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Notes

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December 26, 2021

1 卡门–豪沃思方程

N–S

$$\frac{\partial R_{ij}}{\partial t} - \frac{\partial}{\partial r_m}\left(S_{imj} + S_{jmi}\right) = 2\nu \nabla^2 R_{ij} \tag{1.1}$$

K–H

$$\frac{\partial\left(\left\langle u^2\right\rangle f\right)}{\partial t} - \left\langle u^2\right\rangle^{\frac{3}{2}}\left(k' + \frac{4k}{r}\right) = 2\nu\left\langle u^2\right\rangle\left(f'' + \frac{4f'}{r}\right) \tag{1.2}$$

自模拟假设（自保持, 自相似）$f(r,t)$, $k(r,t)$, $\left\langle u^2\right\rangle(t)$ 引入（微尺度）$\lambda(t)$

设 $f(r,t) = F\left(\frac{r}{\lambda(t)}\right)$, $\quad k(r,t) = K\left(\frac{r}{\lambda(t)}\right)$

$$\frac{\partial\left(r^4\left\langle u^2\right\rangle f\right)}{\partial t} - \left\langle u^2\right\rangle^{\frac{3}{2}}\left(r^4 k\right)' = 2\nu\left\langle u^2\right\rangle\left(r^4 f'\right)' \tag{1.3}$$

$$\frac{\partial\int_0^{+\infty}\left(r^4\left\langle u^2\right\rangle f\right)\mathrm{d}r}{\partial t} - \left\langle u^2\right\rangle^{\frac{3}{2}}\left(r^4 k\right)\Big|_0^{+\infty} = 2\nu\left\langle u^2\right\rangle\left(r^4 f'\right)\Big|_0^{+\infty} \tag{1.4}$$

Loitsansky 积分不变量

$$\left\langle u^2\right\rangle\int_0^\infty r^4 f\,\mathrm{d}r = \Lambda_0 \tag{1.5}$$

与时间无关的假设, $r^4 f' \to 0$. Taylor 展开结果:

$$f = 1 - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2 + \frac{f''''(0)}{24}r^4 + \cdots \tag{1.6}$$

$$k = \frac{k'''(0)}{6}r^3 + \cdots \tag{1.7}$$

$$\frac{\partial\left\langle u^2\right\rangle f}{\partial t} = \frac{\mathrm{d}\left\langle u^2\right\rangle}{\mathrm{d}t} - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2\frac{\mathrm{d}\left\langle u^2\right\rangle}{\mathrm{d}t} + \frac{r^2}{\lambda^3}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \cdots \tag{1.8}$$

$$k' + \frac{4k}{r} = \frac{k'''(0)}{2}r^2 + \frac{2}{3}k'''(0)r^2 + \cdots = \frac{7}{6}k'''(0)r^2 + \cdots \tag{1.9}$$

$$f'' + \frac{4f'}{r} = -\frac{1}{\lambda^2} - \frac{4}{\lambda^2} + \cdots = -\frac{5}{\lambda^2} + \cdots \tag{1.10}$$

$$\frac{\mathrm{d}\langle u^2 \rangle}{\mathrm{d}t} = -10\nu\frac{\langle u^2 \rangle}{\lambda^2} \tag{1.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \omega^2 \right\rangle = \frac{7}{3\sqrt{15}}\left\langle \omega^2 \right\rangle^{\frac{3}{2}}\left(S - \frac{2G}{R_\lambda}\right) \tag{1.12}$$

$$S \equiv -\lambda^3 k'''(0), \quad G \equiv \lambda^4 f''''(0), \quad R_\lambda \equiv \frac{\left\langle u^2 \right\rangle^{\frac{1}{2}}\lambda}{\nu} \tag{1.13}$$

2 HIT 衰变后期规律

k 忽略（惯性项近似为 0 ）

$$\frac{\partial\left\langle u^2\right\rangle f}{\partial t} = 2\nu\left\langle u^2\right\rangle\left(f'' + \frac{4f'}{r}\right) \tag{2.1}$$

设 $f = F\left(\frac{r}{\lambda(t)}\right)$, 则 $\xi \equiv \frac{r}{\lambda(t)}$

$$\frac{\mathrm{d}\langle u^2\rangle}{\mathrm{d}t}F + \left\langle u^2\right\rangle F'\left(-\frac{\xi}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right) = 2\nu\left\langle u^2\right\rangle\left(\frac{F''}{\lambda^2} + \frac{4F'}{r\lambda}\right) \tag{2.2}$$

$$-10\nu\frac{\langle u^2\rangle}{\lambda^2}\left(-\xi\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)F' \tag{2.3}$$

$$-10\nu F = 2\nu\left(F'' + \frac{4F'}{\xi}\right) + F'\xi\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t} \tag{2.4}$$

$$F'' + \frac{4F'}{\xi} + F'\xi\frac{\lambda}{2\nu}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + 5F = 0 \tag{2.5}$$

