PEKING UNIVERSITY

计算流体力学上机作业 1

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编写一维完全气体 Euler 方程组的 LF 格式, MacCormack 格式, 和一阶精度的显式迎风格式 (Roe 格式) 的程序, 并计算讲义 (CFDLect04-com01_cn.pdf) 的第 101-102 页的问题 2 和问题 4.

$$\begin{cases} \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix}_x = 0, \\ p = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right), \ \gamma = 1.4. \end{cases} \tag{0.1}$$

计算动图可点击 https://www.bilibili.com/video/bv17N411f7tX 查看.

代码可点击 https://github.com/circlelq/Computational-Fluid-Dynamics/tree/main/code1 查看.

1

初始条件

$$\boldsymbol{U} = \begin{cases} (1, 0, 2.5)^{\mathrm{T}}, & x < 0.3, \\ (0.125, 0, 0.25)^{\mathrm{T}}, & x > 0.3. \end{cases} \tag{1.1}$$

计算区间为 $[0,1]$, 输出时刻 $t = 0.2$.

如图 1.1 所示为初始条件, 密度 ρ 和压强 p 有一个初始间断, 速度 u 都为 0. 之后的格式都采用固定 CFL 数, 根据 CFL 数来求 $\mathrm{d}t$. 其中 N 是 x 的网格数. 之后的输出结果都是 $t = 0.2$ 时刻的结果, 除非震荡剧烈, 无法继续计算.

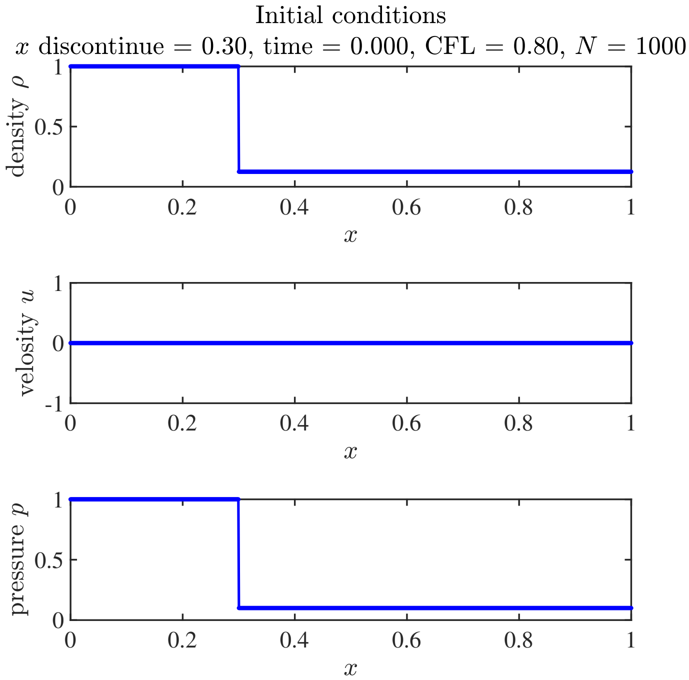

图 1.1. 初始条件.

LF 格式

$$\boldsymbol{u}_j^{n+1}=\frac{1}{2}\left(\boldsymbol{u}_{j+1}^n+\boldsymbol{u}_{j-1}^n\right)-\frac{1}{2}r\left(\boldsymbol{f}_{j+1}^n-\boldsymbol{f}_{j-1}^n\right). \tag{1.2}$$

其中 $r=\tau/h$, 稳定性条件为

$$|\boldsymbol{u}|_{\max}\frac{\tau}{h}\leqslant 1. \tag{1.3}$$

如图 1.2 所示, LF 格式是 TVD 的, 所以无震荡. 可以发现, 计算结果为左边是一个稀疏波, 中间是一个接触间断, 右边是一个激波.

MacCormack 格式

MacCormack 格式 [R.W. MacCormack, AIAA Paper No. 1969-354 (1969)]

$$\begin{aligned} u\left(x_j,t_{n+1}\right)=&u\left(x_j,t_n\right)+\tau\left(u_t\right)_j^n+\frac{1}{2}\tau^2\left(u_{tt}\right)_j^n+\mathcal{O}\left(\tau^3\right)\\ =&\frac{1}{2}u\left(x_j,t_n\right)+\frac{1}{2}\left(\bar{u}+\tau\bar{u}_t\right)_j^n+\mathcal{O}\left(\tau^3\right),\quad \bar{u}:=u+\tau u_t \end{aligned} \tag{1.4}$$

$$\left\{\begin{array}{l} \bar{u}_j^*=u_j^n-\frac{\tau}{h}\left(f\left(u_{j+1}^n\right)-f\left(u_j^n\right)\right), \\ u_j^{n+1}=\frac{1}{2}\left(u_j^n+\bar{u}_j^*\right)-\frac{\tau}{2h}\left(f\left(\bar{u}_j^*\right)-f\left(\bar{u}_{j-1}^*\right)\right). \end{array}\right. \tag{1.5}$$

