A Brief Introduction to Integral Equations and Singular Kernel Evaluation

E. Wind-Andersen[1]

[1]Department of Mathematics
New Jersey Institute of Technology

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Outline

The Helmholtz equation is given by the follow

$$\Delta u + k^2 u = 0$$

We aim to solve this equation on two domain, the interior and exterior of the circle with the following boundary equation:

$$u(R, \theta) = \sum_{n=0}^{N} f_n e^{in\theta}.$$

By separation of variables

$$\begin{cases} \eta''(\theta) + \lambda\eta(\theta) = 0 \\ r^2 R''(r) + rR'(r) + (k^2 r^2 - \lambda)R(r) = 0 \end{cases}$$

The first of these has solution:

$$\eta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta).$$

The second equation is the Bessel equation, and the most general solution to this type of equation is given by:

$$H_n^{(1)}(kr) = J_n(kr) + iY_n(kr) \qquad H_n^{(2)}(kr) = J_n(kr) - iY_n(kr)$$

$$H_n^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{\pi}{4} - \frac{n\pi}{2})} \qquad H_n^{(2)}(kr) \sim \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \frac{\pi}{4} - \frac{n\pi}{2})}$$

The problem we aim to solve is a two point boundary method. For the interior and exterior problem we impose the following conditions:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{for } |x| > R \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) u(r,\theta) = 0 \\ u(R,\theta) = e^{in\theta} \end{cases}$$

With solution

$$u(r,\theta) = \sum_{n=0} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(KR)} f_n e^{in\theta}.$$

The interior problem is

$$\begin{cases} \Delta u + k^2 u = 0 & \text{for } |x| < R \\ u(0,\theta) < \infty \\ u(R,\theta) = e^{in\theta} \end{cases}$$

with solution being

$$u(r,\theta) = \sum_{n=0} \frac{J_n(kr)}{J_n(kR)} e^{in\theta}.$$

A different approach is to find the Greens function for the differential equation. Consider

$$-\Delta E - k^2 E = \delta(x)$$

Requiring that there is angular symmetry we can reduce the equation to a Bessel type equation:

$$r^2 u''(r) + r u'(r) + k^2 r^2 u(r) = 0$$

Which has general solutions as a Hankel functions of the first and second kind. By the decay at infinity we find that $u(r) = cH_0^{(1)}(kr)$. Using the Divergence theorem we reduce the Laplace operator

$$\int_D \Delta E dx = \int_\sigma \frac{\partial}{\partial r}(E) dS = -1$$

from the previous slides we know that $E = cH_0^{(1)}(kr)$ from solving the Bessel equation. For small values of r we have the asymptotic form

$$H_n^{(1)}(r) \sim -\frac{i2^n(n-1)!}{\pi} r^{-n}$$

Using

$$lim_{r\to 0} r S_n(1) \frac{\partial}{\partial r}(E) = -1$$

We can solve for c and find that

$$E = \frac{i}{4} H_0^{(1)}(kr)$$

Meaning that a solution of the inhomogenous Helmholtz equation can be given as follows

$$u(x,y) = \int_D \frac{i}{4} \phi(y) H_0^{(1)}(k|x-y|) dy(s)$$

Usually this presents an issue as the Hankel function is singular at the origin. It has a log type singularity. We therefore need an approach that allows us to deal with these types of integrals. If we can overcome this we can solve the usual types of integral equations found on the next slide.

$$\sigma(x) + \int_0^T k(x, x')\sigma(x')dx' = f(x), \quad x \in [0, T]$$

To be able to solve this numerically we need to start with an underlying quadrature scheme on the given above interval. We partition up our interval as $0 \leq x_0 < \cdots < x_n < T$ and corresponding weights $\{w_i\}_i^n$ meaning that

$$\int_0^T g(x)dx = \sum_{i=1}^{N} w_i g(x_i)$$

The Nystrom method for discretizing the above integral equation constructs a linear system that relates a given data vector $\vec{f} = \{f_i\}_i^N$, where $f_i = f(x_i)$ to a unknown solution vector $\vec{\sigma} = \{\sigma_i\}_i = 1, \ldots, N$

