_0 \le \bar{x} \le A_0 \\ \dfrac{N}{\sigma^2}A_0\bar{x} - \dfrac{NA_0^2}{2\sigma^2} & \bar{x} > A_0 \end{cases}$$

or we decide $\mathcal{H}_1$ if $\left(\gamma' = \frac{2\sigma^2}{N}\ln\gamma\right)$

$$-2A_0\bar{x} - A_0^2 > \gamma' \qquad \bar{x} < -A_0$$
$$\bar{x}^2 > \gamma' \qquad -A_0 \le \bar{x} \le A_0$$
$$2A_0\bar{x} - A_0^2 > \gamma' \qquad \bar{x} > A_0$$

As $A_0 \to \infty$ we decide $\mathcal{H}_1$ if
$\bar{x}^2 > \gamma'$ or $|\bar{x}| > \sqrt{\gamma'}$

7.5) $$L(\underline{x}) = \frac{\int p(\underline{x} \mid A; \mathcal{H}_1)\, p(A)\, dA}{p(\underline{x}; \mathcal{H}_0)}$$

$$= \frac{\int_{-A_0}^{A_0} \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum (x[n]-A)^2} \frac{1}{2A_0}\, dA}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum x^2[n]}}$$

$$= \int_{-A_0}^{A_0} e^{-\frac{1}{2\sigma^2}(-2NA\bar{x} + NA^2)} \frac{1}{2A_0}\, dA$$

$$= \int_{-A_0}^{A_0} e^{-\frac{1}{2\sigma^2/N}(A-\bar{x})^2} e^{\frac{1}{2\sigma^2/N}\bar{x}^2} \frac{1}{2A_0}\, dA$$

$$= \frac{\sqrt{2\pi\sigma^2/N}}{2A_0} e^{\frac{N\bar{x}^2}{2\sigma^2}} \int_{-A_0}^{A_0} \frac{1}{\sqrt{2\pi\sigma^2/N}} e^{-\frac{1}{2\sigma^2/N}(A-\bar{x})^2}\, dA$$

As $A_0 \to \infty$, integral $\to 1$ $\Rightarrow$ decide $\mathcal{H}_1$ if

$$\frac{\sqrt{2\pi\sigma^2/N}}{2A_0} e^{\frac{N\bar{x}^2}{2\sigma^2}} > \gamma$$

or $|\bar{x}|^2 > \gamma'$ or $|\bar{x}| > \sqrt{\gamma'}$

7.6) Under $\mathcal{H}_0$ $\underline{x} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I})$

Under $\mathcal{H}_1$ $\underline{x} \sim \mathcal{N}(\underline{0}, (\sigma_s^2+\sigma^2)\underline{I})$

This is just the detection of a white Gaussian signal in WGN. From Example 5.1 we decide $\mathcal{H}_1$ if

$$\sum_{n=0}^{N-1} x^2(n) > \gamma'$$

With no knowledge of the signal we should examine just the energy of $x(n)$. In other words, even though the signal is deterministic, our lack of knowledge leads to an incoherent detector.

7.7) GLRT follows from (7.12) with $N=1$, $s(0)=1$ or we decide $\mathcal{H}_1$ if $x^2(0) > \gamma'$ or $|x(0)| > \sqrt{\gamma'}$

For NP we have

$$x(0) \sim \mathcal{N}(0, \sigma^2) \quad \text{under } \mathcal{H}_0$$
$$\mathcal{N}(0, \sigma_A^2+\sigma^2) \quad \text{under } \mathcal{H}_1$$

From Prob 7.6 we have that

$$x^2(0) > \gamma' \quad \text{or} \quad |x(0)| > \sqrt{\gamma'}$$

Tests are identical. Note that thresholds are also the same since PDF under $\mathcal{H}_0$ is the same in each case.

$$P_D = P_r\{|x[0]| > \sqrt{\gamma'} ; \mathcal{H}_1\}$$
$$= P_r\{|A + w[0]| > \sqrt{\gamma'} ; \mathcal{H}_1\}$$
$$= \int_{-\infty}^{\infty} P_r\{|A + w[0]| > \sqrt{\gamma'} \mid A = a\} p_A(a)\, da$$
$$= \int_{-\infty}^{\infty} P_r\{|a + w[0]| > \sqrt{\gamma'}\} p_A(a)\, da$$

since A and $w[0]$ are independent

$$= \int_{-\infty}^{\infty} P_D(a) p_A(a)\, da$$
$$= \int_{-\infty}^{\infty} P_D(A) \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{1}{2\sigma_A^2}A^2}\, dA$$

7.8) $\mathrm{mse}_1 = \mathrm{var}(T_1) + (E(T_1) - A^2)^2$

$$T_1 = \bar{x}^2 \qquad \bar{x} \sim \mathcal{N}(A, \sigma^2/N)$$
$$\Rightarrow \mathrm{var}(T_1) = 4A^2\sigma^2/N + 2\sigma^4/N^2$$
$$E(T_1) = A^2 + \sigma^2/N$$
$$\mathrm{mse}_1 = 4A^2\sigma^2/N + 2\sigma^4/N^2 + \sigma^4/N^2$$
$$= 4A^2\sigma^2/N + 3\sigma^4/N^2$$

$$T_2 = 1/N \sum x^2[n]$$

$$E(T_2) = E(x^2[n]) = A^2 + \sigma^2$$

$$\mathrm{var}(T_2) = 1/N^2 \sum \mathrm{var}(x^2[n])$$

$$= \frac{1}{N}\mathrm{var}(x^2[n])$$

$$= \frac{4A^2\sigma^2 + 2\sigma^4}{N}$$

$$\mathrm{mse}_2 = 4A^2\sigma^2/N + 2\sigma^4/N + \sigma^4$$

$$= \mathrm{mse}_1$$

$$-3\sigma^4/N^2 + 2\sigma^4/N + \sigma^4$$

$$= \mathrm{mse}_1 + \sigma^4 \underbrace{(1 + 2/N - 3/N^2)}_{>0 \text{ for } N \geq 2}$$

