Homework 1

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Exercise 1

Question 16.5.1: Find the second partial derivative of the following function
(2) $u = xy + \frac{y}{x}$
(4) $u = (xy)^z$
Answer:
(2) $\frac{\partial^2 u}{\partial x^2} = \frac{2y}{x^3}$; $\frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{1}{x^2}$; $\frac{\partial^2 u}{\partial y^2} = 0$
(4) $\frac{\partial^2 u}{\partial x^2} = \frac{uz(z-1)}{x^2}$; $\frac{\partial^2 u}{\partial y^2} = \frac{uz(z-1)}{y^2}$; $\frac{\partial^2 u}{\partial z^2} = u(\ln x + \ln y)^2$; $\frac{\partial^2 u}{\partial x \partial y} = \frac{uz^2}{xy}$; $\frac{\partial^2 u}{\partial y \partial z} = \frac{uz}{y}(\ln x + \ln y) + \frac{u}{y}$; $\frac{\partial^2 u}{\partial x \partial z} = \frac{uz}{x}(\ln x + \ln y) + \frac{u}{x}$.

Exercise 2

Question 16.5.2: Find the specified partial derivative of the following function:
(3) $u = x \ln(xy)$, $\frac{\partial^3 u}{\partial x^2 \partial y}$;
(7) $u = \ln \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}$, $\frac{\partial^4 u}{\partial x \partial y \partial \xi \partial \eta}$
Answer:
(3) 0
(7) $\frac{48(y-\eta)^2(x-\xi)^2}{((x-\xi)^2+(y-\eta)^2)^4} - \frac{6}{((x-\xi)^2+(y-\eta)^2)^2}$

Exercise 3

Question 16.5.7: Find the second partial derivative of the following function
(1) $u = f(ax, by)$;
(2) $u = f(x + y, x - y)$;
(3) $u = f(x + y, xy)$;
(4) $u = f(x + y + z, x^2 + y^2 + z^2)$;
(5) $u = f(\frac{x}{y}, \frac{y}{z})$;
(6) $u = f(x^2 + y^2 + z^2)$.
Answer:
(1) $\frac{\partial^2 u}{\partial x^2} = a^2 f_{11}$;
$\frac{\partial^2 u}{\partial x \partial y} = abf_{12}$;

$\frac{\partial^2 u}{\partial y^2} = b^2 f_{22}$

(2) $\frac{\partial^2 u}{\partial x^2} = f_{11} + 2f_{12} + f_{22}$;
$\frac{\partial^2 u}{\partial x \partial y} = f_{11} - f_{22}$;
$\frac{\partial^2 u}{\partial y^2} = f_{11} - 2f_{12} + f_{22}$

(3) $\frac{\partial^2 u}{\partial x^2} = f_{11} + 2yf_{12} + y^2 f_{22}$;
$\frac{\partial^2 u}{\partial x \partial y} = f_{11} + f_2 + (x+y)f_{12} + xyf_{22}$;
$\frac{\partial^2 u}{\partial y^2} = f_{11} + 2xf_{12} + x^2 f_{22}$

(4) $\frac{\partial^2 u}{\partial x^2} = f_{11} + 4xf_{12} + 2f_2 + 4x^2 f_{22}$;
$\frac{\partial^2 u}{\partial y^2} = f_{11} + 4yf_{12} + 2f_2 + 4y^2 f_{22}$;
$\frac{\partial^2 u}{\partial z^2} = f_{11} + 4zf_{12} + 2f_2 + 4z^2 f_{22}$;
$\frac{\partial^2 u}{\partial x \partial y} = f_{11} + 2(x+y)f_{12} + 4xyf_{22}$;
$\frac{\partial^2 u}{\partial y \partial z} = f_{11} + 2(y+z)f_{12} + 4yzf_{22}$;
$\frac{\partial^2 u}{\partial x \partial z} = f_{11} + 2(x+z)f_{12} + 4xzf_{22}$.

(5) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{y^2} f_{11}$;
$\frac{\partial^2 u}{\partial y^2} = \frac{2x}{y^3} f_1 + \frac{x^2}{y^4} f_{11} - \frac{2x}{zy^2} f_{12} + \frac{1}{z^2} f_{22}$;
$\frac{\partial^2 u}{\partial z^2} = \frac{2y}{z^3} f_2 + \frac{y^2}{z^4} f_{22}$;
$\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{y^2} f_1 - \frac{x}{y^3} f_{11} + \frac{1}{zy} f_{12}$;
$\frac{\partial^2 u}{\partial y \partial z} = -\frac{1}{z^2} f_2 + \frac{x}{yz^2} f_{12} - \frac{y}{z^3} f_{22}$;
$\frac{\partial^2 u}{\partial x \partial z} = -\frac{1}{z^2} f_{12}$.

