PEKING UNIVERSITY

计算流体力学作业 7

College of Engineering   2001111690   袁磊祺

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1

可以.

例如松弛系统的一阶半离散迎风近似

$$\frac{\partial}{\partial t}\boldsymbol{u}_j+\frac{1}{2h_j}\left(\boldsymbol{v}_{j+1}-\boldsymbol{v}_{j-1}\right)-\frac{1}{2h_j}\boldsymbol{A}^{1/2}\left(\boldsymbol{u}_{j+1}-2\boldsymbol{u}_j+\boldsymbol{u}_{j-1}\right)=0, \tag{1.1}$$

$$\frac{\partial}{\partial t}\boldsymbol{v}_j+\frac{1}{2h_j}\boldsymbol{A}\left(\boldsymbol{u}_{j+1}-\boldsymbol{u}_{j-1}\right)-\frac{1}{2h_j}\boldsymbol{A}^{1/2}\left(\boldsymbol{v}_{j+1}-2\boldsymbol{v}_j+\boldsymbol{v}_{j-1}\right)=-\frac{1}{\varepsilon}\Big(\boldsymbol{v}_j-F\left(\boldsymbol{u}_j\right)\Big). \tag{1.2}$$

考虑一维空间变量的守恒系统

$$\frac{\partial}{\partial t}\boldsymbol{u}+\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u})=0, \quad (x,t)\in\mathbb{R}^1\times\mathbb{R}, \quad \boldsymbol{u}\in\mathbb{R}^n, \tag{1.3}$$

其中 $\boldsymbol{F}(\boldsymbol{u})\in\mathbb{R}^n$ 是一个光滑的向量值函数, 我们引入一个关于式 (1.3) 的松弛系统

$$\frac{\partial}{\partial t}\boldsymbol{u}+\frac{\partial}{\partial x}\boldsymbol{v}=0, \quad \boldsymbol{v}\in\mathbb{R}^n, \tag{1.4}$$

$$\frac{\partial}{\partial t}\boldsymbol{v}+\boldsymbol{A}\frac{\partial}{\partial x}\boldsymbol{u}=-\frac{1}{\varepsilon}(\boldsymbol{v}-F(\boldsymbol{u})), \quad \varepsilon>0. \tag{1.5}$$

其中

$$\boldsymbol{A}=\operatorname{diag}\left\{a_1,a_2,\ldots,a_n\right\}, \tag{1.6}$$

是一个需要确定的对角矩阵. 对于小的 ε, 应用 ChapmanEnskog 展开得下面的近似

$$\frac{\partial}{\partial t}\boldsymbol{u}+\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u})=\varepsilon\frac{\partial}{\partial x}\left(\left(\boldsymbol{A}-\boldsymbol{F}'(\boldsymbol{u})^2\right)\frac{\partial}{\partial x}\boldsymbol{u}\right) \tag{1.7}$$

其中 $\boldsymbol{F}'(\boldsymbol{u})$ 是 $\boldsymbol{F}$ 的雅可比矩阵. 式 (1.7) 控制松弛系统的一阶行为. 为了确保式 (1.7) 的耗散性, 则需要

$$\boldsymbol{A} - \boldsymbol{F}'(\boldsymbol{u})^2 \geqslant 0 \quad \text{对所有的} \quad \boldsymbol{u}. \tag{1.8}$$

我们假设 $\boldsymbol{A}$ 具有

$$\boldsymbol{A} = a\boldsymbol{I}, \quad a > 0, \tag{1.9}$$

的形式, 其中 $\boldsymbol{I}$ 是单位矩阵. 如果

$$\frac{\lambda\mu}{a} \leqslant 1. \tag{1.10}$$

则系统是耗散的, 即 $\boldsymbol{A}$ 的一个充分条件.

2

对 Hamilton-Jacobi 方程

$$\phi_x + H(\phi_x) = 0, \tag{2.1}$$

的半离散格式可以写成

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_i = -\hat{H}\left(\frac{\Delta^+\phi_i}{h}, \frac{\Delta^-\phi_i}{h}\right), \tag{2.2}$$

其中 $\hat{H}$ 称为数值哈密顿量, 它是所有参数的 Lipschitz 连续函数, 并且与 PDE 中的哈密顿量 H 一致.

