Cramér–Rao Bound Analysis of Distributed Positioning in Sensor Networks

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***Abstract*—In future applications of sensor networking technology, it is envisioned that nodes will be able to determine their geographical position by measuring the range differences between one another in a collaborative fashion. The aim of this letter is to provide quantitative expressions that can be used both to facilitate an understanding for why such distributed positioning works and to assess the ultimately achievable accuracy in practice. Specifically, we compute the Cramér–Rao bound on the positioning accuracy, under different assumptions on the network synchronization. Numerical examples illustrate our results.**

***Index Terms*—Ad hoc networks, Cramér–Rao bound (CRB), global positioning system (GPS), radio-location, sensor networks.**

I. Introduction

FUTURE WIRELESS communications infrastructure is anticipated to comprise small, self-organizing *ad hoc* networks, which consist of multiple identical or nearly identical nodes. Apart from communicating with each other, it is envisioned in some applications that these nodes will also be able to obtain the geographical position (i.e., the physical coordinates) of themselves by performing range (and possibly angular) measurements on one another. Clearly, determining the absolute coordinates of all devices in a network is impossible unless the position of at least some reference nodes are known, but this problem can be easily solved by equipping some "anchor nodes" with an absolute location reference such as a global positioning system (GPS) receiver.

The "collaborative," or "distributed," positioning scheme discussed in the previous paragraph is a fairly new research topic; yet, it is expected to become an important component of many future ad hoc networks. Examples where positioning is important include the control, coordination, and navigation for unmanned aerial and underwater vehicles, self-calibration and self-configuration of sensor networks in industrial applications, and location-based routing in ad hoc networks. Recently, several research groups have made important contributions to the development of positioning algorithms and efficient protocols for distributed radiolocation e.g., see [1]–[6].

The objective of this letter is to demonstrate via estimation-theoretic analysis why collaborative positioning via range measurements should be a very sensible thing to do in practice. We illustrate, via numerical examples, that collaboration between nodes can almost entirely eliminate the difficulties with poor measurement geometry, which are known to pose major problems in many conventional radiolocation and ranging-based navigation applications.

II. Model

Consider a sensor network comprising N nodes located at the coordinates (x_n, y_n), $n = 1, \ldots, N$. We assume that there are N_k anchor nodes whose positions are known (e.g., obtained via GPS or some other "absolute" reference) and that the remaining $N_u = N - N_k$ nodes are located at unknown positions. Without loss of generality, we may assume that the positions of the nodes $1, \ldots, N_u$ are unknown, whereas the locations of the nodes $N_u + 1, \ldots, N$ are known. Let

$$\begin{aligned} \boldsymbol{x} &= [\, x_1 \quad \ldots \quad x_{N_u} \,]^T \\ \boldsymbol{y} &= [\, y_1 \quad \ldots \quad y_{N_u} \,]^T \end{aligned} \tag{1}$$

be vectors that contain the coordinates of the nodes whose position is unknown, where $(\cdot)^T$ stands for the transpose.

We assume that all nodes cooperate to determine their unknown coordinates by first performing range measurements on each other, and then solving the corresponding geolocation problem—sharing all measurement data with one another. Denote by

$$d_{n,k} = \sqrt{(x_n - x_k)^2 + (y_n - y_k)^2} \tag{2}$$

the distance between node n and node k, and let

$$t_{n,k} = \frac{d_{n,k}}{c} + \tau_n - \tau_k \tag{3}$$

be the perceived propagation delay between node n and k, where c is the speed of light and τ_k is the *clock bias* of node k, i.e., the value of its internal clock relative to absolute time.

