PEKING UNIVERSITY

计算流体力学作业 2

College of Engineering 2001111690 袁磊祺

March 24, 2021

动量方程和能量方程推导

动量方程推导

$\tau_{zy}, \tau_{zy}, \tau_{zz}$ 是作用在与 xoy 坐标平面平行的表面上的三个粘性应力分量.

下面考虑应力的 y 分量: 压力 p, 应力 $\tau_{xy}, \tau_{yy}, \tau_{zy}$. 单位面积上压力:

$$p\left(x, y \pm \frac{1}{2}\delta y, z, t\right) = p(\boldsymbol{x}, t) \pm \frac{1}{2}\delta y \frac{\partial p}{\partial y}(\boldsymbol{x}, t) \tag{1.1}$$

单位面积上粘性应力:

$$\tau_{xy} \pm \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\delta x, \quad \tau_{yy} \pm \frac{\partial \tau_{yy}}{\partial y}\frac{1}{2}\delta y, \quad \tau_{zy} \pm \frac{\partial \tau_{zy}}{\partial z}\frac{1}{2}\delta z. \tag{1.2}$$

作用在前表面和后表面上的 y 方向净应力为

$$\left[\left(p - \frac{\partial p}{\partial y}\frac{1}{2}\delta y\right) - \left(\tau_{yy} - \frac{\partial \tau_{yy}}{\partial y}\frac{1}{2}\delta y\right)\right]\delta x \delta z \tag{1.3}$$

$$+\left[-\left(p + \frac{\partial p}{\partial y}\frac{1}{2}\delta y\right) + \left(\tau_{yy} + \frac{\partial \tau_{yy}}{\partial y}\frac{1}{2}\delta y\right)\right]\delta x \delta z \tag{1.4}$$

$$= \left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y}\right)\delta x \delta y \delta z. \tag{1.5}$$

作用在左表面和右表面上的 y 方向净应力为

$$-\left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\delta x\right)\delta y \delta z + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}\delta x\right)\delta y \delta z = \frac{\partial \tau_{xy}}{\partial x}\delta x \delta y \delta z. \tag{1.6}$$

最后在上表面和下表面上的 y 方向净应力为

$$-\left(\tau_{zy}-\frac{\partial\tau_{zy}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y+\left(\tau_{zy}+\frac{\partial\tau_{zy}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y=\frac{\partial\tau_{zy}}{\partial z}\delta x\delta y\delta z. \tag{1.7}$$

因此由于这些表面应力在流体上的单位体积的总应力是

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\left(-p+\tau_{yy}\right)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}. \tag{1.8}$$

不考虑体积力的细节, 将它们的整体结果 (或影响) 用单位时间内单位体积的 $y-$ 方向的动量源 S_x 来体现. 因此, y 方向的动量方程:

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t}=-\frac{\partial p}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+S_y \tag{1.9}$$

下面考虑应力的 z 分量: 压力 p, 应力 $\tau_{xz},\tau_{zz},\tau_{yz}$. 单位面积上压力:

$$p\left(x,z\pm\frac{1}{2}\delta z,y,t\right)=p(\boldsymbol{x},t)\pm\frac{1}{2}\delta z\frac{\partial p}{\partial z}(\boldsymbol{x},t) \tag{1.10}$$

单位面积上粘性应力:

$$\tau_{xz}\pm\frac{\partial\tau_{xz}}{\partial x}\frac{1}{2}\delta x,\quad \tau_{zz}\pm\frac{\partial\tau_{zz}}{\partial z}\frac{1}{2}\delta z,\quad \tau_{yz}\pm\frac{\partial\tau_{yz}}{\partial y}\frac{1}{2}\delta y. \tag{1.11}$$

作用在上表面和下表面上的 z 方向净应力为

$$\left[\left(p-\frac{\partial p}{\partial z}\frac{1}{2}\delta z\right)-\left(\tau_{zz}-\frac{\partial\tau_{zz}}{\partial z}\frac{1}{2}\delta z\right)\right]\delta x\delta y \tag{1.12}$$

$$+\left[-\left(p+\frac{\partial p}{\partial z}\frac{1}{2}\delta z\right)+\left(\tau_{zz}+\frac{\partial\tau_{zz}}{\partial z}\frac{1}{2}\delta z\right)\right]\delta x\delta y \tag{1.13}$$

$$=\left(-\frac{\partial p}{\partial z}+\frac{\partial\tau_{zz}}{\partial z}\right)\delta x\delta z\delta y. \tag{1.14}$$

作用在左表面和右表面上的 z 方向净应力为

$$-\left(\tau_{xz}-\frac{\partial\tau_{xz}}{\partial x}\frac{1}{2}\delta x\right)\delta z\delta y+\left(\tau_{xz}+\frac{\partial\tau_{xz}}{\partial x}\frac{1}{2}\delta x\right)\delta z\delta y=\frac{\partial\tau_{xz}}{\partial x}\delta x\delta z\delta y. \tag{1.15}$$

