

# 计算流体力学作业 4

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1

1a

完全气体一维守恒形式的 Euler 方程组为

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{pmatrix} = 0 \quad (1.1)$$

和  $p = (\gamma - 1) \left( E - \frac{1}{2} \rho u^2 \right)$ ,  $\gamma = C_p / C_v$  为常数,  $E = \rho e + \frac{1}{2} \rho u^2$ ,  $p = (\gamma - 1) \rho e$ . 声速  $a = \sqrt{\gamma p / \rho}$ .

假设

$$x = \rho, \quad y = \rho u, \quad z = E. \quad (1.2)$$

Jacobi 矩阵

$$\frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3y^2}{2x^2} + \gamma \frac{y^2}{2x^2} & \frac{3y}{x} - \frac{\gamma y}{x} & \gamma - 1 \\ (\gamma - 1) \frac{y^3}{x^3} - \frac{y}{x^2} \gamma z & \frac{\gamma z}{x} - \frac{(\gamma - 1) 3y^2}{2x^2} & \gamma \frac{y}{x} \end{pmatrix}, \quad (1.3)$$

再代入式 (1.2) 得

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma - 3}{2} u^2 & (3 - \gamma) u & \gamma - 1 \\ \frac{\gamma - 2}{2} u^3 - \frac{a^2 u}{\gamma - 1} & \frac{3 - 2\gamma}{2} u^2 + \frac{a^2}{\gamma - 1} & \gamma u \end{pmatrix}, \quad (1.4)$$

根据

$$\det|A - \lambda \mathbf{I}| = 0, \quad (1.5)$$

可求出特征值

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a, \quad (1.6)$$

设总焓  $H = (E + p)/\rho$ ,

$$\mathbf{R} = (\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \mathbf{R}^{(3)}) = \begin{pmatrix} 1 & 1 & 1 \\ u - a & u & u + a \\ H - ua & \frac{1}{2}u^2 & H + ua \end{pmatrix}. \quad (1.7)$$

$$\nabla_U \lambda_1(\mathbf{U}) \cdot \mathbf{R}^{(1)}(\mathbf{U}) \quad (1.8)$$

$$= \left( -\frac{u}{\rho} - \frac{\gamma}{2a} \frac{(\gamma-1)u^2\rho/2 - p}{\rho^2}, \frac{1}{\rho} + \frac{\gamma}{2a} \frac{(\gamma-1)u}{\rho}, -\frac{\gamma}{2a} \frac{(\gamma-1)}{\rho} \right) \cdot \begin{pmatrix} 1 \\ u - a \\ H - ua \end{pmatrix} \quad (1.9)$$

$$= -\frac{\gamma-1}{\rho^2}p - \frac{a}{\rho} < 0, \quad (1.10)$$

所以第一个特征场是真正非线性的.

$$\nabla_U \lambda_2(\mathbf{U}) \cdot \mathbf{R}^{(2)}(\mathbf{U}) \quad (1.11)$$

$$= \left( -\frac{u}{\rho}, \frac{1}{\rho}, 0 \right) \cdot \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix} \quad (1.12)$$

$$= 0, \quad (1.13)$$

所以第二个特征场是非线性退化的.

$$\nabla_U \lambda_3(\mathbf{U}) \cdot \mathbf{R}^{(3)}(\mathbf{U}) \quad (1.14)$$

$$= \left( -\frac{u}{\rho} + \frac{\gamma}{2a} \frac{(\gamma-1)u^2\rho/2 - p}{\rho^2}, \frac{1}{\rho} - \frac{\gamma}{2a} \frac{(\gamma-1)u}{\rho}, \frac{\gamma}{2a} \frac{(\gamma-1)}{\rho} \right) \cdot \begin{pmatrix} 1 \\ u + a \\ H + ua \end{pmatrix} \quad (1.15)$$

$$= \frac{\gamma-1}{\rho^2}p + \frac{a}{\rho} > 0, \quad (1.16)$$

所以第三个特征场是真正非线性的.

## 1b

一维原始变量形式的 Euler 方程组为

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \frac{1}{\rho}p_x = 0, \\ p_t + \rho a^2 u_x + up_x = 0. \end{cases} \quad (1.17)$$

它属于非守恒形式, 又可以写成矩阵向量形式

$$\mathbf{W}_t + \tilde{\mathbf{A}}(\mathbf{W})\mathbf{W}_x = 0, \quad \mathbf{W} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}, \quad \tilde{\mathbf{A}} = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{pmatrix} \quad (1.18)$$

其中  $\mathbf{W}$  和  $U$  的关系为

$$U_x = \frac{\partial U}{\partial \mathbf{W}} \mathbf{W}_x, \quad U_t = \frac{\partial U}{\partial \mathbf{W}} \mathbf{W}_t. \quad (1.19)$$