$$\alpha = \frac{\lambda}{2\nu}\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{1}{4\nu}\frac{\mathrm{d}\lambda^2}{\mathrm{d}t} \implies \text{常数} \tag{2.6}$$

$$\lambda^2 = 4\nu(t - t_0)\alpha \tag{2.7}$$

$$\frac{\mathrm{d}\langle u^2\rangle}{\mathrm{d}t} = -10\nu\frac{\langle u^2\rangle}{4\nu\alpha(t - t_0)} \tag{2.8}$$

$$\left\langle u^2\right\rangle = A(t - t_0)^{-\frac{5}{2\alpha}} \tag{2.9}$$

$$\left\langle u^2\right\rangle\int_0^{+\infty} r^4 f\,\mathrm{d}r = \left\langle u^2\right\rangle\lambda^5\int_0^{+\infty}\xi^5 F(\xi)\,\mathrm{d}\xi = \Lambda_0 \implies \alpha = 1. \tag{2.10}$$

$$\left\langle u^2\right\rangle \sim (t - t_0)^{\frac{5}{2}} \tag{2.11}$$

$$F'' + \left(\xi + \frac{4}{\xi}\right)F' + 5F = 0 \tag{2.12}$$

设 $x=\xi^2, \quad F(\xi)=y(x)$

$$F' = y' 2\xi \tag{2.13}$$

$$F'' = 2y' + y'' 4x \tag{2.14}$$

$$4xy'' + \left(\xi + \frac{4}{\xi}\right) 2\xi y' + 5y = 0 \tag{2.15}$$

$$4xy'' + (2x+10)y' + 5y = 0 \tag{2.16}$$

设 $y = \mathrm{e}^{`x`}$

$$y' = \beta y, \quad y'' = \beta^2 y \tag{2.17}$$

$$4x\beta^2 y + (2x+8)\beta y + 5y = 0 \tag{2.18}$$

$$4x\beta^2 y + 2x\beta + 10\beta + 5 = 0 \tag{2.19}$$

$$4\beta^2 + 2\beta = 0 \implies \beta = -\frac{1}{2} \tag{2.20}$$

$$F(\xi) = C\mathrm{e}^{-\frac{\xi^2}{2}} \tag{2.21}$$

$$f(r,t) = C\mathrm{e}^{-\frac{1}{2}\frac{r^2}{4\nu(t-t_0)}} = C\mathrm{e}^{-\frac{r^2}{8\nu(t-t_0)}} \implies f = \mathrm{e}^{-\frac{r^2}{8\nu(t-t_0)}} \tag{2.22}$$

谱空间

$$\frac{\partial}{\partial t}\Phi_{ij} - \mathrm{i}m_m\left(\Gamma_{imj} + \Gamma_{jmi}\right) = -2\nu k^2 \Phi_i \tag{2.23}$$

缩并 i,

$$\Phi_{ij} = \frac{E}{4\pi k^4}(k^2\delta_{ij} - k_i k_j) \tag{2.24}$$

$$\Gamma_{ijl} = \mathrm{i}\Gamma\left(k_i k_j k_l - \frac{k^2}{2}(k_i\delta_{jl} + k_j\delta_{il})\right) \tag{2.25}$$

$$\Gamma_{imi} = \mathrm{i}\Gamma\left(k^2 k_m - \frac{k^2}{2}(k_m + 3k_m)\right) = -\mathrm{i}k^2\Gamma k_m \tag{2.26}$$

$$\frac{1}{2\pi k^2}\frac{\partial E}{\partial t} - 2k^4\Gamma = -2\nu k^2 \frac{E}{2\pi k^2} \tag{2.27}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)E = 4\pi k^6 \Gamma \equiv T(k,t) \tag{2.28}$$

$$\frac{\partial}{\partial t}\int_0^k E(k,t)\,\mathrm{d}k - \int_0^k T(k,t)\,\mathrm{d}k = -2\nu\int_0^k k^2 E\,\mathrm{d}k \equiv \Pi(k,t) = \int_k^\infty T\,\mathrm{d}k \tag{2.29}$$

T 忽略

$$\frac{\partial E}{\partial t} = -2\nu k^2 E \implies E(k,t) = E_0(k)\mathrm{e}^{-2\nu k^2 t} \tag{2.30}$$

