思考题 6

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1

设可微函数 $u=f(x,y)$ 满足方程 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0$.

证明: $f(x,y)$ 在极坐标系里除原点的全空间只是 θ 的函数.

证明. 令 $x=r\cos\theta,\ y=r\sin\theta$,

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta=\frac{1}{r}\left(\frac{\partial f}{\partial x}r\cos\theta+\frac{\partial f}{\partial y}r\sin\theta\right)=\frac{1}{r}\left(\frac{\partial f}{\partial x}x+\frac{\partial f}{\partial y}y\right)=0. \tag{1}$$

$\therefore f(x,y)$ 是 θ 的函数.

若题中 $u=f(x,y)\in C^1(D)$, D 为含原点的凸区域, 则 $f(x,y)$ 在 D 上为一常数. 由 $u=f(x,y)\in C^1(D)$ 可得 $r=0$ 时, $\frac{\partial f}{\partial r}=0$.

由有限增量定理

$$f(x,y)=f(r\cos\theta,r\sin\theta)=g(r,\theta)=g_0+g_r(\lambda r,\theta)r=g_0=f(0,0), \tag{2}$$

其中 $g(r,\theta)=f(r\cos\theta,r\sin\theta)$, $\frac{\partial g}{\partial r}=\frac{\partial f}{\partial r}=0$, $r=0$ 时, $g(r,\theta)=g_0=f(0,0)$.

□

2

设二元函数 $F(x,y)=f(x)g(y)$, 在极坐标系可表示为 $F(x,y)=S(r)$, 求 $F(x,y)$.

解: 令 $x = r\cos\theta, y = r\sin\theta$

$\because F(x,y) = S(r)$,

$\therefore$

$$\frac{\partial F}{\partial \theta} = -\frac{\partial F}{\partial x} r\sin\theta + \frac{\partial F}{\partial y} r\cos\theta = -y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y} = 0. \tag{3}$$

即 $yf'(x)g(y) = xf(x)g'(y)$

$$\frac{f'(x)}{xf(x)} = \frac{g'(y)}{yg(y)} = C, \tag{4}$$

可得 $f(x) = C_1 \mathrm{e}^{\frac{C}{2}x^2}$, $g(y) = C_2 \mathrm{e}^{\frac{C}{2}y^2}$, $F(x,y) = f(x)g(y) = C_3 \mathrm{e}^{C_4\left(x^2+y^2\right)}$.

3

函数 u 满足 $uu_{xy} = u_x u_y$.

求证: $u(x,y) = f(x)g(y)$.

证明. 由已知

$$\frac{\partial u_x}{u_x \partial y} = \frac{\partial u}{u \partial y}, \tag{5}$$

$$\frac{\partial \ln u_x}{\partial y} = \frac{\partial \ln u}{\partial y}, \tag{6}$$

$$\ln u_x = \ln u + c(x), \tag{7}$$

$$u_x = uC(x), \tag{8}$$

$$\frac{\partial \ln u}{\partial x} = C(x), \tag{9}$$

$$\ln u = F(x) + G(y), \tag{10}$$

$$u(x,y) = f(x)g(y). \tag{11}$$

$\square$