On the Theory of Statistical and Isotropic Turbulence

Author(s): W. Heisenberg

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On the theory of statistical and isotropic turbulence

BY W. HEISENBERG, *Göttingen*

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The statistical theory of turbulence, initiated by Taylor (1935) and v. Kármán & Howarth (1938), has recently been developed so far that a satisfactory explanation of the spectral distribution of energy among the turbulent eddies can be given. In fact Kolmogoroff (1941 *a, b*) and independently Onsager (1945) and v. Weizsaecker (1948) have introduced a similarity hypothesis, which allows a determination of the spectrum for eddies with large Reynolds numbers, and the author (Heisenberg 1948) has extended these calculations to include those frequency components which have small Reynolds numbers. Since the distribution of energy among the largest eddies must be a geometrical and not a statistical problem, one may say that the statistical part of the spectrum is now well understood. Recently Batchelor & Townsend (1947, 1948 *a, b*) have studied the decay of turbulent motion caused by a mesh grating in a wind tunnel, and the following discussions will apply the statistical theory to this problem.

For the calculations the notation of Heisenberg (1948) will be used. If $\rho F(k)\,dk$ denotes the energy contained between the wave numbers k and $k+dk$, the following equation for the dissipation of energy was given (Heisenberg 1948, equation (13)):

$$S_k = \left\{\mu + \rho\kappa \int_k^\infty \sqrt{\left(\frac{F(k'')}{k''^3}\right)}\,dk''\right\} \int_0^k 2F(k')\,k'^2\,dk'. \tag{1}$$

S_k means the total loss of the energy of that part of the spectrum that is contained between $k = 0$ and k. The physical assumption underlying equation (1) is that for all eddies between 0 and k the action of the smaller eddies can be represented by an additional viscosity, since the way in which these smaller eddies transfer momentum is similar to the action of ordinary friction (Prandtl 1945). This additional viscosity must depend upon the intensity $F(k)$ of the small eddies, i.e. on the part of the spectrum with large k. If one admits this hypothesis, then the value

$$\eta_k = \rho\kappa \int_k^\infty \sqrt{\left(\frac{F(k'')}{k''^3}\right)} dk'' \tag{2}$$

for the turbulent viscosity follows as the simplest assumption from purely dimensional reasons, since η_k must be the product of density, velocity and length. κ is a numerical constant, which from the experimental data seems to have a value of about 0·8. Equation (2) is of course just a simple mathematical expression of the similarity hypothesis introduced by Kolmogoroff, Onsager and v. Weizsaecker.

In Heisenberg (1948) the shape of $F(k)$ has been discussed for a stationary turbulent motion. It has been shown that for smaller k (i.e. for large Reynolds numbers of the eddies) $F(k) \sim k^{-\frac{5}{3}}$, while for large k (small Reynolds numbers of the eddies) $F(k) \sim k^{-7}$. Both results follow from the underlying similarity hypothesis. In fact, if one denotes by v_k the characteristic velocity of an eddy of diameter π/k, then for medium values of k ($k \gg k_0$, where k_0 characterizes the biggest eddies, but $v_k/k\nu \gg 1$) the product of the turbulent viscosity $\left(\sim \rho \frac{v_k}{k}\right)$ and the average value $\overline{\mathrm{curl}^2\, v}$ for this part of the spectrum ($\sim v_k^2 k^2$) must be constant, since S_k means just the total energy flow and must therefore be independent of k; that leads to

$$\frac{v_k}{k} v_k^2 k^2 = \text{const.}, \quad v_k \sim k^{-\frac{1}{3}}, \tag{3}$$

which is equivalent to $F(k) \sim k^{-\frac{5}{3}}$. For very large k, on the other hand, ($v_k/k\nu \ll 1$) the total flow of energy from all larger eddies into these small ones is given by $\sim \rho \frac{v_k}{k} \overline{\mathrm{curl}^2\, v}$, where the average $\overline{\mathrm{curl}^2\, v}$ can now be taken practically over the whole spectrum and does in this approximation not depend upon k. This energy must equal the energy that is consumed in the small eddies by ordinary viscosity, i.e. $\sim \mu v_k^2 k^2$. Therefore

$$\rho \frac{v_k}{k} \overline{\mathrm{curl}^2\, v} \sim \mu v_k^2 k^2, \quad v_k \sim k^{-3}, \tag{4}$$

which is equivalent to $F(k) \sim k^{-7}$.

