计算流体力学作业 3

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March 30, 2021

Cauchy 问题的解

利用特征线理论分析问题

$$\begin{cases} u_t + a(u)u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

并给出 (光滑的) 解.

解: $u_t + a(u)u_x = 0, u(x,0) = u_0(x)$. 问题转化为

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a(u), \\ x(0) = x_0. \end{cases} \qquad \begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = 0, \\ u(0) = u_0\left(x_0\right). \end{cases} \tag{1.2}$$

由上述 ODE 初值问题得

$$u(x,t) = u_0(x_0), \tag{1.3}$$

$$x = x_0 + a(u_0(x_0))t = x_0 + a(u(x,t))t, \tag{1.4}$$

x_0 依赖于给定的点 (x,t),

$$u(x,t) = u_0\left(x - a(u_0(x_0))t\right) = u_0\left(x - a(u_0(x_0(x,t)))t\right) = u_0\left(x - a(u(x,t))t\right). \tag{1.5}$$

由式 (1.1) 得

$$u_t = u_0'\left(x_0\right)\frac{\partial x_0}{\partial t}, \quad u_x = u_0'\left(x_0\right)\frac{\partial x_0}{\partial x} \tag{1.6}$$

将式 (1.4) 的第一等号两端分别对 t 和 x 求导, 得

$$a\left(u_0\left(x_0\right)\right)+\left[1+a'\cdot u_0'\left(x_0\right)\cdot t\right]\frac{\partial x_0}{\partial t}=0 \tag{1.7}$$

$$\left(1+a'u_0'\left(x_0\right)t\right)\frac{\partial x_0}{\partial x}=1 \tag{1.8}$$

从式 (1.7) and (1.8) 得

$$\frac{\partial x_0}{\partial t}=-\frac{a\left(u_0\left(x_0\right)\right)}{1+\left(a'u_0'\right)_{x_0}t},\quad \frac{\partial x_0}{\partial x}=\frac{1}{1+\left(a'u_0'\right)_{x_0}t}. \tag{1.9}$$

将其代入式 (1.6) 知

$$u_t+a(u)u_x=0. \tag{1.10}$$

$t=0$ 时

$$u(0)=u_0\left(x_0\right), \tag{1.11}$$

所以式 (1.5) 满足式 (1.1).

无黏 Burgers 方程的定解问题

$$\begin{cases} u_t+(0.5u^2)_x=0, \\ u_0(x)=\cos(\pi x), \quad x\in[-1,1]. \end{cases} \tag{2.1}$$

此时 $a(u)=u$, 爆破点

$$t^*=-\frac{1}{a'u_0'}=\frac{1}{\pi\sin(\pi x_0)}, \tag{2.2}$$

只在 $x>0$ 的部分会出现爆破, 最快达到爆破的点为 $x_0=0.5$, 经历时间 $t_0^*=\frac{1}{\pi}$.

初始图像如图 2.1所示, 当其运动到爆破的时候如图 2.2 所示, 可以发现右边的线已经和 x 轴垂直了.

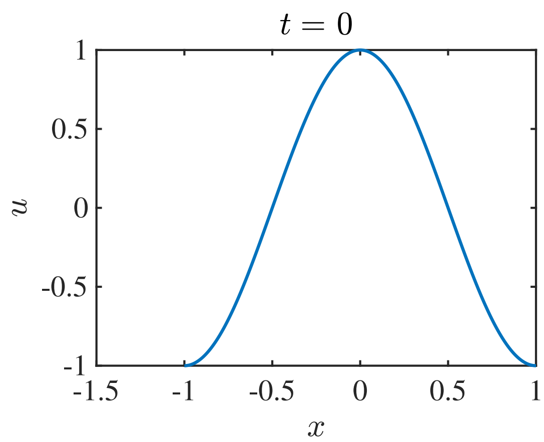

图 2.1. $t = 0$.

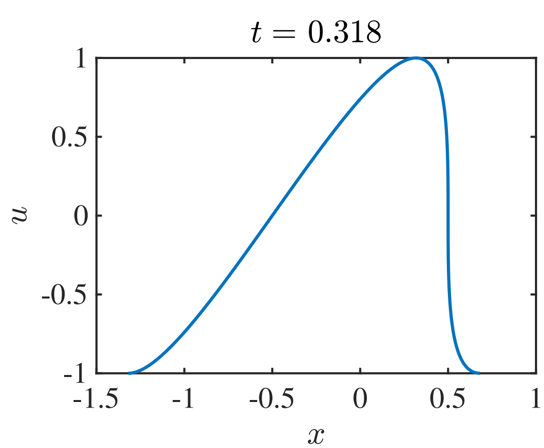

图 2.2. $t = \frac{1}{\pi}$. 爆破时刻.

