## 4.18. MATRICES

Let us consider a set of simultaneous equations,

$$x + 2y + 3z + 5t = 0$$
$$4x + 2y + 5z + 7t = 0$$
$$3x + 4y + 2z + 6t = 0.$$

Now we write down the coefficients of $x, y, z, t$ of the above equations and enclose them within brackets and then we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \end{bmatrix}$$

The above system of numbers, arranged in a rectangular array in rows and columns and bounded by the brackets, is called a matrix.

It has got 3 rows and 4 columns and in all $3 \times 4 = 12$ elements. It is termed as $3 \times 4$ matrix, to be read as [3 by 4 matrix]. In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, $a_{ij}$ lies in the $i$th row and $j$th column.

## 4.19 VARIOUS TYPES OF MATRICES

(*a*) **Row Matrix.** If a matrix has only one row and any number of columns, it is called a Row matrix, *e.g.*,

$$[2\ 7\ 3\ 9]$$

(*b*) **Column Matrix.** A matrix, having one column and any number of rows, is called a Column matrix, e.g., $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(*c*) **Null Matrix or Zero Matrix.** Any matrix, in which all the elements are zeros, is called a Zero matrix or Null matrix *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(*d*) **Square Matrix.** A matrix, in which the number of rows is equal to the number of columns, is called a square matrix *e.g.*,

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

(*e*) **Diagonal Matrix.** A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero *e.g.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(*f*) **Unit or Identity Matrix.** A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(g) **Symmetric Matrix.** A square matrix will be called symmetric, if for all values of i and j, i.e. $a_{ij} = a_{ji}$ or $A' = A$ e.g.,

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

(h) **Skew Symmetric Matrix.** A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ji}$ for all values of i and j, or $A' = -A$

(2) All diagonal elements are zero, e.g.,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(i) **Triangular Matrix (Echelon form).** A square matrix, all of whose elements below the leading diagonal are zero, is called an upper triangular matrix. A square matrix, all of whose elements above the leading diagonal are zero, is called a lower triangular matrix e.g.,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

(j) **Transpose of a Matrix.** If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and is denoted by $A'$ or $A^T$ e.g.,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

(k) **Orthogonal Matrix.** A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix $A'$ is an identity matrix e.g.,

$$A \cdot A' = I$$

if $|A| = 1$, matrix A is proper.

(*l*) **Conjugate of a Matrix.**

Let $$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

Conjugate of matrix $A$ is $\bar{A}$

$$\therefore \quad \bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

(m) **Matrix $A^\theta$.** Transpose of the conjugate of a matrix A is denoted by $A^\theta$.

Let
$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$
$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & +i & 3+2i \end{bmatrix}$$
$$(\bar{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$
or
$$A^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

(*n*) **Unitary Matrix.** A square matrix $A$ is said to be unitary if
$$A^{\theta} A = I$$
e.g.
$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad A^{\theta} = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}, \quad A \cdot A^{\theta} = I$$

(*o*) **Hermitian Matrix.** A square matrix $A = (a_{ij})$ is called **Hermitian matrix, if every** *i*-*j*th element of $A$ is equal to conjugate complex *j*-*i*th element of $A$.

In other words $a_{ij} = \bar{a}_{ji}$.

e.g.
$$\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix $A$ to be Hermitian is that $A = A^{\theta}$ i.e. conjugate transpose of $A$

or $A = (\bar{A})'$.

(*p*) **Skew Hermitian Matrix.** A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every *i*-*j*th element of $A$ is equal to negative conjugate complex of *j*-*i*th element of $A$.

In other words $a_{ij} = -\bar{a}_{ji}$

All the elements in the principal diagonal will be of the form
$$a_{ii} = -\bar{a}_{ii} \quad \text{or} \quad a_{ii} + \bar{a}_{ii} = 0$$
If $a_{ii} = a + ib$ then $\bar{a}_{ii} = a - ib$
$$(a+ib) + (a-ib) = 0 \quad \text{or} \quad 2a = 0 \text{ or } a = 0$$
So $a_{ii}$ is pure imaginary or $a_{ii} = 0$

Hence all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

e.g.
$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix $A$ to be Skew Hermitian is that
$$A^{\theta} = -A$$
$$(\bar{A})' = -A$$

(*q*) **Idempotent Matrix.** A matrix, such that $A^2 = A$ is called Idempotent Matrix.

e.g. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$

(*r*) **Periodic Matrix.** A matrix A will be called a Periodic Matrix, if
$$A^{k+1} = A$$
where k is a +ve integer. If k is the least +ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose $k = 1$ we get $A^2 = A$ and we call it to be idempotent matrix.

