计算流体力学作业 8

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考虑 1D 双曲守恒律方程

$$u_t + f(u)_x = 0,\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$u(x,0) = u_0(x)$, 和均匀网格 $\left\{x_j : x_j = jh,\ j \in \mathbb{Z}\right\}$. 试写出上述方程的守恒形式的 Engquist-Osher 格式. 它是否是单调格式? 如果是, 是否可以证明它满足极值原理? 如果应用这些格式于具体问题的数值计算, 则时间步长如何选取?

迎风格式的特征是: 根据信息传播的方向离散微分方程. 考虑标量方程

$$u_t + f_x = 0, \tag{1.2}$$

如果 $f'(u) > 0,\ \forall u \in \mathbb{R}$, 则

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_j + \frac{1}{h}\left(f_j - f_{j-1}\right) = 0, \tag{1.3}$$

或如果 $f'(u) < 0,\ \forall u \in \mathbb{R}$, 则

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_j + \frac{1}{h}\left(f_{j+1} - f_j\right) = 0, \tag{1.4}$$

一般地, 式 (1.2) 可改写为

$$u_t + f_x^+ + f_x^- = 0, \tag{1.5}$$

其中 $f^+(u)+f^-(u)=f(u)$, $\frac{\mathrm{d}f^+}{\mathrm{d}u}\geq 0$, $\frac{\mathrm{d}f^-}{\mathrm{d}u}\leq 0$, 因而可离散为

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_j+\frac{1}{h}\left(f_j^+-f_{j-1}^+\right)+\frac{1}{h}\left(f_{j+1}^--f_j^-\right)=0, \tag{1.6}$$

进一步离散为

$$\begin{aligned} u_j^{n+1} &= u_j^n-\frac{\Delta t}{\Delta x}\left(\Delta_+f^-\left(u_j^n\right)+\Delta_-f^+\left(u_j^n\right)\right) &(1.7)\\ &= H(u_{j-1}^n,u_j^n,u_{j+1}^n), &(1.8)\end{aligned}$$

其中

$$f^+(u)=\int_0^u \max\left(0,f'(\xi)\right)\mathrm{d}\xi+f(0),\quad f^-(u)=\int_0^u \min\left(0,f'(\xi)\right)\mathrm{d}\xi+f(0). \tag{1.9}$$

考虑格式的单调性,

$$\frac{\partial u_j^{n+1}}{\partial u_j^n}=1+r\min\left(0,f'(u_j^n)\right)-r\max\left(0,f'(u_j^n)\right)=1-r\left|f'(u_j^n)\right|. \tag{1.10}$$

其中

$$r=\frac{\Delta t}{\Delta x}. \tag{1.11}$$

易得另外两个偏导

$$\frac{\partial u_j^{n+1}}{\partial u_{j+1}^n}=-r\min\left(0,f'(u_{j+1}^n)\right)\geq 0,\quad \frac{\partial u_j^{n+1}}{\partial u_{j-1}^n}=r\max\left(0,f'(u_{j-1}^n)\right)\geq 0. \tag{1.12}$$

当满足 CFL

$$1\geq r\left|f'(u)\right| \tag{1.13}$$

条件时, 格式是单调的.

对于局部极值原理, 根据式 (1.8) 有

$$H(u,u,u)=u, \tag{1.14}$$

又根据之前得到的格式的单调性有

$$\begin{aligned} H(u_{j-1},u_j,u_{j+1})\leq H(u,u,u)=u,\quad u=\max\left\{u_{j-1},u_j,u_{j+1}\right\}, &\quad(1.15)\\ H(u_{j-1},u_j,u_{j+1})\geq H(u,u,u)=u,\quad u=\min\left\{u_{j-1},u_j,u_{j+1}\right\}. &\quad(1.16)\end{aligned}$$

所以格式在式 (1.13) 条件下是满足局部极值原理的.

