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湍流 3

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代码等作业内容可在 https://github.com/circlelq/Turbulence 查看.

1

解:

对于归一化的一维光滑平稳高斯过程 $X(t)$, 即满足 $\langle X\rangle = 0, \left\langle X^2\right\rangle = 1$, 记 $\dot{X} = \mathrm{d}X/\,\mathrm{d}t$, 证明如下两个条件平均关系: $\langle \ddot{X} \mid X = x\rangle = -\left\langle \dot{X}^2\right\rangle x$; $\left\langle \dot{X}^2 \mid X = x\right\rangle = \left\langle \dot{X}^2\right\rangle$. 提示: 应该首先说明或者证明 $\dot{X}(t)$ 和 $\ddot{X}(t)$ 也是高斯过程.

2

对于不可压均匀各向同性湍流, 试给出两空间点的涡量速度关联张量的最简表达式.

3

对于不可压均匀各向同性湍流, 根据不可压条件 (连续性方程) 和湍流统计量与构型 (configuration) 的方向无关的特点, 通过标架旋转证明一点的速度梯度满足统计关系:

$$\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = 2\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{3.1}$$

提示: 可参阅 G.I. Taylor 在 1935 年发表的关于均匀向同性湍流的有关论文.

为了书写方便, 记

$$u = u_1,\ v = u_2,\ w = u_3,\ x = x_1,\ y = x_2,\ z = x_3. \tag{3.2}$$

由不可压有

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.3}$$

即

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}, \tag{3.4}$$

两边平方得

$$\begin{aligned}\left(\frac{\partial u}{\partial x}\right)^2 &= \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \\ &= \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.\end{aligned} \tag{3.5}$$

取系综平均可得

$$\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle = \left\langle\left(\frac{\partial v}{\partial y}\right)^2\right\rangle + \left\langle\left(\frac{\partial w}{\partial z}\right)^2\right\rangle + \left\langle 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right\rangle \tag{3.6}$$

由各向同性湍流可得 [1]

$$\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle = \left\langle\left(\frac{\partial v}{\partial y}\right)^2\right\rangle = \left\langle\left(\frac{\partial w}{\partial z}\right)^2\right\rangle \tag{3.7}$$

所以式 (3.6) 化为

$$\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle = -2\left\langle\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right\rangle \tag{3.8}$$

对坐标进行 45° 可得

$$\begin{cases}\sqrt{2}x' & = x + y \\ \sqrt{2}y' & = -x + y \\ z' = z\end{cases} \tag{3.9}$$

$$\begin{cases}\sqrt{2}u' & = u + v \\ \sqrt{2}v' & = -u + v \\ v' = v\end{cases} \tag{3.10}$$

Hence

$$\begin{cases}\frac{\partial u}{\partial x} & = \frac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial v}{\partial x} & = \frac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} - \frac{\partial u'}{\partial y'}\right) \\ \frac{\partial w}{\partial x} & = \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} - \frac{\partial w'}{\partial y'}\right)\end{cases} \tag{3.11}$$

$$
\begin{cases}
\frac{\partial u}{\partial y} &= \frac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial y'}\right) \\
\frac{\partial v}{\partial y} &= \frac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\
\frac{\partial w}{\partial y} &= \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} + \frac{\partial w'}{\partial y'}\right)
\end{cases} \tag{3.12}
$$

$$
\begin{cases}
\frac{\partial u}{\partial z} &= \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} - \frac{\partial v'}{\partial z'}\right) \\
\frac{\partial v}{\partial z} &= \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial z'}\right) \\
\frac{\partial w}{\partial z} &= \frac{\partial w'}{\partial z'}
\end{cases} \tag{3.13}
$$

表 3.1. 记号说明.