(*s*) **Nilpotent Matrix.** A matrix will be called a Nilpotent matrix, if $A^k = 0$ (null matrix) where k is a +ve integer ; if however k is the least +ve integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

e.g., $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, $A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

A is nilpotent matrix whose index is 2.

(*t*) **Involuntary Matrix.** A matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always $\therefore$ Unit matrix is involuntary.

(*u*) **Equal Matrices.** Two matrices are said to be equal if

(*i*) They are of the same order.

(*ii*) The elements in the corresponding positions are equal.

Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Here $A = B$

(*v*) **Singular Matrix :** If the determinant of the matrix is zero, then the matrix is known as singular matrix. e.g. If $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$, then A is a singular matrics

**Example 1.** *Find the values of x, y, z and 'a' which satisfy the matrix equation.*
$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

**Solution.** As the given matrices are equal, so their corresponding elements are equal.
$$x + 3 = 0 \quad \text{or} \quad x = -3 \qquad \text{....(1)}$$
$$2y + x = -7 \qquad \text{...(2)}$$
$$z - 1 = 3 \quad \text{or} \quad z = 4 \qquad \text{...(3)}$$
$$4a - 6 = 2a \text{ or } a = 3 \qquad \text{...(4)}$$
Putting the value of $x = -3$ from (1) into (2), we have
$$2y - 3 = -7 \text{ or } y = -2$$
$$x = -3, \; y = -2, \; z = 4, \; a = 3$$ **Ans.**

## 20 ADDITION OF MATRICES

If A and B be two matrices of the same order, then their sum, $A + B$ is defined as the matrix, each element of which is the sum of the corresponding elements of A and B.

Thus if $A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

then $A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 1+3 & 3+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 4 & 4 & -2 \end{bmatrix}$

If $A = [a_{ij}]$, $B = [b_{ij}]$ then $A + B = [a_{ij} + b_{ij}]$

**Example 2.** *Show that any square matrix can be expressed as the sum of two matrices, one symmetric and the other anti-symmetric.*

**Solution.** Let $A$ be a given square matrix.

Then $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Now $$(A + A')' = A' + A = A + A'.$$

$\therefore$ $A + A'$ is a symmetric matrix.

Also $$(A - A')' = A' - A = -(A - A')$$

$\therefore$ $A - A'$ or $\frac{1}{2}(A - A')$ is an anti-symmetric matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

= Symmetric matrix + antisymmetric matrix **Proved.**

**Example 3.** *Write matrix (A) given below as the sum of a symmetric and a skew symmetric matrix.*

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{pmatrix}$$

**Solution:** $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$ On transposing we get $A' = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix}$

On adding $A$ and $A'$, we have

$$A + A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} \quad ...(1)$$

On subtracting $A'$ from $A$ we get

$$A - A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix} \quad ...(2)$$

On adding (1) and (2), we have

$$2A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 5 & \frac{9}{2} \\ \frac{3}{2} & \frac{9}{2} & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & \frac{5}{2} \\ -2 & 0 & -\frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

= [Symmetric matrix] + [Skew symmetric matrix.] **Ans.**



4. If $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Find (i) $2A + 3B$ (ii) $3A - 4B$. **Ans.** (i) $= \begin{bmatrix} 3 & 10 & 3 \\ 8 & 3 & 6 \\ 2 & 2 & 13 \end{bmatrix}$, (ii) $= \begin{bmatrix} -4 & -2 & -4 \\ -5 & -4 & 9 \\ 3 & 3 & -6 \end{bmatrix}$

5. If A is a skew-symmetric matrix of odd order, then the determinant of A is

(a) $-1$. (b) 0. (c) 1. (d) a real number. **Ans.** (b)

6. Let A be a non-singular matrix. Then the inverse of the matrix $AA^T$

(a) is symmetric (b) is skew – symmetric (c) does not exist (d) equals $A^{-1}(A^{-1})^T$

*(A.M.I.E.T.E., Summer 2005)* **Ans.** (a)

## 4.24. MULTIPLICATION

The product of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the product AB of these matrices is an $m \times p$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \dots + a_{in} b_{nj}$$

## 4.25. $(AB)' = B'A'$

If A and B are two matrices conformal for product AB, then show that $(AB)' = B'A'$, where dash represents transpose of a matrix.

**Solution.** Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be $n \times p$ matrix.

Since AB is an $m \times p$ matrix, $(AB)'$ is a $p \times m$ matrix.

Further $B'$ is $p \times n$ matrix and $A'$ an $n \times m$ matrix and therefore $B'A'$ is a $p \times m$ matrix.

