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# Notes

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## 1 卡门–豪沃思方程

N–S

$$\frac{\partial R_{ij}}{\partial t} - \frac{\partial}{\partial r_m}\left(S_{imj} + S_{jmi}\right) = 2\nu \nabla^2 R_{ij} \tag{1.1}$$

K–H

$$\frac{\partial\left(\left\langle u^2\right\rangle f\right)}{\partial t} - \left\langle u^2\right\rangle^{\frac{3}{2}}\left(k' + \frac{4k}{r}\right) = 2\nu\left\langle u^2\right\rangle\left(f'' + \frac{4f'}{r}\right) \tag{1.2}$$

自模拟假设（自保持, 自相似）$f(r,t)$, $k(r,t)$, $\left\langle u^2\right\rangle(t)$ 引入（微尺度）$\lambda(t)$

设 $f(r,t) = F\left(\frac{r}{\lambda(t)}\right)$, $\quad k(r,t) = K\left(\frac{r}{\lambda(t)}\right)$

$$\frac{\partial\left(r^4\left\langle u^2\right\rangle f\right)}{\partial t} - \left\langle u^2\right\rangle^{\frac{3}{2}}\left(r^4 k\right)' = 2\nu\left\langle u^2\right\rangle\left(r^4 f'\right)' \tag{1.3}$$

$$\frac{\partial \int_0^{+\infty}\left(r^4\left\langle u^2\right\rangle f\right)\mathrm{d}r}{\partial t} - \left\langle u^2\right\rangle^{\frac{3}{2}}\left(r^4 k\right)\Big|_0^{+\infty} = 2\nu\left\langle u^2\right\rangle\left(r^4 f'\right)\Big|_0^{+\infty} \tag{1.4}$$

Loitsansky 积分不变量

$$\left\langle u^2\right\rangle\int_0^\infty r^4 f\,\mathrm{d}r = \Lambda_0 \tag{1.5}$$

与时间无关的假设, $r^4 f' \to 0$. Taylor 展开结果:

$$f = 1 - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2 + \frac{f''''(0)}{24}r^4 + \cdots \tag{1.6}$$

$$k = \frac{k'''(0)}{6}r^3 + \cdots \tag{1.7}$$

$$\frac{\partial\left\langle u^2\right\rangle f}{\partial t} = \frac{\mathrm{d}\left\langle u^2\right\rangle}{\mathrm{d}t} - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2\frac{\mathrm{d}\left\langle u^2\right\rangle}{\mathrm{d}t} + \frac{r^2}{\lambda^3}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \cdots \tag{1.8}$$

$$k' + \frac{4k}{r} = \frac{k'''(0)}{2}r^2 + \frac{2}{3}k'''(0)r^2 + \cdots = \frac{7}{6}k'''(0)r^2 + \cdots \tag{1.9}$$

$$f'' + \frac{4f'}{r} = -\frac{1}{\lambda^2} - \frac{4}{\lambda^2} + \cdots = -\frac{5}{\lambda^2} + \cdots \tag{1.10}$$

$$\frac{\mathrm{d}\langle u^2\rangle}{\mathrm{d}t} = -10\nu\frac{\langle u^2\rangle}{\lambda^2} \tag{1.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \omega^2\right\rangle = \frac{7}{3\sqrt{15}}\left\langle \omega^2\right\rangle^{\frac{3}{2}}\left(S - \frac{2G}{R_\lambda}\right) \tag{1.12}$$

$$S \equiv -\lambda^3 k'''(0), \quad G \equiv \lambda^4 f''''(0), \quad R_\lambda \equiv \frac{\left\langle u^2\right\rangle^{\frac{1}{2}}\lambda}{\nu} \tag{1.13}$$

# 2 HIT 衰变后期规律

$k$ 忽略（惯性项近似为 0 ）

$$\frac{\partial\left\langle u^2\right\rangle f}{\partial t} = 2\nu\left\langle u^2\right\rangle\left(f'' + \frac{4f'}{r}\right) \tag{2.1}$$

设 $f = F\left(\frac{r}{\lambda(t)}\right)$, 则 $\xi \equiv \frac{r}{\lambda(t)}$

$$\frac{\mathrm{d}\langle u^2\rangle}{\mathrm{d}t}F + \left\langle u^2\right\rangle F'\left(-\frac{\xi}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right) = 2\nu\left\langle u^2\right\rangle\left(\frac{F''}{\lambda^2} + \frac{4F'}{r\lambda}\right) \tag{2.2}$$

$$-10\nu\frac{\langle u^2\rangle}{\lambda^2}\left(-\xi\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)F' \tag{2.3}$$

$$-10\nu F = 2\nu\left(F'' + \frac{4F'}{\xi}\right) + F'\xi\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t} \tag{2.4}$$

$$F'' + \frac{4F'}{\xi} + F'\xi\frac{\lambda}{2\nu}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + 5F = 0 \tag{2.5}$$

$$\alpha = \frac{\lambda}{2\nu}\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{1}{4\nu}\frac{\mathrm{d}\lambda^2}{\mathrm{d}t} \implies \text{常数} \tag{2.6}$$

$$\lambda^2 = 4\nu(t - t_0)\alpha \tag{2.7}$$

$$\frac{\mathrm{d}\langle u^2\rangle}{\mathrm{d}t} = -10\nu\frac{\langle u^2\rangle}{4\nu\alpha(t - t_0)} \tag{2.8}$$

