PEKING UNIVERSITY

参考答案 12

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21.2.1

计算下列第一型曲线积分:

(3)

$\int_C xyz\,\mathrm{d}s$, C 为螺线:

$x = a\cos t,\quad y = a\sin t,\quad z = bt,\quad 0 < a < b,\quad 0 \le t \le 2\pi.$

$$\mathrm{d}s = \sqrt{a^2+b^2}\,\mathrm{d}t, \tag{1}$$

$$\begin{aligned}\int_c xyz\,\mathrm{d}s &= \int_0^{2\pi} a^2b\sqrt{a^2+b^2}t\sin t\cos t\,\mathrm{d}t \\ &= a^2b\sqrt{a^2+b^2}\int_0^{2x} t\sin t\cos t\,\mathrm{d}t. \\ &= -\frac{a^2b\sqrt{a^2+b^2}\pi}{2}\end{aligned} \tag{2}$$

(4)

$\int_C \sqrt{x^2+y^2}\,\mathrm{d}s$, C 为圆周 $x^2+y^2=ax$.

设

$$\begin{cases} x = \frac{a}{2}\cos\theta + \frac{a}{2} \\ y = \frac{a}{2}\sin\theta \end{cases} \tag{3}$$

$\mathrm{d}s = \frac{a}{2}\,\mathrm{d}\theta$

$$\begin{aligned}\int_0^{2\pi} a\sqrt{\frac{1}{2}\cos\theta+\frac{1}{2}}\frac{a}{2}\,\mathrm{d}\theta &= \frac{a^2}{2}\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|\mathrm{d}\theta \\ &= 2a^2\end{aligned} \tag{4}$$

(6)

$\int_C xy\,\mathrm{d}s,\ C$ 为球面 $x^2+y^2+z^2=a^2$ 与平面 $x+y+z=0$ 之交线.

$$\begin{aligned}\int_c xy\,\mathrm{d}s &= \int_c \frac{(x+y+z)^2-\left(x^2+y^2+z^2\right)}{6}\,\mathrm{d}s \\ &= \int_c \frac{0-a^2}{6}\,\mathrm{d}s \\ &= -\frac{\pi a^3}{3}\end{aligned} \tag{5}$$

21.2.4

方法一

换用极坐标有

$$\begin{aligned}u(x,y) &= -\frac{\rho_0}{2}\oint \ln\left(R^2+r^2-2Rr\cos\varphi\right)R\,\mathrm{d}\varphi \\ &= -\rho_0 R\int_0^{\pi}\ln R^2\left[1+\left(\frac{r^2}{R^2}\right)-2\frac{r}{R}\cos\varphi\right]\mathrm{d}\varphi \\ &= -\rho_0 R\int_0^{\pi}\ln R^2\,\mathrm{d}\varphi-\rho_0 R\int_0^{\pi}\ln\left[1+\left(\frac{r^2}{R^2}\right)-2\frac{r}{R}\cos\varphi\right]\mathrm{d}\varphi \\ &= -2\pi R\rho_0\ln R-\rho_0 R\int_0^{\pi}\ln\left[1+\left(\frac{r^2}{R^2}\right)-2\frac{r}{R}\cos\varphi\right]\mathrm{d}\varphi\end{aligned} \tag{6}$$

设 $a=\frac{r}{R}$,

$$J(a) = \int_0^{\pi}\ln\left(a^2-2a\cos\varphi+1\right)\mathrm{d}\varphi \tag{7}$$

对 J 求导

$$J'(a)=2\int_0^\pi \frac{(a-\cos\varphi)\,\mathrm{d}\varphi}{1-2a\cos\varphi+a^2}=\left(\frac{x}{2a}+\frac{\left(a^2-1\right)\arctan\left(\frac{|a+1|\tan\left(\frac{x}{2}\right)}{|a-1|}\right)}{a\,|a-1|\,|a+1|}\right)\Bigg|_0^\pi \tag{8}$$

$$=\begin{cases}\frac{\pi}{a}, & a\geq 1\\ 0, & a<1\end{cases}$$

又 $J(0)=0$, 所以

$$u(x,y)=\begin{cases}2\pi\rho_0 R\ln\frac{1}{R}, & r=\sqrt{x^2+y^2}<R\\ 2\pi\rho_0 R\ln\frac{1}{r}, & r=\sqrt{x^2+y^2}>R\end{cases} \tag{9}$$

