Peking University

计算流体力学作业 5

College of Engineering   2001111690   袁磊祺

April 15, 2021

1

参见讲义 (CFDLect04-com01_cn.pdf) 的第 43 页. 证明: 一维完全气体动力学方程组 (Euler 方程组) 的第 3 个波 ($\lambda_3 = u + a$) 的左右状态满足: $p_L > p_R, u_L > u_R$, 激波; $p_L < p_R, u_L < u_R$, 稀疏波.

第 2 个波 ($\lambda_2 = u$) 的左右状态满足:$p_L = p_R, u_L = u_R$, 接触间断.

设 3 波是激波, 则有熵条件 (Lax 激波条件)

$$\lambda_{i-1}(\boldsymbol{U}(x-0,t)) \leq s < \lambda_i(\boldsymbol{U}(x-0,t)), \tag{1.1}$$

$$\lambda_i(\boldsymbol{U}(x+0,t)) < s \leq \lambda_{i+1}(\boldsymbol{U}(x+0,t)). \tag{1.2}$$

即

$$u_L + a_L > s > u_R + a_R, \quad s > u_L, \tag{1.3}$$

由此得 ($v := s - u$)

$$a_L > v_L > 0, \quad 0 < a_R < v_R. \tag{1.4}$$

由第三个间断跳跃条件知

$$\frac{\gamma+1}{\gamma-1}v_L^2 < \frac{2a_L^2}{\gamma-1} + v_L^2 = \frac{2a_R^2}{\gamma-1} + v_R^2 < \frac{\gamma+1}{\gamma-1}v_R^2, \tag{1.5}$$

因此, $a_L < a_R$. 再次利用第三个间断跳跃条件, 得

$$0 > \frac{2a_R^2}{\gamma-1} - \frac{2a_L^2}{\gamma-1} = v_L^2 - v_R^2. \tag{1.6}$$

注意 $v_L > 0, v_R > 0$, 所以有

$$v_L < v_R \Longleftrightarrow u_L > u_R. \tag{1.7}$$

由第一和第二个间断跳跃条件知

$$p_L - p_R = \rho_R v_R^2 - \rho_L v_L^2 = \rho_L v_L \left(v_R - v_L\right) > 0, \tag{1.8}$$

所以

$$p_L > p_R. \tag{1.9}$$

设 3 波是稀疏波. “熵” 条件

$$\lambda_3 \left(\boldsymbol{U}_L\right) = u_L + a_L < u_R + a_R, \tag{1.10}$$

表示波头比波尾快. 由 3-Riemann 不变量给出

$$\begin{aligned} u_L + a_L - \frac{\gamma+1}{\gamma-1} a_L = u_L - \frac{2a_L}{\gamma-1} = u_R - \frac{2a_R}{\gamma-1} = u_R + a_R - \frac{\gamma+1}{\gamma-1} a_R \quad &(1.11)\\ > u_L + a_L - \frac{\gamma+1}{\gamma-1} a_R, \quad &(1.12) \end{aligned}$$

因此, $a_L < a_R$, 并且 $u_L < u_R$. 再由另一个 3-Riemann 不变量给出

$$p_L \rho_L^{-\gamma} = p_R \rho_R^{-\gamma}, \tag{1.13}$$

则

$$\frac{p_R}{p_L} = \left(\frac{\rho_R}{\rho_L}\right)^{\gamma} = \left(\frac{a_R}{a_L}\right)^{2\gamma/(\gamma-1)}. \tag{1.14}$$

由此得 $p_L < p_R$.

而对接触间断, p, u 是 Riemann 不变量, 故结论成立.

$$p_L = p_R, \quad u_L = u_R. \tag{1.15}$$

□

2

已知一维标量守恒律方程 $u_t + f(u)_x = 0$ 的差分格式

$$u_j^{n+1} = u_j^n - \frac{\tau}{h}\left(\hat{f}\left(u_j^n, u_{j+1}^n\right) - \hat{f}\left(u_{j-1}^n, u_j^n\right)\right), \tag{2.1}$$

其中数值通量 $\hat{f}(u,v)$ 是一个连续可微的二元函数, 且

$$\frac{\partial \hat{f}(u,v)}{\partial u} \geq 0, \quad \frac{\partial \hat{f}(u,v)}{\partial v} \leq 0. \tag{2.2}$$

试分析和给出该格式满足 TVD(总变差不增) 性质和局部极值原理的 (最优) 条件.

