DISCRETE STRUCTURES, SPRING 2017 - PROBLEM SET 4

Name: ____

Use this worksheet as the cover sheet for your write-up: write your name on this page, and staple this sheet to the front of your homework packet.

You will receive no credit for submitting solutions that the grader cannot read and understand—be sure to write legibly!

NOTE: Before going into the actual homework assignment for this week, we will all need to agree on how to show specific mathematical expressions are injections or bijections. Below is an example which you should serve as a guide, but argumentation should NOT be copied verbatim to the problems below. This is only meant to serve as a guide.

Example: Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is injective and surjective. What does that imply about the function?

Solution:
Injective: Let $a, b \in \mathbb{R}$. We will suppose that

$$\begin{aligned} f(a) &= f(b) \\ 2(a) + 1 &= 2(b) + 1 \\ 2a &= 2b \\ a &= b \end{aligned}$$

This shows that the function $f(x)$ is injective.

Surjective:
To prove this, we let $y \in \mathbb{R}$. We need to find an $x \in R$, such that $f(x) = y$, per the definition of injectivity. I claim that $x = \frac{y-1}{2}$ satisfies this. We now check that this is indeed the case:

$$\begin{aligned} f(x) &= f\left(\frac{y-1}{2}\right) \\ &= 2\left[\frac{y-1}{2}\right] + 1 \\ &= y - 1 + 1 \\ &= y \end{aligned}$$

Now, what do we know about functions that are both surjective and injective? I leave this for you to think about as you will need it for the questions below.

Problem 1. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Give an example of a function $f : A \to B$ that is neither injective nor surjective.

Problem 2. Define $f : \mathbb{N} \to \mathbb{Z}$ by the following equation:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{1-n}{2} & \text{if n is odd} \end{cases}$$

(1) Show that f is a bijection.
(2) Find a formula for the inverse f^{-1}.

Date: Friday March 24th, 2017.

Problem 3. Answer the following:

(1) Why is f in each of the cases below not a functions from $\mathbb{R}$ to $\mathbb{R}$
  (a) $f(x) = \frac{1}{x}$?
  (b) $f(x) = \frac{1}{\sqrt{x}}$?
  (c) $\pm\sqrt{x^2+1}$?

(2) What would need to be done to make each of the above a function?

Problem 4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as

$$f(x) = \begin{cases} x+3 & \text{if x is odd} \\ x-3 & \text{if x is even} \end{cases}$$

(1) Show that this function is a bijection.
(2) Is it possible to find f^{-1}? Find it if you think it is.

Problem 5. Define a function $g : \mathbb{Z} \to \mathbb{N}$ defined by $g(z) = z^2 + 1$.

(1) Prove that g is not injective.
(2) Prove that g is not surjective.
(3) What does this imply about f^{-1}?

Problem 6 (Bonus)**.** If we have a function $f : A \to B$ with $S, T \subseteq A$, prove that $f(S \cup T) = f(S) \cup f(T)$.