PEKING UNIVERSITY

Advanced Fluid Mechanics Homework 5

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$$\frac{u}{U} = 1 - \exp\left(-\frac{v_0 y}{\nu}\right) \tag{1.1}$$

边界层位移厚度

$$\delta^* = \int_0^\infty \frac{U-u}{U}\,\mathrm{d}y = \int_0^\infty \exp\left(-\frac{v_0 y}{\nu}\right)\mathrm{d}y = \frac{\nu}{v_0}. \tag{1.2}$$

粘性系数

$$\mu = \nu\rho, \tag{1.3}$$

表面摩擦力

$$\tau_0 = \mu\frac{\mathrm{d}u}{\mathrm{d}y}\bigg|_{y=0} = \rho U v_0. \tag{1.4}$$

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先统一一下符号标记

$$x = z, \tag{2.1}$$

$$\sigma = r, \tag{2.2}$$

$$v = u_\theta. \tag{2.3}$$

由 *Vortical Flows* 中的 (6.1.8b), 对于无粘情况,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_\theta}{r}\right) = \frac{1}{r^4}\frac{\partial C^2}{\partial z} = \frac{2C}{r^4}\frac{\partial C}{\partial z}, \tag{2.4}$$

其中

$$C = vr. \tag{2.5}$$

又根据 (6.1.4a)

$$\omega_r = -\frac{1}{r}\frac{\partial C}{\partial z}, \tag{2.6}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_\theta}{r}\right) = -\frac{2v\omega_r}{r^2}. \tag{2.7}$$

□

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在轴对称假设下, 流体的动能为

$$T = \pi\rho \int_{-\infty}^{+\infty} \int_0^{+\infty} \left(\psi\omega_\theta + V_\theta^2 r\right) \mathrm{d}r\, \mathrm{d}z. \tag{3.1}$$

对于 Hill 球涡,

$$V_\theta = 0, \tag{3.2}$$

$$\omega_\theta(r,z) = \begin{cases} -\frac{15}{2}\frac{U}{a^2}r, & R = \sqrt{r^2+z^2} < a, \\ 0, & R = \sqrt{r^2+z^2} > a, \end{cases} \tag{3.3}$$

$$\psi(r,z) = \begin{cases} -\frac{3}{4}Ur^2\left(1-\frac{R^2}{a^2}\right) - \frac{3}{4}Ur^2\left(1-\frac{R^2}{a^2}\right) - \frac{1}{2}Ur^2, & R = \sqrt{r^2+z^2} < a, \\ \frac{1}{2}Ur^2\left(1-\frac{a^3}{R^3}\right) - \frac{3}{4}Ur^2\left(1-\frac{R^2}{a^2}\right) - \frac{1}{2}Ur^2, & R = \sqrt{r^2+z^2} > a. \end{cases} \tag{3.4}$$

积分可得

$$T = \pi\rho \int_{-a}^{a} \int_0^{+\sqrt{a^2-z^2}} (\psi\omega_\theta)\, \mathrm{d}r\, \mathrm{d}z = \frac{10}{7}\pi\rho U^2 a^3. \tag{3.5}$$

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在极坐标下, 方程

$$\nabla^2\psi = c\psi \tag{4.1}$$

写为

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = c\psi. \tag{4.2}$$

把式子

$$\psi^{(i)} = cJ_1(kr)\sin\theta \tag{4.3}$$

代入式 (4.2) , 并假设 $c<0,\ \sin\theta\neq 0,\ x=kr$ 可得

$$x^2J_1''+xJ_1'-\left(\frac{c}{k^2}x^2+1\right)J_1=0. \tag{4.4}$$

当 $k=\sqrt{-c}$ 时, 式 (4.3) 正好是 $m=1$ 的贝塞尔方程的解.

在极坐标下, 流函数和速度的关系为

$$v_\theta^{(i)}=-\frac{\partial\psi^{(i)}}{\partial r}=-ckJ_1'(kr)\sin\theta,\quad v_r^{(i)}=\frac{1}{r}\frac{\partial\psi^{(i)}}{\partial\theta}=\frac{c}{r}J_1(kr)\cos\theta. \tag{4.5}$$

$v_r=0$ 时

$$J_1(kr_0)=0. \tag{4.6}$$

假设 kr_0 是 J_1 的第一个零点, 则

$$v_\theta^{(i)}=-ckJ_1'(kr_0)\sin\theta. \tag{4.7}$$

均匀来流的流函数

$$\psi^{(o)}=U_\infty\left(1-\frac{a^2}{r^2}\right)r\sin\theta,\quad a=r_0. \tag{4.8}$$

速度

$$v_\theta^{(o)}=-\frac{\partial\psi^{(o)}}{\partial r}=-U_\infty\left(1+\frac{a^2}{r^2}\right)\sin\theta,\quad v_r^{(o)}=\frac{1}{r}\frac{\partial\psi^{(o)}}{\partial\theta}=U_\infty\left(1-\frac{a^2}{r^2}\right)\cos\theta. \tag{4.9}$$

匹配

$$v_\theta^{(o)}(r=a)=v_\theta^{(i)}(r=a) \tag{4.10}$$

得

$$U_\infty=ckJ_1'(kr_0)/2. \tag{4.11}$$

得内外流函数

$$\psi=\begin{cases}cJ_1(kr)\sin\theta, & r<a,\\ \frac{1}{2}ckJ_1'(kr_0)\left(1-\frac{a^2}{r^2}\right)r\sin\theta, & r>a.\end{cases} \tag{4.12}$$

其中 $k=\sqrt{-c},\ a=r_0$.