对于满足局部极值原理, 这里给出一个充分条件, 即

$$1 + r\left(\frac{\partial \hat{f}(u,v)}{\partial v} - \frac{\partial \hat{f}(u,v)}{\partial u}\right) \geq 0, \tag{2.3}$$

其中

$$r = \tau/h. \tag{2.4}$$

由拉格朗日中值定理,

$$\hat{f}\left(u_j^n, u_{j+1}^n\right) = \hat{f}\left(u_j^n, u_j^n\right) + \hat{f}_v\left(u_j^n, \xi\right)\left(u_{j+1}^n - u_j^n\right), \tag{2.5}$$

$$\hat{f}\left(u_{j-1}^n, u_j^n\right) = \hat{f}\left(u_j^n, u_j^n\right) + \hat{f}_u\left(\eta, u_j^n\right)\left(u_{j-1}^n - u_j^n\right), \tag{2.6}$$

其中

$$\xi \in \left[u_j^n, u_{j+1}^n\right], \quad \eta \in \left[u_{j-1}^n, u_j^n\right]. \tag{2.7}$$

记

$$p = \hat{f}_v\left(u_j^n, \xi\right) \leq 0, \quad q = \hat{f}_u\left(\eta, u_j^n\right) \geq 0. \tag{2.8}$$

所以

$$u_j^{n+1} = (1 + rp - rq)u_j^n - rpu_{j+1}^n + rqu_{j-1}^n. \tag{2.9}$$

上式又可以写成

$$u_j^{n+1} = Au_{j-1}^n + Bu_j^n + Cu_{j+1}^n \tag{2.10}$$

其中

$$A = rq, B = (1 + rp - rq), C = -rp, \tag{2.11}$$

由于在条件式 (2.3) 下, $A, B, C \geq 0$, 且 $A + B + C = 1$, 所以有

$$\min\left\{u_{j-1}^n, u_j^n, u_{j+1}^n\right\} \leq u_j^{n+1} \leq \max\left\{u_{j-1}^n, u_j^n, u_{j+1}^n\right\}. \tag{2.12}$$

而对于 TVD 性质, 由于 $A, B, C \geq 0$, 可得格式是守恒型单调格式, 根据 `Lec 5 P7`, 守恒型单调格式是 l_1 压缩的, 又根据 l_1 压缩是 TVD 的, 即得证.

3

考虑二维双曲型守恒律方程组

$$\frac{\partial}{\partial t}\boldsymbol{U}+\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U})+\frac{\partial}{\partial y}\boldsymbol{G}(\boldsymbol{U})=0$$

其中 $\boldsymbol{U}\in\mathbb{R}^m$, 和笛卡尔网格 $\left\{\left(x_j,y_k\right):x_j=jh_x,y_k=kh_y,j,k\in\mathbb{Z}\right\}$. 试着写出它的 LF 格式, LW 格式, MacCormack 格式, 和一阶精度的显式迎风格式 (Roe 格式) (参见讲义 (CFDLect04-com01_cn.pdf) 的第 74-76, 81 页), 并说明时间步长的选取准则.

Lax-Friedrichs 格式

$$\boldsymbol{U}_j^{n+1}=\frac{\boldsymbol{U}_{j+1}^n+\boldsymbol{U}_{j-1}^n}{2}-\frac{\lambda_x}{2}\left(\boldsymbol{F}\left(\boldsymbol{U}_{j+1}^n\right)-\boldsymbol{F}\left(\boldsymbol{U}_{j-1}^n\right)\right)-\frac{\lambda_y}{2}\left(\boldsymbol{G}\left(\boldsymbol{U}_{j+1}^n\right)-\boldsymbol{G}\left(\boldsymbol{U}_{j-1}^n\right)\right). \tag{3.1}$$

其中 $\lambda_x=\tau/h_x$, $\lambda_y=\tau/h_y$.

MacCormack

$$\begin{cases}\bar{\boldsymbol{U}}_j^*=\boldsymbol{U}_j^n-\frac{\tau}{h_x}\left(\boldsymbol{F}\left(\boldsymbol{U}_{j+1}^n\right)-\boldsymbol{F}\left(\boldsymbol{U}_j^n\right)\right)-\frac{\tau}{h_y}\left(\boldsymbol{G}\left(\boldsymbol{U}_{j+1}^n\right)-\boldsymbol{G}\left(\boldsymbol{U}_j^n\right)\right)\\ \boldsymbol{U}_j^{n+1}=\frac{1}{2}\left(\boldsymbol{U}_j^n+\bar{\boldsymbol{U}}_j^*\right)-\frac{\tau}{2h_y}\left(\boldsymbol{F}\left(\bar{\boldsymbol{U}}_j^*\right)-\boldsymbol{F}\left(\bar{\boldsymbol{U}}_{j-1}^*\right)\right)-\frac{\tau}{2h_y}\left(\boldsymbol{G}\left(\bar{\boldsymbol{U}}_j^*\right)-\boldsymbol{G}\left(\bar{\boldsymbol{U}}_{j-1}^*\right)\right)\end{cases} \tag{3.2}$$