7.9) $P_D = Q(Q^{-1}(P_{FA}) - \sqrt{d^2})$

$$0.99 = Q(Q^{-1}(10^{-4}) - \sqrt{d^2})$$

$$d^2 = (Q^{-1}(10^{-4}) - Q^{-1}(0.99))^2$$

$$= 36.54$$

$$10 \log_{10} \eta_{MF} = 10 \log_{10} 36.54 - 20$$

$$= -4.37$$

$$10 \log_{10} \eta_{ED} = 5 \log_{10} 36.54 + 1.5 - 10$$

$$= -0.68$$

Loss = 3.7 dB

7.10) From (7.9) we decide $\mathcal{H}_1$ if

$$\sum_{n=0}^{N-1} x[n]s[n] > \gamma''$$

↑ does not depend on A as long as $A > 0$

$\Rightarrow$ UMP test

$$T = \sum x[n]s[n]$$

$$\sim \mathcal{N}(0, \sigma^2 \sum s^2[n]) \quad \mathcal{H}_0$$

$$\mathcal{N}(A \sum s^2[n], \sigma^2 \sum s^2[n]) \quad \mathcal{H}_1$$

$\Rightarrow$ $P_D = Q(Q^{-1}(P_{FA}) - \sqrt{d^2})$ see Chapter 3

where

$$d^2 = \frac{(A \sum s^2[n])^2}{\sigma^2 \sum s^2[n]}$$

$$= \frac{A^2 \sum s^2[n]}{\sigma^2}$$

7.11) $$p(\underline{x}; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum (x[n] - As[n])^2}$$

To find MLE maximize $p(\underline{x}; A)$ over A or minimize $J(A) = \sum (x[n] - As[n])^2$

$$dJ/dA = 2 \sum (x[n] - As[n])(-s[n]) = 0$$

$$\Rightarrow \hat{A} = \sum x[n]s[n] / \sum s^2[n]$$

7.12) $T'(\underline{x}) = \dfrac{\sigma_A^2}{2\sigma^2(\sigma^2+\sigma_A^2\underline{s}^T\underline{s})}(\underline{x}^T\underline{s})^2$

$$\underline{x} \sim \begin{cases} N(\underline{0}, \sigma^2\underline{I}) & \mathcal{H}_0 \\ N(\underline{0}, \sigma_A^2\underline{s}\underline{s}^T + \sigma^2\underline{I}) & \mathcal{H}_1 \end{cases}$$

$$\underline{x}^T\underline{s} \sim \begin{cases} N(0, \sigma^2\underline{s}^T\underline{s}) & \mathcal{H}_0 \\ N(0, \sigma_A^2(\underline{s}^T\underline{s})^2 + \sigma^2\underline{s}^T\underline{s}) & \mathcal{H}_1 \end{cases}$$

$$\begin{aligned} P_D &= P_r\{|\underline{x}^T\underline{s}| > \gamma'' ; \mathcal{H}_1\} \\ &= 2\,P_r\{\underline{x}^T\underline{s} > \gamma'' ; \mathcal{H}_1\} \\ &= 2\,Q(\gamma''/\sigma_1) \qquad \sigma_1^2 = \sigma_A^2(\underline{s}^T\underline{s})^2 + \sigma^2\underline{s}^T\underline{s} \end{aligned}$$

Similarly letting $\sigma_A^2 = 0$,

$$P_{FA} = 2\,Q(\gamma''/\sigma_0) \qquad \sigma_0^2 = \sigma^2\underline{s}^T\underline{s}$$

7.13) $z = \max(x, y)$ $\quad x \sim N(0,1),\ y \sim N(0,1)$ and independent

$$\begin{aligned} F_z(z) &= P_r\{z \le z\} \\ &= P_r\{\max(x,y) \le z\} \\ &= P_r\{x \le z, y \le z\} \\ &= P_r\{x \le z\}\,P_r\{y \le z\} \quad \text{due to independence} \\ &= (1 - Q(z))^2 \end{aligned}$$

$$p_z(z) = \frac{d}{dz}(1 - Q(z))^2$$

$$= -2(1-Q(z))\frac{dQ(z)}{dz}$$

$$Q(z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2 t^2} dt$$

$$= 1 - \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$dQ(z)/dz = -\frac{1}{\sqrt{2\pi}} e^{-1/2 z^2}$$

$$p_Z(z) = 2(1-Q(z))\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

7.14)
$$\sum_{n=-\infty}^{\infty} x(n)y(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)Y^*(f)\,df$$

$$\Rightarrow \sum_{n=n_0}^{n_0+M-1} x(n)s(n-n_0) = \sum_{n=-\infty}^{\infty} x(n)s(n-n_0)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\left(S(f)e^{-j2\pi f n_0}\right)^* df$$

7.15) For n_0 unknown the GLRT is

$$\ln L_G(\underline{x}) = \ln \frac{p(\underline{x};\hat{A},\hat{n}_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$= \ln \frac{\max_{n_0} p(\underline{x};\hat{A},n_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$= \ln \max_{n_0} \frac{p(\underline{x};\hat{A},n_0,\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

$$= \max_{n_0} \ln \frac{p(\underline{x}; \hat{A}, n_0, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)}$$

From Section 7.4.1 for $n_0 = 0$

$$\ln \frac{p(\underline{x}; \hat{A}, n_0, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} = \frac{\hat{A}^2 \sum_{n=0}^{N-1} s^2[n]}{2\sigma^2} \quad \text{See (7.12)}$$

For $n_0 \neq 0$ and replacing N by M

$$\ln \frac{p(\underline{x}; \hat{A}, n_0, \mathcal{H}_1)}{p(\underline{x}; \mathcal{H}_0)} = \frac{\hat{A}^2 \sum_{n=n_0}^{n_0+M-1} s^2[n-n_0]}{2\sigma^2}$$

$$\text{where } \hat{A} = \frac{\sum_{n=n_0}^{n_0+M-1} x[n] s[n-n_0]}{\sum_{n=n_0}^{n_0+M-1} s^2[n-n_0]}$$

$$\ln L_G(\underline{x}) = \max_{n_0} \frac{\left(\sum_{n=n_0}^{n_0+M-1} x[n] s[n-n_0] \right)^2}{2\sigma^2 \underbrace{\sum_{n=n_0}^{n_0+M-1} s^2[n-n_0]}_{\mathcal{E}}}$$

or we decide $\mathcal{H}_1$ if

$$\max_{n_0 \in [0, N-M]} \left| \sum_{n=n_0}^{n_0+M-1} x[n] s[n-n_0] \right| > \gamma'$$

7.16) $L_G(\underline{x}) = \dfrac{p(\underline{x};\hat{A},\hat{\underline{\theta}},\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$

$$= \max_{\underline{\theta}} \frac{p(\underline{x};\hat{A},\underline{\theta},\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)}$$