所以式 (1.1) 变为

$$\frac{\partial U}{\partial \mathbf{W}} \mathbf{W}_x + \mathbf{A}(U) \frac{\partial U}{\partial \mathbf{W}} \mathbf{W}_t = 0. \quad (1.20)$$

即

$$\mathbf{W}_x + \left( \frac{\partial U}{\partial \mathbf{W}} \right)^{-1} \mathbf{A}(U) \frac{\partial U}{\partial \mathbf{W}} \mathbf{W}_t = 0. \quad (1.21)$$

所以矩阵  $\tilde{\mathbf{A}}(\mathbf{W})$  与  $\mathbf{A}(U)$  相似,  $\tilde{\mathbf{A}}(\mathbf{W}) = \left( \frac{\partial U}{\partial \mathbf{W}} \right)^{-1} \mathbf{A}(U) \frac{\partial U}{\partial \mathbf{W}}$ . 由相似性, 两矩阵的特征值相等, 特征向量有一个乘矩阵的变化, 即

$$\mathbf{R} = \left( \frac{\partial U}{\partial \mathbf{W}} \right) \tilde{\mathbf{R}}, \quad (1.22)$$

矩阵  $\tilde{\mathbf{A}}$  的特征值和 (左右) 特征向量矩阵分别为

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a, \quad (1.23)$$

$$\tilde{\mathbf{L}} = \begin{pmatrix} 0 & 1 & -\frac{1}{\rho a} \\ 1 & 0 & -\frac{1}{a^2} \\ 0 & 1 & \frac{1}{\rho a} \end{pmatrix}, \quad \tilde{\mathbf{R}} = \begin{pmatrix} -\frac{\rho}{2a} & 1 & -\frac{\rho}{2a} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2}\rho a & 0 & \frac{1}{2}\rho a \end{pmatrix}. \quad (1.24)$$

同时, 三个特征场的真正非线性和非线性退化的性质保持不变.

- 第一个特征场是真正非线性的.
- 第二个特征场是非线性退化的.
- 第三个特征场是真正非线性的.

**1c**

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad \mathbf{R} = (\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \mathbf{R}^{(3)}) = \begin{pmatrix} 1 & 1 & 1 \\ u - a & u & u + a \\ H - ua & \frac{1}{2}u^2 & H + ua \end{pmatrix}. \quad (1.25)$$

$$\frac{du_1}{r_{1j}(\mathbf{U})} = \frac{du_2}{r_{2j}(\mathbf{U})} = \frac{du_3}{r_{3j}(\mathbf{U})} \quad (1.26)$$

当  $j = 1$  时,

$$\frac{d\rho}{1} = \frac{d\rho u}{u - a} = \frac{dE}{H - ua} \quad (1.27)$$

化简可得

$$\begin{cases} \rho du + a d\rho = 0, \\ \gamma p d\rho = \rho dp. \end{cases} \quad (1.28)$$

解得

$$\begin{cases} u = -\frac{2\sqrt{\gamma c_1}}{\gamma-1} \rho^{\frac{\gamma-1}{2}} + c_2, \\ p = c_1 \rho^\gamma. \end{cases} \quad (1.29)$$

当  $j = 2$  时,

$$\frac{d\rho}{1} = \frac{d\rho u}{u} = \frac{dE}{\frac{1}{2}u^2} \quad (1.30)$$

化简可得

$$\begin{cases} \rho du = 0, \\ \gamma p d\rho = \rho dp. \end{cases} \quad (1.31)$$

解得

$$\begin{cases} u = c_2, \\ p = c_1 \rho^\gamma. \end{cases} \quad (1.32)$$

当  $j = 3$  时,

$$\frac{d\rho}{1} = \frac{d\rho u}{u+a} = \frac{dE}{H+ua} \quad (1.33)$$

化简可得

$$\begin{cases} \rho du + a d\rho = 0, \\ \gamma p d\rho = \rho dp. \end{cases} \quad (1.34)$$

解得

$$\begin{cases} u = \frac{2\sqrt{\gamma c_1}}{\gamma-1} \rho^{\frac{\gamma-1}{2}} + c_2, \\ p = c_1 \rho^\gamma. \end{cases} \quad (1.35)$$

其中  $c_1, c_2$  为常数.

$$r_{1j}(\mathbf{U}) \frac{\partial W}{\partial u_1} + r_{2j}(\mathbf{U}) \frac{\partial W}{\partial u_2} + r_{3j}(\mathbf{U}) \frac{\partial W}{\partial u_3} = 0, \quad (1.36)$$

其中  $W = W(u_1, u_2, u_3) \in \mathbb{R}$ .

