The Integral equations of Sound-Soft and Sound-Hard Scattering

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Outline

The goal of this talk is to present methods that can solve the exterior Helmholtz equation:

$$\begin{cases} \Delta u + k^2 u = 0 & x \in \mathbb{R}^2/D \\ u = f & x \in \partial D = \Gamma \\ \lim_{|x|\to\infty} \sqrt{x}\left(\frac{\partial}{\partial x} - ik\right) u = 0. & \end{cases}$$

for some arbitrary, possibly non-smooth, domain D. From potential theory, we are able to represent the solution of the above problem in terms of finding an unknown density:

$$u(x) = \int_\Gamma G_k(x-y)\phi(y)dS_y = \int_\Gamma \frac{i}{4} H_0^{(1)}(k|x-y|)\phi(y)dS_y \quad x \in \mathbb{R}^2/D$$

$$u(x) = \int_\Gamma \frac{\partial G_k(x-y)}{\partial n_y}\psi(y)dS_y = \int_\Gamma \frac{ik}{4} H_1^{(1)}(k|x-y|)\frac{\langle (x-y), n_y\rangle}{|x-y|}\psi(y)dS_y \quad x \in \mathbb{R}^2/D$$

We therefore solve a Boundary Integral Equation (BIE) of the Single-Layer (SL) or Double-Layer (DL):

$$\int_\Gamma G_k(x-y)\phi(x)dS_y = f(x) \quad x \in \Gamma$$

$$\frac{\psi(x)}{2} + \int_\Gamma \frac{\partial G_k(x-y)}{\partial n_y}\psi(y)dS_y = f(x) \quad x \in \Gamma.$$

Given a integral equation

$$b\sigma(\boldsymbol{x}) + \int_\Gamma K(\boldsymbol{x},\boldsymbol{x}')\sigma(\boldsymbol{x}')dl(\boldsymbol{x}') = f(\boldsymbol{x}) \quad \in \Gamma,$$

we start with a set of nodes $\{x_i\}_{i=1}^N$ such that $0 \le x_1 \le \cdots \le x_N < T$ and construct a linear system that related the given data vector $\boldsymbol{f} = \{f_i\}_{i=1}^N$, where $f(x_i) = f_i$, to $\{\sigma_i\}_{i=1}^N$ where $\sigma_i \approx \sigma(x_i)$. We then use the nodes $\{x_i\}_{i=1}^N$ as collocation points where we enforce:

$$b\sigma(x_i) + \int_0^T K(x_i,x')\sigma(x')dx' = f(x_i) \quad i = 1,\dots,N.$$

We can then approximate the integral if we have appropriate quadrature rule designed to handle the kernel with singularities

$$\int_0^T K(x_i,x')\sigma(x')dx' \approx \sum_{j=1} a_{i,j}\sigma(x_j),$$

from which we will get the following linear system:

$$b\sigma_i + \sum_{j=1}^N a_{i,j}\sigma_j = f_i \quad i = 1,\dots,N$$

$$b\boldsymbol{\sigma} + A\boldsymbol{\sigma} = \boldsymbol{f}$$

To determine the appropriate quadrature rule, we observe the behavior of the free space solution:

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x) \quad H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x)$$

$$J_\alpha(x) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^\alpha \quad \alpha \geq 0 \quad x \to 0^+$$

$$Y_\alpha(x) \sim \frac{1}{\pi}\left(\frac{z}{2}\right)^{-\alpha}\sum_{k=0}^{\alpha-1}\frac{(\alpha-k-1)!}{k!}\left(\frac{x^2}{4}\right)^k + \frac{2}{\pi}J_\alpha(x)\ln\left(\frac{x}{2}\right) \quad x \to 0^+$$

Therefore, to handle scattering problems in both the Single-Layer and Double-Layer case we need quadrature rules that can integrate log-like kernels.

Suppose, a parameterization $x(t)$, we can take advantage of the asymptotic expansions of the Hankel function to write the single layer kernel

$$M_k(t,\tau) = H_0^{(1)}(k|t-\tau|)$$

in its following split form:

$$\begin{aligned} M_k(t,\tau) &= M_{k,1}(t,\tau)\ln\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) + M_{k,2}(t,\tau) \\ M_{k,1}(t,\tau) &= \frac{-1}{4\pi}J_0(k|t-\tau|) \\ M_{k,2}(t,\tau) &= M_k(t,\tau) - M_{k,1}(t,\tau)\ln\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) \end{aligned}$$

with the following limiting values

$$M_{k,1}(t,t) = \frac{-1}{4\pi} \qquad M_{k,2}(t,t) = \frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\ln\left(\frac{k|x'(t)|}{2}\right).$$

We can then build our quadrature Matrix and linear system through the following

$$\begin{aligned} t_k &= \frac{\pi k}{n} \\ S(i,j) &= R^{(n)}_{|i-k|}M_{k,1}(t_i,t_k) + \frac{\pi}{n}M_{k,2}(t_i,t_k) \qquad i,j = 0,\cdots,2n-1. \\ R_k^{(n)} &= -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\left(\frac{mk\pi}{n} - \frac{(-1)^k\pi}{n^2}\right). \end{aligned}$$

Kapur-Rokhlin Quadrature

We will assume that a logarithmic singularity exists for the kernel $k(x, x')$ when $x = x'$. The mth-rder Kapur-Rohklin quadrature rule T_m^{N+1} which corrects for a log singularity at both left and right endpoints is

$$\int_0^T g(x)dx \approx T_m^{N+1} + O(h^m);$$

where

$$\begin{aligned} T_m^{N+1} &= h\left(g(h) + \cdots + g(T-h)\right) \\ &+ \left(\sum_{l=1}^{m} (\gamma_l + \gamma_{-l}) g(lh)\right) + \left(\sum_{l=-m}^{-1} (\gamma_l + \gamma_{-l}) g(T + lh)\right) \end{aligned}$$