From Section 7.4.1 with $\underline{\theta}$ known

$$\frac{p(\underline{x};\hat{A},\underline{\theta},\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} = \frac{\hat{A}^2 \sum s^2(n;\underline{\theta})}{2\sigma^2}$$

$$\text{where } \hat{A} = \frac{\sum_{n=0}^{N-1} x(n)s(n;\underline{\theta})}{\sum_{n=0}^{N-1} s^2(n;\underline{\theta})}$$

If $\sum_{n=0}^{N-1} s^2(n;\underline{\theta}) = \mathcal{E}$ or the energy does not depend on $\underline{\theta}$

$$L_G(\underline{x}) = \max_{\underline{\theta}} \frac{\left(\sum_{n=0}^{N-1} x(n)s(n;\underline{\theta})\right)^2}{2\sigma^2\mathcal{E}}$$

or we decide $\mathcal{H}_1$ if

$$\max_{\underline{\theta}} \left(\sum_{n=0}^{N-1} x(n)s(n;\underline{\theta})\right)^2 > \gamma'$$

7.17)
$$s(n) = A\cos(2\pi f_0 n + \phi)$$
$$= \underbrace{A\cos\phi}_{\alpha_1}\cos 2\pi f_0 n \underbrace{- A\sin\phi}_{\alpha_2}\sin 2\pi f_0 n$$

$$\Rightarrow \quad \underline{\theta} = (\alpha_1\ \alpha_2)^T$$

$$\underline{H} = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1) \end{bmatrix}$$

$$\mathcal{H}_0 : \underline{\theta} = \underline{0}$$
$$\mathcal{H}_1 : \underline{\theta} \neq \underline{0}$$

From Theorem 7.1, $\underline{A} = \underline{I}$, $\underline{b} = \underline{0}$ and

$$\underline{H}^T\underline{H} \approx N/2\ \underline{I}$$
$$\underline{\hat{\theta}}_1 = 2/N\ \underline{H}^T\underline{x} = \begin{bmatrix} 2/N \sum x(n)\cos 2\pi f_0 n \\ 2/N \sum x(n)\sin 2\pi f_0 n \end{bmatrix}$$

$$T(\underline{x}) = \frac{\underline{\hat{\theta}}_1^T\underline{H}^T\underline{H}\underline{\hat{\theta}}_1}{\sigma^2} = \frac{N}{2\sigma^2}(\hat{\alpha}_1^2 + \hat{\alpha}_2^2)$$

$$= \frac{N}{2\sigma^2}\frac{4}{N^2}\left[\left(\sum x(n)\cos 2\pi f_0 n\right)^2 + \left(\sum x(n)\sin 2\pi f_0 n\right)^2\right]$$

$$= \frac{2}{\sigma^2} I(f_0)$$

Decide $\mathcal{H}_1$ if $\frac{2}{\sigma^2} I(f_0) > \gamma'$

or $I(f_0) > \frac{\sigma^2}{2}\gamma' = \gamma''$

$$P_{FA} = Pr\{ I(f_0) > \gamma''; \mathcal{H}_0 \}$$
$$= Pr\{ \underbrace{\frac{2I(f_0)}{\sigma^2}}_{T(\underline{x})} > \underbrace{\frac{2\gamma''}{\sigma^2}}_{\gamma'}; \mathcal{H}_0 \}$$

$$= Q_{\chi^2_2}(2\gamma''/\sigma^2) \quad \text{from Theorem 7.1}$$

$$= e^{-\gamma''/\sigma^2}$$

$$P_D = Pr\{ T(\underline{x}) > 2\gamma''/\sigma^2; \mathcal{H}_1 \}$$
$$= Q_{\chi'^2_2(\lambda)}(2\gamma''/\sigma^2) \quad \text{from Theorem 7.1}$$

But $\gamma''/\sigma^2 = \ln 1/P_{FA}$

$$P_D = Q_{\chi'^2_2(\lambda)}(2 \ln 1/P_{FA})$$

where $\lambda = \dfrac{\underline{\theta}_1^T \underline{H}^T \underline{H} \underline{\theta}_1}{\sigma^2}$

$$= \frac{N}{2\sigma^2} \underline{\theta}_1^T \underline{\theta}_1 = \frac{N}{2\sigma^2}(\alpha_1^2 + \alpha_2^2)$$

$$= NA^2/2\sigma^2$$

7.18) Under $\mathcal{H}_0$

$$E(\xi_1) = \frac{1}{\sqrt{N}} \sum E(w[n]) \cos 2\pi f_0 n = 0$$
$$E(\xi_2) = \frac{1}{\sqrt{N}} \sum E(w[n]) \sin 2\pi f_0 n = 0$$

$$\mathrm{var}(z_1) = E(z_1^2)$$
$$= 1/N \sum_m \sum_n E(w(m)w(n)) \cos 2\pi f_0 m \cos 2\pi f_0 n$$
$$= 1/N \sum_m \sum_n \sigma^2 \delta(m-n) \cos 2\pi f_0 m \cos 2\pi f_0 n$$
$$= \sigma^2/N \sum_n \cos^2 2\pi f_0 n$$
$$= \frac{\sigma^2}{2N} \sum_n (1 + \cos 4\pi f_0 n)$$
$$\simeq \sigma^2/2$$

and similarly for $\mathrm{var}(z_2) \simeq \sigma^2/2$

$$\mathrm{cov}(z_1, z_2) = 1/N \sum_m \sum_n E(w(m)w(n)) \cos 2\pi f_0 n \sin 2\pi f_0 n$$
$$= \sigma^2/N \sum_n \cos 2\pi f_0 n \sin 2\pi f_0 n$$
$$= \sigma^2/2N \sum_n \sin 4\pi f_0 n \simeq 0$$

Under H_1 all the variances and the covariances are the same since we first subtract out the mean (i.e., the signal) before computing the moments.

$$E(z_1) = \frac{1}{\sqrt{N}} \sum_n A \cos(2\pi f_0 n + \phi) \cos 2\pi f_0 n$$
$$= \frac{1}{\sqrt{N}} \sum_n (\alpha_1 \cos 2\pi f_0 n + \alpha_2 \sin 2\pi f_0 n) \cdot \cos 2\pi f_0 n$$

where $\alpha_1 = A \cos\phi$, $\alpha_2 = -A \sin\phi$

$$= \frac{1}{\sqrt{N}} \sum (\alpha_1 \cos^2 2\pi f_0 n + \alpha_2 \sin 2\pi f_0 n \cos 2\pi f_0 n)$$

$$= \frac{1}{\sqrt{N}} \sum \left[\frac{\alpha_1}{2}(1 + \cos 4\pi f_0 n) + \alpha_2/2 \sin 4\pi f_0 n\right)$$

$$\approx \frac{1}{\sqrt{N}} \sum \alpha_1/2 = \sqrt{N}\,\alpha_1/2 = \frac{\sqrt{N}}{2} A\cos\phi$$

and similarly

$$E(J_2) = -\frac{\sqrt{N}}{2} A \sin\phi$$

7.19) $$p(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\{(x-\mu_x)^2 + (y-\mu_y)^2\}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2 - 2(\mu_x x + \mu_y y) + (\mu_x^2 + \mu_y^2))}$$