Roe

$$\hat{\boldsymbol{F}}\left(\boldsymbol{U}_j,\boldsymbol{U}_{j+1}\right)=\frac{\boldsymbol{F}\left(\boldsymbol{U}_j\right)+\boldsymbol{F}\left(\boldsymbol{U}_{j+1}\right)}{2}-\frac{1}{2}\left|\hat{\boldsymbol{A}}_{j+1/2}\right|\left(\boldsymbol{U}_{j+1}-\boldsymbol{U}_j\right), \tag{3.3}$$

其中 $|\hat{\boldsymbol{A}}|$ 定义为: $|\hat{\boldsymbol{A}}|=\boldsymbol{R}|\hat{\boldsymbol{\Lambda}}|\boldsymbol{R}^{-1}|$, $|\hat{\boldsymbol{\Lambda}}\ |=\operatorname{diag}\left\{\left|\hat{\lambda}_1\right|,\cdots,\left|\hat{\lambda}_m\right|\right\}$, $\boldsymbol{R}$ 为 $\hat{\boldsymbol{A}}$ 的右特征向量矩阵, $\hat{\boldsymbol{\Lambda}}=\operatorname{diag}\left\{\hat{\lambda}_1,\cdots,\hat{\lambda}_m\right\}$, 即 $\boldsymbol{R}^{-1}\hat{\boldsymbol{A}}\boldsymbol{R}=\hat{\boldsymbol{\Lambda}}$.

$$\hat{\boldsymbol{G}}\left(\boldsymbol{U}_j, \boldsymbol{U}_{j+1}\right) = \frac{\boldsymbol{G}\left(\boldsymbol{U}_j\right) + \boldsymbol{G}\left(\boldsymbol{U}_{j+1}\right)}{2} - \frac{1}{2}\left|\hat{\boldsymbol{B}}_{j+1/2}\right|\left(\boldsymbol{U}_{j+1} - \boldsymbol{U}_j\right), \tag{3.4}$$

其中 $|\hat{\boldsymbol{B}}|$ 定义为: $|\hat{\boldsymbol{B}}| = \boldsymbol{R}|\hat{\boldsymbol{\Lambda}}|\boldsymbol{R}^{-1}|$, $|\hat{\boldsymbol{\Lambda}}\ | = \operatorname{diag}\left\{\left|\hat{\lambda}_1\right|, \cdots, \left|\hat{\lambda}_m\right|\right\}$, $\boldsymbol{R}$ 为 $\hat{\boldsymbol{B}}$ 的右特征向量矩阵, $\hat{\boldsymbol{\Lambda}} = \operatorname{diag}\left\{\hat{\lambda}_1, \cdots, \hat{\lambda}_m\right\}$, 即 $\boldsymbol{R}^{-1}\hat{\boldsymbol{B}}\boldsymbol{R} = \hat{\boldsymbol{\Lambda}}$.

$$\boldsymbol{U}_{j+1} = \boldsymbol{U}_j - \frac{\tau}{h_x}\left(\hat{\boldsymbol{F}}\left(\boldsymbol{U}_j, \boldsymbol{U}_{j+1}\right) - \hat{\boldsymbol{F}}\left(\boldsymbol{U}_{j-1}, \boldsymbol{U}_j\right)\right) - \frac{\tau}{h_y}\left(\hat{\boldsymbol{G}}\left(\boldsymbol{U}_j, \boldsymbol{U}_{j+1}\right) - \hat{\boldsymbol{G}}\left(\boldsymbol{U}_{j-1}, \boldsymbol{U}_j\right)\right). \tag{3.5}$$

Lax-Wendroff

$$\begin{aligned}
\boldsymbol{U}\left(x_j, t_{n+1}\right) =& \boldsymbol{U}\left(x_j, t_n\right) + \tau\left(\boldsymbol{U}_t\right)_j^n + \frac{1}{2}\tau^2\left(\boldsymbol{U}_{tt}\right)_j^n + \mathcal{O}\left(\tau^3\right) && (3.6)\\
=& \boldsymbol{U}\left(x_j, t_n\right) - \tau\left(\boldsymbol{F}_x + \boldsymbol{G}_y\right)_j^n && (3.7)\\
&+ \frac{1}{2}\tau^2\left[\partial_x\left(\boldsymbol{A}\left(\boldsymbol{F}_x + \boldsymbol{G}_y\right)\right) + \partial_y\left(\boldsymbol{B}\left(\boldsymbol{F}_x + \boldsymbol{G}_y\right)\right)\right] + \mathcal{O}\left(\tau^3\right) && (3.8)\\
=& \boldsymbol{U}\left(x_j, t_n\right) - \tau\left(\boldsymbol{F}_x + \boldsymbol{G}_y\right)_j^n && (3.9)\\
&+ \frac{1}{2}\tau^2\left[\partial_x\left(\boldsymbol{A}\left(\boldsymbol{A}\boldsymbol{U}_x + \boldsymbol{B}\boldsymbol{U}_y\right)\right) + \partial_y\left(\boldsymbol{B}\left(\boldsymbol{A}\boldsymbol{U}_x + \boldsymbol{B}\boldsymbol{U}_y\right)\right)\right] + \mathcal{O}\left(\tau^3\right). && (3.10)
\end{aligned}$$