设 $E = V^2 l F(kl)$ 用 Loistansky 不变量可得 $E_0(k) = ck^4$

2.1 Kolmogorov 1941 理论

1. 展示湍流（一般湍流）在 Re 极大时, 在局部为均匀各向同性（远离边界, 奇点）, 在增量意义下. n 点联合 p.d.f（增量）只依赖于 n 点构型形状、大小与位置、时刻及方位无关. 由此引入结构函数的概念.
2. 在 Re 极大时, 小尺度范围为普适平衡. $\frac{\partial}{\partial t}\langle\cdot\rangle \approx 0$, 统计特性 $\langle\cdot\rangle$ 只依赖于 $\langle\varepsilon\rangle, \nu$
3. 在 Re 极大时, 小尺度范围的低波数段, 统计量只依赖于 $\langle\varepsilon\rangle$（第二相似性假设）

$$S_2(r) = B(\varepsilon, \nu, r) \tag{2.31}$$

$$\eta \equiv \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}, \quad v \equiv (\varepsilon\eta)^{\frac{1}{3}} \tag{2.32}$$

$$S_2(r) = v^2 F\left(\frac{r}{\eta}\right), \quad S_3(r) = v^3 G(\frac{r}{\eta}), \quad \frac{\eta v}{\nu} = 1 \tag{2.33}$$

$$S_2 = B(r, \varepsilon), \quad v \sim (\varepsilon r)^{\frac{1}{3}}, \quad S_2 = C_2(\varepsilon r)^{\frac{2}{3}} \tag{2.34}$$

衰变后期 $\lambda^2 = 4\nu(t - t_0)$

纵串 (L. F. Richardson 1922) cascade 大涡发展成小涡, 非线性作用（惯性）

$$\begin{aligned}\langle\varepsilon\rangle &= \nu\left\langle\|\nabla\boldsymbol{u}\|^2\right\rangle \\ &= \nu\left\langle\omega^2\right\rangle \\ &= 15\nu\frac{\left\langle u^2\right\rangle}{\lambda^2}\end{aligned} \tag{2.35}$$

$$f(r) = 1 - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2 + o(r^2) \tag{2.36}$$

1. 局部各向同性
2. 第一相似性 ε, ν 平衡范围
3. 第二相似性 ε

纵向

$$S_2 \equiv \left\langle\left((\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x}))\frac{\boldsymbol{r}}{r}\right)^2\right\rangle = \left\langle(u_2 - u_1)^2\right\rangle = (\varepsilon\eta)^{\frac{2}{3}} B_2(\frac{r}{\eta}) \tag{2.37}$$

$$S_3 \equiv \left\langle(u_2 - u_1)^3\right\rangle = (\varepsilon\eta) B_3\left(\frac{r}{\eta}\right) \tag{2.38}$$

$$S_2(r)=C_2(\varepsilon r)^{\frac{2}{3}}\quad \text{实验上基本成立} \tag{2.39}$$

$$S_3(r)=C_3(\varepsilon r) \tag{2.40}$$

C_2, C_3 是普适常数.

Obukhov (1941) 在平衡范围

$$E(k)=\varepsilon^{\frac{1}{4}}\nu^{\frac{5}{4}}F(k\eta) \tag{2.41}$$

惯性范围 $E(k)=C_k\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$, $-\frac{5}{3}$ 定律 $E(k)\sim k^{-\frac{5}{3}}$

K41 推广:p 阶矩 ($p\in\mathbb{N}$)

$$S_p(r)\sim(\varepsilon r)^{\frac{p}{3}}C_p \implies p_r(\delta u)=? \tag{2.42}$$

$$S_p(r)=\left(\frac{r}{L}\right)^{\frac{p}{3}}S_p(L) \implies p_r(\delta u)=p_L\left(\delta u\left(\frac{r}{L}\right)^{-\frac{1}{3}}\right)\left(\frac{r}{L}\right)^{-\frac{1}{3}} \tag{2.43}$$

K-H:

$$\frac{\partial\left(\left\langle u^2\right\rangle f\right)}{\partial t}-\left\langle u^2\right\rangle^{\frac{3}{2}}\left(k'+\frac{4k}{r}\right)=2\nu\left\langle u^2\right\rangle\left\langle f''+\frac{4f'}{r}\right\rangle \tag{2.44}$$

$f,k\to(S_2,S_3)$, $S_3=\left\langle u^2\right\rangle^{\frac{3}{2}}k$

$$2\left\langle u^2\right\rangle-S_2=2\left\langle u^2\right\rangle f \tag{2.45}$$

$$2\frac{\partial\left\langle u^2\right\rangle}{\partial t}-\frac{\partial S_2}{\partial t}-\frac{1}{3}\left(S_3'+\frac{4S_3}{r}\right)=-2\nu\left(S_2''+\frac{4S_2'}{r}\right) \tag{2.46}$$

$$-\frac{4}{3}\varepsilon-\frac{1}{3}\left(S_3'+\frac{4S_3}{r}\right)=-2\nu\left(S_2''+\frac{4S_2'}{r}\right) \tag{2.47}$$

乘 r^4

$$-\frac{r}{3}\varepsilon r^4-\frac{1}{3}\left(r^4S_3\right)'=-2\nu\left(S_2'r^4\right)' \tag{2.48}$$

积分 r

$$-\frac{4}{15}\varepsilon r^5-\frac{1}{3}r^4S_3=-2\nu S_2'r^4 \tag{2.49}$$

$$S_3=-\frac{4}{5}\varepsilon r+6\nu r\frac{\mathrm{d}S_2}{\mathrm{d}r} \tag{2.50}$$