In terms of the kernel integral we create a shift from our original interval to the following:

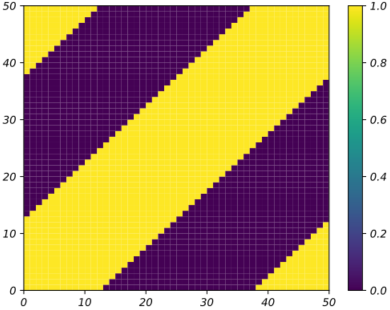
$$\begin{aligned} \int_0^T k(x_i, x')\sigma(x')dx' &= \int_{x_i}^{x_i+T} k(x_i, x')\sigma(x')dx' \\ &\approx h \sum_{j=1+i}^{i+N-1} k(x_i, x_j)\sigma(x_j) + h \sum_{\substack{l=-m \\ l\neq 0}}^{m} (\gamma_l + \gamma_{-l}) k(x_i, x_j)\sigma(x_{i+l}) \end{aligned}$$

$$a_{i,j} = \begin{cases} 0 & \text{if } i = j \\ hk(x_i, x_j) & \text{if } x_i \text{ and } x_j \text{ are well seperated} \\ h(1 + \gamma_{l(i,j)} + \gamma_{-l(i,j)})k(x_i, x_j) & \text{if } x_i \text{ and } x_j \text{ are close and } x_i \neq x_j \end{cases}$$

where the notion of "seperateness" is given by the function

$$l(i,j) = (j-1)(\text{mod N}),$$

which visually does the following



Alpert Quadrature

With the same setup as the with the Kapur-Rokhlin scheme, the quadrature for this method is given as follows:

$$S^N_{\chi^m} = h\sum_{p=1}^{m} w_p g(\chi_p)h + h\sum_{j=0}^{N-2a} g(ah+jh) + h\sum_{p=1}^{m} w_p g(b-\chi_p h)$$

The above rule is based on the Trapezoidal rule, in which a small number of nodes and weights at the endpoints of the integration interval are modified. These nodes and weights work for any T periodic, log-singular function.

$$\begin{aligned}\int_0^T k(x_i,x')dx' \approx h&\sum_{p=0}^{N-2a} k(x_i,x_i+ah+ph)\sigma(x_i+ah+ph)\\ &+ h\sum_{p=1}^{m} w_p k(x_i,x_i+\chi_p h)\sigma(x_i+\chi_p h)\\ &+ h\sum_{p=1}^{m} w_p k(x_i,x_i+T-\chi_p h)\sigma(x_i+T-\chi_p h)\end{aligned}$$

The endpoint correction nodes χ_p are usually not integers, meaning that we would be evaluating σ outside of our equispaced mesh.

Alpert Quadrature

To remedy this, we interpolate the points off from σ.

$$\sigma(x) = \sum_{q=0}^{m=1} L_q^{(x_i)}(x)\sigma(x_i + qh) \qquad \text{where} \qquad L_q^{(x_i)}(x) = \prod_{r=0} \frac{x - (x_i + rh)}{(x_i + qh) - (x_i + rh)}$$

Meaning that we can now shift the evaluation from σ on to the Lagrange basis to form the Nyström linear system:

$$\sigma(x_i + \chi_p h) \approx \sum_{q=0}^{m+3} L_q^{(x_i)}(x_i + \chi_p h)\sigma(x_i + qh)$$

$$\sigma(x_i + T - \chi_p h) \approx \sum_{q=0}^{m+3} L_q^{(x_i)}(x_i + T - \chi_p h)\sigma(x_i + T - qh)$$

Alpert Quadrature

$$
\begin{aligned}
\int_0^T k(x_i, x')\sigma(x')dx' \approx\ & h\sum_{p=0}^{m+3} k(x_i, x_i + ah + ph)\sigma(x_i + ah + ph) \\
& + h\sum_{q=0}^{m+3}\left(\sum_{p=1} w_p k(x_i, x_i + \chi_p h)L_q^{(x_i)}(x_i + \chi_p)\right)\sigma(x_i + qh) \\
& + h\sum_{q=0}^{m+3}\left(\sum_{p=1} w_p k(x_i, x_i + T - \chi_p)L_q^{(x_i+T)}(x_i + T - \chi_p h)\right)\sigma(x_i + T - qh)
\end{aligned}
$$

Based on this last equation we decompose the matrix as follows:

$$
b_{i,j} = \begin{cases} 0 & \text{if } |l(i,j)| < a \\ hk(x_i, x_j) & \text{if } |l(i,j)| \geq a \end{cases}
$$

$$
c_{i,j} = \begin{cases} 0 & \text{if } |l(i,j)| > m+3 \\ h\sum_{p=1} w_p \left(k(x_i, x_i + \chi_p h)L_{l(i,j)}^{(x_i)}(x_i + \chi_p h) + k(x_i, x_i + T - \chi_p h)L_{l(i,j)}^{(x_i+T)}(x_i + T - \chi_p h)\right) & \text{if } |l(i,j)| \leq m+3 \end{cases}
$$

$$
l(i,j) = (j-1)(\text{mod N}).
$$

Making $a_{i,j} = b_{i,j} + c_{i,j}$ yields the linear system matrix.

