POD with Compressed Modes

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Outline

The Compressed Modes algorithm aims to solve the following objective function:

$$\Psi_N = \min_{\Psi \in \mathbb{R}^{n \times N}} \frac{1}{\mu} |\Psi| + Tr(\Psi^T H \Psi) \quad \text{s.t} \;\; \Psi^T \Psi = I,$$

where $|\Psi|$ is the l_1 norm of the matrix Ψ.

procedure INITIALIZE($\Psi_N^0 = P^0 = Q^0, b^0 = B^0 = 0$)
  while "not converged" **do**
    $\Psi_N^k \leftarrow \arg\min_\Psi Tr(\Psi^T \hat{H} \Psi) + \frac{\lambda}{2}\|\Psi - Q^{k-1} + b^{k-1}\|_F^2 + \frac{r}{2}\|\Psi - P^{k-1} + B^{k-1}\|_F^2$
    $Q^k \leftarrow \arg\min_Q \frac{1}{\mu}|Q| + \frac{\lambda}{2}\|\Psi_N^k - Q + b^{k-1}\|_F^2$
    $P^k \leftarrow \arg\min_P \frac{r}{2}\|\Psi_N^k - P + B^{k-1}\|_F^2$ s.t $P^T P = I$
    $b^k \leftarrow b^{k-1} + \Psi_N^k - Q^k$
    $B^k \leftarrow B^{k-1} + \Psi_N^k - P^k$
  end while
end procedure

where the variables P, Q, B, b come from introducing auxiliary variables by setting $\Psi = Q$ and $P = \Psi$ as explained in.

Previous slide is messy! Can simplify the algorithm considerably:

$$
\begin{aligned}
(2H + \lambda + r)\Psi_N^k &= r(P^{k-1} - B^{k-1}) + \lambda(Q^{k-1} - b^{k-1}) \qquad (1)\\
Q^k &= Shrink(\Psi_N^k + b^{k-1}, 1/(\lambda r))\\
P^k &= (\Psi_N^k + B^{k-1})U\Lambda^{-1/2}S^T\\
b^k &= b^{k-1} + \Psi_N^k - Q^k\\
B^k &= B^{k-1} + \Psi_N^k - P^k
\end{aligned}
$$

Where,

$$Shrink\,(u, \delta) = sgn(u)\max\left(0, |u| - \delta\right),$$

and

$$U\Lambda S^T = svd((\Psi^k + B^{k-1})^T(\Psi^k + B^{k-1}))$$

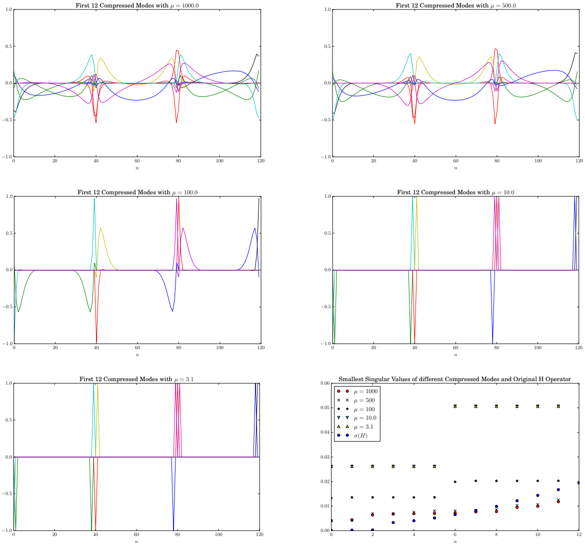


Figure: Above we see the effect of running Osher's sparsifying algorithm with

Right, so while this is nice, it does not help us with trying to do some sort of POD with this algorithm. To help us with this, we want to turn the objective function

$$\Psi_N = \min_{\Psi \in \mathbb{R}^{n \times N}} \frac{1}{\mu}|\Psi| + Tr(\Psi^T H \Psi) \quad \text{s.t} \;\; \Psi^T \Psi = I,$$

into:

$$\Psi_N = \max_{\Psi \in \mathbb{R}^{n \times N}} \frac{1}{\mu}|\Psi| + Tr(\Psi^T H \Psi) \quad \text{s.t} \;\; \Psi^T \Psi = I,$$