If the turbulent motion is not maintained by means of external forces, it must decay and the rate of decay can be directly derived from (1), which now can be put into the form ($\nu = \mu/\rho$)

$$\frac{\partial}{\partial t} \int_0^k F(k,t)\, dk = -\left\{\nu + \kappa \int_k^\infty \sqrt{\left(\frac{F(k'',t)}{k''^3}\right)} dk''\right\} \int_0^k 2F(k',t)\, k'^2 dk'. \tag{5}$$

If the initial spectral distribution $F(k,0)$ is given, then (5) determines the spectrum for any later time. Here again we may look for solutions with simple similarity properties. One sees at once, that a special group of solutions can be given by assuming

$$F(k,t) = \frac{1}{\sqrt{t}} f(k\sqrt{t}). \tag{6}$$

Equation (5) then goes over into

$$\int_0^x f(x)\,dx - \tfrac{1}{2}xf(x) = \left\{\nu + \kappa\int_x^\infty \sqrt{\left(\frac{f(x'')}{x''^3}\right)}\,dx''\right\}\int_0^x 2f(x')\,x'^2 dx'. \tag{7}$$

The physical meaning of the similarity hypothesis (6) can be most easily expressed by saying that the total spectrum is determined essentially by one length, which we may call π/k_0, the diameter of the 'largest' eddies, and their velocity v_0 only. Then it follows from dimensional reasoning that

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{1}{k_0}\right) \sim v_0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{1}{v_0}\right) \sim k_0, \\ \text{which gives at once} \qquad &k_0 \sim v_0 \sim 1/\sqrt{t}. \end{aligned}\right\} \tag{8}$$

It is important to note that the similarity (6) is not destroyed by the presence of the viscosity μ, and that the Reynolds number of the turbulent motion stays constant throughout the whole decay, since this number is essentially $\rho v_0/\mu k_0$, $\sim$ const.

The solution of (7) will be discussed first in the case $\nu = 0$, i.e. for very large Reynolds numbers. Then $f(x)$ seems to be uniquely determined except again for a similarity transformation; for if $f = g(x)$ is a solution of (7) for $\nu = 0$, then $f = \alpha^3 g(\alpha x)$ is also a solution.

As essential properties of $f(x)$ one finds from (7):

$$\left.\begin{aligned} &f(x) \sim \text{const.}\, x \quad \text{for very small } x, \\ \text{and} \qquad &f(x) \sim \text{const.}\, x^{-\frac{5}{3}} \quad \text{for very large } x. \end{aligned}\right\} \tag{9}$$

The second property is of course just the result of the Kolmogoroff hypothesis. If one puts $\kappa = 0{\cdot}8$, then an approximate solution of (7) is given by

$$f(x) = \frac{x\alpha^4}{\left[1 + \left(\dfrac{x\alpha}{1{\cdot}1}\right)^{\frac{4}{3}}\right]^2}. \tag{10}$$

This solution is correct for small and large values of x and is a reasonably good approximation for intermediate values; α is an arbitrary constant of integration.

If one characterizes the initial state $t = t_0$ by two constants v_0 and k_0 one gets finally from (6) and (10)

$$F(k,t) = \frac{v_0^2}{k_0^2}\frac{k}{\left[1 + \left(\dfrac{k}{1{\cdot}1}\right)^{\frac{4}{3}}\left(\dfrac{tv_0}{k_0}\right)^{\frac{2}{3}}\right]^2}. \tag{11}$$

The reason for $F(k,t)$ being proportional to k at small values of k is that energy dissipation in the very big eddies is very small, and if there is to be similarity in the sense of equation (6), it can only be by a linear dependence of $F(k,t)$ on k.

In the case of finite but small values of ν the spectrum (11) can only be valid for not too large values of k and will then be continued by the k^{-7} law as discussed in Heisenberg (1948).

In the opposite case of very large values of ν, i.e. very small Reynolds numbers of the total turbulent motion one can again derive an approximate solution of (7).

This solution varies for small values of x as $x\,\mathrm{e}^{-2\nu x^2}$ and for large values according to the x^{-7} law. To a reasonable approximation it is given by ($\beta \ll 1$),

$$F(k,t) = \beta^2\nu^2 k\left[\mathrm{e}^{-k^2\nu t} + \frac{\kappa\beta}{4}\frac{k^2\nu t}{(1+2^{\frac{1}{3}}k^2\nu t)^3}\right]^2, \tag{12}$$

where $\beta \ll 1$ measures essentially the Reynolds number of the motion.