We assume that the ranging measurements are performed by transmitting and receiving a signal with known shape. If there is no multipath propagation (which could bias the range measurements), we can model the measured propagation delays as $\hat{t}_{n,k} = t_{n,k} + e_{n,k}$ where $e_{n,k}$ is zero-mean measurement noise with variance $\sigma^2_{n,k}$. If we assume that all range estimation errors $\{e_{n,k}\}$ are independent of one another, which is a reasonable assumption for most real physical systems, then the positions of all nodes can be *jointly* determined by minimizing (with respect to

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all elements in $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{\tau} = [\tau_1 \quad \cdots \quad \tau_{N_u}]^T$) a multidimensional nonlinear criterion function, typically of a least squares form

$$\min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\tau}} \quad \sum_{k=1}^{N} \sum_{l=1, k\neq l}^{N} \frac{1}{\sigma_{k,l}^2} \cdot \left(\hat{t}_{k,l} - \frac{d_{k,l}}{c} - \tau_k + \tau_l \right)^2. \quad (4)$$

In practice, the dimension of the parameter space $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\tau})$ can be reduced by observing that the criterion in (4) is quadratic in $\boldsymbol{\tau}$; hence $\boldsymbol{\tau}$ is easily eliminated from (4).

III. Cramér–Rao Bound on the Location Accuracy

For any unbiased estimator $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\tau}})$ of the node locations $(\boldsymbol{x}, \boldsymbol{y})$ and the clock biases $\boldsymbol{\tau}$, it follows from the Cramér–Rao bound (CRB) theorem [9, Th. 3.2] that

$$\operatorname{cov}\begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\tau}} \end{bmatrix} \geq \left(\underbrace{\begin{bmatrix} \boldsymbol{I}_{x,x} & \boldsymbol{I}_{x,y} & \boldsymbol{I}_{x,\tau} \\ \boldsymbol{I}_{x,y}^T & \boldsymbol{I}_{y,y} & \boldsymbol{I}_{y,\tau} \\ \boldsymbol{I}_{x,\tau}^T & \boldsymbol{I}_{y,\tau}^T & \boldsymbol{I}_{\tau,\tau} \end{bmatrix}}_{\boldsymbol{I}} \right)^{-1} \quad (5)$$

where $\boldsymbol{I}$ is the *Fisher information matrix* for the problem under study (here the matrix inequality $\boldsymbol{A} \geq \boldsymbol{B}$ indicates that $\boldsymbol{A} - \boldsymbol{B}$ is positive semidefinite). In particular, the second-order statistics for the location error of node n must satisfy

$$\operatorname{cov}\begin{bmatrix} \hat{x}_n \\ \hat{y}_n \end{bmatrix} \geq \begin{bmatrix} [\boldsymbol{I}^{-1}]_{n,n} & [\boldsymbol{I}^{-1}]_{n,n+N_u} \\ [\boldsymbol{I}^{-1}]_{n+N_u,n} & [\boldsymbol{I}^{-1}]_{n+N_u,n+N_u} \end{bmatrix}. \quad (6)$$

The eigenvalue-decomposition of the covariance matrix in (6) determines an ellipse within which the positioning error will lie with a given probability and is usually referred to as the *geometric dilution of precision* in the radio-navigation literature. (Similar error bounds can be constructed for the synchronization errors $\hat{\boldsymbol{\tau}}$.)

If the measurement errors $\{e_{n,k}\}$ are Gaussian (which can be justified by using large-sample properties of the maximum-likelihood time-delay estimator), then $\boldsymbol{I}$ can be computed via the Slepian–Bang's formula [9, Sec. 3.9]

$$[\boldsymbol{I}_{\mu,\nu}]_{m,n} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{1}{\sigma_{k,l}^2} \frac{\partial t_{k,l}}{\partial \mu_n} \cdot \frac{\partial t_{k,l}}{\partial \nu_m} \quad (7)$$

