Introduction to Convolution Quadrature

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Problem

Let Ω be a bounded domain with boundary Γ. The problem we want to solve is the exterior wave equation, with $\Omega^+ = \mathbb{R}^2/\bar{\Omega}$:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) - \Delta_x u(t,x) = 0 & x \in \Omega^+ \\ u(0,x) = \frac{\partial u(0,x)}{\partial t} = 0 & x \in \Omega^+ \\ u(t,x) = g(t,x) & x \in \Gamma. \end{cases}$$

And we assume that Γ is smooth.

Another approach is write to write the wave equation in weak form:

$$\int_{\Omega} u_{tt} v + \nabla u \cdot \nabla v dx - \int_{\partial\Omega} v \nabla u \cdot n dS = 0$$

Supposing that we can expand u in terms of basis functions:

$$u(t, x) \approx \sum_{i=0}^{N} \xi_i(t) \varphi_i(x)$$

The weak form above can be written as:

$$\boldsymbol{\xi}'' \boldsymbol{A} + \boldsymbol{\xi} \boldsymbol{S} - \boldsymbol{\xi} \boldsymbol{B} = 0$$

where

$$A_{ij} = \int_{\Omega} \varphi_i(x) \varphi_j(x) dx \quad S_{ij} = \int_{\Omega} \nabla \varphi_i(x) \cdot \nabla \varphi_j(x) dx$$
$$B_{ij} = \int_{\Omega} \varphi_i(x) \cdot \nabla \varphi_j(x) dx.$$

Note that:

- Note that the partial derivatives need to be discritized over all of $\mathbb{R}^2$.
- A,B,C needs to be evaluated over all of Ω.

In the origonal setup of the wave equation, $\Omega^+ = \mathbb{R}^2/\bar{\Omega}$ where $\bar{\Omega}$ is a smooth and compact domain.

Meaning that if we were to use a finite difference or finite element method, we need to discetize and evaluate over a truncated domain in $\mathbb{R}^2$.

Further, both finite difference and finite element method are known to suffer from dispersion error.

Back to our Original Problem

Let Ω be a bounded domain with boundary Γ. The problem we want to solve is the exterior wave equation, with $\Omega^+ = \mathbb{R}^2/\bar{\Omega}$:

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And we assume that Γ is smooth.

Convolution Quadrature

We sample the above solution $u(t, x)$ in time. Meaning, we assume a sequence of equisampled approximations in time:

$$u_d(t_0, x), u_d(t_1, x), \dots, u_d(t_n, x), \dots$$

Where $u(t_n, x) \approx u_d(t_n, x)$, where $u(t_n, x)$ is the true solution at the point x at time t. We now also introduce the Z-transform:

$$U_d(z, x) = \sum_{n=0}^{\infty} u_d(t_n, x) z^n. \tag{2}$$

At this point it should also be noted that we can invert the transform by a simple contour integral:

$$u_d(t_n, x) = \int_{|z|=\lambda} \frac{U_d(z, x)}{z^{n+1}} dz. \tag{3}$$

Manipulating the Wave Equation

To begin our derivation, we first write the wave equation as a first order linear equation:

$$\begin{cases} \frac{\partial Y(t,x)}{\partial t} = LY(t,x) & x \in \Omega^+ \\ Y(0,x) = 0 & x \in \Omega^+ \\ BY(t,x) = F(t,x) & x \in \Gamma \end{cases} \tag{4}$$

where we have that·

$$Y(t,x) = \begin{bmatrix} u(t,x) \\ \frac{\partial u(t,x)}{\partial t} \end{bmatrix}, L = \begin{bmatrix} 0 & I \\ \Delta_x & 0 \end{bmatrix}, B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, F(t,x) = \begin{bmatrix} g(t,x) \\ 0 \end{bmatrix}. \tag{5}$$

To this we want to apply a multistep scheme in time.

