Numerical Analysis Final Examination
Thursday, December 12th, 2013

You are permitted the use of a calculator for performing elementary operations.

1. Solve the nonlinear equation
$$x + x^5 = 1$$
by Newton's method with initial guess $x^0 = 0.7$.

   a) Let α be the solution (up to 8 or 9 decimal places). Make a table of the values $e_n = x_n - \alpha$.

   b) Using your table of values, illustrate the asymptotic, quadratic convergence relationship
$$\frac{e_{n+1}}{e_n^2} \approx \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}.$$

2. Perform one iteration of Newton's method for solving the system of nonlinear equations
$$\begin{cases} (x_1)^2 + (x_2)^2 = 4 \\ (x_1)^2 + (x_2)^3 = 1 \end{cases}$$
with the initial guess $x^0 = (1, -1)^T$.

3. Solve the system of linear equations
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$$
by first finding and verifying an LU factorization for the coefficient matrix and then using forward and backward substitution.

4. Prove for $1 \le p < \infty$ that

   a) $\|x\|_\infty \le \|x\|_p \le n^{1/p}\|x\|_\infty$ for all $x \in \mathbb{R}^n$,

   b) $n^{-1/p}\|A\|_\infty \le \|A\|_p \le n^{1/p}\|A\|_\infty$ for all $A \in \mathbb{R}^{n\times n}$ (Hint: use part a),

   c) Give an example of $x \in \mathbb{R}^n$ for which $\|x\|_\infty = \|x\|_p$ and another example for which $\|x\|_p = n^{1/p}\|x\|_\infty$.

5. Consider the system of linear equations

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

which has solution $x = (1,1)^T$.

a) Indicate which properties of the coefficient matrix A guarantee that the Jacobi iteration converges to the solution for any initial guess.

b) Calculate $\rho(N^{-1}P)$ where $A = N - P$ is the splitting for the Jacobi method.

c) Find a closed form expression for the iterations x^n and error $e^n = x - x^n$ using the Jacobi method with initial guess $x^0 = (1,0)^T$.

d) Compare your findings in part c) with the result from part b).

Repeat the analysis from parts a) through d), replacing Jacobi with Gauss-Seidel.

6. Let $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$.

a) Evaluate $\rho(A)$, $\|A\|_2$, and $\|A\|_\infty$.

b) Find a nearly diagonal decomposition of A.

c) Specify a norm $\|\cdot\|_N$ on $\mathbb{R}^2$ for which $\|A\|_N < \rho(A) + 10^{-5}$.

7. (Bonus) Show that $\mathbb{R}^2$ equipped with the norm $\|\cdot\|_1$ is not an inner product space, i.e. show that there is no inner product $(\cdot,\cdot)$ on $\mathbb{R}^2$ with the property that $\sqrt{(x,x)} = \|x\|_1$ for all $x \in \mathbb{R}^2$.