Peking University

Advanced Fluid Mechanics
Homework 4

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假设板宽为 l, 在图 1.1中, 板和水的相互作用力垂直与板, 虚线下的水和虚线上的水的相互作用力为竖直方向. 如图 1.2, 单位时间内水的动量改变量为

$$\Delta \boldsymbol{p} = l\rho U d\Delta t(\boldsymbol{v_2} - \boldsymbol{v_1}), \tag{1.1}$$

其中 $\boldsymbol{v_2}, \boldsymbol{v_1}$ 的大小为 U, 方向和 $\boldsymbol{p_2}, \boldsymbol{p_1}$ 相同. 由图 1.2,

$$\Delta \boldsymbol{p} = \Delta \boldsymbol{F}\Delta t, \tag{1.2}$$

其中

$$\Delta \boldsymbol{F} = \boldsymbol{F_2} - \boldsymbol{F_1}. \tag{1.3}$$

由几何关系可算得

$$F = l\rho U^2 d\cot\frac{\alpha}{2}, \tag{1.4}$$

所以滑板受到的水的总的垂直向上的支持力

$$N = F\cos\alpha = l\rho U^2 d\cot\frac{\alpha}{2}\cos\alpha. \tag{1.5}$$

单位宽度滑板受到的水的总的垂直向上的支持力

$$N' = \rho U^2 d\cot\frac{\alpha}{2}\cos\alpha. \tag{1.6}$$

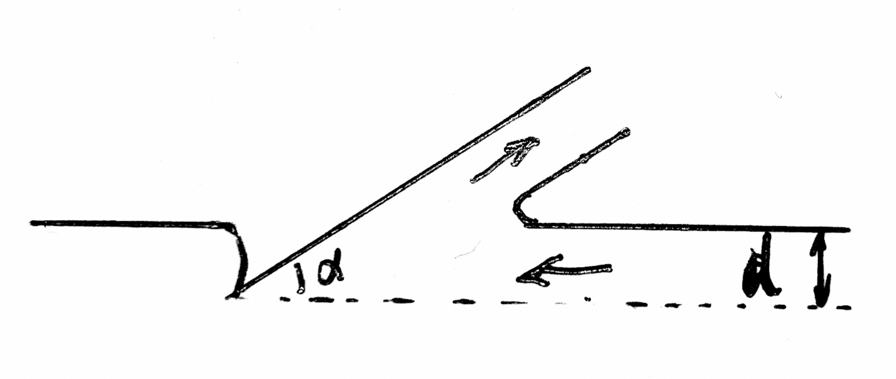

图 1.1. 流动示意图

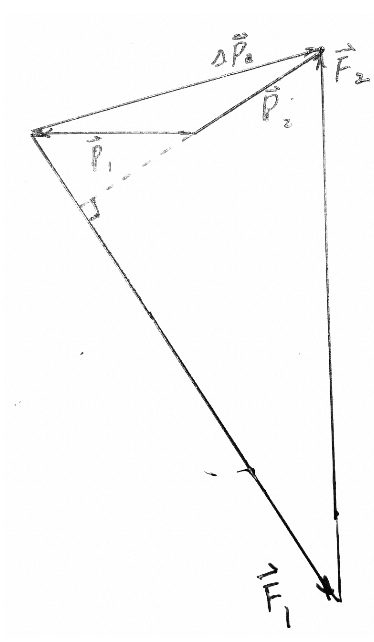

图 1.2. 受力示意图

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(1)

设液体密度为 ρ, 假设半径相同处的流体的流速基本相同. 取半径为 r 的物质体, 物质体内流体的总质量守恒,

$$0 = \mathrm{d}\left(\rho h \pi r^2\right) = r^2 \,\mathrm{d}h + 2hr \,\mathrm{d}r, \tag{2.1}$$

所以半径为 r 处流体的流速

$$v(r) = \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{r}{2h}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{Ur}{2h}. \tag{2.2}$$

半径从 r 到 $r + \Delta r$ 处的流体受到上下板的水平方向的阻力近似为

$$\mathrm{d}f(r) = \frac{\mu v}{h} 2\pi r \,\mathrm{d}r = \frac{\mu \pi r^2 U}{h^2} \,\mathrm{d}r. \tag{2.3}$$

