计算流体力学上机作业 3

College of Engineering 2001111690 袁磊祺

June 11, 2021

写程序计算 2D Euler 方程组的双马赫反射问题或前台阶问题（见讲义【CFDLect06-com03_cn.pdf】的第 106-108 页）. 数值格式: “Roe 或 HLLC 或 HLL 解法器或 KFVS” + “线性重构” + “显示的 Runge-Kutta 时间离散”.

计算动图可点击 https://www.bilibili.com/video/BV1cp4y1t7Bc 查看.

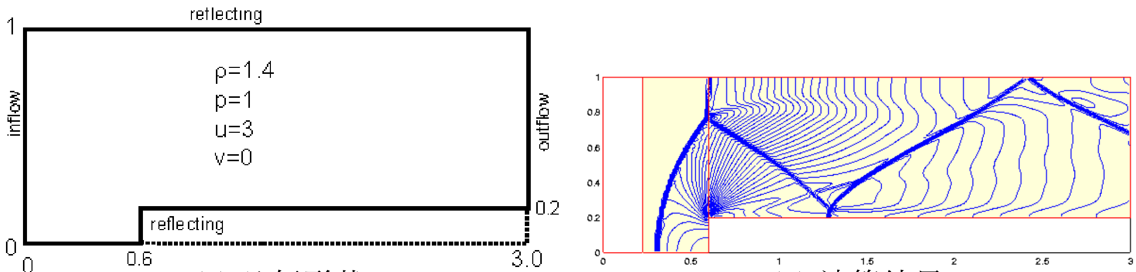
代码可点击 https://github.com/circlelq/Computational-Fluid-Dynamics/tree/main/code2 查看.

问题描述

前台阶问题

几何形状和结果 (密度等值线) 见示意图图 1.1.

初始时刻, 区域内充满均匀流 $(\rho, u, v, p) = (1.4, 3, 0, 1)$. 输出时间 $t = 4$. 上下边界为反射边界; 左边界为入流边界 (即左边界处流动为初始均匀流); 右边界为出流.



(a) 几何形状. (b) 计算结果.

图 1.1. 几何形状和计算结果.

控制方程

二维无黏流动 Euler 方程组

采用理想气体模型. 不考虑体力、外部热源和流体源 (汇), 二维 Euler 方程组:

$$\boldsymbol{u}_t + \boldsymbol{f}_x + \boldsymbol{g}_y = \boldsymbol{0}, \tag{2.1}$$

其中

$$\boldsymbol{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}. \tag{2.2}$$

其中 ρ 为流体密度, p 为压力, u, v 为 x 和 y 方向上速度分量, E 为单位体积流体总能. 理想气体状态方程:

$$p = (\gamma - 1)\rho e = (\gamma - 1)\left[E - \frac{1}{2}\rho\left(u^2 + v^2\right)\right]. \tag{2.3}$$

焓

$$H = (E+p)/\rho = \frac{\gamma p}{\rho(\gamma - 1)} + \frac{1}{2}\left(u^2 + v^2\right), \tag{2.4}$$

声速

$$a = \sqrt{\gamma p/\rho} = \sqrt{(\gamma - 1)\left[H - \frac{1}{2}(u^2 + v^2)\right]}. \tag{2.5}$$

非线性 Jacobian 系数矩阵、特征值和特征矢量矩阵

把二维 Euler 方程组式 (2.1) 写成非守恒型形式:[张德良, 2010, P289]

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial y} = \boldsymbol{0} \tag{2.6}$$

其中 $\boldsymbol{A}$, $\boldsymbol{B}$ 为非守恒型方程组的非线性 Jacobian 系数矩阵, 由于流通量矢量 $\boldsymbol{f}$, $\boldsymbol{g}$ 是流动量矢量 $\boldsymbol{u}$ 的一次齐次函数, 则有:

$$\boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} = \frac{\partial \boldsymbol{f}}{\partial x}, \quad \boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial y} = \frac{\partial \boldsymbol{g}}{\partial y}. \tag{2.7}$$

(1) 非线性 Jacobian 系数矩阵

$$\boldsymbol{A}(\boldsymbol{u})=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2+\frac{\gamma-1}{2}v^2 & (3-\gamma)u & (1-\gamma)v & \gamma-1 \\ -uv & v & u & 0 \\ u\left[\frac{\gamma-2}{2}\left(u^2+v^2\right)-\frac{a^2}{\gamma-1}\right] & \frac{3-2\gamma}{2}u^2+\frac{1}{2}v^2+\frac{a^2}{\gamma-1} & (1-\gamma)uv & \gamma u \end{pmatrix} \tag{2.8}$$

$$\boldsymbol{B}(\boldsymbol{u})=\begin{pmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \frac{\gamma-1}{2}u^2+\frac{\gamma-3}{2}v^2 & (1-\gamma)u & (3-\gamma)v & \gamma-1 \\ v\left[\frac{\gamma-2}{2}\left(u^2+v^2\right)-\frac{a^2}{\gamma-1}\right] & (1-\gamma)uv & \frac{1}{2}u^2+\frac{3-2\gamma}{2}v^2+\frac{a^2}{\gamma-1} & \gamma v \end{pmatrix} \tag{2.9}$$