Let $x = r\cos\theta$, $y = r\sin\theta$ $\quad r > 0$, $0 \le \theta \le 2\pi$

$$p(r, \theta) = p(x(r,\theta), y(r,\theta))\, r$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[r^2 - 2(\mu_x r\cos\theta + \mu_y r\sin\theta) + (\mu_x^2 + \mu_y^2)]}\, r$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[r^2 - 2r\sqrt{\mu_x^2+\mu_y^2}\cos(\theta-\psi) + (\mu_x^2+\mu_y^2)]} \cdot r$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+\alpha^2)}\, e^{\frac{r}{\sigma^2}\alpha\cos(\theta-\psi)}$$

$$p(r) = \frac{r}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+\alpha^2)} \int_0^{2\pi} e^{\frac{r}{\sigma^2}\alpha\cos(\theta-\psi)}\, d\theta$$

Since $e^{\frac{r}{\sigma^2}\alpha\cos(\theta-\psi)}$ is periodic in θ with period 2π, the integral is the same if

We integrate from ψ to $\psi + 2\pi$ $\Rightarrow$

$$p(r) = \frac{r}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2+\alpha^2)} \underbrace{\int_0^{2\pi} e^{r\alpha/\sigma^2 \cos\theta'} d\theta'}_{2\pi I_0(\frac{\alpha}{\sigma^2} r)}$$

or $$p(z) = \frac{z}{\sigma^2} e^{-\frac{1}{2\sigma^2}(z^2+\alpha^2)} I_0\left(\frac{\alpha z}{\sigma^2}\right) \quad z > 0$$

7.20) $$P_D = Q_{\chi_2'^2(\lambda)}(2\gamma'/\sigma^2)$$

where $\lambda = NA^2/2\sigma^2$

$$P_D = P_r\{\chi_2'^2(\lambda) > 2\gamma'/\sigma^2\}$$
$$= P_r\{\sqrt{\chi_2'^2(\lambda)} > \sqrt{2\gamma'/\sigma^2}\}$$

But $\sqrt{\chi_2'^2(\lambda)}$ is a Rician random variable with $\sigma^2 = 1$ and $\alpha^2 = \lambda$

$$\Rightarrow P_D = \int_{\sqrt{2\gamma'/\sigma^2}}^{\infty} z e^{-\frac{1}{2}(z^2+\lambda)} I_0(\sqrt{\lambda} z)\, dz$$

from Prob. 7.19

$$= Q_M\left(\sqrt{\lambda}, \sqrt{2\gamma'/\sigma^2}\right)$$

$$= Q_M\left(\sqrt{\frac{NA^2}{2\sigma^2}}, \sqrt{\frac{2\gamma'}{\sigma^2}}\right)$$

7.21) $$\frac{p(\underline{x};\hat{\theta},\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

$$\frac{\max_{\underline{\theta}} p(\underline{x};\underline{\theta},\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

by definition of the MLE. Now $p(\underline{x};\mathcal{H}_0)$ does not depend on $\underline{\theta}$ and $p(\underline{x};\mathcal{H}_0) \geq 0$

$$\max_{\underline{\theta}} \frac{p(\underline{x};\underline{\theta},\mathcal{H}_1)}{p(\underline{x};\mathcal{H}_0)} > \gamma$$

$$\max_{\underline{\theta}} L(\underline{x};\underline{\theta}) > \gamma$$

$$\ln \max_{\underline{\theta}} L(\underline{x};\underline{\theta}) > \ln \gamma$$

since $\ln(x)$ is a monotonically increasing function of x

But $\ln \max_{\underline{\theta}} L(\underline{x};\underline{\theta}) = \max_{\underline{\theta}} \ln L(\underline{x};\underline{\theta})$

since $g(x)$ and $\ln g(x)$ will be maximized for the same value of x, again due to the monotonicity of $\ln$.

7.22) $$L(\underline{x}) = \frac{\int_0^{2\pi} p(\underline{x} \mid \phi; \mathcal{H}_1)\, p(\phi)\, d\phi}{p(\underline{x}; \mathcal{H}_0)}$$

$$= \frac{\int_0^{2\pi} \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum [x(n) - A\cos(2\pi f_0 n + \phi)]^2} \frac{1}{2\pi} d\phi}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum x^2(n)}}$$

$$= \int_0^{2\pi} e^{-\frac{1}{2\sigma^2}\left[-2A\sum x(n)\cos(2\pi f_0 n + \phi) + A^2 \sum \cos^2(2\pi f_0 n + \phi)\right]} \frac{d\phi}{2\pi}$$

But $$\sum_{n=0}^{N-1} \cos^2(2\pi f_0 n + \phi) = \frac{1}{2}\sum_{n=0}^{N-1}\left(1 + \cos(4\pi f_0 n + 2\phi)\right) \approx N/2$$

$$L(\underline{x}) = e^{-\frac{NA^2}{4\sigma^2}} \int_0^{2\pi} e^{A/\sigma^2 \sum x(n)\cos(2\pi f_0 n + \phi)} \frac{d\phi}{2\pi}$$

$$\sum x(n)\cos(2\pi f_0 n + \phi) =$$

$$\cos\phi \sum x(n)\cos 2\pi f_0 n - \sin\phi \sum x(n) \sin 2\pi f_0 n$$

$$= \beta_1 \cos\phi + \beta_2 \sin\phi$$

Let $\beta_1 = r\cos\psi$

$\beta_2 = r\sin\psi$

or change $\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix}$ to polar coordinates

$$= r\cos\psi\cos\phi + r\sin\psi\sin\phi$$

$$= r\cos(\phi - \psi)$$

where $r = \sqrt{\beta_1^2 + \beta_2^2}$ $\quad \psi = \arctan \beta_2/\beta_1$

$$L(\underline{x}) = e^{-NA^2/4\sigma^2} \int_0^{2\pi} e^{A/\sigma^2(\beta_1 \cos\phi + \beta_2 \sin\phi)} \frac{d\phi}{2\pi}$$

$$= e^{-NA^2/4\sigma^2} \int_0^{2\pi} e^{\frac{A}{\sigma^2}\sqrt{\beta_1^2+\beta_2^2}\cos(\phi-\psi)} \frac{d\phi}{2\pi}$$