At this point we apply a multistep scheme to the (4), resulting in:·

$$\frac{1}{\Delta t}\sum_{j=0}^{n}\gamma_{n-j}Y_d(t_j,x)=LY_d(t_n,x), \tag{6}$$

$Y_d(t_n,x)$ is the discrete approximation to $Y(t_n,x)$, and γ_{n-j} are the coefficients to the multistep scheme. We now apply the Z-transform to (6):

$$\sum_{n=0}^{\infty}\left[\frac{1}{\Delta t}\sum_{j=0}^{n}\gamma_{n-j}Y_d(t_j,x)\right]z^n=\sum_{n=0}^{\infty}\left[LY_d(t_n,x)\right]z^n.$$

Which after some rearranging we can write as:·

$$\frac{1}{\Delta t}\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\gamma_{n-j}Y_d(t_j,x)\right]z^n = L\sum_{n=0}^{\infty}Y_d(t_n,x)z^n. \tag{7}$$

We now define the following variables:

$$\gamma(z) = \sum_{n=0}^{\infty}\gamma_n z^n$$

$$Y_d(z,x) = \sum_{n=0}^{\infty}Y_d(t_n,x)z^n$$

Notice that (7) between γ and Y_d is a discrete convolution. Convolutions and Z-transform will become a product in z-space. We then get the following:

$$\frac{1}{\Delta t}\gamma(z)Y_d(z,x) = LY_d(z,x)$$

Taking the equation and translating it back into second order form we then get the following equation:·

$$\begin{cases} \left(\frac{\gamma(z)}{\Delta t}\right)^2 U_d(z,x) - \Delta_x U_d(z,x) = 0 & x \in \Omega^+ \\ U_d(z,x) = G(x,z) & x \in \Gamma. \end{cases} \tag{8}$$

Where we note that

$$U_d(z,x) = \sum_{n=0}^{\infty} u_d(t_n,x) z^n$$
$$G(z,x) = \sum_{n=0}^{\infty} g(t_n,x) z^n.$$

Which is the Z-transform of the original variables of the wave equation in (1). It should also be noticed that (8) is now the modified Helmholts equation with $k = i\frac{\gamma(z)}{\Delta t}$. In the specific case that the multistepping is the BDFM2 method, $\gamma(z) = 0.5(z^2 - 4z + 3)$

Stated differently, this means we have to solve

$$\begin{cases} \left(\frac{\gamma(z_i)}{\Delta t}\right)^2 U_d(z_i,x) - \Delta_x U_d(z_i,x) = 0 & x \in \Omega^+ \\ U_d(z_i,x) = G(x,z_i) & x \in \Gamma. \end{cases}$$

satisfying the Sommerfeld radiation condition at infinity.

$$\lim_{|x|->\infty} |x| \left(\frac{\partial}{\partial |x|} - ik\right) u(x) = 0. \tag{9}$$

This can be solved by supposing that that the solution U_d takes the form of a single, double, or combined layer potential

$$U_d(z_i,x) = \int_\Gamma K(is_l|x-y|)\varphi(z_i,y)dS(y) \quad (SL)$$

$$U_d(z_i,x) = \int_\Gamma \frac{\partial K(is_l|x-y|)}{\partial n(y)}\sigma(z_i,y)dS(y) \quad (DL)$$

$$U_d(z_i,x) = \int_\Gamma \left(\frac{\partial K(is_l|x-y|)}{\partial n(y)} - is_l K_{s_l}(x,y)\right)\eta(z_i,y)dS(y) \quad (CL)$$

Where in each of the potentials above, $x \notin \Gamma$.

Note:

$$s_l = \left(\frac{\gamma(z_l)}{\Delta t}\right)^2$$

$$K(x) = \frac{i}{4} H_0^{(1)}(|x|).$$

Each of the cases above leads to the following equations:

$$\begin{aligned}
(SL\varphi)(x, z_i) &= G(x, z_i) \quad x \in \Gamma \\
(DL\sigma)(x, z_i) + \frac{\sigma(x, z_i)}{2} &= G(x, z_i) \quad x \in \Gamma \\
(CL\eta)(x, z_i) + \frac{\eta(x, z_i)}{2} &= G(x, z_i) \quad x \in \Gamma
\end{aligned}$$

Where the densities needs to be solved for.