Fourier Alpert

$$\int_0^T k(x_i, x')\sigma(x')dx' \approx h\sum_{p=1}^{N-a} k(x_i, x_i + ph)\sigma(x') + h\sum_{p=1}^{2m} k(x_i, x_i + \chi_p h)\sigma(x_i + \chi_p h)$$

The first sum on the right is on our grid, so that part is to be left alone. If we look at the last sum and suppose that

$$\sigma(x) \approx \frac{1}{2\pi}\sum_{n=-\frac{N}{2}+1}^{N/2} \hat{\sigma}_n e^{inx}$$

where

$$\begin{bmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \vdots \\ \hat{\sigma}_n \end{bmatrix} = DFT \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}$$

$$\sigma(x_i + \chi_p h) \approx \frac{1}{2\pi}\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} \left(\sum_{j=1}^{N} DFT_{nj}\sigma_j\right) e^{inx_i} e^{in\chi_p h}$$

Throughout the rest of this presentation we will consider the following three representations of the solution of the Helmholtz equation:

$$u(x) = \int_\Gamma G_k(x-y)\phi(y)dS_y = \int_\Gamma \frac{i}{4}H_0^{(1)}(|x-y|)\phi(y)dS_y$$

$$u(x) = \int_\Gamma \frac{\partial G_k(x-y)}{\partial n_y}\phi(y)dS_y = \int_\Gamma \frac{ik}{4}H_1^{(1)}(|x-y|)\frac{\langle x-y, n_y\rangle}{|x-y|}\phi(y)dS_y$$

$$u(x) = \int_\Gamma \left(\frac{\partial G_k(x-y)}{\partial n_y} - ikG_k(x-y)\right)\phi(y)dS_y$$

which are respectively the Single-Layer, Double-Layer, and Combined Field Integral Equation. We test the above quadrature rules with above representations and the following problem:

$$\begin{cases} \Delta u + k^2 u = 0 \quad \text{for } x \in \mathbb{R}^2/D \\ u = \frac{i}{4}H_0^{(1)}(k|x-x_0|) \quad \text{for } x \in \Gamma, x_0 \in D \\ u \text{ radiative} \end{cases}$$

On the boundary the integral equations we have to solve are:

$$\int_\Gamma G_k(x-y)\phi(y)dS_y = \frac{i}{4}H_0^{(1)}(|x-x_0|)$$

$$\frac{\psi(x)}{2} + \int_\Gamma \frac{\partial G_k(x-y)}{\partial n_y}\psi(y)dS_y = \frac{i}{4}H_0^{(1)}(|x-x_0|)$$

$$\frac{\rho(x)}{2} + \int_\Gamma \left(\frac{\partial G_k(x-y)}{\partial n_y} - ikG_k(x-y)\right)\rho(y)dS_y = \frac{i}{4}H_0^{(1)}(|x-x_0|)$$

Respectively for each of the layer representations.

When we have solved for ϕ , we can numerically solve for the exterior everywhere by

$$u_{appr}(z) \approx h \sum_{p=1}^{N} G_k(z, y_p)\phi_p.$$

or

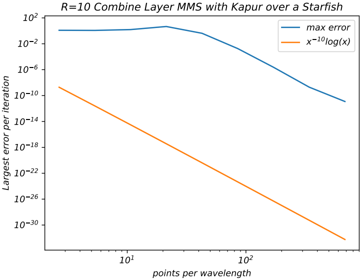
$$u_{appr}(z) \approx h \sum_{p=1}^{N} \frac{\partial G_k(z, y_p)}{\partial n_{y_p}} \phi_p.$$

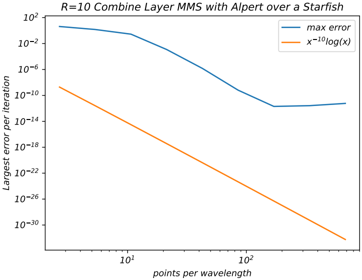
with actual solution being

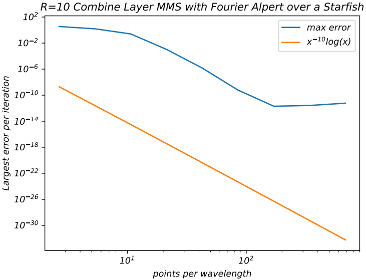
$$\frac{i}{4} H_0^{(1)}(|x - x_0|).$$

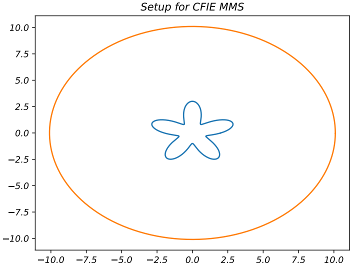
We test this on sample points $\vec{z}_j = 10(\cos\theta_j, \sin\theta_j)$, thereby giving

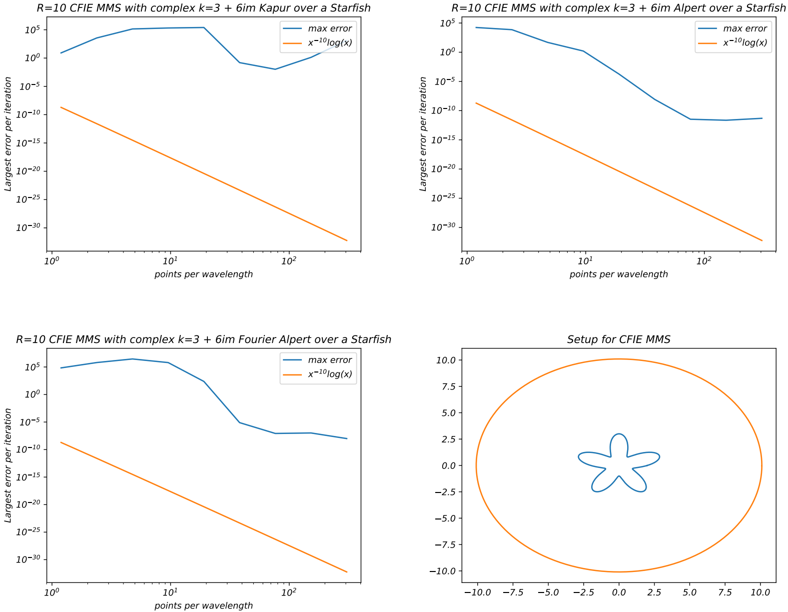
$$error = max_{z_j} \left| h \sum_{p=1}^{N} G_k(z_j, y_p)\phi_p - \frac{i}{4} H_0^{(1)}(|z_j - x_0|) \right| .$$