We then simply consider the integral equations for each point i:

$$\sigma(x_i) + \int_0^T k(x_i, x')\sigma(x')dx' = f(x_i) \quad i = 1, \ldots, N$$

We can then approximate the integral by the employing any appropriate quadrature rule by

$$\int_0^T k(x_i, x')\sigma(x')dx' \approx \sum_{j=1} a_{i,j}\sigma(x_j)$$

where if the kernel $k(x, x')$ is smooth then the matrix elements obtained from the above simply become $a_{i,j} = w_j k(x_i, x_j)$. This leads to the linear system

$$\boldsymbol{\sigma} + A\boldsymbol{\sigma} = \boldsymbol{f}$$

However, if the kernel is not smooth, as is often the case, it is not obvious how to construct the matrix A such that it holds up to high accuracy. This relies on modification of existing quadrature techniques to be able to handle functions with singularities in them.

Kapur-Rokhlin

We will assume that a singularity exists for the kernel $k(x, x')$ which is periodic and occurs at the endpoints 0 and T. The Kapur-Rohklin scheme is able to handle only singularities which posses logarithmic type singularities.
The mth-rder Kapur-Rohklin quaadrature rule T_m^{N+1} which corrects for a log singularity at both left and right endpoints is

$$T_m^{N+1} = h\left[\left(\sum_{l=-m}^{m}\gamma_l g(lh)\right) + g(h) + \cdots + g(T-h) + \left(\sum_{l=-m}^{m}\gamma_{-l} g(T+lh)\right)\right]$$

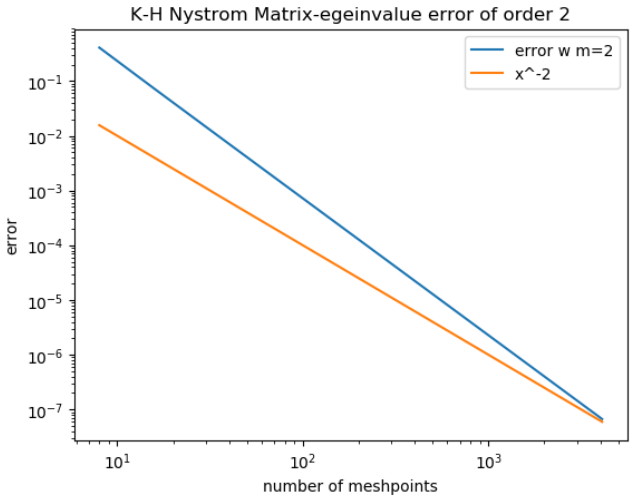
Which can be rewritten slightly differently

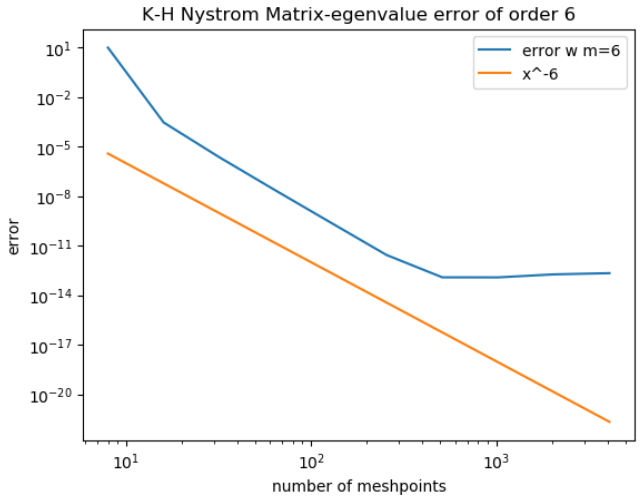
$$T_m^{N+1} = h\left[\sum_{l=1}^{m}(\gamma_l + \gamma - l)g(hl) + g(h) + \cdots + g(T-h) + \sum_{l=-m}^{-1}(\gamma_l + \gamma_{-l})g(T+hl)\right]$$

