

高等应用数学作业 4

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Ch. 11 Sec. 11.3 Prom. 3

$$\ddot{y} + \epsilon \dot{y} + y = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1$$

忽略小量 ϵ , 则 $y = \sin(t)$ 。

故 $f^{(0)}(t, \tau) = A(\tau) \sin(t)$ 且 $A(0) = 1$

$$y(t, \tau) = f^{(0)}(t, \tau) + \epsilon f^{(1)}(t, \tau) + O(\epsilon^2), \quad \tau = \epsilon t$$

$$\dot{y}(t, \tau) = f_1^{(0)}(t, \tau) + \epsilon(f_1^{(1)}(t, \tau) + f_2^{(0)}(t, \tau)) + O(\epsilon^2)$$

$$\ddot{y}(t, \tau) = f_{11}^{(0)}(t, \tau) + \epsilon(f_{11}^{(1)}(t, \tau) + 2f_{12}^{(0)}(t, \tau)) + O(\epsilon^2)$$

$$\ddot{y} + \epsilon \dot{y} + y = f_{11}^{(0)}(t, \tau) + f^{(0)}(t, \tau) + \epsilon(f_{11}^{(1)}(t, \tau) + 2f_{12}^{(0)}(t, \tau) + f_1^{(0)}(t, \tau) + f^{(1)}(t, \tau)) + O(\epsilon^2) = 0$$

初始边界条件 $f^{(0)}(0, 0) = f^{(1)}(0, 0) = \dots = 0$, $f_1^{(0)}(0, 0) = 1$, $f_1^{(1)}(0, 0) + f_2^{(0)}(0, 0) = 0$, \dots

方程展开式条件:

$$\begin{cases} 0 = f_{11}^{(0)}(t, \tau) + f^{(0)}(t, \tau) = -A(\tau) \sin(t) + A(\tau) \sin(t) \\ 0 = f_{11}^{(1)}(t, \tau) + 2f_{12}^{(0)}(t, \tau) + f_1^{(0)}(t, \tau) + f^{(1)}(t, \tau) \\ \quad = f_{11}^{(1)}(t, \tau) + f^{(1)}(t, \tau) + A(\tau)^3 \cos^3(t) + 2A'(\tau) \cos(t) \end{cases} \quad (1.1)$$

消除共振项 $A(\tau)^3 \cos^3(t) + 2A'(\tau) \cos(t) = 0 = (3A^3/4 + 2A') \cos(t) + A^3/4 \cos(3t)$

$3A^3/4 + 2A' = 0$ 且 $A(0) = 1$ 。

得 $A(\tau) = (1 + 3\tau/4)^{-0.5}$, $y \approx (1 + 3\epsilon t/4)^{-0.5} \sin(t) + O(\epsilon)$

解释: \ddot{y} 项系数小, 针对快变量小尺度时间忽略该项, 得到解主要受 \dot{y} 与 y 项影响的周期函数, 在大尺度时间下慢变量的影响反映在解的幅值随时间逐渐下降。

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Ch. 11 Sec. 11.3 Prob.7

3.1

$$\varepsilon \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0; \quad y(0) = 0, \quad y(1) = 1, \quad 0 < x < 1, \quad 0 < \varepsilon \ll 1 \quad (3.1)$$

假设

$$y = f(x, X, \varepsilon) = f^{(0)}(x, X) + \varepsilon f^{(1)}(x, X) + \dots \quad (3.2)$$

其中 $X = x/\varepsilon$, 那么

$$\frac{d}{dx} f(x, X, \varepsilon) = \frac{1}{\varepsilon} f_2^{(0)} + f_1^{(0)} + f_2^{(1)} + \varepsilon f_1^{(1)} \quad (3.3)$$

$$\frac{d^2}{dx^2} f(x, X, \varepsilon) = \frac{1}{\varepsilon^2} f_{22}^{(0)} + \frac{1}{\varepsilon} \left[2f_{21}^{(0)} + f_{22}^{(1)} \right] + f_{11}^{(0)} + 2f_{12}^{(1)} + \varepsilon f_{12}^{(1)} \quad (3.4)$$

把式 (2.1) to (2.3) 代入式 (2.1) 可得

$$\frac{1}{\varepsilon} f_{22}^{(0)} + 2f_{21}^{(0)} + f_{22}^{(1)} + \varepsilon \left[f_{11}^{(0)} + 2f_{12}^{(1)} \right] + \varepsilon^2 f_{12}^{(1)} + \frac{2}{\varepsilon} f_2^{(0)} + 2f_1^{(0)} + 2f_2^{(1)} + 2\varepsilon f_1^{(1)} + f^{(0)} + \varepsilon f^{(1)} = 0 \quad (3.5)$$

$O(\varepsilon)$ 项为 0, 所以

$$f_{22}^{(0)} + 2f_2^{(0)} = 0 \quad (3.6)$$

求解式 (2.6) 可得

$$f^{(0)}(x, X) = C_1(x) + C_2(x) \exp(-2X). \quad (3.7)$$

3.2

$O(1)$ 项为 0, 所以

$$2f_{21}^{(0)} + f_{22}^{(1)} + 2f_1^{(0)} + 2f_2^{(1)} = 0 \quad (3.8)$$

式 (2.7) 代入式 (2.8) 可得

$$-2C_2' \exp(-2X) + f_{22}^{(1)} + 2C_1' + 2f_2^{(1)} + C_1 + C_2 \exp(-2X) = 0 \quad (3.9)$$

即

$$f_{22}^{(1)} + 2f_2^{(1)} = (2C_2' - C_2) \exp(-2X) - (2C_1' + C_1) \quad (3.10)$$

因为对于任意的 ε ，都要有 $O(1) = 0$ ，所以有

$$2C_1' + C_1 = 0 \quad (3.11)$$

$(2C_2' - C_2) \exp(-2X)$ 非共振项，所以并不要求 $(2C_2' - C_2) \exp(-2X) = 0$ 。由式 (2.11) 可以解得

$$C_1 = \exp \left[A - \frac{1}{2}x \right] \quad (3.12)$$

由初始条件可得

$$C_1(1) = 1 \quad (3.13)$$

所以

$$C_1 = \exp \left[\frac{1}{2}(1 - x) \right] \quad (3.14)$$

又根据初始条件 $y(0) = 0$

$$C_1(0) + C_2(0) = 0 \quad (3.15)$$

所以

$$e^{\frac{1}{2}} + C_2(0) = 0 \quad (3.16)$$

$$C_2(0) = -e^{\frac{1}{2}} \quad (3.17)$$

式 (2.12) 和 9.2.5 远离边界层 $x = 0$ 的解是一样的。

$$y = e^{\frac{1}{2}} \left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}} \right) \quad (3.18)$$

3.3

$$\begin{cases} C_1(0) + C_2(0) = 0 \\ C_1(1) = 0, \quad C_2(1)e^{-\frac{2}{\varepsilon}} = 1 \end{cases} \quad (3.19)$$

$$C_1 = 0, \quad C_2(1) = e^{2/\varepsilon} \quad (3.20)$$

当 $C_2(x)$ 取 $e^{2/\varepsilon}$ 时，

$$y(x, \varepsilon) = e^{(2/\varepsilon)(1-x)} \quad (3.21)$$