PEKING UNIVERSITY

参考答案 8

袁磊祺

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18.3.4

求下列函数的极大值点和极小值点:

(2)

$f(x,y)=xy\left(x^2+y^2-1\right)$.

$\frac{\partial f}{\partial x}=y\left(x^2+y^2-1\right)+xy(2x)=0$,

$\frac{\partial^2 f}{\partial x^2}=6xy$,

$\frac{\partial f}{\partial y}=x\left(x^2+y^2-1\right)+xy(2y)=0$.

$\frac{\partial^2 f}{\partial y^2}=6xy$.

$(x,y)=(0,0)$, $\left(\pm\frac{1}{2},\pm\frac{1}{2}\right)$, $(0,\pm1)$, $(\pm1,0)$ 可能为极值点.

极大值: $(x,y)=\left(-\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},-\frac{1}{2}\right)$,

极小值: $(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(-\frac{1}{2},-\frac{1}{2}\right)$.

(5)

$f(x,y)=\sin x+\sin y+\sin(x+y)$.

$\frac{\partial f}{\partial x}=\cos x+\cos(x+y)=0$,

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0.$

$(x,y) = (0,0)$, $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$, $(0,\pm 1)$, $(\pm 1, 0)$ 可能为极值点.

极大值: $(x,y) = \left(\frac{\pi}{3} + 2n\pi, \frac{\pi}{3} + 2m\pi\right)$, 其中 $n,m \in \mathbb{Z}$, $H_f = -\frac{\sqrt{3}}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, 负定.

极小值: $(x,y) = \left(-\frac{\pi}{3} + 2n\pi, -\frac{\pi}{3} + 2m\pi\right)$, 其中 $n,m \in \mathbb{Z}$, $H_f = \frac{\sqrt{3}}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, 正定.

18.3.5

求下列函数的极大值点和极小值点:

(1)

$f(x,y,z) = x^2 + y^2 + z^2 - 4xy + 6x + 2z.$

$\frac{\partial f}{\partial x} = 2x - 4y + 6 = 0,$

$\frac{\partial f}{\partial y} = 2y - 4x = 0,$

$\frac{\partial f}{\partial z} = 2z + 2 = 0.$

$\therefore x = 1,\ y = 2,\ z = -1.$

$H_f = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 不定, 无极值点.

(2)

$f(x,y,z) = (x+y+z)\mathrm{e}^{-x^2-y^2-z^2}.$

$\frac{\partial f}{\partial x} = [-2x(x+y+z)+1]\mathrm{e}^{-x^2-y^2-z^2} = 0,$

$\frac{\partial f}{\partial y} = [-2y(x+y+z)+1]\mathrm{e}^{-x^2-y^2-z^2} = 0,$

$\frac{\partial f}{\partial z} = [-2z(x+y+z)+1]\mathrm{e}^{-x^2-y^2-z^2} = 0.$

$\therefore x = y = z = \pm\frac{\sqrt{6}}{6}.$

$H_f(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6}) = -\frac{\sqrt{6}e^{-\frac{1}{2}}}{3}\begin{pmatrix}4 & 1 & 1\\ 1 & 4 & 1\\ 1 & 1 & 4\end{pmatrix}$ 负定, 为极大值点.

$H_f(\frac{-\sqrt{6}}{6},\frac{-\sqrt{6}}{6},\frac{-\sqrt{6}}{6}) = \frac{\sqrt{6}e^{-\frac{1}{2}}}{3}\begin{pmatrix}4 & 1 & 1\\ 1 & 4 & 1\\ 1 & 1 & 4\end{pmatrix}$ 正定, 为极小值点.