where μ and ν represent any pair of x, y, or τ. It follows from (2) and (3) that

$$\frac{\partial t_{k,l}}{\partial x_n} = \begin{cases} \frac{1}{c} \cdot \frac{x_k - x_l}{\sqrt{(x_k-x_l)^2 + (y_k-y_l)^2}}, & n = k \\ \frac{1}{c} \cdot \frac{x_l - x_k}{\sqrt{(x_k-x_l)^2 + (y_k-y_l)^2}}, & n = l \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{\partial t_{k,l}}{\partial y_n} = \begin{cases} \frac{1}{c} \cdot \frac{y_k - y_l}{\sqrt{(x_k-x_l)^2 + (y_k-y_l)^2}}, & n = k \\ \frac{1}{c} \cdot \frac{y_l - y_k}{\sqrt{(x_k-x_l)^2 + (y_k-y_l)^2}}, & n = l \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{\partial t_{k,l}}{\partial \tau_n} = \begin{cases} 1, & n = k \\ -1, & n = l \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Note that it does not make any sense for a node to measure on itself, and therefore we can set $\sigma_{n,n}^2 = \infty$ for all n; then the terms for which $k = l$ disappear from (7).

A. Absolute Time Reference Available

In the special case when all nodes are synchronized relativel to one another (i.e., when they have an absolute time reference—which may not be very realistic in practice), then we can assume that $\{\tau_k\}$ are known. In this case, the blocks $\boldsymbol{I}_{x,\tau}$, $\boldsymbol{I}_{y,\tau}$, and $\boldsymbol{I}_{\tau,\tau}$ of $\boldsymbol{I}$ become irrelevant, and the CRB for $(\boldsymbol{x}, \boldsymbol{y})$ is found simply by inverting the upper $2N_u \times 2N_u$ block of $\boldsymbol{I}$.

B. No Absolute Time Reference Available

If synchronization uncertainties are present, $\{\tau_k\}$ are unknown, and the CRB formula (5) is applicable. However, in this case, we also must assume that $\sigma_{k,l}^2 = \infty$ whenever $k \geq l$. Otherwise, if both $\hat{t}_{k,l}$ and $\hat{t}_{l,k}$ are available, then

$$\begin{aligned} &\frac{1}{2}(\hat{t}_{k,l} + \hat{t}_{l,k}) \\ &= \frac{1}{2}\left(\frac{d_{k,l}}{c} + \tau_k - \tau_l + e_{k,l} + \frac{d_{k,l}}{c} - \tau_k + \tau_l + e_{l,k} \right) \\ &= \frac{d_{k,l}}{c} + \frac{e_{k,l} + e_{l,k}}{2} \end{aligned} \quad (9)$$

even in the presence of a clock bias for each node (i.e., regardless of τ_k and τ_l), and hence unbiased estimates (with finite variance) of $\{d_{k,l}\}$, free of $\{\tau_k, \tau_l\}$, would be available. The physical interpretation of this observation is that the availability of both $\hat{t}_{k,l}$ and $\hat{t}_{l,k}$ effectively implies the availability of a round-trip measurement, in which the clock biases are canceled.

IV. Numerical Examples

Here, we study a sample network with $N = 33$ nodes located on a uniform grid. Among these nodes, $N_k = 3$ arbitrarily chosen anchor devices are located at known positions, whereas the coordinates of the remaining $N_u = 30$ nodes are to be estimated. In general, due to the propagation attenuation, $\sigma_{n,k}^2$ should increase with increasing $d_{n,k}$. For illustration purposes, in this example we assume that the signal-to-noise ratio (SNR) $\text{SNR}_{n,k}$ on the radio channel used to obtain $\hat{t}_{n,k}$ follows a log-distance propagation model with exponent η, i.e., assume that $\text{SNR}_{n,k} \propto d_{n,k}^{\eta}$ [7], [8]. Then, it is reasonable to assume that $\sigma_{n,k}^2$ is proportional to $d_{n,k}^{\eta}$ as well (this can be justified by a large-sample analysis of the maximum-likelihood time-delay estimator)

$$\sigma_{n,k}^2 = \sigma_0^2 \cdot d_{n,k}^{\eta} = \sigma_0^2 \cdot \left((x_n - x_k)^2 + (y_n - y_k)^2 \right)^{\eta/2} \quad (10)$$

where σ_0^2 is a constant.