Collocation and Nystrom Quadrature

To solve for the densities numerically we need to discretize the integral operator naively, by a trapezoidal rule:

$$\begin{aligned}(SL\varphi)(x_{ii}) &= \int_\Gamma K(is_l|x_{ii}-y|)\varphi(yi,z_i)dS(y)\\ &\approx h\sum_{jj=0}^{M-1} K(is_l|x_{ii}-y_{jj}|)\varphi(y_{jj},z_i)dS(y_{jj})\end{aligned}$$

If we discretize the boundary condition G also, the equation we have to solve becomes:

$$\begin{bmatrix} K(i|x_0-y_0|) & \dots & K(i|x_0-y_{M-1}|)\\ K(i|x_1-y_0|) & \dots & K(i|x_0-y_{M-1}|)\\ \vdots & \dots & \vdots\\ K(i|x_{M-2}-y_0|) & \dots & K(i|x_{M-2}-y_{M-1}|)\\ K(i|x_{M-1}-y_0|) & \dots & K(i|x_{M-1}-y_{M-1}|)\end{bmatrix}\begin{bmatrix}\varphi(y_0,z_i)\\ \varphi(y_1,z_i)\\ \vdots\\ \varphi(y_{M-2},z_i)\\ \varphi(y_{M-1},z_i)\end{bmatrix} = \begin{bmatrix} G(x_0,z_i)\\ G(x_1,z_i)\\ \vdots\\ G(x_{M-1},z_i)\\ G_(x_{M-1},z_i)\end{bmatrix}$$

What are the discritizations of z

$$U_d(z,x) = \sum_{n=0}^{\infty} u_d(t_n,x) z^n.$$

$$u_d(t_n,x) = \int_{|z|=\lambda} \frac{U_d(z,x)}{z^{n+1}} dz.$$

Let T be the final time, then if we discritize as follows:

$$z_k = \lambda e^{\frac{-2\pi ik}{N}}$$

The forward $Z-$transform then becomes

$$U_d(z_k,x) = \sum_{n=0}^{N} \lambda^n u_d(t_n,x) e^{-\frac{2\pi ink}{N}}$$

$$u_d(t_n,x) = \int_C \frac{U_d(z_k,x)}{z_k^{n+1}} dz \approx \frac{\lambda^{-(n+1)}}{N} \sum_{k=0}^{N} U_d(z_k,x) e^{\frac{2\pi ikn}{N}}.$$

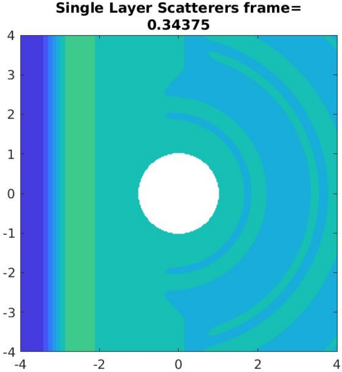
Which is a λ-scaled forward and backwards DFT transform.

Simulation Setup

We suppose our incoming plane wave is given by:

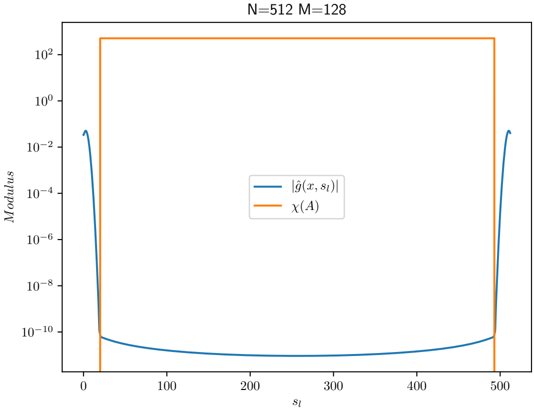
$$g(x,t) = cos(5t - x\cdot(1,0))\exp\left(-1.5(5t - x\cdot(1,0) - 5)^5\right) \quad x \in \Gamma, \quad (10)$$

and that Γ is the unit circle.



