Homework 1

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Exercise 1

Question: Verify:

$$\lim_{n\longrightarrow\infty}\int_n^{n+p}\frac{\sin x}{x}dx = 0 \ \ (p>0)$$

Proof: Because $\frac{1}{x}$ is monotonously decreasing and positive in $[n, n+p]$, by **theorem** II_1 **in page 91**, there is a $c \in [n, n+p]$ such that

$$\left|\int_n^{n+p}\frac{\sin x}{x}dx\right| = \left|\frac{1}{n}\int_n^c \sin x dx\right| \le \frac{2}{n}$$

Let $n \longrightarrow \infty$, original integral tend to zero.

Exercise 2

Question: Let

$$f(x) = \int_x^{x+1}\sin t^2 dt$$

Verify: when $x > 0$,

$$|f(x)| < \frac{1}{x}$$

Proof:

$$|f(x)| = \left|\int_x^{x+1}\sin t^2 dt\right| = \left|\int_{x^2}^{(x+1)^2}\frac{\sin t}{2\sqrt{t}}dt\right|$$

By **theorem** II_1 **in page 91**, because $\frac{1}{\sqrt{t}}$ is monotonously decreasing and positive, there is a $c \in [x^2, (x+1)^2]$ such that

$$|f(x)| = \frac{1}{2x}\left|\int_{x^2}^c \sin t dt\right| = \frac{\left|\cos c - \cos x^2\right|}{2x} \le \frac{1}{x}$$

Now, it needs to proof equality cannot be established. If there exists a x_0 such that

$$|f(x_0)| = \frac{1}{x_0}$$

Then

$$\cos x_0^2 = \pm 1$$

$$x_0^2 = k_1\pi \quad k_1 \in Z^+$$

Let

$$g(s) = \int_{x_0^2}^{s} \frac{\sin t}{2\sqrt{t}} dt$$

then

$$|g(s)| = \frac{1}{2x_0}\left|\int_{x^2}^{c} \sin t dt\right| \le \frac{1}{x_0}$$

. And since $|g((x_0+1)^2) = \frac{1}{x_0}$, $g(s)$ must reach its extreme at $(x_0+1)^2$, namely

$$0 = g'((x_0+1)^2) = \frac{\sin(x_0+1)^2}{2(x_0+1)}$$

$$(x_0+1)^2 = k_2\pi$$

$$\left(\frac{x_0+1}{x_0}\right)^2 = \frac{k_2}{k_1}$$

$$\pi = \frac{1}{k_1 - 2\sqrt{k_1k_2} + k_2}$$

It is incompatible with that π is a irrational number.

Exercise 3

Question: Let $f(x)$ monotonously decrease in $[-\pi, \pi]$. Verify:

$$b_{2n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin 2nx dx \ge 0$$

$$b_{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(2n+1)x dx \le 0$$

$n \ge 1$.
Proof: For $n \ge 1$,

$$b_{2n} = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x) - f(\pi)) \sin 2nx dx$$

By **theorem** II_1 **in page 91**, because $f(x) - f(\pi)$ is monotonously decreasing and positive, there is a $c \in [-\pi, \pi]$ such that

$$b_{2n} = \frac{f(-\pi) - f(\pi)}{\pi}\int_{-\pi}^{c} \sin 2nx dx = \frac{f(-\pi) - f(\pi)}{2n\pi}(1 - \cos 2nc) \ge 0$$

For $n \ge 1$,

$$b_{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x) - f(\pi)) \sin(2n+1)x dx$$

By **theorem** II_1 **in page 91**, because $f(x) - f(\pi)$ is monotonously decreasing and positive, there is a $c \in [-\pi, \pi]$ such that

$$b_{2n+1} = \frac{f(-\pi) - f(\pi)}{\pi}\int_{-\pi}^{c} \sin(2n+1)x dx = \frac{f(-\pi) - f(\pi)}{(2n+1)\pi}(-1 - \cos 2nc) \le 0$$

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Exercise 4

Question: Let $f(x)$ be a convex function in $[-\pi, \pi]$, and $f'(x)$ be bounded. Verify:

$$a_{2n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2nx dx \geq 0$$

$$a_{2n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2n+1)x dx \leq 0$$

$n \geq 1$.
Proof:

$$a_{2n} = -\frac{1}{2n\pi}\int_{-\pi}^{\pi} f'(x)\sin 2nx dx$$

$$a_{2n+1} = -\frac{1}{(2n+1)\pi}\int_{-\pi}^{\pi} f'(x)\sin(2n+1)x dx$$

Because $f(x)$ is a convex function, $-f'(x)$ is monotonously increasing. And because $f'(x)$ is bounded, $f'(\pi)$ and $f'(-\pi)$ exist. Then by the conclusions of the Exercise 3, $a_{2n} \geq 0$, $a_{2n+1} \leq 0$.

Exercise 5

Question: Let $e^2 < a < b$. Verify:

$$\int_a^b \frac{dx}{\ln x} < \frac{2b}{\ln b}$$

.

Proof:

$$\int_a^b \frac{dx}{\ln x} = \int_a^b \frac{\sqrt{x}}{2\ln\sqrt{x}}\frac{dx}{\sqrt{x}}$$

Let

$$f(x) = \frac{\sqrt{x}}{2\ln\sqrt{x}}$$

then

$$f'(x) = \frac{\ln x - 2}{2\sqrt{x}\ln^2 x}$$

For $x > e^2$, $f'(x) > 0$. By **theorem** II_2 **in page 94**, because $f(x)$ is monotonously decreasing and positive, there is a $c \in [a, b]$ such that

$$\int_a^b \frac{dx}{\ln x} = \frac{\sqrt{b}}{2\ln\sqrt{b}}\int_c^b \frac{dx}{\sqrt{x}} = \frac{\sqrt{b}}{\ln\sqrt{b}}(\sqrt{b} - \sqrt{c}) < \frac{2b}{\ln b}$$