$\overline{\left(\frac{\partial u}{\partial x}\right)^2}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}$	$\overline{\left(\frac{\partial u}{\partial y}\right)^2}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}$
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}

对式 (3.9) 求平方, 并利用各向同性可得以下等式

$$a_4 = a_7 = a_2 = a_5 = a_{10} = a_9 = 0. \tag{3.14}$$

$$a_1 = -2a_6, \tag{3.15}$$

$$a_1 - a_3 - a_6 - a_8 = 0, \tag{3.16}$$

$$a_1 - a_3 - a_6 - a_8 = 0, \tag{3.17}$$

$$a_1 + 2a_8 = 0, \tag{3.18}$$

$$a_1 = -2a_6 \tag{3.19}$$

$$a_1 = \frac{1}{2}a_3. \tag{3.20}$$

□

4

将不可压均匀各向同性湍流中两点纵向速度关联函数与一维能谱之间的傅立叶积分变换关系

$$\left\langle u^2 \right\rangle f(r) = \int_{-\infty}^{\infty} \phi_1(k) \mathrm{e}^{\mathrm{i}kr}\,\mathrm{d}k \tag{4.1}$$

代入如下两点纵向速度关联函数与三维能谱之间的积分变换关系

$$E(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left\langle u^2 \right\rangle f(r)(kr)^2 \left(\frac{\sin kr}{kr} - \cos kr\right) \mathrm{d}r, \tag{4.2}$$

直接通过计算推出

$$E(k)=k^3\frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{1}{k}\frac{\mathrm{d}\phi_1(k)}{\mathrm{d}k}\right]. \tag{4.3}$$

或者, 将

$$\left\langle u^2\right\rangle f(r)=2\int_{-\infty}^{\infty}E(k)(kr)^{-2}\left(\frac{\sin kr}{kr}-\cos kr\right)\mathrm{d}k \tag{4.4}$$

代入傅立叶逆变换关系

$$\phi_1(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle u^2\right\rangle f(r)\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r, \tag{4.5}$$

进行积分, 推出

$$\phi_1(k)=\frac{1}{2}\int_k^{\infty}\left(1-\frac{k^2}{\lambda^2}\right)\frac{E(\lambda)}{\lambda}\,\mathrm{d}\lambda. \tag{4.6}$$

两个推导过程任选其一完成即可. 提示: 第一个推导利用 δ 函数及其导数的性质; 第二个推导用到由如下积分关系

$$\frac{1}{2}\int_0^1\left(1-k^2\right)\cos kx\,\mathrm{d}k=\frac{1}{x^2}\left(\frac{\sin x}{x}-\cos x\right) \tag{4.7}$$

带来的余弦变换.

将式 (4.4) 带入式 (4.5) 可得

$$\begin{aligned}\phi_1(k)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle u^2\right\rangle f(r)\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\int_{-\infty}^{\infty}E(\lambda)(\lambda r)^{-2}\left(\frac{\sin\lambda r}{\lambda r}-\cos\lambda r\right)\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r\end{aligned} \tag{4.8}$$

5

用不同于教材中的方法证明教材 (2.3.42) 式:

$$\left\langle u^2\right\rangle^{\frac{3}{2}}k(r)=8\pi\int_0^{\infty}\frac{k^5\Gamma(k)}{(kr)^4}\left(3\sin kr-3kr\cos kr-(kr)^2\sin kr\right)\mathrm{d}k. \tag{5.1}$$

6

自己从网上或者课题组内部获取实验测量或者直接数值模拟得到的湍流脉动速度信号的一段长时间的时间序列或者一定样本容量的空间序列. 完成下列两项任务:

1. 计算两点纵向的二阶和三阶速度关联系数 $f(r)$, $k(r)$ 并画出曲线 (可以用泰勒冻结假设);
2. 根据 $f(r)$ 计算一维能谱 $\phi_1(k)$ 和三维能谱 $E(k)$ (在假设各向同性的情况下利用教材公式 (2.3.23)), 并画出曲线.

参考文献

[1] G. I. Taylor. Statistical theory of turbulence. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 151(873):421–444, 1935. 2