18.3.6

用隐函数微分法求隐函数 $z = z(x, y)$ 的极大值和极小值:

(3)

$x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0.$

一次微分: $2x\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z - x\,\mathrm{d}z - z\,\mathrm{d}x - y\,\mathrm{d}z - z\,\mathrm{d}y + 2\,\mathrm{d}x + 2\,\mathrm{d}y + 2\,\mathrm{d}z = 0,$

$\mathrm{d}z = 0$ 得 $x = y = -3 \pm \sqrt{6}.$

两次微分: $2\,\mathrm{d}x\,\mathrm{d}x + 2\,\mathrm{d}y\,\mathrm{d}y + 2z\mathrm{d}^2z - y\mathrm{d}^2z - x\mathrm{d}^2z + 2\mathrm{d}^2z = 0.$

$\mathrm{d}^2z = -\frac{2}{z+4}(\mathrm{d}x\,\mathrm{d}x + \mathrm{d}y\,\mathrm{d}y).$

$\therefore H_z = -\frac{2}{z+4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$

$x = y = -3 + \sqrt{6}$ 时, H_z 负定, 为极大值, $z = 2\sqrt{6} - 4$, ($z = -4$ 舍弃)

$x = y = -3 - \sqrt{6}$ 时, H_z 正定, 为极小值, $z = -2\sqrt{6} - 4.$

(4)

$z^2 + xyz - x^2 - xy^2 - 9 = 0.$

一次微分: $2z\,\mathrm{d}z + xy\,\mathrm{d}z + xz\,\mathrm{d}y + zy\,\mathrm{d}x - 2x\,\mathrm{d}x - y^2\,\mathrm{d}x - 2xy\,\mathrm{d}y = 0,$

$\mathrm{d}z = 0$ 得 $x = 1,\ z = 2y = \pm 2\sqrt{2}.$

二次微分: $2z\mathrm{d}^2z + xy\mathrm{d}^2z + 2z\,\mathrm{d}x\,\mathrm{d}y - 2\,\mathrm{d}x\,\mathrm{d}x - 2y\,\mathrm{d}x\,\mathrm{d}y - 2x\,\mathrm{d}y\,\mathrm{d}y - 2y\,\mathrm{d}x\,\mathrm{d}y = 0.$

$\therefore H_z = \frac{1}{5y}\begin{pmatrix} 2 & y \\ y & 2 \end{pmatrix}$.

$y = \sqrt{2}$ 时, H_z 正定, 为极小值, $z = 2\sqrt{2}$,

$y = -\sqrt{2}$ 时, H_z 负定, 为极大值, $z = -2\sqrt{2}$.

18.3.7

设 $f(x,y) = 3x^2y - x^4 - 2y^2$. 证明: $(0,0)$ 不是它的极值点, 但沿过 $(0,0)$ 点的每条直线,$(0,0)$ 都是它的极大值点.

证明. $f(x,y) = 3x^2y - x^4 - 2y^2$.

$f(0,0) = 0$, 若 $y = \frac{3}{4}x^2$, $f(x,y) = \frac{1}{8}x^4 > 0$, 若 $x = 0$, $f(x,y) = -2y^2 < 0$, 所以不为极值点.

若 $y = kx$, $f(x,y) = 3kx^3 - x^4 - 2k^2x^2 = g(x)$, $g'(0) = 0$, $g''(0) = -4k^2 < 0$ $(k \neq 0)$, 所以极大.

若 $k = 0$ 或 $x = 0$, 易得极大.

□

18.3.8

求证:

(3)

$f(x,y,z) = (ax+by+cz)\mathrm{e}^{-(x^2+y^2+z^2)}$ 在 $\mathbb{R}^3$ 有最大值和最小值, 其中 $a^2+b^2+c^2 > 0$.

证明. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, 则 $x_0 = ak,\ y_0 = bk,\ z_0 = ck,\ k = \pm\frac{1}{\sqrt{2(a^2+b^2+c^2)}}$.

$\therefore f(x_0,y_0,z_0) = \pm\frac{\sqrt{2}}{2}\sqrt{a^2+b^2+c^2}\mathrm{e}^{-\frac{1}{2}}$, 令 $p = \frac{\sqrt{2}}{2}\sqrt{a^2+b^2+c^2}\mathrm{e}^{-\frac{1}{2}}$.