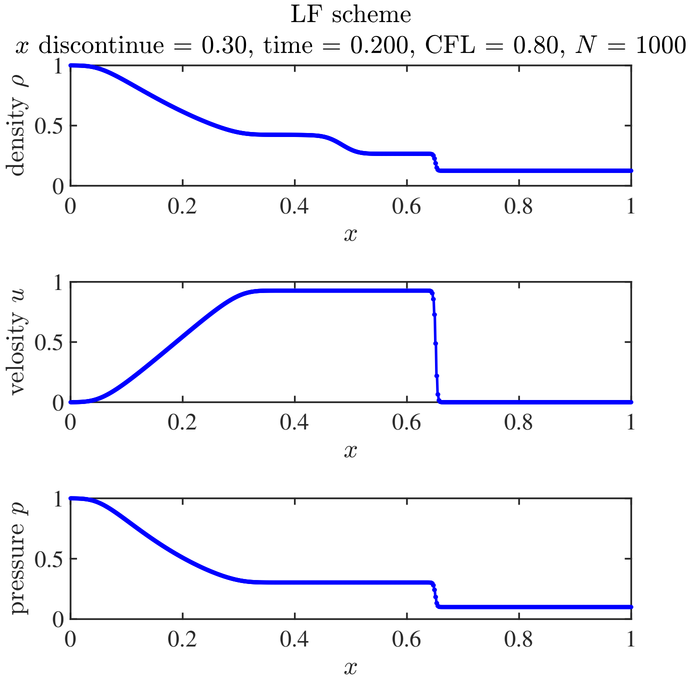

图 1.2. LF 格式计算结果.

或

$$
\begin{cases}
\bar{u}_j^* = u_j^n - \frac{\tau}{h}\left(f\left(u_j^n\right) - f\left(u_{j-1}^n\right)\right), \\
u_j^{n+1} = \frac{1}{2}\left(u_j^n + \bar{u}_j^*\right) - \frac{\tau}{2h}\left(f\left(\bar{u}_{j+1}^*\right) - f\left(\bar{u}_j^*\right)\right).
\end{cases}
\tag{1.6}
$$

如图 1.3 所示, MacCormack 格式震荡严重, 导致 dt 只能取非常小的值, 所以经过 300 步的计算后还是停留在 $t = 0.001$.

Roe 格式

(P.L. Roe, JCP, 43, $1981, 357-372/135, 1997, 250-258$)

$$
\hat{\boldsymbol{F}}\left(\boldsymbol{U}_j, \boldsymbol{U}_{j+1}\right) = \frac{\boldsymbol{F}\left(\boldsymbol{U}_j\right) + \boldsymbol{F}\left(\boldsymbol{U}_{j+1}\right)}{2} - \frac{1}{2}\left|\hat{\boldsymbol{A}}_{j+1/2}\right|\left(\boldsymbol{U}_{j+1} - \boldsymbol{U}_j\right). \tag{1.7}
$$

其中 $|\hat{\boldsymbol{A}}|$ 定义为: $|\hat{\boldsymbol{A}}| = \boldsymbol{R}|\hat{\boldsymbol{\Lambda}}|\boldsymbol{R}^{-1}|, |\hat{\boldsymbol{\Lambda}}\,| = \mathrm{diag}\left\{\left|\hat{\lambda}_1\right|, \cdots, \left|\hat{\lambda}_m\right|\right\}$, $\boldsymbol{R}$ 为 $\hat{\boldsymbol{A}}$ 的右特征向量矩阵, $\hat{\Lambda} = \mathrm{diag}\left\{\hat{\lambda}_1, \cdots, \hat{\lambda}_m\right\}$, 即 $\boldsymbol{R}^{-1}\hat{\boldsymbol{A}}\boldsymbol{R} = \hat{\boldsymbol{\Lambda}}$,

$$
\boldsymbol{R} = \left(\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \boldsymbol{R}^{(3)}\right) = \begin{pmatrix} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{pmatrix}. \tag{1.8}
$$