Figs. 1 and 2 show the true position of the nodes, along with scatter plots of the location error predicted by the CRB analysis, for four different scenarios (described in the next two paragraphs). The scatter plots indicate the shape of the error ellipse in (6), and they are obtained by plotting 500 independent realizations of a Gaussian random vector $[\hat{x}_n \quad \hat{y}_n]^T$ with mean $[x_n \quad y_n]^T$ and covariance matrix equal to the CRB (6). We show results both for the case when round-trip measurements

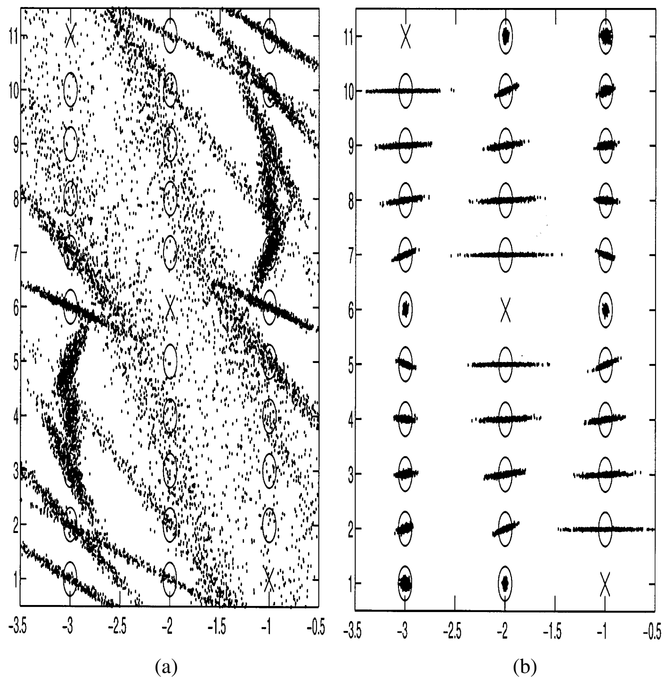

Fig. 1. Scatter plot of the positioning errors for *conventional positioning* (only measurements on anchor nodes are possible) with log-distance fading exponent $\eta = 2$. "$\times$" denotes an anchor node (with known position), and "o" denotes a node whose position is to be estimated. (a) With clock bias, i.e., τ_k is unknown for all nodes. (b) With absolute time reference, i.e., τ_k is known for all nodes.

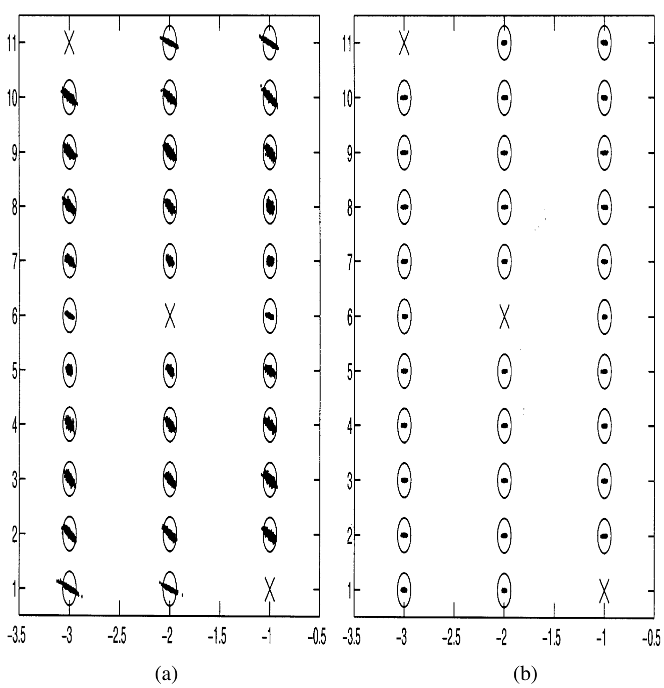

Fig. 2. Scatter plot of the positioning errors for *collaborative positioning* (all nodes can measure on all other nodes) with $\eta = 2$. (a) With clock bias, i.e., τ_k is unknown for all nodes. (b) With absolute time reference, i.e., τ_k is known for all nodes.