$\because \lim\limits_{x^2+y^2+z^2\to+\infty} f = 0$, $\therefore \exists R > 0$ s.t. 当 $r \geq R$ 时, $-p/2 < f < p/2$, 其中 $r = \sqrt{x^2+y^2+z^2}$

对于有界闭集 $\{(x,y,z)|r \leq R\}$, 其最值点在边界或 (x_0,y_0,z_0).

$\because r \geq R$ 时, $-p/2 < f < p/2$, $\therefore$ 最值点为 (x_0,y_0,z_0). □

18.3.15

在椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 的内接长方体中, 求体积为最大的那个长方体.

设 $x,y,z>0,\ V=8xyz,\ P(x)=8xyz,\ Q(x)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1.$

$F(x,y,z)=P(x)+\lambda Q(x).$

$\frac{\partial F}{\partial x}=8yz+\frac{2x\lambda}{a^2}=0,\ \frac{\partial F}{\partial y}=8xz+\frac{2y\lambda}{b^2}=0,\ \frac{\partial F}{\partial z}=8yx+\frac{2z\lambda}{c^2}=0.$

$x=\frac{a}{\sqrt{3}},\ y=\frac{b}{\sqrt{3}},\ z=\frac{c}{\sqrt{3}}.$

$V=\frac{8\sqrt{3}}{9}abc.$

18.4.5

求下列条件极大值和条件极小值:

(3)

$\left(x^2+y^2+z^2\right)^2=a^2x^2+b^2y^2+c^2z^2,\ lx+my+nz=0$, 求 $f(x,y,z)=x^2+y^2+z^2$ 的极值;

设 $\varphi=x^2+y^2+z^2,$

$g(x,y,z,\varphi)=\varphi+\lambda_1\left[\varphi^2-\left(a^2x^2+b^2y^2+c^2z^2\right)\right]+\lambda_2(lx+my+nz)+\lambda_3\left[\varphi-\left(x^2+y^2+z^2\right)\right]$

$\frac{\partial g}{\partial \varphi}=1+2\lambda_1\varphi+\lambda_3=0,$

$\frac{\partial g}{\partial x}=-2\lambda_1a^2x+\lambda_2l-2\lambda_3x=0,$

$\frac{\partial g}{\partial y}=-2\lambda_1b^2y+\lambda_2m-2\lambda_3y=0,$

$\frac{\partial g}{\partial z}=-2\lambda_1c^2z+\lambda_2n-2\lambda_3z=0.$

$\frac{\partial g}{\partial x}x+\frac{\partial g}{\partial y}y+\frac{\partial g}{\partial z}z=-2\lambda_1\varphi^2-2\lambda_3\varphi=0.$

$\therefore \lambda_1\varphi+\lambda_3=0,\ \lambda_3=1,\ \varphi=-\frac{1}{\lambda_1}.$

$x=\frac{\lambda_2l}{2+2\lambda_1a^2},\ y=\frac{\lambda_2m}{2+2\lambda_1b^2},\ z=\frac{\lambda_2n}{2+2\lambda_1c^2}.$

$\therefore \frac{l^2}{1+\lambda_1a^2}+\frac{m^2}{1+\lambda_1b^2}+\frac{n^2}{1+\lambda_1c^2}=0.$

解得

$\varphi = \begin{pmatrix} \frac{\sigma_2}{\sigma_4+\sigma_3-\sigma_1+\sigma_6+\sigma_5} \\ \frac{\sigma_2}{\sigma_4+\sigma_3+\sigma_1+\sigma_6+\sigma_5} \end{pmatrix}$

其中

$\sigma_1 = \sqrt{a^4m^4 + 2a^4m^2n^2 + a^4n^4 + 2a^2b^2l^2m^2 - 2a^2b^2l^2n^2 - 2a^2b^2m^2n^2 - 2a^2b^2n^4}$

