DISCRETE STRUCTURES, SPRING 2017 - FINAL

Name: ______________________________

Write your name on this page. Please indicate clearly which problems you have worked on. You will receive no credit for submitting solutions that I cannot read and understand—be sure to write legibly! LEGIBLY !!!!

Problem 1. In class we showed how to prove one version of De Morgan's Law. Replicate the proof, and prove the other version as well. In other words, if A and B are non-empty sets, prove the following equalities to be true:

(1) $(A \bigcup B)' = A' \bigcap B'$
(2) $(A \bigcap B)' = A' \bigcup B'$

Problem 2. If A, B, and C are nonempty sets, draw the Venn diagrams for each of the following sets:

(1) $(A \bigcup B \bigcup C) - [(A \bigcap B) \bigcup (A \bigcap C) \bigcup (B \bigcap C)]$.
(2) If $A \subseteq B$ and $A \bigcap C = \emptyset$, $B \bigcap C \neq \emptyset$, draw $(B \bigcap A') - C$.

Problem 3. If $\{B_i\}_{i=0}^n$ is a collection of non-empty sets (meaning that we have $B_1, B_2, \ldots, B_n$ sets) prove the following equalities to be true:

(1)
$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

(2)
$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i)$$

Problem 4. Answer the following:

(1) Why is f in each of the cases below not a functions from $\mathbb{R}$ to $\mathbb{R}$
  (a) $f(x) = \frac{1}{x}$?
  (b) $f(x) = \frac{1}{\sqrt{x}}$?
  (c) $\pm\sqrt{x^2+1}$?
(2) What would need to be done to make each of the above a function?

Problem 5. Let R be a relation on the set of ordered pars of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

Problem 6. Suppose $f : A \to B$ and $g : B \to C$ are both bijective functions.

(1) Prove that $g \circ f$ is also a bijection.

Problem 7. Bonus
Fifteen children together gathered 100 nuts. Prove that some pair of children gathered the same number of nuts.

Problem 8. Decide whether or not the following pairs of statements are logically equivalent. Use Truth tables to determine it.

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(1) $(P \wedge Q)$ and $\neg(\neg P \vee \neg Q)$.
(2) $(\neg P) \wedge (P \implies Q)$ and $\neg(Q \implies P)$.

Problem 9. We are going to assume that there are 20 million Texans in Texas. Let $A = \{x | x \text{ is a Texan}\}$. It is also a biological fact that there are fewer than a million hair on the human head.
Let $B = \{x | x \leq 1000000 \ \& \ x \in \mathbb{N}\}$. Prove that there are at least two Texans with exactly the same number of hairs on their head.

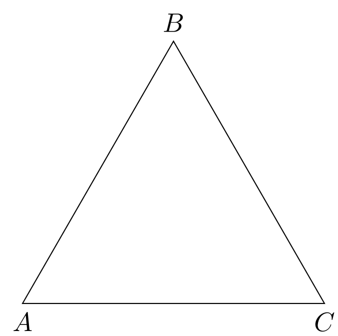
Problem 10. Prove the following identity:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Problem 11. Evaluate the following expression:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

Problem 12. Three ants are sitting on the vertices of a triangle, point A,B,C. Each of the ants can only walk along the edges of the triangle. They can only go left or right. Once they have picked a direction, they keep walking along that direction.



Whats the probability that any two ants will collide?

Problem 13. Show that if $P(A) = 1$, then $P(B|A) = P(B)$.

Problem 14. Three missiles are fired at a target and hit independently with probability 0.7, 0.8, and 0.9 respectively. What is the probability that the target is hit?

Problem 15. A box contains 20 fuses, of which 5 are defective. What is the expected number of defective items among three randomly chosen fuses?

Problem 16. Bonus.
If p is the probability of success in a single binomial experiment, the probability of x successes and $n - x$ failures in n independent repeated trails of experiment, knows as the **binomial probability**, is

$$P(\text{x successes in n trails}) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

Show that the expectation of X is:

$$\mathbb{E}(X) = np.$$

Problem 17 (Bonus)**.** If we have a function $f : A \to B$ with $S, T \subseteq A$, prove that $f(S \cup T) = f(S) \cup f(T)$.