$$\left\langle u^2\right\rangle = A(t - t_0)^{-\frac{5}{2\alpha}} \tag{2.9}$$

$$\left\langle u^2\right\rangle\int_0^{+\infty} r^4 f\,\mathrm{d}r = \left\langle u^2\right\rangle\lambda^5\int_0^{+\infty}\xi^5 F(\xi)\,\mathrm{d}\xi = \Lambda_0 \implies \alpha = 1. \tag{2.10}$$

$$\left\langle u^2\right\rangle \sim (t - t_0)^{\frac{5}{2}} \tag{2.11}$$

$$F'' + \left(\xi + \frac{4}{\xi}\right)F' + 5F = 0 \tag{2.12}$$

设 $x=\xi^2, \quad F(\xi)=y(x)$

$$F'=y'2\xi \tag{2.13}$$

$$F''=2y'+y''4x \tag{2.14}$$

$$4xy''+\left(\xi+\frac{4}{\xi}\right)2\xi y'+5y=0 \tag{2.15}$$

$$4xy''+(2x+10)y'+5y=0 \tag{2.16}$$

设 $y=\mathrm{e}^{`x`}$

$$y'=\beta y, \quad y''=\beta^2 y \tag{2.17}$$

$$4x\beta^2 y+(2x+8)\beta y+5y=0 \tag{2.18}$$

$$4x\beta^2 y+2x\beta+10\beta+5=0 \tag{2.19}$$

$$4\beta^2+2\beta=0 \implies \beta=-\frac{1}{2} \tag{2.20}$$

$$F(\xi)=C\mathrm{e}^{-\frac{\xi^2}{2}} \tag{2.21}$$

$$f(r,t)=C\mathrm{e}^{-\frac{1}{2}\frac{r^2}{4\nu(t-t_0)}}=C\mathrm{e}^{-\frac{r^2}{8\nu(t-t_0)}} \implies f=\mathrm{e}^{-\frac{r^2}{8\nu(t-t_0)}} \tag{2.22}$$

谱空间

$$\frac{\partial}{\partial t}\Phi_{ij}-\mathrm{i}m_m\left(\Gamma_{imj}+\Gamma_{jmi}\right)=-2\nu k^2\Phi_i \tag{2.23}$$

缩并 $i$,

$$\Phi_{ij}=\frac{E}{4\pi k^4}(k^2\delta_{ij}-k_ik_j) \tag{2.24}$$

$$\Gamma_{ijl}=\mathrm{i}\Gamma\left(k_ik_jk_l-\frac{k^2}{2}(k_i\delta_{jl}+k_j\delta_{il})\right) \tag{2.25}$$

$$\Gamma_{imi}=\mathrm{i}\Gamma\left(k^2k_m-\frac{k^2}{2}(k_m+3k_m)\right)=-\mathrm{i}k^2\Gamma k_m \tag{2.26}$$

$$\frac{1}{2\pi k^2}\frac{\partial E}{\partial t}-2k^4\Gamma=-2\nu k^2\frac{E}{2\pi k^2} \tag{2.27}$$

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E=4\pi k^6\Gamma\equiv T(k,t) \tag{2.28}$$

$$\frac{\partial}{\partial t}\int_0^k E(k,t)\,\mathrm{d}k-\int_0^k T(k,t)\,\mathrm{d}k=-2\nu\int_0^k k^2E\,\mathrm{d}k\equiv\Pi(k,t)=\int_k^\infty T\,\mathrm{d}k \tag{2.29}$$

$T$ 忽略

$$\frac{\partial E}{\partial t}=-2\nu k^2E \implies E(k,t)=E_0(k)\mathrm{e}^{-2\nu k^2t} \tag{2.30}$$

设 $E=V^2lF(kl)$ 用 Loistansky 不变量可得 $E_0(k)=ck^4$

Kolmogorov 1941 理论

1. 展示湍流（一般湍流）在 Re 极大时, 在局部为均匀各向同性（远离边界, 奇点）, 在增量意义下. $n$ 点联合 p.d.f（增量）只依赖于 $n$ 点构型形状、大小与位置、时刻及方位无关. 由此引入结构函数的概念.

2. 在 Re 极大时, 小尺度范围为普适平衡. $\frac{\partial}{\partial t}\langle\cdot\rangle \approx 0$, 统计特性 $\langle\cdot\rangle$ 只依赖于 $\langle\varepsilon\rangle, \nu$

3. 在 Re 极大时, 小尺度范围的低波数段, 统计量只依赖于 $\langle\epsilon\rangle$（第二相似性假设）

$$S_2(r) = B(\epsilon, \nu, r) \tag{2.31}$$

$$\eta \equiv \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}, \quad v \equiv (\epsilon\eta)^{\frac{1}{3}} \tag{2.32}$$

$$S_2(r) = v^2 F\left(\frac{r}{\eta}\right), \quad S_3(r) = v^3 G(\frac{r}{\eta}), \quad \frac{\eta v}{\nu} = 1 \tag{2.33}$$

$$S_2 = B(r, \epsilon), \quad v \sim (\epsilon r)^{\frac{1}{3}}, \quad S_2 = C_2(\epsilon r)^{\frac{2}{3}} \tag{2.34}$$