are performed (or equivalently, the nodes have an absolute time reference), and for the—perhaps more practical—case when all nodes have an unknown clock bias. For simplicity, in this example, all propagation is modeled as log-distance with $\eta = 2$, which corresponds to free-space propagation. In all figures, "$\times$" denotes an anchor node (with known position), and "o" denotes a node whose position is to be estimated.

A. Conventional Positioning (Fig. 1)

The nodes can only measure on the anchor nodes. Each unit computes its own position without any interaction with other nodes; therefore, the positioning is not collaborative. (In the above derivations, this is easily modeled by letting $\sigma_{n,k}^2 = \infty$ whenever none of n or k is a node with known coordinates, i.e., whenever $n \leq N_u$ and $k \leq N_u$.) In Fig. 1(a), the nodes have an unknown clock bias; in this case, positioning is simply impossible for most nodes due to poor measurement geometry. In Fig. 1(b), the absolute time is known at each node; the situation here is somewhat better, but for many of the nodes accurate location is still extremely difficult.

B. Collaborative Positioning (Fig. 2)

All nodes measure on one another, and they compute their positions by cooperation. In Fig. 2(a), there is an unknown clock bias at each node (this is probably the most practical case), whereas in Fig. 2(b) the absolute time is unknown at each node. In either case, almost all of the nodes can be located accurately.

V. Discussion

Experiments not detailed here have indicated that the above-presented CRB is often achievable to some extent by minimizing a criterion of the form (4), at least when no synchronization uncertainties exist, and provided that the network is reasonably homogenous. In these experiments, we used a standard conjugate-gradient steepest-descent-type search algorithm, and an initialization technique inspired by [6] as follows. Whenever round-trip measurements are available, the equalities in (2) and (3) can be relaxed to obtain a set of *inequalities* of the form (neglecting for now the measurement errors $\{e_{n,k}\}$)

$$(x_n - x_k)^2 + (y_n - y_k)^2 \leq c^2 \cdot (t_{n,k} - \tau_n + \tau_k)^2 \qquad (11)$$

for $n = 1, \ldots, N$ and $k = 1, \ldots, N$, where the right-hand side of (11) is *known*. Following [6], the problem of finding an initial solution guess to (4) can be formulated as that of finding feasible solutions to a problem of the form (11). As suggested in [6], this problem can in turn be cast as a semidefinite programming problem; while this is a clever idea, it leads to a computationally rather burdensome optimization procedure. We found that by further relaxation of (11), the problem can be transformed into a *linear* program

$$\begin{aligned} x_k - x_n &\leq c \cdot |t_{n,k} - \tau_n + \tau_k| \\ y_k - y_n &\leq c \cdot |t_{n,k} - \tau_n + \tau_k| \\ x_n - x_k &\leq c \cdot |t_{n,k} - \tau_n + \tau_k| \\ y_n - y_k &\leq c \cdot |t_{n,k} - \tau_n + \tau_k| \end{aligned} \qquad (12)$$

for $n = 1, \ldots, N$ and $k = 1, \ldots, N$, which can be very efficiently solved also for networks of much larger size.

We believe that a study of fundamental limits is important to establish an understanding for the positioning accuracies that are ultimately possible to achieve. Future work in this direction may include establishing asymptotic results for the behavior of $\boldsymbol{I}^{-1}$ when $N \to \infty$, which would facilitate an understanding for the phenomena that occur when the size of a network increases. A statistical analysis of the positioning accuracy for randomly distributed nodes would also be interesting. Finally, estimation-theoretic bounds that take into account the possibilities of ambiguities (such as the Barankin bound), could also be relevant to examine.

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