R=10 Combine Layer MMS with Kapur over a Starfish
max error
$x^{-10}log(x)$
Largest error per iteration
points per wavelength


R=10 Combine Layer MMS with Alpert over a Starfish
max error
$x^{-10}log(x)$
Largest error per iteration
points per wavelength


R=10 Combine Layer MMS with Fourier Alpert over a Starfish
max error
$x^{-10}log(x)$
Largest error per iteration
points per wavelength


Setup for CFIE MMS


R=10 CFIE MMS with complex k=3 + 6im Kapur over a Starfish
R=10 CFIE MMS with complex k=3 + 6im Alpert over a Starfish
R=10 CFIE MMS with complex k=3 + 6im Fourier Alpert over a Starfish
Setup for CFIE MMS
max error
$x^{-10}log(x)$
Largest error per iteration
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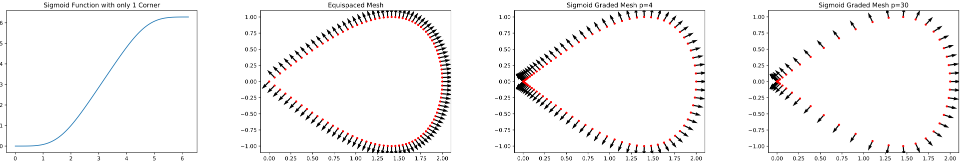
- Domain D with corners at $x_1, \cdots, x_p$ with respective angles, measurered from the intrior of the domain, is $\gamma_1, \cdots, \gamma_p$
- Assume also that the the curve is piecewise smooth between each of the corners.
- We let $(x_1(t), x_2(t))$ be a 2π-parametrization of the domain with a corner occuring at $t = T_j$.

We introduce the sigmoid transform:

$$w(s) = \frac{T_{j+1}[v(s)]^p + T_j[1 - v(s)]^p}{[v(s)]^p + [1 - v(s)]^p} \quad T_j \leq s \leq T_{j+1}$$

$$v(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{T_j + T_{j+1} - 2s}{T_{j+1} - T_j}\right)^3 + \frac{1}{p}\left(\frac{2s - T_j - T_{j+1}}{T_{j+1} - T_j}\right) + \frac{1}{2}$$



We then aim to solve an integral equation that has the representation

$$\int_\Gamma K(x(t_w) - x(\tau_w))\phi(x(\tau_w))|x'(w(\tau))|w'(\tau)d\tau = f(t) \quad t \in \Gamma$$

where $x(t_w) = x(w(t))$. We solve for a weighted density by multiplying $w'(t)$ in the equation above:

$$\int_\Gamma K(x(t_w) - x(\tau_w))\phi(x(\tau_w))|x'(w(\tau))|w'(\tau)w'(t)d\tau = w'(t)f(t) \quad t \in \Gamma.$$

We now let $w'(t)\phi(t_w)) = \phi^w(t)$, $w'(\tau)\phi(w(\tau)) = \phi^w(\tau)$, and $w'(t)f(w(t)) = f^w(t)$. The Single-Layer integral equations can then be written in the following form

$$\int_\Gamma G_k(x(t_w) - x(\tau_w))\phi^w(\tau))|x'(w(\tau))|w'(t)d\tau = f^w(t)$$

Where we solve for the weighted density ϕ^w by a Nyström Method with any of the quadrature methods we have explored previously.

For the Double layer formulation we need to pay attention to points near the corner. Letting $x(t)$ and $x(\tau)$ approach the corner from two opposite directions leads to a singularity of $|x(t)-x(\tau)|^{-1}$. We add and substract Double-Layer Laplacian:

$$\begin{aligned} D_k\phi(t) &= \int_0^{2\pi} \frac{\partial G_k(x(t_w)-x(\tau_w))}{\partial n_\tau}\phi^w(\tau)|x'(\tau)|w'(t)d\tau \\ &= \int_0^{2\pi} \frac{\partial (G_k(x(t_w)-x(\tau_w)) - G_0(x(t_w)-x(\tau_w)))}{\partial n_\tau}\phi^w(\tau)|x'(\tau)|w'(t)d\tau \\ &+ \phi^w(t)\int_0^{2\pi} \frac{\partial G_0(x(t_w)-x(\tau_w))}{\partial n_\tau}|x'(\tau)|w'(t)d\tau \\ &+ \int_0^{2\pi} \frac{\partial G_0(x(t_w)-x(\tau_w))}{\partial n_\tau}(\phi^w(\tau)-\phi^w(t))|x'(\tau)|w'(t)d\tau, \end{aligned}$$

where we have that

$$G_0(x) = \frac{-1}{2\pi}\log(|x|)$$
$$\frac{\partial G_0(x)}{\partial n} = \frac{1}{2\pi}\frac{\langle x, n\rangle}{|x|^2}.$$

Since,

$$\int_0^{2\pi} \frac{\partial G_0(x(t_w) - x(\tau_w))}{\partial n_\tau}|x'(\tau)|w'(\tau)d\tau = \begin{cases} \frac{-1}{2} & t \in [0, 2\pi]/\{T_q, \cdots, T_p\} \\ \frac{-\gamma_j}{2\pi} & t = T_j, 1 \leq j \leq P \end{cases}$$

on Γ the boundary integral equation simply becomes:

$$f^w(t) = \int_0^{2\pi} \frac{\partial(G_k(x(t_w) - x(\tau_w)) - G_0(x(t_w) - x(\tau_w)))}{\partial n_\tau}\phi^w(\tau)|x'(\tau)|w'(t)d\tau$$
$$+ \int_0^{2\pi} \frac{\partial G_0(x(t_w) - x(\tau_w))}{\partial n_\tau}(\phi^w(\tau) - \phi^w(t))|x'(\tau)|w'(t)d\tau.$$

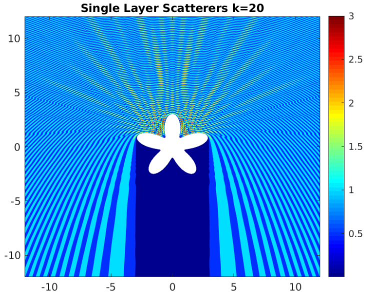
The second integral above is regular enough that we can approximate it by an trapezoidal rule.

