计算流体力学作业 13

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分别推导出二维可压缩 Navier-Stokes（NS）方程 (守恒形式) 和二维不可压缩 NS 方程 (原始变量形式) 在可逆变换 $x = x(\xi,\eta), y = y(\xi,\eta)$ 下的形式（散度形式和非散度形式）. 如果考虑极坐标变换 $(x,y) \to (r,\theta)$, 则给出在 (r,θ) 坐标下方程的形式. 举例说明, 相应的边界条件变换后的形式.

假设坐标 (ξ,η) 是与时间无关的. 对于可逆变换有雅可比行列式

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0. \tag{1.1}$$

根据矩阵关系

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} \tag{1.2}$$

可得

$$\begin{aligned} x_\xi &= \frac{\eta_y}{J}, \qquad x_\eta = -\frac{\xi_y}{J}, \\ y_\xi &= -\frac{\eta_x}{J}, \qquad y_\eta = \frac{\xi_x}{J}. \end{aligned} \tag{1.3}$$

考虑 (x,y) 坐标下的任意守恒型方程组

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0, \tag{1.4}$$

在坐标变换下有

$$\begin{aligned} \frac{\partial E}{\partial x} &= \frac{\partial E}{\partial \xi}\xi_x + \frac{\partial E}{\partial \eta}\eta_x, \\ \frac{\partial F}{\partial y} &= \frac{\partial F}{\partial \xi}\xi_y + \frac{\partial F}{\partial \eta}\eta_y. \end{aligned} \tag{1.5}$$

代入式 (1.4) 可得

$$\frac{\partial U}{\partial t}+\left(\frac{\partial E}{\partial \xi}\xi_x+\frac{\partial F}{\partial \xi}\xi_y\right)+\left(\frac{\partial E}{\partial \eta}\eta_x+\frac{\partial F}{\partial \eta}\eta_y\right)=0 \tag{1.6}$$

为将上式变为守恒形式, 利用式 (1.3) 可得

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(\frac{E\xi_x+F\xi_y}{J}\right)&=\frac{1}{J}\left(\frac{\partial E}{\partial \xi}\xi_x+\frac{\partial F}{\partial \xi}\xi_y\right)+E\frac{\partial}{\partial \xi}\left(\frac{1}{J}\xi_x\right)+F\frac{\partial}{\partial \xi}\left(\frac{1}{J}\xi_y\right)\\ \frac{\partial}{\partial \eta}\left(\frac{E\eta_x+F\eta_y}{J}\right)&=\frac{1}{J}\left(\frac{\partial E}{\partial \eta}\eta_x+\frac{\partial F}{\partial \eta}\eta_y\right)+E\frac{\partial}{\partial \eta}\left(\frac{1}{J}\eta_x\right)+F\frac{\partial}{\partial \eta}\left(\frac{1}{J}\eta_y\right)\end{aligned} \tag{1.7}$$

由式 (1.3) 可得

$$\begin{aligned}\frac{\partial}{\partial \xi}\left(\frac{1}{J}\xi_x\right)+\frac{\partial}{\partial \eta}\left(\frac{1}{J}\eta_x\right)&=0,\\ \frac{\partial}{\partial \xi}\left(\frac{1}{J}\xi_y\right)+\frac{\partial}{\partial \eta}\left(\frac{1}{J}\eta_y\right)&=0.\end{aligned} \tag{1.8}$$

将式 (1.7) 中的两个式子相加, 代入式 (1.6) 得到

$$\frac{\partial}{\partial t}\left(\frac{U}{J}\right)+\frac{\partial}{\partial \xi}\left(\frac{E\xi_x+F\xi_y}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{E\eta_x+F\eta_y}{J}\right)=0 \tag{1.9}$$

即为 (ξ,η) 坐标下守恒形式的方程组. [2, P28]

二维可压缩 NS 方程 (守恒形式)

下面具体考虑二维可压缩 NS 方程组. 其中,

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho\varepsilon\end{pmatrix},\quad E=\begin{pmatrix}\rho u\\ \rho u^2+p-\tau_{xx}\\ \rho uv-\tau_{xy}\\ (\rho\varepsilon+p)u-u\tau_{xx}-v\tau_{xy}+q_x\end{pmatrix},\quad F=\begin{pmatrix}\rho v\\ \rho uv-\tau_{xy}\\ \rho v^2+p-\tau_{yy}\\ (\rho\varepsilon+p)v-u\tau_{xy}-v\tau_{yy}+q_y\end{pmatrix}. \tag{1.10}$$