Thus $(AB)'$ and $B'A'$ are matrices of the same order.

Now the $(j, i)$th element of $(AB)'$ = $(i, j)$th element of (AB) $= \sum_{k=1}^{n} a_{ik} b_{kj}$ ...(1)

Also the jth row of $B'$ is $b_{1j}, b_{2j} \dots b_{jm}$ and ith column of $A'$ is $a_{i1}, a_{i2}, a_{i3} \dots a_{in}$.

$\therefore$ (j, i)th element of $B'A' = \sum_{k=1}^{n} b_{kj} a_{ik}$ ...(2)

From (1) and (2) we have (j, i)th element of $(AB)' = (j, i)$th element of $B'A'$.

As the matrices $(AB)'$ and $B'A'$ are of the same order and their corresponding elements are equal, we have $(AB)' = B'A'$. Proved

## 4.26. PROPERTIES OF MATRIX MULTIPLICATION

1. Multiplication of matrices is not commutative.

$$AB \neq BA$$

2. Matrix multiplication is associative, if confirmability is assured.

$$A(BC) = (AB)C$$

3. Matrix multiplication is distributive with respect to addition.

$$A(B + C) = AB + AC$$

4. Multiplication of matrix A by unit matrix.

$$AI = IA = A$$

5. Multiplicative inverse of a matrix exists if $|A| \neq 0$.

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

$\therefore$ The product $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$ **Proved.**

**Example 15.** *Verify that*

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \text{ is orthogonal.}$$

**Solution.**

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \quad \therefore \ A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence $A$ is an orthogonal matrix. **Verified.**

**Example 16.** *Determine the values of $\alpha, \beta, \gamma$ when*

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \text{ is orthogonal.}$$

**Solution.** Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

On transposing $A$, we have

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

If $A$ is orthogonal, then $AA' = I$

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$\left.\begin{array}{l} 4\beta^2+\gamma^2 = 1 \\ 2\beta^2-\gamma^2 = 0 \end{array}\right\} \Rightarrow \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}$$

$$\alpha^2+\beta^2+\gamma^2 = 1 \quad \text{as} \quad \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}, \alpha = \pm\frac{1}{\sqrt{2}} \quad \textbf{Ans.}$$



where $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2, \quad A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1c_3 + b_3c_1$

$$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1, \quad B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2c_3 + a_3c_2$$

$$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1, \quad B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1c_2 + a_2c_1$$

$$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2, \quad C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1b_3 + a_3b_1$$

$$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Then the transpose of the matrix of co-factors

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

is called the adjoint of the matrix $A$ and is written as adj $A$.

## 4.28 PROPERTY OF ADJOINT MATRIX

*The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant*

If $A$ be a square matrix, then $(\text{Adjoint}\, A)\cdot A = A\cdot(\text{Adjoint}\, A) = |A|\cdot I$

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ and adj. $A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

$$A\cdot(\text{adj.}\, A) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1+a_2A_2+a_3A_3 & a_1B_1+a_2B_2+a_3B_3 & a_1C_1+a_2C_2+a_3C_3 \\ b_1A_1+b_2A_2+b_3A_3 & b_1B_1+b_2B_2+b_3B_3 & b_1C_1+b_2C_2+b_3C_3 \\ c_1A_1+c_2A_2+c_3A_3 & c_1B_1+c_2B_2+c_3B_3 & c_1C_1+c_2C_2+c_3C_3 \end{bmatrix}$$

$$\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I$$

## 4.29 INVERSE OF A MATRIX

(A.M.I.E. Summer 2004)

If $A$ and $B$ are two square matrices of the same order, such that

$$AB = BA = I$$

then $B$ is called the inverse of $A$ *i.e.* $B = A^{-1}$ and $A$ is the inverse of $B$. ($I$ = unit matrix)

*Condition for a square matrix A to possess an inverse is that matrix A is non-singular, i.e.,* $|A| \neq 0$

If $A$ is a square matrix and $B$ be its inverse, then $AB = I$

Taking determinant of both sides

$$|AB| = |I| \quad \text{or} \quad |A||B| = I$$

From this relation it is clear that $|A| \neq 0$

$\therefore$ the matrix $A$ is non-singular.

