计算流体力学作业 3

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Cauchy 问题的解

利用特征线理论分析问题

$$\begin{cases} u_t + a(u)u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

并给出 (光滑的) 解.

解: $u_t + a(u)u_x = 0, u(x,0) = u_0(x)$. 问题转化为

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a(u), \\ x(0) = x_0. \end{cases} \qquad \begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = 0, \\ u(0) = u_0\left(x_0\right). \end{cases} \tag{1.2}$$

由上述 ODE 初值问题得

$$u(x,t) = u_0(x_0), \tag{1.3}$$

$$x = x_0 + a(u_0(x_0))t = x_0 + a(u(x,t))t, \tag{1.4}$$

x_0 依赖于给定的点 (x,t),

$$u(x,t) = u_0\left(x - a(u_0(x_0))t\right) = u_0\left(x - a(u_0(x_0(x,t)))t\right) = u_0\left(x - a(u(x,t))t\right). \tag{1.5}$$

由式 (1.1) 得

$$u_t = u_0'\left(x_0\right)\frac{\partial x_0}{\partial t}, \quad u_x = u_0'\left(x_0\right)\frac{\partial x_0}{\partial x} \tag{1.6}$$

将式 (1.4) 的第一等号两端分别对 t 和 x 求导, 得

$$a\left(u_0\left(x_0\right)\right)+\left[1+a'\cdot u_0'\left(x_0\right)\cdot t\right]\frac{\partial x_0}{\partial t}=0 \tag{1.7}$$

$$\left(1+a'u_0'\left(x_0\right)t\right)\frac{\partial x_0}{\partial x}=1 \tag{1.8}$$

从式 (1.7) and (1.8) 得

$$\frac{\partial x_0}{\partial t}=-\frac{a\left(u_0\left(x_0\right)\right)}{1+\left(a'u_0'\right)_{x_0}t},\quad \frac{\partial x_0}{\partial x}=\frac{1}{1+\left(a'u_0'\right)_{x_0}t}. \tag{1.9}$$

将其代入式 (1.6) 知

$$u_t+a(u)u_x=0. \tag{1.10}$$

$t=0$ 时

$$u(0)=u_0\left(x_0\right), \tag{1.11}$$

所以式 (1.5) 满足式 (1.1).

无黏 Burgers 方程的定解问题

$$\begin{cases} u_t+(0.5u^2)_x=0, \\ u_0(x)=\cos(\pi x), \quad x\in[-1,1]. \end{cases} \tag{2.1}$$

此时 $a(u)=u$, 爆破点

$$t^*=-\frac{1}{a'u_0'}=\frac{1}{\pi\sin(\pi x_0)}, \tag{2.2}$$

只在 $x>0$ 的部分会出现爆破, 最快达到爆破的点为 $x_0=0.5$, 经历时间 $t_0^*=\frac{1}{\pi}$.

Burgers 方程 Riemann 问题的弱解

弱解满足的方程为

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t\boldsymbol{U}+\phi_x\boldsymbol{F}(\boldsymbol{U})\right]\mathrm{d}x\,\mathrm{d}t=-\int_{-\infty}^{+\infty}\phi(x,0)\boldsymbol{U}(x,0)\,\mathrm{d}x. \tag{3.1}$$

对 Burgers 方程有

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t=-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. \tag{3.2}$$

激波弱解

$$u(x,t)=\begin{cases}u_L, & x<st,\\ u_R, & x>st.\end{cases} \tag{3.3}$$

其中

$$s=\frac{u_L+u_R}{2}. \tag{3.4}$$

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t && (3.5)\\
=&\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_t u\,\mathrm{d}x\,\mathrm{d}t+\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_x\frac{1}{2}u^2\,\mathrm{d}x\,\mathrm{d}t && (3.6)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}x && (3.7)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.8)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+s\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}(x/s) && (3.9)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.10)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. && (3.11)
\end{aligned}$$

$u_L<u_R$ 弱解

$$u(x,t)=\begin{cases}u_L, & x<st,\\ u_R, & x>st.\end{cases} \tag{3.12}$$

其中

$$s=\frac{u_L+u_R}{2}. \tag{3.13}$$

同样的

$$
\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t && (3.14)\\
=&\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_t u\,\mathrm{d}x\,\mathrm{d}t+\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi_x\frac{1}{2}u^2\,\mathrm{d}x\,\mathrm{d}t && (3.15)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}x && (3.16)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.17)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+s\int_0^{+\infty}\phi(x,x/s)\left(u_L-u_R\right)\mathrm{d}(x/s) && (3.18)\\
&+\frac{1}{2}\int_0^{+\infty}\phi(st,t)\left(u_L^2-u_R^2\right)\mathrm{d}t && (3.19)\\
=&-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x. && (3.20)
\end{aligned}
$$

稀疏波弱解

$$
u(x,t)=\begin{cases}u_L, & x<s_mt\\ u_m, & s_mt\le x\le u_mt\\ \frac{x}{t}, & u_mt\le x\le u_Rt\\ u_R, & x>u_Rt\end{cases} \tag{3.21}
$$

也是一个弱解, 其中 $u_m\in[u_L,u_R]$ 为任意, $s_m=\frac{u_L+u_m}{2}$.

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_t u+\phi_x\frac{1}{2}u^2\right]\mathrm{d}x\,\mathrm{d}t \tag{3.22}$$

$$=-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\left(-u_L\phi\left(x,\frac{x}{s_m}\right)+u_m\phi\left(x,\frac{x}{s_m}\right)\right. \tag{3.23}$$

$$\left.-u_m\phi\left(x,\frac{x}{u_m}\right)+\int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\phi_t\frac{x}{t}\,\mathrm{d}t+u_R\phi\left(x,\frac{x}{u_R}\right)\right)\mathrm{d}x \tag{3.24}$$

$$+\int_0^{+\infty}-\frac{1}{2}\left(u_R^2\phi\left(u_Rt,t\right)+\int_{u_mt}^{u_Rt}\frac{x^2}{t^2}\phi_x(x,t)\,\mathrm{d}x+u_m^2\phi\left(u_mt,t\right)+\left(u_L^2-u_m^2\right)\phi\left(s_mt,t\right)\right)\mathrm{d}t. \tag{3.25}$$

$$=-\int_{-\infty}^{+\infty}\phi(x,0)u(x,0)\,\mathrm{d}x+\int_0^{+\infty}\left(-u_m\phi\left(x,\frac{x}{u_m}\right)+\int_{\frac{x}{u_R}}^{\frac{x}{u_m}}\phi_t\frac{x}{t}\,\mathrm{d}t+u_R\phi\left(x,\frac{x}{u_R}\right)\right)\mathrm{d}x \tag{3.26}$$

$$+\int_0^{+\infty}-\frac{1}{2}\left(u_R^2\phi\left(u_Rt,t\right)+\int_{u_mt}^{u_Rt}\frac{x^2}{t^2}\phi_x(x,t)\,\mathrm{d}x+u_m^2\phi\left(u_mt,t\right)\right)\mathrm{d}t. \tag{3.27}$$