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Turbulent Thermal Convection at Arbitrary Prandtl Number

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The mixing-length theory of turbulent thermal convection in a gravitationally unstable fluid is extended to yield the dependence of Nusselt number H/H_0 on both Prandtl number σ and Rayleigh number Ra. The analysis assumes a layer of Boussinesq fluid contained between infinite, horizontal, perfectly conducting, rigid plates. Also obtained is the dependence of mean temperature deviation $\bar{T}(z)$, rms temperature fluctuation $\tilde{\psi}(z)$, and rms velocity upon height z above the bottom plate. The theory gives $H/H_0 \propto \mathrm{Ra}^{1/3}$ (high σ), $H/H_0 \propto (\sigma\,\mathrm{Ra})^{1/3}$ (low σ), and $H/H_0 \sim 1$ (very low σ). The boundaries of the several σ ranges are determined. At one intermediate Prandtl number only, the behavior of $\bar{T}(z)$ and $\tilde{\psi}(z)$ reduces to that previously found by Priestley. At high σ, there is a range of z, outside the molecular conduction region, where $\bar{T}(z) \propto z^{-1}$, $\tilde{\psi}(z) \propto z^{-1}$. The results at very low σ reduce to those of Ledoux, Schwarzschild, and Spiegel. The dynamics are found to be importantly modified at extremely large Ra because of the stirring action of small-scale turbulence generated in shear boundary layers attached to the eddies of largest scale. The consequent corrected asymptotic law of heat transport at fixed σ is $H/H_0 \propto [\mathrm{Ra}/(\ln \mathrm{Ra})^3]^{1/2}$.

I. INTRODUCTION

TURBULENT thermal convection in a gravitationally unstable layer of fluid has been studied both experimentally[1–6] and theoretically.[4–8] The present paper is an extension of the Prandtl-type mixing-length treatment of this phenomenon developed by Priestley[4] and Böhm-Vitense.[7] We generalize Priestley's theory so as to obtain the dependence of heat transport, velocity structure, and temperature structure on both Prandtl and Rayleigh numbers. In addition, we consider the effects, upon heat transport, of small-scale turbulence generated in the shear boundary layers of the large-scale turbulent eddies. All of the analysis is based upon the Boussinesq approximation for convection in a thin layer.[9] Within the limits of this approximation, we hope that the present treatment has some qualitative validity over the ranges of Prandtl and Rayleigh numbers which are realizable in the laboratory or in meteorological and astrophysical situations.

The motivation of this study is twofold. We hope, first, that the inclusion of Prandtl-number dependence will permit a more meaningful comparison of the predictions of mixing-length theory with experiment. In this connection, we shall discuss the laboratory measurements made recently by Globe and Dropkin[3] over a wide range of Prandtl and Rayleigh numbers. Also, we shall reassess the disagreement reported by Townsend[5] between Priestley's theory and laboratory experiments in air. The second motivation is to provide a qualitative physical framework within which to discuss more deductive theories based upon systematic mathematical approximations to the equations of motion.

Priestley's theory is concerned with statistically steady-state turbulent convection in a horizontally infinite layer of fluid. It is assumed that at sufficiently large Rayleigh number most of the change in mean temperature across the layer occurs in thin boundary regions, at the surfaces, where molecular heat conduction and molecular viscosity are dominant. Elsewhere in the fluid, it is assumed that convective heat transport and eddy viscosity are dominant. The theory leads to qualitative predictions for the total heat transport and for the structure of the mean temperature field, fluctuating temperature field, and velocity field. All of the predicted functional forms and orders of magnitude are independent of Prandtl number.

In the present treatment, two asymptotic cases are distinguished: high and low Prandtl numbers. For high Prandtl number, we admit, in addition to the two regions considered by Priestley, an intermediate region where convective heat transport dominates over molecular conduction but where momentum transfer still is dominated by molecular viscosity. Conversely, for low Prandtl number we

[1] M. Jakob, *Heat Transfer* (John Wiley & Sons, Inc., New York, 1949), Vol. 1.
[2] P. L. Silveston, Forsch. Gebiete Ingenieurw. **24**, 29, 59 (1958).
[3] S. Globe and D. Dropkin, J. Heat Transfer **81**, 24 (1959).
[4] C. H. B. Priestley, *Turbulent Transfer in the Lower Atmosphere* (University of Chicago Press, Chicago, 1959). This book contains many further references.
[5] A. A. Townsend, J. Fluid Mech. **5**, 209 (1959).
[6] W. V. R. Malkus, Proc. Roy. Soc. (London) **A225**, 185, 196 (1954); Nuovo cimento Suppl. **22**, 376 (1961).
[7] E. Böhm-Vitense, Z. Astrophys. **46**, 108 (1958).
[8] P. Ledoux, M. Schwarzschild, and E. Spiegel, Astrophys. J. **133**, 184 (1961); E. A. Spiegel, J. Geophys. Research **67**, 3063 (1962).
[9] See reference 8, for example.

admit an intermediate region where molecular heat transport and eddy viscosity are dominant. In either case, the boundaries of the several regions are determined by the values of the Reynolds and Peclet numbers formed from the local root-mean-square (rms) vertical velocity and the local dominant spatial scale of the turbulence. This scale, or mixing length, is assumed to be the order of the height from the boundary surface, as in the Priestley theory. The qualitative behavior of the total heat transport and of the temperature and velocity fields is determined from the requirement of continuity of heat transport and from the qualitative balance equations for production and dissipation of temperature and velocity fluctuations.

At a transition Prandtl number, which we estimate with low confidence as about 0.1, our treatments for high and low Prandtl numbers both give qualitative results that coincide with those of Priestley. For higher Prandtl numbers, we obtain Priestley's law of total heat transport but a mean temperature gradient that falls off more rapidly with height above the conduction region than does Priestley's. For Prandtl numbers less than the transition value, we find that the heat transport for given Rayleigh number decreases with Prandtl number. In the extreme case where the Prandtl number is so small that the conduction region extends over the entire fluid, our results reduce to those of Ledoux, Schwarzschild, and Spiegel.[8] The formulas that express all of these results are summarized in Sec. VIII.

In the second half of the present paper, we introduce an assumption that there are turbulent shear boundary layers associated with those eddies whose spatial scale is the order of the total layer thickness. When the Rayleigh number is sufficiently high, we find that there then should be substantially stronger horizontal velocities near a nonslip boundary surface than would be predicted if shear-boundary-layer effects were ignored. At still higher Rayleigh numbers, we find that turbulence of small spatial scale, generated locally in the boundary layers of the big eddies, can enhance heat convection near the boundary surfaces sufficiently to alter the asymptotic law of dependence of heat transport on Rayleigh number.

II. BOUSSINESQ EQUATIONS OF MOTION

Consider a horizontally infinite layer of fluid of thickness D confined between rigid, perfectly conducting top and bottom surfaces at which all components of velocity vanish. We shall discuss other boundary conditions later. Let the fluid be heated from below so that a steady temperature difference ΔT is maintained between top and bottom surfaces. The Boussinesq equations of motion are[9]

$$(\partial/\partial t - \kappa\nabla^2)T = -\mathbf{u}\cdot\nabla T, \tag{2.1}$$

$$(\partial/\partial t - \nu\nabla^2)\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho_m^{-1}\nabla p + \mathbf{n}\gamma T, \tag{2.2}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.3}$$

where κ is the thermometric conductivity, T the temperature, $\mathbf{u}$ the velocity vector, ν the kinematic viscosity, ρ_m the mean density, p the pressure, $\mathbf{n}$ the unit vector pointing vertically upward, and γ the product of gravitational acceleration and volume coefficient of thermal expansion. We take the z axis vertical with z measured upward from the bottom surface. The zero of temperature is taken as the mean temperature at the midplane of the fluid, $z = \frac{1}{2}D$.

It follows from dimensional analysis of (2.1)–(2.3) that the structure of the thermal convection depends upon two dimensionless parameters, the Prandtl number

$$\sigma = \nu/\kappa \tag{2.4}$$

and the Rayleigh number

$$\mathrm{Ra} = \gamma\Delta T D^3/\kappa\nu. \tag{2.5}$$

In particular, for steady or statistically steady convection the mean thermometric rate of vertical heat transport per unit area of surface must be expressible in the form

$$H = H_0 f(\mathrm{Ra}, \sigma), \tag{2.6}$$

where

$$H_0 = \kappa\Delta T/D \tag{2.7}$$

is the transport which would exist if there were no motion. The functional form f depends, in general, on the boundary conditions.

Let us write

$$T = \bar{T} + \psi, \tag{2.8}$$

where $\bar{T}$ is the ensemble mean of T and ψ then is the temperature fluctuation. In the present steady-state, horizontally infinite case, we assume that an ensemble can be formed such that ensemble averages are equivalent to averages either over time or over the horizontal. Then we have $\bar{T} = \bar{T}(z)$, and our convention for the zero of temperature gives $\bar{T}(\frac{1}{2}D) = 0$. An assumption of statistical isotropy in the horizontal, together with (2.3), implies that the velocity field has zero ensemble mean.

Equations (2.1) and (2.2) may be decomposed into the following equations for $\bar{T}$, ψ, and $\mathbf{u}$.

$$\kappa(d^2\bar{T}/dz^2) = d\langle w\psi\rangle/dz, \tag{2.9}$$

$$(\partial/\partial t - \kappa\nabla^2)\psi = -w(d\bar{T}/dz) - [\mathbf{u}\cdot\nabla\psi - \langle\mathbf{u}\cdot\nabla\psi\rangle], \tag{2.10}$$

$$(\partial/\partial t - \nu\nabla^2)\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho_m^{-1}\nabla p + \mathbf{n}\gamma\psi. \tag{2.11}$$

Here $\langle\ \rangle$ denotes ensemble mean and w is the vertical velocity component. In writing (2.9) and (2.10), we have made use of the assumption of isotropy in the horizontal, the steady-state condition $\partial\bar{T}(z)/\partial t = 0$, and (2.3). [Also, the part of p that represents steady hydrostatic pressure has been omitted in (2.11), together with the counterbalancing mean buoyancy term $\gamma\mathbf{n}\bar{T}$.] Equation (2.9) may be integrated once to yield

$$H = -\kappa(d\bar{T}/dz) + \langle w\psi\rangle, \tag{2.12}$$

where H now appears as the constant of integration.

