17.3.9 对下列函数求出单叶性区域，并计算

$$\frac{\partial(u,v)}{\partial(x,y)},\quad \frac{\partial(x,y)}{\partial(u,v)}.$$

(1) $u=xy,\quad v=\dfrac{x}{y}$;

(2) $u=\dfrac{x}{x^2+y^2},\ v=\dfrac{y}{x^2+y^2}$;

(3) $u=x^2+y^2,\ v=2xy$.

(1). $\begin{cases} x_1y_1=x_2y_2 \\ \dfrac{x_1}{y_1}=\dfrac{x_2}{y_2} \end{cases}$ 得 $\begin{cases} x_1=x_2 \\ y_1=y_2 \end{cases}$ $\begin{cases} x_1=-x_2 \\ y_1=-y_2 \end{cases}$

又. x轴上 $u=v=0$. y轴无定义. ∴单叶区域:四个象限.

$\dfrac{\partial u}{\partial x}=y\quad \dfrac{\partial u}{\partial y}=x\quad \dfrac{\partial v}{\partial x}=\dfrac{1}{y}\quad \dfrac{\partial v}{\partial y}=\dfrac{-x}{y^2}$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix}=\frac{-x}{y}-\frac{x}{y}=\frac{-2x}{y}.$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{-y}{2x}=\frac{-1}{2v}.$$

(2) $u=\dfrac{x}{x^2+y^2}\quad v=\dfrac{y}{x^2+y^2}\Rightarrow u=\dfrac{\cos\theta}{r}\quad v=\dfrac{\sin\theta}{r}$. 由 $\begin{pmatrix}u_1\\v_1\end{pmatrix}=\begin{pmatrix}u_2\\v_2\end{pmatrix}$ 得.

$\begin{matrix}\dfrac{\cos\theta_1}{r_1}=\dfrac{\cos\theta_2}{r_2} \\ \dfrac{\sin\theta_1}{r_1}=\dfrac{\sin\theta_2}{r_2}\end{matrix}$ ∴ $\sin\theta_1\cos\theta_2=\sin\theta_2\cos\theta_1$

$\theta_1=\theta_2$.

∴ $r_1=r_2$. ∴ 于 $\mathbb{R}^2/\{0\}$ 单值.

$u=\frac{x}{x^2+y^2} \qquad v=\frac{y}{x^2+y^2}$

$u_x=\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2} \qquad u_y=\frac{-2xy}{(x^2+y^2)^2}$

$v_x=\frac{-xy}{(x^2+y^2)^2} \qquad v_y=\frac{1}{(x^2+y^2)}-\frac{2y^2}{(x^2+y^2)}$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{pmatrix} \frac{1}{r^2}-\frac{2\cos^2\theta}{r^2} & \frac{-2\sin\theta\cos\theta}{r^2} \\ \frac{-2\sin\theta\cos\theta}{r^2} & \frac{1}{r^2}-\frac{2\sin^2\theta}{r^2} \end{pmatrix}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)}=-(x^2+y^2)^2=\frac{-1}{(u^2+v^2)^2}$$

(3)

$u=x^2+y^2 \qquad u=r^2$

$v=2xy \qquad v=2r^2\sin\theta\cos\theta.$

$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}=\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \Rightarrow$

$r_1^2=r_2^2. \qquad 2r_1^2\sin\theta_1\cos\theta_1=2r_2^2\sin\theta_2\cos\theta_2$

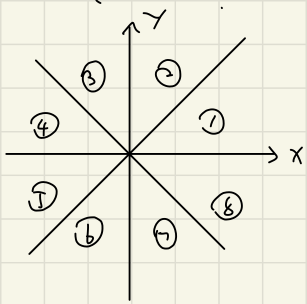
即 $\sin\theta_1\cos\theta_1=\sin\theta_2\cos\theta_2.$

$\sin(2\theta_1)=\sin(2\theta_2).$

$$\begin{cases} \theta_1=\theta_2 & (舍) \\ \theta_1=\theta_2+\pi \\ \theta_1+\theta_2=\frac{\pi}{2} \end{cases}$$

即 $\begin{cases} y_2=-y_1 \\ x_2=-x_1 \end{cases}$ 及 $\begin{cases} y_2=x_1 \\ x_2=y_1 \end{cases}$.

∴单叶区域：



共8个.