设

$$u^*=u\xi_x+v\xi_y \tag{1.11}$$

$$v^*=u\eta_x+v\eta_y. \tag{1.12}$$

代入式 (1.9), 可以得到 (ξ,η) 坐标下的方程组为

$$\frac{\partial}{\partial t}\hat{U}+\frac{\partial}{\partial \xi}\hat{E}+\frac{\partial}{\partial \eta}\hat{F}=0, \tag{1.13}$$

其中,

$$\hat{U}=\frac{1}{J}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho\varepsilon\end{pmatrix},\quad \hat{E}=\frac{1}{J}\begin{pmatrix}\rho u^*\\ \rho uu^*+p\xi_x-\tau_{xx}\xi_x-\tau_{xy}\xi_y\\ \rho vu^*+p\xi_y-\tau_{xy}\xi_x-\tau_{yy}\xi_y\\ (\rho\varepsilon+p)u^*-\left(u\tau_{xx}+v\tau_{xy}\right)\xi_x-\left(u\tau_{xy}+v\tau_{yy}\right)\xi_y+q_x\xi_x+q_y\xi_y\end{pmatrix}, \tag{1.14}$$

$$\hat{F}=\frac{1}{J}\begin{pmatrix}\rho v^*\\ \rho uv^*+p\eta_x-\tau_{xx}\eta_x-\tau_{xy}\eta_y\\ \rho vv^*+p\eta_y-\tau_{xy}\eta_x-\tau_{yy}\eta_y\\ (\rho\varepsilon+p)v^*-\left(u\tau_{xx}+v\tau_{xy}\right)\eta_x-\left(u\tau_{xy}+v\tau_{yy}\right)\eta_y+q_x\eta_x+q_y\eta_y\end{pmatrix}. \tag{1.15}$$

$$\begin{aligned}
\tau_{xx}&=\frac{2}{3}\mu\left(2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)=\frac{2}{3}\mu\left[2\left(u_\xi\xi_x+u_\eta\eta_x\right)-\left(v_\xi\xi_y+v_\eta\eta_y\right)\right],\\
\tau_{xy}&=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\mu\left(u_\xi\xi_y+u_\eta\eta_y+v_\xi\xi_x+v_\eta\eta_x\right),\\
\tau_{yy}&=\frac{2}{3}\mu\left(2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)=\frac{2}{3}\mu\left[2\left(v_\xi\xi_y+v_\eta\eta_y\right)-\left(u_\xi\xi_x+u_\eta\eta_x\right)\right],\\
q_x&=-k\frac{\partial T}{\partial x}=-k\left(\frac{\partial T}{\partial\xi}\xi_x+\frac{\partial T}{\partial\eta}\eta_x\right),\\
q_y&=-k\frac{\partial T}{\partial y}=-k\left(\frac{\partial T}{\partial\xi}\xi_y+\frac{\partial T}{\partial\eta}\eta_y\right).
\end{aligned} \tag{1.16}$$

二维不可压缩 NS 方程 (原始变量形式)

如果考虑二维不可压缩 NS 方程, 假设 ρ 为常数, 暂时不考虑体力项, 当体力有势时, 体力项可以与压力项合并在一起, 定义变量

$$\tilde{p}=\frac{p}{\rho} \tag{1.17}$$

不考虑热传导, 式 (1.4) 中的各变量为

$$U=\begin{pmatrix}0\\ u\\ v\end{pmatrix},\quad E=\begin{pmatrix}u\\ u^2+\tilde{p}-\tau'_{xx}\\ uv-\tau'_{xy}\end{pmatrix},\quad F=\begin{pmatrix}v\\ uv-\tau'_{xy}\\ v^2+\tilde{p}-\tau'_{yy}\end{pmatrix}. \tag{1.18}$$

根据式 (1.9) 可以得到, 坐标 (ξ,η) 下的守恒型方程组仍为式 (1.13) 的形式, 其中

$$\hat{U}=\frac{1}{J}\begin{pmatrix}0\\ u\\ v\end{pmatrix},\quad \hat{E}=\frac{1}{J}\begin{pmatrix}u^*\\ uu^*+\tilde{p}\xi_x-\tau'_{xx}\xi_x-\tau'_{xy}\xi_y\\ vu^*+\tilde{p}\xi_y-\tau'_{xy}\xi_x-\tau'_{yy}\xi_y\end{pmatrix} \tag{1.19}$$

$$\hat{F} = \frac{1}{J}\begin{pmatrix} v^* \\ \rho u v^* + \tilde{p}\eta_x - \tau'_{xx}\eta_x - \tau'_{xy}\eta_y \\ \rho v v^* + \tilde{p}\eta_y - \tau'_{xy}\eta_x - \tau'_{yy}\eta_y \end{pmatrix}. \tag{1.20}$$