不考虑流体的惯性力, r 到 $r + \Delta r$ 的水平方向压力差和水平阻力相等

$$-2\pi r h \,\mathrm{d}p(r) = \mathrm{d}f(r) = \frac{\mu \pi r^2 U}{h^2} \,\mathrm{d}r. \tag{2.4}$$

$$\mathrm{d}p(r) = -\frac{\mu r U}{2h^3} \,\mathrm{d}r. \tag{2.5}$$

假设流体外部压强为 0, 则

$$p(r) = \frac{\mu U}{4h^3}\left(R^2 - r^2\right). \tag{2.6}$$

所以平板受到的阻力为

$$F = \int_0^R 2\pi r p(r) \,\mathrm{d}r = \int_0^R 2\pi r \frac{\mu U}{4h^3}\left(R^2 - r^2\right) \mathrm{d}r = \frac{\pi \mu U R^4}{8h^3}. \tag{2.7}$$

(2)

考虑到液体表面张力的效应, 液体和空气的压强差

$$\Delta p = \sigma \frac{1}{R}, \tag{2.8}$$

当 R 足够小时, Δp 足够大, 液体可以支撑起平板.

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当流动为层流

在考虑平板边界层时, 边界层是一个强剪切层, 当 Re 趋于无穷时, 边界层就成为了面涡, 所以这里直接使用边界层厚度的结论

$$\delta \approx \sqrt{\frac{\nu x}{U}} = \sqrt{\frac{x^2 \rho}{\mathrm{Re}}}, \tag{3.1}$$

其中 x 是特征尺度, U 是特征速度.

当流动为湍流

$$\frac{\delta}{x} = 0.37\,\mathrm{Re}_x^{-1/5}, \tag{3.2}$$

其中

$$\mathrm{Re}_x = \frac{U_\infty x}{\nu}. \tag{3.3}$$

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引入这样一个环量函数, 假如我们考虑的点是面涡上的点 A, 让环量函数穿过点 A 和面涡上的另一确定的点 B, 其它路径任意, 则

$$\oint_C \nabla\phi \cdot \mathrm{d}\boldsymbol{x} = [\![\phi_A]\!] - [\![\phi_B]\!] \equiv \Gamma, \tag{4.1}$$

所以

$$\gamma = \boldsymbol{n} \times \nabla_\pi(\Gamma + [\![\phi_B]\!]) = \boldsymbol{n} \times \nabla_\pi \Gamma. \tag{4.2}$$

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根据 Biot-Savart 公式,

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2(n-1)\pi} \int_S \frac{\boldsymbol{\gamma}\left(\boldsymbol{x}'\right) \times \boldsymbol{r}}{r^n} \,\mathrm{d}S\left(\boldsymbol{x}'\right). \tag{5.1}$$

对于二维情况,

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2\pi} \int_S \frac{\boldsymbol{\gamma}\left(\boldsymbol{x}'\right) \times \boldsymbol{r}}{r^2} \,\mathrm{d}l\left(\boldsymbol{x}'\right). \tag{5.2}$$

假设 $\boldsymbol{\gamma}$ 的方向为垂直平面向上为 $\boldsymbol{z}$, $\boldsymbol{\theta}$ 的方向为 $\boldsymbol{z} \times \boldsymbol{r}$ 的方向. 由轴对称性, 速度只有 $\boldsymbol{\theta}$ 方向的分量, 在环外

$$u_\theta(r) 2\pi r = \gamma 2\pi a, \tag{5.3}$$

$$u_\theta(r) = \frac{\gamma a}{r}. \tag{5.4}$$

其中 a 是面涡的半径, 在环内, $\boldsymbol{u} = 0$.