Reduction of the system and Optimizing Serial

- Discretize in time and approximate the Z-transform smartly is same as taking the DFT in time.
- Meaning, if we have a real incoming plane wave, we only need to solve for half the densities, since: $U_d(\hat{z}, x) = \hat{U}_d(z, x)$.
- If a real plane wave has nearly compact support (as in $g(x,t)$), then we don't solve for all frequencies!



Algorithm for solving Densities and Total field

Algorithm 1: Algorithm For Calculating Densities with BDFM2 CQ

Choose a $T > 0$ and the equispaced mesh $t_n = n\Delta t, \Delta t = \frac{T}{N}$;
Choose a λ in accordance with [3];
Use a mesh $\vec{x}(\theta_m)$ for $0 < \theta_m \leq 2\pi$.

1. Sample $g(t_n, x_m)$ for $0 \leq n \leq N$ and construct the FFT of $\{\lambda^n g(t_n, x_m)\}_{0 \leq n \leq N}$ for all grid points x_m. This can be done at once and it produces $\{\hat{g}_l(x_m)\}_{0 \leq l \leq N}$ for all grid points $\{x_m\}_{1 \leq m \leq M}$.
2. In parallel Solve for only half of the densities:

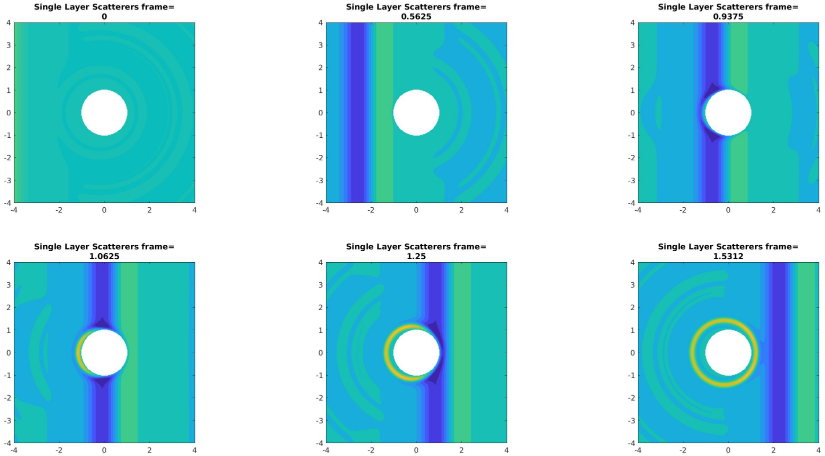
$$\frac{i}{4}\int_\Gamma H_0^{(1)}(is_l|x-y|)\varphi(z_l, y)ds(y) = G(z_l, x)$$

   for $0 \leq l \leq N$. When the data is $g(t, x)$ is real, solve only those problems where the frequency is not within a neighborhood of zero.
3. Do the IFFT of

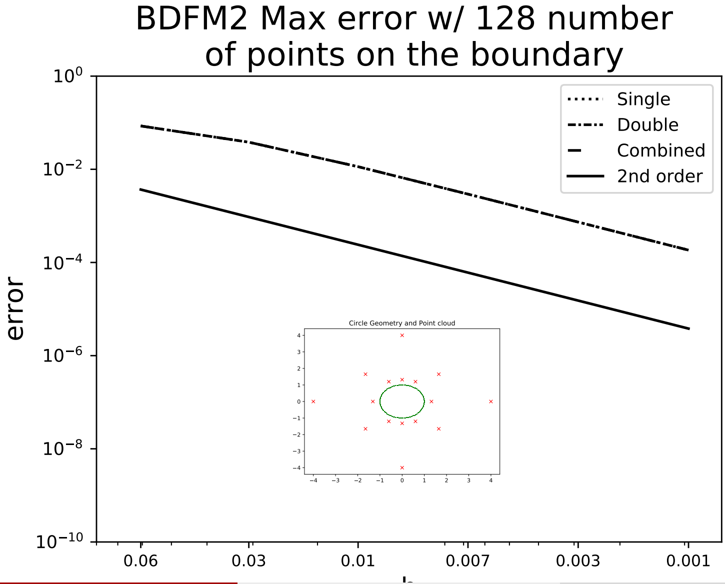
$$\{\varphi(z_l, x_m)\}_{0 \leq l \leq N}$$

   and for $1 \leq m \leq M$ and then multiply it by λ^{-l} to get an approximation of $\varphi(t_l, x_m)$ on all of its spatial and time grid points.