We now have to turn to the question whether the actual turbulent motion should during decay approach the simple solution (11). This solution describes a state of equilibrium. The smaller eddies are produced by the bigger ones and lose their energy to still smaller ones. The energy is derived always from the largest eddies. One may describe the similarity hypotheses (6) and (8) also by saying that the biggest eddies grow partly larger and partly smaller. When they become smaller they get very quickly into the general Kolmogoroff equilibrium. When they become larger they must have approximately the same angular momentum as they had before, since even if new parts of the fluid are included in the larger eddy, on the average no angular momentum will have been added. So v_0/k_0 for the largest eddies will approximately stay constant, which again leads to (8).

Now the question arises whether the actual turbulent motion will approach this equilibrium. The experiments of Batchelor & Townsend (1948*a*) show that after a short time the decay actually takes place according to (8), the energy decreasing as $1/t$, but after a certain period the decay becomes faster, the Reynolds number of the turbulence decreases, and finally the inertia terms play no appreciable role in the turbulent motion, as has been studied in detail by Batchelor & Townsend (1948*b*) and the energy decreases at $t^{-\frac{5}{2}}$.

The actual turbulent motion will certainly for large k values, i.e. the small eddies, very soon come into a state of equilibrium. The real problem arises from the region of the large eddies. In Heisenberg (1948) it had been assumed that the size of the largest eddies should be approximately constant; this is certainly not true and would never follow from any reasonable spectrum according to (5). But neither is it possible that the real spectrum ultimately approaches (11) for small k values, because the real spectrum must approach zero for very small values of k much faster than (11). Batchelor (1948) has shown that for very small k, $F(k)$ should vary as

$$F(k) \sim k^4\,\text{const.} \tag{13}$$

Usually the real spectrum will, in a certain range of not too small k be well represented by a function like (11). But there will be a region of very small k in which $F(k)$ from the beginning is much smaller than (11); the very biggest eddies can never contain as much energy as indicated by (11). Therefore the linear decay of the energy ($v^2 \sim 1/t$) will take place until finally the maximum of the curve (11) is shifted to values of k which approach this region. Then the biggest eddies no longer contain enough energy to supply the $1/t$ law, the Reynolds number of the turbulent motion drops and finally gets so low that the inertia terms become unimportant, as has been shown by Batchelor & Townsend (1948*b*).

The point at which the $1/t$ law is replaced by some faster decay depends on the initial conditions, especially on the amount of energy contained in Fourier components belonging to wave-lengths that are larger than the mesh width of the grating.

This whole description applies even if the intial Reynolds number of the turbulent motion is not large, since the similarity hypothesis (6) leads to solutions of (5) for any value of the Reynolds number. The essential condition for the occurrence of a $1/t$ law for the decay is the existence of a rather large range of the initial spectrum (or its creation in the very first establishment of turbulent equilibrium) in which $F(k) \sim k$, and the $1/t$ law is limited only by the breakdown of this law $F(k) \sim k$ at very small values of k.

A few remarks may be added with regard to the physical picture of turbulence presented in the recent papers. In the earlier years one thought that turbulence was caused by viscosity. This seemed to be true since without viscosity the liquid could theoretically perform all the classical laminar motions, where the liquid glides along the walls; it is only through the viscosity that rotational motions are produced near the walls. At present we know that it is almost the other way round. The liquid without friction is a system with an infinite number of degrees of freedom. It is extremely improbable that only those few degrees of freedom which a laminar motion represents should be excited. As soon as one puts energy into a liquid without friction, this energy will be distributed among all degrees of freedom, and what finally results is a certain equilibrium distribution, corresponding to the Maxwellian distribution in gases, and represented by the spectrum found by Kolmogoroff, Onsager and v. Weizsaecker. It is the viscosity that reduces the number of degrees of freedom, since it damps very quickly all motions in the very small eddies. Therefore only through viscosity is a laminar motion at all possible. Turbulence is an essentially statistical problem of the same type as one meets in statistical mechanics, since it is the problem of distribution of energy among a very large number of degrees of freedom. Just as in Maxwell theory this problem can be solved without going into any details of the mechanical motion, so it can be solved here by simple considerations of similarity. Equation (1) is just a very simple way of describing these similarity properties.

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