相容条件

$$\hat{H}(u, u) = H(u). \tag{2.3}$$

3

Vinokur 已经证明: 微分类算法（有限差分算法等）和及分类算法（有限体积算法等）仅仅在集合处理上有差别, 这种差别会影响到算法到计算精度和效率, 但在算法本质上没有根本的差别. 所以这里给出二阶精度的 Lax-Wendroff 格式. [1, P229]

Lax-Wendroff

$$
\begin{aligned}
\boldsymbol{U}\left(x_j, y_k, t_{n+1}\right) =& \boldsymbol{U}\left(x_j, y_k, t_n\right)+\tau\left(\boldsymbol{U}_t\right)_{j,k}^n+\frac{1}{2}\tau^2\left(\boldsymbol{U}_{tt}\right)_{j,k}^n+\mathcal{O}\left(\tau^3\right) && (3.1)\\
=& \boldsymbol{U}\left(x_j, y_k, t_n\right)-\tau\left(\boldsymbol{F}_x+\boldsymbol{G}_y\right)_{j,k}^n && (3.2)\\
&+\frac{1}{2}\tau^2\left[\partial_x\left(\boldsymbol{A}\left(\boldsymbol{F}_x+\boldsymbol{G}_y\right)\right)+\partial_y\left(\boldsymbol{B}\left(\boldsymbol{F}_x+\boldsymbol{G}_y\right)\right)\right]_{j,k}^n+\mathcal{O}\left(\tau^3\right) && (3.3)\\
=& \boldsymbol{U}\left(x_j, y_k, t_n\right)-\tau\left(\boldsymbol{F}_x+\boldsymbol{G}_y\right)_{j,k}^n && (3.4)\\
&+\frac{1}{2}\tau^2\left[\partial_x\left(\boldsymbol{A}\left(\boldsymbol{A}\boldsymbol{U}_x+\boldsymbol{B}\boldsymbol{U}_y\right)\right)+\partial_y\left(\boldsymbol{B}\left(\boldsymbol{A}\boldsymbol{U}_x+\boldsymbol{B}\boldsymbol{U}_y\right)\right)\right]_{j,k}^n+\mathcal{O}\left(\tau^3\right). && (3.5)
\end{aligned}
$$

利用中心差商代替空间微商, 略去高阶项, 并用 $\boldsymbol{U}_{j,k}^n$ 代替 $\boldsymbol{U}\left(x_j, y_k, t_n\right)$, 得

$$
\begin{aligned}
\boldsymbol{U}_{j,k}^{n+1} =& \boldsymbol{U}_{j,k}^n-\frac{\lambda_x}{2}\left(\boldsymbol{F}\left(\boldsymbol{U}_{j+1,k}^n\right)-\boldsymbol{F}\left(\boldsymbol{U}_{j-1,k}^n\right)\right)-\frac{\lambda_y}{2}\left(\boldsymbol{G}\left(\boldsymbol{U}_{j,k+1}^n\right)-\boldsymbol{G}\left(\boldsymbol{U}_{j,k-1}^n\right)\right) && (3.6)\\
&+\frac{\lambda_x^2\boldsymbol{A}^2\left(\boldsymbol{U}_{j+\frac{1}{2},k}^n\right)}{2}\left(\boldsymbol{U}_{j+1,k}^n-\boldsymbol{U}_{j,k}^n\right)-\frac{\lambda_x^2\boldsymbol{A}^2\left(\boldsymbol{U}_{j-\frac{1}{2},k}^n\right)}{2}\left(\boldsymbol{U}_{j,k}^n-\boldsymbol{U}_{j-1,k}^n\right) && (3.7)\\
&+\frac{\lambda_x\lambda_y\boldsymbol{A}\boldsymbol{B}\left(\boldsymbol{U}_{j,k}^n\right)}{8}\left(\boldsymbol{U}_{j+1,k+1}^n-\boldsymbol{U}_{j-1,k+1}^n\right)-\frac{\lambda_x\lambda_y\boldsymbol{A}\boldsymbol{B}\left(\boldsymbol{U}_{j,k}^n\right)}{8}\left(\boldsymbol{U}_{j+1,k-1}^n-\boldsymbol{U}_{j-1,k-1}^n\right) && (3.8)\\
&+\frac{\lambda_x\lambda_y\boldsymbol{B}\boldsymbol{A}\left(\boldsymbol{U}_{j,k}^n\right)}{8}\left(\boldsymbol{U}_{j+1,k+1}^n-\boldsymbol{U}_{j+1,k-1}^n\right)-\frac{\lambda_x\lambda_y\boldsymbol{B}\boldsymbol{A}\left(\boldsymbol{U}_{j,k}^n\right)}{8}\left(\boldsymbol{U}_{j-1,k+1}^n-\boldsymbol{U}_{j-1,k-1}^n\right) && (3.9)\\
&+\frac{\lambda_y^2\boldsymbol{B}^2\left(\boldsymbol{U}_{j,k+\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_{j,k+1}^n-\boldsymbol{U}_{j,k}^n\right)-\frac{\lambda_y^2\boldsymbol{B}^2\left(\boldsymbol{U}_{j,k-\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_{j,k}^n-\boldsymbol{U}_{j,k-1}^n\right). && (3.10)
\end{aligned}
$$