Snapshots of Solution



Convergence



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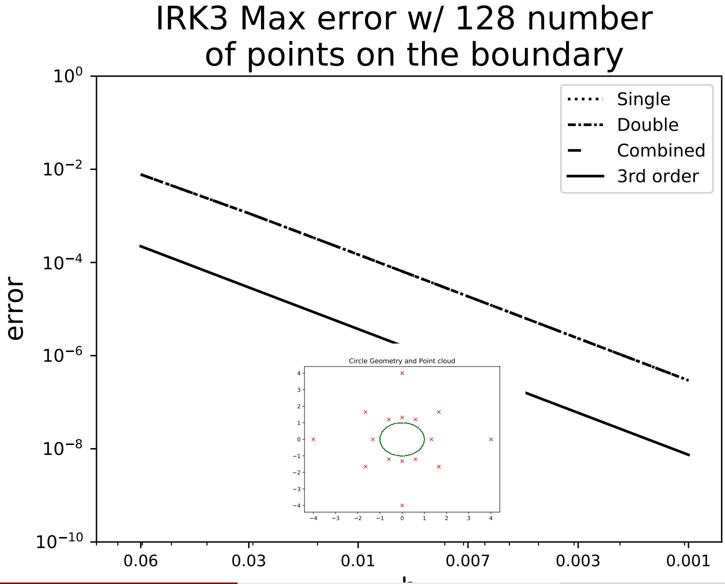

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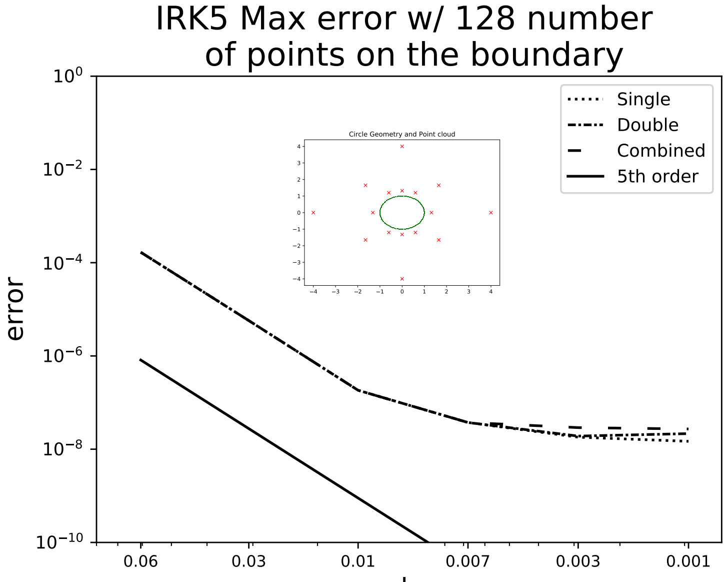

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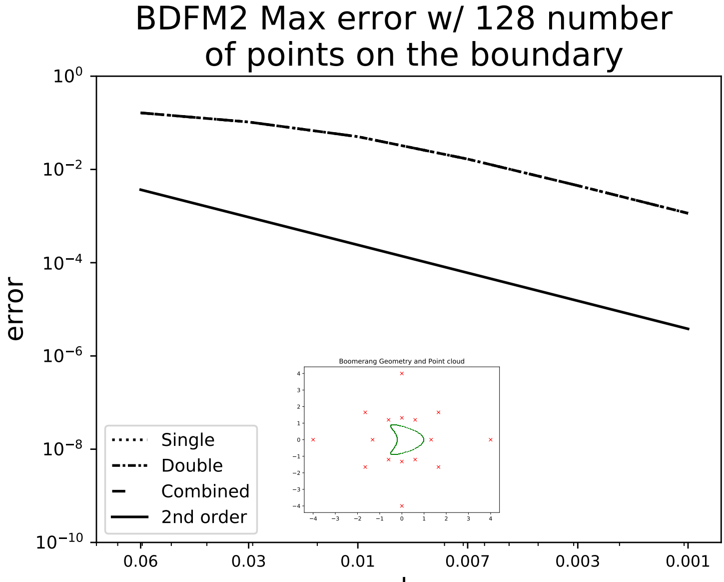