$u_x=2x \qquad v_x=2y \qquad \frac{\partial(u,v)}{\partial(x,y)}=\begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}=4(x^2-y^2) \qquad \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{4(x^2-y^2)}=\frac{1}{4\sqrt{(u+v)}\sqrt{u-v}}$

$u_y=2y \qquad v_y=2x$

17.3.10 求反函数的偏导数:

(1) 设 $u=x\cos\frac{y}{x}, v=x\sin\frac{y}{x}$. 求 $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$;

(2) 设 $u=e^x+x\sin y, v=e^x-x\cos y$. 求 $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

$$\begin{cases} x=\sqrt{u^2+v^2} \\ y=\sqrt{u^2+v^2}\operatorname{arctg}\frac{v}{u} \end{cases} \quad \therefore \quad \begin{cases} x_u=\frac{u}{\sqrt{u^2+v^2}} \\ y_u=\frac{u}{\sqrt{u^2+v^2}}\operatorname{arctg}\frac{v}{u}-\frac{v}{\sqrt{u^2+v^2}} \end{cases}$$

$$\begin{cases} x_v=\frac{v}{\sqrt{u^2+v^2}} \\ y_v=\frac{v}{\sqrt{u^2+v^2}}\operatorname{arctg}\frac{v}{u}+\frac{u}{\sqrt{u^2+v^2}} \end{cases}$$

(2) $$\begin{cases} u=e^x+x\sin y \\ v=e^x-x\cos y \end{cases} \quad \begin{cases} x_u e^x+x_u\sin y+x y_u\cos y=1 \\ x_u e^x-x_u\cos y+x y_u\sin y=0 \end{cases}$$

$$\Rightarrow \begin{cases} x_u=\frac{\sin y}{e^x(\sin y-\cos y+1)} \\ y_u=\frac{e^x-\cos y}{x(e^x(\sin y-\cos y)+1)} \end{cases}$$

同理.

$$x_v=\frac{\cos y}{e^x(\sin y-\cos y)+1}$$

$$y_v=\frac{e^x+\sin y}{x(e^x(\sin y-\cos y)+1)}$$

(本问无显式解)

17.4.3 由下列方程组求$\frac{dy}{dx}$, $\frac{dz}{dx}$和$\frac{d^2y}{dx^2}$, $\frac{d^2z}{dx^2}$.

(1) $\begin{cases} x+y+z=0, \\ x^2+y^2+z^2=1; \end{cases}$ (2) $\begin{cases} x^3+y^3+z^3=3xyz, \\ x+y+z=a. \end{cases}$

(1) $\begin{cases} 1+y'+z'=0 \\ 2x+2yy'+2zz'=0 \end{cases}$

解得 $z'=\frac{y-x}{z-y}$ $y'=\frac{z-x}{y-z}$

$z''=\frac{1}{(z-y)^2}\left[(x-2z+y)-(y-x)\frac{(y+z-2x)}{z-y}\right]$

$y''=\frac{1}{(z-y)^2}\left[(x-2y+z)-(z-x)\frac{(z+y-2x)}{y-z}\right]$

(2)

$\begin{cases} x^3+y^3+z^3=3xyz \\ x+y+z=a \end{cases}$

$\begin{cases} 3x^2+3y^2y'+3z^2z'=3yz+3xy'z+3xyz' \\ 1+y'+z'=0 \end{cases}$

$\therefore \frac{dy}{dx}=\frac{z-x}{y-z}$

$\frac{dz}{dx}=-\frac{y-x}{y-z}$

$y''=\frac{1}{(y-z)^2}\left((z'-1)(y-z)-(z-x)(y'-z')\right)$

$=\frac{1}{(y-z)^2}\left[(x+z-2y)-\frac{(z-x)(z+y-2x)}{(y-z)}\right]$

$z''=\frac{1}{(y-z)^2}\left[(y'-1)(z-y)-(y-x)(z'-y')\right)$

$=\frac{1}{(y-z)^2}\left[(x+y-2z)-\frac{(y-x)(y+z-2x)}{(z-y)}\right]$

17.4.5 设

$$\begin{cases} x=\cos\varphi\cos\psi, \\ y=\cos\varphi\sin\psi, \\ z=\sin\varphi. \end{cases}$$

$x^2+y^2+z^2=1$. $z=z(x,y)$.

$2z\dfrac{\partial z}{\partial x}+2x=0 \qquad \dfrac{\partial z}{\partial x}=\dfrac{-x}{z}$

$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{-1}{z}+\dfrac{x}{z}\dfrac{\partial z}{\partial x} = \dfrac{-1}{z}-\dfrac{x^2}{z^3}$

17.4.7 求由下列方程组所确定的函数 $u=u(x,y)$ 的所有二阶偏导数.

