湍流 3

College of Engineering 2001111690 袁磊祺

November 25, 2021

2021 年 11 月 27 日前交电子版.

代码等作业内容可在 https://github.com/circlelq/Turbulence 查看.

1

对于归一化的一维光滑平稳高斯过程 $X(t)$, 即满足 $\langle X\rangle = 0$, $\left\langle X^2\right\rangle = 1$, 记 $\dot{X} = \frac{\mathrm{d}X}{\mathrm{d}t}$, 证明如下两个条件平均关系: $\langle \ddot{X} \mid X = x\rangle = -\left\langle \dot{X}^2\right\rangle x$; $\left\langle \dot{X}^2 \mid X = x\right\rangle = \left\langle \dot{X}^2\right\rangle$. 提示: 应该首先说明或者证明 $\dot{X}(t)$ 和 $\ddot{X}(t)$ 也是高斯过程.

根据 [2, P45] 可得正态随机过程的导数也符合正态率, 并且 $X(t), \dot{X}(t)$ 是独立的, 所以

$$\left\langle \dot{X}^2 \middle| X = x\right\rangle = \left\langle \dot{X}^2\right\rangle . \tag{1.1}$$

写出 $X_1(-\tau), X_2(0), X_3(\tau)$ 三个随机变量的概率函数有

$$f(X_1, -\tau; X_2, 0; X_3, \tau) = \frac{(2\pi)^{-\frac{3}{2}}}{\sigma^3\sqrt{D}} \exp\left[-\frac{1}{2D\sigma^2}\sum_{i,k}^{3} D_{ik}x_i x_k\right] \tag{1.2}$$

其中 D 是矩阵

$$R_{ik} = \begin{pmatrix} 1 & R(\tau) & R(2\tau) \\ R(\tau) & 1 & R(\tau) \\ R(2\tau) & R(\tau) & 1 \end{pmatrix} \tag{1.3}$$

矩阵的行列式,D_{ik} 是元素 R_{ik} 的余子式. 在假设 $R'(0) = 0$ 的情况下, 根据

$$f''(t) = \lim_{\tau\to 0} \frac{f(t+\tau) - 2f(t) + f(t-\tau)}{\tau^2} \tag{1.4}$$

可计算得

$$
\begin{aligned}
\langle \ddot{X} \mid X = x \rangle &= \lim_{\tau\to 0} \left\langle \frac{X_1(t+\tau) - 2X_2(t) + X_3(t-\tau)}{\tau^2} \right\rangle \\
&= \lim_{\tau\to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{X_1(t+\tau) - 2X_2(t) + X_3(t-\tau)}{\tau^2} f(X_1, -\tau; X_2, 0; X_3, \tau) \, \mathrm{d}X_1 \, \mathrm{d}X_3 \\
&= -\left\langle \dot{X}^2 \right\rangle x
\end{aligned}
\tag{1.5}
$$

□

2

对于不可压均匀各向同性湍流, 试给出两空间点的涡量速度关联张量的最简表达式.

解:

定义二元涡量速度关联函数

$$
R(\boldsymbol{r}, \boldsymbol{a}, \boldsymbol{b}) \equiv \left\langle \omega_a u_b' \right\rangle = R_{ij} a_i b_j. \tag{2.1}
$$

由各向同性可得

$$
R_{ij}(\boldsymbol{r}) = A_1(r^2) r_i r_j + B_1(r^2)\delta_{ij} \tag{2.2}
$$

$$
R_{11} = \left\langle \omega_1 u_1' \right\rangle \equiv \langle \omega u \rangle f(r) = A_1 r^2 + B_1 \tag{2.3}
$$

$$
R_{22} = \left\langle \omega_2 u_2' \right\rangle \equiv \langle \omega u \rangle g(r) = B_1 \tag{2.4}
$$

其中 $\langle \omega u \rangle = \frac{1}{3} \langle \boldsymbol{\omega} \cdot \boldsymbol{u} \rangle$. 由于速度有无散条件

$$
\nabla \cdot \boldsymbol{u} = 0, \tag{2.5}
$$

所以有

$$
\frac{\partial R_{ij}}{\partial r_j} = 0, \tag{2.6}
$$

得

$$
R_{ij} = \langle \omega u \rangle \left[-\frac{1}{2r}\frac{\partial f}{\partial r} r_i r_j + \left(f + \frac{r}{2}\frac{\partial f}{\partial r} \right) \delta_{ij} \right]. \tag{2.7}
$$

其中 f 是关于 r 的函数.