Burgers 方程 Riemann 问题的弱解

弱解满足的方程为

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t\boldsymbol{U}+\phi_x\boldsymbol{F}(\boldsymbol{U})\right]\mathrm{d}x\,\mathrm{d}t=-\int_{-\infty}^{+\infty}\phi(x,0)\boldsymbol{U}(x,0)\,\mathrm{d}x. \tag{3.1}$$

对 Burgers 方程有

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t=-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. \tag{3.2}$$

激波弱解

$$u(x,t)=\begin{cases}u_L, & x<st,\\ u_R, & x>st.\end{cases} \tag{3.3}$$

其中

$$s=\frac{u_L+u_R}{2}. \tag{3.4}$$

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t && (3.5)\\
=&\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_t u\,\mathrm{d}x\,\mathrm{d}t+\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_x\frac{1}{2}u^2\,\mathrm{d}x\,\mathrm{d}t && (3.6)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}x && (3.7)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.8)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+s\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}(x/s) && (3.9)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.10)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. && (3.11)
\end{aligned}$$

$u_L < u_R$ 弱解

$$u(x,t)=\begin{cases}u_L, & x<st,\\ u_R, & x>st.\end{cases} \tag{3.12}$$

其中

$$s=\frac{u_L+u_R}{2}. \tag{3.13}$$

同样的

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t && (3.14)\\
=&\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_t u\,\mathrm{d}x\,\mathrm{d}t+\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_x\frac{1}{2}u^2\,\mathrm{d}x\,\mathrm{d}t && (3.15)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}x && (3.16)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.17)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+s\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}(x/s) && (3.18)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.19)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. && (3.20)
\end{aligned}$$

稀疏波弱解

$$u(x,t)=\begin{cases}u_L, & x<s_m t\\ u_m, & s_m t\le x\le u_m t\\ \frac{x}{t}, & u_m t\le x\le u_R t\\ u_R, & x>u_R t\end{cases} \tag{3.21}$$

也是一个弱解, 其中 $u_m \in [u_L, u_R]$ 为任意, $s_m = \frac{u_L+u_m}{2}$. 不妨设 $u_L > 0,\ u_R > 0$.

$$
\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u + \phi_x \frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t && (3.22)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x + \int_0^{+\infty}\left(-u_L\phi\left(x,\frac{x}{s_m}\right) + u_m\phi\left(x,\frac{x}{s_m}\right)\right. && (3.23)\\
&\left.-u_m\phi\left(x,\frac{x}{u_m}\right) + \int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\phi_t\frac{x}{t}\,\mathrm{d}t + u_R\phi\left(x,\frac{x}{u_R}\right)\right)\mathrm{d}x && (3.24)\\
&+\int_0^{+\infty}\frac{1}{2}\left(-u_R^2\phi\left(u_R t,t\right) + \int_{u_m t}^{u_R t}\frac{x^2}{t^2}\phi_x(x,t)\,\mathrm{d}x + u_m^2\phi\left(u_m t,t\right) + \left(u_L^2-u_m^2\right)\phi\left(s_m t,t\right)\right)\mathrm{d}t. && (3.25)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x + \int_0^{+\infty}\left(-u_m\phi\left(x,\frac{x}{u_m}\right) + \int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\phi_t\frac{x}{t}\,\mathrm{d}t + u_R\phi\left(x,\frac{x}{u_R}\right)\right)\mathrm{d}x && (3.26)\\
&+\int_0^{+\infty}\frac{1}{2}\left(-u_R^2\phi\left(u_R t,t\right) + \int_{u_m t}^{u_R t}\frac{x^2}{t^2}\phi_x\,\mathrm{d}x + u_m^2\phi\left(u_m t,t\right)\right)\mathrm{d}t. && (3.27)
\end{aligned}
$$

其中

$$
\begin{aligned}
\int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\phi_t\frac{x}{t}\,\mathrm{d}t =& \left(\frac{x}{t}\phi\right)\Bigg|_{t=\frac{x}{u_R}}^{t=\frac{x}{u_m}} + \int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\frac{x}{t^2}\phi\,\mathrm{d}t && (3.28)\\
=&u_m\phi(x,\frac{x}{u_m}) - u_R\phi(x,\frac{x}{u_R}) + \int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\frac{x}{t^2}\phi\,\mathrm{d}t, && (3.29)
\end{aligned}
$$

$$
\begin{aligned}
\int_{u_m t}^{u_R t}\frac{x^2}{t^2}\phi_x\,\mathrm{d}x =& \frac{x^2}{t^2}\phi\Bigg|_{x=u_m t}^{x=u_R t} - \int_{u_m t}^{u_R t}\frac{2x}{t^2}\phi\,\mathrm{d}x && (3.30)\\
=&u_R^2\phi(u_R t,t) - u_m^2\phi(u_m t,t) - \int_{u_m t}^{u_R t}\frac{2x}{t^2}\phi\,\mathrm{d}x. && (3.31)
\end{aligned}
$$