At any height z, the two terms on the right side of (2.12) represent, respectively, the contributions to H from molecular conduction and from convection. For the purposes of mixing-length analysis, we shall attribute the following characteristic roles to the various terms in (2.10) and (2.11): The term $-w(d\bar{T}/dz)$ in (2.10) represents creation (or destruction) of temperature fluctuations by action of the velocity field on the mean temperature gradient. The terms

$$-\kappa\nabla^2\psi \quad \text{and} \quad -[\mathbf{u}\cdot\nabla\psi - \langle\mathbf{u}\cdot\nabla\psi\rangle]$$

represent destruction (smoothing) of temperature fluctuations by molecular and eddy conduction, respectively. In (2.11), the term $\mathbf{n}\gamma\psi$ represents buoyancy, which is the prime mover for the velocity field. The terms $-\nu\nabla^2\mathbf{u}$ and $-(\mathbf{u}\cdot\nabla)\mathbf{u}$ represent molecular and eddy damping of the velocity field, respectively. The pressure force $-\nabla p$ is the reaction force which maintains incompressibility of the flow. It is a means by which the vertical buoyancy force can result in horizontal as well as vertical motion.

III. TRANSITION REYNOLDS AND PECLET NUMBERS

Let local Reynolds and Peclet numbers be defined by

$$\mathrm{Re}\,(z) = \tilde{w}(z)z/\nu \tag{3.1}$$

and

$$\mathrm{Pe}\,(z) = \tilde{w}(z)z/\kappa, \tag{3.2}$$

respectively, where $\tilde{w}(z)$ is the rms value of w at height z. Following Priestley, we shall assume that the effective mixing length at height z is of order z, as in Prandtl's mixing length theory of turbulent shear flow over a wall.[10] The assumption implies that the local coefficients of eddy conductivity and eddy viscosity are measured by $\tilde{w}z$. Thus Re and Pe should provide measures of the relative importance of eddy transport of momentum and heat over molecular transport.

Let us denote by z_κ the height z at which convection accounts for just half the total mean heat transport. Similarly, let us denote by z_ν the height at which eddy viscosity effects, appropriately measured, become equal to molecular viscosity effects.[11] We may then define transition Peclet and Reynolds numbers by

$$\mathrm{Pe_T} = \tilde{w}(z_\kappa)z_\kappa/\kappa, \tag{3.3}$$

$$\mathrm{Re_T} = \tilde{w}(z_\nu)z_\nu/\nu. \tag{3.4}$$

The analysis in the following sections is based on the assumption that $\mathrm{Pe_T}$ and $\mathrm{Re_T}$ take numerical values which are independent of σ and Ra. It is unlikely that this is strictly true. In any given case, the actual values plausibly should depend in part upon detailed statistical properties of the flow structures, involving spatial form, degree of isotropy, etc. These in turn will not be wholly independent of σ and Ra.

Some support, however, is provided by the validity of analogous assumptions in shear flows. In flows with a mixing layer—for example, jets and boundary layers—turbulent breakdown occurs when the Reynolds number based on layer thickness and velocity change across the layer reaches a critical range of values which appears to vary little with overall flow Reynolds number, or with the type of flow. For a variety of jet, wake, and boundary-layer flows, this critical range is of order 200–300.[12] At first guess, we might take these numbers as an estimate for $\mathrm{Re_T}$ in the thermal convection problem also. Actually, $\mathrm{Re_T}$ may be substantially lower. The reason is that in the convection problem the relevant mixing layers will be between eddies rather than, say, between still air and a jet. In the author's

[10] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1959), Chap. 4.

[11] One definition of z_ν could be the height at which the mean square transport of vertical momentum across vertical surface elements is half by Reynolds stresses and half by viscosity.

[12] A. A. Townsend, *The Structure of Turbulent Shear Flow* (Cambridge University Press, Cambridge, England, 1956).

opinion, this suggests a higher degree of instability for the convection case, and, consequently, lower minimum Reynolds numbers for turbulent breakdown of the large-scale flow structures.

Possibly a lower limit to Re_T could be estimated from data on the decay of the energy range of isotropic turbulence. Such turbulence should constitute an optimum kind of flow for turbulent breakdown of the large-scale motions. Measurements on grid-produced isotropic turbulence indicate that the decay time of the energy range due to eddy cascade is approximately $L/\tilde{u}$, where $\tilde{u}$ is the rms value of any velocity component and L is the integral scale.[13] If this result is extrapolated down to low turbulent intensities, then the value of the Reynolds number $\tilde{u}L/\nu$ that makes $L/\tilde{u}$ equal to the time for direct viscous decay of the energy range is of order 20. The exact value obtained depends on the form assumed for the spectrum in the energy range.

The author has not found experimental data from which Pe_T can be estimated directly. However, a theoretical estimate of Pe_T can be made on the basis of the analysis of finite-amplitude thermal convection given by Malkus and Veronis.[14] These authors employ a modified perturbation method to treat convection by roll-form cells in a layer of fluid with slip boundary conditions (i.e., zero temperature fluctuation, vertical velocity, and horizontal stress on the boundaries). Using their lowest-order finite-amplitude results, one finds that the mean heat transfer across the midplane of the fluid is half by convection and half by conduction when the Rayleigh number is such that $H/H_0 = 4/3$. At this H/H_0, the higher-order perturbation corrections appear to be small. The result for the velocity field then gives $\mathrm{Pe}_\mathrm{T} \sim 1.7$. We regard this value of Pe_T as probably an underestimate for the turbulent convection regime with nonslip boundary conditions. It seems plausible that the velocity fields which occur in the latter case should have a less efficient form for heat transfer than the very regular fields encountered in the finite-amplitude cellular regime.

For the purposes of the present paper, we shall use the nominal values

$$\mathrm{Re}_\mathrm{T} \sim 30, \qquad \mathrm{Pe}_\mathrm{T} \sim 3 \tag{3.5}$$

when numerical values of Re_T and Pe_T are needed. It should be clear from the discussion above that these must be regarded as *only* nominal values. Experience with shear flows suggests that reliable numerical values must wait upon direct measurements. We wish to remark that the value of Pe_T plays a larger role than that of Re_T in the results to be obtained in the present paper.

Despite the large uncertainties in the estimates we have presented, it appears indicated that the ratio $\mathrm{Re}_\mathrm{T}/\mathrm{Pe}_\mathrm{T}$ is substantially greater than 1. We may ask why this should be. One reason may lie in an important intrinsic difference between the transport of heat by the velocity field and the transport by the field of its own momentum. Transport of momentum down momentum gradients perpendicular to the large-scale velocity field typically involves, first, the generation of small eddies which provide velocity components pointed crosswise to those of the large-scale field. In heat transport, on the other hand, the large-scale velocity field itself can have major components pointing up and down the mean temperature gradient, as in the cellular regime.

We wish at this point to describe the mode of presentation adopted in the sections which follow. The results to be obtained for the various ranges of Prandtl and Rayleigh numbers, and for the several z regions, combine two related kinds of predictions. The first consists of qualitative laws (typically power laws) for the functional dependence of of H/H_0 upon σ and Ra, and of $\bar{\psi}$, $\bar{T}$, and $\bar{w}$ upon z. The second involves determination of approximate values for the numerical coefficients which occur in the laws of functional dependence. If only the first kind of prediction were desired, it would suffice to express our basic assumption about effective mixing length by simply replacing space derivatives with z^{-1} in (2.10)–(2.12). Instead, we frequently shall take particular, highly idealized models of the effective flow pattern at height z which embody our mixing-length assumption and which, in addition, permit the estimation of the numerical coefficients. Since the flow models to be chosen are both artificial and arbitrary, the numerical values to be obtained must be regarded as crude estimates only.

IV. HIGH PRANDTL NUMBER

When σ is large and Ra is sufficiently high, we anticipate that the flow will consist of three regions:

$$\text{I.}\quad z < z_\kappa,$$

$$\text{II.}\quad z_\kappa < z < z_\nu,$$

$$\text{III.}\quad z_\nu < z < \tfrac{1}{2}D.$$

[13] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, England, 1953), Chap. 6.

[14] W. V. R. Malkus and G. Veronis, J. Fluid Mech. 4, 225 (1958).

In Region I, molecular conduction and viscosity both dominate over the corresponding eddy processes. In Region II, eddy conductivity and molecular viscosity are dominant while in Region III eddy conductivity and eddy viscosity are dominant. If σ is large enough for given Ra, we expect that Region III will be nonexistent and that molecular viscosity will dominate over the entire flow. A criterion for this to occur will be obtained presently.

