计算流体力学作业 4

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1a

完全气体一维守恒形式的 Euler 方程组为

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{pmatrix}=0 \tag{1.1}$$

和 $p=(\gamma-1)\left(E-\frac{1}{2}\rho u^2\right), \gamma=C_p/C_v$ 为常数, $E=\rho e+\frac{1}{2}\rho u^2, p=(\gamma-1)\rho e$. 声速 $a=\sqrt{\gamma p/\rho}$.

假设

$$x=\rho,\quad y=\rho u,\quad z=E. \tag{1.2}$$

Jacobi 矩阵

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}=\boldsymbol{A}(\boldsymbol{U})=\begin{pmatrix}0 & 1 & 0\\ -\frac{3y^2}{2x^2}+\gamma\frac{y^2}{2x^2} & \frac{3y}{x}-\frac{\gamma y}{x} & \gamma-1\\ (\gamma-1)\frac{y^3}{x^3}-\frac{y}{x^2}\gamma z & \frac{\gamma z}{x}-\frac{(\gamma-1)3y^2}{2x^2} & \gamma\frac{y}{x}\end{pmatrix}, \tag{1.3}$$

再代入式 (1.2) 得

$$\boldsymbol{A}(\boldsymbol{U})=\begin{pmatrix}0 & 1 & 0\\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1\\ \frac{\gamma-2}{2}u^3-\frac{a^2u}{\gamma-1} & \frac{3-2\gamma}{2}u^2+\frac{a^2}{\gamma-1} & \gamma u\end{pmatrix}, \tag{1.4}$$

根据

$$\det|A-\lambda\boldsymbol{I}|=0, \tag{1.5}$$

可求出特征值

$$\lambda_1=u-a,\quad \lambda_2=u,\quad \lambda_3=u+a, \tag{1.6}$$

设总焓 $H=(E+p)/\rho$,

$$\boldsymbol{R}=\left(\boldsymbol{R}^{(1)},\boldsymbol{R}^{(2)},\boldsymbol{R}^{(3)}\right)=\begin{pmatrix}1 & 1 & 1\\ u-a & u & u+a\\ H-ua & \frac{1}{2}u^2 & H+ua\end{pmatrix}. \tag{1.7}$$

$$\begin{aligned}
&\nabla_{\boldsymbol{U}}\lambda_1(\boldsymbol{U})\cdot\boldsymbol{R}^{(1)}(\boldsymbol{U}) && (1.8)\\
=&\left(-\frac{u}{\rho}-\frac{\gamma}{2a}\frac{(\gamma-1)u^2\rho/2-p}{\rho^2},\frac{1}{\rho}+\frac{\gamma}{2a}\frac{(\gamma-1)u}{\rho},-\frac{\gamma}{2a}\frac{(\gamma-1)}{\rho}\right)\cdot\begin{pmatrix}1\\ u-a\\ H-ua\end{pmatrix} && (1.9)\\
=&-\frac{\gamma-1}{\rho^2}p-\frac{a}{\rho}<0, && (1.10)
\end{aligned}$$

所以第一个特征场是真正非线性的.

$$\begin{aligned}
&\nabla_{\boldsymbol{U}}\lambda_2(\boldsymbol{U})\cdot\boldsymbol{R}^{(2)}(\boldsymbol{U}) && (1.11)\\
=&\left(-\frac{u}{\rho},\frac{1}{\rho},0\right)\cdot\begin{pmatrix}1\\ u\\ \frac{1}{2}u^2\end{pmatrix} && (1.12)\\
=&0, && (1.13)
\end{aligned}$$

所以第二个特征场是非线性退化的.

$$\begin{aligned}
&\nabla_{\boldsymbol{U}}\lambda_3(\boldsymbol{U})\cdot\boldsymbol{R}^{(3)}(\boldsymbol{U}) && (1.14)\\
=&\left(-\frac{u}{\rho}+\frac{\gamma}{2a}\frac{(\gamma-1)u^2\rho/2-p}{\rho^2},\frac{1}{\rho}-\frac{\gamma}{2a}\frac{(\gamma-1)u}{\rho},\frac{\gamma}{2a}\frac{(\gamma-1)}{\rho}\right)\cdot\begin{pmatrix}1\\ u+a\\ H+ua\end{pmatrix} && (1.15)\\
=&\frac{\gamma-1}{\rho^2}p+\frac{a}{\rho}>0, && (1.16)
\end{aligned}$$

所以第三个特征场是真正非线性的.

