Peking University

思考题 11

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1

设一元函数 $f(x)$ 在 $[a,b]$ 上可积. 在 $[a,b]\times[a,b]$ 上定义 $F(x,y)=[f(x)-f(y)]^2$

(1) 将重积分 $\iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y$ 化为累次积分;

(2) 证明: $\left[\int_a^b f(x)\,\mathrm{d}x\right]^2 \leqslant (b-a)\int_a^b f^2(x)\,\mathrm{d}x$.

(1) **解:**

$$\begin{aligned}\iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b\int_a^b \left[f^2(x)+f^2(y)-2f(x)f(y)\right]\mathrm{d}x\,\mathrm{d}y \\ &= \int_a^b\int_a^b f^2(x)\,\mathrm{d}x\,\mathrm{d}y+\int_a^b\int_a^b f^2(y)\,\mathrm{d}x\,\mathrm{d}y-2\int_a^b\int_a^b f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 2(b-a)\int_a^b f^2(x)\,\mathrm{d}x-2\int_a^b f(x)\,\mathrm{d}x\int_a^b f(y)\,\mathrm{d}y \\ &= 2(b-a)\int_a^b f^2(x)\,\mathrm{d}x-2\left[\int_a^b f(x)\,\mathrm{d}x\right]^2.\end{aligned} \tag{1}$$

(2) **证明.** $F(x,y)=[f(x)-f(y)]^2\geqslant 0$, $\therefore \iint_D F(x,y)\,\mathrm{d}x\,\mathrm{d}y\geqslant 0$, 由 (1) 即有

$$\left[\int_a^b f(x)\,\mathrm{d}x\right]^2 \leqslant (b-a)\int_a^b f^2(x)\,\mathrm{d}x. \tag{2}$$

□

2

计算三重积分 $I=\int_0^1 \mathrm{d}x \int_x^1 \mathrm{d}y \int_y^1 y\sqrt{1+z^4}\,\mathrm{d}z$.

解:

$$
\begin{aligned}
I &= \int_0^1 \mathrm{d}z \int_0^z \mathrm{d}y \int_0^y y\sqrt{1+z^4}\,\mathrm{d}x && (3)\\
&= \int_0^1 \mathrm{d}z \int_0^z y^2\sqrt{1+z^4}\,\mathrm{d}y && (4)\\
&= \frac{1}{3}\int_0^1 z^3\sqrt{1+z^4}\,\mathrm{d}z && (5)\\
&= \frac{1}{12}\int_0^1 \sqrt{1+z^4}\,\mathrm{d}\big(1+z^4\big) && (6)\\
&= \frac{1}{12}\cdot\frac{2}{3}\left(1+z^4\right)^{\frac{3}{2}}\Big|_0^1 && (7)\\
&= \frac{2\sqrt{2}-1}{18}. && (8)
\end{aligned}
$$

3

化重积分为累次计分 $\iiint_V f\,\mathrm{d}V$ ，其中 V 是 $x^2+y^2=1,\ y^2+z^2=1,\ x^2+z^2=1$ 围成的区域.

解:

$$
\begin{aligned}
\iiint_V f\,\mathrm{d}V = &\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}y \int_{|y|}^{\sqrt{1-y^2}} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f\,\mathrm{d}z + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}y \int_{-\sqrt{1-y^2}}^{-|y|} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f\,\mathrm{d}z \\
&+ \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}x \int_{|x|}^{\sqrt{1-x^2}} \mathrm{d}y \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f\,\mathrm{d}z + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{-|x|} \mathrm{d}y \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f\,\mathrm{d}z. \qquad (9)
\end{aligned}
$$