(6) $\frac{\partial^2 u}{\partial x^2} = 2f' + 4x^2 f''$;
$\frac{\partial^2 u}{\partial y^2} = 2f' + 4y^2 f''$;
$\frac{\partial^2 u}{\partial z^2} = 2f' + 4z^2 f''$;
$\frac{\partial^2 u}{\partial x \partial y} = 4xyf''$;
$\frac{\partial^2 u}{\partial y \partial z} = 4yzf''$;
$\frac{\partial^2 u}{\partial x \partial z} = 4xzf''$.

Exercise 4

Question 16.5.16: Let $x = e^{\xi},\ y = e^{\eta}$, transform equation:

$$ax^2 \frac{\partial^2 u}{\partial x^2} + 2bxy \frac{\partial^2 u}{\partial x \partial y} + cy^2 \frac{\partial^2 u}{\partial y^2} = 0$$

Answer:

$$\frac{\partial u}{\partial x} = \frac{1}{x} \frac{\partial u}{\partial \xi}, \quad \frac{\partial u}{\partial y} = \frac{1}{y} \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{x^2}\left(\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial u}{\partial \xi}\right),\quad \frac{\partial^2 u}{\partial y^2}=\frac{1}{y^2}\left(\frac{\partial^2 u}{\partial \eta^2}-\frac{\partial u}{\partial \eta}\right),\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{1}{xy}\frac{\partial^2 u}{\partial \xi\partial \eta}$$

$$a\left(\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial u}{\partial \xi}\right)+2b\frac{\partial^2 u}{\partial \xi\partial \eta}+c\left(\frac{\partial^2 u}{\partial \eta^2}-\frac{\partial u}{\partial \eta}\right)=0$$

Exercise 5

Question 16.5.20: Verify: No function satisfies

$$\frac{\partial f}{\partial x}=y,\quad \frac{\partial f}{\partial y}=x^2$$

Proof: Because

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=2x\neq 1=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Exercise 6

Question 16.6.2: Find the Taylor expansion of $f(x,y)=x^2+xy+y^2+3x-2y+4$ at (-1,1).
Answer:

$$f(x,y)=(x+1-1)^2+(x+1-1)(y-1+1)+(y-1+1)^2+3(x+1-1)-2(y-1+1)+4$$

$$=(x+1)^2-2(x+1)+1+(x+1)(y-1)-(y-1)+(x+1)-1+(y-1)^2+2(y-1)+1+3(x+1)-3-2(y-1)-2+4$$

$$=2(x+1)-(y-1)+(x+1)^2+(x+1)(y-1)+(y-1)^2$$

Exercise 7

Question 16.6.3: Find the Taylor expansion of $f(x,y,z)=x^3+y^3+z^3-3xyz$ at (1,1,1).
Answer:

$$f=3(x-1)^2-3(x-1)(z-1)-3(y-1)(z-1)-3(x-1)(y-1)$$

$$+(x-1)^3+3(y-1)^2+(y-1)^3+3(z-1)^2+(z-1)^3-3(x-1)(y-1)(z-1)$$

Exercise 8

Question 16.6.6: Find the forth-order Taylor expansion of following functions at (0,0)
(1) $u=\sin(x^2+y^2)$;
(2) $u=\sqrt{1+x^2+y^2}$;
(3) $u=\ln(1+x)\ln(1+y)$;
(4) $u=e^x\cos y$.
Answer: (1)

$$u=x^2+y^2+o(||h||^4)$$

(2)

$$u=1+\frac{x^2}{2}+\frac{y^2}{2}-\frac{x^4}{8}-\frac{y^4}{8}-\frac{x^2y^2}{4}+o(h^4)$$

(3)

$$u = xy - \frac{xy^2}{2} - \frac{x^2y}{2} + \frac{x^3y}{3} + \frac{xy^3}{3} + \frac{x^2y^2}{4} + o(h^4)$$

(4)

$$u = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2} + \frac{x^4}{24} - \frac{x^2y^2}{4} + \frac{y^4}{24} + o(h^4)$$