3

对于不可压均匀各向同性湍流, 根据不可压条件 (连续性方程) 和湍流统计量与构型（configuration）的方向无关的特点, 通过标架旋转证明一点的速度梯度满足统计关系:

$$\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = 2\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{3.1}$$

提示: 可参阅 G.I. Taylor 在 1935 年发表的关于均匀向同性湍流的有关论文.

为了书写方便, 记

$$u = u_1,\ v = u_2,\ w = u_3,\ x = x_1,\ y = x_2,\ z = x_3. \tag{3.2}$$

由不可压有

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.3}$$

即

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}, \tag{3.4}$$

两边平方得

$$\begin{aligned}\left(\frac{\partial u}{\partial x}\right)^2 &= \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \\ &= \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.\end{aligned} \tag{3.5}$$

取系综平均可得

$$\left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle = \left\langle \left(\frac{\partial v}{\partial y}\right)^2 \right\rangle + \left\langle \left(\frac{\partial w}{\partial z}\right)^2 \right\rangle + \left\langle 2\frac{\partial v}{\partial y}\frac{\partial w}{\partial z} \right\rangle \tag{3.6}$$

由各向同性湍流可得 [1]

$$\left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle = \left\langle \left(\frac{\partial v}{\partial y}\right)^2 \right\rangle = \left\langle \left(\frac{\partial w}{\partial z}\right)^2 \right\rangle \tag{3.7}$$

所以式 (3.6) 化为

$$\left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle = -2\left\langle \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} \right\rangle \tag{3.8}$$

对坐标进行 45° 可得

$$\begin{cases} \sqrt{2}x' &= x + y \\ \sqrt{2}y' &= -x + y \\ z' &= z \end{cases} \tag{3.9}$$

$$
\begin{cases}
\sqrt{2}u' & = u+v \\
\sqrt{2}v' & = -u+v \\
v' & = v
\end{cases}
\tag{3.10}
$$

Hence

$$
\begin{cases}
\frac{\partial u}{\partial x} & = \frac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\
\frac{\partial v}{\partial x} & = \frac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} - \frac{\partial u'}{\partial y'}\right) \\
\frac{\partial w}{\partial x} & = \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} - \frac{\partial w'}{\partial y'}\right)
\end{cases}
\tag{3.11}
$$

$$
\begin{cases}
\frac{\partial u}{\partial y} & = \frac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial y'}\right) \\
\frac{\partial v}{\partial y} & = \frac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\
\frac{\partial w}{\partial y} & = \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} + \frac{\partial w'}{\partial y'}\right)
\end{cases}
\tag{3.12}
$$

$$
\begin{cases}
\frac{\partial u}{\partial z} & = \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} - \frac{\partial v'}{\partial z'}\right) \\
\frac{\partial v}{\partial z} & = \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial z'}\right) \\
\frac{\partial w}{\partial z} & = \frac{\partial w'}{\partial z'}
\end{cases}
\tag{3.13}
$$

表 3.1. 记号说明.