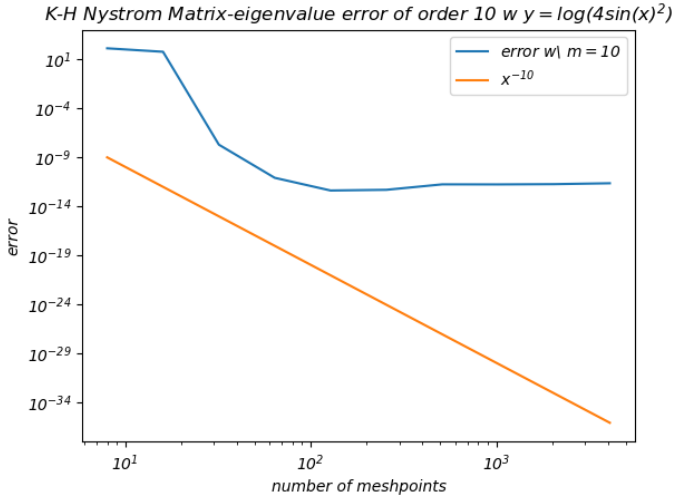
In terms of the kernel integral we create a shift from our original interval to the following:

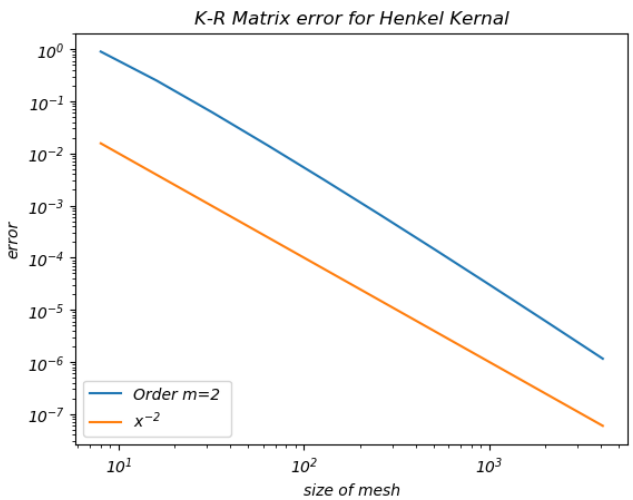
$$\int_0^T k(x_i, x')\sigma(x')dx' = \int_{x_i}^{x_i+T} k(x_i, x')\sigma x' dx' \approx h\sum_{j=1+i}^{i+N-1} k(x_i, x_j)\sigma(x_j) + h\sum_{l=-m}^{m}(\gamma_l + \gamma_{-l}k(x_i, x_j)\sigma(x_i + l))$$

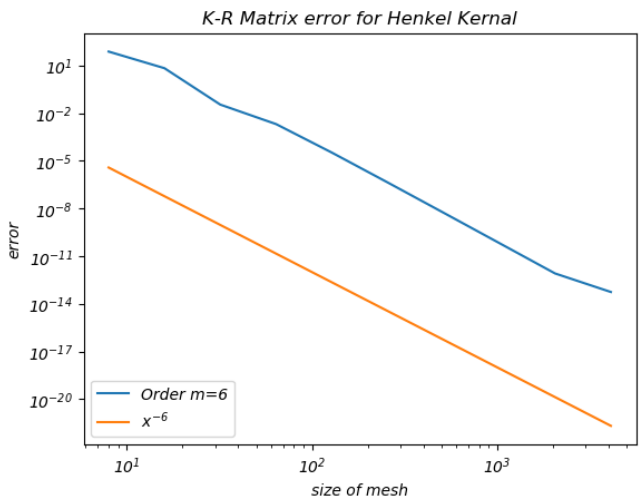
$$a_{i,j} = \begin{cases} 0 & \text{if } i = j \\ hk(x_i, x_j) & \text{if } x_i \text{ and } x_j \text{ are well seperated} \\ h(1 + \gamma_{l(i,j)} + \gamma_{-l(i,j)})k(x_i, x_j) & \text{if } x_i \text{ and } x_j \text{ are close and } x_i \neq x_j \end{cases} \quad (1)$$