对于面涡上的点,

$$u_\theta = \frac{\gamma a}{2r}. \tag{5.5}$$

即圆形面涡绕圆心转圈, 所以该面涡的形状将不发生改变. □

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$\boldsymbol{u}^+ = u_1^+ \boldsymbol{e}_1 + u_2^+ \boldsymbol{e}_2$ but $\boldsymbol{u}^- = u_1^- \boldsymbol{e}_1$ for the velocities on the outer and inner sides of the sheet, respectively,

$$[\![\boldsymbol{u}]\!] = \boldsymbol{u}^+ - \boldsymbol{u}^- = \left(u_1^+ - u_1^-\right) \boldsymbol{e}_1 + u_2^+ \boldsymbol{e}_2, \tag{6.1}$$

$$\overline{\boldsymbol{u}} = \frac{1}{2}\left(\boldsymbol{u}^+ + \boldsymbol{u}^-\right) = \frac{1}{2}\left[\left(u_1^+ + u_1^-\right) \boldsymbol{e}_1 + u_2^+ \boldsymbol{e}_2\right], \tag{6.2}$$

$$\gamma = \boldsymbol{e}_3 \times [\![\boldsymbol{u}]\!] = u_2^+ \boldsymbol{e}_1 + \left(u_1^- - u_1^+\right) \boldsymbol{e}_2, \tag{6.3}$$

where

$$\boldsymbol{e}_3 = \boldsymbol{e}_2 \times \boldsymbol{e}_1. \tag{6.4}$$

$$q^{+2} = 2\overline{\boldsymbol{u}} \cdot [\![\boldsymbol{u}]\!] + u_1^{-2} = u_1^{+2} + u_2^{+2} = u_1^{-2}. \tag{6.5}$$

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Birkhoff-Rott Equation

Given vorticity, $\omega(\boldsymbol{x})$, one can calculate the stream function, $\psi(\boldsymbol{x})$ as follows:

$$\Delta\psi(\boldsymbol{x}) = \omega(\boldsymbol{x}) \tag{7.1}$$

Now this can be solved to get

$$\psi(\boldsymbol{x}) = \frac{1}{2\pi} \int_D \omega(\boldsymbol{x}) \ln |\boldsymbol{x} - \boldsymbol{x}'| \tag{7.2a}$$

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{2\pi} \int_D \frac{\hat{\boldsymbol{z}} \times (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2} \omega(\boldsymbol{x}') d\boldsymbol{x}' \tag{7.2b}$$

Using $\omega(\boldsymbol{x})d\boldsymbol{x} = d\Gamma(\boldsymbol{x})$ and restricting the domain to be a curve C we obtain

$$\frac{\partial x(\alpha,t)}{\partial t} = \frac{-1}{2\pi}\int\limits_C \frac{(y(\alpha,t)-y(\alpha',t))}{((x(t)-x(\alpha',t))^2+(y(\alpha,t)-y(\alpha',t))^2}\Gamma'(\alpha')d\alpha' \tag{7.2c}$$

$$\frac{\partial y(\alpha,t)}{\partial t} = \frac{1}{2\pi}\int\limits_C \frac{(x(t)-x(\alpha',t))}{((x(\alpha,t)-x(\alpha',t))^2+(y(\alpha,t)-y(\alpha',t))^2}\Gamma'(\alpha')d\alpha' \tag{7.2d}$$

Desingularization of the Birkhoff-Rott equation

Given $\delta > 0$, called a smoothing parameter, we consider the equations:

$$\frac{\partial x(\alpha,t)}{\partial t} = \frac{-1}{2\pi}\int\limits_C \frac{(y(\alpha,t)-y(\alpha',t))}{((x(t)-x(\alpha',t))^2+(y(\alpha,t)-y(\alpha',t))^2+\delta^2}\Gamma'(\alpha')d\alpha' \tag{7.3a}$$

$$\frac{\partial y(\alpha,t)}{\partial t} = \frac{1}{2\pi}\int\limits_C \frac{(x(\alpha,t)-x(\alpha',t))}{((x(\alpha,t)-x(\alpha',t))^2+(y(\alpha,t)-y(\alpha',t))^2+\delta^2}\Gamma'(\alpha')d\alpha' \tag{7.3b}$$

These are called the δ-equations, which is an approximation to the Birkhoff-Rott equation. Our goal here is to solve the δ-equations numerically and observe its behaviour as δ approaches 0.