对于不可压缩流体,

$$\begin{aligned} \tau'_{xx} &= 2\nu\frac{\partial u}{\partial x} = 2\nu\left(u_\xi \xi_x + u_\eta \eta_x\right), \\ \tau'_{xy} &= \nu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \nu\left(u_\xi \xi_y + u_\eta \eta_y + v_\xi \xi_x + v_\eta \eta_x\right), \\ \tau'_{yy} &= 2\nu\frac{\partial v}{\partial y} = 2\nu\left(v_\xi \xi_y + v_\eta \eta_y\right). \end{aligned} \tag{1.21}$$

若考虑非守恒形式的二维可压缩 NS 方程组, 暂时不考虑能量方程, 则有

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + v\dfrac{\partial \rho}{\partial y} + \rho\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0, \\ \rho\dfrac{\partial u}{\partial t} + \rho u\dfrac{\partial u}{\partial x} + \rho v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y}, \\ \rho\dfrac{\partial v}{\partial t} + \rho u\dfrac{\partial v}{\partial x} + \rho v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y}. \end{cases} \tag{1.22}$$

将坐标变换关系代入可得

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u\left(\dfrac{\partial \rho}{\partial \xi}\xi_x + \dfrac{\partial \rho}{\partial \eta}\eta_x\right) + v\left(\dfrac{\partial \rho}{\partial \xi}\xi_y + \dfrac{\partial \rho}{\partial \eta}\eta_y\right) + \rho\left(\dfrac{\partial u}{\partial \xi}\xi_x + \dfrac{\partial u}{\partial \eta}\eta_x + \dfrac{\partial v}{\partial \xi}\xi_y + \dfrac{\partial v}{\partial \eta}\eta_y\right) = 0 \\ \rho\dfrac{\partial u}{\partial t} + \rho u\left(\dfrac{\partial u}{\partial \xi}\xi_x + \dfrac{\partial u}{\partial \eta}\eta_x\right) + \rho v\left(\dfrac{\partial u}{\partial \xi}\xi_y + \dfrac{\partial u}{\partial \eta}\eta_y\right) = -\dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial \xi}\xi_x + \dfrac{\partial p}{\partial \eta}\eta_x\right) + \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} \\ \rho\dfrac{\partial v}{\partial t} + \rho u\left(\dfrac{\partial v}{\partial \xi}\xi_x + \dfrac{\partial v}{\partial \eta}\eta_x\right) + \rho v\left(\dfrac{\partial v}{\partial \xi}\xi_y + \dfrac{\partial v}{\partial \eta}\eta_y\right) = -\dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial \xi}\xi_y + \dfrac{\partial p}{\partial \eta}\eta_y\right) + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} \end{cases} \tag{1.23}$$

若考虑非守恒形式的二维不可压 NS 方程组, 并假设 ν 为常数, 则有

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} = \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{1.24}$$