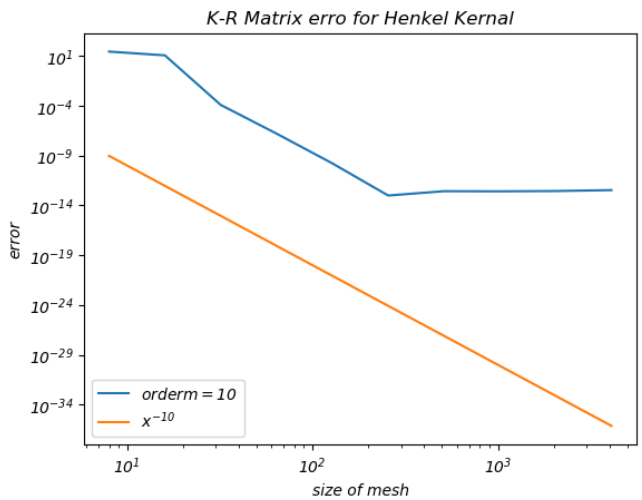

K-H Nystrom Matrix-egeinvalue error of order 2
error w m=2
x^-2
error
number of meshpoints


K-H Nystrom Matrix-egenvalue error of order 6
error w m=6
x^-6
error
number of meshpoints


K-H Nystrom Matrix-eigenvalue error of order 10 w y = log(4sin(x)^2)
error w\ m = 10
x^-10
error
number of meshpoints


K-R Matrix error for Henkel Kernal
Order m=2
x^-2
error
size of mesh


K-R Matrix error for Henkel Kernal
Order m=6
x^-6
error
size of mesh


K-R Matrix erro for Henkel Kernal
orderm = 10
x^-10
error
size of mesh

Alpert Quadrature

With the same setup as the with the Kapur-Rokhlin scheme, the quadrature for this method is given as follows:

$$S^N_{\chi m} = h\sum_{p=1}^{m} w_p g(\chi_p)h + h\sum_{j=0}^{N-2a} g(ah+jh) + h\sum_{p=1}^{m} g(b-\chi_p h)$$

The above rule is based on the Trapezoidal rule, in which a small number of nodes and weights at the endpoints of the integration interval are modified. The nodes and weights for this method needs to be given and are independent of interval of integration. These nodes and weights should work for any T periodic, endpoint singular function. Like with the R-K kernel integral, we compute the integral point by point with this new quadrature.

$$\int_0^T k(x_i,x')dx' \approx h\sum_{p=0}^{N-2a} k(x_i,x_i+ah+ph)\sigma(x_i+ah+ph) + h\sum_{p=1}^{m} w_p k(x_i,x_i+\chi_p h)\sigma(x_i+\chi_p h)$$
$$+ h\sum_{p=1}^{m} w_p k(x_i,x_i+T-\chi_p h)\sigma(x_i+T-\chi_p h)$$

We must address one issue with this sort of kernel evaluation. The, the endpoint correction nodes χ_p are usually not integers, meaning that we would be evaluating σ outside of our equispaced mesh. This posses a problem for when we want to form a linear system.

