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参考答案 4

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16.1.2

(1) $u = \sqrt[z]{\frac{x}{y}}$ at $(1,1,1)$.

$\frac{\partial u}{\partial x}(1,1,1) = \frac{\mathrm{d}x}{\mathrm{d}x}\big|_{x=1} = 1.$

$\frac{\partial u}{\partial y}(1,1,1) = \left(\frac{1}{y}\right)'\big|_{y=1} = -1.$

$\frac{\partial u}{\partial z}(1,1,1) = \frac{\mathrm{d}1}{\mathrm{d}z}\big|_{z=1} = 0.$

(2) $z = x + (y-1)\arcsin\frac{x}{y}$ at $(0,1)$.

$\frac{\partial z}{\partial x}(0,1) = \frac{\mathrm{d}x}{\mathrm{d}x}\big|_{x=0} = 1.$

$\frac{\partial z}{\partial y}(0,1) = \frac{\mathrm{d}0}{\mathrm{d}y}\big|_{y=1} = 0.$

(3) $u = \arctan\frac{x+y+z-xyz}{1-xy-xz-yz}$ at $(0,0,0)$.

$\frac{\partial u}{\partial x}(0,0,0) = (\arctan x)'\big|_{x=0} = 1.$

$\frac{\partial u}{\partial y}(0,0,0) = (\arctan y)'\big|_{y=0} = 1.$

$\frac{\partial u}{\partial z}(0,0,0) = (\arctan z)'\big|_{z=0} = 1.$

16.1.3

$u = \ln(1 + x_1 + x_2^2 + x_3^2).$

$u_{x_1} = \frac{1}{1+x_1+x_2^2+x_3^2}$,

$u_{x_2} = \frac{2x_2}{1+x_1+x_2^2+x_3^2}$,

$u_{x_3} = \frac{2x_3}{1+x_1+x_2^2+x_3^2}$,

$\therefore \sum\limits_{i=1}^{3} u_{x_i}(1,1,1) = \frac{5}{4}$.

16.1.7

$z = x^n f\left(\frac{y}{x^2}\right)$,

$\frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x^2}\right) - 2\frac{yx^n}{x^3} f'\left(\frac{y}{x^2}\right)$,

$\frac{\partial z}{\partial y} = x^n f\left(\frac{y}{x^2}\right)\frac{1}{x^2}$,

$x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x^2}\right) = nz$. $\square$

16.2.4

$u(x+h, y+k) - u(x,y) = Ah + Bk + o(\sqrt{h^2+k^2})$,

$v(x+h, y+k) - v(x,y) = Ch + Dk + o(\sqrt{h^2+k^2})$.

$$\begin{aligned}\Delta uv &= u(x+h,y+k)v(x+h,y+k) - u(x,y)v(x,y) \\ &= [u(x+h,y+k)v(x+h,y+k) - u(x,y)v(x+h,y+k)] + [u(x,y)v(x+h,y+k) - u(x,y)v(x,y)] \\ &= [(Ah+Bk)v(x+h,y+k) + v(x+h,y+k)o(\sqrt{h^2+k^2})] + [u(x,y)(Ch+Dk) + u(x,y)o(\sqrt{h^2+k^2})],\end{aligned}$$

$\because \lim\limits_{\sqrt{h^2+k^2}\to 0} v(x+h,y+k) = v(x,y)$,

$\therefore \mathrm{d}(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$. $\square$

16.2.15

(1) $\mathrm{d}u = ax\,\mathrm{d}x + by\,\mathrm{d}y$,

$u = \frac{1}{2}ax^2 + \frac{1}{2}by^2 + c$.

(2) $\mathrm{d}u = (x^2+2xy-y^2)\,\mathrm{d}x + (x^2-2xy-y^2)\,\mathrm{d}y$,

$u = \frac{1}{3}x^3 + x^2y - y^2x - \frac{1}{3}y^3 + c$.

(3) $\mathrm{d}u = (5x^4+3xy^2-y^3)\,\mathrm{d}x + (3x^2y-3xy^2+y^2)\,\mathrm{d}y$,

$u = x^5 + \frac{3}{2}x^2y^2 - y^3x + \frac{1}{3}y^3 + c$.

16.3.2

(1) $u = f(\sqrt{x^2+y^2})$,

$\frac{\partial u}{\partial x} = f'(\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}}$,

$\frac{\partial u}{\partial y} = f'(\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}}$.