将坐标变换关系代入可以得到

$$
\begin{cases}
\dfrac{\partial u}{\partial \xi}\xi_x + \dfrac{\partial u}{\partial \eta}\eta_x + \dfrac{\partial v}{\partial \xi}\xi_y + \dfrac{\partial v}{\partial \eta}\eta_y = 0 \\
\dfrac{\partial u}{\partial t} + u\left(\dfrac{\partial u}{\partial \xi}\xi_x + \dfrac{\partial u}{\partial \eta}\eta_x\right) + v\left(\dfrac{\partial u}{\partial \xi}\xi_y + \dfrac{\partial u}{\partial \eta}\eta_y\right) + \dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial \xi}\xi_x + \dfrac{\partial p}{\partial \eta}\eta_x\right) = \\
\nu\Bigg[\left(\dfrac{\partial^2 u}{\partial \xi^2}\xi_x^2 + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta}\xi_x\eta_x + \dfrac{\partial^2 u}{\partial \eta^2}\eta_x^2\right) + \left(\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi_x}{\partial \xi} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta_x}{\partial \xi}\right)\xi_x + \left(\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi_\eta}{\partial \xi} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta_\eta}{\partial \xi}\right)\eta_x \\
+\left(\dfrac{\partial^2 u}{\partial \xi^2}\xi_y^2 + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta}\xi_y\eta_y + \dfrac{\partial^2 u}{\partial \eta^2}\eta_y^2\right) + \left(\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi_y}{\partial \xi} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta_y}{\partial \xi}\right)\xi_y + \left(\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi_\eta}{\partial \xi} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta_\eta}{\partial \xi}\right)\eta_y\Bigg] \\
\dfrac{\partial v}{\partial t} + u\left(\dfrac{\partial v}{\partial \xi}\xi_x + \dfrac{\partial v}{\partial \eta}\eta_x\right) + v\left(\dfrac{\partial v}{\partial \xi}\xi_y + \dfrac{\partial v}{\partial \eta}\eta_y\right) + \dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial \xi}\xi_y + \dfrac{\partial p}{\partial \eta}\eta_y\right) = \\
\nu\Bigg[\left(\dfrac{\partial^2 u}{\partial \xi^2}\xi_x^2 + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta}\xi_x\eta_x + \dfrac{\partial^2 u}{\partial \eta^2}\eta_x^2\right) + \left(\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi_x}{\partial \xi} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta_x}{\partial \xi}\right)\xi_x + \left(\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi_\eta}{\partial \xi} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta_\eta}{\partial \xi}\right)\eta_x \\
+\left(\dfrac{\partial^2 v}{\partial \xi^2}\xi_y^2 + 2\dfrac{\partial^2 v}{\partial \xi \partial \eta}\xi_y\eta_y + \dfrac{\partial^2 v}{\partial \eta^2}\eta_y^2\right) + \left(\dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi_y}{\partial \xi} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta_y}{\partial \xi}\right)\xi_y + \left(\dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi_\eta}{\partial \xi} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta_\eta}{\partial \xi}\right)\eta_y\Bigg]
\end{cases}
\tag{1.25}
$$

极坐标变换

考虑极坐标 (r, θ), 极坐标变换下有如下关系

$$
r_x = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, \quad r_y = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \tag{1.26}
$$

$$
\theta_x = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}, \tag{1.27}
$$

$$
J = \frac{1}{r}. \tag{1.28}
$$

将上述关系代入式 (1.9) 可得到极坐标下二维可压缩 NS 方程的守恒形式

$$
\hat{U} = r\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho\varepsilon \end{pmatrix}, \tag{1.29}
$$

$$\hat{E}=r\begin{pmatrix}\rho u^* \\ \rho u u^*+p\cos\theta-\tau_{xx}\cos\theta-\tau_{xy}\sin\theta \\ \rho v u^*+p\sin\theta-\tau_{xy}\cos\theta-\tau_{yy}\sin\theta \\ (\rho\varepsilon+p)u^*-\big(u\tau_{xx}+v\tau_{xy}\big)\cos\theta-\big(u\tau_{xy}+v\tau_{yy}\big)\sin\theta-kT_\xi\end{pmatrix}, \tag{1.30}$$

$$\hat{F}=r\begin{pmatrix}\rho v^* \\ \rho u v^*-p\sin\theta+\tau_{xx}\sin\theta-\tau_{xy}\cos\theta \\ \rho v v^*+p\cos\theta+\tau_{xy}\sin\theta-\tau_{yy}\cos\theta \\ (\rho\varepsilon+p)v^*-\big(u\tau_{xx}-v\tau_{xy}\big)\sin\theta-\big(u\tau_{xy}+v\tau_{yy}\big)\cos\theta-kT_\eta\end{pmatrix}. \tag{1.31}$$

其中,

$$u^*=u\cos\theta+v\sin\theta, \tag{1.32}$$

$$v^*=-u\sin\theta+v\cos\theta, \tag{1.33}$$

$$\tau_{xx}=\frac{2}{3}\mu\left[2\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)-\left(\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial\theta}\frac{\cos\theta}{r}\right)\right], \tag{1.34}$$

$$\tau_{xy}=\mu\left[\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}+\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial\theta}\frac{\sin\theta}{r}\right], \tag{1.35}$$

$$\tau_{yy}=\frac{2}{3}\mu\left[2\left(\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial\theta}\frac{\cos\theta}{r}\right)-\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\right]. \tag{1.36}$$