当 $\nu\to 0$ 时

$$S_3(r)=-\frac{4}{5}\varepsilon r \tag{2.51}$$

设一个 S_2 的函数, 计算 $E(k)$

$$S_2=C_2\left(\varepsilon r\right)^{\frac{2}{3}},\quad E(k)=C_k\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}} \tag{2.52}$$

都是 $-\frac{5}{3}$ 的关系

$$\phi_1(k) = C_1 \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{2.53}$$

$$C_k = C_1 \left(-\frac{5}{3}\right) \left(-\frac{5}{3} - 2\right) \tag{2.54}$$

一维能谱 $\phi_1(k)$

$$\begin{aligned} \left\langle u^2 \right\rangle f(r) &= \int_{-\infty}^{+\infty} \phi_1 \mathrm{e}^{\mathrm{i}kr} \,\mathrm{d}k \\ &= 2\int_0^{+\infty} \phi_1 \cos kr \,\mathrm{d}k \end{aligned} \tag{2.55}$$

$$S_2 = 2\left\langle u^2 \right\rangle - 4\int_0^{\infty} \phi_1 \cos kr \,\mathrm{d}k \tag{2.56}$$

$$\begin{aligned} S_2 &= 4\int_0^{\infty} \phi_1 (1-\cos kr) \,\mathrm{d}k \\ &= 4\int_0^{\infty} C_1 \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} (1-\cos kr) \,\mathrm{d}k \\ &= 4C_1 \varepsilon^{\frac{2}{3}} \int_0^{\infty} k^{-\frac{5}{3}} (1-\cos kr) \,\mathrm{d}k \end{aligned} \tag{2.57}$$

用复变函数积分得

$$\int_0^{\infty} k^{-\frac{5}{3}} \cos kr \,\mathrm{d}k = \frac{\Gamma\left(\frac{1}{3}\right)}{2\sqrt[3]{r}} \tag{2.58}$$

$$C_2 = 3\Gamma\left(\frac{1}{3}\right) C_1 \tag{2.59}$$

能谱方程

$$\frac{\partial E}{\partial t} = T - 2\nu k^2 E \tag{2.60}$$

$T(k,t)$ 输运, 普适平衡

$$\begin{aligned} \frac{\partial}{\partial t}\int_0^k E \,\mathrm{d}k &= \int_0^k T \,\mathrm{d}k - 2\nu \int_0^k k^2 E \,\mathrm{d}k \\ &= -\int_k^{\infty} T \,\mathrm{d}k - \left(\varepsilon - 2\nu \int_k^{\infty} k^2 E \,\mathrm{d}k\right) \end{aligned} \tag{2.61}$$

$$\frac{\partial}{\partial x}\int_0^k E \,\mathrm{d}k \approx \frac{\partial}{\partial t}\int_0^{\infty} E \,\mathrm{d}k \tag{2.62}$$

能流

$$-S \equiv -\int_k^{\infty} T \,\mathrm{d}k \tag{2.63}$$

$$\Pi \equiv \int_k^{\infty} E \,\mathrm{d}k \tag{2.64}$$

封闭方程

$$S = 2\nu \int_k^{\infty} k^2 E \,\mathrm{d}k \tag{2.65}$$

2.2 封闭方法

Obukhov (1940)

$$S = \gamma \int_k^\infty E \, \mathrm{d}k \sqrt{2 \int_0^k k^2 E \, \mathrm{d}k} \tag{2.66}$$

Millsaps (1955)

解法: 令 $2\int_0^k k^2 E \, \mathrm{d}k = Z^2 \implies k^2 E = Z \cdot Z'$

$$\gamma \int_k^\infty EZ = 2\nu \int_k^\infty k^2 E \implies \varepsilon - 2\nu \int_0^k k^2 E = \gamma Z \int_k^\infty E \implies \varepsilon - \nu Z^2 = \gamma Z \int_k^\infty E \tag{2.67}$$

$$\frac{\varepsilon}{Z} - \nu Z = g \int_k^\infty E \tag{2.68}$$

$$\left(-\frac{\varepsilon}{Z^2} - \nu \right) Z' = -\gamma E = -\gamma \frac{Z}{k^2} \tag{2.69}$$

$$\frac{\varepsilon}{Z^2} + \nu = \gamma \frac{Z}{k^2} \tag{2.70}$$

其中

$$k^2 = \frac{\gamma Z^3}{\varepsilon + \nu Z^2} \tag{2.71}$$

注意到 $Z \in \left[0, \sqrt{\frac{\varepsilon}{\nu}}\right]$ 则 $k^2 \leq \frac{\gamma}{2\varepsilon} \left(\frac{\varepsilon}{\nu}^{\frac{3}{2}}\right) \equiv k_{\max}$ 在 $k \leq k_{\max}$ 内估算（当 k 很大时）

$$k^2 \approx \frac{\gamma}{2\varepsilon} Z^3 \tag{2.72}$$

$$Z \approx \left(\frac{2\varepsilon}{\nu} \right)^{\frac{1}{3}} k^{\frac{2}{3}} \tag{2.73}$$

$$E = \frac{ZZ'}{k^2} = \left(\frac{2\varepsilon}{\nu} \right)^{\frac{2}{3}} k^{-\frac{5}{3}} \tag{2.74}$$