$\overline{-2a^2c^2l^2m^2 + 2a^2c^2l^2n^2 - 2a^2c^2m^4 - 2a^2c^2m^2n^2 + b^4l^4 + 2b^4l^2n^2 + b^4n^4 - 2b^2c^2l^4}$

$\overline{-2b^2c^2l^2m^2 - 2b^2c^2l^2n^2 + 2b^2c^2m^2n^2 + c^4l^4 + 2c^4l^2m^2 + c^4m^4}$, $\sigma_2 = 2a^2b^2n^2 + 2a^2c^2m^2 + 2b^2c^2l^2$, $\sigma_3 = c^2\left(l^2 + m^2\right)$, $\sigma_4 = n^2\left(a^2 + b^2\right)$, $\sigma_5 = b^2l^2$, $\sigma_6 = a^2m^2$.

对于有界闭集, 最大最小值存在, 所以 φ 第一个分量为极大值, 第二个分量为极小值. $(0,0,0)$ 的某个去心领域内无定义.

(4)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\ lx + my + nz = 0$, 求 $f(x,y,z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ 的极值;

设 $\varphi = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$,

$g(x,y,z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + \lambda_1(lx + my + nz)$

$\frac{\partial g}{\partial x} = \lambda_1 l + \frac{2\lambda_2 x}{a^2} + \frac{2x}{a^4} = 0,$

$\frac{\partial g}{\partial y} = \lambda_1 m + \frac{2\lambda_2 y}{b^2} + \frac{2y}{b^4} = 0,$

$\frac{\partial g}{\partial z} = \lambda_1 n + \frac{2\lambda_2 z}{c^2} + \frac{2z}{c^4} = 0,$

$\frac{\partial g}{\partial x}x + \frac{\partial g}{\partial y}y + \frac{\partial g}{\partial z}z = 2\lambda_2 + 2\varphi = 0,\ \varphi = -\lambda_2.$

$x = -\frac{a^4\lambda_1 l}{2+2\lambda_2 a^2},\ y = -\frac{b^4\lambda_1 m}{2+2\lambda_2 b^2},\ z = -\frac{c^4\lambda_1 n}{2+2\lambda_2 c^2}.$

$\therefore \frac{a^4l^2}{1+\lambda_2 a^2} + \frac{b^4m^2}{1+\lambda_2 b^2} + \frac{c^4n^2}{1+\lambda_2 c^2} = 0.$

解得 $\varphi = \begin{pmatrix} \frac{\sigma_8-\sigma_1+\sigma_7+\sigma_6+\sigma_5+\sigma_4+\sigma_3}{\sigma_2} \\ \frac{\sigma_1+\sigma_8+\sigma_7+\sigma_6+\sigma_5+\sigma_4+\sigma_3}{\sigma_2} \end{pmatrix}$,

其中

$\sigma_1 = \sqrt{a^8b^4l^4 - 2a^8b^2c^2l^4 + a^8c^4l^4 + 2a^6b^6l^2m^2 - 2a^6b^4c^2l^2m^2 - 2a^6b^2c^4l^2n^2}$

$\overline{+2a^6c^6l^2n^2 + a^4b^8m^4 - 2a^4b^6c^2l^2m^2 + 2a^4b^4c^4l^2m^2 + 2a^4b^4c^4l^2n^2 + 2a^4b^4c^4m^2n^2}$

$\overline{-2a^4b^2c^6l^2n^2 + a^4c^8n^4 - 2a^2b^8c^2m^4 - 2a^2b^6c^4m^2n^2 - 2a^2b^4c^6m^2n^2 - 2a^2b^2c^8n^4}$

$\overline{+b^8c^4m^4 + 2b^6c^6m^2n^2 + b^4c^8n^4}$, $\sigma_2 = 2a^2b^2c^2\left(a^2l^2 + b^2m^2 + c^2n^2\right)$, $\sigma_3 = b^2c^4n^2$, $\sigma_4 = b^4c^2m^2$, $\sigma_5 = a^2c^4n^2$, $\sigma_6 = a^4c^2l^2$, $\sigma_7 = a^2b^4m^2$, $\sigma_8 = a^4b^2l^2$.