But $\beta_1^2 + \beta_2^2 = N\, I(f_0)$

$$L(\underline{x}) = e^{-NA^2/4\sigma^2} \underbrace{\int_0^{2\pi} e^{\sqrt{NA^2 I(f_0)/\sigma^4}\cos(\phi-\psi)} \frac{d\phi}{2\pi}}_{I_0\left(\sqrt{\frac{NA^2 I(f_0)}{\sigma^4}}\right)}$$

That $I_0(x)$ is monotonically increasing with x follows from (see (2.14))

$$I_0(u) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}u)^{2k}}{k!\,\Gamma(k+1)}$$

7.23) $\underline{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix}$ $\quad \underline{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$

$\mathcal{H}_0 : B = 0 \quad \Rightarrow \quad \underline{A} = [0\ 1]$

$\mathcal{H}_1 : B \neq 0 \qquad\quad \underline{b} = \underline{0}$

$$T(\underline{x}) = \frac{1}{\sigma^2} [\hat{\underline{\theta}}_1]_2 \left(\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T\right)^{-1} [\hat{\underline{\theta}}_1]_2$$

But $\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T = [(\underline{H}^T\underline{H})^{-1}]_{22}$

$$T(\underline{x}) = \frac{[\hat{\underline{\theta}}_1]_2^2}{\sigma^2 [(\underline{H}^T \underline{H})^{-1}]_{22}}$$

$$\underline{H}^T \underline{H} = \begin{bmatrix} N & \Sigma n \\ \Sigma n & \Sigma n^2 \end{bmatrix}$$

$$(\underline{H}^T \underline{H})^{-1} = \frac{\begin{bmatrix} \Sigma n^2 & -\Sigma n \\ -\Sigma n & N \end{bmatrix}}{N \Sigma n^2 - (\Sigma n)^2} \qquad \underline{H}^T \underline{x} = \begin{bmatrix} \Sigma x(n) \\ \Sigma n x(n) \end{bmatrix}$$

$$T(\underline{X}) = \frac{\left[\dfrac{-\Sigma n \, \Sigma x(n) + N \, \Sigma n x(n)}{N \Sigma n^2 - (\Sigma n)^2} \right]^2}{\sigma^2 \dfrac{N}{N \Sigma n^2 - (\Sigma n)^2}}$$

$$= \frac{(N \Sigma n x(n) - \Sigma n \, \Sigma x(n))^2}{N \sigma^2 [N \Sigma n^2 - (\Sigma n)^2]}$$

For no noise $\Sigma x(n) = \Sigma (A + Bn)$

$$= NA + B \, \Sigma n$$

$$\Sigma n x(n) = \Sigma n (A + Bn)$$

$$= A \, \Sigma n + B \, \Sigma n^2$$

$$T(\underline{x}) = \frac{(NA \, \Sigma n + NB \, \Sigma n^2 - \Sigma n (NA + B \, \Sigma n))^2}{N \sigma^2 [N \, \Sigma n^2 - (\Sigma n)^2]}$$

$$= \frac{(NB\sum n^2 - B(\sum n)^2)^2}{N\sigma^2(N\sum n^2 - (\sum n)^2)}$$

$$= B^2 \frac{N\sum n^2 - (\sum n)^2}{N\sigma^2}$$

7.24) $\underline{A} = \underline{I},\ \underline{b} = \underline{0}$

$$\Rightarrow T(\underline{x}) = \frac{\hat{\underline{\theta}}_1^T \underline{H}^T\underline{H}\hat{\underline{\theta}}_1}{\sigma^2}$$

$$= \frac{((\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x})^T \underline{H}^T\underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}{\sigma^2}$$

$$= \frac{\underline{x}^T\underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{H}(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}{\sigma^2}$$

$$= \frac{\underline{x}^T\underline{H}\overbrace{(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}^{\theta_1}}{\sigma^2}$$

$$= \frac{\underline{x}^T\underline{H}\hat{\underline{\theta}}_1}{\sigma^2} = \frac{\underline{x}^T\hat{\underline{s}}}{\sigma^2}$$

7.25) Here $\underline{A} = [\underline{I}_r\ \underline{0}] = [r\times r\ \ r\times s]$, $\underline{b} = \underline{0}$

so that

$$\underline{A}\underline{\theta} = \underbrace{[\underline{I}_r\ \underline{0}]}_{r\times(r+s)}\begin{bmatrix}\underline{\theta}_r\\ \underline{\theta}_s\end{bmatrix} = \underline{\theta}_r$$

Also, $\underline{A}\,\hat{\underline{\theta}}_1 = \hat{\underline{\theta}}_r$

$$\underline{A}\,(\underline{H}^T\underline{H})^{-1}\underline{A}^T =$$

$$[\underline{I}_r\ \ \underline{0}]\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix}\begin{bmatrix} \underline{I}_r \\ \underline{0} \end{bmatrix}$$

$$= [\underline{I}_r\ \ \underline{0}]\begin{bmatrix} \underline{B}_{11} \\ \underline{B}_{21} \end{bmatrix} = \underline{B}_{11}$$

$$= [(\underline{H}^T\underline{H})^{-1}]_{rr}$$

$$\Rightarrow T(\underline{x}) = \frac{(\underline{A}\,\underline{\theta}_1)^T[\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T]^{-1}\underline{A}\,\underline{\theta}_1}{\sigma^2}$$

$$= \frac{\hat{\underline{\theta}}_r^T([(\underline{H}^T\underline{H})^{-1}]_{rr})^{-1}\hat{\underline{\theta}}_r}{\sigma^2}$$

$$= \hat{\underline{\theta}}_r^T\,\underline{C}_{\hat{\theta}_r}^{-1}\,\hat{\underline{\theta}}_r$$

Also

$$\lambda = \frac{(\underline{A}\,\underline{\theta}_1)^T(\underline{A}(\underline{H}^T\underline{H})^{-1}\underline{A}^T)^{-1}\underline{A}\,\underline{\theta}_1}{\sigma^2}$$

$$= \frac{\underline{\theta}_r^T([(\underline{H}^T\underline{H})^{-1}]_{rr})^{-1}\underline{\theta}_r}{\sigma^2}$$