Heisonberg–Weizsacker 模型

$$S = 2\gamma \int_k^\infty \left(k^{-3} E\right)^{\frac{1}{2}} \mathrm{d}k \int_0^k k^2 E \, \mathrm{d}k \tag{2.75}$$

$k < k_d : \; E \approx \left(\frac{8\varepsilon}{9\gamma}\right)^{\frac{2}{3}} k^{-\frac{5}{7}}$

$k \geq k_d : E \approx \left(\frac{\gamma\varepsilon}{2\nu^2}\right)^2 k^{-7}$

Ellison 修正

$$S = \alpha k E \sqrt{2 \int_0^k k^2 E} \tag{2.76}$$

Hinge 修正

$$S = \alpha \int_0^k \sqrt{kE} \int_k^\infty E \tag{2.77}$$

Karman 模型 (1948)

$$S = 2\gamma \int_k^\infty E^\alpha k^\beta \int_0^k E^{\frac{3}{2}-\alpha} k^{\frac{1}{2}-\beta} \tag{2.78}$$

Kovasznay 模型

$$S = \beta k^{\frac{5}{2}} E^{\frac{3}{2}} \tag{2.79}$$

$$\begin{cases} E \sim k^{-\frac{5}{3}}, & k < k_d \\ E = 0, & k \geq k_d \end{cases} \tag{2.80}$$

Pao YH 模型 (1965)

$$S = \sigma(k)E(k) = \alpha^{-1}\varepsilon^{\frac{1}{3}} k^{\frac{5}{3}} E(k) \tag{2.81}$$

$$\alpha^{-1}\varepsilon^{\frac{1}{3}} k^{\frac{5}{3}} E = 2\nu \int_k^\infty k^2 E \tag{2.82}$$

$$\alpha^{-1}\varepsilon^{\frac{1}{3}} \left[\frac{5}{3} k^{\frac{2}{3}} E + k^{\frac{5}{3}} \frac{\mathrm{d}E}{\mathrm{d}k}\right] = -2\nu k^2 E \tag{2.83}$$

$$\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}k} = -2\nu\alpha\varepsilon^{-\frac{1}{3}} k^{\frac{1}{3}} - \frac{5}{3} k^{-1} \tag{2.84}$$

$$E = C_k \varepsilon^{\frac{1}{3}} k^{-\frac{5}{3}} \mathrm{e}^{-\frac{3}{2}\alpha\nu\varepsilon^{-\frac{1}{3}} k^{\frac{4}{3}}} \tag{2.85}$$

若 $\boldsymbol{v}$ 解析, $k \to \infty, E \sim \mathrm{e}^{-\beta k}$

3 HIT 衰变早期的相似性解

考虑相似性解: 引入 v, l 都只是 t 的函数, $E = V^2 l F(x)$, $x = kl$, $T = V^3 W(x)$

$$\frac{\partial E}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(V^2 l\right) F(x) + V^2 l F' k \frac{\mathrm{d}l}{\mathrm{d}t} = V^3 W - 2\nu k^2 V^2 l F \tag{3.1}$$

$$V^2\frac{\mathrm{d}l}{\mathrm{d}t}xF' + \left[\frac{\mathrm{d}}{\mathrm{d}t}(V^2 l) + 2\nu V^2 l^{-1}x^2\right]F - V^3 W = 0 \tag{3.2}$$

同除 V^3

$$\frac{1}{V}\frac{\mathrm{d}l}{\mathrm{d}t}xF' + \left[\frac{1}{V^3}\frac{\mathrm{d}}{\mathrm{d}t}(V^2 l) + 2\nu\frac{1}{lV}l^{-1}x^2\right]F - W = 0 \tag{3.3}$$

要求: $\frac{1}{V}\frac{\mathrm{d}l}{\mathrm{d}t} = \alpha$

$$\frac{1}{V^3}\frac{\mathrm{d}}{\mathrm{d}t}(V^2 l) = \frac{1}{V}\frac{\mathrm{d}l}{\mathrm{d}t} + \frac{2l}{V^2}\frac{\mathrm{d}V}{\mathrm{d}t}, \quad \frac{\nu}{lV} = \alpha_3 \tag{3.4}$$