We then need to modify the algorithm to take account for the new objective. We doing so yields:

$$
\begin{aligned}
2H^{-1} + \lambda I + rI &= U diag\left\{\frac{2}{\sigma_i}\right\} U^T + rUU^T + \lambda UU^T \\
&= U\left[diag\left\{\frac{2}{\sigma_i}\right\} + r + \lambda\right] U^T \\
&= U diag\left\{\frac{2 + \sigma_i(r+\lambda)}{\sigma_i}\right\} U^T,
\end{aligned}
$$

$$
(2H^{-1} + \lambda I + rI)^{-1} = U diag\left\{\frac{\sigma_i}{2 + \sigma_i(r+\lambda)}\right\} U^T.
$$

So by considering H^{-1} instead of H, we keep the minimization algo, but as a result we get the largest CM and corresponding CM eigs.

To do POD, the simple dynamical system we consider is:

$$u_t = \kappa u_{xx}$$

We solve the above, by Crank-Nicholson:

$$\left(I - \frac{k}{2h^2}\hat{L}\right) U^{n+1} = \left(I + \frac{k}{2h^2}\hat{L}\right) U^n + G$$

$$\begin{aligned} \hat{L} &= \text{Discrete 2-d Laplacian with boundary condition.} \\ I &= \text{identity of size} N \times N. \\ k &= \text{time step.} \\ h &= \text{spatial discritization} \\ G &= \text{Boundary Conditions.} \\ U^n &= \text{Mesh at a particular time step.} \end{aligned}$$

$$Y = (y(t_1), \ldots, y(t_m)) \in \mathbb{R}^{n\times m} \quad \text{Snapshot matrix}$$

$$H = Y^* Y$$

$$u_j = \frac{1}{\sqrt{\lambda_j}} Y v_j$$

$$I(d) = \frac{\sum_{i=1}^{d} \lambda_i}{\sum_{i=1}^{n} \lambda_i}$$

Projection

$$\left(I - \frac{k}{2h^2}\hat{L}\right) U^{n+1} = \left(I + \frac{k}{2h^2}\hat{L}\right) U^n + G \tag{2}$$

$$U^n \approx V\alpha(n),$$

where $\alpha \in \mathbb{R}^d$

$$V^T \left(I - \frac{k}{2h^2}\hat{L}\right) V\alpha(n+1) = V^T \left(I + \frac{k}{2h^2}\hat{L}\right) V\alpha(n) + V^T G. \tag{3}$$