(1) $u=yz+zx+xy,\ x^2+y^2+z^2=1$;

$\begin{cases} u_x=yz_x+z+xz_x+y \\ u_y=z+yz_y+xz_y+x \end{cases} \qquad \begin{cases} 2x+2zz_x=0 \\ 2y+2zz_y=0 \end{cases} \Rightarrow \begin{cases} z_x=\dfrac{-x}{z} \\ z_y=\dfrac{-y}{z} \end{cases} \qquad \begin{cases} z_{xx}=\dfrac{-1}{z}-\dfrac{x^2}{z^3} \\ z_{yy}=\dfrac{-1}{z}-\dfrac{y^2}{z^3} \\ z_{xy}=\dfrac{-xy}{z^3} \end{cases}$

$u_{xx}=yz_{xx}+2z_x+xz_{xx}=\dfrac{-y}{z}-\dfrac{x^2y}{z^3}-\dfrac{3x}{z}-\dfrac{x^3}{z^3}$

$u_{yy}=2z_y+yz_{yy}+xz_{yy}=\dfrac{-x}{z}-\dfrac{y^2x}{z^3}-\dfrac{3y}{z}-\dfrac{y^3}{z^3}$

$u_{xy}=z_x+yz_{xy}+z_y+xz_{xy}+1$

$=1-\dfrac{x}{z}-\dfrac{y}{z}-\dfrac{xy^2}{z^3}-\dfrac{x^2y}{z^3}$

17.4.7 求由下列方程组所确定的函数 $u=u(x,y)$ 的所有二阶偏导数.

(1) $u=yz+zx+xy, x^2+y^2+z^2=1$;

(2) $u=xyz, x^2+y^2+z^2=1$.

(2) $u_x = yz + xyz_x + y$

$u_y = xz + xyz_y + x$

$$\begin{cases} u_{xx} = 2yz_x + xyz_{xx} = \dfrac{-3yx}{z} - \dfrac{x^3y}{z^3} \\ u_{yy} = 2xz_y + xyz_{yy} = -\dfrac{3yx}{z} - \dfrac{y^3x}{z^3} \\ u_{xy} = yz_y + z + xz_x + xyz_{xy} + 1 = \dfrac{-x^2}{z} - \dfrac{y^2}{z} + z - \dfrac{x^2y^2}{z^3} + 1 \end{cases}$$

17.4.9 设

$yz_y + z$

$xz_x + \cdot$

$$\begin{cases} u=f(x,y,z,t), \\ g(y,z,t)=0, \\ h(z,t)=0. \end{cases}$$

什么条件下 u 是 x,y 的函数? 并求 $\dfrac{\partial u}{\partial x}$ 和 $\dfrac{\partial u}{\partial y}$.

即 $z=z(y)$ $\quad t=t(y)$.

$\therefore \dfrac{\partial(g,h)}{\partial(z,t)} \neq 0$. 且 $\begin{pmatrix} z' \\ t' \end{pmatrix} = \left(\dfrac{\partial(g,h)}{\partial(z,t)}\right)^{-1} \begin{pmatrix} g_y \\ 0 \end{pmatrix}$.

$u_x = f_x$ $\qquad u_y = f_y + f_z z' + f_t t'$.

18.1.3 在曲线 $y=x^2, z=x^3$ 上求出一点，使此点的切线平行于平面 $x+2y+z=4$。

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}.$$

$x+2y+z=4$ 法向量：$(1,2,1)$.

$\therefore (1,2x,3x^2)\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=0$ 即 $3x^2+4x+1=0$ $(x+1)(3x+1)=0$

$\therefore x=-1$ 或 $x=-\frac{1}{3}$

$\therefore (-1,1,-1)$ 或 $\left(\frac{-1}{3},\frac{1}{9},\frac{-1}{27}\right)$.

18.1.7 设有一曲线 $f(r,\varphi)=0$，其中 r,φ 为极坐标，又设 f'_r、f'_φ 不同时为零，求其曲率公式.

设曲线充分光滑.

$f_r \neq 0 \Rightarrow r = r(\theta)$.