对于二维不可压 NS 方程, 可得到其守恒形式为

$$\hat{U}=r\begin{pmatrix}0\\ u\\ v\end{pmatrix}, \tag{1.37}$$

$$\hat{E}=r\begin{pmatrix}u^* \\ uu^*+\tilde{p}\cos\theta-\tau'_{xx}\cos\theta-\tau'_{xy}\sin\theta \\ vu^*+\tilde{p}\sin\theta-\tau'_{xy}\cos\theta-\tau'_{yy}\sin\theta\end{pmatrix}, \tag{1.38}$$

$$\hat{F}=r\begin{pmatrix}v^* \\ uv^*-\tilde{p}\sin\theta+\tau'_{xx}\sin\theta-\tau'_{xy}\cos\theta \\ vv^*+\tilde{p}\cos\theta+\tau'_{xy}\sin\theta-\tau'_{yy}\cos\theta\end{pmatrix}. \tag{1.39}$$

其中,

$$u^*=u\cos\theta+v\sin\theta, \tag{1.40}$$

$$v^* = -u\sin\theta + v\cos\theta, \tag{1.41}$$

$$\tau'_{xx} = 2\nu\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right), \tag{1.42}$$

$$\tau'_{xy} = \nu\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r} + \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial\theta}\frac{\sin\theta}{r}\right), \tag{1.43}$$

$$\tau'_{yy} = 2\nu\left(\frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial\theta}\frac{\cos\theta}{r}\right). \tag{1.44}$$

为了得到极坐标下的速度分量 (u_r, u_θ) 的方程, 做替换

$$u = u_r\cos\theta - u_\theta\sin\theta, \tag{1.45}$$

$$v = u_r\sin\theta + u_\theta\cos\theta. \tag{1.46}$$

最终可以得到

$$\begin{cases} \dfrac{\partial(ru_r)}{\partial r} + \dfrac{\partial u_\theta}{\partial\theta} = 0, \\ \dfrac{\partial u_r}{\partial t} + u_r\dfrac{\partial u_r}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial u_r}{\partial\theta} - \dfrac{u_\theta^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + \nu\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_r}{\partial r}\right) - \dfrac{u_r}{r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u_r}{\partial\theta^2} - \dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial\theta}\right], \\ \dfrac{\partial u_\theta}{\partial t} + u_r\dfrac{\partial u_\theta}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r u_\theta}{r} = -\dfrac{1}{\rho r}\dfrac{\partial p}{\partial\theta} + \nu\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u_\theta}{\partial r}\right) - \dfrac{u_\theta}{r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u_\theta}{\partial\theta^2} + \dfrac{2}{r^2}\dfrac{\partial u_r}{\partial\theta}\right]. \end{cases} \tag{1.47}$$

对于可压缩 NS 方程, 利用拉梅系数, 通过分析各算子在曲线坐标下的形式可得 [1, P190]

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \dfrac{1}{r}\dfrac{\partial(\rho r u_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial(\rho u_\theta)}{\partial\theta} = 0, \\ \rho\left(\dfrac{\mathrm{D}u_r}{\mathrm{D}t} - \dfrac{u_\theta^2}{r}\right) = \dfrac{1}{r}\left[\dfrac{\partial(rP_{rr})}{\partial r} + \dfrac{P_{r\theta}}{\partial\theta} - P_{\theta\theta}\right], \\ \rho\left(\dfrac{\mathrm{D}u_\theta}{\mathrm{D}t} + \dfrac{u_r u_\theta}{r}\right) = \dfrac{1}{r}\left[\dfrac{\partial(rP_{r\theta})}{\partial r} + \dfrac{P_{\theta\theta}}{\partial\theta} + P_{r\theta}\right]. \end{cases} \tag{1.48}$$

其中

$$\begin{aligned} P_{rr} &= -p + 2\mu\left(\frac{\partial u_r}{\partial r} - \frac{1}{3}\nabla\cdot\boldsymbol{u}\right) \\ P_{\theta\theta} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} - \frac{1}{3}\nabla\cdot\boldsymbol{u}\right) \\ P_{r\theta} &= \mu\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right) \\ \frac{\mathrm{D}}{\mathrm{D}t} &= \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} \\ \nabla\cdot\boldsymbol{u} &= \frac{1}{r}\left[\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta}\right] \end{aligned} \tag{1.49}$$

边界条件

对于无穷远边界条件

$$r \to \infty, \quad \boldsymbol{u} = \boldsymbol{u}_\infty, \quad \rho = \rho_\infty, \quad p = p_\infty \tag{1.50}$$

对于一般的边界条件, 例如对于某一个面, 法向为 $\boldsymbol{n}$, 切向为 $\boldsymbol{\tau}$ 则把速度投影到法向和切向

$$\boldsymbol{u}_n = \boldsymbol{n} \cdot \boldsymbol{u}, \quad \boldsymbol{u}_\tau = \boldsymbol{\tau} \cdot \boldsymbol{u}. \tag{1.51}$$

然后按照无滑移或无穿透边界条件给定边界条件.