$\overline{\left(\frac{\partial u}{\partial x}\right)^2}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}$	$\overline{\left(\frac{\partial u}{\partial y}\right)^2}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}$
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}

对式 (3.9) 求平方, 并利用各向同性可得以下等式

$$a_4 = a_7 = a_2 = a_5 = a_{10} = a_9 = 0. \tag{3.14}$$

$$a_1 = -2a_6, \tag{3.15}$$

$$a_1 - a_3 - a_6 - a_8 = 0, \tag{3.16}$$

$$a_1 - a_3 - a_6 - a_8 = 0, \tag{3.17}$$

$$a_1 + 2a_8 = 0, \tag{3.18}$$

$$a_1 = -2a_6 \tag{3.19}$$

$$a_1 = \frac{1}{2}a_3. \tag{3.20}$$

□

4

将不可压均匀各向同性湍流中两点纵向速度关联函数与一维能谱之间的傅立叶积分变换关系

$$\left\langle u^2\right\rangle f(r)=\int_{-\infty}^{\infty}\phi_1(k)\mathrm{e}^{\mathrm{i}kr}\,\mathrm{d}k \tag{4.1}$$

代入如下两点纵向速度关联函数与三维能谱之间的积分变换关系

$$E(k)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left\langle u^2\right\rangle f(r)(kr)^2\left(\frac{\sin kr}{kr}-\cos kr\right)\mathrm{d}r, \tag{4.2}$$

直接通过计算推出

$$E(k)=k^3\frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{1}{k}\frac{\mathrm{d}\phi_1(k)}{\mathrm{d}k}\right]. \tag{4.3}$$

或者, 将

$$\left\langle u^2\right\rangle f(r)=2\int_{-\infty}^{\infty}E(k)(kr)^{-2}\left(\frac{\sin kr}{kr}-\cos kr\right)\mathrm{d}k \tag{4.4}$$

代入傅立叶逆变换关系

$$\phi_1(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle u^2\right\rangle f(r)\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r, \tag{4.5}$$

进行积分, 推出

$$\phi_1(k)=\frac{1}{2}\int_{k}^{\infty}\left(1-\frac{k^2}{\lambda^2}\right)\frac{E(\lambda)}{\lambda}\,\mathrm{d}\lambda. \tag{4.6}$$

两个推导过程任选其一完成即可. 提示: 第一个推导利用 δ 函数及其导数的性质; 第二个推导用到由如下积分关系

$$\frac{1}{2}\int_{0}^{1}\left(1-k^2\right)\cos kx\,\mathrm{d}k=\frac{1}{x^2}\left(\frac{\sin x}{x}-\cos x\right) \tag{4.7}$$

带来的余弦变换.

将式 (4.4) 带入式 (4.5) 可得

$$\begin{aligned}
\phi_1(k)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle u^2\right\rangle f(r)\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\int_{-\infty}^{\infty}E(\lambda)(\lambda r)^{-2}\left(\frac{\sin\lambda r}{\lambda r}-\cos\lambda r\right)\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r\\
&=\frac{1}{\pi}\int_{-\infty}^{\infty}E(\lambda)\lambda^{-2}\int_{-\infty}^{\infty}r^{-2}\left(\frac{\sin\lambda r}{\lambda r}-\cos\lambda r\right)\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{d}r\,\mathrm{d}\lambda\\
&=\frac{1}{\pi}\int_{-\infty}^{\infty}E(\lambda)\lambda^{-2}\int_{-\infty}^{\infty}r^{-2}\left(\frac{\sin\lambda r}{\lambda r}-\cos\lambda r\right)\cos(kr)\,\mathrm{d}r\,\mathrm{d}\lambda
\end{aligned} \tag{4.8}$$

其中

$$
\begin{aligned}
&4\lambda \int_{-\infty}^{\infty} r^{-2}\left(\frac{\sin\lambda r}{\lambda r}-\cos\lambda r\right)\cos(kr)\,\mathrm{d}r\\
=&(\lambda-k)(\lambda+k)\operatorname{Si}\big((\lambda+k)r\big)+(\lambda-k)(\lambda+k)\operatorname{Si}\big((\lambda-k)r\big)\\
&+\frac{2k\sin(kr)\sin(\lambda r)+2\lambda\cos(kr)\cos(\lambda r)}{r}-\frac{2\cos(kr)\sin(\lambda r)}{r^2}
\end{aligned}
\tag{4.9}
$$

