10.1.1. (3) $\int_1^{+\infty}\frac{\ln x}{(1+x)^2}dx$ 记 $y=\frac{1}{x}$ 则 $dy=-\frac{1}{x^2}dx$.

$\therefore \frac{\ln x}{(1+x)^2}dx=\frac{-\ln y}{(1+\frac{1}{y})^2}(-y^2)dy=\frac{\ln y}{(1+y)^2}dy$.

$\therefore \int_1^{+\infty}\frac{\ln x}{(1+x)^2}dx=-\int_0^1\frac{\ln y}{(1+y)^2}dy$.

$\therefore \int_0^{+\infty}\frac{\ln x}{(1+x)^2}dx=0$.

(10) 设 $x=\tan t$, 则 $dx=\frac{1}{\cos^2 t}dt$.

$\therefore \int_0^{+\infty}\frac{dx}{(2x^2+1)\sqrt{x^2+1}}=\int_0^{\frac{\pi}{2}}\frac{dt}{\cos^2 t\frac{1}{\cos t}\frac{1+\sin^2 t}{\cos^2 t}}=\int_0^{\frac{\pi}{2}}\frac{\cos t}{1+\sin^2 t}dt$.

$=\arctan x\Big|_0^1=\frac{\pi}{4}$

当$a\neq 0$时, 不妨$a>0$.

(11) 记 $x=a\tan t$ 则 $dx=\frac{a}{\cos^2 t}dt$.

$\therefore \int_0^{+\infty}\frac{x\,dx}{(x^2+a^2)^{\frac{3}{2}}}=\int_0^{\frac{\pi}{2}}\frac{\frac{a}{\cos^2 t}\cdot a\tan t}{a^3\frac{1}{\cos^3 t}}dt=\int_0^{\frac{\pi}{2}}\frac{\sin t}{a}dt=\frac{1}{a}$.

当$a=0$时

原式 $=\int_0^{+\infty}\frac{dx}{x^2}$ 不可积.

$\therefore$ 原式 $=\frac{1}{|a|}$ $(a\neq 0)$

10.1.5. $\Rightarrow$: 对$\forall X>a$ $\int_a^X x f'(x)dx=xf(x)\Big|_a^X-\int_a^X f(x)dx$.

$\because \int_a^{+\infty}f(x)dx$收敛 $\therefore \lim_{x\to+\infty}xf(x)=0$

$\therefore \lim_{X\to+\infty}\int_a^X xf'(x)dx=\lim_{X\to+\infty}xf(x)\Big|_a^X-\lim_{X\to+\infty}\int_a^X f(x)dx$.

$\therefore \int_a^{+\infty}xf'(x)dx$ 收敛.

$\Leftarrow$：$\forall t>x$，$\because f(t)$单调减 $\therefore f'(t)<0$ $\therefore tf'(t)<xf'(t)$

$\therefore \int_x^{x+p} tf'(t)dt<\int_x^{x+p} xf'(t)\,dt = x(f(x+p)-f(x))$

又 $\because \int_a^{+\infty} xf'(x)dx$ 收敛 $\therefore \forall \varepsilon>0, \exists x_0, \forall x>x_0, \forall p>0$ 均有

$\left|\int_x^{x+p} tf'(t)\,dt\right|<\varepsilon$. $\therefore x|f(x+p)-f(x)|<\varepsilon$.

固定 x，再 $p\to+\infty$，则 $|xf(x)|<\varepsilon$ 即 $xf(x)\to 0$.

由($\Rightarrow$)中可知 $\int_a^{+\infty} f(x)\,dx$ 收敛.

10.2.1. (2). $\because \lim\limits_{x\to+\infty}\dfrac{\frac{1}{x^3\sqrt{1+x^2}}}{\frac{1}{x^3}}=1$ 且 $\dfrac{1}{x^3\sqrt{1+x^2}}>0$ $\therefore$ 收敛.

(4). $p\le 1$ 时发散 $p>1$ 时收敛.

(8). $n\le 0$ 时发散 $n>0$ 时由Dirichlet判别法收敛.

(10). $\ln(1+\frac{1}{x})=\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2})\ (x\to+\infty)$ $\therefore \ln(1+\frac{1}{x})-\frac{1}{x+1}=\frac{1}{2x^2}+o(\frac{1}{x^2})\ (x\to+\infty)$

$\therefore$ 收敛.