Alpert Quadrature

To remedy this, we interpolate the points off from σ.

$$\sigma(x) = \sum_{q=0}^{m=1} L_q^{(x_i)}(x)\sigma(x_i + qh) \qquad \text{where} \qquad L_q^{(x_i)}(x) = \prod_{r=0} \frac{x - (x_i + rh)}{(x_i + qh) - (x_i + rh)}$$

Meaning that we can now shift the evaluation of from sigma and on to the Lagrange basis to form the linear system as we would like.

$$\sigma(x_i + \chi_p h) \approx \sum_{q=0}^{m+3} L_q^{(x_i)}(x_i + chi_p h)\sigma(x_i + qh) \tag{2}$$

$$\sigma(x_i + T - \chi_p h) \approx \sum_{q=0}^{m+3} L_q^{(x_i)}(x_i + T - chi_p h)\sigma(x_i + T - qh) \tag{3}$$

Alpert Quadrature

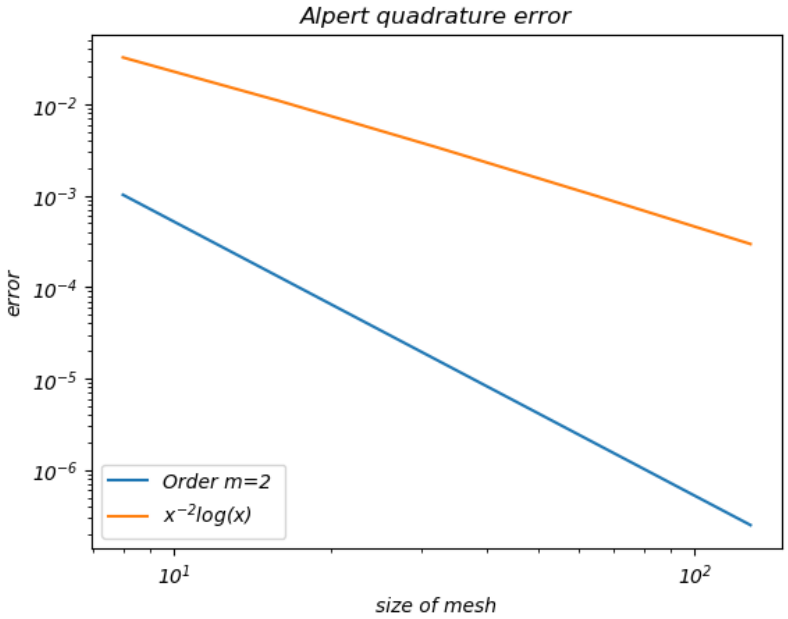
$$\begin{aligned}\int_0^T k(x_i, x')\sigma(x')dx' \approx\ & h\sum_{p=0}^{m+3} k(x_i, x_i + ah + ph)\sigma(x_i + ah + ph) \\ & + h\sum_{q=0}^{m+3}\left(\sum_{p=1} w_p k(x_i, x_i + \chi_p h)L_q^{(x_i)}(x_i + \chi_p)\right)\sigma(x_i + qh) \\ & + h\sum_{q=0}^{m+3}\left(\sum_{p=1} w_p k(x_i, x_i + T - \chi_p)L_q^{(x_i+T)}(x_i + T - \chi_p h)\right)\sigma(x_i + T - qh)\end{aligned}$$