有界闭集有最值, 所以 φ 第一个分量为极小值, 第二个分量为极大值.

18.4.6

$x^2 + y^2 + z^2 \le 1$, 求 $x^3 + y^3 + z^3 - 2xyz$ 的最大值和最小值.

设 $x^2 + y^2 + z^2 = k$, $k \in [0, 1]$, 由齐次性, 用放缩法可知, 取最值时 $k = 1$.

$g(x, y, z) = x^3 + y^3 + z^3 - 2xyz + \lambda(x^2 + y^2 + z^2 - 1)$

$\frac{\partial g}{\partial x} = 3x^2 - 2yz + 2\lambda x = 0$,

$\frac{\partial g}{\partial y} = 3y^2 - 2xz + 2\lambda y = 0$,

$\frac{\partial g}{\partial z} = 3z^2 - 2xy + 2\lambda z = 0$.

可以解得 $f(x, y, z) = \left(\frac{19\sqrt{6}}{54}, \frac{19\sqrt{6}}{54}, -\frac{19\sqrt{6}}{54}, -\frac{19\sqrt{6}}{54}, \frac{\sqrt{3}}{9}, -\frac{\sqrt{3}}{9}, \frac{19\sqrt{6}}{54}, -\frac{19\sqrt{6}}{54}, 1, -1, 1, -1, 1, -1\right)$, 其中每一个值为可能的解对应的值,

$\therefore$ 最大值为 1, 最小值为 -1.

18.4.11

证明: 椭圆的哪接三角形中, 面积最大的三角形的一顶点的椭圆发现必与三角形的该顶点的对边垂直; 并求椭圆中面积最大的内接三角形.

证明. 设三点为 (x_1, y_1), (x_2, y_2), (x_3, y_3).

三角形面积 $2S = \begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_1 - x_3 & y_1 - y_3 \end{vmatrix} = (x_1 - x_2)(y_1 - y_3) - (y_1 - y_2)(x_1 - x_3)$.

$f = (x_1 - x_2)(y_1 - y_3) - (y_1 - y_2)(x_1 - x_3) + \sum_{i=1}^{3} \lambda_i \left(\frac{x_i^2}{a^2} + \frac{y_i^2}{a^2} - 1\right)$,

$\frac{\partial f}{\partial x_1} = y_2 - y_3 + \frac{2\lambda_1}{a^2} x_1 = 0$, $\frac{\partial f}{\partial y_1} = x_3 - x_2 + \frac{2\lambda_1}{b^2} y_1 = 0$.

(x_1, y_1) 处的法线 $\boldsymbol{n} = \left(\frac{2x_1}{a^2}, \frac{2y_1}{b^2}\right)$, 由上面两式知 $\boldsymbol{n} \cdot (x_2 - x_3, y_2 - y_3) = 0$.

$\square$

圆内接三角形为正三角形时面积最大, 由缩放关系得 $S_{\max}=\frac{3\sqrt{3}}{4}ab$.