方法二

此题可看作无穷长带电圆柱, 可以参考电磁学教程. 根据对称性和高斯定理可得电场分布是: 圆柱外的电场径向向外, 圆柱内无电场, 圆柱电荷面密度

$$\sigma_e=2\pi\varepsilon_0\rho_0. \tag{10}$$

当 $\sqrt{x^2+y^2}>R$ 时, 等效为一无穷长直线产生的电场, 线密度

$$\eta_e=2\pi R\sigma_e=4\pi^2 R\varepsilon_0\rho_0. \tag{11}$$

产生的电场为

$$E(x,y)=\frac{\eta_e}{2\pi\varepsilon_0\sqrt{x^2+y^2}}=\frac{2\pi R\rho_0}{\sqrt{x^2+y^2}} \tag{12}$$

电势

$$u(x,y)=-\int E\,\mathrm{d}\sqrt{x^2+y^2}=-2\pi R\rho_0\ln\left(\sqrt{x^2+y^2}\right). \tag{13}$$

当 $\sqrt{x^2+y^2}<R$ 时, 由于内部无电场, 则

$$u(x,y)=\oint_L \rho_0\ln\frac{1}{r}\,\mathrm{d}s=-2\pi R\rho_0\ln R. \tag{14}$$

22.2.2

$$
\begin{aligned}
\vec{r} &= ((b + a\cos\psi)\cos\varphi, (b + a\cos\psi)\sin\varphi, a\sin\psi) && (15)\\
\vec{r}_\psi &= (-a\sin\psi\cos\varphi, -a\sin\psi\sin\varphi, a\cos\psi) && (16)\\
\vec{r}_\varphi &= (-(b + a\cos\psi)\sin\varphi, (b + a\cos\psi)\cos\varphi, 0) && (17)
\end{aligned}
$$

$$
E = \vec{r}_\psi \cdot \vec{r}_\psi = a^2, \quad F = \vec{r}_\psi \cdot \vec{r}_\varphi = 0, \quad G = \vec{r}_\varphi \cdot \vec{r}_\varphi = (b + \cos\psi)^2 \tag{18}
$$

$$
\begin{aligned}
S &= \int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi \int_{\psi_1}^{\psi_2} \sqrt{EG - F^2}\,\mathrm{d}\psi \\
&= \int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi \int_{\psi_1}^{\psi_2} a(b + a\cos\psi)\,\mathrm{d}\psi \\
&= ab\,(\varphi_2 - \varphi_1)\,(\psi_2 - \psi_1) + a^2\,(\varphi_2 - \varphi_1)\int_{\psi_1}^{\psi_2} \cos\psi\,\mathrm{d}\psi \\
&= ab\,(\varphi_2 - \varphi_1)\,(\psi_2 - \psi_1) + a^2\,(\varphi_2 - \varphi_1)\,(\sin\psi_2 - \sin\psi_1)
\end{aligned} \tag{19}
$$

$$
S_{\text{all}} = 4\pi^2 ab. \tag{20}
$$

22.2.5

$$
\frac{\partial z}{\partial x} = \sqrt{\frac{y}{2x}} \quad \frac{\partial z}{\partial y} = \sqrt{\frac{x}{2y}} \tag{21}
$$

$$
\begin{aligned}
\int_0^1 \mathrm{d}x \int_{1-x}^1 \sqrt{\frac{y}{2x} + \frac{x}{2y} + 1}\,\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_{1-x}^1 \frac{x + y}{\sqrt{2xy}}\,\mathrm{d}y \\
&= \int_0^1 \mathrm{d}x \int_{1-x}^1 \left(\sqrt{\frac{x}{2y}} + \sqrt{\frac{y}{2x}}\right)\mathrm{d}y \\
&= \int_0^1 \left(\sqrt{\frac{x}{2}}2\sqrt{y} + \sqrt{\frac{1}{2x}}\frac{2}{3}y^{\frac{3}{2}}\right)\Bigg|_{1-x}^1 \mathrm{d}x \\
&= \frac{4\sqrt{2}}{3} - \frac{\sqrt{2}}{4}\pi
\end{aligned} \tag{22}
$$

22.2.6

$$3\left(x^2+y^2\right)=(2a-x-y)^2 \Rightarrow x^2+y^2-xy+2a(x+y)=2a^2 \tag{23}$$

$$\text{平面部分}\quad S_1=\iint_D \sqrt{1+z_x^2+z_y^2}\,\mathrm{d}x\,\mathrm{d}y=2\iint_D \mathrm{d}x\,\mathrm{d}y \tag{24}$$

$$\text{曲面部分}\quad S_2=\iint_D \sqrt{1+z_x^2+z_y^2}\,\mathrm{d}x\,\mathrm{d}y=\sqrt{3}\iint_D \mathrm{d}x\,\mathrm{d}y \tag{25}$$