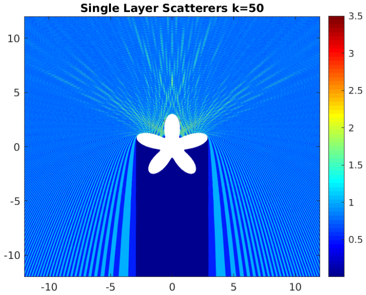
Scattering

To put these techniques to use, we now try to solve a simple scattering problem:

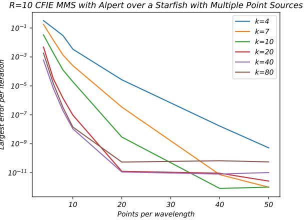
$$\begin{cases} \Delta u^{scat} + k^2 u^{scat} = 0 & x \in \mathbb{R}^2/D \\ u^{scat} = -u^{inc} & x \in \Gamma \\ u^{scat} \text{ radiative at infinity} \end{cases}$$

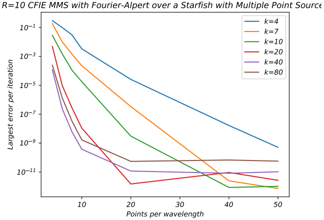
Where the incoming plane wave is charecterized by $u^{inc}(x) = e^{ik\langle x,d\rangle}$ and $\|d\| = 1$ unit vector describing the angle of incident.





Scattering


R=10 CFIE MMS with Alpert over a Starfish with Multiple Point Sources
k=4
k=7
k=10
k=20
k=40
k=80
Points per wavelength


R=10 CFIE MMS with Fourier-Alpert over a Starfish with Multiple Point Source
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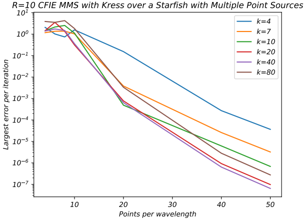
Largest error per iteration
Points per wavelength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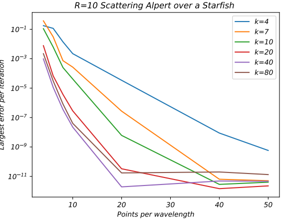
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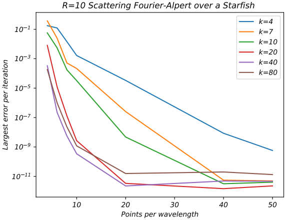
R=10 Scattering Fourier-Alpert over a Starfish
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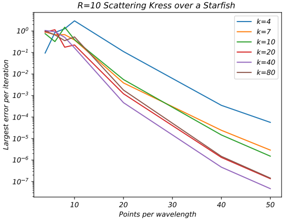
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k=4
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Largest error per iteration
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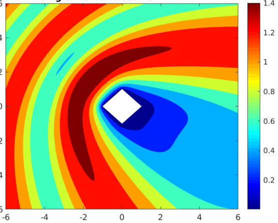
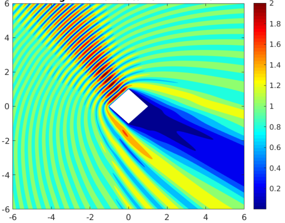
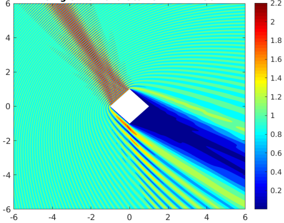
Single Diamond Scatter with k=1



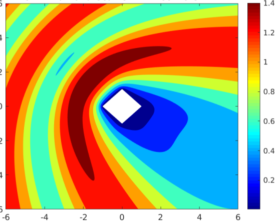
Single Diamond Scatter with k=10



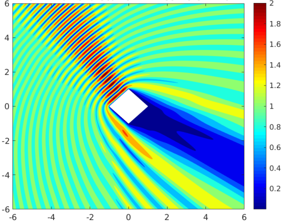
Single Diamond Scatter with k=30



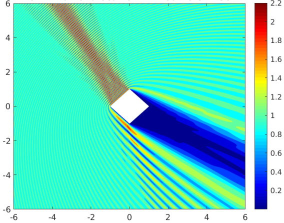
CFIE Diamond Scatter with Corners k=1



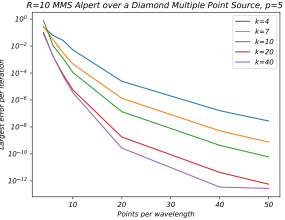
CFIE Diamond Scatter with Corners k=10

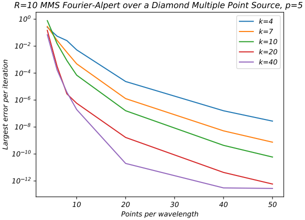


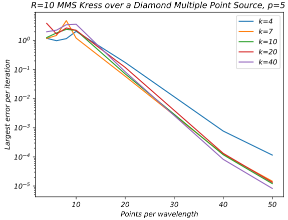
CFIE Diamond Scatter with Corners k=30

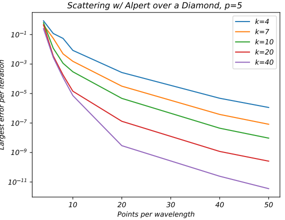


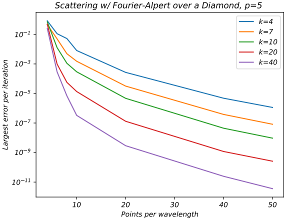
Scattering

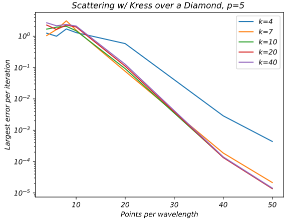

R=10 MMS Alpert over a Diamond Multiple Point Source, p=5
Largest error per iteration
Points per wavelength
k=4
k=7
k=10
k=20
k=40