则

$$\alpha_1 xF' + \left[\alpha_1 + \alpha_2 + \alpha_3 x^2\right]F - W = 0 \tag{3.5}$$

$$\frac{1}{V}\frac{\mathrm{d}l}{\mathrm{d}t} = \alpha_1 \implies 2V\frac{\mathrm{d}l}{\mathrm{d}t} = 2V^2\alpha_1 \tag{3.6}$$

$$\frac{2l}{V^2}\frac{\mathrm{d}V}{\mathrm{d}t} = \alpha_2 \implies 2l\frac{\mathrm{d}V}{\mathrm{d}t} = V^2\alpha_2 \tag{3.7}$$

$$2\frac{\mathrm{d}(lV)}{\mathrm{d}t} = V^2(2\alpha_1 + \alpha_2) \implies 2\alpha_1 + \alpha_2 = 0 \tag{3.8}$$

$$\frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\frac{\nu}{\alpha_3}}{V}\right) = \alpha_1 \tag{3.9}$$

$$\frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{V}\right) = \frac{\alpha_1\alpha_3}{\nu} \tag{3.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{V}\right)^2 = \frac{2\alpha_1\alpha_3}{\nu} \tag{3.11}$$

$$\left(\frac{1}{V}\right)^2 = \frac{2\alpha_1\alpha_3}{\nu}t + C \tag{3.12}$$

$$l^2 = \left(\frac{\nu}{\alpha_3 V}\right)^2 = \frac{\nu^2}{\alpha_3^2}\left[\frac{2\alpha_1\alpha_3}{\nu}t + C\right] = \frac{2\nu\alpha_1}{\alpha_3}t + C' \tag{3.13}$$

$$\frac{3}{2}\left\langle u^2\right\rangle = \int_0^\infty E\,\mathrm{d}k = V^2\int_0^\infty F(x)\,\mathrm{d}x \sim V^2 \sim (t+C)^{-1} \tag{3.14}$$

$$\frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle u^2\right\rangle = -\varepsilon \sim (t+C)^{-2} \tag{3.15}$$

$$\frac{\mathrm{d}\frac{3}{2}\left\langle u^2\right\rangle}{\mathrm{d}t}\frac{1}{\frac{3}{2}\left\langle u^2\right\rangle} = -(t+C)^{-1} = \frac{-15\nu\frac{\left\langle u^2\right\rangle}{\lambda^2}}{\frac{3}{2}\left\langle u^2\right\rangle} = -10\frac{\nu}{\lambda^2} \tag{3.16}$$

$$R_0 \equiv \frac{\sqrt{\left\langle u^2\right\rangle|_{t=0}}\lambda_0}{\nu} \tag{3.17}$$

$$\int_0^\infty 2x^2F(x)\,\mathrm{d}x = \frac{\varepsilon t^2}{\nu} = \frac{3}{20}R_0^2 \equiv R \tag{3.18}$$

Landau 质疑 (1944)

$$S_2(r) = C_2 \varepsilon^{\frac{2}{3}} r^{\frac{2}{3}}, \quad \eta \ll r \leq L \tag{3.19}$$

C_2 不唯一.

K62 理论

$v \sim \varepsilon r \implies S_2(r) \propto (\varepsilon_r r)^{\frac{2}{3}}$ 粗粒化的耗散率 $\varepsilon_r \equiv \frac{1}{V}\int_{B(r)} \varepsilon \, \mathrm{d}V$

设 $\ln \varepsilon_r$ 为正态分布

$$\langle \ln \varepsilon \rangle = a(r) = c \ln \frac{r_0}{r} + a_0 \tag{3.20}$$

$$\left\langle (\ln \varepsilon_r - a)^2 \right\rangle = \sigma(r) = \mu \ln \frac{r_0}{r} + A \tag{3.21}$$

$$p(\varepsilon_r) = \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{\varepsilon_r} \mathrm{e}^{-\frac{(\ln \varepsilon_r - a)^2}{2\sigma^2}} \tag{3.22}$$

$$\int_0^\infty x^n p(x) \, \mathrm{d}x = \int_0^\infty x^n \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} \mathrm{e}^{-\frac{(\ln x - a)^2}{2\sigma^2}} \, \mathrm{d}x = \mathrm{e}^{na + \frac{\sigma^2 n^2}{2}} \tag{3.23}$$

$$\langle \varepsilon_r^p \rangle = (\mathrm{e}^a)^n \left(\mathrm{e}^{\frac{\sigma^2}{2}} \right)^{n^2} \propto \left(\frac{r_0}{r} \right)^{cn} \left(\frac{r_0}{r} \right)^{\frac{\mu}{2} n^2} = \left(\frac{r}{r_0} \right)^{-cn - \frac{\mu}{2} n^2} \tag{3.24}$$

$$\xi_p = \frac{p}{3} + \frac{\mu}{2} \left(\frac{p}{3} - \frac{p^2}{9} \right) \tag{3.25}$$

$$\left\langle \varepsilon_l^p \right\rangle \sim l^{\tau(p)} \tag{3.26}$$

$$\delta u_l \sim (\varepsilon_l l)^{\frac{1}{3}} \tag{3.27}$$

$$\xi(p) = \frac{p}{3} + \tau \left(\frac{p}{3} \right) \tag{3.28}$$

$P(\delta u_l)$ 与 $P(\varepsilon_l)$ 有简单的关系

$$\tau \left(\frac{p}{3} \right) = \frac{\mu}{18} p(3 - p) \tag{3.29}$$

$$\tau\left(p\right) = \frac{\mu}{2} p(1 - p) \tag{3.30}$$

$\xi(p) < 0$?