Discretization

We apply standard discretization techniques to solve the δ-equations.Let z_j(t)=x_j(t)+iy_j(t) be an approximation to the vortex sheet's position $z(\Gamma(\alpha_j),t)$ at equidistant parameter values $\alpha_j{=}\pi(j-1)/2N$, $j = 1,2.....2N+1$. The integral is evaluated by the trapezoidal rule, yielding a system of coupled ordinary differential equations for the blob's motions:

$$\frac{dx_j}{dt} = \sum_{k=1}^{2N+1} \frac{-(y_j(t)-y_k(t))}{((x_j(t)-x_k(t))^2+(y_j(t)-y_k(t))^2+\delta^2)}w_k \tag{7.4a}$$

$$\frac{dy_j}{dt} = \sum_{k=1}^{2N+1} \frac{(x_j(t)-x_k(t))}{((x_j(t)-x_k(t))^2+(y_j(t)-y_k(t))^2+\delta^2)}w_k \tag{7.4b}$$

where $w_k = \Gamma'(\alpha_k)\frac{1}{2\pi N}, k = 1,2.....2N+1$ refer to the quadrature weights. We use the fourth-order Runge Kutta scheme to integrate the above system of equations.

Periodic Vortex Sheet

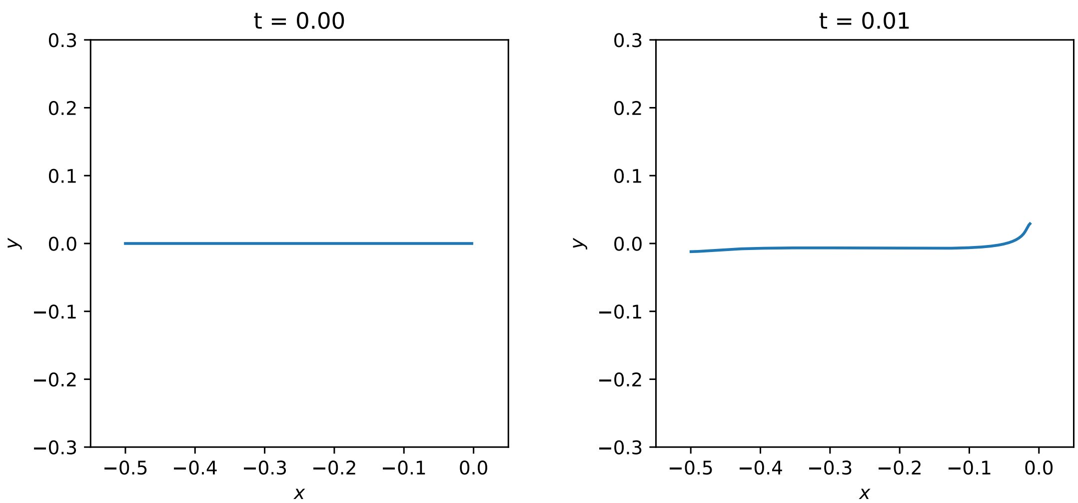
The δ-equations for a periodic vortex-sheet are given by:

$$\frac{dx_j}{dt} = \frac{-1}{2N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\sinh 2\pi(y_j(t) - y_k(t))}{(\cosh 2\pi(x_j(t) - x_k(t)) - \cos 2\pi(y_j(t) - y_k(t)) + \delta^2)} w_k \tag{7.5a}$$