7.26) $$I(k) = \sum_{n=0}^{N-1} i(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} B\cos(2\pi \ell/_N\, n + \phi)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{B}{2} \sum_{n=0}^{N-1} \left(e^{j(\frac{2\pi}{N}\ell n+\phi)} + e^{-j(\frac{2\pi}{N}\ell n+\phi)}\right) e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{B}{2} e^{j\phi} \sum_{n=0}^{N-1} e^{j2\pi/N(\ell-k)n}$$

$$+ \frac{B}{2} e^{-j\phi} \sum_{n=0}^{N-1} e^{j2\pi/N(-\ell-k)n}$$

$$= B/_2\, e^{j\phi} \sum_{n=0}^{N-1} e^{j2\pi/N(\ell-k)n}$$

$$+ \frac{B}{2} e^{-j\phi} \sum_{n=0}^{N-1} e^{j2\pi/N(N-\ell-k)n}$$

since $e^{j2\pi/N\, N} = 1$

For $k \neq \ell, N-\ell \Rightarrow I(k) = 0$

Chapter 8

8.1) $$\ln p = -\ln 2\pi - \frac{1}{2}\ln\det(\underline{C}_s + \sigma^2\underline{I}) - \frac{1}{2}\underline{x}^T(\underline{C}_s + \sigma^2\underline{I})^{-1}\underline{x}$$

$$\underline{C}_x = \underline{C}_s + \sigma^2\underline{I} = \begin{bmatrix} r_{ss}(0) + \sigma^2 & \rho r_{ss}(0) \\ \rho r_{ss}(0) & r_{ss}(0) + \sigma^2 \end{bmatrix}$$

$$\det \underline{C}_x = (r_{ss}(0) + \sigma^2)^2 - \rho^2 r_{ss}^2(0)$$

$$\underline{C}_x^{-1} = \frac{\begin{bmatrix} r_{ss}(0) + \sigma^2 & -\rho r_{ss}(0) \\ -\rho r_{ss}(0) & r_{ss}(0) + \sigma^2 \end{bmatrix}}{(r_{ss}(0) + \sigma^2)^2 - \rho^2 r_{ss}^2(0)}$$

Must minimize $J(r_{ss}(0)) =$

$$\ln\det \underline{C}_x + \underline{x}^T \underline{C}_x^{-1}\underline{x}$$

$$J(r_{ss}(0)) = \ln\left[(r_{ss}(0) + \sigma^2)^2 - \rho^2 r_{ss}^2(0)\right] + \frac{\left[(r_{ss}(0) + \sigma^2)(x^2(0) + x^2(1)) - 2\rho r_{ss}(0)\, x(0)x(1)\right]}{(r_{ss}(0) + \sigma^2)^2 - \rho^2 r_{ss}^2(0)}$$

8.2) $$J(P_0) = N\ln(P_0\lambda + \sigma^2) + \frac{\underline{x}^T\underline{x}}{P_0\lambda + \sigma^2}$$

$$P_0^+ = \frac{\frac{1}{N}\sum x^2(n) - \sigma^2}{\lambda}$$

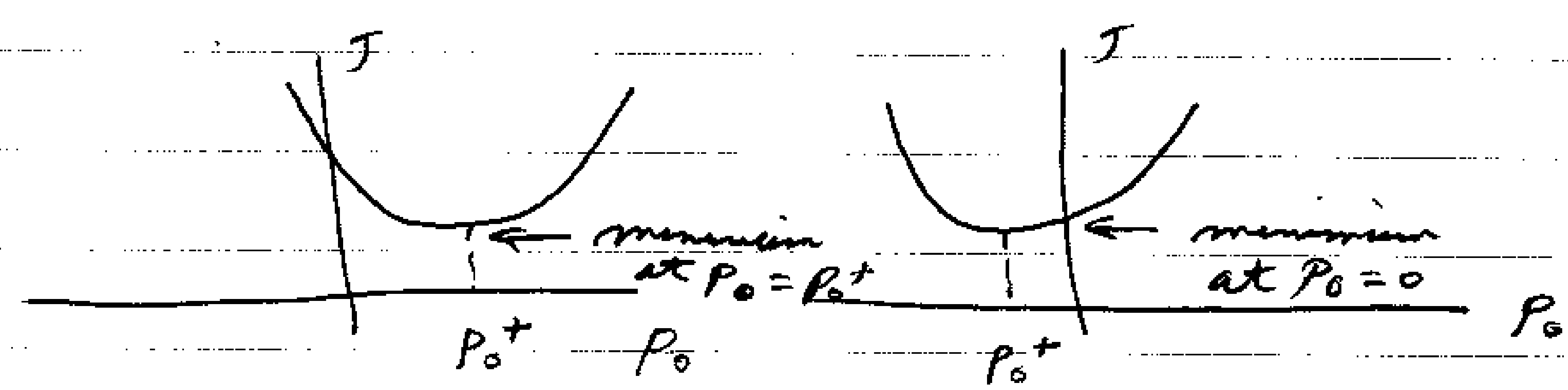
$$\frac{dJ}{dP_0} = \frac{N\lambda}{P_0\lambda+\sigma^2} - \frac{\underline{x}^T\underline{x}\,\lambda}{(P_0\lambda+\sigma^2)^2}$$

$$= \frac{N\lambda}{P_0\lambda+\sigma^2}\left(1 - \frac{\underline{x}^T\underline{x}/N}{P_0\lambda+\sigma^2}\right)$$

$$= \frac{N\lambda}{P_0\lambda+\sigma^2}\left[1 - \frac{P_0^+\lambda+\sigma^2}{P_0\lambda+\sigma^2}\right)$$

> 0 for $P_0 < P_0^+$

< 0 for $P_0 > P_0^+$



8.3) $$L_G(\underline{x}) = \frac{\max_{P_0} p(\underline{x}; P_0, \mathcal{H}_1)}{p(\underline{x}; P_0 = 0, \mathcal{H}_1)}$$

Due to the maximization we will have $L_G(\underline{x}) \geq 1$ for all $\underline{x}$. If $\gamma \leq 1$, we will always declare $\mathcal{H}_1$, even under $\mathcal{H}_0$. Thus, $P_{FA} = 1$.

