NUMERICAL ANALYSIS MIDTERM II, MONDAY, NOVEMBER 18th, 2013

1. Compute $\|A\|_2$ and $\|A\|_\infty$ for $A = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

   (a) Find the range values of t for which $I - tA$ is convergent.

   (b) Find a Schur decomposition for A.

   (c) Find a matrix C with $C^2 = A$.

3. Suppose A is a 10 by 10 matrix with distinct, real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_{10}$. Given that $\|A\|_\infty = 4$, show that at least two of the eigenvalues lie within a distance 1 of each other, i.e. there are $i \neq j$ for which $|\lambda_i - \lambda_j| \leq 1$.

4. Let V be a vector space with inner product $(\cdot, \cdot)$ and let W be a subspace of V.

   (a) Show that if $(v, w) = 0$ for all $w \in V$, then $v = 0$.

   (b) Given $v \in V$, show that there can be at most one element $w_v \in W$ with the property

$$(v - w_v, w) = 0, \quad \forall w \in W.$$

5. Let $\mathbf{P}_n$ denote the vector space of nth degree polynomials in t. Let $D : \mathbf{P}_n \to \mathbf{P}_n$ be the operator corresponding to taking one derivative in t.

   (a) Give the matrix representation of D with respect to the basis $(1, t, t^2, \ldots, t^n)$.

   (b) Show that $\rho(D) = 0$.

   (c) Compute $\|D\|_2$.

   (d) Show that $D^m = 0$ for some m sufficiently large.