当  $j = 1$  时,

$$\frac{\partial W}{\partial u_1} + (u-a) \frac{\partial W}{\partial u_2} + (H-ua) \frac{\partial W}{\partial u_3} = 0, \quad (1.37)$$

当  $j = 2$  时,

$$\frac{\partial W}{\partial u_1} + u \frac{\partial W}{\partial u_2} + \frac{1}{2} u^2 \frac{\partial W}{\partial u_3} = 0, \quad (1.38)$$

当  $j = 3$  时,

$$\frac{\partial W}{\partial u_1} + (u+a) \frac{\partial W}{\partial u_2} + (H+ua) \frac{\partial W}{\partial u_3} = 0, \quad (1.39)$$

当方程的变量满足式 (1.26) 时, 假设式 (1.26) 是一函数的微分, 即  $dt$ , 那么  $W$  是关于  $t$  的不变量

$$\frac{DW}{Dt} = \sum_{i=1}^3 \frac{\partial u_i}{\partial t} \frac{\partial W}{\partial u_i} = \sum_{i=1}^3 r_{ij} \frac{\partial W}{\partial u_i} = 0. \quad (1.40)$$

## 2

考虑二维 Euler 方程组

$$\frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial x} \mathbf{F}(\mathbf{U}) + \frac{\partial}{\partial y} \mathbf{G}(\mathbf{U}) = 0 \quad (2.1)$$

其中

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{pmatrix}, \quad \mathbf{G}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{pmatrix}, \quad (2.2)$$

$$p = (\gamma - 1)\rho e, \quad E = \rho e + \frac{1}{2}\rho(u^2 + v^2). \quad \text{声速 } a = \sqrt{\gamma p / \rho}.$$

**2a**

$$\frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{3y^2}{2x^2} + \gamma \frac{y^2}{2x^2} & \frac{3y}{x} - \frac{\gamma y}{x} & -\frac{(\gamma-1)m}{x} & \gamma - 1 \\ -\frac{ym}{x^2} & \frac{m}{x} & \frac{y}{x} & 0 \\ (\gamma-1)\frac{(y^3+ym^2)}{x^3} - \frac{y}{x^2}\gamma z & \frac{\gamma z}{x} - \frac{(\gamma-1)(3y^2+m^2)}{2x^2} & -\frac{y(\gamma-1)m}{x^2} & \gamma \frac{y}{x} \end{pmatrix}, \quad (2.3)$$

假设

$$x = \rho, \quad y = \rho u, \quad z = E, \quad m = v. \quad (2.4)$$

再代入式 (2.4) 得

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{\gamma-1}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ \frac{\gamma-2}{2}u(u^2+v^2) - \frac{a^2u}{\gamma-1} & \frac{1}{2}(u^2+v^2) - (\gamma-1)u^2 + \frac{a^2}{\gamma-1} & -(\gamma-1)uv & \gamma u \end{pmatrix} \quad (2.5)$$

求得特征值为

$$\lambda_1 = u, \quad \lambda_2 = u, \quad \lambda_3 = u + a, \quad \lambda_4 = u - a. \quad (2.6)$$

特征向量矩阵为

$$\begin{pmatrix} -\frac{2v}{\sigma_3} & \frac{2}{\sigma_3} & \frac{2(\gamma-1)}{\sigma_1} & \frac{2(\gamma-1)}{\sigma_2} \\ -\frac{2uv}{\sigma_3} & \frac{2u}{\sigma_3} & \frac{2(a+u)(\gamma-1)}{\sigma_1} & -\frac{2(a-u)(\gamma-1)}{\sigma_2} \\ 1 & 0 & \frac{\sigma_4}{\sigma_1} & \frac{\sigma_4}{\sigma_2} \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad (2.7)$$

其中

$$\sigma_1 = \gamma u^2 - 2au + \gamma v^2 + 2a^2 - u^2 - v^2 + 2a\gamma u, \quad (2.8)$$

$$\sigma_2 = 2au + \gamma u^2 + \gamma v^2 + 2a^2 - u^2 - v^2 - 2a\gamma u, \quad (2.9)$$

$$\sigma_3 = u^2 - v^2, \quad (2.10)$$

$$\sigma_4 = 2v(\gamma - 1). \quad (2.11)$$

经验证可得  $\lambda_1, \lambda_2$  对应的特征场是非线性退化的.  $\lambda_3, \lambda_4$  对应的特征场是真正非线性的.

## 2b

可以. 由对称性, 交换  $u, v$  符号即可, 可得特征值为

$$\lambda_1 = u, \quad \lambda_2 = v, \quad \lambda_3 = u + a, \quad \lambda_4 = v - a. \quad (2.12)$$

同样的,  $\lambda_1, \lambda_2$  对应的特征场是非线性退化的.  $\lambda_3, \lambda_4$  对应的特征场是真正非线性的.