Peking University

思考题 5

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2021 年 4 月 6 日

1

设 $f_x(x,y)$ 在 (x_0,y_0) 存在, $f_y(x,y)$ 在 (x_0,y_0) 连续.

求证: $f(x,y)$ 在 (x_0,y_0) 可微.

证明.

$$
\begin{aligned}
f\left(x_0+\Delta x, y_0+\Delta y\right)-f\left(x_0, y_0\right)= & f\left(x_0+\Delta x, y_0+\Delta y\right)-f\left(x_0+\Delta x, y_0\right) & (1)\\
& +f\left(x_0+\Delta x, y_0\right)-f\left(x_0, y_0\right) & (2)\\
= & f_y\left(x_0+\Delta x, y_0+\theta \Delta y\right) \Delta y+f_x\left(x_0, y_0\right) \Delta x+o(\Delta x) & (3)\\
= & f_y\left(x_0, y_0\right) \Delta y+\alpha \Delta y+f_x\left(x_0, y_0\right) \Delta x+o(\Delta x), & (4)
\end{aligned}
$$

其中 $\alpha=f_y\left(x_0+\Delta x, y_0+\theta \Delta y\right)-f_y\left(x_0, y_0\right), \quad 0<\theta<1$

当 $\Delta x, \Delta y \rightarrow 0$ 时, $\alpha \rightarrow 0$,

$\therefore \lim\limits_{\Delta x \rightarrow 0, \Delta y \rightarrow 0} \alpha \Delta y / \sqrt{\Delta x^2+\Delta y^2}=0$.

$\therefore \alpha \Delta y=o\left(\sqrt{\Delta x^2+\Delta y^2}\right)$.

$\therefore f\left(x_0+\Delta x, y_0+\Delta y\right)-f\left(x_0, y_0\right)=f_y\left(x_0, y_0\right) \Delta y+f_x\left(x_0, y_0\right) \Delta x+o\left(\sqrt{\Delta x^2+\Delta y^2}\right)$.

$\square$

附: 二元函数 $f(x,y)$ 在 (x_0,y_0) 连续, 则在 (x_0,y_0) 临域内由

$$\begin{aligned}\left|f\left(x_0+\Delta x,y_0+\Delta y\right)-f\left(x_0+\Delta x,y_0\right)\right| \le &\left|f\left(x_0+\Delta x,y_0+\Delta y\right)-f\left(x_0,y_0\right)\right| &(5)\\ &+\left|f\left(x_0+\Delta x,y_0\right)-f\left(x_0,y_0\right)\right| &(6)\end{aligned}$$

可得 $f\left(x_0+\Delta x,y\right)$ 在 $\left(x_0+\Delta x,y_0\right)$ 关于 y 连续.

2

若函数 $f(x,y,z)$ 对任意正实数 t 满足关系 $f(tx,ty,tz)=t^nf(x,y,z)$, 则称 $f(x,y,z)$ 为 n 次齐次函数. 设 $f(x,y,z)$ 可微.

证明: $f(x,y,z)$ 为 n 次齐次函数的充要条件是 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z)$.

证明.

(1) 必要性对任意固定参数 X,Y,Z, 设 $x=Xt,\ y=Yt,\ z=Zt$,

$$\begin{aligned}f(x,y,z)=f(Xt,Yt,Zt)=t^nf(X,Y,Z), &\quad(7)\\ \frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial x}X+\frac{\partial f}{\partial y}Y+\frac{\partial f}{\partial z}Z=nt^{n-1}f(X,Y,Z), &\quad(8)\\ \frac{\partial f}{\partial x}Xt+\frac{\partial f}{\partial y}Yt+\frac{\partial f}{\partial z}Zt=nt^nf(X,Y,Z), &\quad(9)\end{aligned}$$

即

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z). \quad (10)$$

(2) 充分性对任意固定 x,y,z

设$X=xt,\ Y=yt,\ Z=zt,$

$$\begin{aligned}f(X,Y,Z)=&f(tx,ty,tz), &(11)\\ \frac{\mathrm{d}f(X,Y,Z)}{\mathrm{d}t}&=\frac{\partial f}{\partial X}x+\frac{\partial f}{\partial Y}y+\frac{\partial f}{\partial Z}z, &(12)\\ &=\frac{\partial f}{\partial X}\frac{X}{t}+\frac{\partial f}{\partial Y}\frac{Y}{t}+\frac{\partial f}{\partial Z}\frac{Z}{t}, &(13)\\ &=\frac{1}{t}nf(X,Y,Z). &(14)\\ \therefore f(X,Y,Z)=Ct^n. && (15)\end{aligned}$$

令 $t=1$ 可得 $C=f(x,y,z)$, 即 $f(xt,yt,zt)=t^nf(x,y,z)$.

$\square$

3

设 $f(x,y)$ 在区域 D 上满足 $f_x(x,y)\equiv 0$.

问: $f(x,y)$ 在 D 上能否表示为 $\varphi(y)$.

不能. 可举反例:

$$f(x,y)=\begin{cases}\operatorname{sgn}(x)y^2 & (x\neq 0, y>0)\\ 0 & (y\leq 0)\end{cases} \tag{16}$$

若对于区域内任意 $y=y_0$, x 的定义域是连续的（凸区域即满足此条件）, 则可由 $f(x,y)=f\left(x_0,y\right)+f_x\left(x_0+\theta\Delta x,y\right)\Delta x=f\left(x_0,y\right)$ 得 $f(x,y)$ 在 D 上能表示为 $\varphi(y)$.