(11). $\cos\frac{1}{x}+\sin\frac{1}{x}=1-\frac{1}{2x^2}+\frac{1}{x}+o(\frac{1}{x^2})$ $x\to+\infty$

$\ln(\cos\frac{1}{x}+\sin\frac{1}{x})=\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2})-\frac{1}{2}[-\frac{1}{2x^2}+\frac{1}{x}+o(\frac{1}{x^2})]^2+o(\frac{1}{x^2})$

$=\frac{1}{x}-\frac{1}{x^2}+o(\frac{1}{x^2})$

$\therefore \lim\limits_{x\to+\infty}\dfrac{\ln(\cos\frac{1}{x}+\sin\frac{1}{x})}{\frac{1}{x}}=1$

$\therefore$ 发散

(12) 分析 $\int_0^1 \frac{1}{x^2}\ln(1-\frac{\sin^2 x}{2})^{-1}dx$ 与 $\int_1^{+\infty}\dfrac{\ln(1-\frac{\sin^2 x}{2})^{-1}}{x^2}dx$.

$\because \dfrac{\ln(1-\frac{\sin^2 x}{2})^{-1}}{x^2}\ge 0$ 恒成立 且 $\dfrac{\ln(1-\frac{\sin^2 x}{2})^{-1}}{x^2}\le\dfrac{\ln 2}{x^2}$ 恒成立 $\therefore \int_1^{+\infty}\dfrac{\ln(1-\frac{\sin^2 x}{2})^{-1}}{x^2}dx$ 收敛.

下分析 $\int_0^1\dfrac{\ln(1-\frac{\sin^2 x}{2})^{-1}}{x^2}dx$ $\because \lim\limits_{x\to 0}\dfrac{\frac{\ln(1-\frac{\sin^2 x}{2})^{-1}}{x^2}}{\frac{1}{2}}=1$ $\therefore$ 收敛.

证毕.

10.2.3 $\because f(x)$单调下降 $\therefore f'(x)\le 0$恒成立.

$\therefore \left|\int_0^{+\infty} f'(x)\sin^2 x\,dx\right| \le \left|\int_0^{+\infty} f'(x)dx\right| = f(x)\big|_{+\infty}^{0} = f(0)$

$\because f'(x)\sin^2 x \le 0$ $\therefore$ 原式收敛

10.2.4 $\max(f(x),0) = \frac{f(x)+|f(x)|}{2}$

$\min(f(x),0) = \frac{f(x)-|f(x)|}{2}$

$\because \int_0^{+\infty} f(x)\,dx$收敛 $\int_0^{+\infty}|f(x)|dx$发散.

$\therefore \int_0^{+\infty}\frac{f(x)+|f(x)|}{2}dx$ 与 $\int_0^{+\infty}\frac{f(x)-|f(x)|}{2}dx$ 均发散.

10.2.6. (1) 由A氏判别法 结论显然.

(2) 构造一个递增序列 $\{x_n\}$ $a<x_1<\cdots<x_n<\cdots$

对 $\forall n\in N_+$. 均$\exists a\le y_n<+\infty$. st $\int_a^{x_n} f(x)g(x)dx = f(a)\int_a^{y_n} g(x)dx$

若$\{y_n\}$有界, 则$\exists$一个子列 $\{y_{n_k}\}$, st $\lim\limits_{k\to+\infty} y_{n_k} = y_0$ 且 $y_0\in[a,+\infty)$

且 $\int_a^{+\infty} f(x)g(x)dx = f(a)\int_a^{y_0} g(x)dx$.

若 $\{y_n\}$无界, 则$\exists$一个递增子列. $\{y_{n_k}\}$, st $a<y_{n_1}<\cdots<y_{n_k}<\cdots$

且 $\lim\limits_{k\to+\infty} y_{n_k} = +\infty$ $\therefore \lim\limits_{k\to+\infty}\int_a^{x_{n_k}} f(x)g(x)dx = \lim\limits_{k\to+\infty} f(a)\int_a^{y_{n_k}} g(x)dx$.

$\therefore \int_a^{+\infty} f(x)g(x)dx = f(a)\int_a^{+\infty} g(x)dx$

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