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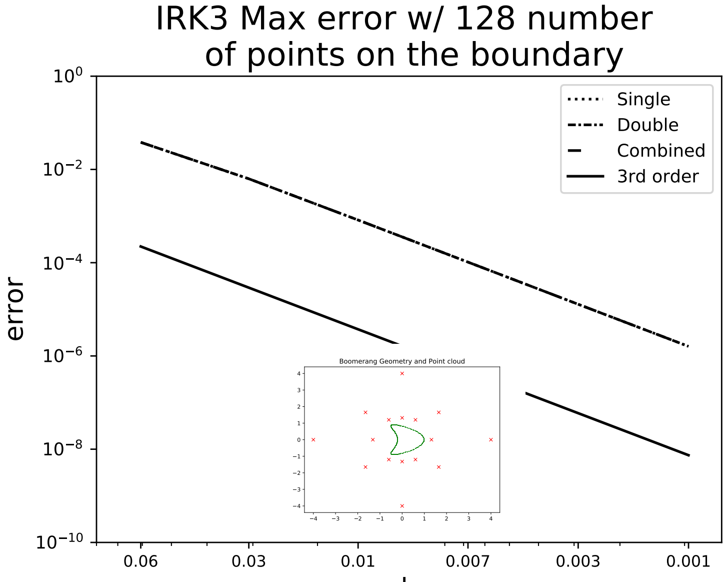

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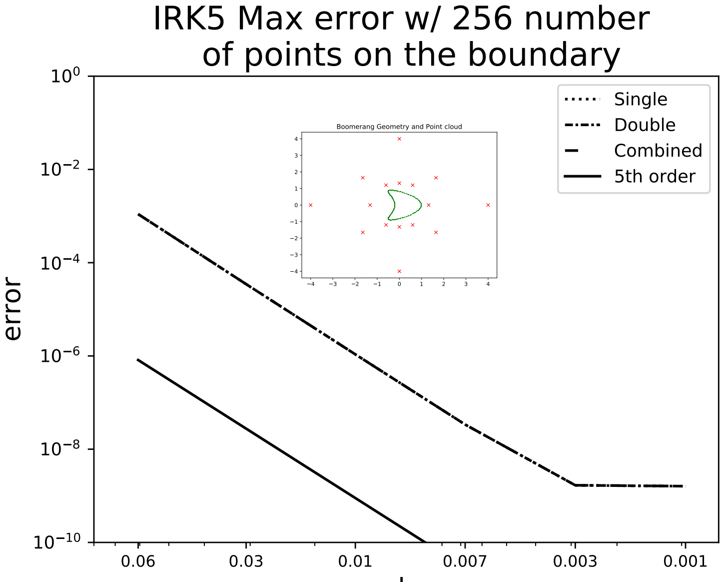

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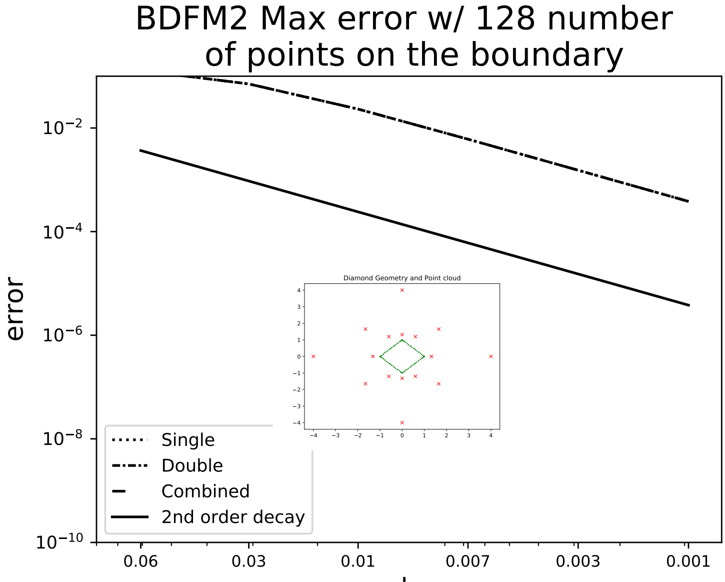