时间步长需要满足式 (1.13). 对每一步来说

$$\tau\leq\frac{h}{\max_j\left\{\left|f'(u_j^n)\right|\right\}}. \tag{1.17}$$

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上述格式可以看成是对双曲守恒律方程先空间离散为

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{1}{h}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right) =: L\left(u_j(t)\right), \tag{1.18}$$

[属于线方法或半离散方法], 然后再对上式中的时间导数采用显式 Euler 方法离散. 如果把时间离散换成下列 Runge-Kutta 方法

$$u^{(1)} = u^n + \Delta t L\left(u^n\right) \tag{1.19}$$

$$u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L\left(u^{(1)}\right) \tag{1.20}$$

或

$$u^{(1)} = u^n + \Delta t_n L\left(u^n\right) \tag{1.21}$$

$$u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}\left(u^{(1)} + \Delta t_n L\left(u^{(1)}\right)\right) \tag{1.22}$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}\left(u^{(2)} + \Delta t_n L\left(u^{(2)}\right)\right) \tag{1.23}$$

前述结果又如何?

定义

$$H\left(u^n, j\right) = H\left(u_{j-1}, u_j, u_{j+1}\right). \tag{1.24}$$

$$u_j^{(1)} = u_j^n + \Delta t_n L\left(u_j^n\right) = H(u^n, j), \tag{1.25}$$

$$\begin{aligned} u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L\left(u^{(1)}\right) \\ &= \frac{1}{2}u^n + \frac{1}{2}H\left(u_j^{(1)}\right) \\ &= \frac{1}{2}u^n + \frac{1}{2}H\left(H\left(u_{j-2}^n, u_{j-1}^n, u_j^n\right), H\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right), H\left(u_j^n, u_{j+1}^n, u_{j+2}^n\right)\right), \end{aligned} \tag{1.26}$$

定义

$$H\left(H\left(u_{j-2}^n, u_{j-1}^n, u_j^n\right), H\left(u_{j-1}^n, u_j^n, u_{j+1}^n\right), H\left(u_j^n, u_{j+1}^n, u_{j+2}^n\right)\right) =: H^{(1)}\left(u^n, j\right). \tag{1.27}$$

在式 (1.13) 条件下 H 是单调的, 所以复合函数也是单调的, 故格式是单调的.

根据式 (1.14) 易知

$$H\left(H(u,u,u),H(u,u,u),H(u,u,u)\right)=H(u,u,u)=u. \tag{1.28}$$

同理可得格式是满足局部极值原理的.

而对于

$$u^{(1)}=u^n+\Delta t_n L\left(u^n\right) \tag{1.29}$$

$$u^{(2)}=\frac{3}{4}u^n+\frac{1}{4}\left(u^{(1)}+\Delta t_n L\left(u^{(1)}\right)\right) \tag{1.30}$$

$$u^{n+1}=\frac{1}{3}u^n+\frac{2}{3}\left(u^{(2)}+\Delta t_n L\left(u^{(2)}\right)\right), \tag{1.31}$$

$$u^{(1)}=u^n+\Delta t_n L\left(u^n\right)=H\left(u^n,j\right). \tag{1.32}$$

$$u^{(2)}=\frac{3}{4}u^n+\frac{1}{4}\left(u^{(1)}+\Delta t_n L\left(u^{(1)}\right)\right)=\frac{3}{4}u^n+\frac{1}{4}H\left(u^{(1)},j\right). \tag{1.33}$$

$$u^{n+1}=\frac{1}{3}u^n+\frac{2}{3}\left(u^{(2)}+\Delta t_n L\left(u^{(2)}\right)\right)=\frac{1}{3}u^n+\frac{2}{3}H\left(u^{(2)},j\right). \tag{1.34}$$

由此可见, 在式 (1.13) 的条件下该格式同样有单调性, 且有式 (1.14) 的性质, 所以满足局部极值原理.