最后在前表面和后表面上的 z 方向净应力为

$$-\left(\tau_{yz}-\frac{\partial\tau_{yz}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z+\left(\tau_{yz}+\frac{\partial\tau_{yz}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z=\frac{\partial\tau_{yz}}{\partial y}\delta x\delta z\delta y. \tag{1.16}$$

因此由于这些表面应力在流体上的单位体积的总应力是

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial\left(-p+\tau_{zz}\right)}{\partial z}+\frac{\partial \tau_{yz}}{\partial y}. \tag{1.17}$$

不考虑体积力的细节, 将它们的整体结果 (或影响) 用单位时间内单位体积的 $z-$ 方向的动量源 S_x 来体现. 因此, z 方向的动量方程:

$$\rho \frac{\mathrm{D} w}{\mathrm{D} t}=-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{zz}}{\partial z}+\frac{\partial \tau_{yz}}{\partial y}+S_z \tag{1.18}$$

能量方程推导

由 ppt 上的动量方程 (2.15)-(2.17)

$$\rho \frac{\mathrm{D} \boldsymbol{u}}{\mathrm{D} t}=-\nabla p+\nabla \cdot \boldsymbol{\tau}+\boldsymbol{S}_{\boldsymbol{M}}, \tag{1.19}$$

其中 $\boldsymbol{\tau}$ 是张量, 两边左点乘 $\boldsymbol{u}$ 得

$$\rho \frac{\mathrm{D} \frac{1}{2}|\boldsymbol{u}|^2}{\mathrm{D} t}=-\boldsymbol{u} \cdot \nabla p+\boldsymbol{u} \cdot(\nabla \cdot \boldsymbol{\tau})^T+\boldsymbol{u} \cdot \boldsymbol{S}_{\boldsymbol{M}}. \tag{1.20}$$

将上式加上 ppt 上的 (2.31) 可得

$$\begin{aligned}\rho \frac{\mathrm{D} \varepsilon}{\mathrm{D} t}=-\operatorname{div}(p \boldsymbol{u})+\Bigg[&\frac{\partial\left(u \tau_{xx}\right)}{\partial x}+\frac{\partial\left(u \tau_{yx}\right)}{\partial y}+\frac{\partial\left(u \tau_{zx}\right)}{\partial z} \\ &+\frac{\partial\left(v \tau_{xy}\right)}{\partial x}+\frac{\partial\left(v \tau_{yy}\right)}{\partial y}+\frac{\partial\left(v \tau_{zy}\right)}{\partial z}+\frac{\partial\left(w \tau_{xz}\right)}{\partial x} \\ &+\frac{\partial\left(w \tau_{yz}\right)}{\partial y}+\frac{\partial\left(w \tau_{zz}\right)}{\partial z}\Bigg]+\operatorname{div}(k \operatorname{grad} T)+S_E,\end{aligned} \tag{1.21}$$

为了证明

$$\begin{aligned}\frac{\partial\left(\rho h_0\right)}{\partial t}+\operatorname{div}\left(\rho h_0 \boldsymbol{u}\right)=\operatorname{div}(k \operatorname{grad} T)+\frac{\partial p}{\partial t}+\Bigg[&\frac{\partial\left(u \tau_{xx}\right)}{\partial x}+\frac{\partial\left(u \tau_{yx}\right)}{\partial y} \\ &+\frac{\partial\left(u \tau_{zx}\right)}{\partial z}+\frac{\partial\left(v \tau_{xy}\right)}{\partial x}+\frac{\partial\left(v \tau_{yy}\right)}{\partial y}+\frac{\partial\left(v \tau_{zy}\right)}{\partial z} \\ &+\frac{\partial\left(w \tau_{xz}\right)}{\partial x}+\frac{\partial\left(w \tau_{yz}\right)}{\partial y}+\frac{\partial\left(w \tau_{zz}\right)}{\partial z}\Bigg]+S_h,\end{aligned} \tag{1.22}$$

把 $h_0=\varepsilon+p/\rho$ 代入式 (1.21) 并减去式 (1.22), 即证

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(h_0-p/\rho)+\nabla\cdot(p\boldsymbol{u})-\frac{\partial\left(\rho h_0\right)}{\partial t}-\operatorname{div}\left(\rho h_0\boldsymbol{u}\right)+\frac{\partial p}{\partial t}=0. \tag{1.23}$$