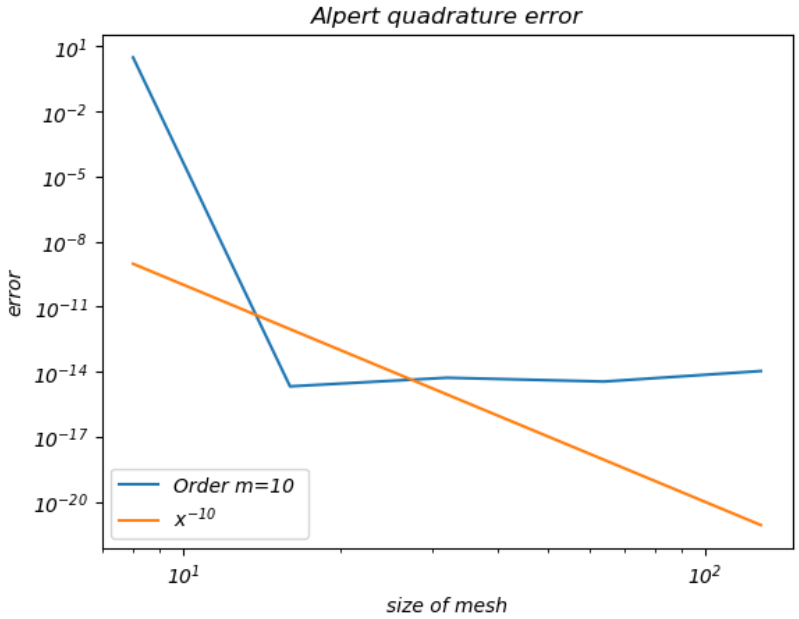
Based on this last equation we decompose the matrix as follows:

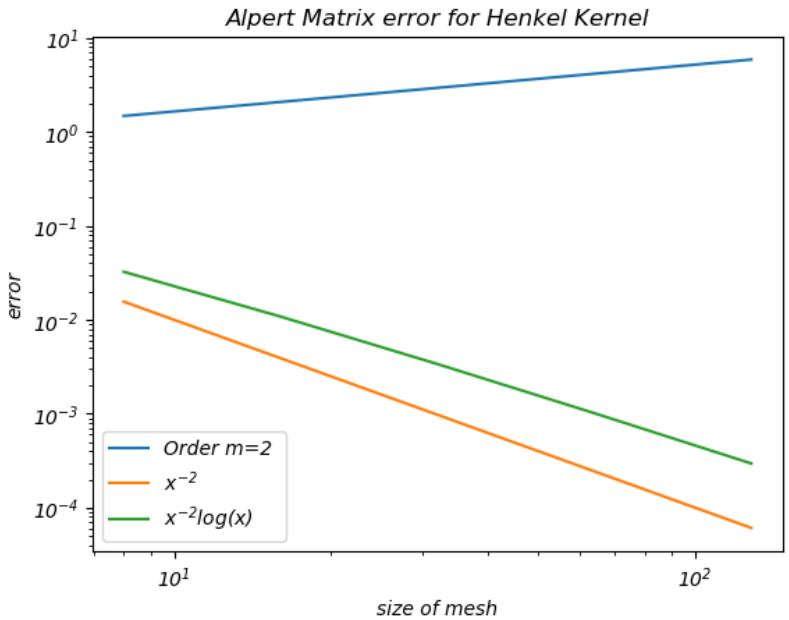
$$b_{i,j} = \begin{cases} 0 & \text{if } |I(i,j)| < a \\ hk(x_i, x_j) & \text{if } |I(i,j)| \geq a \end{cases}$$

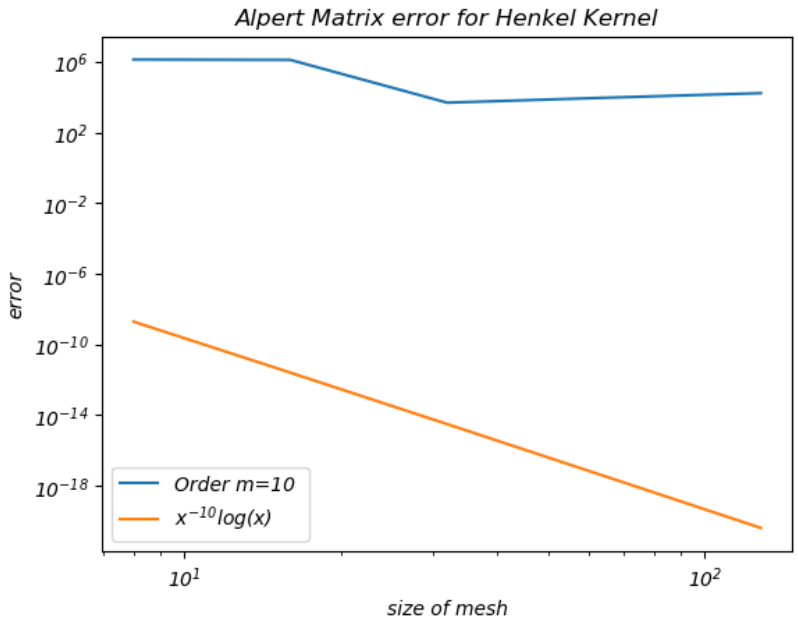
$$c_{i,j} = \begin{cases} 0 & \text{if } |I(i,j)| > m+3 \\ h\sum_{p=1} w_p \left(k(x_i, x_i + \chi_p h)L_{I(i,j)}^{(x_i)} + k(x_i, x_i + T - \chi_p h)L_{I(i,j)}^{(x_i+T)}(x_i + T - \chi_p h)\right) & \text{if } |I(i,j)| \leq m+3 \end{cases}$$