B

给定三维不可压 Navier-Stokes（INS）方程的原始变量形式的定解问题（讲义“CFDLect08-incom_cn.pdf”中第 7 页）, 引入向量势函数 $\boldsymbol{A}$ 和涡向量函数 $\boldsymbol{\omega}$, 推导三维 INS 方程的涡向量势公式（讲义“CFDLect09-incom_cn.pdf”中第 35 页）.

三维 INS 方程的原始变量形式定解问题为

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla \dfrac{p}{\rho} + \nu \nabla^2 \boldsymbol{u} + \dfrac{\boldsymbol{f}_B}{\rho}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_b, \quad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0. \end{cases} \tag{2.1}$$

上式中已经考虑 ρ 为常数. 引入向量势函数 $\boldsymbol{A}$ 和涡向量函数 $\boldsymbol{\omega}$

$$\boldsymbol{u} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}. \tag{2.2}$$

利用关系式

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \frac{|\boldsymbol{u}|^2}{2} + \boldsymbol{\omega} \times \boldsymbol{u}, \tag{2.3}$$

三维 INS 方程组的动量方程可以化为兰姆-葛罗米柯形式

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \frac{|\boldsymbol{u}|^2}{2} + \boldsymbol{\omega} \times \boldsymbol{u} = -\nabla \frac{p}{\rho} + \nu \nabla^2 \boldsymbol{u} + \frac{\boldsymbol{f}_B}{\rho} \tag{2.4}$$

对上式取旋度, 假设体力 $\boldsymbol{f}_B$ 有势, 并利用

$$\nabla \times \nabla^2 \boldsymbol{u} = \nabla^2 \boldsymbol{\omega}, \tag{2.5}$$

则

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{\omega}. \tag{2.6}$$

三维 INS 方程的涡向量势公式

$$\begin{cases} \dfrac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{\omega}, \\ \boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0, \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_b, \quad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0. \end{cases} \tag{2.7}$$

C

考虑一维网格生成问题. 设逻辑区域 (也称为参考区域) $\Omega_c = \{\xi : 0 \leq \xi \leq 1\}$ 到物理区域 $\Omega_p = \{x : a \leq x \leq b\}$ 的坐标变换 $\xi = \xi(x)$ 满足:

$$\xi_{xx} = P, \tag{3.1}$$

P 为常数. 分析: 右端项 P 对生成物理区域 Ω_p 的网格的影响.

若 $P = 0$, 则网格是均匀的. 对式 (3.1) 积分一次可得

$$\xi_x = C, \tag{3.2}$$

其中 C 是常数, 且不等于 0, 否则无法划分网格. 再积分一次得

$$\xi = Cx + D, \tag{3.3}$$

其中 D 也为常数. 根据边界条件, 端点处重合, 所以

$$\begin{cases} C \times a + D = 0, \\ C \times b + D = 1. \end{cases} \quad \text{or} \quad \begin{cases} C \times b + D = 0, \\ C \times a + D = 1. \end{cases} \tag{3.4}$$

解得

$$\begin{cases} D = -\frac{a}{b-a}, \\ C = \frac{1}{b-a}. \end{cases} \quad \text{or} \quad \begin{cases} D = \frac{b}{b-a}, \\ C = \frac{1}{a-b}. \end{cases} \tag{3.5}$$

若 P 不为 0, 则两次积分后是二次函数

$$\xi = \frac{1}{2}Px^2 + Cx + D. \tag{3.6}$$

必须要求此二次函数是单调的, 否则会出现一个 x 点对应不同的 ξ. 极值点为 $x_0 = -\frac{C}{P}$,

$$x_0 < a, \quad \text{or} \quad x_0 > b. \tag{3.7}$$

然后要满足在端点处重合

$$\begin{cases} \frac{1}{2}Pa^2 + C \times a + D = 0, \\ \frac{1}{2}Pb^2 + C \times b + D = 1. \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{2}Pb^2 + C \times b + D = 0, \\ \frac{1}{2}Pa^2 + C \times a + D = 1. \end{cases} \tag{3.8}$$

P 越大, 式 (3.6) 的非线性性越强, 网格越不均匀, 反之, P 越小, 则越接近 $P = 0$ 的线性情况. 另外,P 的正负将影响到网格是在哪边更密, 哪边更稀疏.

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