Peking University

思考题 13

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$f(x) \in C[-h,h],\ h=\sqrt{\alpha^2+\beta^2+\gamma^2}$, 证明:

$$\iint_S f(\alpha x+\beta y+\gamma z)\,\mathrm{d}S = 2\pi\int_{-1}^{1} f(hu)\,\mathrm{d}u, \tag{1}$$

其中 $S = x^2+y^2+z^2=1$.

证明.

$$\begin{cases} \xi = a_1x+b_1y+c_1z \\ \eta = a_2x+b_2y+c_2z \\ \zeta = \dfrac{1}{h}(\alpha x+\beta y+\gamma z) \end{cases} \tag{2}$$

其中

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \frac{\alpha}{h} & \frac{\beta}{h} & \frac{\gamma}{h} \end{pmatrix} \tag{3}$$

是单位正交矩阵. 令

$$\xi=\cos\theta\cos\varphi,\quad \eta=\sin\theta\sin\varphi,\quad \zeta=\sin\varphi.\qquad D:\begin{cases} 0\le\theta\le 2\pi \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2} \end{cases} \tag{4}$$

$$
\begin{aligned}
\iint_S f(\alpha x+\beta y+\gamma z)\,\mathrm{d}S &= \iint_D f(h\zeta)\,\mathrm{d}S \\
&= \iint_D f(h\zeta)\cos\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \int_0^{2\pi}\mathrm{d}\theta\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(h\sin\varphi)\cos\varphi\,\mathrm{d}\varphi \\
&= 2\pi\int_{-1}^{1} f(hu)\,\mathrm{d}u,
\end{aligned}
\tag{5}
$$

□

2

在无穷大三维空间中, 半径为 R 的球面上均匀分布着电荷密度为 ρ 的电荷, 求任一空间点的电势.

$$
\begin{cases}
x = R\cos\theta\cos\varphi \\
y = R\sin\theta\cos\varphi \\
z = R\sin\varphi
\end{cases}
\tag{6}
$$

$$
\begin{aligned}
W(0,0,a) &= \iint_S \frac{\rho\,\mathrm{d}S}{\sqrt{x^2+y^2+(z-a)^2}} \\
&= \rho R^2\int_0^{2\pi}\mathrm{d}\theta\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\varphi\,\mathrm{d}\varphi}{\sqrt{R^2+a^2-2R\sin\varphi}} \\
&= \rho R^2\int_0^{2\pi}\mathrm{d}\theta\left(-\frac{1}{2Ra}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathrm{d}\big(R^2+a^2-2R\sin\varphi\big)}{\sqrt{R^2+a^2-2R\sin\varphi}} \\
&= \frac{2\pi\rho R}{a}\sqrt{R^2+a^2-2R\sin\varphi}\,\Big|_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \\
&= \frac{2\pi\rho R}{a}\big(R+a-|R-a|\big) = \begin{cases} 4\pi R\rho, & 0<a<R \\ \frac{4\pi R^2}{a}\rho, & a\ge R \end{cases}
\end{aligned}
\tag{7}
$$

3

设 $f(x,y)$ 连续, L 是一封闭的分段光滑简单曲线, 设

$$u(x,y)=\oint_L f(\xi,\eta)\ln\left(\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\right)\mathrm{d}s, \tag{8}$$

证明: $\lim\limits_{x\to\infty,y\to\infty} u(x,y)=0$ 的充要条件是 $\oint_L f(\xi,\eta)\,\mathrm{d}s=0$.

证明. $f(x,y)$ 在 L 上有界, $\left|f(x,y)\right|\le k$, 设 L 的长度为 S. 固定一点 $(\xi_0,\eta_0)\in\alpha$.

$$\left|\ln\left(\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\right)-\ln\left(\frac{1}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}\right)\right|=\left|\ln\left(\frac{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\right)\right| \tag{9}$$

趋于 0.

$$u(x,y)=\oint f(\xi,\eta)\ln\left(\frac{1}{\sqrt{(x-\xi_0)^2+(y-\eta_0)^2}}\right)+f(\xi,\eta)\ln\left(\frac{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\right)\mathrm{d}s \tag{10}$$

所以

$$\left|u(x,y)-\oint f(\xi,\eta)\ln\left(\frac{1}{\sqrt{(x-\xi_0)^2+(y-\eta_0)^2}}\right)\mathrm{d}s\right|\le kS\varepsilon \tag{11}$$

又

$$\oint f(\xi,\eta)\ln\left(\frac{1}{\sqrt{(x-\xi_0)^2+(y-\eta_0)^2}}\right)\mathrm{d}s=\ln\left(\frac{1}{\sqrt{(x-\xi_0)^2+(y-\eta_0)^2}}\right)\oint f(\xi,\eta)\,\mathrm{d}s \tag{12}$$

所以 $\lim\limits_{x\to\infty,y\to\infty} u(x,y)=0$ 的充要条件是 $\oint_L f(\xi,\eta)\,\mathrm{d}s=0$.

$\square$