其中

$$
\operatorname{Si}(x)=\int_0^x \frac{\sin(t)}{t}\,\mathrm{d}t,\quad \operatorname{Si}(+\infty)=\frac{\pi}{2}. \tag{4.10}
$$

只有当 $(\lambda+k)$, $(\lambda-k)$ 同号时, 积分才不为 0. 又考虑到 $k>0$, 所以 $\lambda>k$, 则

$$
\phi_1(k)=\frac{1}{2}\int_k^{\infty}\left(1-\frac{k^2}{\lambda^2}\right)\frac{E(\lambda)}{\lambda}\,\mathrm{d}\lambda. \tag{4.11}
$$

□

5

用不同于教材中的方法证明教材 (2.3.42) 式:

$$
\langle u^2\rangle^{\frac{3}{2}} k(r)=8\pi\int_0^{\infty}\frac{k^5\Gamma(k)}{(kr)^4}\big(3\sin kr-3kr\cos kr-(kr)^2\sin kr\big)\,\mathrm{d}k. \tag{5.1}
$$

教材并没有明说 (2.3.42) 是怎么得到的, 假设它是由 (2.3.41) 的反变换得到的, 那么下面给出另外一种证明方法. (2.3.41) 上面的一个式子为:

$$
\frac{1}{r^2}\frac{\partial}{\partial r}[r^3K(r)]=8\pi\int_0^{+\infty}k^6\Gamma(k)\frac{\sin kr}{kr}\,\mathrm{d}k. \tag{5.2}
$$

把 r^2 乘过去再积分得

$$
\begin{aligned}
r^3K(r)&=8\pi\int_0^{+\infty}\int_0^r x^2k^6\Gamma(k)\frac{\sin kx}{kx}\,\mathrm{d}x\,\mathrm{d}k\\
&=8\pi\int_0^{+\infty}k^3\Gamma(k)\big(\sin(kr)-kr\cos(kr)\big)\,\mathrm{d}k
\end{aligned}
\tag{5.3}
$$

带入

$$
K(r)=\langle u^2\rangle^{\frac{3}{2}}\frac{1}{r^4}\frac{\partial}{\partial r}(r^4k) \tag{5.4}
$$

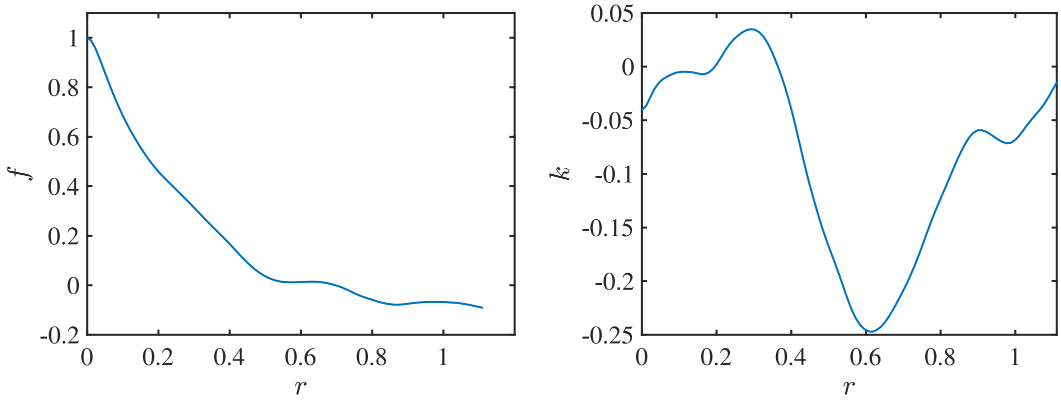
得

$$
\langle u^2\rangle^{\frac{3}{2}}\frac{1}{r}\frac{\partial}{\partial r}(r^4k)=8\pi\int_0^{+\infty}k^3\Gamma(k)\big(\sin(kr)-kr\cos(kr)\big)\,\mathrm{d}k. \tag{5.5}
$$

再次积分可得

$$
\begin{aligned}
\langle u^2\rangle^{\frac{3}{2}} r^4k&=8\pi\int_0^{+\infty}\int_0^r xk^3\Gamma(k)\big(\sin(kx)-kx\cos(kx)\big)\,\mathrm{d}x\,\mathrm{d}k\\
&=8\pi\int_0^{+\infty}k\Gamma(k)(3\sin(kr)-3kr\cos(kr)-k^2r^2\sin(kr))\,\mathrm{d}k.
\end{aligned}
\tag{5.6}
$$

□



(a) 二元速度关联函数 f.　(b) 三元速度关联函数 k.