$$S_p(l) = \left(\frac{l}{l_0} \right)^{\xi(p)} S_p(l_0), \quad \frac{l}{l_0} \leq 1 \tag{3.31}$$

若 δu_l 有最大值 $U_{\max}$, 若某个 $\xi_p < 0, l \to 0, S_p + \left\langle |\delta u_l|^p \right\rangle \leq (2U_{\max})^p$

ξ_p 是凸的 $\xi_p'' < 0$

$$\ln S_p(l) = \xi(p) \ln \left(\frac{l}{l_0} \right) + \ln S_p(l_0) \tag{3.32}$$

对 p 求导

$$\frac{S_p'}{S_p} = \xi_p' \ln\left(\frac{l}{l_0}\right) + \frac{S_p'(l_0)}{S_p(l_0)} \tag{3.33}$$

再求导

$$\frac{S_p''S_p - (S_p')^2}{S_p^2} = \xi_p'' \ln\left(\frac{l}{l_0}\right) + \frac{S_p''(l_0)S_p(l_0) - [S_p'(l_0)]^2}{S_p(l_0)^2} \tag{3.34}$$

由施瓦尔兹不等式得结论, 最左边小于 0, 最右边大于 0

$$S_p = \int_0^\infty x^p P_l(x)\,\mathrm{d}x \tag{3.35}$$

$$S_p' = \int_0^\infty x^p \ln x P_l(x)\,\mathrm{d}x \tag{3.36}$$

$$S_p'' = \int_0^\infty x^p \ln^2 x P_l(x)\,\mathrm{d}x \tag{3.37}$$

分布意义下

$$\delta u_l \equiv W_{ll_0} \delta u_{l_0}, \quad l \le l_0, \tag{3.38}$$

其中 W_{ll_0} 是随机乘子

$$\begin{aligned} \varepsilon_l &\equiv W_{ll_1}\varepsilon_{l_1} \\ &= W_{ll_1}W_{l_1l_2}\varepsilon_{l_2} \\ &= W_{ll_1}W_{l_1l_2}W_{l_1l_2}\varepsilon_{l_2} \end{aligned} \tag{3.39}$$

$$\ln W_{ll_1} = \sum_{i=1}^{n} \ln W_{l_il_{i+1}} \tag{3.40}$$

W_{ll_1} 的分布只与 $\frac{l}{l_1}$ 有关.

$$\begin{aligned} \left\langle \varepsilon_l^p \right\rangle &= \left\langle W_{ll_1}^p \varepsilon_{l_1}^p \right\rangle \\ &= \left\langle W_{ll_1^p} \right\rangle \left\langle \varepsilon_{l_1}^p \right\rangle \end{aligned} \tag{3.41}$$

$$\left\langle W_{ll_1}^p \right\rangle = \left(\frac{l}{l_1}\right)^{\tau(p)} \tag{3.42}$$

$$\frac{l}{l_1} = \frac{l}{l_2}\frac{l_2}{l_3}\cdots\frac{l_{n_1}}{l_1} = \left(\frac{l}{l_2}\right)^{n-1} \tag{3.43}$$

分形几何方法,β-模型, 大涡 l_0 小涡 $\lambda l_0, \lambda < 1$ 体积 $\beta l_0^3 \neq l^3$ n 次 $l = \lambda^n l_0, V = \beta^n l_0^3$ 小涡占有体积比率

个数 $N = \frac{\beta^n l_0^3}{l^3} = \lambda^{n\frac{\ln\beta}{\ln\lambda}}$

4 数值模拟

三维 $\boldsymbol{u}(\boldsymbol{x},t)$, 周期边界条件.

$$\boldsymbol{u}(\boldsymbol{x},t)=\sum_{\boldsymbol{k}}\hat{\boldsymbol{u}}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4.1}$$

$$\nabla\cdot\boldsymbol{u}=0 \implies \sum_{k}\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{u}}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}=0 \implies \boldsymbol{k}\cdot\hat{\boldsymbol{u}}(\boldsymbol{k},t)=0, \tag{4.2}$$

$$\frac{\partial\boldsymbol{u}}{\partial t}+\nabla\cdot(\boldsymbol{u}\boldsymbol{u})=-\nabla\left(\frac{p}{\rho}\right)+\nu\nabla^2\boldsymbol{u}+\boldsymbol{f}. \tag{4.3}$$