B

考虑 2D 双曲守恒律方程

$$u_t+f(u)_x+g(u)_y=0, x,y\in\mathbb{R}, t>0, \tag{2.1}$$

$f=g=\frac{1}{2}u^2,\ u(x,y,0)=u_0(x,y)$, 和均匀的矩形网格 $\{(x_j,y_k):x_j=jh_x,\ y_k=kh_y,\ j,\ k\in\mathbb{Z}\}$. 请详细写出上述方程的如下形式的 Godunov 格式

$$\begin{aligned}\bar{u}_{j,k}^{n+1}=\bar{u}_{j,k}^n&-\frac{\tau}{h_x}\left[f\left(\omega\left(0;\bar{u}_{j,k}^n,\bar{u}_{j+1,k}^n\right)\right)-f\left(\omega\left(0;\bar{u}_{j-1,k}^n,\bar{u}_{j,k}^n\right)\right)\right]\\&-\frac{\tau}{h_y}\left[g\left(\omega\left(0;\bar{u}_{j,k}^n,\bar{u}_{j,k+1}^n\right)\right)-g\left(\omega\left(0;\bar{u}_{j,k-1}^n,\bar{u}_{j,k}^n\right)\right)\right],\end{aligned} \tag{2.2}$$

其中 $\omega\left(\frac{x-x_{j+\frac{1}{2}}}{t-t_n};\bar{u}_{j,k}^n,\bar{u}_{j+1,k}^n\right)$ 是

$$u_t+f(u)_x=0,\ u\left(x,t_n\right)=\begin{cases}\bar{u}_{j,k}^n, & x-x_{j+\frac{1}{2}}<0,\\ \bar{u}_{j+1,k}^n, & x-x_{j+\frac{1}{2}}>0,\end{cases}\tag{2.3}$$

的精确解; $\omega\left(\frac{y-y_{k+\frac{1}{2}}}{t-t_n};\bar{u}_{j,k}^n,\bar{u}_{j,k+1}^n\right)$ 是

$$u_t+g(u)_y=0,\ u\left(y,t_n\right)=\begin{cases}\bar{u}_{j,k}^n, & y-y_{k+\frac{1}{2}}<0,\\ \bar{u}_{j,k+1}^n, & y-y_{k+\frac{1}{2}}>0,\end{cases}\tag{2.4}$$

的精确解.也就是给出局部 1D Riemann 问题精确解, 再完整地给出 $f(\omega(0;\bar{u}_{j,k}^n,\bar{u}_{j+1,k}^n))$ 和 $g(\omega(0;\bar{u}_{j,k}^n,\bar{u}_{j,k+1}^n))$ 的计算. [提示: 参照课堂上写的 1D Burgers 方程的 Godunov 格式]

对于 1D Burgers 方程的黎曼问题

$$u_t+uu_x=0,\ u\left(x,0\right)=\begin{cases}u_l, & x<0,\\ u_r, & x>0,\end{cases}\tag{2.5}$$

的精确解为

1. $u_l>u_r$, 为激波解

$$u\left(x,0\right)=\begin{cases}u_l, & x<ct,\\ u_r, & x>ct,\end{cases}\tag{2.6}$$

   其中 $c=\frac{u_l+u_r}{2}$.

2. $u_l<u_r$, 为稀疏波解

$$u\left(x,0\right)=\begin{cases}u_l, & x<u_lt,\\ x/t, & u_lt<x<u_rt,\\ u_r, & x>u_rt.\end{cases}\tag{2.7}$$

下面分类讨论来求 $x=0$ 的精确解, $\omega\left(0;u_l,u_r\right)=:\varphi(u_l,u_r)$.

1. $u_l>u_r$,

   (a) $c>0,\ \omega\left(0;u_l,u_r\right)=u_l$.

(b) $c<0,\ \omega\left(0; u_l, u_r\right)=u_r$.

2. $u_l<u_r$,

   (a) $u_l>0,\ \omega\left(0; u_l, u_r\right)=u_l$.

   (b) $u_r<0,\ \omega\left(0; u_l, u_r\right)=u_r$.