做变量替换

$$u=\frac{x+y}{\sqrt{2}},\quad v=\frac{x-y}{\sqrt{2}} \Rightarrow \frac{(u+2\sqrt{2}a)^2}{12a^2}+\frac{v^2}{4a^2}=1 \tag{26}$$

x, y 平面内的面积

$$\iint_D \mathrm{d}x\,\mathrm{d}y=4\sqrt{3}xa^2 \tag{27}$$

总面积

$$S=4\pi a^2(2\sqrt{3}+3) \tag{28}$$

体积

$$V=\frac{1}{3}S_2\cdot H=\frac{8\sqrt{3}}{3}\pi a^3 \tag{29}$$

22.2.7

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{30}$$

$$\frac{\partial z}{\partial x}=-\frac{c}{a^2}\frac{x}{z},\quad \frac{\partial z}{\partial y}=-\frac{c^2}{b^2}\frac{y}{z} \tag{31}$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{2y}\right)^2}=\sqrt{1+\frac{c^2}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)}=A \tag{32}$$

$$S_1=\iint_{D_1} A\,\mathrm{d}x\,\mathrm{d}y\quad S_2=\iint_{D_2} A\,\mathrm{d}x\,\mathrm{d}y \tag{33}$$

其中

$$\begin{aligned} D_1&=\left\{(x,y)\,\middle|\,\tfrac{x^2}{a^2}+\tfrac{y^2}{b^2}+\tfrac{z^2}{c^2}=1,\quad lx+my+nz\geqslant p\right\}\\ D_2&=\left\{(x,y)\,\middle|\,\tfrac{x^2}{a^2}+\tfrac{y^2}{b^2}+\tfrac{z^2}{c^2}=1,\quad lx+my+nz\leqslant p\right\}\end{aligned} \tag{34}$$

22.3.1

求下列积分

(1)

$$z=\sqrt{x^2+y^2}\Rightarrow z_x=\frac{x}{\sqrt{x^2+y^2}},\quad z_y=\frac{y}{\sqrt{x^2-2y^2}} \tag{35}$$

$$\sqrt{z_x^2+z_y^2+1}=\sqrt{2}. \tag{36}$$

$$\begin{aligned}\iint_S\left(x^2+y^2\right)\mathrm{d}s=&\iint_{x^2+y^2\leq 1}\sqrt{2}\left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y+\iint_{x^2+y^2\leq 1}\left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y\\ =&\int_0^{2z}\mathrm{d}\theta\int_0^1(\sqrt{2}+1)r^3\,\mathrm{d}r\\ =&\frac{\pi}{2}(\sqrt{2}+1)\end{aligned} \tag{37}$$

(2)

$$\frac{\partial z}{\partial x}=2x\quad\frac{\mathrm{d}z}{\partial y}=2y \tag{38}$$

$$\begin{aligned}\iint_s\left|zx^3y^2\right|\mathrm{d}s=&\iint_{x^2+y^2\leq 1}\left|x^3-y^2\right|\left(x^2+y^2\right)\sqrt{4x^2+4y^2+1}\,\mathrm{d}x\,\mathrm{d}y\\ =&\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^8\sqrt{4r^2+1}\left|\cos^3\theta\sin^2\theta\right|\mathrm{d}r\\ =&\frac{8}{15}\times\frac{40262\sqrt{5}+21\,\mathrm{arcsinh}(2)}{393216}\\ =&\frac{40262\sqrt{5}+21\,\mathrm{arcsinh}(2)}{737280}\end{aligned} \tag{39}$$

(3)

$$x_u=\cos v,\quad y_u=\sin v\quad z_u=0 \tag{40}$$

$$x_v=-u\sin v,\quad y_v=u\cos v,\quad z_v=1 \tag{41}$$

$$E=\vec{r}_u\cdot\vec{r}_u=1\quad F=\vec{r}_u\cdot\vec{r}_v=0\quad G=\vec{r}_v\cdot\vec{r}_v=u^2+1 \tag{42}$$

$$
\begin{aligned}
\iint_S z^2 \,\mathrm{d}s = & \int_0^a \mathrm{d}u \int_0^{2\pi} \sqrt{u^2+1} v^2 \,\mathrm{d}v \\
= & \frac{8\pi^3}{3} \int_0^a \sqrt{u^2+1} \,\mathrm{d}u \\
= & \frac{4\pi^3}{3} \left[a\sqrt{1+a^2} + \ln\left(a + \sqrt{1+a^2} \right) \right]
\end{aligned} \tag{43}
$$

(4)

$$
\begin{aligned}
\iint_S \left(x^2 + y^2 \right) \mathrm{d}s = & \frac{2}{3} \iint_S x^2 + y^2 + z^2 \,\mathrm{d}s \\
= & \frac{8}{3}\pi R^4
\end{aligned} \tag{44}
$$