**To find the inverse matrix with the help of adjoint matrix**

We know that $\quad A \cdot (Adj.\, A) = |A| I$

$$A \cdot (Adj) \cdot \frac{A}{|A|} = I \qquad \text{Provided } |A| \neq 0$$

or

and

$$A \cdot A^{-1} = I \quad \therefore \quad A^{-1} = \frac{1}{|A|}(Adj \cdot A)$$

**Example 18.** *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, *find* $A^{-1}$. (A.M.I.E. Summer 2004)

**Solution.** $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = 3(-3+4) + 3(2-0) + 4(-2-0) = 3 + 6 - 8 = 1$$

The co-factors of elements of various rows of $|A|$ are

$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$

Therefore the matrix formed by the co-factors of $|A|$ is

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}, \; Adj.\, A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} Adj.\, A = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \qquad \textbf{Ans.}$$

**Example 19.** *If* $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, *prove that* $A^{-1} = A'$, $A'$ *being the transpose of A.* (A.M.I.E., Winter 2000)

**Solution.** If

$$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}, \quad A' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } AA' = I$$

$A' = A^{-1}$ **Proved**



**Example 20.** *If a matrix A satisfies a relation* $A^2 + A - I = 0$ *prove that* $A^{-1}$ *exists and that* $A^{-1} = I + A$, *I being an identity matrix.* (AMIE Winter 2003)

**Solution.** Here $A^2 + A - I = 0 \quad$ or $\quad A^2 + AI = I \quad$ or $\quad A(A + I) = I$

$\therefore \quad |A||A + I| = |I|$

$\therefore \quad |A| \neq 0$ and so $A^{-1}$ exists.

Again $\quad A^2 + A - I = 0 \quad$ or $\quad A^2 + A = I \qquad$ ...(1)

Multiplying (1) by $A^{-1}$, we get

$A^{-1}(A^2 + A) = A^{-1} I \quad$ or $\quad A + I = A^{-1}$

$\Rightarrow \quad A^{-1} = I + A \qquad$ **Proved**

**Example 21.** *If A and B are non-singular matrices of the same order then,*

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

Hence, prove that $(A^{-1})^m = (A^m)^{-1}$ for any positive integer m.

**Solution.** We know that

$$(AB) \cdot (B^{-1}A^{-1}) = [(AB)B^{-1}] \cdot A^{-1} = [A(BB^{-1})] \cdot A^{-1}$$

$$= [AI] \cdot A^{-1} = A \cdot A^{-1} = I$$

Also $\quad B^{-1}A^{-1} \cdot (AB) = B^{-1}[A^{-1} \cdot (AB)] = B^{-1}[(A^{-1}A) \cdot B]$

$$= B^{-1}[I \cdot B] = B^{-1} \cdot B = I$$

By definition of the inverse of a matrix, $B^{-1}A^{-1}$ is inverse of AB.

$\Rightarrow \quad B^{-1}A^{-1} = (AB)^{-1} \qquad$ **Prove**

$$(A^m)^{-1} = [A \cdot A^{m-1}]^{-1} = (A^{m-1})^{-1} A^{-1}$$

$$= (A \cdot A^{m-2})^{-1} \cdot A^{-1} = [(A^{m-2})^{-1} \cdot A^{-1}] \cdot A^{-1} = (A^{m-2})^{-1} (A^{-1}$$

$$= [A \cdot A^{m-3}]^{-1} (A^{-1})^2 = [(A^{m-3})^{-1} \cdot A^{-1}](A^{-1})^2 = (A^{m-3})^{-1} (A$$

$$= A^{-1}(A^{-1})^{m-1} = (A^{-1})^m$$ **Prov**

**Example 22.** *Prove that the inverse of a matrix is unique.*

**Solution.** We suppose that B and C are two inverse matrices of a given matrix say

Then $\quad AB = BA = I \qquad$ B is inverse

and $\quad AC = CA = I \qquad$ C is inverse

But $\quad C \cdot (AB) = (CA) \cdot B \qquad$ (Associative

$\Rightarrow \quad C \cdot I = I \cdot B \quad$ or $\quad C = B$

Hence, the inverse of matrix A is unique. **Pr**

**Example 23.** *Find A satisfying the Matrix equation.*

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

**Solution.** $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

Both sides of the equation are pre-multiplied by the inverse of $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ i.e. $\Big[\begin{matrix} 2 \\ -3 \end{matrix}$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}$$

$$A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix}$$

Again both sides are post-multiplied by the inverse of $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ *i.e.* $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$$A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 9 \\ 12 & -14 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$

$$A = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix} \quad \textbf{Ans.}$$

**Example 24.** *Given* $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

*find C such that* $BCA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

**Solution.** $BCA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = -\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = -\begin{bmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{bmatrix}$$

$$C = -\begin{bmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}^{-1}$$

$$C = \begin{bmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix} \quad \textbf{Ans.}$$