(3) $u = f(\frac{xz}{y})$,

$\frac{\partial u}{\partial x} = f'(\frac{xz}{y})\frac{z}{y}$,,

$\frac{\partial u}{\partial y} = -f'(\frac{xz}{y})\frac{xz}{y^2}$,

$\frac{\partial u}{\partial z} = f'(\frac{xz}{y})\frac{x}{y}$.

(5) $u = f(x, xy, xyz)$,

$\frac{\partial u}{\partial x} = f_x(x,xy,xyz) + f_y(x,xy,xyz)y + f_z(x,xy,xyz)yz$,

$\frac{\partial u}{\partial y} = f_y(x,xy,xyz)x + f_z(x,xy,xyz)xz$,

$\frac{\partial u}{\partial z} = f_z(x,xy,xyz)xy$.

(7) $u = f(x+y+z, x^2+y^2+z^2)$,

$\frac{\partial u}{\partial x} = f_x(x+y+z, x^2+y^2+z^2) + f_y(x+y+z, x^2+y^2+z^2)2x$,

$\frac{\partial u}{\partial y} = f_x(x+y+z, x^2+y^2+z^2) + f_y(x+y+z, x^2+y^2+z^2)2y$,

$\frac{\partial u}{\partial z} = f_x(x+y+z, x^2+y^2+z^2) + f_y(x+y+z, x^2+y^2+z^2)2z$.

(9) $u = f(x^2+y^2, x^2-y^2, 2xy)$,

$\frac{\partial u}{\partial x} = f_x(x^2+y^2, x^2-y^2, 2xy)2x + f_y(x^2+y^2, x^2-y^2, 2xy)2x + f_z(x^2+y^2, x^2-y^2, 2xy)2y$,

$\frac{\partial u}{\partial y} = f_x(x^2+y^2, x^2-y^2, 2xy)2y - f_y(x^2+y^2, x^2-y^2, 2xy)2y + f_z(x^2+y^2, x^2-y^2, 2xy)2x$.

16.4.1

$\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(x_0,y_0)}$.

(1) $\frac{\partial u}{\partial x}\Big|_{(1,1)} = 2x\Big|_{(1,1)} = 2,$

$\frac{\partial u}{\partial y}\Big|_{(1,1)} = -2y\Big|_{(1,1)} = -2,$

$\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(1,1)} = 2\cos\frac{\pi}{3} - 2\cos\frac{\pi}{6} = 1 - \sqrt{3}.$

(2) $\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(1,1)} = \frac{\partial u}{\partial x}\Big|_{(1,1)}\cos\frac{\pi}{3} \pm \frac{\partial u}{\partial y}\Big|_{(1,1)}\cos\frac{\pi}{6} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$

(3) $\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(1,1)} = \frac{\partial u}{\partial x}\Big|_{(1,1)}\cos\frac{\pi}{4} + \frac{\partial u}{\partial y}\Big|_{(1,1)}\cos\frac{\pi}{4} = 2\mathrm{e}\frac{\sqrt{2}}{2} + \mathrm{e}\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}\mathrm{e}.$

16.4.2

$\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(x_0,y_0,z_0)}$.

(1) $\cos\alpha = \frac{1}{\sqrt{6}},\ \cos\beta = -\frac{1}{\sqrt{6}},\ \cos\gamma = \frac{2}{\sqrt{6}}.$

$\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(1,1,0)} = \frac{\partial u}{\partial x}\Big|_{(1,1,0)}\cos\alpha + \frac{\partial u}{\partial y}\Big|_{(1,1,0)}\cos\beta + \frac{\partial u}{\partial z}\Big|_{(1,1,0)}\cos\gamma = \frac{2}{\sqrt{6}} - \frac{4}{\sqrt{6}} = -\frac{\sqrt{6}}{3}.$

(2) $\cos\alpha = \frac{2}{\sqrt{6}},\ \cos\beta = \frac{1}{\sqrt{6}},\ \cos\gamma = -\frac{1}{\sqrt{6}}.$

$\frac{\partial u}{\partial \boldsymbol{l}}\Big|_{(1,1,1)} = \frac{\partial u}{\partial x}\Big|_{(1,1,1)}\cos\alpha + \frac{\partial u}{\partial y}\Big|_{(1,1,1)}\cos\beta + \frac{\partial u}{\partial z}\Big|_{(1,1,1)}\cos\gamma = \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}.$