R=10 MMS Fourier-Alpert over a Diamond Multiple Point Source, p=5
Largest error per iteration
Points per wavelength
k=4
k=7
k=10
k=20
k=40


R=10 MMS Kress over a Diamond Multiple Point Source, p=5
Largest error per iteration
Points per wavelength
k=4
k=7
k=10
k=20
k=40


Scattering w/ Alpert over a Diamond, p=5
Largest error per iteration
Points per wavelength
k=4
k=7
k=10
k=20
k=40


Scattering w/ Fourier-Alpert over a Diamond, p=5
Largest error per iteration
Points per wavelength
k=4
k=7
k=10
k=20
k=40


Scattering w/ Kress over a Diamond, p=5
Largest error per iteration
Points per wavelength
k=4
k=7
k=10
k=20
k=40

Suppose we have a multiply simply connected domain $D = D_1 \bigcup D_2$, and we seek an a solution:

$$u(x) = \int_{\Gamma} K(x,y)\phi(y)dS_y = \int_{\Gamma_1} K(x,y)\phi^1(y)dS_y + \int_{\Gamma_2} K(x,y)\phi^2(y)dS_y.$$

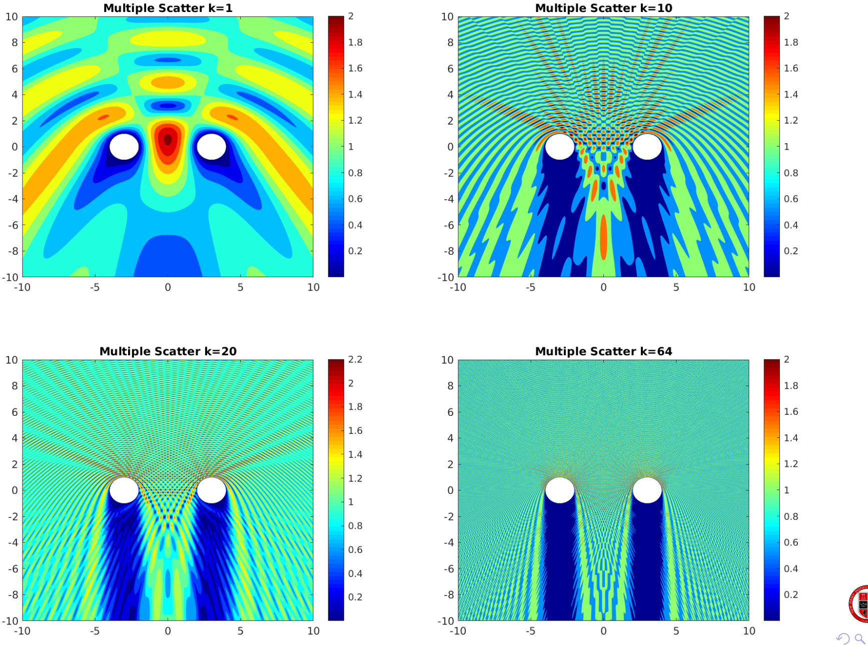
With an incoming plane wave $u^{inc}(x) = e^{ik\langle x,d\rangle}$ on the boundary Γ yields the following system that needs solving:

$$\begin{aligned} \int_{\Gamma_1} K(x,y)\phi^1(y)dS_y + \int_{\Gamma_2} K(x,y)\phi^2(y)dS_y &= -e^{ik\langle x,d\rangle} \quad x \in \Gamma_1 \\ \int_{\Gamma_1} K(x,y)\phi^1(y)dS_y + \int_{\Gamma_2} K(x,y)\phi^2(y)dS_y &= -e^{ik\langle x,d\rangle} \quad x \in \Gamma_2 \end{aligned}$$

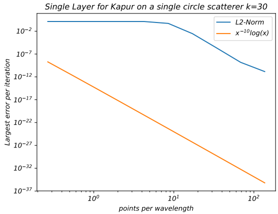
Which leads to the following system that needs solving:

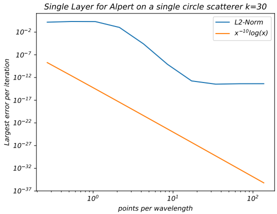
$$\begin{bmatrix} S_{\Gamma_1}^{x\in\Gamma_1} & S_{\Gamma_2}^{x\in\Gamma_1} \\ S_{\Gamma_1}^{x\in\Gamma_2} & S_{\Gamma_2}^{x\in\Gamma_2} \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} = \begin{bmatrix} -e^{ik\langle x^{\in\Gamma_1},d\rangle} \\ -e^{ik\langle x^{\in\Gamma_2},d\rangle} \end{bmatrix}.$$

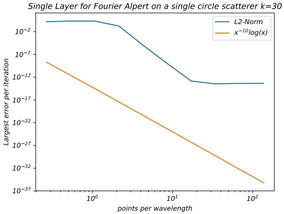
Notice that the off diagonal terms do not contain a singularity, so they are smooth and we can evaluate with simple trapezoidal rule.