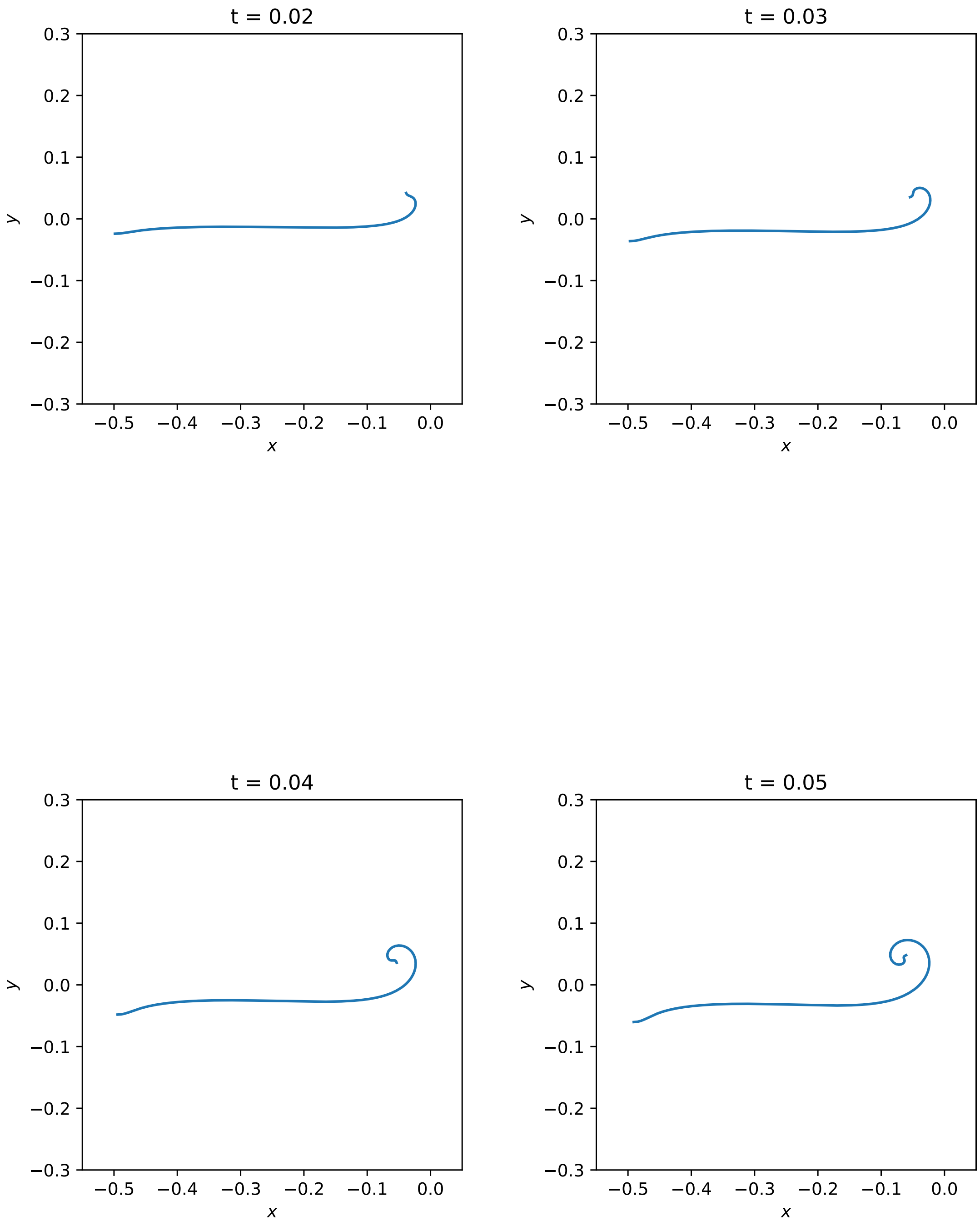
$$\frac{dy_j}{dt} = \frac{1}{2N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\sin 2\pi(x_j(t) - x_k(t))}{(\cosh 2\pi(x_j(t) - x_k(t)) - \cos 2\pi(y_j(t) - y_k(t)) + \delta^2)} w_k \tag{7.5b}$$

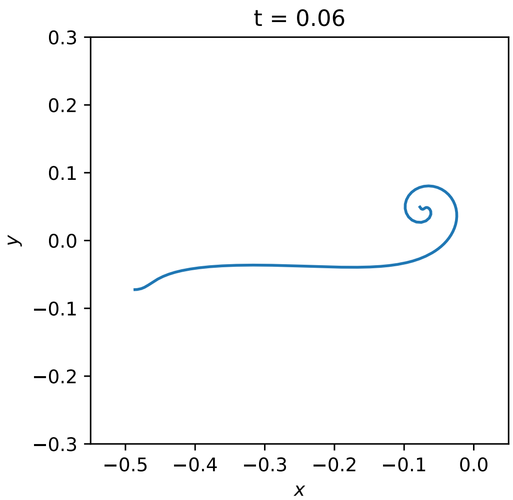
The initial conditions are given by $x_j(0) = \Gamma_j + 0.01 \sin 2\pi\Gamma_j$and $y_j(0) = -0.01 \sin 2\pi\Gamma_j$
.

