

参考答案 12

袁磊祺

2021 年 6 月 8 日

21.2.1

计算下列第一型曲线积分:

(3)

$\int_C xyz \, ds$, C 为螺线:

$$x = a \cos t, \quad y = a \sin t, \quad z = bt, \quad 0 < a < b, \quad 0 \leq t \leq 2\pi.$$

$$ds = \sqrt{a^2 + b^2} \, dt, \tag{1}$$

$$\begin{aligned} \int_C xyz \, ds &= \int_0^{2\pi} a^2 b \sqrt{a^2 + b^2} t \sin t \cos t \, dt \\ &= a^2 b \sqrt{a^2 + b^2} \int_0^{2\pi} t \sin t \cos t \, dt. \\ &= -\frac{a^2 b \sqrt{a^2 + b^2} \pi}{2} \end{aligned} \tag{2}$$

(4)

$\int_C \sqrt{x^2 + y^2} \, ds$, C 为圆周 $x^2 + y^2 = ax$.

设

$$\begin{cases} x = \frac{a}{2} \cos \theta + \frac{a}{2} \\ y = \frac{a}{2} \sin \theta \end{cases} \tag{3}$$

$$ds = \frac{a}{2} d\theta$$

$$\begin{aligned} \int_0^{2\pi} a \sqrt{\frac{1}{2} \cos \theta + \frac{1}{2} \frac{a}{2}} d\theta &= \frac{a^2}{2} \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right| d\theta \\ &= 2a^2 \end{aligned} \quad (4)$$

(6)

$\int_C xy \, ds$, C 为球面 $x^2 + y^2 + z^2 = a^2$ 与平面 $x + y + z = 0$ 之交线.

$$\begin{aligned} \int_C xy \, ds &= \int_c \frac{(x + y + z)^2 - (x^2 + y^2 + z^2)}{6} \, ds \\ &= \int_c \frac{0 - a^2}{6} \, ds \\ &= -\frac{\pi a^3}{3} \end{aligned} \quad (5)$$

21.2.4

方法一

换用极坐标有

$$\begin{aligned} u(x, y) &= -\frac{\rho_0}{2} \oint \ln(R^2 + r^2 - 2Rr \cos \varphi) R \, d\varphi \\ &= -\rho_0 R \int_0^\pi \ln R^2 \left[1 + \left(\frac{r^2}{R^2} \right) - 2 \frac{r}{R} \cos \varphi \right] d\varphi \\ &= -\rho_0 R \int_0^\pi \ln R^2 \, d\varphi - \rho_0 R \int_0^\pi \ln \left[1 + \left(\frac{r^2}{R^2} \right) - 2 \frac{r}{R} \cos \varphi \right] d\varphi \\ &= -2\pi R \rho_0 \ln R - \rho_0 R \int_0^\pi \ln \left[1 + \left(\frac{r^2}{R^2} \right) - 2 \frac{r}{R} \cos \varphi \right] d\varphi \end{aligned} \quad (6)$$

设 $a = \frac{r}{R}$,

$$J(a) = \int_0^\pi \ln(a^2 - 2a \cos \varphi + 1) \, d\varphi \quad (7)$$

对 J 求导

$$J'(a) = 2 \int_0^\pi \frac{(a - \cos \varphi) d\varphi}{1 - 2a \cos \varphi + a^2} = \left(\frac{x}{2a} + \frac{(a^2 - 1) \arctan \left(\frac{|a+1| \tan(\frac{x}{2})}{|a-1|} \right)}{a|a-1||a+1|} \right) \Big|_0^\pi \quad (8)$$

$$= \begin{cases} \frac{\pi}{a}, & a \geq 1 \\ 0, & a < 1 \end{cases}$$

又 $J(0) = 0$, 所以

$$u(x, y) = \begin{cases} 2\pi\rho_0 R \ln \frac{1}{R}, & r = \sqrt{x^2 + y^2} < R \\ 2\pi\rho_0 R \ln \frac{1}{r}, & r = \sqrt{x^2 + y^2} > R \end{cases} \quad (9)$$