亥姆霍兹分解 $\boldsymbol{a}=\nabla\phi+\boldsymbol{a}_\perp$

$$-k^2\hat{\phi}=\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{a}}(\boldsymbol{k}),\quad \hat{\phi}=-\frac{\mathrm{i}\boldsymbol{k}}{k^2}\cdot\hat{\boldsymbol{a}}(\boldsymbol{k}) \tag{4.4}$$

$$\hat{\boldsymbol{a}}_\perp=\hat{\boldsymbol{a}}-\mathrm{i}\boldsymbol{k}\hat{\phi}=\hat{\boldsymbol{a}}+\mathrm{i}\boldsymbol{k}\frac{\mathrm{i}\boldsymbol{k}}{k^2}\cdot\hat{\boldsymbol{a}}=\hat{\boldsymbol{a}}-\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\cdot\hat{\boldsymbol{a}}=\left(I-\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right)\cdot\hat{\boldsymbol{a}}=P\cdot\hat{\boldsymbol{a}} \tag{4.5}$$

其中

$$P_{ij}=\delta_{ij}-\frac{k_ik_j}{k^2} \tag{4.6}$$

ODE:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu k^2\right)\hat{u}_i(\boldsymbol{k},t)=-\mathrm{i}k_mP_{ij}(\boldsymbol{k})\cdot\sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}}u_j(\boldsymbol{p},t)u_m(\boldsymbol{q},t)+\hat{f}(\boldsymbol{k},t) \tag{4.7}$$

$$\sum_{k-\frac{1}{2}\leq|\boldsymbol{k}|\leq k+\frac{1}{2}}\left\langle|\hat{u}_i\hat{u}_i|\right\rangle=E(k) \tag{4.8}$$

Kolmogorov 尺度 $\frac{L}{N}\sim\eta$

$$\eta\equiv\left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \tag{4.9}$$

螺旋波分解

$$\begin{cases} u=\sin z\\ v=\cos z\\ w=0.\end{cases} \tag{4.10}$$

$$\boldsymbol{\omega}=\boldsymbol{u} \tag{4.11}$$

$$\begin{cases} u=-\sin z\\ v=\cos z\\ w=0.\end{cases} \tag{4.12}$$

$$\boldsymbol{\omega} = -\boldsymbol{u} \tag{4.13}$$

$$\begin{cases} u = \sin kz \\ v = \cos kz \\ w = 0. \end{cases} \implies \begin{aligned} \boldsymbol{V}_1^+ &= (\sin kz, \cos kz, 0) \\ \boldsymbol{V}_1^- &= (-\sin kz, \cos kz, 0) \end{aligned} \tag{4.14}$$

$$\begin{aligned} \boldsymbol{V}_2^+ &= (-\sin kz, \cos kz, 0) \\ \boldsymbol{V}_2^- &= (-\sin kz, -\cos kz, 0) \end{aligned} \tag{4.15}$$

$kz \to \boldsymbol{k} \cdot \boldsymbol{x}$

$$\begin{cases} \boldsymbol{V}^+ = \boldsymbol{V}_1^+ + \mathrm{i}\boldsymbol{V}_2^+ \\ \boldsymbol{V}^- = \boldsymbol{V}_1^- + \mathrm{i}\boldsymbol{V}_2^- \end{cases} \tag{4.16}$$

$$\boldsymbol{u} = \sum_{\boldsymbol{k}} c_k \hat{\boldsymbol{V}}^{\pm}(\boldsymbol{k}, t) \tag{4.17}$$

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = \sum_{m,n} Q_{kmn} c_m c_n + \nu k^2 c_k + f_k \tag{4.18}$$

三波 $\boldsymbol{m} + \boldsymbol{n} + \boldsymbol{k} = 0.$

吉田善章 1990

$$\begin{cases} \nabla \times \boldsymbol{B}_n = \lambda_n \boldsymbol{B}_n \\ \boldsymbol{n} \cdot \boldsymbol{B}_n = 0 \\ \boldsymbol{n} \cdot \nabla \times \boldsymbol{B}_n = 0 \end{cases} \tag{4.19}$$

4.1 湍流数值模拟

Reynold 应力, $R_{ij} \equiv \left\langle u_i' u_j' \right\rangle$ 湍流模型.

二阶矩方法

$$\frac{\partial \left\langle u_i' u_j' \right\rangle}{\partial t} = P + D_1 + D_2 \tag{4.20}$$

右端分别是产生, 扩散和耗散.

涡粘性方法

$$\left\langle u_i' u_j' \right\rangle = 2\nu_T S_{ij} + \frac{2}{3}\delta_{ij} k, \quad k \equiv -\frac{1}{2}\left\langle u_i' u_j' \right\rangle \tag{4.21}$$

$$\frac{\mathrm{d}\nu_T}{\mathrm{d}t} = \cdots \tag{4.22}$$

Calay–Hamilton 定理