PEKING UNIVERSITY

高等应用数学作业 4

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$\ddot{y} + \epsilon\dot{y}^3 + y = 0,\ y(0) = 0,\ \dot{y}(0) = 1$

忽略小量 ϵ，则 $y = \sin(t)$。

故 $f^{(0)}(t,\tau) = A(\tau)\sin(t)$ 且 $A(0) = 1$

$y(t,\tau) = f^{(0)}(t,\tau) + \epsilon f^{(1)}(t,\tau) + O(\epsilon^2)$，$\tau = \epsilon t$

$\dot{y}(t,\tau) = f_1^{(0)}(t,\tau) + \epsilon(f_1^{(1)}(t,\tau) + f_2^{(0)}(t,\tau)) + O(\epsilon^2)$

$\ddot{y}(t,\tau) = f_{11}^{(0)}(t,\tau) + \epsilon(f_{11}^{(1)}(t,\tau) + 2f_{12}^{(0)}(t,\tau)) + O(\epsilon^2)$

$\ddot{y}+\epsilon\dot{y}^3+y = f_{11}^{(0)}(t,\tau)+f^{(0)}(t,\tau)+\epsilon(f_{11}^{(1)}(t,\tau)+2f_{12}^{(0)}(t,\tau)+f_1^{(0)}(t,\tau)+f^{(1)}(t,\tau))^3+O(\epsilon^2) = 0$

初始边界条件 $f^{(0)}(0,0) = f^{(1)}(0,0) = \cdots = 0$，$f_1^{(0)}(0,0) = 1$，$f_1^{(1)}(0,0) + f_2^{(0)}(0,0) = 0$，$\cdots$

方程展开式条件:

$$\begin{cases} 0 & = f_{11}^{(0)}(t,\tau) + f^{(0)}(t,\tau) = -A(\tau)\sin(t) + A(\tau)\sin(t) \\ 0 & = f_{11}^{(1)}(t,\tau) + 2f_{12}^{(0)}(t,\tau) + f_1^{(0)3}(t,\tau) + f^{(1)}(t,\tau) \\ & = f_{11}^{(1)}(t,\tau) + f^{(1)}(t,\tau) + A(\tau)^3\cos^3(t) + 2A'(\tau)\cos(t) \end{cases} \tag{1.1}$$

消除共振项 $A(\tau)^3\cos^3(t) + 2A'(\tau)\cos(t) = 0 = (3A^3/4 + 2A')\cos(t) + A^3/4\cos(3t)$

$3A^3/4 + 2A' = 0$ 且 $A(0) = 1$。

得 $A(\tau) = (1 + 3\tau/4)^{-0.5}$，$y \approx (1 + 3\epsilon t/4)^{-0.5}\sin(t) + O(\epsilon)$

解释：$\dot{y}$ 项系数小，针对快变量小尺度时间忽略该项，得到解主要受 $\ddot{y}$ 与 y 项影响的周期函数，在大尺度时间下慢变量的影响反映在解的幅值随时间逐渐下降。

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3.1

$$\varepsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+y=0;\quad y(0)=0,\quad y(1)=1,\quad 0<x<1,\quad 0<\varepsilon\ll 1 \tag{3.1}$$

假设

$$y=f(x,X,\varepsilon)=f^{(0)}(x,X)+\varepsilon f^{(1)}(x,X)+\cdots \tag{3.2}$$

其中 $X=x/\varepsilon$, 那么

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x,X,\varepsilon)=\frac{1}{\varepsilon}f_2^{(0)}+f_1^{(0)}+f_2^{(1)}+\varepsilon f_1^{(1)} \tag{3.3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x,X,\varepsilon)=\frac{1}{\varepsilon^2}f_{22}^{(0)}+\frac{1}{\varepsilon}\left[2f_{21}^{(0)}+f_{22}^{(1)}\right]+f_{11}^{(0)}+2f_{12}^{(1)}+\varepsilon f_{12}^{(1)} \tag{3.4}$$

把式 (3.1) to (3.3) 代入式 (3.1) 可得

$$\frac{1}{\varepsilon}f_{22}^{(0)}+2f_{21}^{(0)}+f_{22}^{(1)}+\varepsilon\left[f_{11}^{(0)}+2f_{12}^{(1)}\right]+\varepsilon^2f_{12}^{(1)}+\frac{2}{\varepsilon}f_2^{(0)}+2f_1^{(0)}+2f_2^{(1)}+2\varepsilon f_1^{(1)}+f^{(0)}+\varepsilon f^{(1)}=0 \tag{3.5}$$

$O(\varepsilon)$ 项为 0, 所以

$$f_{22}^{(0)}+2f_2^{(0)}=0 \tag{3.6}$$

求解式 (3.6) 可得

$$f^{(0)}(x,X)=C_1(x)+C_2(x)\exp\left(-2X\right). \tag{3.7}$$

3.2

$O(1)$ 项为 0, 所以

$$2f_{21}^{(0)}+f_{22}^{(1)}+2f_1^{(0)}+2f_2^{(1)}=0 \tag{3.8}$$

式 (3.7) 代入式 (3.8) 可得

$$-2C_2'\exp\left(-2X\right)+f_{22}^{(1)}+2C_1'+2f_2^{(1)}+C_1+C_2\exp\left(-2X\right)=0 \tag{3.9}$$

即

$$f_{22}^{(1)}+2f_{2}^{(1)}=(2C_2'-C_2)\exp(-2X)-(2C_1'+C_1) \tag{3.10}$$

因为对于任意的 ε，都要有 $O(1)=0$，所以有

$$2C_1'+C_1=0 \tag{3.11}$$

$(2C_2'-C_2)\exp(-2X)$ 非共振项，所以并不要求 $(2C_2'-C_2)\exp(-2X)=0$。由式 (3.11) 可以解得

$$C_1=\exp\left[A-\frac{1}{2}x\right] \tag{3.12}$$

由初始条件可得

$$C_1(1)=1 \tag{3.13}$$

所以

$$C_1=\exp\left[\frac{1}{2}(1-x)\right] \tag{3.14}$$

又根据初始条件 $y(0)=0$

$$C_1(0)+C_2(0)=0 \tag{3.15}$$

所以

$$\mathrm{e}^{\frac{1}{2}}+C_2(0)=0 \tag{3.16}$$

$$C_2(0)=-\mathrm{e}^{\frac{1}{2}} \tag{3.17}$$

式 (3.12) 和 9.2.5 远离边界层 $x=0$ 的解是一样的。

$$y=\mathrm{e}^{\frac{1}{2}}\left(\mathrm{e}^{-\frac{x}{2}}-\mathrm{e}^{-\frac{2x}{\varepsilon}}\right) \tag{3.18}$$

3.3

$$\begin{cases} C_1(0)+C_2(0)=0 \\ C_1(1)=0, \quad C_2(1)\mathrm{e}^{-\frac{2}{\varepsilon}}=1 \end{cases} \tag{3.19}$$

$$C_1=0, \quad C_2(1)=\mathrm{e}^{2/\varepsilon} \tag{3.20}$$

当 $C_2(x)$ 取 $\mathrm{e}^{2/\varepsilon}$ 时，

$$y(x,\varepsilon)=\mathrm{e}^{(2/\varepsilon)(1-x)} \tag{3.21}$$