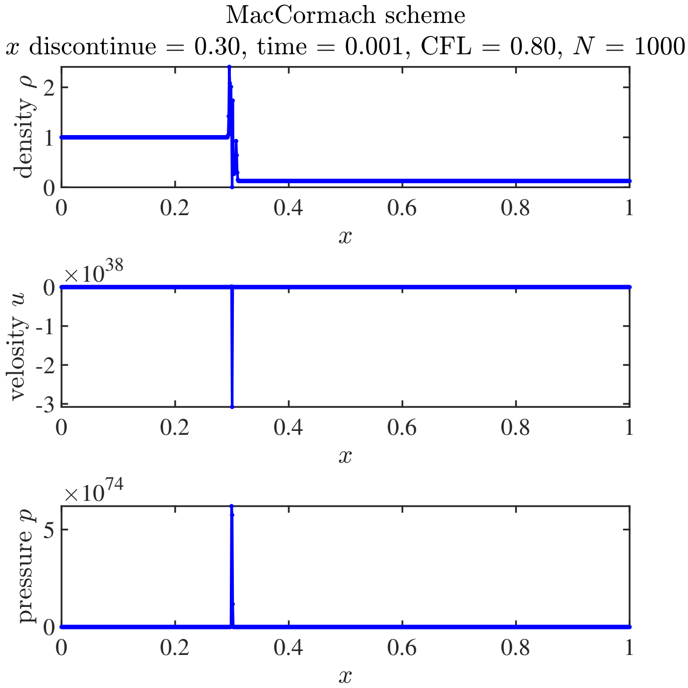

图 1.3. Mac 格式计算结果.

$$\boldsymbol{\Lambda} = \begin{pmatrix} u-a & & \\ & u & \\ & & u+a \end{pmatrix}, \tag{1.9}$$

$$\hat{\boldsymbol{A}}_{j+1/2}\left(\boldsymbol{U}_{j+1}-\boldsymbol{U}_j\right) = \boldsymbol{F}\left(\boldsymbol{U}_{j+1}\right) - \boldsymbol{F}\left(\boldsymbol{U}_j\right). \tag{1.10}$$

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - r\left(\hat{\boldsymbol{F}}_{j+\frac{1}{2}}^n - \hat{\boldsymbol{F}}_{j-\frac{1}{2}}^n\right). \tag{1.11}$$

对于理性气体, 通过如下方法构造 $\hat{\boldsymbol{A}}$:

$$\begin{cases} \bar{\rho} = \dfrac{\sqrt{\rho_r}\rho_\ell + \sqrt{\rho_\ell}\rho_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}} = \sqrt{\rho_\ell\rho_r} = \left[\dfrac{1}{2}\left(\sqrt{\rho_r}+\sqrt{\rho_\ell}\right)\right]^2, \\ \bar{u} = \dfrac{\sqrt{\rho_\ell}u_\ell + \sqrt{\rho_r}u_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}}, \\ \bar{H} = \dfrac{\sqrt{\rho_\ell}H_\ell + \sqrt{\rho_r}H_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}}. \end{cases} \tag{1.12}$$

$$\boldsymbol{A}(\boldsymbol{U}) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ \frac{\gamma-2}{2}u^3 - \frac{a^2u}{\gamma-1} & \frac{3-2\gamma}{2}u^2 + \frac{a^2}{\gamma-1} & \gamma u \end{pmatrix}, \tag{1.13}$$

$$a = \sqrt{\gamma p/\rho}, \quad H = (E+p)/\rho, \quad p = (\gamma-1)\left(E - \frac{1}{2}\rho u^2\right). \tag{1.14}$$

$$\hat{\boldsymbol{A}} = \boldsymbol{A}(\bar{\boldsymbol{U}}). \tag{1.15}$$

因为

$$E = \frac{1}{\gamma}\left[H\rho + (\gamma-1)\frac{1}{2}\rho u^2\right], \quad p = H\rho - E, \tag{1.16}$$

所以

$$\bar{a}^2 = \frac{\gamma\bar{p}}{\bar{\rho}} = \gamma\frac{\bar{H}\bar{\rho} - \bar{E}}{\bar{\rho}} = \gamma\frac{\bar{H}\bar{\rho} - \frac{1}{\gamma}\left[\bar{H}\bar{\rho} + (\gamma-1)\frac{1}{2}\bar{\rho}\bar{u}^2\right]}{\bar{\rho}} = (\gamma-1)\left(\bar{H} - \frac{1}{2}\bar{u}^2\right). \tag{1.17}$$

如图 1.4 所示, Roe 格式也无震荡.