图 6.1. 近似均匀各向同性同性湍流中的关联函数.

6

自己从网上或者课题组内部获取实验测量或者直接数值模拟得到的湍流脉动速度信号的一段长时间的时间序列或者一定样本容量的空间序列. 完成下列两项任务:

1. 计算两点纵向的二阶和三阶速度关联系数 $f(r)$, $k(r)$ 并画出曲线 (可以用泰勒冻结假设);
2. 根据 $f(r)$ 计算一维能谱 $\varphi_1(k)$ 和三维能谱 $E(k)$ (在假设各向同性的情况下利用教材公式 (2.3.23)), 并画出曲线.

解:

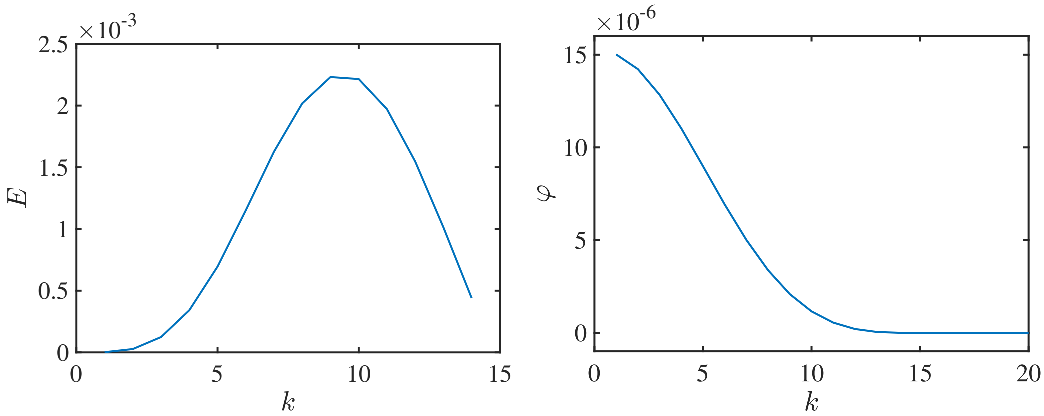
这里选取的是杨延涛课题组的 RB 对流算例中的流场数据. 上板冷却, 下板加热, $\mathrm{Ra} = 1 \times 10^8$ 中间区域近似为各向同性均匀湍流, 利用泰勒冻结假设, 对某一水平方向算相关函数取平均后可得图 1(a) 和 1(b). 三维能谱 $E(k)$ 和 $f(r)$ 的关系是

$$E(k) = \frac{1}{\pi} \int_0^{+\infty} \langle u^2 \rangle f(r)(kr)^2 \left(\frac{\sin kr}{kr} - \cos kr \right) \mathrm{d}r. \tag{6.1}$$

一维能谱 $\varphi(k)$ 和 $E(k)$ 的关系是

$$\varphi_1(k_1) = \frac{1}{2} \int_{k_1}^{+\infty} \left(1 - \frac{k_1^2}{k^2} \frac{E(k)}{k} \right) \mathrm{d}k. \tag{6.2}$$

画图可得图 2(a) 和 2(b).



(a) 三维能谱 $E(k)$.　(b) 一维能谱 $\varphi(k)$.

图 6.2. 能谱图.

参考文献

[1] G. I. Taylor. Statistical theory of turbulence. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 151(873):421–444, 1935. 3

[2] S. 潘契夫. 随机函数和湍流. 科学出版社, 1976. 1