18.4.17

证明椭球面 $ax^2+by^2+cz^2+2dxy+2exz+2fyz=1$ 的最大轴长 l 为如下方程之最大实根:

$$\begin{vmatrix} a-\frac{1}{l^2} & d & e \\ d & b-\frac{1}{l^2} & f \\ e & f & c-\frac{1}{l^2} \end{vmatrix}=0.$$

证明. 设 $l^2=x^2+y^2+z^2$,

$f(x,y,z)=2l+\lambda_1\left(ax^2+by^2+cz^2+2dxy+2exz+2fyz\right)+\lambda_2\left(l^2-x^2+y^2+z^2\right),\ (l>0).$

$\frac{\partial f}{\partial l}=2+2l\lambda_2,\ \therefore\lambda_2=-\frac{1}{l}.$

$\frac{\partial f}{\partial x}=2a\lambda_1x+2\lambda_1dy+2\lambda_1ez-2\lambda_2x=0.$

$\frac{\partial f}{\partial x}x+\frac{\partial f}{\partial y}y+\frac{\partial f}{\partial z}z=l+\lambda_1=0,\ \lambda_1=-l$, 代入上式得

$l\left[\left(a-\frac{1}{l^2}\right)x+dy+ez\right]=0,\ \because l>0,$

$\therefore A\cdot(x,y,z)^{\mathrm{T}}=0.$

$\because$ 存在三个极值点的非平凡解,

$\therefore \det A=0.$ $\square$

18.2

设 $A=\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$.

用求条件极值的方法证明: $|A|\le\left(\sum\limits_{i=1}^{3}x_i^2\right)\left(\sum\limits_{i=1}^{3}y_i^2\right)\left(\sum\limits_{i=1}^{3}z_i^2\right)$.

证明. 不失一般性, 固定 $a=\sqrt{\sum\limits_{i=1}^{3}x_i^2},\ b=\sqrt{\sum\limits_{i=1}^{3}y_i^2},\ c=\sqrt{\sum\limits_{i=1}^{3}z_i^2},$

$$f = A + \lambda_1\left(a^2 - \sum_{i=1}^{3} x_i^2\right) + \lambda_2\left(b^2 - \sum_{i=1}^{3} y_i^2\right) + \lambda_3\left(c^2 - \sum_{i=1}^{3} z_i^2\right).$$

$$\frac{\partial f}{\partial x_1} = \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - 2\lambda_1 x_1 = 0.$$

$\therefore \boldsymbol{x} \parallel (\boldsymbol{y} \times \boldsymbol{z})$.

$\therefore \boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z}$ 两两正交, 又有界闭集上有最大最小值,

$$\therefore -\left(\sum_{i=1}^{3} x_i^2\right)\left(\sum_{i=1}^{3} y_i^2\right)\left(\sum_{i=1}^{3} z_i^2\right) \le A \le \left(\sum_{i=1}^{3} x_i^2\right)\left(\sum_{i=1}^{3} y_i^2\right)\left(\sum_{i=1}^{3} z_i^2\right),$$

$$\therefore |A| \le \left(\sum_{i=1}^{3} x_i^2\right)\left(\sum_{i=1}^{3} y_i^2\right)\left(\sum_{i=1}^{3} z_i^2\right).$$

□

18.6

设 $\sum\limits_{i,j=1}^{n} a_{ij}\xi_i\xi_j$ 是正定二次型, $u(x) \in C\left(\bar{\Omega}, \mathbb{R}^1\right), \Omega$ 是 $\mathbb{R}^n$ 中的有界开区域. 若 $u \in C^{(2)}(\Omega), u$ 在 $\bar{\Omega}$ 的最小值于 $x_0 \in \Omega$ 取到, 求证: $\sum\limits_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\Bigg|_{x=x_0} \geqslant 0.$

证明. 设 $A = a_{ij},\ B = \frac{\partial^2 u}{\partial x_i \partial x_j}\Big|_{x=x_0},\ A,\ B$ 对称.

A 正定, 可表示成 $A = P^{\mathrm{T}}P,\ B$ 半正定, $\det(P) \neq 0$.

即证 $\operatorname{tr}(AB) \ge 0$,

$\because \operatorname{tr}(AB) = \operatorname{tr}(BA)$(迹的性质),

$\therefore \operatorname{tr}(AB) = \operatorname{tr}(P^{\mathrm{T}}PB) = \operatorname{tr}(PBP^{\mathrm{T}}) \ge 0.$

□