其中 $\boldsymbol{U}_{j+\frac{1}{2},k}^n=\frac{1}{2}\left(\boldsymbol{U}_{j,k}^n+\boldsymbol{U}_{j+1,k}^n\right)$.

说明: 其中的 $\boldsymbol{A}\left(\boldsymbol{U}_{j+\frac{1}{2},k}\right)$ 可以替代为

$$\boldsymbol{A}_{j+\frac{1}{2},k}=\begin{cases}\frac{\boldsymbol{F}\left(\boldsymbol{U}_{j+1,k}\right)-\boldsymbol{F}\left(\boldsymbol{U}_{j,k}\right)}{\boldsymbol{U}_{j+1,k}-\boldsymbol{U}_{j,k}}, & \boldsymbol{U}_{j+1,k}\neq\boldsymbol{U}_{j,k},\\ \boldsymbol{A}\left(\boldsymbol{U}_{j,k}\right), & \boldsymbol{U}_{j+1,k}=\boldsymbol{U}_{j,k}.\end{cases} \tag{3.11}$$

$\boldsymbol{B}$ 同理.

4

一维情况

这里考虑标量的情况.

$$\begin{gathered}\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}v=0,\quad v\in\mathbb{R}^1,\\ \frac{\partial}{\partial t}v+a\frac{\partial}{\partial x}u=-\frac{1}{\varepsilon}(v-F(u)),\quad \varepsilon>0.\end{gathered} \tag{4.1}$$

松弛系统式 (4.1) 的均匀网格的简单一阶守恒格式可以写成

$$\begin{gathered}\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{v_{j+1/2}^n-v_{j-1/2}^n}{h}=0,\\ \frac{v_j^{n+1}-v_j^n}{\tau}+a\frac{u_{j+1/2}^n-u_{j-1/2}^n}{h}=-\frac{1}{\varepsilon}\left(v_j^{n+1}-f\left(u_j^{n+1}\right)\right).\end{gathered} \tag{4.2}$$

式 (4.1) 的一阶迎风格式是

$$\begin{aligned}u_{j+1/2}^n=&\frac{1}{2}\left(u_j^n+u_{j+1}^n\right)-\frac{1}{2\sqrt{a}}\left(v_{j+1}^n-v_j^n\right),\\ v_{j+1/2}^n=&\frac{1}{2}\left(v_j^n+v_{j+1}^n\right)+\frac{\sqrt{a}}{2}\left(u_{j+1}^n-u_j^n\right).\end{aligned} \tag{4.3}$$

应用式 (4.2) 得

$$\frac{u_j^{n+1}-u_j^n}{k}+\frac{1}{2h}\left(v_{j+1}^n-v_{j-1}^n\right)-\frac{\sqrt{a}}{2h}\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)=0. \tag{4.4}$$

$$\frac{v_j^{n+1}-v_j^n}{k}+\frac{a}{2h}\left(u_{j+1}^n-u_{j-1}^n\right)-\frac{1}{2\sqrt{a}h}\left(v_{j+1}^n-2v_j^n+v_{j-1}^n\right)=-\frac{1}{\varepsilon}\left(v_j^n-f\left(u_j^n\right)\right). \tag{4.5}$$