$$\begin{aligned}
&\rho\frac{\mathrm{D}}{\mathrm{D}t}(h_0-p/\rho)+\nabla\cdot(p\boldsymbol{u})-\frac{\partial\left(\rho h_0\right)}{\partial t}-\operatorname{div}\left(\rho h_0\boldsymbol{u}\right)+\frac{\partial p}{\partial t} && (1.24)\\
=&\ \rho\frac{\partial}{\partial t}h_0+\frac{p}{\rho}\frac{\partial\rho}{\partial t}+\rho\boldsymbol{u}\cdot\nabla h_0-\rho\boldsymbol{u}\cdot\nabla\frac{p}{\rho}+\nabla\cdot(p\boldsymbol{u})-\frac{\partial\left(\rho h_0\right)}{\partial t}-\nabla\cdot\left(\rho h_0\boldsymbol{u}\right) && (1.25)\\
=&\ -h_0\frac{\partial\rho}{\partial t}+\frac{p}{\rho}\frac{\partial\rho}{\partial t}+\frac{p}{\rho}\boldsymbol{u}\cdot\nabla\rho-\boldsymbol{u}\cdot\nabla p+\nabla\cdot(p\boldsymbol{u})-h_0\nabla\cdot(\rho\boldsymbol{u}) && (1.26)\\
=&\ \frac{p}{\rho}\frac{\partial\rho}{\partial t}+\frac{p}{\rho}\boldsymbol{u}\cdot\nabla\rho+p\nabla\cdot\boldsymbol{u} && (1.27)\\
=&\ 0. && (1.28)
\end{aligned}$$

推导过程用到了质量守恒

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0. \tag{1.29}$$

拟一维喷管 (nozzle) 流动的控制方程组

连续性方程

对于控制体内的气体, 质量的变化率等于质量的流走率

$$\frac{\partial\rho A\,\mathrm{d}x}{\partial t}=\rho AV-(\rho+\mathrm{d}\rho)(A+\mathrm{d}A)(V+\mathrm{d}V), \tag{2.1}$$

忽略高阶小量, 两边除以 $\mathrm{d}x$ 得

$$\frac{\partial(\rho A)}{\partial t}+\rho A\frac{\partial V}{\partial x}+\rho V\frac{\partial A}{\partial x}+VA\frac{\partial\rho}{\partial x}=0. \tag{2.2}$$

动量方程

对于控制体内的气体, 动量的变化率等于两边的压力差加上动量的流走率

$$\frac{\partial\rho VA\,\mathrm{d}x}{\partial t}=\rho A-(\rho+\mathrm{d}\rho)(A+\mathrm{d}A)+\rho AV^2-(\rho+\mathrm{d}\rho)(A+\mathrm{d}A)(V+\mathrm{d}V)^2+p\,\mathrm{d}A, \tag{2.3}$$

注意要考虑壁面的压力, 忽略高阶小量, 两边除以 $\mathrm{d}x$, 代入式 (2.2) 得

$$\rho\frac{\partial V}{\partial t}+\rho V\frac{\partial V}{\partial x}=-\frac{\partial p}{\partial x}. \tag{2.4}$$

又根据

$$p = \rho RT, \tag{2.5}$$

得

$$\rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} = -R\left(\rho\frac{\partial T}{\partial x} + T\frac{\partial \rho}{\partial x}\right). \tag{2.6}$$

能量方程

对于控制体内的气体, 动量的变化率等于两边的压力差加上动量的流走率

$$\frac{\partial(\frac{1}{2}\rho V^2 A\,\mathrm{d}x + c_v T\rho A\,\mathrm{d}x)}{\partial t} \tag{2.7}$$

$$= pVA - (V+\mathrm{d}V)(A+\mathrm{d}A)(p+\mathrm{d}p) + \frac{1}{2}\rho AV^3 - \frac{1}{2}(\rho+\mathrm{d}\rho)(A+\mathrm{d}A)(V+\mathrm{d}V)^3 \tag{2.8}$$

$$+ c_v T\rho AV - c_v(T+\mathrm{d}T)(\rho+\mathrm{d}\rho)(A+\mathrm{d}A)(V+\mathrm{d}V), \tag{2.9}$$

代入式 (2.2) 和式 (2.6), 忽略高阶小量得

$$A\rho c_v\frac{\partial T}{\partial t} + c_v A\rho V\frac{\partial T}{\partial x} + \frac{1}{2}V^2\frac{\partial \rho A}{\partial t} + \rho AV\frac{\partial V}{\partial t} \tag{2.10}$$

$$= pVA - (V+\mathrm{d}V)(A+\mathrm{d}A)(p+\mathrm{d}p) + \frac{1}{2}\rho AV^3 - \frac{1}{2}(\rho+\mathrm{d}\rho)(A+\mathrm{d}A)(V+\mathrm{d}V)^3, \tag{2.11}$$

$$A\rho c_v\frac{\partial T}{\partial t} + c_v A\rho V\frac{\partial T}{\partial x} + V^2\rho A\frac{\partial V}{\partial x} + \rho AV\frac{\partial V}{\partial t} = pVA - (V+\mathrm{d}V)(A+\mathrm{d}A)(p+\mathrm{d}p), \tag{2.12}$$

$$A\rho c_v\frac{\partial T}{\partial t} + c_v A\rho V\frac{\partial T}{\partial x} = -p\left(A\frac{\partial V}{\partial x} + V\frac{\partial A}{\partial x}\right), \tag{2.13}$$

$$\rho c_{\mathrm{v}}\frac{\partial T}{\partial t} + \rho V c_{\mathrm{v}}\frac{\partial T}{\partial x} = -\rho RT\left[\frac{\partial V}{\partial x} + V\frac{\partial(\ln A)}{\partial x}\right]. \tag{2.14}$$