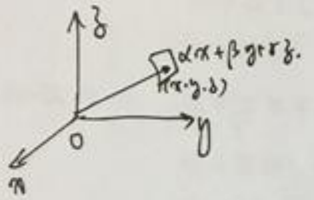
例: $f(x)\in C[-h,h]$. $h=\sqrt{\alpha^2+\beta^2+\gamma^2}$

证明. $\iint_S f(\alpha x+\beta y+\gamma z)\,dS = 2\pi\int_{-1}^{1} f(h\cdot u)\,du$.

这里: $S = x^2+y^2+z^2=1$.



$$\begin{cases} \xi = a_1x+b_1y+c_1z. \\ \eta = a_2x+b_2y+c_2z. \\ \zeta = \frac{1}{h}(\alpha x+\beta y+\gamma z). \end{cases}$$

证明:

这里: $(a_1\ b_1\ c_1)$, $(a_2\ b_2\ c_2)$, $\frac{1}{h}(\alpha,\beta,\gamma)$ > 单位向量. 两两正交.

令. $\xi = \cos\theta\cos\varphi$. $\eta = \sin\theta\sin\varphi$. $\zeta = \sin\varphi$. D: $0\le\theta\le 2\pi$, $-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$$\iint_S f(\alpha x+\beta y+\gamma z)\,ds = \iint_D f(h\zeta)\,ds = \iint_D f(h\zeta)\cos\varphi\, d\theta\, d\varphi.$$

∴ $u(x,y)\to 0$, 当 $(x,y)\to(+\infty,+\infty)$

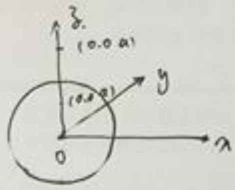
$\Longleftrightarrow$ $\oint f(\xi,\eta)\,ds = 0$.

$$\iint_S f(\alpha x+\beta y+\gamma z)\,dS = \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(h\sin\varphi)\cos\varphi\, d\varphi$$

$$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(h\sin\varphi)\, d\sin\varphi \overset{u=\sin\varphi}{=\!=\!=} 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(hu)\,du$$

例 谁位势 $\frac{1}{r}$.

圆面的电势或引力势. (假设电荷或质量分布是均匀的).



两种情形: ① $0<a<R$

② $0<R<a$.

位势 $W(0.0a) = \iint_S \frac{\rho\, dS}{\sqrt{x^2+y^2+(z-a)^2}}$.

S: $x = R\cos\theta\cos\varphi$ $\qquad 0\le\theta\le 2\pi$

$y = R\sin\theta\cos\varphi$. $\qquad -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$.

$z = R\sin\varphi$. $\qquad dS = R^2\cos\varphi\, d\theta\, d\varphi$.

$$W(0.0a) = \rho R^2\int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\varphi\, d\theta}{\sqrt{R^2+a^2-2Ra\sin\varphi}}$$

$$= \rho R^2\int_0^{2\pi} d\theta\left(-\frac{1}{2Ra}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\,(R^2+a^2-2Ra\sin\varphi)}{\sqrt{R^2+a^2-2Ra\sin\varphi}}$$

球面内部等势体

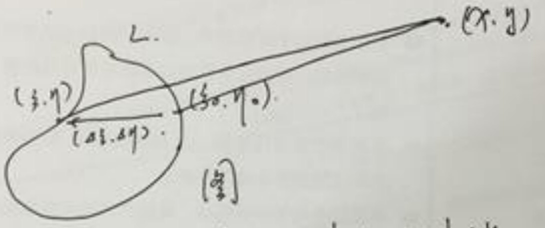
$$= \frac{2\pi\rho R}{a}\sqrt{R^2+a^2-2Ra\sin\varphi}\,\Bigg|_{\frac{\pi}{2}}^{-\frac{\pi}{2}} = \frac{2\pi\rho R}{a}\{R+a-|R-a|\} = \begin{cases} 4\pi R\cdot\rho & 0<a<R \\ \frac{4\pi R^2}{a}\rho. & a\ge R. \end{cases}$$

设 $f(x,y)$ 连续. L - 封闭的逐段光滑简单曲线.

证明: $$u(x,y) = \oint f(\xi,\eta)\ln\left(\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\right)ds.$$

当 $\begin{matrix} x\to\infty \\ y\to\infty \end{matrix}$ 时. 趋于零的充要条件是 $\oint_L f(\xi,\eta)\,ds = 0$

证明:



首先. $f(x,y)$ 在 L 上有界. $|f(x,y)| \le K.$ 设 L 的长度为 $S.$

固定一点 $(\xi_0,\eta_0)\in L.$ 考察 L 上任一点 $(\xi,\eta)\in L.$ 如图所示

$$\left|\ln\left(\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\right) - \ln\frac{1}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}\right| = \left|\ln\underbrace{\frac{\sqrt{(\xi-x)^2+(\eta-y)^2}}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}}_{(x,y)\to(\infty,\infty)\text{ 趋于1}}\right| \begin{matrix} <\varepsilon \\ \to 0 \\ \text{对于任意 }\varepsilon \end{matrix}$$

$$u(x,y) = \oint f(\xi,\eta)\left(\ln\frac{1}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}\right) + f(\xi,\eta)\left(\ln\frac{\sqrt{(\xi-x)^2+(\eta-y)^2}}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}\right).ds.$$

$$\therefore \quad \left|u(x,y) - \oint f(\xi,\eta)\ln\frac{1}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}\,ds\right| \le KS\varepsilon.$$

$$\ln\frac{1}{\sqrt{(\xi_0-x)^2+(\eta_0-y)^2}}\oint f(\xi,\eta)\,ds$$

$\therefore \quad u(x,y)\to 0$ 当 $(x,y)\to(+\infty,+\infty)$

$\Longleftrightarrow \oint f(\xi,\eta)\,ds = 0$