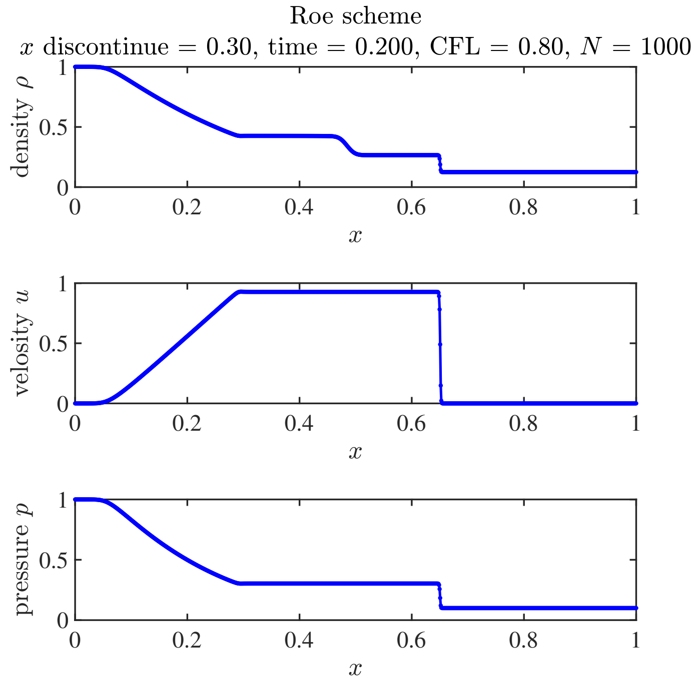

图 1.4. Roe 格式计算结果.

2

初始条件

$$(\rho, u, p)(x, 0) = \begin{cases} (3.857143, 2.629369, 10.33333), & x < -4, \\ (1 + 0.2\sin(5x), 0, 1), & x \geq -4. \end{cases} \tag{2.1}$$

计算区间为 $[-5, 5]$, 其中在 $x = \pm 5$ 边界处 $\partial_x \rho = \partial_x u = \partial_x p = 0$. 输出时刻为 $t = 1.8$.

如图 2.1 所示为初始条件.

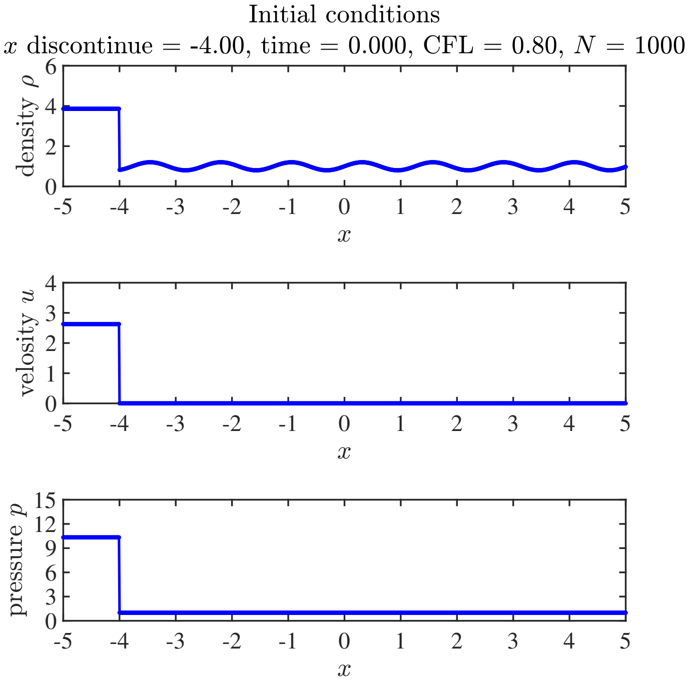

图 2.1. 初始条件.

LF 格式

如图 2.2 所示. 可以发现, 计算结果为一激波向右传播, 耗散性较强, 磨平了初始的密度浮动.

MacCormack 格式

如图 2.3 所示, MacCormack 格式稍微有一些震荡, 尤其是在间断处间断明显, 但是能完成计算. 此格式的耗散性较弱.

Roe 格式

如图 2.4 所示, Roe 格式也无震荡.