We shall now derive approximate expressions for H/H_0 and for the behavior of ψ, $\bar{T}$, and w in the several regions. First consider Region II. If the second part of the inequality $z_\kappa < z < z_\nu$ is strong enough, we may approximate (2.11) by

$$-\nu\nabla^2\mathbf{u} \sim -\rho_m^{-1}\nabla p + \mathbf{n}\gamma\psi, \qquad (4.1)$$

where the Reynolds stress term is omitted. The term $\partial\mathbf{u}/\partial t$ is omitted also because the characteristic time for variation of the driving term $\mathbf{n}\gamma\psi$ is expected to be the order of the local eddy circulation time, which is large compared to the time for viscous decay if Re (z) is small. Thus we expect

$$|\partial\mathbf{u}/\partial t| \ll |\nu\nabla^2\mathbf{u}|.$$

In order to obtain an approximate relation between w and ψ, let us consider a hypothetical situation in which the nonslip boundary at $z = 0$ is replaced by one where horizontal stress rather than horizontal velocity vanishes. Assume that w and ψ at height z exhibit approximately sinusoidal spatial variation with vertical wavenumber $k_z = \pi/2z$ (vertical half-wavelength $= 2z$) and horizontal wavenumber $k_h = k_z$. Then it follows from (4.1) and (2.3) that ψ and w are in phase with each other and that their rms values are related by

$$\tilde{w} \sim \pi^{-2}\nu^{-1}z^2\gamma\tilde{\psi}. \qquad (4.2)$$

Our flow model is artificial, and consequently no significance is to be attached to the precise numerical coefficient π^{-2} which appears in (4.2). However, we believe that this value is more realistic than the value one which might be suggested by the crudest order-of-magnitude analysis.[15]

Under the assumption that molecular conduction is negligible in Region II, (2.12) reduces to

$$H \sim \tilde{w}\tilde{\psi}, \qquad (4.3)$$

where we use the result that w and ψ are in phase. Finally, let us assume

$$H \sim \kappa(\tfrac{1}{2}\Delta T)/z_\kappa. \qquad (4.4)$$

[15] If the actual nonslip boundary condition at $z = 0$ were used we would have obtained a lower rms value of w. On the other hand, the value could have been raised by choosing a more optimal value of k_z/k_h.

It follows from the definition in Sec. III that z_κ is the height at which $|d\bar{T}/dz|$ has decreased to one-half its value at $z = 0$. Also, we may note that an exact expression for H is

$$H = -\kappa(d\bar{T}/dz)_{z=0}. \qquad (4.5)$$

Thus (4.4) says that if the slope $|d\bar{T}/dz|$ did not diminish as z increased, then $\bar{T}$ would rise to the value zero at the height z_κ. Again, no significance can be attached to the precise numerical constant (one) which we have chosen in (4.4).

We may now obtain the following relations by manipulation of (2.5), (2.7), (3.3), (4.2), (4.3), and (4.4):

$$\tilde{\psi} \sim (2\,\mathrm{Pe_T})^{-1}\Delta T(z/z_\kappa)^{-1} \qquad (z_\kappa < z < z_\nu), \qquad (4.6)$$

$$\tilde{w} \sim \mathrm{Pe_T}\,\kappa z_\kappa^{-1}(z/z_\kappa) \qquad (z_\kappa < z < z_\nu), \qquad (4.7)$$

$$\gamma\Delta T z_\kappa^3/\kappa\nu \sim 2\pi^2(\mathrm{Pe_T})^2, \qquad (4.8)$$

$$H/H_0 \sim D/2z_\kappa \sim [\mathrm{Ra}/16\pi^2(\mathrm{Pe_T})^2]^{\frac{1}{3}}. \qquad (4.9)$$

To complete the analysis of Region II, we must find the qualitative behavior of $\bar{T}(z)$. Under our assumptions, statistical stationarity of ψ requires a balance of the form

$$|\mathbf{u}\cdot\nabla\psi - \langle\mathbf{u}\cdot\nabla\psi\rangle| \sim |w\, d\bar{T}/dz|. \qquad (4.10)$$

It can be seen from (4.6) that if this balance is to hold when the inequalities $z_\kappa < z < z_\nu$ are strong, we must have $\bar{T} \propto z^{-1}$. We therefore shall take

$$\bar{T} \sim \tilde{\psi} \qquad (z_\kappa < z < z_\nu). \qquad (4.11)$$

A constant of proportionality in (4.11) different from one might be indicated by more detailed analysis.

If we extrapolate (4.7) to $z = z_\nu$, we have

$$z_\nu/z_\kappa \sim (\sigma\,\mathrm{Re_T}/\mathrm{Pe_T})^{\frac{1}{2}}, \qquad (4.12)$$

by (3.3) and (3.4). Then, by (4.9), $z_\nu \sim \frac{1}{2}D$ when

$$[\mathrm{Ra}/16\pi^2(\mathrm{Pe_T})^2]^{\frac{1}{3}} \sim (\sigma\,\mathrm{Re_T}/\mathrm{Pe_T})^{\frac{1}{2}}. \qquad (4.13)$$

If Ra is less than the value determined by (4.13), Region III is nonexistent and molecular viscosity dominates throughout the flow. If Ra exceeds this value, $z_\nu < \frac{1}{2}D$ and Region III exists.

In Region III, the balance of input and dissipation for velocity and temperature fluctuations requires the respective order-of-magnitude relations

$$|(\mathbf{u}\cdot\nabla)\mathbf{u}| \sim |\gamma\psi|, \qquad (4.14)$$

$$|\mathbf{u}\cdot\nabla\psi| \sim |w\, d\bar{T}/dz|, \qquad (4.15)$$

while the constancy of mean heat flux requires

$$\langle w\psi\rangle \sim H. \qquad (4.16)$$

Relations (4.14)–(4.16) can be satisfied if

$$\tilde{w} \propto z^{\frac{1}{3}}, \qquad \tilde{\psi} \propto z^{-\frac{1}{3}}, \qquad \bar{T} \propto z^{-\frac{1}{3}}, \qquad (4.17)$$

which are Priestley's similarity laws.

In order to estimate the numerical coefficients in the power laws for $\tilde{w}$ and $\tilde{\psi}$, we make two, more specific, dynamical assumptions which embody (4.14)–(4.17). First, we take

$$\tfrac{1}{2}\tilde{w}^2 \sim \tfrac{1}{2}\gamma z\tilde{\psi}. \qquad (4.18)$$

This implies that a typical fluid element which reaches the height z attains a vertical kinetic energy equal to one-half the mean potential energy that would be available from buoyancy if the element had started at $z = 0$ with a temperature excess $\tilde{\psi}$ over its surroundings. Second, we replace (4.16) by

$$\tfrac{1}{2}\tilde{w}\tilde{\psi} \sim H. \qquad (4.19)$$

The factor $\frac{1}{2}$ expresses an assumption that w and ψ have only a 50% phase correlation in Region III. From (4.4), (4.8), (4.12), (4.18), and (4.19) we have

$$\tilde{\psi} \sim (2\pi^2 \, \mathrm{Pe}_\mathrm{T}^2)^{-\frac{1}{3}}\sigma^{-\frac{1}{3}}\Delta T(z/z_\kappa)^{-\frac{1}{3}} \quad (z_\nu < z < \tfrac{1}{2}D), \qquad (4.20)$$

$$\tilde{w} \sim (\gamma\kappa\Delta T)^{\frac{1}{3}}(z/z_\kappa)^{\frac{1}{3}} \quad (z_\nu < z < \tfrac{1}{2}D). \qquad (4.21)$$

To complete the equations for Region III, we assume that (4.11) is still approximately valid. This is consistent with (4.17).

As a check on the internal consistency of the analysis, we may compare the values of $\tilde{w}(z_\nu)$ and $\tilde{\psi}(z_\nu)$ obtained from the Region II laws (4.6) and (4.7) with those obtained from (4.20) and (4.21). We find

$$\begin{aligned} [\tilde{\psi}(z_\nu)]_{\mathrm{III}}/[\tilde{\psi}(z_\nu)]_{\mathrm{II}} &= 2(\mathrm{Re}_\mathrm{T}/2\pi^2)^{\frac{1}{3}}, \\ [\tilde{w}(z_\nu)]_{\mathrm{III}}/[\tilde{w}(z_\nu)]_{\mathrm{II}} &= (2\pi^2/\mathrm{Re}_\mathrm{T})^{\frac{1}{3}}. \end{aligned} \qquad (4.22)$$

Taking the nominal value (3.5), we find qualitative but inexact agreement. The precise degree of quantitative agreement is not significant, since we have made many arbitrary choices of constants of proportionality in obtaining the laws of dependence. In any event, it is not to be expected that the asymptotic laws for the two regions should give an exact join if extrapolated to $z = z_\nu$. Instead, a transition region is reasonable where neither law is exactly valid.

The following related reservation should be noted concerning the validity of the procedures used in this section. We have neglected completely any effect of the $(\mathbf{u}\cdot\nabla)\mathbf{u}$ term in (2.2) for Re $(z) < \mathrm{Re}_\mathrm{T}$. However, it is known in other flow situations that important dynamical effects of this term are felt at Reynolds numbers far below those required for turbulent breakdown to occur and for turbulent viscosity to be dominant. The implication in the present problem might be that a rather gradual transition is to be expected between Regions II and III.

A safe criterion for neglecting the Reynolds stresses might be Re $(z) \ll 1$. From (4.7), (4.9), and (3.3), we find that this condition should be satisfied throughout the flow if

$$H/H_0 \sim [\mathrm{Ra}/16\pi^2 \, \mathrm{Pe}_\mathrm{T}^2]^{\frac{1}{3}} \ll (\sigma/\mathrm{Pe}_\mathrm{T})^{\frac{1}{2}}. \qquad (4.23)$$

It should be noted that Re $(z) \ll 1$ everywhere does not imply absence of turbulence. Disordered motion still is to be expected, when Pe (z) is large, because of the stirring of the temperature field into complicated patterns and the consequent generation of complicated velocity fields by buoyancy forces.

In conclusion, we note from (4.12) that $z_\nu > z_\kappa$ implies

$$\sigma > \mathrm{Pe}_\mathrm{T}/\mathrm{Re}_\mathrm{T}, \qquad (4.24)$$

which gives a limit to the applicability of the high σ analysis developed in Sec. IV. The low σ case, in which (4.24) is not satisfied, will be treated in the next section.

V. LOW PRANDTL NUMBER

We expect to find two principal regions for small σ and sufficiently large Ra:

$$\mathrm{I.} \quad z < z_\kappa,$$

$$\mathrm{II.} \quad z_\kappa < z < \tfrac{1}{2}D.$$

Molecular conductivity and eddy viscosity should dominate throughout most of Region I if the inequality

$$\sigma < \mathrm{Pe}_\mathrm{T}/\mathrm{Re}_\mathrm{T}$$

is strong. In Region II, both eddy conductivity and eddy viscosity should dominate. If σ is small enough, for given Ra, we expect that Region II will be nonexistent and that molecular conductivity will dominate over the entire flow. The condition for this will be obtained shortly.