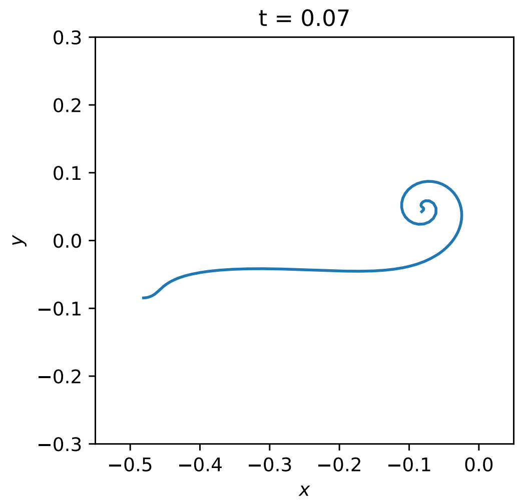
The figures below show a trignometric interpolating polynomial, in the variable Γ. It interpolates the perturbation quantities $p_j(t) = x_j(t) - \Gamma_j + iy_j(t)$, the coefficients of the polynomial are the fourier coefficients of p_j and are calculated using the Fast Fourier Transform. The plotted image is $\Gamma + P(\Gamma, t)$, where P is the polynomial described above.

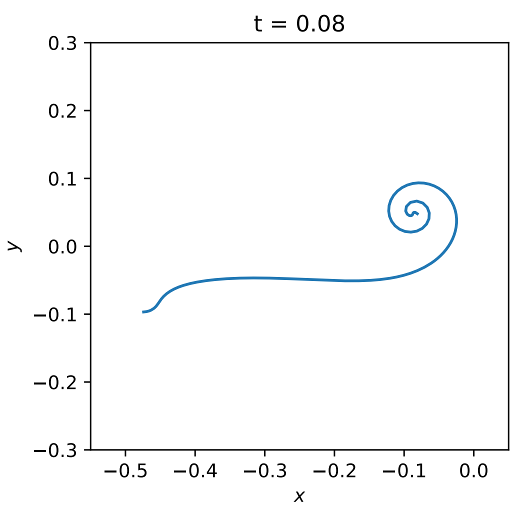


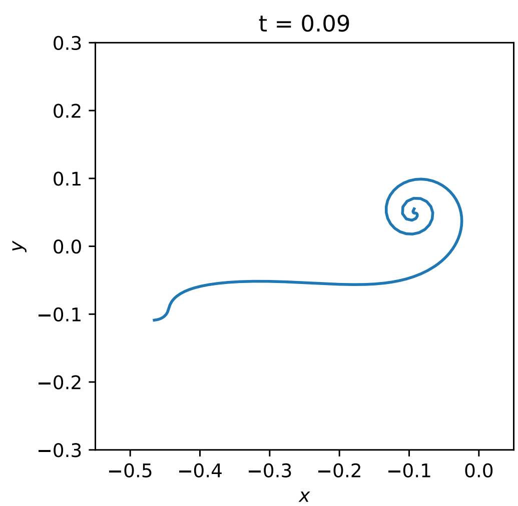

```
import numpy as np
from math import cos, sin, sinh, cosh, pi, log
import matplotlib.pyplot as plt

N = 200
delta = 0.1
xmax = 0.5
n = float(N)
s = (np.arange(N) / n - 1) * xmax
dt = 0.005
T = 0.05