$f(r(\theta),\theta)=0$

$f_r r' + f_\theta = 0 \qquad r'(\theta) = \dfrac{-f_\theta}{f_r}$

$f_{rr}(r')^2 + f_r r'' + 2f_{r\theta} r' + f_{\theta\theta} = 0$

$$\therefore r'' = \frac{-f_{\theta\theta} - 2f_{r\theta}r' - f_{rr}(r')^2}{f_r} = \frac{-f_r^2 f_{\theta\theta} + 2f_{r\theta} f_\theta f_r - f_{rr}f_\theta^2}{f_r^3}$$

设 $x = r(\theta)\cos\theta = x(\theta)$

$y = r(\theta)\sin\theta = y(\theta)$ $\qquad ds = \sqrt{(x')^2+(y')^2}\,d\theta$

$$d\alpha = d\,\mathrm{arctg}\left(\frac{dy}{dx}\right) = d\left(\mathrm{arctg}\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)\right)$$

$$= \frac{x'y''-x''y'}{[(x')^2+(y')^2]}\,d\theta$$

$$\therefore k = \left|\frac{d\alpha}{ds}\right| = \left|\frac{x'y''-x''y'}{((x')^2+(y')^2)^{\frac{3}{2}}}\right| = \left|\frac{r^2+2(r')^2+rr''}{(r^2+(r')^2)^{\frac{3}{2}}}\right|$$

代入 r', r'' 即可.

$$\left|\frac{r^2 f_r^3 + 2f_\theta^2 f_r + r(-f_r^2 f_{\theta\theta} + 2f_{r\theta} f_r f_\theta - f_{rr} f_\theta^2)}{(r^2 f_r^2 + f_\theta^2)}\right|$$

$f_\theta \neq 0 \Rightarrow \theta = \theta(r)$

$\theta'(r) = \dfrac{-f_r}{f_\theta}$

$$\theta''(r) = \frac{-f_{rr}f_\theta^2 + 2f_{r\theta} f_r f_\theta - f_{\theta\theta} f_r^2}{f_\theta^3}$$

$x = r\cos(\theta(r)) = x(r)$

$y = r\sin\theta(r) = y(r)$

同理.

$$k = \frac{x'y''-x''y'}{((x')^2+(y')^2)^{\frac{3}{2}}},$$

$x' = \cos\theta - r\theta'\sin\theta \qquad x'' = 2\theta'\sin\theta - r\theta''\sin\theta - r(\theta')^2\cos\theta$

$y' = \sin\theta + r\theta'\cos\theta \qquad y'' = 2\theta'\cos\theta + r\theta''\cos\theta - r(\theta')^2\sin\theta$

$x'y''-x''y' = 2\theta' + r\theta'' + r^2(\theta')^3$

$(x')^2+(y')^2 = 1 + r^2(\theta')^2$

$$\therefore k(r) = \left|\frac{2\dot\theta + r\ddot\theta + r^2\dot\theta^3}{(1+r^2\dot\theta^2)^{\frac{3}{2}}}\right|$$ 代入 θ', θ'' 即可.

$$= \left|\frac{2f_r f_\theta^2 + r^2 f_r^3 + r(-f_{rr}f_\theta^2 + 2f_{r\theta} f_r f_\theta - f_{\theta\theta} f_r^2)}{(f_\theta^2 + r^2 f_r^2)^{\frac{3}{2}}}\right|.$$

左右形式一致，可见与 $f_r=0$ 还是 $f_\theta=0$ 无关.

(4) $z=y+\ln\frac{x}{z}$, $\quad P_0(1,1,1)$.

求法线.切平面.

$F(x,y,z)= \quad z-y-\ln x+\ln z=0$

$\frac{\partial F}{\partial x}=\frac{-1}{x}$, $\frac{\partial F}{\partial y}=-1$ $\frac{\partial F}{\partial z}=1+\frac{1}{z}$

$\therefore$ 法向: $\begin{pmatrix}-1\\-1\\2\end{pmatrix}$. 切平面: $-(x-1)-(y-1)+2(z-1)=0$.

18.2.2 求下列曲面在指定点的切平面方程:

(1) $x=a\cos\varphi\cos\theta, y=b\cos\varphi\sin\theta, z=c\sin\varphi$, 于$M_0(\theta_0,\varphi_0)$处;

(2) $x=u\cos v, y=u\sin v, z=av$, 于$M_0(u_0,v_0)$处.

$\begin{cases}x=x(u,v)\\y=y(u,v)\\z=z(u,v)\end{cases}$ $\begin{cases}\frac{\partial x}{\partial u}=\cos v\\\frac{\partial x}{\partial v}=-u\sin v\end{cases}$ $\begin{cases}\frac{\partial y}{\partial u}=\sin v\\\frac{\partial y}{\partial v}=u\cos v\end{cases}$ $\begin{cases}\frac{\partial z}{\partial u}=0\\\frac{\partial z}{\partial v}=a\end{cases}$

$\therefore$ 法向: $(\cos v,\sin v,0)\times(-u\sin v,u\cos v,a)=(a\sin v,-a\cos v,u)$

$\therefore$ 切平面: $a\sin v_0(x-u_0\cos v_0)-a\cos v_0(y-u_0\sin v_0)+u_0(z-av_0)=0$.