方法二

此题可看作无穷长带电圆柱, 可以参考电磁学教程. 根据对称性和高斯定理可得电场分布是: 圆柱外的电场径向向外, 圆柱内无电场, 圆柱电荷面密度

$$\sigma_e = 2\pi\varepsilon_0\rho_0. \quad (10)$$

当 $\sqrt{x^2 + y^2} > R$ 时, 等效为一无穷长直线产生的电场, 线密度

$$\eta_e = 2\pi R\sigma_e = 4\pi^2 R\varepsilon_0\rho_0. \quad (11)$$

产生的电场为

$$E(x, y) = \frac{\eta_e}{2\pi\varepsilon_0\sqrt{x^2 + y^2}} = \frac{2\pi R\rho_0}{\sqrt{x^2 + y^2}} \quad (12)$$

电势

$$u(x, y) = - \int E d\sqrt{x^2 + y^2} = -2\pi R\rho_0 \ln \left(\sqrt{x^2 + y^2} \right). \quad (13)$$

当 $\sqrt{x^2 + y^2} < R$ 时, 由于内部无电场, 则

$$u(x, y) = \oint_L \rho_0 \ln \frac{1}{r} ds = -2\pi R\rho_0 \ln R. \quad (14)$$

22.2.2

$$\vec{r} = ((b + a \cos \psi) \cos \varphi, (b + a \cos \psi) \sin \varphi, a \sin \psi) \quad (15)$$

$$\vec{r}_\psi = (-a \sin \psi \cos \varphi, -a \sin \psi \sin \varphi, a \cos \psi) \quad (16)$$

$$\vec{r}_\varphi = (-(b + a \cos \psi) \sin \varphi, (b + a \cos \psi) \cos \varphi, 0) \quad (17)$$

$$E = \vec{r}_\psi \cdot \vec{r}_\psi = a^2, \quad F = \vec{r}_\psi \cdot \vec{r}_\varphi = 0, \quad G = \vec{r}_\varphi \cdot \vec{r}_\varphi = (b + \cos \psi)^2 \quad (18)$$

$$\begin{aligned} S &= \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\psi_1}^{\psi_2} \sqrt{EG - F^2} d\psi \\ &= \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\psi_1}^{\psi_2} a(b + a \cos \psi) d\psi \end{aligned} \quad (19)$$

$$\begin{aligned} &= ab(\varphi_2 - \varphi_1)(\psi_2 - \psi_1) + a^2(\varphi_2 - \varphi_1) \int_{\psi_1}^{\psi_2} \cos \psi d\psi \\ &= ab(\varphi_2 - \varphi_1)(\psi_2 - \psi_1) + a^2(\varphi_2 - \varphi_1)(\sin \psi_2 - \sin \psi_1) \end{aligned}$$

$$S_{\text{all}} = 4\pi^2 ab. \quad (20)$$

22.2.5

$$\frac{\partial z}{\partial x} = \sqrt{\frac{y}{2x}} \quad \frac{\partial z}{\partial y} = \sqrt{\frac{x}{2y}} \quad (21)$$

$$\begin{aligned} \int_0^1 dx \int_{1-x}^1 \sqrt{\frac{y}{2x} + \frac{x}{2y} + 1} dy &= \int_0^1 dx \int_{1-x}^1 \frac{x+y}{\sqrt{2xy}} dy \\ &= \int_0^1 dx \int_{1-x}^1 \left(\sqrt{\frac{x}{2y}} + \sqrt{\frac{y}{2x}} \right) dy \\ &= \int_0^1 \left(\sqrt{\frac{x}{2}} 2\sqrt{y} + \sqrt{\frac{1}{2x}} \frac{2}{3} y^{\frac{3}{2}} \right) \bigg|_{1-x}^1 dy \\ &= \frac{4\sqrt{2}}{3} - \frac{\sqrt{2}}{4} \pi \end{aligned} \quad (22)$$

22.2.6

$$3(x^2 + y^2) = (2a - x - y)^2 \Rightarrow x^2 + y^2 - xy + 2a(x + y) = 2a^2 \quad (23)$$

$$S_1 = \iint_D \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy = 2 \iint_D dx \, dy \quad (24)$$

$$S_2 = \iint_D \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy = \sqrt{3} \iint_D dx \, dy \quad (25)$$