```


t = 0.02
t = 0.03
t = 0.04
t = 0.05
x
y


t = 0.06
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
y
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
x


t = 0.07
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
y
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
x


t = 0.08
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
y
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
x


t = 0.09
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
y
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
x

```python
x = s
y = 0.0 * abs(np.flipud(s))

gamma = 1 / np.sqrt(-s)
plt.plot(gamma)
time = 0

def f1(x, y):
    dx = np.zeros(N)
    for j in range(N):
        for k in range(N):
            if k != j:
                dx[j] += sinh(2 * pi * (y[j] - y[k])) / (
                    (cosh(2 * pi * (x[j] - x[k])) -
                        cos(2 * pi * (y[j] - y[k])) + delta**2)) * gamma[k]
    return (-0.5 / n) * dx


def f2(x, y):
    dy = np.zeros(N)
    for j in range(N):
        for k in range(N):
            if k != j:
                dy[j] += sin(2 * pi * (x[j] - x[k])) / (
                    (cosh(2 * pi * (x[j] - x[k])) -
                        cos(2 * pi * (y[j] - y[k])) + delta**2)) * gamma[k]
    return (0.5 / n) * dy


xk1, yk1 = np.ones(N), np.ones(N)
xk2, yk2 = np.ones(N), np.ones(N)
xk3, yk3 = np.ones(N), np.ones(N)
xk4, yk3 = np.ones(N), np.ones(N)

Finaltime = int(T / dt)

# data = np.vstack((x, y))
for t in range(Finaltime):
    if (Finaltime - t) % 2 == 0:
        plt.figure(t)
        plt.figure(figsize=(4, 4))
        plt.xlabel(r'$x$')
        plt.ylabel(r'$y$')
        plt.title("t = %.2f" % (time))
        plt.plot(x, y)
        print(Finaltime - t)
        plt.axis([-1.1 * xmax, 0.1 * xmax, -0.6 * xmax, 0.6 * xmax])

        plt.savefig("%.2f.pdf" % (time), bbox_inches='tight')
```

```
    xk1, yk1 = x, y
    xk2 = x + 0.5 * dt * f1(xk1, yk1)
    yk2 = y + 0.5 * dt * f2(xk1, yk1)
    xk3 = x + 0.5 * dt * f1(xk2, yk2)
    yk3 = y + 0.5 * dt * f2(xk2, yk2)
    xk4 = x + dt * f1(xk3, yk3)
    yk4 = y + dt * f2(xk3, yk3)
    x = x + (dt / 6.0) * (f1(xk1, yk1) + 2 * f1(xk2, yk2) + 2
        * f1(xk3, yk3) + f1(xk4, yk4))
    y = y + (dt / 6.0) * (f2(xk1, yk1) + 2 * f2(xk2, yk2) + 2
        * f2(xk3, yk3) + f2(xk4, yk4))
    time += dt
```

8

试用理论分析上一题面涡的卷绕

$$\gamma = Cx^{-1/2}, \quad \Gamma = 2Cx^{1/2}, \tag{8.1}$$

其中 C 是一常数, 量纲为 $L^{3/2}T^{-1}$, 面涡出事状态

$$\xi(\Gamma, 0) = \frac{\Gamma^2}{4C^2}, \quad \Gamma \in (0, +\infty), \tag{8.2}$$

由量纲分析

$$\xi(\Gamma, t, C) = (Ct)^{2/3} f(\tau), \tag{8.3}$$

其中

$$\tau = \Gamma C^{-4/3} t^{-1/3}, \tag{8.4}$$

经过足够长时间后, 取以螺旋中心为原点的极坐标系, 通过量纲分析,

$$\Gamma(r) = 2C(\lambda r)^{1/2}, \tag{8.5}$$

其中 λ 是无量纲常数,

$$V_\theta = \frac{\Gamma}{2\pi r} = \frac{C\lambda^{1/2}}{\pi r^{1/2}}, \tag{8.6}$$

对于 Lagrange 坐标为 Γ_p 的某一流体质点,

$$r_p = \frac{\Gamma_p^2}{4\lambda C^2}, \tag{8.7}$$

又

$$V_\theta = r_p \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_P, \tag{8.8}$$

$$r \approx \left(\frac{C^2\lambda}{\pi^2}\right)^{1/3} \left(\frac{t}{\theta - \theta_0}\right)^{2/3}. \tag{8.9}$$