18.2.5 求曲面 $x^2+2y^2+3z^2=21$ 的平行于平面 $x+4y+6z=0$ 的各切平面。

$\frac{\partial F}{\partial x}=2x \quad \frac{\partial F}{\partial y}=4y \quad \frac{\partial F}{\partial z}=6z$，法向：$\begin{pmatrix} x \\ 2y \\ 3z \end{pmatrix}$

$$\therefore \begin{cases} \frac{x}{1}=\frac{2y}{4}=\frac{3z}{6} \\ x^2+2y^2+3z^2=21 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=2 \\ z=2 \end{cases} \text{ or } \begin{cases} x=-1 \\ y=-2 \\ z=-2 \end{cases}$$

$$\therefore \text{切面：} \begin{cases} (x-1)+4(y-2)+6(z-2)=0 \\ (x+1)+4(y+2)+6(z+2)=0. \end{cases}$$

18.2.9 求椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 上点的法向量与 x 轴、z 轴成等角的点的轨迹

解：$\vec{n}=\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$

$\vec{n}\cdot\vec{i}=\vec{n}\cdot\vec{k}$，$\therefore \frac{x}{a^2}=\frac{z}{c^2}$

即 $\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \\ \frac{x}{a^2}=\frac{z}{c^2} \end{cases}$ 交线，为一椭圆。

18.2.11 证明：曲面

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$$

的切平面在坐标轴上割下的诸线段之和为常量.

法向：$\left(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}, \frac{1}{\sqrt{z}}\right)$,

切平面：$\frac{1}{\sqrt{x_0}}(x-x_0)+\frac{1}{\sqrt{y_0}}(y-y_0)+\frac{1}{\sqrt{z_0}}(z-z_0)=0$.

于 x 轴截距：$\frac{1}{\sqrt{x_0}}(x-x_0)=\sqrt{y_0}+\sqrt{z_0}$

$$x=x_0+\sqrt{x_0}(\sqrt{y_0}+\sqrt{z_0})$$

∴截距和：$x_0+y_0+z_0+2\sqrt{x_0}\sqrt{y_0}+2\sqrt{y_0}\sqrt{z_0}+2\sqrt{z_0}\sqrt{x_0}$

$=(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2=a$. 为与 (x_0,y_0,z_0) 无关的常数,

□.

18.2.16 求下列曲线在给定点处的切线方程：

(1) $x^2+y^2+z^2=6, x+y+z=0$, 于$(1,-2,1)$处；

(2) $x^2+z^2=10, y^2+z^2=10$, 于$(1,1,3)$处；

(3) $z=x^2+y^2, 2x^2+2y^2-z^2=0$, 于$(1,1,2)$处。

$$\begin{cases} f(x,y,z)=0 \\ g(x,y,z)=0 \end{cases} \qquad \begin{cases} f_x\,dx+f_y\,dy+f_z\,dz=0 \\ g_x\,dx+g_y\,dy+g_z\,dz=0 \end{cases} \qquad \therefore (dx,dy,dz) \parallel \nabla f\times\nabla g.$$

(1). $\nabla f=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}=\begin{pmatrix}2\\-4\\2\end{pmatrix}$. $\nabla g=\begin{pmatrix}1\\1\\1\end{pmatrix}$. $\therefore \vec{\tau}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$

切线：$\dfrac{x-1}{-1}=\dfrac{y+2}{0}=\dfrac{z-1}{1}$.

(2). $\nabla f=\begin{pmatrix}2x\\0\\2z\end{pmatrix}=\begin{pmatrix}2\\0\\6\end{pmatrix}$. $\nabla g=\begin{pmatrix}0\\2y\\2z\end{pmatrix}=\begin{pmatrix}0\\2\\6\end{pmatrix}$. $\therefore \vec{\tau}=\begin{pmatrix}-3\\-3\\1\end{pmatrix}$

切线：$\dfrac{x-1}{-3}=\dfrac{y-1}{-3}=\dfrac{z-3}{1}$

(3). $\nabla f=\begin{pmatrix}2x\\2y\\-1\end{pmatrix}=\begin{pmatrix}2\\2\\-1\end{pmatrix}$. $\nabla g=\begin{pmatrix}4x\\4y\\-2z\end{pmatrix}=\begin{pmatrix}4\\4\\-4\end{pmatrix}$ $(1,1,2)$处 $\vec{\tau}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\therefore$ 切线：$\dfrac{x-1}{-1}=\dfrac{y-1}{1}=\dfrac{z-2}{0}$