所以

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t \tag{3.32}$$

$$=-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\frac{x}{t^2}\phi\,\mathrm{d}t\,\mathrm{d}x-\int_0^{+\infty}\int_{u_m t}^{u_R t}\frac{x}{t^2}\phi\,\mathrm{d}x\,\mathrm{d}t \tag{3.33}$$

$$=-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. \tag{3.34}$$

□

线化气体动力学方程

线化气体动力学方程组

$$\begin{cases}\frac{\partial\rho}{\partial t}+\rho_0\frac{\partial u}{\partial x}=0,\\ \frac{\partial u}{\partial t}+\frac{a_0^2}{\rho_0}\frac{\partial\rho}{\partial x}=0.\end{cases} \tag{4.1}$$

一般初值问题的解

$$\boldsymbol{U}_t+\boldsymbol{A}\boldsymbol{U}_x=0,\quad \boldsymbol{U}=\begin{pmatrix}\rho\\ u\end{pmatrix},\quad \boldsymbol{A}=\begin{pmatrix}0&\rho_0\\ a_0^2/\rho_0&0\end{pmatrix} \tag{4.2}$$

其中

$$\boldsymbol{R}^{(1)}=\begin{pmatrix}\rho_0\\ -a_0\end{pmatrix},\quad \boldsymbol{R}^{(2)}=\begin{pmatrix}\rho_0\\ a_0\end{pmatrix},\quad \begin{matrix}\lambda_1=-a_0\\ \lambda_2=a_0\end{matrix} \tag{4.3}$$

$$\boldsymbol{R}^{-1}=\frac{1}{2\rho_0 a_0}\begin{pmatrix}a_0&-\rho_0\\ a_0&\rho_0\end{pmatrix} \tag{4.4}$$

为了给出初值问题的解, 将它转化为相应的规范形

$$\begin{cases}\boldsymbol{W}_t+\boldsymbol{\Lambda}\boldsymbol{W}_x=0\\ \boldsymbol{W}(x,0)=\boldsymbol{W}^{(0)}=\boldsymbol{R}^{-1}\boldsymbol{U}^{(0)}(x)\end{cases} \tag{4.5}$$

其中

$$\mathbf{\Lambda} = \begin{pmatrix} -a_0 & 0 \\ 0 & a_0 \end{pmatrix}. \tag{4.6}$$

该问题的解是

$$\boldsymbol{W}(x,t) = \left(W_1^{(0)}\left(x-\lambda_1 t\right), W_2^{(0)}\left(x-\lambda_2 t\right), \cdots, W_m^{(0)}\left(x-\lambda_m t\right)\right)^T, \tag{4.7}$$

现在返回到表示式 $\boldsymbol{U} = \boldsymbol{RW}$, 即

$$\boldsymbol{U}(x,t) = \sum_{i=1}^{m} W_i(x,t)\boldsymbol{R}^{(i)} = \sum_{i=1}^{m} W_i^{(0)}\left(x-\lambda_i t\right)\boldsymbol{R}^{(i)}. \tag{4.8}$$

Riemann 问题的解

令 (U_R 类似处理)

$$\boldsymbol{U}_L = \begin{pmatrix} \rho_L \\ u_L \end{pmatrix} = \alpha_1 \begin{pmatrix} \rho_0 \\ -a_0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \rho_0 \\ a_0 \end{pmatrix} \tag{4.9}$$

不难得

$$\alpha_1 = \frac{a_0\rho_L - \rho_0 u_L}{2a_0\rho_0},\ \alpha_2 = \frac{a_0\rho_L + \rho_0 u_L}{2a_0\rho_0} \tag{4.10}$$

$$\beta_1 = \frac{a_0\rho_R - \rho_0 u_R}{2a_0\rho_0},\ \beta_2 = \frac{a_0\rho_R + \rho_0 u_R}{2a_0\rho_0} \tag{4.11}$$

在星形域 ($\lambda_1 t < x < \lambda_2 t$) 内的解

$$U^* = \begin{pmatrix} \rho^* \\ u^* \end{pmatrix} = \beta_1 \begin{pmatrix} \rho_0 \\ -a_0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \rho_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\rho_L + \rho_R\right) - \frac{1}{2}\left(u_R - u_L\right)\rho_0/a_0 \\ \frac{1}{2}\left(u_L + u_R\right) - \frac{1}{2}\left(\rho_R - \rho_L\right)a_0/\rho_0 \end{pmatrix}. \tag{4.12}$$

$x < \lambda_1 t$ 的解为 $\boldsymbol{U}_L$, $x > \lambda_2 t$ 的解为 $\boldsymbol{U}_R$.