Let us assume (4.4) as before. The present Region II should be dynamically similar to Region III for the high σ case, and therefore we assume that (4.14)–(4.19) are valid. Then from (3.3), (4.4), (4.18), and (4.19) we find

$$\tilde{\psi} \sim (\mathrm{Pe}_\mathrm{T})^{-1}\Delta T(z/z_\kappa)^{-\frac{1}{3}}, \qquad (5.1)$$

$$\tilde{w} \sim \mathrm{Pe}_\mathrm{T}\kappa z_\kappa^{-1}(z/z_\kappa)^{\frac{1}{3}}, \qquad (5.2)$$

$$\gamma\Delta T z_\kappa^3/\kappa\nu \sim \sigma^{-1} \, \mathrm{Pe}_\mathrm{T}^3, \qquad (5.3)$$

$$H/H_0 \sim D/2z_\kappa \sim [\sigma \,\mathrm{Ra}/8\mathrm{Pe}_T^3]^{\frac{1}{5}}. \tag{5.4}$$

We shall complete the equations in this region with (4.11), the argument being the same as before. From (5.4) we see that the condition $z_\kappa < \frac{1}{2}D$ for the existence of Region II (i.e., the condition for the existence of thermal boundary-layering) is

$$[\sigma \,\mathrm{Ra}/8\mathrm{Pe}_T^3]^{\frac{1}{5}} > 1. \tag{5.5}$$

When (5.5) is not satisfied, the molecular conduction regime extends over the entire fluid. The extreme case

$$z_\nu \ll \tfrac{1}{2}D, \qquad z_\kappa \gg \tfrac{1}{2}D$$

has been treated by Ledoux, Schwarzschild, and Spiegel,[8] who used Heisenberg's expression for eddy viscosity. We shall, instead, use (4.18) to express the balance between buoyant input and eddy dissipation. First, we note that the assumption of dominant molecular conductivity everywhere implies

$$-d\bar{T}/dz \sim \Delta T/D$$

and

$$H \sim H_0. \tag{5.6}$$

Equation (2.10) then reduces to

$$-\kappa\nabla^2\psi \sim (\Delta T/D)w. \tag{5.7}$$

The time-derivative term is neglected for the same reasons as in (4.1). It follows from (5.7) that the spatial Fourier components of ψ and w are in phase. If we assume that the structure of w in the mid-regions ($z \sim \frac{1}{2}D$) is typified by a vertical wavenumber $k_z = 2\pi/D$ and an equal horizontal wavenumber, (5.7) gives

$$\tilde{\psi}(\tfrac{1}{2}D) \sim (8\pi^2\kappa)^{-1}D\Delta T\tilde{w}(\tfrac{1}{2}D). \tag{5.8}$$

Now applying (4.18) at $z = \frac{1}{2}D$ and using (5.8), we find

$$\tilde{\psi}(\tfrac{1}{2}D) \sim (128\pi^4)^{-1}\Delta T\sigma \,\mathrm{Ra}, \tag{5.9}$$

$$\tilde{w}(\tfrac{1}{2}D) \sim (16\pi^2)^{-1}\kappa D^{-1}\sigma \,\mathrm{Ra}, \tag{5.10}$$

$$\tilde{w}(\tfrac{1}{2}D)\tilde{\psi}(\tfrac{1}{2}D) \sim \tfrac{1}{2}(\sigma \,\mathrm{Ra}/32\pi^3)^2H_0. \tag{5.11}$$

Equation (5.11) gives the mean convective heat flux at $z = \frac{1}{2}D$. Apart from somewhat different values of the numerical constants, (5.9)–(5.11) reproduce results of Ledoux, Schwarzschild, and Spiegel.[8]

From (5.10) we see that the Reynolds number at $z = \frac{1}{2}D$ is

$$\mathrm{Re}\,(\tfrac{1}{2}D) \sim (32\pi^2)^{-1}\,\mathrm{Ra}. \tag{5.12}$$

A necessary condition for validity of the assumption that eddy viscosity dominates is that this Reynolds number exceed Re_T. Hence we must have

$$\mathrm{Ra} > 32\pi^2\,\mathrm{Re}_T. \tag{5.13}$$

Equation (5.13) together with

$$[\sigma \,\mathrm{Ra}/8\mathrm{Pe}_T^3]^{\frac{1}{5}} < 1, \tag{5.14}$$

the opposite inequality to (5.5), constitute conditions for the validity of the analysis for the present subcase.

It can be seen from (5.11) that for fixed ν, κ, and ΔT, the predicted convective heat transport is $\propto D^5k_z^{-6}$, where k_z is the typical vertical wavenumber chosen in solving (5.7). This extreme sensitivity suggests that the numerical constants of proportionality in (5.9)–(5.13) are very unreliable. A similar sensitivity is present in the Ledoux, Schwarzschild, Spiegel analysis.

VI. SHEAR BOUNDARY-LAYER EFFECTS

In all of the foregoing analysis, we have assumed that the significant velocity and temperature fluctuations encountered at heights $z \le \frac{1}{2}D$ had characteristic spatial scales of order z. This is an extension to thermal turbulence of a basic similarity assumption usually made in applying Prandtl's mixing-length theory of shear flow over a wall. However, it is known that this assumption is a serious oversimplification for shear flows, despite the success of the mixing-length theory in predicting mean velocity profiles.

A particular deviation from similarity behavior to which we shall call attention now is the persistence of strong excitation in large horizontal spatial scales at values of z just outside the viscous sublayer in shear flows. This effect is well illustrated by Laufer's measurements of the "one-dimensional" energy spectrum of the flow-wise component of turbulent velocity in fully developed pipe flow.[16] The Reynolds number based upon mean velocity on axis and pipe radius r was 2.5×10^5 in Laufer's study, and the thickness of the viscous sublayer was approximately $(0.003)r$. The measurements showed that the spectrum level at wavenumbers $\sim\pi/2r$ was higher for $z/r = 0.008$, the smallest z measured, than for $z/r = 1$.

This behavior suggests to the present author that the big eddies develop turbulent boundary layers

[16] J. Laufer, NACA Rept. 1174 (1954).

which are analogous to the boundary layer of the mean flow itself. The small-scale turbulence present locally in these boundary layers then would be the momentum-transport mechanism that maintains the large-scale horizontal turbulent motions close to the boundary wall.

We shall now return to the free convection problem, where there are no mean velocities at all, and explore the consequences of an assumption that there, too, the big eddies develop turbulent boundary layers. The big eddies to be considered are those of spatial scale $\sim D$. If they developed boundary layers in strict analogy to the boundary layer of a fully developed mean shear flow, we should expect that the phenomenon could be approximately described by the following equations:

$$\tilde{u}(z) = v_*[A \ln (v_* z/\nu) + B] \quad (z > \delta), \tag{6.1}$$

$$\delta v_*/\nu = \mathrm{Re}_S, \tag{6.2}$$

$$v_* = u_0/\alpha, \tag{6.3}$$

$$\alpha + A \ln \alpha = A \ln \mathrm{Re}_0 + B', \tag{6.4}$$

$$\mathrm{Re}_0 = \tfrac{1}{2} D u_0/\nu. \tag{6.5}$$

Here $\tilde{u}(z)$ is the rms horizontal velocity at height z with $u_0 = \tilde{u}(\frac{1}{2}D)$, v_* is the effective "friction velocity," δ is the thickness of the viscous sublayer, Re_S is the characteristic Reynolds number for the top of the viscous sublayer, Re_0 is the Reynolds number based on u_0 and length $\frac{1}{2}D$, and A, B, B' are numerical constants. These equations are constructed in complete analogy to those usually accepted for fully developed shear boundary layers above a plane surface,[10,12] with the sole change that we have substituted the rms horizontal velocity $\tilde{u}(z)$ for the mean velocity $U(z)$. We shall adopt here the nominal numerical values

$$A = 3.0, \quad B = 5.5, \quad B' = 4.8, \quad \mathrm{Re}_S = 20, \tag{6.6}$$

which are based on Laufer's measurements of fully developed flow in channels.[17] We have no assurance that (6.6) represents an appropriate choice for the present case.

We shall assume that v_* here measures the rms shear stress on the boundary surface in the same way as v_* in pipe or channel flow measures the mean shear stress on the boundary. In further analogy, we shall assume that for $z \geq \delta$ the small-scale turbulence arising locally in the boundary layer has spatial scale of order z and rms velocity components of order v_*.

[17] J. Laufer, NACA Rept. 1053 (1951).

A point to be noted is that the big eddies of present interest have finite horizontal extent and last for finite times. The actual boundary layers developed by these eddies could be expected to exhibit a viscous sublayer thickness that grows from one end of the eddy to the other and which grows with time. We therefore interpret (6.1)–(6.5) as describing an average behavior. We anticipate that the observed viscous sublayer thickness and horizontal velocities should exhibit substantial fluctuations about the values given by (6.1) and (6.2).

What now are the principal implications of these presumed shear boundary-layer phenomena? If Re_0 is large enough, the rms horizontal velocity $\tilde{u}(z)$ predicted by (6.1) will be substantially larger for small z than the value that would be expected solely on the basis of the similarity assumptions employed in Secs. IV and V. The direct effect of the increased horizontal velocities upon heat transport is rather hard to predict, as has been noted by Townsend[5] in connection with steady wind rather then fluctuating horizontal velocities. However, the effect of the small-scale turbulence that arises locally in the shear boundary layer should be to increase heat transport. We assumed that this turbulence has rms velocity of order v_*, which according to (6.1)–(6.5) differs from u_0 by a factor that depends only logarithmically upon z/D. On the other hand, the turbulent velocities found in Secs. IV and V decreased with z/D according to power laws. This suggests that at large enough Re_0 the convective heat transport at small z/D can be profoundly affected by shear effects.

