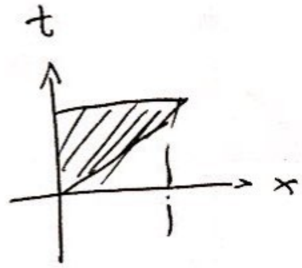
20.4.9 设一元函数 $g(x)$ 在 $[0,1]$ 可积. 证明

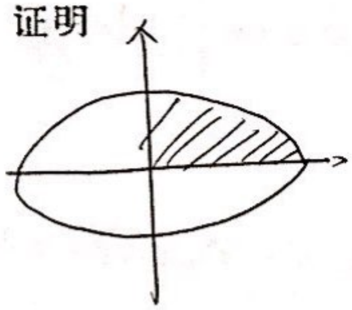
$$\int_0^1 dx\int_x^1 g(t)\,dt=\int_0^1 t\,g(t)\,dt.$$



$$\int_0^1 g(t)\,dt\int_0^t dx=\int_0^1 t\,g(t)\,dt$$

20.4.11 设 m 和 n 是正整数，且都是偶数. 证明

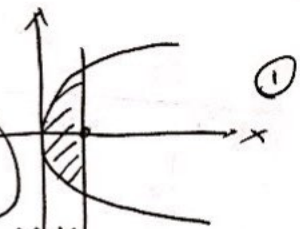
$$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} x^m y^n\,dx\,dy=4\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\\ x\ge 0,\ y\ge 0}} x^m y^n\,dx\,dy.$$



20.4.12 计算下列二重积分:

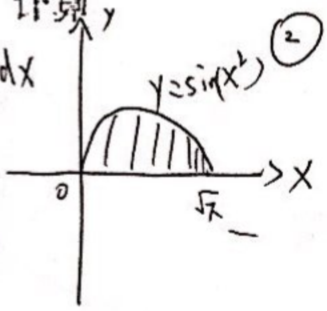
(1) Ω 是由 $y^2=2px\,(p>0)$, $x=\frac{p}{2}$ 围成的区域，计算

$\int_0^{\frac{p}{2}} dx\int_{-\sqrt{2px}}^{\sqrt{2px}} x^m y^k\,dy$ $\qquad \iint\limits_{\Omega} x^m y^k\,dx\,dy\ (m,k>0)$; $\qquad \int_{-p}^{p} dy\int_{\frac{y^2}{2p}}^{\frac{p}{2}} x^m y^n\,dx$



(2) Ω 由 $y=0$, $y=\sin(x^2)$, $x=0$ 和 $x=\sqrt{\pi}$ 围成，计算

$=\frac{1}{2}\int_0^{\sqrt{\pi}}\sin(x^2)\,dx^2=1$ $\qquad \iint\limits_{\Omega} x\,dx\,dy$; $\qquad \int_0^{\sqrt{\pi}} dx\int_0^{\sin(x^2)} x\,dy=\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$



(3) $\Omega=\{(x,y)\,|\,0\le x\le y^2,\ 0\le y\le 2+x,\ x\le 2\}$，计算

$$\iint\limits_{\Omega} x^2y^2\,dx\,dy;$$

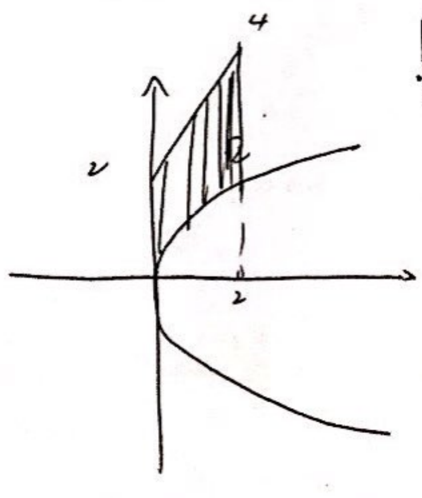
(4) Ω 由 $y=\sqrt{1-x^2}$, $y=0$ 围成，计算 $\iint\limits_{\Omega}(x^2+3xy^2)\,dx\,dy$;