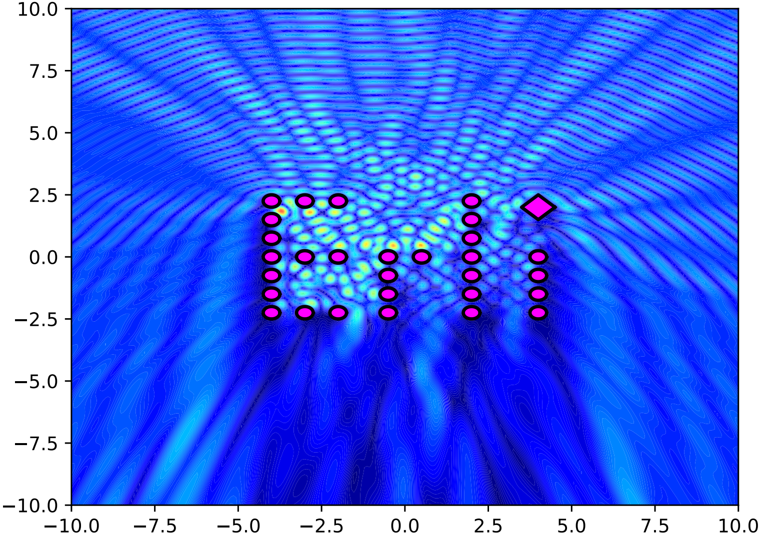

Multiple Scatter k=1
Multiple Scatter k=10
Multiple Scatter k=20
Multiple Scatter k=64
-10
-5
0
5
10
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
2.2

A property of the Single-Layer representation is that the total scattered field is zero on the interior. We can therefore gauge the error by measure how quickly the L_2 norm of the interior of a circle goes to zero,









10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & x \in \Omega \\ \frac{\partial u(x)}{\partial n} = g(x) & x \in \Gamma \end{cases} \tag{1}$$

If we suppose that the solution (1) takes the form of a SL representation

$$u(x) = \int_\Gamma G_k(|x-y|)\varphi_l(y)ds(y),$$

This layer potential leads to the following integral equation:·

$$2(K'\varphi)(x) - \varphi(x) = 2g(x) \quad x \in \Gamma, \tag{2}$$

where

$$(K'\varphi)(x) = \int_\Gamma \frac{\partial G_k(|x-y|)}{\partial n(x)}\varphi(y)ds(y),$$

This is nearly identical to the situation of the DL representation of the Dirichlet problem.

If we now suppose that the solution of (1) takes the form of the DL representation

$$u(x) = \int_\Gamma \frac{\partial G_k(|x-y|)}{\partial n(y)} \varphi(y) ds(y),$$

then we get the following integral equation that needs solving:

$$(N\varphi)(x) = g(x). \tag{3}$$

The operator is defined as:·

$$(N\varphi)(x) = p.v. \int_\Gamma \frac{\partial^2 G_k(|x-y|)}{\partial n(y)\partial n(x)} \varphi(y) ds(y). \tag{4}$$

This operator is hypersingular, meaning that we cannot make a Nyström discritization directly. This is integrable but only in the finite part sense. In addition the eigenvalues of this operator also accumulate at infinity.

We subtract the singularities:

$$(N\varphi)(x) = p.v. \int_\Gamma \frac{\partial^2 G_k(x-y)}{\partial n(y)\partial n(x)} \varphi(y) ds(y).$$

$$= \int_\Gamma \frac{\partial^2 (G_k - G_0)(x-y)}{\partial n(y)\partial n(x)} \varphi(y) ds(y) + p.v. \int_\Gamma \frac{\partial^2 G_0(x-y)}{\partial n(y)\partial n(x)} \varphi(y) ds(y)$$

G_0 is the fundamental solution for the Laplace equation:·

$$G_0(x) = -\frac{1}{2\pi} \log(|x|).$$

$$p.v. \int_\Gamma \frac{\partial^2 G_0(|x-y|)}{\partial n(y)\partial n(x)} \varphi(y) ds(y) = -p.v. \int_\Gamma \frac{\partial^2 G_0(|x-y|)}{\partial t(y)\partial t(x)} \varphi(y) ds(y)$$

$$= +\frac{1}{4\pi |x'(t)|} \int_0^{2\pi} \frac{\partial^2}{\partial t \partial \tau} \log(|x(t) - x(\tau)|^2) \varphi(\tau) d\tau$$

As we did before, we can subtract the singularity of this as well:

$$p.v. \int_\Gamma \frac{\partial^2 G_0(x-y)}{\partial n(y)\partial n(x)} \varphi(y) ds(y) = \frac{1}{4\pi |x'(t)|} \int_0^{2\pi} \frac{\partial^2}{\partial t \partial \tau} \left(\log(|x(t) - x(\tau)|^2) - \log(4 \sin^2(\frac{t-\tau}{2})) \right) \varphi(\tau) d\tau$$

$$+ \frac{1}{4\pi |x'(t)|} \int_0^{2\pi} \frac{\partial^2}{\partial t \partial \tau} \left(\log(4 \sin^2(\frac{t-\tau}{2})) \right) \varphi(\tau) d\tau.$$

For both of the integrals above, we want to integrate by parts to shift one derivative over to the density:·

$$p.v.\int_\Gamma \frac{\partial^2 G_0(x-y)}{\partial n(y)\partial n(x)}\varphi(y)ds(y) = \frac{1}{4\pi|x'(t)|}\int_0^{2\pi}\underbrace{\left(\frac{-2x'(t)\cdot(x(t)-x(\tau))}{|x(t)-x(\tau)|^2} - \cot(\frac{\tau-t}{2})\right)}_{\eta(t,\tau)}\varphi'(\tau)d\tau$$

$$+\frac{1}{4\pi|x'(t)|}\int_0^{2\pi}\cot\left(\frac{\tau-t}{2}\right)\varphi'(\tau)d\tau.$$

$$\eta(t,\tau) = \frac{-2x'(t)\cdot(x(t)-x(\tau))}{|x(t)-x(\tau)|^2} - \cot\left(\frac{\tau-t}{2}\right) \tag{5}$$

$$\lim_{\tau\to t}\eta(t,\tau) = -\frac{x'(t)\cdot x''(t)}{|x'(t)|^2}. \tag{6}$$

$$(N\varphi)(t) = -\int_0^{2\pi} n(t)\cdot\nabla^2_{tt}(G_k-G_0)(x(t)-x(\tau))n(\tau)\varphi(\tau)|x'(\tau)|d\tau + \frac{1}{4\pi|x'(t)|}\int_0^{2\pi}\eta(t,\tau)\varphi'(\tau)d\tau \tag{7}$$

$$+\frac{1}{4\pi|x'(t)|}\int_0^{2\pi}\cot\left(\frac{\tau-t}{2}\right)\varphi'(\tau)d\tau \tag{8}$$

$$D_{ij} = \begin{cases} \frac{1}{2}(-1)^{i-j}\cot\left(\frac{(i-j)h}{2}\right) & (i-j)\%N \neq 0 \\ 0 & (i-j)\%N = 0 \end{cases} \tag{9}$$