(2) 系数矩阵 $\boldsymbol{A}$, $\boldsymbol{B}$ 的特征值矩阵

$$\boldsymbol{\Lambda}_x=\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{L}^{-1}=\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}=\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u-a & 0 \\ 0 & 0 & 0 & u+a \end{pmatrix} \tag{2.10}$$

$$\boldsymbol{\Lambda}_y=\boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{L}^{-1}=\begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{pmatrix}=\begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v-a & 0 \\ 0 & 0 & 0 & v+a \end{pmatrix} \tag{2.11}$$

(3) 系数矩阵 $\boldsymbol{A}$, $\boldsymbol{B}$ 的右特征矢量矩阵

由右特征矢量矩阵 $\boldsymbol{A}\boldsymbol{R}_x=\lambda\boldsymbol{R}_x,\boldsymbol{B}\boldsymbol{R}_y=\mu\boldsymbol{R}_y$ 得到它们的右特征矢量矩阵:

$$\boldsymbol{R}_x=\begin{pmatrix} 1 & 0 & 1 & 1 \\ u & 0 & u-a & u+a \\ 0 & 1 & v & v \\ \frac{1}{2}\left(u^2-v^2\right) & v & h-au & h+au \end{pmatrix} \tag{2.12}$$

$$\boldsymbol{R}_y=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & u & u \\ 0 & v & v-a & v+a \\ u & \frac{1}{2}\left(v^2-u^2\right) & h-av & h+av \end{pmatrix} \tag{2.13}$$

(4) 系数矩阵 A, B 的左特征矢量矩阵

由左特征矢量公式 $\boldsymbol{L}_x\boldsymbol{A}=\lambda\boldsymbol{L}_x$, $\boldsymbol{L}_y\boldsymbol{B}=\mu\boldsymbol{L}_y$, 可得到左特

$$\boldsymbol{L}_x=\begin{pmatrix}1-\frac{\gamma-1}{2a^2}\left(u^2+v^2\right) & \frac{\gamma-1}{a^2}u & \frac{\gamma-1}{a^2}v & \frac{\gamma-1}{a^2}\\ -\frac{\gamma-1}{2a^2}v\left(u^2+v^2\right) & \frac{\gamma-1}{a^2}uv & 1+\frac{\gamma-1}{a^2}v^2 & -\frac{\gamma-1}{a^2}v\\ \frac{\gamma-1}{4a^2}\left(u^2+v^2\right)+\frac{u}{2a} & -\frac{\gamma-1}{2a^2}u-\frac{1}{2a} & -\frac{\gamma-1}{2a^2}v & \frac{\gamma-1}{2a^2}\\ \frac{\gamma-1}{4a^2}\left(u^2+v^2\right)-\frac{u}{2a} & -\frac{\gamma-1}{2a^2}u+\frac{1}{2a} & -\frac{\gamma-1}{2a^2}v & \frac{\gamma-1}{2a^2}\end{pmatrix} \tag{2.14}$$

$$\boldsymbol{L}_y=\begin{pmatrix}-\frac{\gamma-1}{2a^2}u\left(u^2+v^2\right) & 1+\frac{\gamma-1}{a^2}u^2 & \frac{\gamma-1}{a^2}uv & -\frac{\gamma-1}{a^2}u\\ 1-\frac{\gamma-1}{2a^2}\left(u^2+v^2\right) & \frac{\gamma-1}{a^2}u & \frac{\gamma-1}{a^2}v & -\frac{\gamma-1}{a^2}\\ \frac{\gamma-1}{4a^2}\left(u^2+v^2\right)+\frac{1}{2a}v & -\frac{\gamma-1}{2a^2}u & -\frac{\gamma-1}{2a^2}-\frac{1}{2a} & \frac{\gamma-1}{2a^2}\\ \frac{\gamma-1}{4a^2}\left(u^2+v^2\right)-\frac{1}{2a}v & -\frac{\gamma-1}{2a^2}u & -\frac{\gamma-1}{2a^2}v+\frac{1}{2a} & \frac{\gamma-1}{2a^2}\end{pmatrix} \tag{2.15}$$