1b

一维原始变量形式的 Euler 方程组为

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{1}{\rho}p_x = 0, \\ p_t + \rho a^2 u_x + up_x = 0. \end{cases} \tag{1.17}$$

它属于非守恒形式, 又可以写成矩阵向量形式

$$\boldsymbol{W}_t + \widetilde{\boldsymbol{A}}(\boldsymbol{W})\boldsymbol{W}_x = 0, \quad \boldsymbol{W} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}, \quad \widetilde{\boldsymbol{A}} = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{pmatrix} \tag{1.18}$$

其中 $\boldsymbol{W}$ 和 $\boldsymbol{U}$ 的关系为

$$\boldsymbol{U}_x = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\boldsymbol{W}_x, \quad \boldsymbol{U}_t = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\boldsymbol{W}_t. \tag{1.19}$$

所以式 (1.1) 变为

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\boldsymbol{W}_x + \boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\boldsymbol{W}_t = 0. \tag{1.20}$$

即

$$\boldsymbol{W}_x + \left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\right)^{-1}\boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\boldsymbol{W}_t = 0. \tag{1.21}$$

所以矩阵 $\widetilde{\boldsymbol{A}}(\boldsymbol{W})$ 与 $\boldsymbol{A}(\boldsymbol{U})$ 相似, $\widetilde{\boldsymbol{A}}(\boldsymbol{W}) = \left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\right)^{-1}\boldsymbol{A}(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}$. 由相似性, 两矩阵的特征值相等, 特征向量有一个乘矩阵的变化, 即

$$\boldsymbol{R} = \left(\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{W}}\right)\widetilde{\boldsymbol{R}}, \tag{1.22}$$

矩阵 $\widetilde{A}$ 的特征值和 (左右) 特征向量矩阵分别为

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a, \tag{1.23}$$

$$\widetilde{\boldsymbol{L}} = \begin{pmatrix} 0 & 1 & -\frac{1}{\rho a} \\ 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & \frac{1}{\rho a} \end{pmatrix}, \quad \widetilde{\boldsymbol{R}} = \begin{pmatrix} -\frac{\rho}{2a} & 1 & -\frac{\rho}{2a} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2}\rho a & 0 & \frac{1}{2}\rho a \end{pmatrix}. \tag{1.24}$$

同时, 三个特征场的真正非线性和非线性退化的性质保持不变.

- 第一个特征场是真正非线性的.
- 第二个特征场是非线性退化的.
- 第三个特征场是真正非线性的.

1c

$$\boldsymbol{U}=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix},\quad \boldsymbol{R}=\left(\boldsymbol{R}^{(1)},\boldsymbol{R}^{(2)},\boldsymbol{R}^{(3)}\right)=\begin{pmatrix}1 & 1 & 1\\ u-a & u & u+a\\ H-ua & \frac{1}{2}u^2 & H+ua\end{pmatrix}. \tag{1.25}$$

$$\frac{\mathrm{d}u_1}{r_{1j}(\boldsymbol{U})}=\frac{\mathrm{d}u_2}{r_{2j}(\boldsymbol{U})}=\frac{\mathrm{d}u_3}{r_{3j}(\boldsymbol{U})} \tag{1.26}$$

当 $j=1$ 时,

$$\frac{\mathrm{d}\rho}{1}=\frac{\mathrm{d}\rho u}{u-a}=\frac{\mathrm{d}E}{H-ua} \tag{1.27}$$