We shall now estimate the corrections due to shear boundary layer effects for the high Prandtl number case treated in Sec. IV. It is clear that Re_0 must be large if such corrections are to be appreciable. Therefore we shall consider only the case where Region III exists. Let us assume that the rms horizontal and vertical velocities are approximately equal at $z = \frac{1}{2}D$. From (4.9), (4.21), and (6.5) we then find

$$u_0 \sim (\gamma \Delta T \kappa)^{\frac{1}{3}} (\tfrac{1}{2} D z_\kappa)^{\frac{1}{3}}, \tag{6.7}$$

$$\mathrm{Re}_0 \sim \tfrac{1}{2} \sigma^{-\frac{2}{3}} \mathrm{Ra}^{\frac{1}{3}} [\mathrm{Ra}/16\pi^2 \, \mathrm{Pe}_T^2]^{1/9} \tag{6.8}$$

from which we may compute α by (6.4). Then, using (4.7), (4.8), (4.9), (6.3), and (6.7), we find

$$v_*/\tilde{w}(z_\kappa) \sim (2\pi^2/\mathrm{Pe}_T)^{\frac{1}{3}} \alpha^{-1} \sigma^{\frac{1}{3}} [\mathrm{Ra}/16\pi^2 \, \mathrm{Pe}_T^2]^{1/9}, \tag{6.9}$$

which gives a useful measure of v_*. Also, from (3.3), (6.1), and (6.2) we have

$$\delta/z_\kappa = \sigma(\mathrm{Re}_S/\mathrm{Pe}_T)\tilde{w}(z_\kappa)/v_*, \tag{6.10}$$

TABLE I. Shear boundary-layer effects for $\sigma = 1$.

Ra	Re_0	α	$v_*/\tilde{w}(z_\kappa)$	δ/z_κ	$\tilde{u}(\delta)/\tilde{w}(z_\kappa)$	$\tilde{w}(\delta)/\tilde{w}(z_\kappa)$
10^9	2.2×10^3	19.1	0.44	15.3	6.3	4.6
10^{12}	4.8×10^4	27.3	0.66	10.2	9.5	4.1
10^{15}	1.0×10^6	35.5	1.1	6.1	15.8	3.4
10^{18}	2.2×10^7	44.1	1.9	3.5	27.4	2.8
10^{21}	4.8×10^8	52.9	3.4	2.0	49.	2.0
10^{24}	1.0×10^{10}	61.4	6.3	1.1	91.	1.1

$$\tilde{u}(\delta)/\tilde{w}(z_\kappa) = (A \ln \mathrm{Re}_S + B)v_*/\tilde{w}(z_\kappa). \tag{6.11}$$

The last relation gives a useful measure of $\tilde{u}(\delta)$.

If we assign to the various numerical constants the values (3.5) and (6.6), we now have

$$\mathrm{Re}_0 \sim 0.223\sigma^{-\frac{2}{3}}\,\mathrm{Ra}^{4/9}, \tag{6.12}$$

$$v_*/\tilde{w}(z_\kappa) \sim (1.2\alpha)^{-1}\sigma^{\frac{1}{3}}\,\mathrm{Ra}^{1/9}, \tag{6.13}$$

$$\delta/z_\kappa \sim 6.67\sigma\tilde{w}(z_\kappa)/v_*, \tag{6.14}$$

$$\tilde{u}(\delta)/\tilde{w}(z_\kappa) \sim 14.5v_*/\tilde{w}(z_\kappa), \tag{6.15}$$

$$z_\nu/z_\kappa \sim 3.16\sigma^{\frac{1}{2}}. \tag{6.16}$$

The last equation comes from (4.12). Numerical results for $\sigma = 1$ and for several values of Ra are given in Table I. The values of $\tilde{w}(\delta)/\tilde{w}(z_\kappa)$ are determined by (4.7) if $z_\kappa < \delta < z_\nu$ and by (4.21) if $\delta > z_\nu$. They provide a measure of the horizontal velocities that would be expected at $z = \delta$ without shear boundary-layer effects.

Table I indicates that appreciable enhancement of rms horizontal velocities at $z = \delta$ is to be expected for Ra $\sim 10^9$. However, v_* does not exceed $\tilde{w}(z_\kappa)$ until Ra $\sim 10^{15}$, and we do not have $\delta < z_\kappa$ until Ra $> 10^{24}$. We have so far said little about the behavior of the shear boundary layer for $z < \delta$. Let us assume that for values of z/δ which are not too small the approximations

$$\tilde{u}(z) \sim (z/\delta)\tilde{u}(\delta) \qquad (z < \delta), \tag{6.17}$$

$$\tilde{w}'(z) \sim (z/\delta)v_* \qquad (z < \delta) \tag{6.18}$$

are valid, where $w'(z)$ is the rms value of the vertical component of the small-scale turbulence that arises in the boundary layer.[18] If (6.17) and (6.18) are applicable, the table implies appreciable enhancement of rms horizontal velocities at $z = z_\kappa$ for Ra $\gtrsim 10^{12}$ and enhancement of vertical velocities at $z = z_\kappa$ for Ra $> 10^{18}$. The latter effect implies an increase in total heat transport. In this case, the analysis properly should be done with shear boundary-layer effects taken into account from the beginning instead of calculated as a correction. We shall carry this out for extremely high Ra in Sec. VII.

The values of $v_*/\tilde{w}(z_\kappa)$ and δ/z_κ in Table I vary exceedingly slowly with change of Ra. The highest and lowest Ra differ by the ratio 10^{15}, and $v_*/\tilde{w}(z_\kappa)$ changes by a factor of less than 15 over this range. This behavior suggests that changes in the numerical values assumed for the various constants that enter the analysis can make very large changes in the value of Ra at which substantial enhancement of heat transport by boundary-layer stirring is predicted. For this reason, the numerical results in Table I should be taken with considerable reservation.

The analysis in the present section has been concerned with rigid, nonslip boundaries. In the case of slip boundaries, where horizontal stress rather than horizontal velocity vanishes, there is no mechanism for the generation of shear boundary layers and the local turbulence associated therewith. In this case we do not expect enhancement of heat transport as found above. However, it is unlikely here either that the horizontal velocities can be approximated by combining the results of Secs. IV or V with an assumption of isotropy. Instead, it would appear more probable that the rms horizontal velocity everywhere in the fluid, including $z = 0$, takes a value at least as large as its value at $z = \frac{1}{2}D$. The latter value plausibly is $\sim\tilde{w}(\frac{1}{2}D)$.

VII. HEAT FLUX FOR VERY LARGE RAYLEIGH NUMBERS

A. Low Prandtl Number

The analysis in Sec. VI suggests that at extremely high Ra the vertical motion at the edge of the conduction region is dominated by small-scale turbulence originating in the shear boundary layers of the big eddies. We shall now redo some of the work of Secs. IV and V so as to take account of this effect from the beginning.

The analysis is simplest for Prandtl numbers which satisfy

$$\sigma < \mathrm{Pe}_T/\mathrm{Re}_S. \tag{7.1}$$

The significance of this inequality will become clear very soon. We shall now make several assumptions and later find conditions for their consistency. First, we take

$$v_* z_\kappa/\kappa \sim \mathrm{Pe}_T, \tag{7.2}$$

in accord with the assumption that the turbulence at the edge of the conduction region has rms velocity

[18] It can be shown from (2.2) and (2.3) that for rigid, nonslip boundary conditions $\partial^2 w/\partial z^2 = 0$ at $z = 0$. Thus (6.18) cannot be a valid approximation at very small values of z/δ.

v_* and spatial scale z_κ. As in our previous work, we make the approximation

$$H \sim \kappa \Delta T/2z_\kappa. \quad (7.3)$$

From (6.2) and (7.1) we have

$$z_\kappa/\delta \sim \sigma^{-1}\,\mathrm{Pe_T}/\mathrm{Re_S}. \quad (7.4)$$

Thus (7.1) states that z_κ lies outside the viscous sublayer. This actually is a condition implicit in our assumption of (7.2). We shall suppose that (4.18) and (4.19) are valid for $z = \frac{1}{2}D$ so that

$$[\tilde{w}(\tfrac{1}{2}D)]^2 \sim \tfrac{1}{2}D\gamma\tilde{\psi}(\tfrac{1}{2}D) \quad (7.5)$$

and

$$\tfrac{1}{2}\tilde{w}(\tfrac{1}{2}D)\tilde{\psi}(\tfrac{1}{2}D) \sim H. \quad (7.6)$$

Finally, we shall assume that there is approximate isotropy at $z = \frac{1}{2}D$ so that

$$u_0 \sim \tilde{w}(\tfrac{1}{2}D). \quad (7.7)$$

Equations (6.3), (6.4), (6.5), (7.2), (7.3), and (7.5)–(7.7) form a complete set which determine Re_0 and H/H_0 in terms of σ and Ra. In order to obtain the solution in more explicit form, we shall approximate (6.4) by

$$\alpha \sim A \ln \mathrm{Re}_0. \quad (7.8)$$

This will cause only small error if Re_0 is large enough. Then by several eliminations and use of the definitions of σ and Ra, we obtain

$$\mathrm{Re}_0^2 \ln \mathrm{Re}_0 \sim (8A\ \mathrm{Pe_T})^{-1}\sigma^{-1}\,\mathrm{Ra}, \quad (7.9)$$

which determines Re_0, and

$$H/H_0 \sim D/2z_\kappa \sim [\sigma\,\mathrm{Ra}/(2A\ \mathrm{Pe_T}\ln \mathrm{Re}_0)^3]^{\frac{1}{4}}, \quad (7.10)$$

which determines H/H_0. As a simplification when Ra is extremely large and σ is not too small, we may approximate the logarithm of (7.9) by $2 \ln \mathrm{Re}_0 \sim \ln \mathrm{Ra}$ and obtain

$$H/H_0 \sim [\sigma\,\mathrm{Ra}/(A\ \mathrm{Pe_T}\ln \mathrm{Ra})^3]^{\frac{1}{4}}. \quad (7.11)$$

We must now find approximate conditions for the consistency of the assumptions which have been made. One condition, (7.1), has been discussed already. The assumption that the boundary-layer turbulence dominates the heat transport just outside the conduction region requires that H/H_0 given by (7.10) be larger than the value (5.4) which we obtained neglecting shear boundary-layer effects. This condition may be written

$$[\sigma\,\mathrm{Ra}/(2A\ \mathrm{Pe_T}\ln \mathrm{Re}_0)^3]^{\frac{1}{4}} > (A \ln \mathrm{Re}_0)^3. \quad (7.12)$$

An obvious final requirement is that σ be sufficiently large that the Peclet number at $z = \frac{1}{2}D$ exceeds $\mathrm{Pe_T}$. Noting (7.7), we may write this condition as

$$\sigma\,\mathrm{Re}_0 > \mathrm{Pe_T}, \quad (7.13)$$

which is not independent of (7.12).