假设 $c=-1$, 对于一阶贝塞尔函数, $r_0=3.8317$, 去流函数等值面, 可得流线图, 如图 4.1 所示.

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如图 5.1 所示, 从右上角顺时针绕一圈, 箭头先从东北方向向北旋转, 然后返回为东北方向, 然后再重复一遍, 所以该奇点的指标为 0.

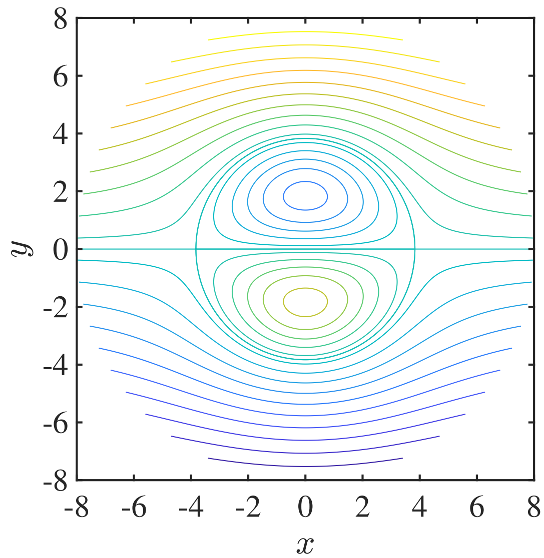

图 4.1. 流线图.

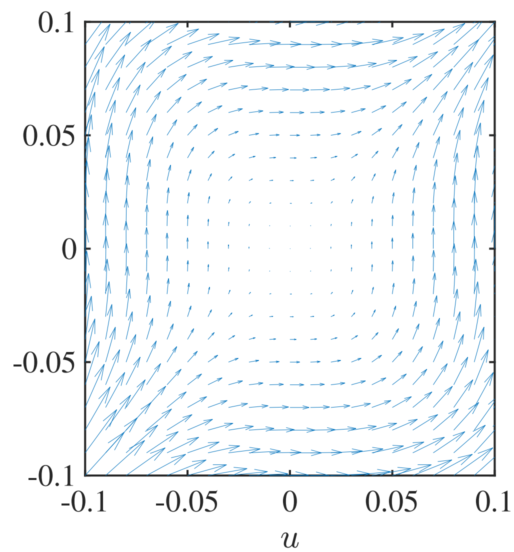

图 5.1. 流场矢量图.

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由于 $\left(\frac{\partial^2 w}{\partial z^2}\right)_0 = -\left[\frac{1}{h_1}\left(\frac{\partial^2 u}{\partial x \partial z}\right) + \frac{1}{h_2}\left(\frac{\partial^2 v}{\partial y \partial z}\right)\right]_0$, 判定流动分离的条件是: 在分离线上

$$\begin{cases} \left(\frac{\partial u}{\partial z}\right)_0 = 0, \\ \left(\frac{\partial^2 u}{\partial x \partial z}\right)_0 < 0, \\ \left(\frac{\partial^2 w}{\partial z^2}\right)_0 > 0 \text{ 或 } \left(\frac{1}{h_1}\frac{\partial^2 u}{\partial x \partial z} + \frac{1}{h_2}\frac{\partial^2 v}{\partial y \partial z}\right)_0 < 0. \end{cases} \tag{6.1}$$

对应于书上的

$$\tau_{x,x} > 0, \tag{6.2}$$
$$\tau_y = 0, \quad \tau_{y,x} = 0, \tag{6.3}$$
$$\tau_{y,y} < 0, \tag{6.4}$$
$$\nabla_\pi \cdot \boldsymbol{\tau} = \tau_{x,x} + \tau_{y,y} < 0. \tag{6.5}$$

注意两书的 x, y 标号是反的.

我从图书馆借了书, 书内容太多, 之前的部分我就不抄了.

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根据三角翼的升力公式

$$L \approx \rho U \int_W y\omega_x \, \mathrm{d}S, \tag{7.1}$$

由于三角翼会卷起一个很强的涡, 并且是延着 x 方向, 即 ω_x 很大, 所以升力也很大.

如图 7.1 所示, 发动机的转子布满一圈三角形的叶片, 图示是正视图, 转动方向如图中箭头所示.

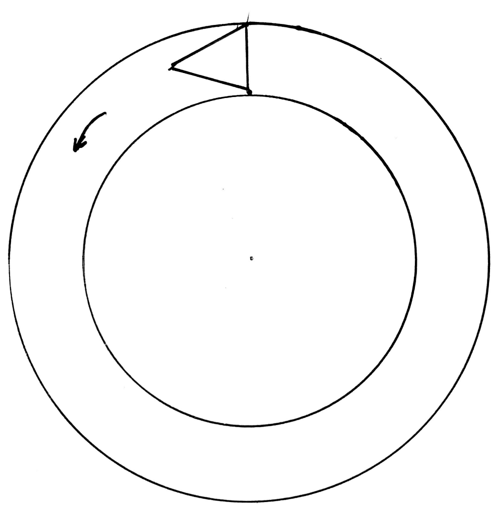

图 7.1. 发动机转子示意图.