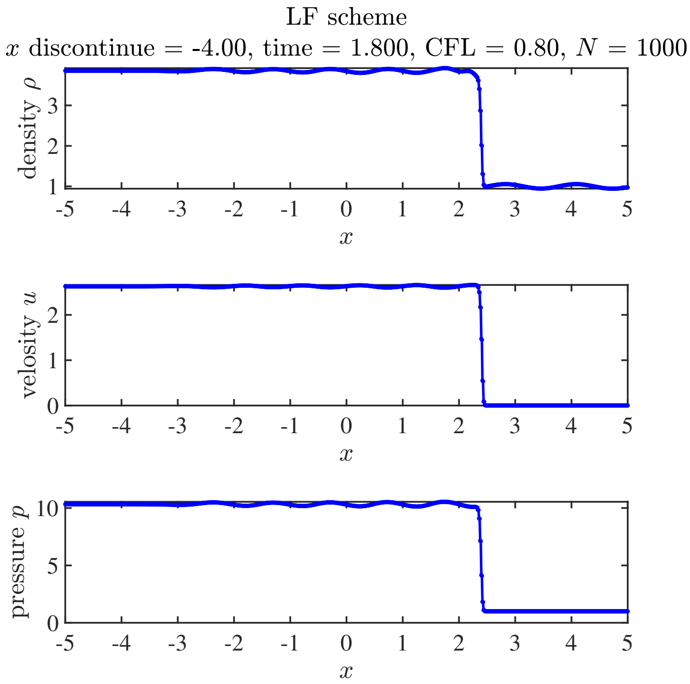

图 2.2. LF 格式计算结果.

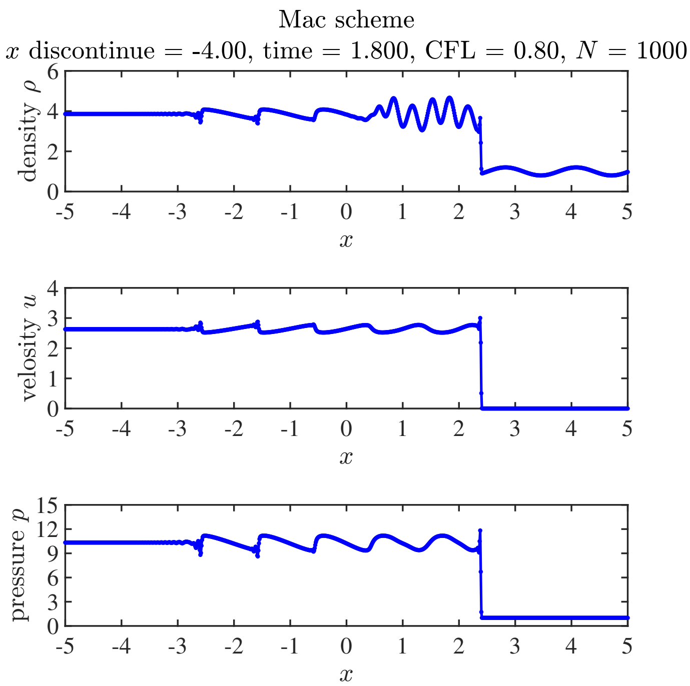

图 2.3. Mac 格式计算结果.

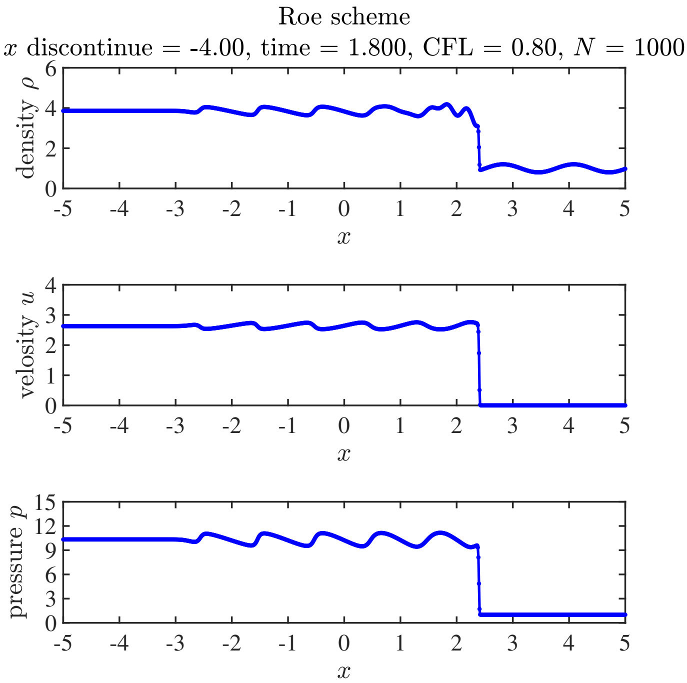

图 2.4. Roe 格式计算结果.