(5) Ω 由 $y=e^x$, $y=1$, $x=0$ 及 $x=1$ 围成，计算

$$\iint\limits_{\Omega}(x+y)\,dx\,dy;$$

$\int_0^2 x^2\,dx\int_{\sqrt{x}}^{2+x} y^2\,dy$

3



$\sqrt{x}=2+x$

$x=4+4x+x^2$

$x^2+3x+4=0$

$=\frac{1}{3}\int_0^2 x^2\left[(2+x)^3-x^{\frac{3}{2}}\right]dx$

$=\frac{592}{15}-\frac{2^{11/2}}{27}$

北京大学

20.4.9. 考虑引理(即 20.4.8) 对于区间 $S \{(x,y) \mid 0 \le y \le x,\ 0 \le x \le a\}$.

$$\iint_S f(x,y)\,dx\,dy = \int_0^a dx \int_0^x f(x,y)\,dy$$

$$\iint_S f(x,y)\,dx\,dy = \int_0^a dy \int_y^a f(x,y)\,dx$$

$$\therefore \int_0^a dx \int_0^x f(x,y)\,dy = \int_0^a dy \int_y^a f(x,y)\,dx.$$

$$\therefore \int_0^1 dx \int_x^1 g(t)\,dt = \int_0^1 dt \int_0^t g(t)\,dx = \int_0^1 t\,g(t)\,dt.$$

20.4.11. $LHS = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} x^m y^n\,dy.$

$$= 4\int_0^a dx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} x^m y^n\,dy.$$

$$= RHS.$$

20.4.12. (1). 原式 $= \int_{-p}^{p} dy \int_{\frac{y^2}{2p}}^{\frac{p}{2}} x^m y^n\,dx.$

$$= \int_{-p}^{p} dy\ \frac{\left(\frac{p}{2}\right)^{m+1} - \left(\frac{y^2}{2p}\right)^{m+1}}{m+1}\, y^n\,dx.$$

$$= \int_{-p}^{p} \left[\frac{p^{m+1}}{2^{m+1}(m+1)}\, y^n - \frac{1}{2^{m+1}p^{m+1}(m+1)}\, y^{n+2m+2}\right] dx.$$

$$= \frac{p^{m+1}\left[p^n - (-p)^n\right]}{(n+1)(m+1)\,2^{m+1}} - \frac{p^{n+2m+3} - (-p)^{n+2m+3}}{2^{m+1}p^{m+1}(m+1)(n+2m+3)}.$$

(2). 原式 $= \int_0^{\sqrt{\pi}} dx \int_0^{\sin x^2} dy = \int_0^{\sqrt{\pi}} x \sin(x^2) dx$.

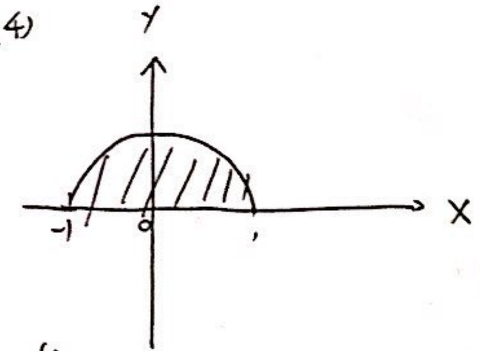
$= 1$.

(3). 原式 $= \int_0^2 x^2 dx \int_{\sqrt{x}}^{2+x} y^2 dy$

$= \frac{1}{3} \int_0^2 x^2 [(2+x)^3 - x^{\frac{3}{2}}] dx$

$= \frac{592}{15} - \frac{2^{\frac{11}{2}}}{27}$.

(4)