Leading POD modes

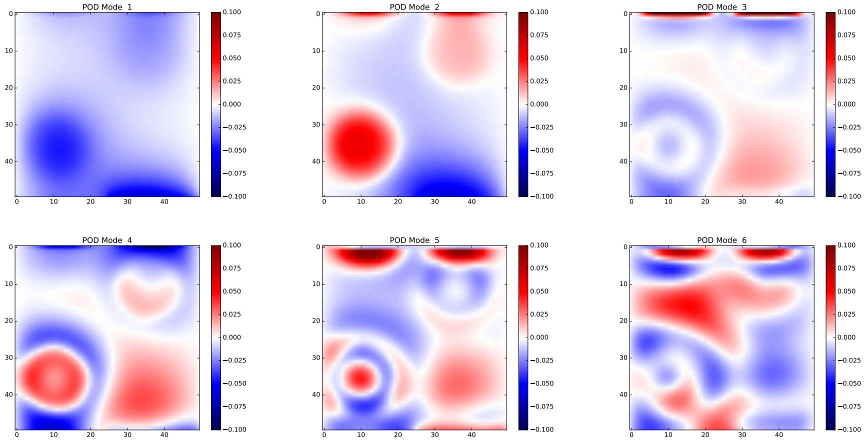
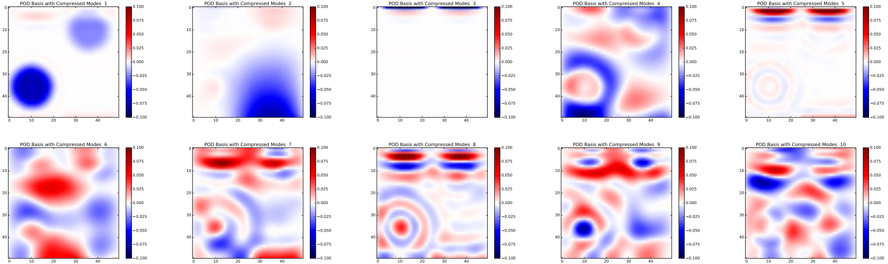
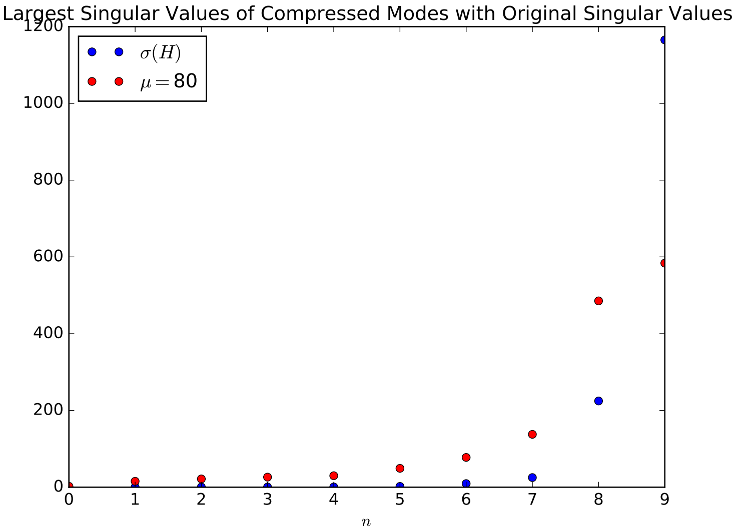


Figure: The leading 6 POD modes.

Leading CM POD modes




Largest Singular Values of Compressed Modes with Original Singular Values
$\sigma(H)$
$\mu = 80$
1200
1000
800
600
400
200
0
0 1 2 3 4 5 6 7 8 9
n

Simulations

Play/Pause

Simulations

Play/Pause

Error in Model Reduction

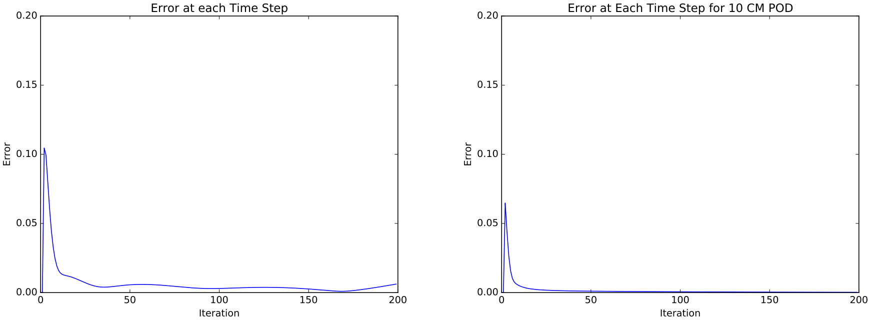
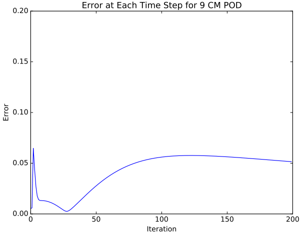
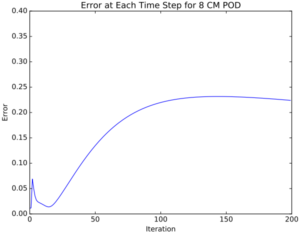
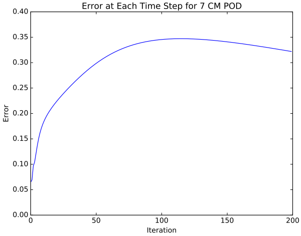


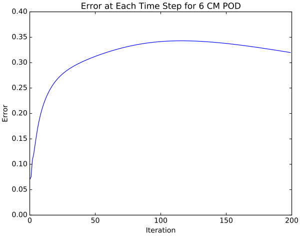
Figure: Here we see the relative error propagation of (left) POD modes against that of (right) CM POD.

Picking Less CM POD









Play/Pause