Finally making $a_{i,j} = b_{i,j} + c_{i,j}$ finally yielding the linear system matrix.

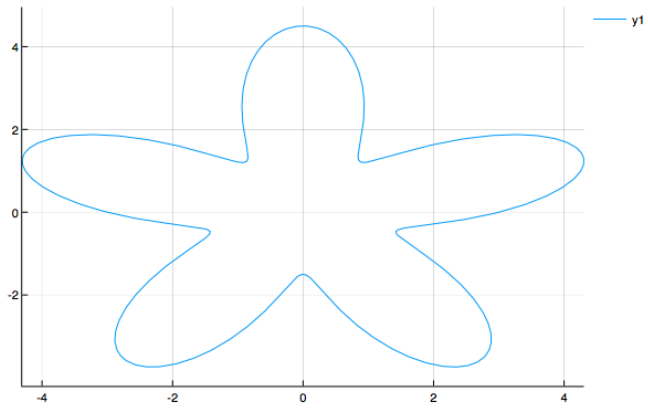

Alpert quadrature error
error
size of mesh
Order m=2
$x^{-2}log(x)$


Alpert quadrature error
error
size of mesh
Order m=10
x^{-10}


Alpert Matrix error for Henkel Kernel
error
size of mesh
Order m=2
x^{-2}
$x^{-2}log(x)$


Alpert Matrix error for Henkel Kernel
error
size of mesh
Order m=10
$x^{-10}log(x)$

So why do we bother with such high accuracy methods?



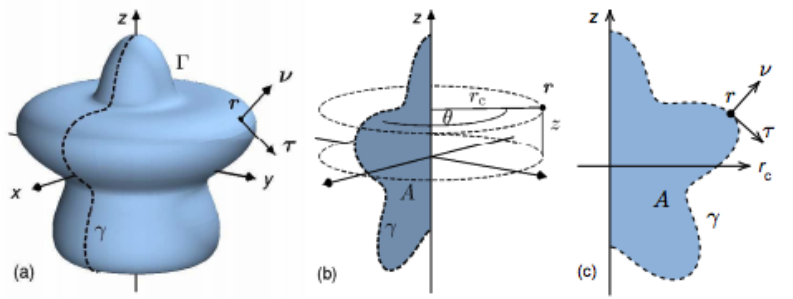


Figure 1: An axially symmetric surface Γ generated by a curve γ. (a) A point $\boldsymbol{r}$ on Γ has outward unit normal $\boldsymbol{\nu}$ and tangent vector $\boldsymbol{\tau}$. (b) $\boldsymbol{r}$ has radial distance r_c, azimuthal angle θ, and height z. The two-dimensional domain A is bounded by γ and the z-axis. (c) Coordinate axes and vectors in the half-plane $\theta = 0$.

1. Fix Alpert Nystrom Method.
2. Generalize the Boundary.
3. Boundary in higher dimensions.
4. Higher order methods.