Let us now consider briefly the behavior of $\tilde{\psi}$, $\tilde{w}$, and $\bar{T}$. For $z_l < z < \frac{1}{2}D$, where z_l is a transition height we shall soon determine, these quantities should obey the similarity laws obtained in Secs. IV and V for a region where both eddy conductivity and eddy viscosity dominate. That is,

$$\tilde{\psi} \propto z^{-\frac{1}{3}},\quad \tilde{w} \propto z^{\frac{1}{3}},\quad \bar{T} \propto z^{-\frac{1}{3}}\quad (z_l < z < \tfrac{1}{2}D). \quad (7.14)$$

The transition height z_l is where (7.14), with proper constants of proportionality, gives $\tilde{w} \sim v_*$. From (6.3), (7.7), (7.8), and (7.14) we find

$$2z_l/D \sim (A \ln \mathrm{Re}_0)^{-3}. \quad (7.15)$$

For $z_\kappa < z < z_l$, the turbulence has spatial scale z and satisfies $\tilde{w} \sim v_*$, according to our assumptions. Then (7.6) implies $\tilde{\psi} \sim$ const in this region. If we assume that ψ as well as w has spatial scale z, then (4.10) implies $|d\bar{T}/dz| \propto z^{-1}$. Thus we have

$$\begin{aligned} &\tilde{\psi} \sim \text{const}, \qquad \tilde{w} \sim \text{const}, \\ &d\bar{T}/dz \propto z^{-1} \qquad (z_\kappa < z < z_l). \end{aligned} \quad (7.16)$$

These relations, which give a logarithmic falloff for $\bar{T}$, are identical with the similarity laws for forced convection in a steady turbulent shear flow described by Priestley.[4] In the present case, however, the shear boundary layer is associated with the large eddies created by the convection itself, rather than with an externally driven steady shear flow.

Approximate values for the factors of proportionality in (7.14) and (7.16) may be found from relations already given together with suitable joining assumptions at $z = z_\kappa$ and $z = z_l$. We shall not carry this out here.

B. Moderate Prandtl Number

When (7.1) is not satisfied, z_κ will lie within the viscous sublayer of the turbulent boundary layer. Thus, it is necessary to make some assumption about the turbulent structure in this sublayer in order to extend our analysis to higher σ. For present purposes, we shall suppose that (6.18) is an admissible approximation. It has already been noted that (6.18) surely overestimates $\tilde{w}'(z)$ for small enough z/δ. We hope, however, that the error at $z = z_\kappa$ will not be large for $\sigma = 1$, the case we shall choose for numerical evaluation.

TABLE II. Heat flux at very high Ra for $\sigma = 1$.

Ra	Re_0	$[H/H_0]_{(7.22)}$	$[H/H_0]_{(4.9)}$
10^{18}	1.8×10^7	4.6×10^4	8.9×10^4
10^{21}	5.2×10^8	1.1×10^6	8.9×10^5
10^{24}	1.5×10^{10}	2.8×10^7	8.9×10^6
10^{27}	4.5×10^{11}	7.2×10^8	8.9×10^7

We proceed as in Sec. A except that (7.2) now is replaced by

$$(z_\kappa/\delta)v_* z_\kappa/\kappa \sim \mathrm{Pe}_\mathrm{T}, \tag{7.17}$$

in accord with (6.18).[19] Equations (6.2), (6.3), (6.5)' (7.3), and (7.5)–(7.8) then form a complete set which may be solved to yield

$$\mathrm{Re}_0^2 \ln \mathrm{Re}_0 \sim (8A)^{-1}(\mathrm{Re}_\mathrm{S}\mathrm{Pe}_\mathrm{T})^{-\frac{1}{2}}\sigma^{-\frac{3}{2}}\,\mathrm{Ra} \tag{7.18}$$

and

$$H/H_0 \sim D/2z_\kappa$$

$$\sim \sigma^{-\frac{1}{4}}[\mathrm{Ra}/(2A\ \mathrm{Pe}_\mathrm{T}^{\frac{1}{2}}\mathrm{Re}_\mathrm{S}^{\frac{1}{2}} \ln \mathrm{Re}_0)^3]^{\frac{1}{2}}. \tag{7.19}$$

The asymptotic form of (7.19), corresponding to (7.11), is

$$H/H_0 \sim \sigma^{-\frac{1}{4}}[\mathrm{Ra}/(A\ \mathrm{Pe}_\mathrm{T}^{\frac{1}{2}}\mathrm{Re}_\mathrm{S}^{\frac{1}{2}} \ln \mathrm{Ra})^3]^{\frac{1}{2}}. \tag{7.20}$$

As in Sec. A, a primary criterion for validity of the analysis is that H/H_0 exceed the value obtained by ignoring shear boundary-layer effects. In the present case, (7.19) must be compared with (4.9). The condition analogous to (7.13) in the present case is simply the obvious requirement that Re_0 be large compared to Re_T.

If the nominal numerical values (3.5) and (6.6) are adopted, (7.18) and (7.19) become

$$\mathrm{Re}_0^2 \ln \mathrm{Re}_0 \sim 5.4 \times 10^{-3}\sigma^{-\frac{3}{2}}\,\mathrm{Ra} \tag{7.21}$$

and

$$H/H_0 \sim 3.2 \times 10^{-3}\sigma^{-\frac{1}{4}}\,\mathrm{Ra}^{\frac{1}{2}}(\ln \mathrm{Re}_0)^{-\frac{3}{2}}. \tag{7.22}$$

From (7.22) we see that for $\sigma > \mathrm{Pe}_\mathrm{T}/\mathrm{Re}_\mathrm{S}$ the enhancement of total heat transport by shear boundary-layer effects decreases as σ increases. The error inherent in (6.18) suggests that for large σ the decrease actually is more rapid than indicated by (7.22).

The values of Re_0 and H/H_0 obtained from (7.21) and (7.22) for $\sigma = 1$ and for several values of Ra are shown in Table II. Also shown are the corresponding values of H/H_0 obtained from (4.9).

[19] Equation (7.17) implies that the spatial scale of turbulence at z is of order z. Actually, for $z/\delta < 1$ it is plausible that the effective mixing length is $> z$. This may partly compensate for overestimation of the rms value of w' by (6.18).

It will be noticed that the values of H/H_0 are comparable at $\mathrm{Ra} = 10^{21}$, which is consistent with the results presented earlier in Table I. For $\mathrm{Ra} \geq 10^{24}$, Table II indicates a marked enhancement of heat transport as a result of shear boundary-layer effects. We wish to repeat, however, the cautions about reliability of the numbers that were stated in connection with Table I.

C. Interpretation

According to the asymptotic formulas (7.11) and (7.20), the heat transport at fixed σ increases as $\mathrm{Ra}^{\frac{1}{2}}$ except for reduction by logarithmic factors. For fixed ΔT, the asymptotic law $H/H_0 \propto \mathrm{Ra}^{\frac{1}{3}}$ would imply that the conduction thickness z_κ and heat transport H were independent of D. The present stronger dependencies therefore imply that H increases with D when D is very large. We wish now to offer a rationalization of this paradoxical behavior.

The basis of the explanation lies in the analysis which led in Sec. IV to the $\mathrm{Ra}^{\frac{1}{3}}$ law. According to (4.21), the mean kinetic energy per unit mass at $z = \frac{1}{2}D$ is of order

$$(\gamma\kappa\Delta T)^{\frac{2}{3}}(D/z_\kappa)^{\frac{2}{3}}.$$

Now if D increased without limit and z_κ approached a constant value, it is clear that the steady-state kinetic energy density would increase without limit. What our present analysis does is to describe a mechanism, the Reynolds stresses, by which this intense kinetic energy excitation in the midregions can react on the boundary regions, vigorously stirring them and enhancing the heat transport. Viewed in this way, the results of Secs. VII A and B do not seem implausible. It should be kept in mind that as D increases, with ΔT fixed, the time required to inject the steady-state level of kinetic energy into the fluid also increases.

Rayleigh numbers of order 10^{21} to 10^{24} would appear to be realizable in atmospheric free convection. One may speculate that under certain conditions of strong convective instability some of the random component of the surface breeze may represent turbulent boundary layers associated with large-scale convective motions.

VIII. DISCUSSION AND SUMMARY

A. Limitations of Mixing-Length Analysis

Before summarizing our results and comparing them with experiment, it seems appropriate to make some remarks on the defects of the mixing-length approach. Throughout this paper we have assumed that the dominant vertical scale of turbulence, and therefore the mixing length, was of order z at height z. This assumption, which also was made by Prandtl and Priestley, seems plausible for the following reasons. Spatial scales much smaller than z should be less effective in transporting heat and momentum, since the eddy transport coefficients are proportional to mixing length. Scales much larger than z are expected to be ineffective for another reason: The continuity equation requires that a velocity structure of scale $\gg z$ have a very small ratio of vertical to horizontal velocity at z. This in turn means high dissipation for a given mean heat transport.