利用中心差商代替空间微商, 略去高阶项, 并用 $\boldsymbol{U}_j^n$ 代替 $\boldsymbol{U}\left(x_j, t_n\right)$, 得

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\lambda_x}{2}\left(\boldsymbol{F}\left(\boldsymbol{U}_{j+1}^n\right) - \boldsymbol{F}\left(\boldsymbol{U}_{j-1}^n\right)\right) - \frac{\lambda_y}{2}\left(\boldsymbol{G}\left(\boldsymbol{U}_{j+1}^n\right) - \boldsymbol{G}\left(\boldsymbol{U}_{j-1}^n\right)\right) \tag{3.11}$$

$$+ \frac{\lambda_x^2 \boldsymbol{A}^2\left(\boldsymbol{U}_{j+\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_{j+1}^n - \boldsymbol{U}_j^n\right) - \frac{\lambda_x^2 \boldsymbol{A}^2\left(\boldsymbol{U}_{j-\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n\right) \tag{3.12}$$

$$+ \frac{\lambda_x\lambda_y \boldsymbol{AB}\left(\boldsymbol{U}_{j+\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_{j+1}^n - \boldsymbol{U}_j^n\right) - \frac{\lambda_x\lambda_y \boldsymbol{AB}\left(\boldsymbol{U}_{j-\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n\right) \tag{3.13}$$

$$+ \frac{\lambda_x\lambda_y \boldsymbol{BA}\left(\boldsymbol{U}_{j+\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_{j+1}^n - \boldsymbol{U}_j^n\right) - \frac{\lambda_x\lambda_y \boldsymbol{BA}\left(\boldsymbol{U}_{j-\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n\right) \tag{3.14}$$

$$+ \frac{\lambda_y^2 \boldsymbol{B}^2\left(\boldsymbol{U}_{j+\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_{j+1}^n - \boldsymbol{U}_j^n\right) - \frac{\lambda_y^2 \boldsymbol{B}^2\left(\boldsymbol{U}_{j-\frac{1}{2}}^n\right)}{2}\left(\boldsymbol{U}_j^n - \boldsymbol{U}_{j-1}^n\right). \tag{3.15}$$

其中 $\boldsymbol{U}_{j+\frac{1}{2}}^n = \frac{1}{2}\left(\boldsymbol{U}_j^n + \boldsymbol{U}_{j+1}^n\right)$.

说明: 其中的 $\boldsymbol{A}\left(\boldsymbol{U}_{j+1/2}\right)$ 可以替代为

$$\boldsymbol{A}_{j+1/2} = \begin{cases} \frac{\boldsymbol{F}\left(\boldsymbol{U}_{j+1}\right) - \boldsymbol{F}\left(\boldsymbol{U}_j\right)}{\boldsymbol{U}_{j+1} - \boldsymbol{U}_j}, & \boldsymbol{U}_{j+1} \neq \boldsymbol{U}_j, \\ \boldsymbol{A}\left(\boldsymbol{U}_j\right), & \boldsymbol{U}_{j+1} = \boldsymbol{U}_j. \end{cases} \tag{3.16}$$

$\boldsymbol{B}$ 同理.

若 $\lambda_x = \lambda_y = r$ 其稳定的必要条件是

$$\lambda \max\left\{\left|\lambda(\boldsymbol{A})\right|, \left|\lambda(\boldsymbol{B})\right|\right\} \leq \frac{1}{2\sqrt{2}}. \tag{3.17}$$

时间步长的选取准则

由于上述格式都是相融的守恒差分格式, 所以需要时间步长取得足够小, 使得 $\delta \to 0$, 那么, 根据 Lax-Wendroff 定理, 如果满足初始条件

$$u_j^0 = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\,\mathrm{d}x. \tag{3.18}$$

的解 $u_\delta(x,t)$ 几乎处处有界且收敛于函数 $u(x,t)$, 则 $u(x,t)$ 是初值问题的一个弱解. 由此能获得一个弱解.