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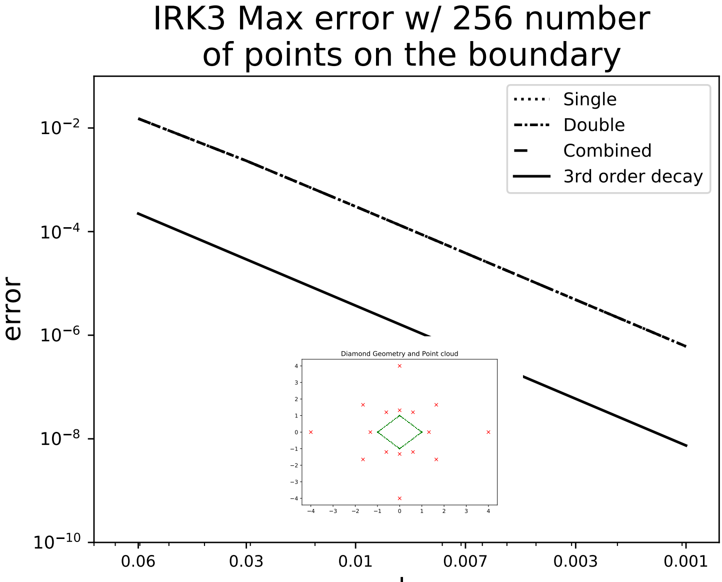

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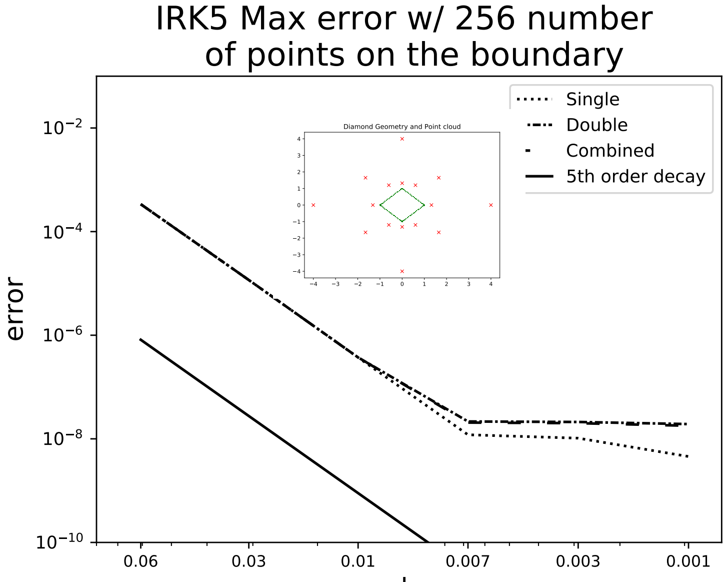