The vague argument just stated can be sharpened by consideration of some simple flow examples. If correct, however, it implies that mixing-length analysis provides an inherently crude description of turbulent thermal convection. In general, mixing-length theories become accurate only when the mixing length is small compared to the other characteristic lengths pertinent to the problem at hand. One cannot hope to describe properly the variations that take place in distances smaller than one mixing length. The results obtained in the preceding sections must be interpreted with this in mind. No matter how ingenious we are at choosing apt values for constants of proportionality, it is unlikely that the results will be accurate except when the inequalities that define the various regimes and regions are strong. The results may be expected to be particularly poor at the joins between different regions of z. Very possibly the actual physical mixing phenomenon is characterized by overshoot effects so that the best choice for the height at which a particular region ends may turn out to be different for $\tilde{w}$, $\tilde{\psi}$, and $\bar{T}$.

B. Summary of Results

For convenience in comparison with experiment, we summarize here the results obtained in Secs. IV, V, and VII for the dependence of the Nusselt number H/H_0 upon Prandtl number σ and Rayleigh number Ra and for the dependence of rms temperature fluctuation $\tilde{\psi}$, mean temperature deviation $\bar{T}$, and rms vertical velocity $\tilde{w}$ upon height z above the lower boundary surface. The nominal values (3.5) are used for the Reynolds and Peclet numbers which characterize the transition from molecular to eddy transport of momentum or heat. It should be kept in mind that the numerical coefficients in the formulas which follow represent approximate estimates based upon idealized models for the flow.

The results obtained for high σ in Sec. IV are summarized by

$$H/H_0 \sim 0.089\,\mathrm{Ra}^{\frac{1}{3}} \qquad (\sigma > 0.1), \tag{8.1}$$

$$\tilde{\psi} \sim 0.17\;\Delta T(z/z_\kappa)^{-1},$$
$$\tilde{w} \sim 3.0\kappa z_\kappa^{-1}(z/z_\kappa),$$
$$\bar{T} \sim \tilde{\psi} \qquad (z_\kappa < z < z_\nu), \tag{8.2}$$

$$\tilde{\psi} \sim 0.18\sigma^{-\frac{1}{3}}\;\Delta T(z/z_\kappa)^{-\frac{1}{3}},$$
$$\tilde{w} \sim (\gamma\kappa\;\Delta T)^{\frac{1}{3}}(z/z_\kappa)^{\frac{1}{3}},$$
$$\bar{T} \sim \tilde{\psi} \qquad (z_\nu < z < \tfrac{1}{2}D). \tag{8.3}$$

Here the conduction thickness z_κ and viscosity thickness z_ν are given by

$$z_\kappa \sim \tfrac{1}{2}(H_0/H)D, \qquad z_\nu \sim 3.2\sigma^{\frac{1}{2}}z_\kappa. \tag{8.4}$$

The condition $z_\nu < \frac{1}{2}D$ is equivalent to

$$\mathrm{Ra}^{\frac{1}{3}} > 35\sigma^{\frac{1}{2}}. \tag{8.5}$$

When (8.5) is not satisfied, the $z^{\frac{1}{3}}$ law region described by (8.3) is nonexistent.

The results obtained in Sec. V for low σ may be summarized by

$$H/H_0 \sim 0.17(\sigma\,\mathrm{Ra})^{\frac{1}{3}} \qquad (\sigma < 0.1), \tag{8.6}$$

$$\tilde{\psi} \sim 0.33\;\Delta T(z/z_\kappa)^{-\frac{1}{3}},$$
$$\tilde{w} \sim 3.0\kappa z_\kappa^{-1}(z/z_\kappa)^{\frac{1}{3}},$$
$$\bar{T} \sim \tilde{\psi} \qquad (z_\kappa < z < \tfrac{1}{2}D), \tag{8.7}$$

$$z_\kappa \sim \tfrac{1}{2}(H_0/H)D. \tag{8.8}$$

The condition $z_\kappa < \frac{1}{2}D$ (thermal boundary layering) is equivalent to

$$(\sigma\,\mathrm{Ra})^{\frac{1}{3}} > 6.0. \tag{8.9}$$

When (8.9) is not satisfied, (8.6)–(8.8) no longer are valid. If

$$(\sigma\,\mathrm{Ra})^{\frac{1}{3}} < 6.0, \qquad \mathrm{Ra} > 9500, \tag{8.10}$$

then

$$H/H_0 \sim 1 \tag{8.11}$$

and the rms temperature fluctuation and velocity at $z = \frac{1}{2}D$ are given by

$$\begin{aligned} \tilde{\psi}(\tfrac{1}{2}D) &\sim 8.0 \times 10^{-5}\;\Delta T\sigma\,\mathrm{Ra}, \\ \tilde{w}(\tfrac{1}{2}D) &\sim 6.3 \times 10^{-3}\kappa\;D^{-1}\sigma\,\mathrm{Ra}. \end{aligned} \tag{8.12}$$

The second inequality in (8.10) is a condition that the flow have a high enough Reynolds number to be turbulent. It was noted in Sec. V that the numerical coefficients in (8.12) are particularly unreliable.

The analysis in Sec. VII indicated that (8.1) and (8.6) should be replaced at extremely high Ra by

$$H/H_0 \sim 0.037\sigma^{\frac{1}{2}}[\mathrm{Ra}/(\ln \mathrm{Ra})^3]^{\frac{1}{2}} \qquad (\sigma < 0.15) \tag{8.13}$$

or by

$$H/H_0 \sim 8.9 \times 10^{-3}\sigma^{-\frac{1}{4}}[\mathrm{Ra}/(\ln \mathrm{Ra})^3]^{\frac{1}{2}} \quad (1 \gtrsim \sigma > 0.15). \tag{8.14}$$

Equations (8.13) and (8.14) take account of the enhanced stirring near the boundaries due to shear boundary layers attached to the large eddies. As discussed in Sec. VII, a necessary condition for the validity of these asymptotic formulas is that they yield values of H/H_0 which exceed by a substantial factor the values obtained from (8.1) or (8.6). The explicit numerical coefficients in (8.13) and (8.14) are based on (3.5) and on the values (6.6) for the parameters in the shear boundary-layer equations.

Apart from uncertainties in the numerical coefficients, it is expected that the various laws of functional dependence summarized above can be accurate only when the qualifying inequalities are all large. For example, the results for $\tilde{\psi}$ given by (8.2) and (8.7) differ by a factor of 2 for $\sigma = 0.1$ and $z = z_\kappa$. This discrepancy may be traced to the fact that we assumed w and ψ were perfectly correlated in deriving (8.2) and only 50% correlated in deriving (8.7). In view of the inaccuracies inherent in the mixing-length approach, we think it would be largely illusory to correct discrepancies of this kind by more careful treatment of the joins between the various asymptotic regions.

C. Comparison with Experiment

It is hard to perform reproducible experiments on convective transport of heat which are suitable for direct comparison with theories based on the Boussinesq equations. A major difficulty is simultaneously to achieve steady states, accurate calorimetry, ideal boundary surfaces, and temperature differences small enough to make the Boussinesq equations accurate. Since fluids undergoing naturally occurring convection do not care whether or not they obey the Boussinesq equations, the onus here really is upon the theorist rather than the experimenter.

A comprehensive recent experimental investigation of turbulent thermal convection between parallel plates is that of Globe and Dropkin[3] on mercury, water, and silicone oils. The measurements cover a range of 1.5×10^5 to 6.76×10^8 for Ra, 0.02 to 8750 for σ, and 1° to 45°C for ΔT. The values of H/H_0 fell between 1.9 and 66.9.

A best power-law fit to the entire set of data was reported to be

Table III. Prandtl-number dependence of H/H_0 according to Globe and Dropkin.

σ	4	15	300	8000
$0.069\sigma^{0.074}$	0.077	0.084	0.105	0.134

$$H/H_0 = 0.069\sigma^{0.074}\,\mathrm{Ra}^{\frac{1}{3}}. \tag{8.15}$$

The typical scatter of individual measured values of H/H_0 about this curve appears to be the order of 30%, the scatter being highest for water and lowest for the silicone oils. This scatter is also the order of the discrepancy between (8.15) and the empirical power laws reported for water and acetone by Malkus[6] and for air by Jakob.[1] Globe and Dropkin suggest that the large scatter for water is due in part to strong dependence of ν and κ on temperature.

The measurements for $\sigma < 1$ were all on mercury, with $\sigma \sim 0.023$ and $1.5 \times 10^5 < \mathrm{Ra} < 4 \times 10^7$. Equation (8.9) is satisfied over this range, so that H/H_0 should obey (8.6), according to our considerations. Inserting $\sigma = 0.023$ in this equation, we have

$$H/H_0 \sim 0.048\,\mathrm{Ra}^{\frac{1}{3}}, \tag{8.16}$$

while Globe and Dropkin report that the best power-law fit to the mercury data is

$$H/H_0 = 0.051\,\mathrm{Ra}^{\frac{1}{3}}. \tag{8.17}$$

It should be noted that the arbitrariness in our choices of numerical constants in the theory makes the extreme closeness of agreement between (8.16) and (8.17) not significant.

The mean values of σ for the measurements in water and in silicone oils of viscosity 1.5, 50, and 1000 centistokes were approximately 4, 15, 300, and 8000, respectively. For these values of σ, the corresponding values of the coefficient $0.069\sigma^{0.074}$ in (8.15) are given in Table III. The range of variation of σ in the measurements on any one of the four fluids was sufficiently small that the corresponding variation in this coefficient is only a few percent. The values in Table III may be compared with the coefficient 0.089 in (8.1).

The comparisons displayed in the two preceding paragraphs indicate a reasonably satisfactory agreement between theory and experiment with regard to the absolute values of H/H_0. The discrepancies are somewhat larger than the experimental scatter, but they are not larger than the uncertainties inherent in our guesses for the various numerical constants that entered the theoretical analysis. However, neither (8.15) nor the actual data points

TABLE IV. Evaluation of (8.18) for the data of Globe and Dropkin.

