刘子源 2022310709

6.19 21
7.2 10 23

6.19 $H=\begin{bmatrix}1 & 0\\ \vdots & \\ \cos 2\pi f_0(n) & \sin 2\pi f_0 n\\ \vdots & \\ \cos 2\pi f_0(N-1) & \sin 2\pi f_0(N-1)\end{bmatrix}$ $\theta=\begin{bmatrix}a\\ b\end{bmatrix}$

GLRT:

$$2\ln L_G(x)=\frac{\hat\theta_1^T H^T H\hat\theta_1}{\sigma^2}$$

$$\hat\theta_1=(H^TH)^{-1}H^Tx$$

$$H^TH=\frac{N}{2}I$$

$$2\ln L_G(x)=\frac{(\frac{2}{N}H^Tx)^T\frac{N}{2}(\frac{2}{N}H^Tx)}{\sigma^2}=\frac{N(\hat a^2+\hat b^2)}{2\sigma^2}$$

$$\hat a=\hat a_{MLE}=\frac{2}{N}\sum_n x[n]\cos 2\pi f_0 n$$

$$\hat b=\hat b_{MLE}=\frac{2}{N}\sum_n x[n]\sin 2\pi f_0 n$$

与Wald和Rao检测的统计量相同

$$2\ln L_G(x)\sim\chi_2^2 \quad H_0$$

$$\chi_2'^2(\lambda) \quad H_1 \qquad \lambda=\frac{N(a^2+b^2)}{2\sigma^2}=\frac{NP}{2\sigma^2}$$

λ可看作信噪比

7.2 $$L(x)=\frac{\int p(x|A;H_1)p(A)\,dA}{p(x|H_0)}$$

$$=\frac{\frac12 p(x|A=1;H_1)+\frac12 p(x|A=-1;H_1)}{p(x;H_0)}$$

$$=\frac{\frac12\left\{\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum_n(x[n]-1)^2}+\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{\sum_n(x[n]+1)^2}{2\sigma^2}}\right\}}{\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum_n x^2[n]}}$$

$$=\frac12 e^{-\frac{1}{2\sigma^2}(-2N\bar x+N)}+\frac12 e^{-\frac{1}{2\sigma^2}(2N\bar x+N)}$$

则 $L(x)>\gamma \Rightarrow e^{\frac{N\bar x}{\sigma^2}}+e^{\frac{-N\bar x}{\sigma^2}}>2\gamma e^{\frac{N}{2\sigma^2}}$

即 $\left|\frac{N\bar x}{\sigma^2}\right|>\gamma'$

$|\bar x|>\gamma''$

6.21 $$T(x)=\frac{\frac{\partial\ln p}{\partial\sigma^2}\Big|_{\sigma^2=\sigma_0^2}}{\sqrt{I(\sigma_0^2)}}$$

$$\cancel{p(x)}=p(x;\sigma^2)=\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum_n x^2[n]}$$

$$\frac{\partial\ln p}{\partial\sigma^2}=-\frac{N}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_n x^2[n]$$

$$I(\sigma^2)=\frac{N}{2\sigma^4}$$

所以 $$T(x)=\frac{-\frac{N}{2\sigma_0^2}+\frac{1}{2\sigma_0^4}\sum_n x^2[n]}{\sqrt{N/2\sigma_0^4}}=\sqrt{\frac{N}{2\sigma_0^4}}\left(\overbrace{\frac1N\sum_n x^2[n]}^{\hat\sigma^2}-\sigma_0^2\right)$$

$$L(x)=\frac{p(x;\sigma_1^2)}{p(x;\sigma_0^2)}>\gamma$$

$$\left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac N2}e^{-\frac12\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum x^2[n]}>\gamma$$

$$-\frac12\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_n x^2[n]>\ln\gamma\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{\frac N2}$$

又 $\sigma_1^2>\sigma_0^2$ 则 $\sum_n x^2[n]>\frac{1}{\frac12\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)}\ln\gamma\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{\frac N2}$

$=\gamma'$

decide H_1 当 $\sum_n x^2[n]>\gamma'$ γ'与σ_1^2无关

UMP检测存在，因为在H_0假设下PDF只与σ_0^2有

7.10 由教材(7.9)式可知判H_1当

$\sum_n x[n]s[n]>\gamma''$ 且 γ''与$A(A>0)$无关

即UMP检验存在

$$T=\sum_n x[n]s[n]\sim N\left(0,\sigma^2\sum_n s^2[n]\right) \quad H$$

$$N\left(A\sum_n s^2[n],\sigma^2\sum_n s^2[n]\right) \quad H$$

$$P_D=Q\left(Q^{-1}(P_{FA})-\sqrt{d^2}\right)$$

其中 $$d^2=\frac{(A\sum s^2[n])^2}{\sigma^2\sum s^2[n]}=\frac{A^2\sum s^2[n]}{\sigma^2}$$

1. ③ $H = \begin{bmatrix} 1 & \cdots\cdots & 1 \\ 0 & 1 \cdots & N-1 \end{bmatrix}^T \quad \theta = \begin{bmatrix} A \\ B \end{bmatrix}$

$H_0: \; B = 0$

$H_1: \; B \neq 0$ 　 则 $A = [0 \;\; 1] \quad b = 0$.

$$T(x) = \frac{1}{\sigma^2}[\hat{\theta}_1]_2 (A(H^TH)^{-1}A^T)^{-1}[\hat{\theta}_1]_2$$

$$= \frac{[\hat{\theta}_1]_2^2}{\sigma^2[(H^TH)^{-1}]_{22}}$$

$$= \frac{(N\sum_n n x[n] - \sum_n n \sum_n x[n])^2}{N\sigma^2[N\sum_n n^2 - (\sum_n n)^2]}$$

当 $w[n] = 0$ 时

$$\sum_n x[n] = NA + B\sum_n n$$

$$\sum_n n x[n] = A\sum_n n + B\sum_n n^2$$

$$T(x) = \frac{(NA\sum n + NB\sum n^2 - \sum n(NA + B\sum n))^2}{N\sigma^2[N\sum n^2 - (\sum n)^2]}$$

$$= B^2 \frac{N\sum n^2 - (\sum n)^2}{N\sigma^2}$$