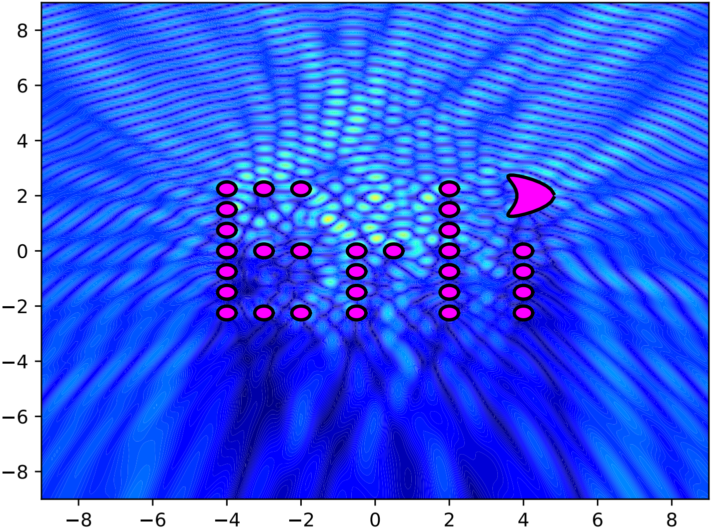
Therefore we can approximate the entire middle integral of (8) as QD . For the last integral, we note the following approximations:·

$$\frac{1}{2\pi}\cot\left(\frac{\tau - t}{2}\right) f'(\tau)d\tau \approx \sum_{j=0}^{2n} T_j^{(n)}(t) f(t_j^{(n)})$$

where

$$T_j^{(m)} = T_j^{(n)}(0) = \begin{cases} \frac{1}{2n\sin^2(t_j^{(n)}/2)} & j \in \text{Odd} \\ 0 & j \in \text{Even} \\ -n/2 & j = 0 \end{cases}$$

Note also that $t_i^{(n)} = (i-1)h$ and $h = \pi/n$.·


8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8


That's all Folks!

Victor Domniguez, Mark Lyon, Catalin Turc. *Well-posed integral equation formulation and Nyström discritization for the solution of Helmholtz transmission problems in two- dimensinoal Lipschitz domains.* [*Journal of Integral Equations and Applications, 2016*].

Carlos Pérez-Arancibia.. *A plane-wave singularity subtraction technique for the classical Dirichlet and Neumann combined field integral equations.* [*Applied Numerical Mathematics, 123:221–240, 2018.*].

S Hao, A H Barnett, P G Martinsson, and P Young. *High-order accurate methods for nystrom discretization of integral equations on smooth curves in the plane.* [*Advances in Computational Mathematics, 40(1):245–272, 2014.*]

Rainer Kress Boundary Integral Equations in Time-Harmonic Acoustic Scattering [*Mathl Comput. Modelling Vol. 15, No.3-5, pp. 229-243, 1991.*]

Ivar Stackgold *Boundary Value Problems of Mathematical Physics, Volume II* [*Classics in Applied Mathematics 29, Siam 2000*]

Rainer Kress *Linear Integral Equations.* [*Vol 82 Third Edition, Springer*]

A different approach is to find the Greens function for the differential equation. Consider

$$-\Delta E - k^2 E = \delta(x)$$

Requiring that there is angular symmetry we can reduce the equation to a Bessel type equation:

$$r^2 u''(r) + ru'(r) + k^2 r^2 u(r) = 0$$

Which has general solutions as a Hankel functions of the first and second kind.

$$u(r) = C_1 H_0^{(1)}(kr) + C_2 H_0^{(2)}(kr)$$

Since we allow k to be complex we discard $H_0^{(2)}(kr)$ as it behaves as follows for large r

$$H_0^{(2)}(kr) \sim \sqrt{\frac{2}{\pi rk}} e^{-i(rk - \frac{\pi}{4})}.$$

Since we only want outgoing waves at infinity and $H_0^{(2)}$ is incoming, we discard it. So,

$$u(r) = C_1 H_0^{(1)}(kr).$$

To find the constant C_1 we want to enforce the conditions that

$$lim_{\epsilon \to 0} \int_{\sigma_\epsilon} \frac{\partial u(r)}{\partial r} dS = -1.$$

To this end, we observe the small argument behavior of the Hankel function:

$$
\begin{aligned}
H_0^{(1)}(kr) &= J_0(kr) + iY_0(kr) \\
&\sim 1 + \frac{2i}{\pi}\left(\ln\left(\frac{kr}{2}\right) + \gamma\right)
\end{aligned}
$$

$$
\frac{\partial H_0^{(1)}(kr)}{\partial r} \sim \frac{2i}{r\pi}.
$$

Meaning then that

$$
lim_{\epsilon \to 0} \int_{\sigma_\epsilon} \frac{\partial u(r)}{\partial r} dS = \frac{2iC_1}{\pi} \int_0^{2\pi} d\theta.
$$

And finally that $C = \frac{i}{4}$. And so the free space solution we where looking for is given by:

$$
G_k(x-y) = \frac{i}{4} H_0^{(1)}(k|x-y|),
$$

With this we can now represent solutions in terms of this kernel entity and an unknown density.