8.4) $\hat{P}_0^+ = \dfrac{\frac{1}{N}\sum x^2(n) - \sigma^2}{\lambda}$

Under $\mathcal{H}_0$ $\quad x(n) \sim N(0, \sigma^2)$ IID

$\Rightarrow \quad \frac{1}{N}\sum x^2(n) \overset{a}{\sim} N(\sigma^2, \mathrm{var}(w^2(n))/N)$

But $\mathrm{var}(w^2(n)) = E(w^4(n)) = \dfrac{2\sigma^4}{N}$

or $\hat{P}_0^+ \overset{a}{\sim} N\left(0, \; 1/\lambda^2 \, \mathrm{var}\left(\frac{1}{N}\sum x^2(n)\right)\right)$

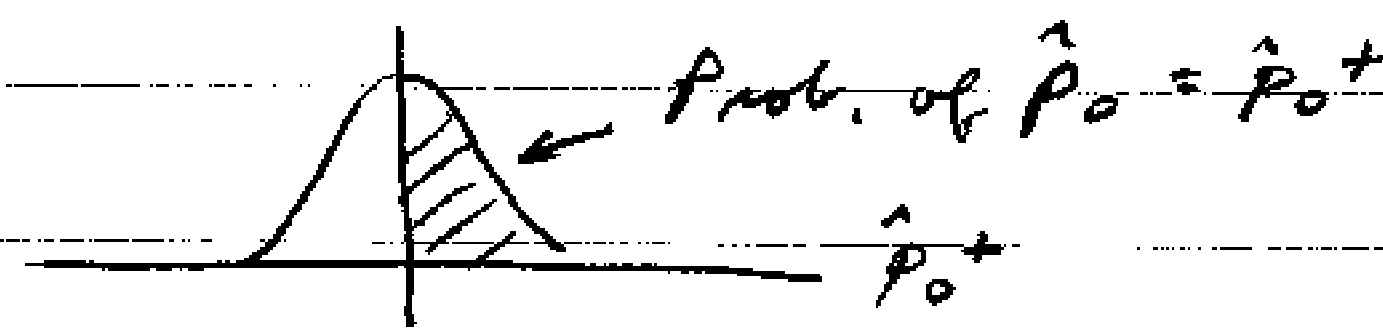
$N(0, \; 2\sigma^4/N\lambda^2)$

$\hat{P}_0 = \max(0, \hat{P}_0^+)$

But $\hat{P}_0 = 0$ with probability $\frac{1}{2}$

$\quad\quad = \hat{P}_0^+$ " " "

~~since~~



$\Rightarrow$ $\hat{P}_0$ has PDF

$$p_{\hat{P}_0}(x) = \frac{1}{2}\,\delta(x) + \frac{1}{2}\,\frac{1}{\sqrt{2\pi\, 2\sigma^4/N\lambda^2}}\, e^{-\frac{1}{\frac{4\sigma^4}{N\lambda^2}} x^2}$$

under $\mathcal{H}_0$

8.5) $\det[r_{ss}(0)] = r_{ss}(0) \neq 0$

$$\begin{bmatrix} r_{ss}(0) & r_{ss}(1) \\ r_{ss}(1) & r_{ss}(0) \end{bmatrix} = r_{ss}^2(0) - r_{ss}^2(1)$$

$$= (\sigma_s^2)^2 - (\sigma_s^2)^2 \cos^2 2\pi f_0$$
$$= (\sigma_s^2)^2 (1 - \cos^2 2\pi f_0) > 0$$
for $0 < f_0 < \frac{1}{2}$

$$\frac{\det(C_s)}{(\sigma_s^2)^3} = \begin{vmatrix} 1 & \cos\theta & \cos 2\theta \\ \cos\theta & 1 & \cos\theta \\ \cos 2\theta & \cos\theta & 1 \end{vmatrix} \qquad \theta = 2\pi f_0$$

$$= (1-\cos^2\theta) - \cos\theta(\cos\theta - \cos\theta\cos 2\theta)$$
$$+ \cos 2\theta(\cos^2\theta - \cos 2\theta)$$
$$= 1 - 2\cos^2\theta - \cos^2 2\theta + 2\cos^2\theta\cos 2\theta$$
$$= 1 - (1+\cos 2\theta) - \cos^2 2\theta + 2\cos 2\theta$$
$$\cdot\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)$$

$$= 0$$

8.6) $\quad \underline{C}_s = \sigma_s^2(\underline{c}\,\underline{c}^T + \underline{s}\,\underline{s}^T)$

$$[\underline{C}_s]_{mn} = \sigma_s^2[(\underline{c})_m[\underline{c}]_n + (\underline{s})_m[\underline{s}]_n$$
$$= \sigma_s^2(\cos 2\pi f_0 m \cos 2\pi f_0 n$$
$$+ \sin 2\pi f_0 m \sin 2\pi f_0 n)$$

$$= \sigma_s^2 \cos 2\pi f_0(m-n)$$

$$\underline{c}^T\underline{s} = \sum_{n=0}^{N-1} \cos 2\pi f_0 n \sin 2\pi f_0 n$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} \sin \underbrace{4\pi f_0}_{\alpha} n$$

$$= \frac{1}{2}\,\mathrm{Im}\left(e^{j(N-1)2\pi f_0}\frac{\sin \pi f_0 N}{\sin \pi f_0}\right)$$

For $f_0 = k/N$ $\quad k \neq 0, N/2 \quad \underline{c}^T \underline{s} = 0$

Can also show $\underline{c}^T \underline{c} = N/2$, $\underline{s}^T \underline{s} = N/2$

$\Rightarrow$ $\underline{v}_1$, $\underline{v}_2$ are eigenvectors of $C_{\underline{s}}$

$\Rightarrow$ $\frac{N}{2}\sigma_s^2$, $\frac{N}{2}\sigma_s^2$ are ~~the~~ eigenvalues since

$$C_{\underline{s}} = \frac{N}{2}\sigma_s^2 \underline{v}_1 \underline{v}_1^T + \frac{N}{2}\sigma_s^2 \underline{v}_2 \underline{v}_2^T$$

8.7) $\quad C_{\underline{s}} = E(\underline{s}\underline{s}^T) = E(A\underline{h}A\underline{h}^T)$

where $\underline{h} = [1 \; r \; \ldots \; r^{N-1}]$

$$C_{\underline{s}} = \sigma_A^2 \underline{h}\underline{h}^T$$

This has rank one since

$$C_{\underline{s}} = \sigma_A^2 \underbrace{\underline{h}^T\underline{h}}_{\lambda_1} \underbrace{\frac{\underline{h}}{\|\underline{h}\|}}_{\underline{v}_1} \frac{\underline{h}^T}{\|\underline{h}\|} = \sigma_A^2 \underline{c}$$