   (c) $u_l<0<u_r,\ \omega\left(0; u_l, u_r\right)=0$.

$$
\begin{aligned}
& f(\omega(0; \bar{u}_{j,k}^n, \bar{u}_{j+1,k}^n))=f(\varphi(\bar{u}_{j,k}^n, \bar{u}_{j+1,k}^n))=\frac{1}{2} \varphi^2(\bar{u}_{j,k}^n, \bar{u}_{j+1,k}^n), && (2.8)\\
& g(\omega(0; \bar{u}_{j,k}^n, \bar{u}_{j,k+1}^n))=g(\varphi(\bar{u}_{j,k}^n, \bar{u}_{j,k+1}^n))=\frac{1}{2} \varphi^2(\bar{u}_{j,k}^n, \bar{u}_{j,k+1}^n). && (2.9)
\end{aligned}
$$

$$
\begin{aligned}
\bar{u}_{j,k}^{n+1}=\bar{u}_{j,k}^n & -\frac{\tau}{h_x}\left(\frac{1}{2} \varphi^2\left(\bar{u}_{j,k}^n, \bar{u}_{j+1,k}^n\right)-\frac{1}{2} \varphi^2\left(\bar{u}_{j-1,k}^n, \bar{u}_{j,k}^n\right)\right) \\
& -\frac{\tau}{h_y}\left(\frac{1}{2} \varphi^2\left(\bar{u}_{j,k}^n, \bar{u}_{j,k+1}^n\right)-\frac{1}{2} \varphi^2\left(\bar{u}_{j,k-1}^n, \bar{u}_{j,k}^n\right)\right).
\end{aligned} \tag{2.10}
$$

C

证明 1D 完全气体 Euler 方程组的 1 激波的关系式, 即讲义 [CFDLect06-com03_cn.pdf] 的第 27 页的 1 激波的关系式.

证明. 以 s_1 速度行进的 1 激波（假设朝左）关系式, 令

$$
\hat{u}_L=u_L-s_1, \quad \hat{u}_*=u_*-s_1, \quad M_L=u_L/a_L, \quad M_S=s_1/a_1. \tag{3.1}
$$

在这个新框架下, 应用 RH 条件, 有

$$
\begin{aligned}
\rho_* \hat{u}_* & =\rho_L \hat{u}_L, \\
\rho_* \hat{u}_*^2+p_* & =\rho_L \hat{u}_L^2+p_L, \\
\hat{u}_*\left(\hat{E}_*+p_*\right) & =\hat{u}_L\left(\hat{E}_L+p_L\right),
\end{aligned} \tag{3.2}
$$

其中 $\hat{E}_*=\rho_* e_*+\frac{1}{2} \rho_* \hat{u}_*^2$. 式 (3.2) 中第三式的左端和右端分别可改写为

$$
\hat{u}_* \rho_*\left[\frac{1}{2} \hat{u}_*^2+\left(e_*+\frac{p_*}{\rho_*}\right)\right], \quad \hat{u}_L \rho_L\left[\frac{1}{2} \hat{u}_L^2+\left(e_L+\frac{p_L}{\rho_L}\right)\right]. \tag{3.3}
$$

应用式 (3.2) 中的第一式和比含 $h=\frac{p}{\rho}+e$, 则有

$$\frac{1}{2}\hat{u}_*^2+h_*=\frac{1}{2}\hat{u}_L^2+h_L, \tag{3.4}$$

又由式 (3.2) 中的第一式和第二式, 得

$$\begin{aligned}&\rho_*\hat{u}_*^2=\left(\rho_L\hat{u}_L\right)\hat{u}_L+p_L-p_* \overset{(3.2)}{\Longrightarrow} \rho_*\hat{u}_*\frac{\rho_*\hat{u}_*}{\rho_L}+p_L-p_*,\\ \Rightarrow\ &\rho_*^2\hat{u}_*^2\left(\frac{\rho_L-\rho_*}{\rho_*\rho_L}\right)=p_L-p_*, \Rightarrow \hat{u}_*^2=\left(\frac{\rho_L}{\rho_*}\right)\left(\frac{p_L-p_*}{\rho_L-\rho_*}\right).\end{aligned} \tag{3.5}$$