$b_1=b_2\frac{\left(u^2+v^2\right)}{2}, b_2=\frac{\gamma-1}{a^2}$, 则式可写成:

$$\boldsymbol{L}_x=\begin{pmatrix}1-b_1 & b_2u & b_2v & -b_2\\ -b_1v & b_2uv & 1+b_2v^2 & -b_2v\\ \frac{1}{2}\left(b_1+\frac{u}{a}\right) & -\frac{1}{2}\left(b_2u+\frac{1}{a}\right) & -\frac{1}{2}b_2v & \frac{1}{2}b_2\\ \frac{1}{2}\left(b_1-\frac{u}{a}\right) & -\frac{1}{2}\left(b_2u-\frac{1}{a}\right) & -\frac{1}{2}b_2v & \frac{1}{2}b_2\end{pmatrix}, \tag{2.16}$$

$$\boldsymbol{L}_y=\begin{pmatrix}-b_1u & 1+b_2u^2 & b_2uv & -b_2u\\ 1-b_1 & b_2u & b_2v & -b_2\\ \frac{1}{2}\left(b_1+\frac{v}{a}\right) & -\frac{1}{2}b_2u & -\frac{1}{2}\left(b_2v+\frac{1}{a}\right) & \frac{1}{2}b_2\\ \frac{1}{2}\left(b_1-\frac{v}{a}\right) & -\frac{1}{2}b_2u & -\frac{1}{2}\left(b_2v-\frac{1}{a}\right) & \frac{1}{2}b_2\end{pmatrix}. \tag{2.17}$$

Roe 算法

对于理性气体, 通过如下方法构造 $\hat{\boldsymbol{A}}$:

$$\begin{cases}\bar{\rho}=\dfrac{\sqrt{\rho_r}\rho_\ell+\sqrt{\rho_\ell}\rho_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}}=\sqrt{\rho_\ell\rho_r}=\left[\dfrac{1}{2}\left(\sqrt{\rho_r}+\sqrt{\rho_\ell}\right)\right]^2,\\ \bar{u}=\dfrac{\sqrt{\rho_\ell}u_\ell+\sqrt{\rho_r}u_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}},\\ \bar{v}=\dfrac{\sqrt{\rho_\ell}v_\ell+\sqrt{\rho_r}v_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}},\\ \bar{H}=\dfrac{\sqrt{\rho_\ell}H_\ell+\sqrt{\rho_r}H_r}{\sqrt{\rho_\ell}+\sqrt{\rho_r}}.\end{cases} \tag{3.1}$$

$$\hat{\boldsymbol{A}}=\boldsymbol{A}(\bar{\boldsymbol{u}}). \tag{3.2}$$

$$\hat{\boldsymbol{f}}\left(\boldsymbol{u}_j,\boldsymbol{u}_{j+1}\right)=\frac{\boldsymbol{f}\left(\boldsymbol{u}_j\right)+\boldsymbol{f}\left(\boldsymbol{u}_{j+1}\right)}{2}-\frac{1}{2}\left|\hat{\boldsymbol{A}}_{j+1/2}\right|\left(\boldsymbol{u}_{j+1}-\boldsymbol{u}_j\right). \tag{3.3}$$

$$\hat{\boldsymbol{g}}\left(\boldsymbol{u}_j, \boldsymbol{u}_{j+1}\right) = \frac{\boldsymbol{g}\left(\boldsymbol{u}_j\right) + \boldsymbol{g}\left(\boldsymbol{u}_{j+1}\right)}{2} - \frac{1}{2}\left|\hat{\boldsymbol{B}}_{j+1/2}\right|\left(\boldsymbol{u}_{j+1} - \boldsymbol{u}_j\right). \tag{3.4}$$

其中 $|\hat{\boldsymbol{A}}|$ 定义为: $|\hat{\boldsymbol{A}}| = \boldsymbol{R}|\hat{\boldsymbol{\Lambda}}|\boldsymbol{R}^{-1}|$, $|\hat{\boldsymbol{\Lambda}}\ | = \operatorname{diag}\left\{\left|\hat{\lambda}_1\right|, \cdots, \left|\hat{\lambda}_m\right|\right\}$, 可以得到半离散格式 [张德良, 2010, P294]

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_{i,j} + \frac{1}{h}\left(\hat{\boldsymbol{f}}_{i+\frac{1}{2},j} - \hat{\boldsymbol{f}}_{i-\frac{1}{2},j}\right) + \frac{1}{h}\left(\hat{\boldsymbol{g}}_{i,j+\frac{1}{2}} - \hat{\boldsymbol{g}}_{i,j-\frac{1}{2}}\right) = 0. \tag{3.5}$$

然后再用三阶 TVD 性质的 RK 时间差分格式

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L\left(u^n\right), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L\left(u^{(1)}\right), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L\left(u^{(2)}\right). \end{aligned} \tag{3.6}$$

其中 L 是空间离散算符.

边界条件

上下边界为反射边界; 左边界为入流边界 (即左边界处流动为初始均匀流); 右边界为出流. 对于刚性壁面, 满足华裔反射边界条件: 即在隔壁面法向方向上各物理量取一阶导数为 0, 这可以在壁面发现方向上采用镜面反射原则取值. 由此克制, 在水平壁面上 $v_w = 0$, 竖直壁面上 $u_w = 0$. 在各壁面切向方向上直接取流场中相邻点物理量的值. [张德良, 2010, P339]

参考文献

张德良. *计算流体力学教程*. 高等教育出版社, 2010. 2, 5