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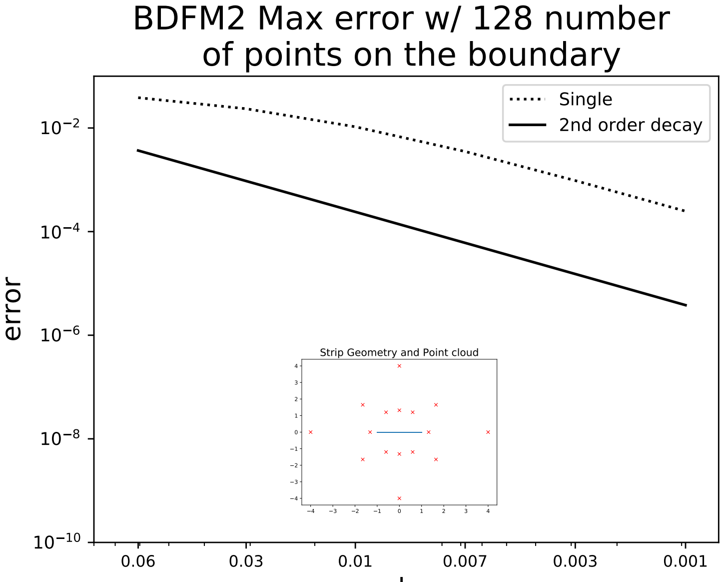

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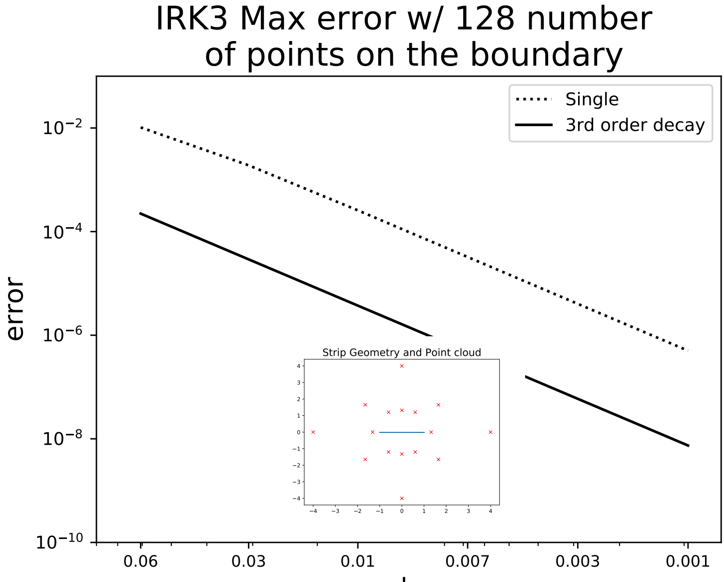

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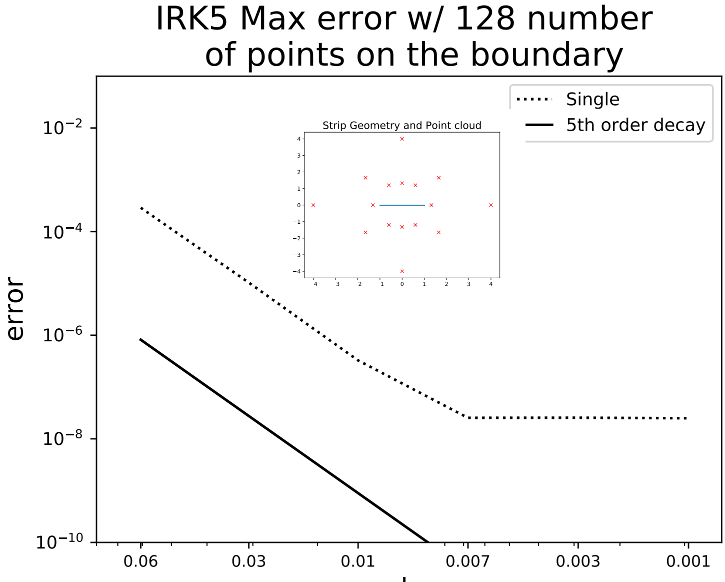

Benifits and Limitations of CQ

- Works in parallel.
- Most pulses are nearly compact, leading to a big reduction in the number of Helmhotz solves we need to perform
- Includes fast alogoritms, like FFT.
- Time-stepping schemes needs to be A-stable. Limiting CQ to be used with only BDFM2 and Implicit Runge-Kutta (3rd and 5th order).
- For closer evaluation on the boundary, another quadrature rule is needed.
- CQ seems to saturate at 10^{-8} digits of accuracy.
- Can handle corners well.

References

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