做变量替换

$$u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{x-y}{\sqrt{2}} \Rightarrow \frac{(u+2\sqrt{2}a)^2}{12a^2} + \frac{v^2}{4a^2} = 1 \quad (26)$$

x, y 平面内的面积

$$\iint_D dx \, dy = 4\sqrt{3}a^2 \quad (27)$$

总面积

$$S = 42a^2(2\sqrt{3} + 3) \quad (28)$$

体积

$$V = \frac{1}{3}S_2 \cdot H = \frac{8\sqrt{3}}{3}\pi a^3 \quad (29)$$

22.2.7

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (30)$$

$$\frac{\partial z}{\partial x} = -\frac{c}{a^2} \frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{c^2}{b^2} \frac{y}{z} \quad (31)$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{c^2}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)} = A \quad (32)$$

$$S_1 = \iint_{D_1} A \, dx \, dy \quad S_2 = \iint_{D_2} A \, dx \, dy \quad (33)$$

其中

$$\begin{aligned} D_1 &= \left\{ (x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad lx + my + nz \geq p \right\} \\ D_2 &= \left\{ (x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad lx + my + nz \leq p \right\} \end{aligned} \quad (34)$$

22.3.1

求下列积分

(1)

$$z = \sqrt{x^2 + y^2} \Rightarrow z_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad z_y = \frac{y}{\sqrt{x^2 + y^2}} \quad (35)$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{2}. \quad (36)$$

$$\begin{aligned} \iint_S (x^2 + y^2) \, ds &= \iint_{x^2 + y^2 \leq 1} \sqrt{2} (x^2 + y^2) \, dx \, dy + \iint_{x^2 + y^2 \leq 1} (x^2 + y^2) \, dx \, dy \\ &= \int_0^{2\pi} d\theta \int_0^1 (\sqrt{2} + 1) r^3 \, dr \\ &= \frac{\pi}{2} (\sqrt{2} + 1) \end{aligned} \quad (37)$$

(2)

$$\frac{\partial z}{\partial x} = 2x \quad \frac{\partial z}{\partial y} = 2y \quad (38)$$

$$\begin{aligned} \iint_S |zx^3y^2| \, ds &= \iint_{x^2 + y^2 \leq 1} |x^3 - y^2| (x^2 + y^2) \sqrt{4x^2 + 4y^2 + 1} \, dx \, dy \\ &= \int_0^{2\pi} d\theta \int_0^1 r^8 \sqrt{4r^2 + 1} |\cos^3 \theta \sin^2 \theta| \, dr \\ &= \frac{8}{15} \times \frac{40262\sqrt{5} + 21 \operatorname{arcsinh}(2)}{393216} \\ &= \frac{40262\sqrt{5} + 21 \operatorname{arcsinh}(2)}{737280} \end{aligned} \quad (39)$$

(3)

$$x_u = \cos v, \quad y_u = \sin v \quad z_u = 0 \quad (40)$$

$$x_v = -u \sin v, \quad y_v = u \cos v, \quad z_v = 1 \quad (41)$$

$$E = \vec{r}_u \cdot \vec{r}_u = 1 \quad F = \vec{r}_u \cdot \vec{r}_v = 0 \quad G = \vec{r}_v \cdot \vec{r}_v = u^2 + 1 \quad (42)$$

$$\begin{aligned}
\iint_S z^2 \, ds &= \int_0^a du \int_0^{2\pi} \sqrt{u^2 + 1} v^2 \, dv \\
&= \frac{8\pi^3}{3} \int_0^a \sqrt{u^2 + 1} \, du \\
&= \frac{4\pi^3}{3} \left[a\sqrt{1 + a^2} + \ln \left(a + \sqrt{1 + a^2} \right) \right]
\end{aligned} \tag{43}$$

(4)

$$\begin{aligned}
\iint_S (x^2 + y^2) \, ds &= \frac{2}{3} \iint_S x^2 + y^2 + z^2 \, ds \\
&= \frac{8}{3} \pi R^4
\end{aligned} \tag{44}$$