类似地, 有

$$\hat{u}_L^2=\left(\frac{\rho_*}{\rho_L}\right)\left[\frac{p_L-p_*}{\rho_L-\rho_*}\right], \tag{3.6}$$

将式 (3.5) 和 (3.6) 代入式 (3.4), 得

$$h_*-h_L=\frac{1}{2}\left(p_*-p_L\right)\left[\frac{\rho_*+\rho_L}{\rho_*\rho_L}\right], \tag{3.7}$$

或

$$e_*-e_L=\frac{1}{2}\left(p_*+p_L\right)\left[\frac{\rho_*-\rho_L}{\rho_*\rho_L}\right], \quad h=e+\frac{p}{\rho}. \tag{3.8}$$

注意: 直到此, 没有用到状态方程 $e=e(p,\rho)$ 的具体形式. 下面将仅考虑完全气体 $e=\frac{p}{\rho(\gamma-1)}$. 应用其及 式 (3.8), 可得

$$\frac{\rho_*}{\rho_L}=\frac{\left(\frac{p_*}{p_L}\right)+\left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{p_*}{p_L}\right)+1}, \tag{3.9}$$

这建立了穿过激波的密度比 $\left(\frac{\rho_*}{\rho_L}\right)$ 和压力比 $\left(\frac{p_*}{p_L}\right)$ 之间的一个有用关系式.

引入 Mach 数

$$M_L=\frac{u_L}{a_L}, \quad \text{激波前流动的马赫数, 在老的框架里,} \tag{3.10}$$

$$M_S=\frac{s_3}{a_L}, \quad \text{激波马赫数.} \tag{3.11}$$

由式 (3.6), (3.8) 和 (3.9), 可以给出穿过激波的密度比和压力比关于相对马赫数 M_L 和激波马赫数 M_S 的差的函数表示式 (激波关系):

$$\frac{\rho_*}{\rho_L}=\frac{(\gamma+1)\left(M_L-M_S\right)^2}{(\gamma-1)\left(M_L-M_S\right)^2+2}, \tag{3.12}$$

$$\frac{p_*}{p_L}=\frac{2\gamma\left(M_L-M_S\right)^2-(\gamma-1)}{(\gamma+1)}. \tag{3.13}$$

又由式 (3.13), 得下列关系式 (根号前的符号由 Lax 激波不等式决定)

$$M_L - M_S = \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}, \tag{3.14}$$

或

$$s_3 = u_L - a_L\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}. \tag{3.15}$$

由式 (3.2) 中的第一式, 有

$$u_* = \left(1 - \frac{\rho_L}{\rho_*}\right)s_3 + \left(\frac{\rho_L}{\rho_*}\right)u_L. \tag{3.16}$$

所以有

$$\frac{\rho_*}{\rho_L} = \frac{\left(\frac{p_*}{p_L}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{p_*}{p_L}\right) + 1}, \tag{3.17}$$

$$\frac{\rho_*}{\rho_L} = \frac{(\gamma+1)(M_L - M_S)^2}{(\gamma-1)(M_L - M_S)^2 + 2}, \tag{3.18}$$

$$\frac{p_*}{p_L} = \frac{2\gamma(M_L - M_S)^2 - (\gamma-1)}{(\gamma+1)}, \tag{3.19}$$

$$u_* = \left(1 - \frac{\rho_L}{\rho_*}\right)s_1 + \left(\frac{\rho_L}{\rho_*}\right)u_L, \tag{3.20}$$

$$s_1 = u_L - a_L\sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p_*}{p_L}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}, \tag{3.21}$$

$$M_L - M_S = +\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p_L}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}. \tag{3.22}$$

□