Using a Hilbert expansion gives the leading order equations (as $\varepsilon \to 0^+$),

$$
\begin{aligned}
&v_j^n = f\left(u_j^n\right),\\
&u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f\left(u_{j+1}^n\right) - f\left(u_{j-1}^n\right)\right) + \frac{\sqrt{a}\lambda}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).
\end{aligned}
\tag{4.6}
$$

where

$$
\lambda = \frac{k}{h}. \tag{4.7}
$$

The scheme in (4.6) is a first-order relaxed scheme which is a Lax-Friedrichs type scheme. A numerical scheme

$$
U_j^{n+1} = \mathbf{H}\left(U^n; j\right) \tag{4.8}
$$

is called a monotone scheme if

$$
\frac{\partial}{\partial U_i^n}\mathbf{H}\left(U^n; j\right) \geqq 0 \quad \text{for all} \quad i, j, U^n. \tag{4.9}
$$

It can be checked easily that (4.6) is a monotone scheme provided the standard CFL condition arising from the discrete convection terms

$$
\sqrt{a}\lambda \leqq 1. \tag{4.10}
$$

where $\lambda = \frac{\tau}{h}$ and the subcharacteristic condition

$$
-\sqrt{a} \leq f'(u) \leq \sqrt{a} \quad \text{for all } u \tag{4.11}
$$

are satisfied. Thus we have the following theorem.

Under the CFL condition (4.10) and the subcharacteristic condition (4.11), the relaxed scheme (4.6) is a monotone scheme.

所以时间步长应该满足式 (4.10).

二维情况

对于二维问题, 考虑标量情况

$$
\begin{aligned}
\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}v + \frac{\partial}{\partial y}w &= 0,\\
\frac{\partial}{\partial t}v + a\frac{\partial}{\partial x}u &= -\frac{1}{\varepsilon}(v - F(u)),\\
\frac{\partial}{\partial t}w + b\frac{\partial}{\partial y}u &= -\frac{1}{\varepsilon}(w - G(u)).
\end{aligned}
\tag{4.12}
$$

松弛系统式 (4.12) 的均匀网格的简单一阶守恒格式可以写成

$$
\begin{aligned}
&\frac{u_{j,k}^{n+1}-u_{j,k}^{n}}{\tau}+\frac{v_{j+\frac{1}{2},k}^{n}-v_{j-\frac{1}{2},k}^{n}}{h_x}+\frac{w_{j,k+\frac{1}{2}}^{n}-w_{j,k-\frac{1}{2}}^{n}}{h_y}=0,\\
&\qquad\qquad\frac{v_{j,k}^{n+1}-v_{j,k}^{n}}{\tau}+a\frac{u_{j+\frac{1}{2},k}^{n}-u_{j-\frac{1}{2},k}^{n}}{h_x}=-\frac{1}{\varepsilon}\left(v_{j,k}^{n+1}-f\left(u_{j,k}^{n+1}\right)\right),\\
&\qquad\qquad\frac{w_{j,k}^{n+1}-w_{j,k}^{n}}{\tau}+b\frac{u_{j,k+\frac{1}{2}}^{n}-u_{j,k-\frac{1}{2}}^{n}}{h_x}=-\frac{1}{\varepsilon}\left(w_{j,k}^{n+1}-f\left(u_{j,k}^{n+1}\right)\right).
\end{aligned}
\tag{4.13}
$$

一阶迎风格式是

$$
\begin{aligned}
u_{j+1/2}^{n}=&\frac{1}{2}\left(u_j^n+u_{j+1}^n\right)-\frac{1}{2\sqrt{a}}\left(v_{j+1}^n-v_j^n\right),\\
v_{j+1/2}^{n}=&\frac{1}{2}\left(v_j^n+v_{j+1}^n\right)+\frac{\sqrt{a}}{2}\left(u_{j+1}^n-u_j^n\right).
\end{aligned}
\tag{4.14}
$$

特征条件

$$-\sqrt{a}\le F'(u)\le\sqrt{a}\quad \text{for all } u, \tag{4.15}$$

$$-\sqrt{b}\le G'(u)\le\sqrt{b}\quad \text{for all } u. \tag{4.16}$$

时间步长需要满足

$$\sqrt{a}\lambda_x\leqq 1, \tag{4.17}$$

$$\sqrt{b}\lambda_y\leqq 1. \tag{4.18}$$

这样格式是单调的.

参考文献

[1] 张德良. 计算流体力学教程. 高等教育出版社, 2010. 2