Fluid	Nominal mean σ	Approximate range of H/H_0	Approximate range of $\sigma^{-\frac{1}{2}}H/H_0$
Water	4	16 to 33	8 to 16.5
1.5 cS silicone	15	30 to 67	7.7 to 17.3
50 cS silicone	300	13 to 28	0.75 to 1.6
1000 cS silicone	8000	8 to 11	0.09 to 0.12

obtained by Globe and Dropkin supports very well the Prandtl number dependence predicted by (8.1) and (8.6). Theory and data agree to the extent that both give an increase of H/H_0 with σ. But the data suggest a very gradual rise of $(H/H_0)/\mathrm{Ra}^{\frac{1}{3}}$ with σ and give no evidence of leveling off at high σ.

The discrepancy in σ dependence may imply a defect in the theory which was suggested at the end of Sec. IV. It is possible that significant dynamical effects of the Reynolds stresses may appear at Reynolds numbers far below that required for eddy viscosity as such to be dominant. If we take $\mathrm{Pe}_T = 3$, then condition (4.23) for Reynolds number $\ll 1$ throughout the flow becomes

$$\sigma^{-\frac{1}{2}}H/H_0 \ll 0.6. \tag{8.18}$$

In Table IV we give the left side of (8.18) evaluated at the nominal mean values of σ for the four fluids and at the highest and lowest Ra for which measurements on each fluid were reported. We see that (8.18) appears to be satisfied only for the 1000-centistoke silicone oil measurements.

The σ dependence indicated by the measurements for $\sigma \gtrsim 300$ is very intriguing, if real. The values in the last column of Table IV suggest that Re (z) is of order unity for such σ only in the midregions of the fluid. Thus any residual σ dependence of H/H_0 would imply that there is a significant dynamical reaction of the midregions on the conduction region.

Some reservations about the interpretation of the experimental data should be noted at this point. Table IV shows that, with the exception of water (which exhibits the largest scatter and large deviations from Boussinesq conditions), the values of H/H_0 used in the experiment decrease with increase of σ. Suppose that the dependence of H/H_0 upon Ra at fixed σ is more gradual at moderate H/H_0 than $\mathrm{Ra}^{\frac{1}{3}}$. The result would be an overestimation of the σ dependence of H/H_0 when the data are fitted to a law of the form $H/H_0 \propto \sigma^n \mathrm{Ra}^{\frac{1}{3}}$.

Very careful measurements of H/H_0 in the range up to about 3.5 have been made by Silveston[2] on heptane, glycol, and two silicone oils. The values of σ for the four fluids fall between approximately 6 and 3000. These measurements, as well as the data of Malkus[6] on water and acetone, suggest that the dependence of H/H_0 on Ra actually is more gradual than $\mathrm{Ra}^{\frac{1}{3}}$ for H/H_0 between 2.0 and 3.5. Comparison of the values of H/H_0 found by Silveston for the four fluids, at Rayleigh numbers very close to 29 000 in each case, appears to support the conclusion that H/H_0 increases with σ at fixed Ra, over the range of σ involved. However, the indicated increase is more gradual than that found by Globe and Dropkin at higher H/H_0. The values of $(H/H_0)/\mathrm{Ra}^{\frac{1}{3}}$ found by Silveston for $\sigma \sim 3000$ are sufficiently lower than the Globe and Dropkin results for $\sigma \sim 8000$ as to suggest strongly the need for further experiments to resolve the discrepancy.

Laboratory data on the behavior of $\bar{T}$, ψ, and w are scanty. Townsend[5] has measured $\bar{T}(z)$ and $\tilde{\psi}(z)$ for convection in air with $\sigma \sim 0.7$ and $\mathrm{Ra}^{\frac{1}{3}}$ the order of 10^3. He found that $\bar{T}(z)$ for $z > z_\kappa$ appeared to fit a power law of the form z^{-1} better than the law $z^{-\frac{1}{3}}$ given by the Priestley theory. The z^{-1} dependence had previously been predicted by Malkus.[6] An adequate fit for $\tilde{\psi}(z)$ was reported to be $\tilde{\psi}(z) \propto z^{-0.6}$. Unfortunately, the container used by Townsend was open at top to the room air instead of being closed by a conducting surface as assumed in the theories. This, together with the departure from Boussinesq conditions, makes it difficult to know how detailed a comparison is justified.

It will be noted that $\sigma = 0.7$ falls within the high σ range of the present analysis provided one chooses the values (3.5) for the transition Reynolds and Peclet numbers. We then have $z_\nu/z_\kappa \sim 2.7$, according to (8.4). This ratio makes the z^{-1} range given by (8.2) much too short for one to expect accurate adherence to the asymptotic functional forms, but it does seem plausible that $\bar{T}$ and $\tilde{\psi}$ should fall off more rapidly than $z^{-\frac{1}{3}}$. Larger values of $\mathrm{Re}_T/\mathrm{Pe}_T$ may be realistic, as we discussed in Sec. III. They would be expected to yield a better-defined z^{-1} region.

If comparison of theory and experiment is justified, it thus appears that the present analysis offers a possibility of resolving the apparent discrepancy between Townsend's measurements and mixing-length theory. Malkus[6] has pointed out that deviation from the $z^{-\frac{1}{3}}$ law for $\bar{T}(z)$ outside the conduction region is evidence that the molecular constants κ and ν are entering somehow into the local dynamics.

In the present treatment this happens because viscous stresses dominate over Reynolds stresses in the range $z_\kappa < z < z_\nu$. The analysis which led in Sec. IV to the z^{-1} law can be considered an embodiment of Townsend's suggestion[5] that convection outside the conduction region is dominated by updraft plumes which penetrate for a considerable, but not unlimited, distance into the fluid before their eventual turbulent dissolution.

The present work suggests that detailed laboratory measurements of $\bar{T}(z)$, $\psi(z)$, and $\mathbf{u}(z)$ with two rigid boundaries should be made for both low and high σ. A point of particular interest would be the experimental determination of the transition Reynolds and Peclet numbers $\mathrm{Re_T}$ and $\mathrm{Pe_T}$ and an investigation of their degree of independence on σ and Ra. It will be recalled that significant features of the theoretical results depend upon the value of the ratio $\mathrm{Re_T/Pe_T}$. If the value turns out to be greater than the nominal value 10 which we have used, the effect on the theory would be to lower the value of σ which separates the low and high Prandtl number ranges and to increase the length of the z range, at high σ, in which $\bar{T}$ and $\bar{\psi}$ vary as z^{-1}. The discussion in Sec. III suggests that values of $\mathrm{Re_T/Pe_T}$ greater than 10 would not be surprising.

Also of interest would be an attempt to obtain laboratory Rayleigh numbers large enough to test the predicted shear boundary layer effects discussed in Secs. VI and VII. We estimated with high uncertainty in Sec. VII that enhancement of H/H_0 might be marked for Ra $\sim 10^{24}$. It was concluded in Sec. VI that enhancement of rms horizontal velocities near the boundaries because of shear boundary layer effects should be appreciable for much lower Ra, possibly $\sim 10^{12}$.

D. Comparison with the Malkus Theory

Certain of the predictions in the present paper reproduce results from the theory of turbulent convection proposed by Malkus.[6] We wish now to comment on the relation between the two approaches. The Malkus theory is based on the stability properties of infinitesimal disturbances of the fully developed mean temperature profile. The thickness of the conduction region, and hence the total heat transport, is determined by the length scale characteristic of disturbances which are marginally stable. The actual form of the mean profile is then chosen from among other possible profiles by an optimizing procedure which maximizes the heat transport subject to the condition that $d\bar{T}/dz$ has the same sign everywhere. The reader is referred to Malkus' papers for a description of the theory and its modifications.[20]

The result obtained for H/H_0 is completely independent of σ. This is because σ does not enter the infinitesimal stability problem. The effects of eddy viscosity on velocity fluctuations and the action of eddy conductivity in smoothing temperature fluctuations are not taken explicitly into account. Instead, it is stated that these effects should act to reduce H/H_0 below the optimum value obtained through the marginal stability investigation. In applications of the theory, it has been assumed that the reduction of H/H_0 below the theoretical optimum is small.

Equation (8.1) agrees with the optimum H/H_0 obtained by Malkus for large Ra, except for a slight difference in the numerical coefficient. Since the optimum is independent of σ, however, it differs drastically at small enough σ from our result (8.6). The analysis in Sec. V suggests that eddy viscosity inhibits the mean transport of heat markedly at very small σ.

The asymptotic results (8.13) and (8.14) for extremely large Ra represent a departure in the other direction from the Malkus theory. In this case, we find that the Reynolds stresses serve to enhance the transport of heat above the $\mathrm{Ra}^{\frac{1}{3}}$ law, as explained in Sec. VII C. At sufficiently high Ra, far above the range presently attained in the laboratory, the predicted transport greatly exceeds the maximum value obtained by Malkus.

The prediction of the profile $\bar{T}(z)$ on the Malkus theory involves a determination of the shape of the laminar eigenmodes on this profile.[21] Malkus' treatment of this very difficult implicit problem is essentially equivalent to approximating the eigenmodes with sine and cosine functions. His result is $\bar{T}(z) \propto z^{-1}$, when the inequalities $z_\kappa < z < \frac{1}{2}D$ are strong. As with the prediction for H/H_0, there is no dependence on σ. It is apparent from the formulas in Sec. VIII B that our analysis agrees with this result only for high σ and only for a restricted range of z.

We have noted previously that the present treatment follows an underlying tenet of the Malkus theory in that it admits significant direct dynamical effects of molecular dissipation processes outside the conduction region. However, unless some undis-

[20] See also the expository papers by E. A. Spiegel and by A. A. Townsend in *Mécanique de la Turbulence* (Centre National de la Recherche Scientifique, Paris, to be published)

[21] E. A. Spiegel, reference 20.

covered connection of a more specific kind exists, it would appear largely fortuitous that the mixing-length analysis should reproduce exactly the Malkus power law for $\bar{T}(z)$, even within restricted ranges of σ, Ra, and z. The assumptions and approximations that lead to the estimation of $\bar{T}(z)$ seem quite different in the two approaches.

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