化简可得

$$\begin{cases}\rho\,\mathrm{d}u+a\,\mathrm{d}\rho=0,\\ \gamma p\,\mathrm{d}\rho=\rho\,\mathrm{d}p.\end{cases} \tag{1.28}$$

解得

$$\begin{cases}u=-\frac{2\sqrt{\gamma c_1}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}+c_2,\\ p=c_1\rho^{\gamma}.\end{cases} \tag{1.29}$$

当 $j=2$ 时,

$$\frac{\mathrm{d}\rho}{1}=\frac{\mathrm{d}\rho u}{u}=\frac{\mathrm{d}E}{\frac{1}{2}u^2} \tag{1.30}$$

化简可得

$$\begin{cases}\rho\,\mathrm{d}u=0,\\ -(\gamma-1)\rho u\,\mathrm{d}u=\mathrm{d}p.\end{cases} \tag{1.31}$$

解得

$$\begin{cases}u=c_2,\\ p=c_1.\end{cases} \tag{1.32}$$

当 $j=3$ 时,

$$\frac{\mathrm{d}\rho}{1}=\frac{\mathrm{d}\rho u}{u+a}=\frac{\mathrm{d}E}{H+ua} \tag{1.33}$$

化简可得

$$\begin{cases}\rho\,\mathrm{d}u+a\,\mathrm{d}\rho=0,\\ \gamma p\,\mathrm{d}\rho=\rho\,\mathrm{d}p.\end{cases} \tag{1.34}$$

解得

$$\begin{cases}u=\frac{2\sqrt{\gamma c_1}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}+c_2,\\ p=c_1\rho^{\gamma}.\end{cases} \tag{1.35}$$

其中 c_1,c_2 为常数.

$$r_{1j}(\boldsymbol{U})\frac{\partial W}{\partial u_1}+r_{2j}(\boldsymbol{U})\frac{\partial W}{\partial u_2}+r_{3j}(\boldsymbol{U})\frac{\partial W}{\partial u_3}=0, \tag{1.36}$$

其中 $W=W\left(u_1,u_2,u_3\right)\in\mathbb{R}$.

当 $j=1$ 时,

$$\frac{\partial W}{\partial u_1}+(u-a)\frac{\partial W}{\partial u_2}+(H-ua)\frac{\partial W}{\partial u_3}=0, \tag{1.37}$$

当 $j=2$ 时,

$$\frac{\partial W}{\partial u_1}+u\frac{\partial W}{\partial u_2}+\frac{1}{2}u^2\frac{\partial W}{\partial u_3}=0, \tag{1.38}$$

当 $j=3$ 时,

$$\frac{\partial W}{\partial u_1}+(u+a)\frac{\partial W}{\partial u_2}+(H+ua)\frac{\partial W}{\partial u_3}=0, \tag{1.39}$$

如果连续可微函数 $W(u_1,u_2,\cdots,u_m)$ 不恒等于常数, 且在 U 空间中沿着式 (1.26) 的任一积分曲线 (即特征线), W 恒为常数, 则称 W 为式 (1.26) 的一个第一积分. 方程组式 (1.26) 的任一个第一积分是式 (1.36) 的解.

方程式 (1.26) 解得出的不变量及其线性组合即为式 (1.36) 的解.

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考虑二维 Euler 方程组

$$\frac{\partial}{\partial t}\boldsymbol{U}+\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U})+\frac{\partial}{\partial y}\boldsymbol{G}(\boldsymbol{U})=0 \tag{2.1}$$

其中

$$\boldsymbol{U}=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ E\end{pmatrix},\quad \boldsymbol{F}(\boldsymbol{U})=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{pmatrix},\quad \boldsymbol{G}(\boldsymbol{U})=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{pmatrix},\tag{2.2}$$

$p=(\gamma-1)\rho e,\ E=\rho e+\frac{1}{2}\rho\left(u^2+v^2\right)$. 声速 $a=\sqrt{\gamma p/\rho}$.