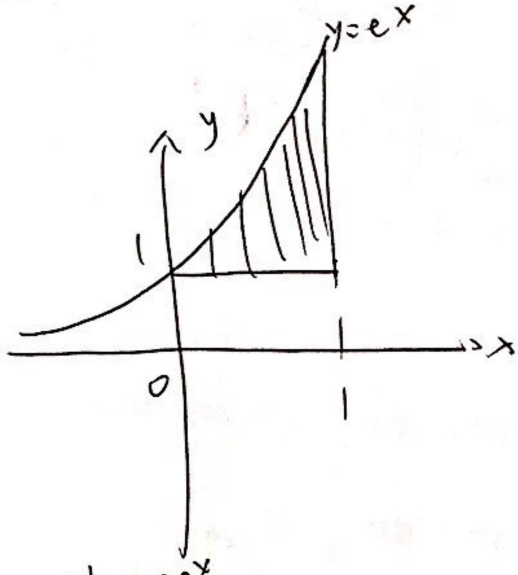
$$\iint (x^2+3xy^2)\,dx\,dy$$

$$\iint (r^2\cos^2\theta+3r^3\cos\theta\sin^2\theta)\,r\,dr\,d\theta=\int_0^{\pi}d\theta\int_0^1(r^2\cos^2\theta+3r^3\cos\theta\sin^2\theta)\,r\,dr$$

$$=\int_0^1 r^3dr\int_0^{\pi}(\cos^2\theta+3\cos\theta\sin^2\theta)\,d\theta=\int_0^{\pi}\left(\frac{1}{4}\cos^2\theta+\frac{3}{5}\cos\theta\sin^2\theta\right)d\theta$$

$$=\pi/8$$

(5)



$$\int_0^1 dx\int_1^{e^x}(x+y)\,dy$$

$$=\int_0^1\left[xe^x-x+\frac{1}{2}e^{2x}-\frac{1}{2}\right)dx$$

$$=\frac{e^2-1}{4}$$

$\frac{abc}{4}(\sin\theta+1$

$+\frac{y^2}{b}$

(6) Ω 是以(2,2),(2,3)和(3,1)为顶点的三角形,计算

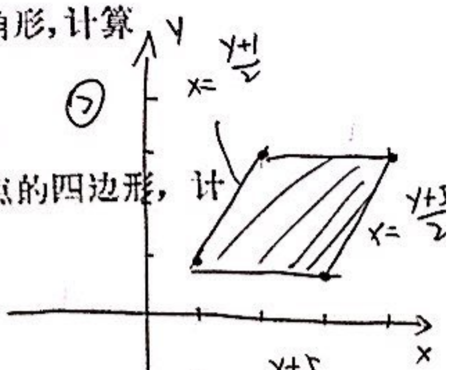
$$\iint_\Omega e^{x+y}dxdy;$$

(7) Ω 是以(1,1),(2,3),(3,1)和(4,3)为顶点的四边形,计算 $\iint_\Omega (x^2+y^2)dxdy$;

(8) Ω 由 $y=x^2$, $y=4x$ 和 $y=4$ 围成,计算

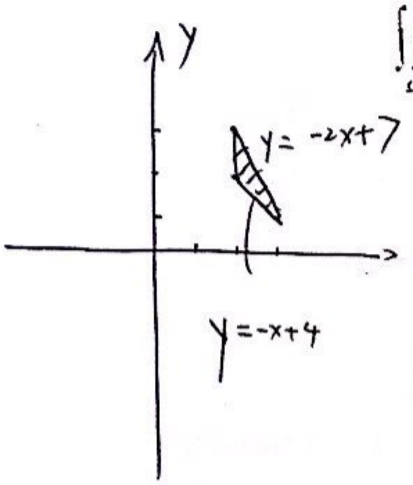
$$\iint_\Omega (\sin nx)dxdy.$$

⑦



$\int_1^3 dy \int_{\frac{y+1}{2}}^{\frac{y+5}{2}} (x^2+y^2)dx$ $=44$

(6)

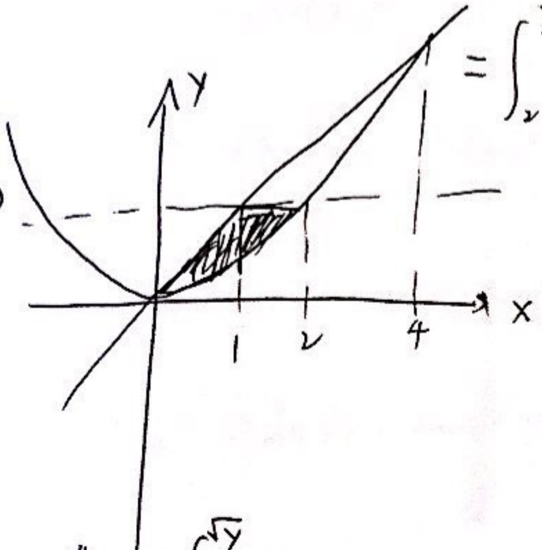


$\int_2^3 dx \int_{-x+4}^{-2x+7} e^{x+y} dy$

$= \int_2^3 e^x dx \int_{-x+4}^{-2x+7} e^y dy$

$= \int_2^3 (e^{-x+7} - e^4) dx = (e-2)e^4$

(8)