Thus (8.13) applies and we decide $\mathcal{H}_1$ if

$$(\underline{v}_1^T \underline{x})^2 > \gamma''$$

$$\frac{(\underline{h}^T\underline{x})^2}{\|\underline{h}\|^2} = \frac{(\underline{h}^T\underline{x})^2}{\underline{h}^T\underline{h}} = \left(\frac{\sum_n x(n) r^n}{\sum_n r^{2n}}\right)^2 > \gamma''$$

8.8) $\quad$ Under $\mathcal{H}_0$ $\quad \bar{x} \sim N(0, \sigma^2/N)$

$\mathcal{H}_1$ $\quad \bar{x} \sim N(0, \sigma_A^2 + \sigma^2/N)$

since $\bar{x} = \frac{1}{N}\sum_n (A + w(n)) = A + \bar{w}$

$$P_{FA} = P_r\{\bar{x}^2 > \gamma''/N ; \mathcal{H}_0\}$$
$$= 2P_r\{\bar{x} > \sqrt{\gamma''/N} ; \mathcal{H}_0\}$$

since PDF is symmetric about zero

$$= 2Q\left(\frac{\sqrt{\gamma''/N}}{\sqrt{\sigma^2/N}}\right)$$

Similarly, $P_D = 2Q\left(\frac{\sqrt{\gamma''/N}}{\sqrt{\sigma_A^2+\sigma^2/N}}\right)$

$$= 2Q\left(\frac{Q^{-1}(P_{FA}/2)\sqrt{\sigma^2/N}}{\sqrt{\sigma_A^2+\sigma^2/N}}\right)$$

or $P_D = 2Q\left(\frac{Q^{-1}(P_{FA}/2)}{\sqrt{1+\sigma_A^2/\sigma^2/N}}\right)$

8.9) From (8.5) with $\lambda_1 = \lambda_2 = \lambda$ and
$\lambda_i = 0 \quad i = 3, 4, \ldots, N$

$$J(P_0) = 2\ln(P_0\lambda+\sigma^2) + \frac{(\underline{v}_1^T\underline{x})^2}{P_0\lambda+\sigma^2} + \frac{(\underline{v}_2^T\underline{x})^2}{P_0\lambda+\sigma^2}$$

$$= 2\ln(P_0\lambda+\sigma^2) + \frac{(\underline{v}_1^T\underline{x})^2+(\underline{v}_2^T\underline{x})^2}{P_0\lambda+\sigma^2}$$

But $(\underline{v}_1^T\underline{x})^2 + (\underline{v}_2^T\underline{x})^2 = \frac{1}{N/2}\left[(\underline{c}^T\underline{x})^2+(\underline{s}^T\underline{x})^2\right]$
$= 2I(f_0)$

$$\frac{J(P_0)}{2} = \ln(P_0\lambda+\sigma^2) + \frac{I(f_0)}{P_0\lambda+\sigma^2}$$

Differentiating produces

$$\frac{\lambda}{P_0\lambda+\sigma^2} - \frac{I(f_0)\,\lambda}{(P_0\lambda+\sigma^2)^2} = 0$$

$$\Rightarrow \hat{P}_0^{+} = \frac{I(f_0)-\sigma^2}{\lambda} = \frac{2}{N}\left(I(f_0)-\sigma^2\right)$$

$$\text{or } \hat{P}_0 = \max\left(0, \frac{2}{N}\left(I(f_0)-\sigma^2\right)\right)$$

8.10) First note that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln\left(\frac{P_{ss}(f;\delta)}{\sigma^2}+1\right) df$$

does not depend on δ since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln\left(\frac{P_{ss}(f;\delta)}{\sigma^2}+1\right) df =$$

$$\int_{-\frac{f_1}{1+\delta}}^{f_1/1+\delta} \ln\left(\frac{P((1+\delta)f)}{\sigma^2}+1\right) df$$

Let $u=(1+\delta)f$

$$= \int_{-f_1}^{f_1} \ln\left(\frac{P(u)}{\sigma^2}+1\right) \frac{du}{1+\delta}$$

$$\approx \int_{-f_1}^{f_1} \ln\left(\frac{P(u)}{\sigma^2}+1\right) du \quad \text{for } |\delta|\ll 1$$

does not depend on δ.

Using (8.19),

$$T(\underline{x}) = \max_{\delta} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P((1+\delta)f)}{P((1+\delta)f)+\sigma^2} I(f)\,df$$

8.11) To find the MLE of P_0 must minimize (8.16) or

$$J(P_0) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \ln\left(\frac{2P_0}{\sigma^2}+1\right) - \frac{2P_0}{2P_0+\sigma^2}\frac{I(f)}{\sigma^2}\,df$$

$$= \frac{1}{2}\ln\left(\frac{2P_0}{\sigma^2}+1\right) - \frac{2P_0}{2P_0+\sigma^2}\frac{1}{\sigma^2}\underbrace{\int_{-\frac{1}{4}}^{\frac{1}{4}} I(f)\,df}_{\mathcal{S}}$$

$$\frac{\partial J}{\partial P_0} = \frac{\frac{1}{2}\,2/\sigma^2}{\frac{2P_0}{\sigma^2}+1} - \frac{2\mathcal{S}}{\sigma^2}\frac{(2P_0+\sigma^2)-P_0 2}{(2P_0+\sigma^2)^2}$$

$$= \frac{1}{2P_0+\sigma^2} - \frac{2\mathcal{S}}{(2P_0+\sigma^2)^2} = 0$$

$$\Rightarrow \quad 2P_0 + \sigma^2 = 2\mathcal{S}$$

$$\hat{P}_0^{+} = \mathcal{S} - \sigma^2/2$$

$$= \int_{-\frac{1}{4}}^{\frac{1}{4}} I(f)\,df - \sigma^2/2$$

or $\hat{P}_0 = \max\left(0, \int_{-\frac{1}{4}}^{\frac{1}{4}} I(f)\,df - \frac{\sigma^2}{2}\right)$

Now from (8.17) we decide $\mathcal{H}_1$ if

$$-\int_{-\frac{1}{4}}^{\frac{1}{4}} \ln\left(\frac{2\hat{P}_0}{\sigma^2}+1\right) - \frac{2\hat{P}_0}{2\hat{P}_0+\sigma^2}\frac{I(f)}{\sigma^2}\,df > \gamma'$$

since $P_{ss}(f; P_0) = 0 \quad |f| > 1/4$

Must have $\gamma' > 0$ or $\ln\gamma > 1$ since we