2a

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}=\boldsymbol{A}(\boldsymbol{U})=\begin{pmatrix}0 & 1 & 0 & 0\\ -\frac{3y^2}{2x^2}+\gamma\frac{y^2}{2x^2} & \frac{3y}{x}-\frac{\gamma y}{x} & -\frac{(\gamma-1)m}{x} & \gamma-1\\ -\frac{ym}{x^2} & \frac{m}{x} & \frac{y}{x} & 0\\ (\gamma-1)\frac{(y^3+ym^2)}{x^3}-\frac{y}{x^2}\gamma z & \frac{\gamma z}{x}-\frac{(\gamma-1)(3y^2+m^2)}{2x^2} & -\frac{y(\gamma-1)m}{x^2} & \gamma\frac{y}{x}\end{pmatrix},\tag{2.3}$$

假设

$$x=\rho,\quad y=\rho u,\quad z=E,\quad m=v.\tag{2.4}$$

再代入式 (2.4) 得

$$\boldsymbol{A}(\boldsymbol{U})=\begin{pmatrix}0 & 1 & 0 & 0\\ \frac{\gamma-3}{2}u^2+\frac{\gamma-1}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1\\ -uv & v & u & 0\\ \frac{\gamma-2}{2}u(u^2+v^2)-\frac{a^2u}{\gamma-1} & \frac{1}{2}(u^2+v^2)-(\gamma-1)u^2+\frac{a^2}{\gamma-1} & -(\gamma-1)uv & \gamma u\end{pmatrix}\tag{2.5}$$

求得特征值为

$$\lambda_1=u-a,\quad \lambda_2=u,\quad \lambda_3=u,\quad \lambda_4=u+a.\tag{2.6}$$

特征向量矩阵为

$$\begin{pmatrix}\frac{2\,(\gamma-1)}{\sigma_2} & -\frac{2\,v}{\sigma_3} & \frac{2}{\sigma_3} & \frac{2\,(\gamma-1)}{\sigma_1}\\ -\frac{2\,(a-u)\,(\gamma-1)}{\sigma_2} & -\frac{2\,u\,v}{\sigma_3} & \frac{2\,u}{\sigma_3} & \frac{2\,(a+u)\,(\gamma-1)}{\sigma_1}\\ \frac{\sigma_4}{\sigma_2} & 1 & 0 & \frac{\sigma_4}{\sigma_1}\\ 1 & 0 & 1 & 1\end{pmatrix},\tag{2.7}$$

其中

$$\sigma_1 = \gamma u^2 - 2au + \gamma v^2 + 2a^2 - u^2 - v^2 + 2a\gamma u, \tag{2.8}$$

$$\sigma_2 = 2au + \gamma u^2 + \gamma v^2 + 2a^2 - u^2 - v^2 - 2a\gamma u, \tag{2.9}$$

$$\sigma_3 = u^2 - v^2, \tag{2.10}$$

$$\sigma_4 = 2v(\gamma - 1). \tag{2.11}$$

经验证可得 λ_2, λ_3 对应的特征场是非线性退化的. λ_1, λ_4 对应的特征场是真正非线性的.

2b

可以. 由对称性, 交换 u, v 符号, 并交换 $\boldsymbol{U}$, $\boldsymbol{F}$ 的二三元素即可, 所以交换 $\boldsymbol{A}(\boldsymbol{U})$ 的二三列和二三行（交换顺序并不影响结果）并交换 u, v 即得 $\boldsymbol{B}(\boldsymbol{U})$, 交换二三列和二三行相当于做了一个相似变换, 特征值不变, 而交换 u, v 后, 其特征值为

$$\lambda_1 = v - a, \quad \lambda_2 = v, \quad \lambda_3 = v, \quad \lambda_4 = v + a. \tag{2.12}$$

交换特征向量矩阵的式 (2.7) 的二三行并交换 u, v 即得 $\boldsymbol{B}(\boldsymbol{U})$ 的特征向量矩阵.

同样的, λ_2, λ_3 对应的特征场是非线性退化的. λ_1, λ_4 对应的特征场是真正非线性的.