$\int_0^4 dy \int_{\frac{y}{4}}^{\sqrt{y}} (\sin nx) dx$ $\frac{4}{n}\sin n$

$= \frac{1}{n}\int_0^4 [(\cos\frac{ny}{4} - \cos(n\sqrt{y}))] dy = \frac{4\sin n}{n^2}\left[1-2\cos n + \frac{\sin n}{n}\right]$

20.4.14 计算下列三重积分:

(1) Ω: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$, 计算 $\iiint\limits_{\Omega}(x+y+z)dxdydz$;

奇 奇 奇

(2) Ω 由曲面 $x^2+y^2+z^2=1, x=0, y=0, z=0$ 围成的位于第一卦限的有界区域, 计算 $\iiint\limits_{\Omega}x^3yzdxdydz$;

$z=r^2$ $x=r\cos\theta$ $y=r\sin\theta$

(3) Ω 由曲面 $z=x^2+y^2, z=1, z=2$ 围成, 计算 $\iiint\limits_{\Omega}zdxdydz$;

$\iiint zrdrd\theta dz$

$=\int_1^2 dz\int_0^{\sqrt{z}} dr\int_0^{2\pi} zr\,d\theta$

$=2\pi\int_1^2 z\cdot\frac{1}{2}z\,dz$

$=\pi\int_1^2 z^2dz$

$=\frac{7}{3}\pi$

(4) Ω 由曲面 $x^2=z^2+y^2, x=2, x=4$ 围成, 计算 $\iiint\limits_{\Omega}(1+x^4)dxdydz$;

(5) Ω 由 $z=16(x^2+y^2), z=4(x^2+y^2), z=64$ 围成, 计算 $\iiint\limits_{\Omega}(x^2+y^2)dxdydz$.

(1) $x=ar\cos\theta$, $y=br\sin\theta\cos\varphi$, $z=cr\sin\theta\sin\varphi$

$I=\iiint(a\cos\theta+b\sin\theta\cos\varphi+c\sin\theta\sin\varphi)abcr^3\sin\theta\,drd\theta d\varphi$

$=\int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\int_0^1 abc\sin\theta(a\cos\theta+\cancel{b\sin\theta\cos\varphi}+\cancel{c\sin\theta\sin\varphi})r^3dr$

$=\frac{1}{4}\int_0^{\pi}a^2bc\sin\theta\cos\theta\,d\theta=0$

(2) $\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^1 r^5\cos^3\theta\sin^2\theta\sin\varphi\cos\varphi\, r^2\sin\theta\,dr$

$=\frac{1}{8}\int_0^{\frac{\pi}{2}}\cos^3\theta\sin^3\theta\,d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\,d\varphi=\frac{1}{16}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^3\theta\,d\theta=\frac{1}{192}$

④

$z = r\cos\theta,\quad y = r\sin\theta,\quad x^2 = r^2 \Rightarrow x = r$

$$\int_2^4 dx \int_0^x dr \int_0^{2\pi} (1+x^4)\, r\, d\theta$$

$$= 2\pi \int_2^4 dx \int_0^x (1+x^4)\, r\, dr$$

$$= 2\pi \int_2^4 (1+x^4)\frac{1}{2}x^2\, dx = \frac{49160}{21}\pi$$

⑤

$$\int_0^{64} dz \int_{\frac{\sqrt{z}}{4}}^{\frac{\sqrt{z}}{2}} dr \int_0^{2\pi} r^3\, d\theta$$

$$= 2\pi \int_0^{64} dz \int_{\frac{\sqrt{z}}{4}}^{\frac{\sqrt{z}}{2}} r^3\, dr$$

$$= \frac{15}{512}\pi \int_0^{64} z^2\, dz = 2560\pi$$