Pedro M. Rosario Barbosa

The Relation Between Formal Science and Natural Science

Third Edition

Revised and Expanded

Underdetermination of Science: Part I

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Third Edition:
This third edition is dedicated to the Asociación de Estudiantes de Filosofía of the University of Puerto Rico in Río Piedras.

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Preface

The subject of underdetermination or subdetermination of science has stimulated me ever since I began reading about philosophy of science. Pierre Duhem's *The Aim and Structure of Physical Theory* and W. V. O. Quine's “Two Dogmas of Empiricism” are the kind of writings that never leave the mind of the philosopher alone. They continually poke the mind sparking a philosophical reflection. Unfortunately, some academics treat these philosophical works as dogmas of faith, doing a disservice to Duhem, Quine, and to philosophy in general. On the contrary, we should question these, and we should follow their example in many ways. Only by doing this, philosophy becomes a fruitful legacy; a source of intellectual richness for future discussions.

Another big stimulus to my philosophical research has been Edmund Husserl's phenomenological doctrine, which, contrary to what is generally expected, is deeply associated with many aspects of analytic philosophy. One cannot cease to be seduced by his doctrine, which has contributed more to twentieth century philosophy than many scholars realize, and much of his legacy still endures, completely unnoticed by analytic and continental philosophers alike.

Philosophy of science and many aspects of husserlian phenomenology deal with two of the most important universal questions that permeate philosophy: “What *is*?” and “How do we *know*?” According to Aristotle, these questions are what have made “love of wisdom” the queen of all sciences. No wonder these are the most important questions. Aristotle says in the first sentence that appears in his *Metaphysics* “All men by nature wish to know”. That sole statement implies that there is always a wish to know something that *exists* but is *unknown*.

Philosophy of science tries to address one specific form of knowledge, that which comes from natural science. According to many authors, “epistemology” is defined as philosophy of science. For them, there is no difference between these disciplines.[1] The only knowledge there is for many of them is scientific knowledge.

Husserl's and Frege's reading, as well as reflections on mathematics and logic have made me think otherwise. That conception of epistemology as philosophy of science is too narrow. It is difficult to say that mathematics and logic are just pure tautologies and that many of the most complex theorems proved in both fields are simply “nothing new”. On the contrary, historically, philosophy has always depended on both fields. Their use has made the difference between sophists and philosophers. Examples of rigor are Socrates' mayeuctic, Plato's dialectics, Aristotle's *Organon*, Descartes' *Meditations*, Kant's *Critique of Pure Reason*, Frege's *The Foundations of Arithmetic*, or Husserl's *Logical Investigations*. The most fascinating of Plato's works involve his thinking on mathematical entities, like his beautiful and difficult work *Timaeus*. We can describe Aristotle's *Metaphysics* as one of the greatest gems of humanity, an example of how far logical thinking can lead a man to make important rational advances.

Most of the rejection to mathematical and logical knowledge is due to the fact that too much attention is paid to natural-scientific knowledge. Sadder still is that such a way of thinking has led most schools and universities to restrict mathematics only to *applied* mathematics. They never give any importance to the wisdom of knowing mathematics in its theoretical aspect, especially as a stimulus to think. Just like philosophy and science, mathematics deserves to be learned for its own sake. The more we know about it, its axioms and theorems, the better our intellect and our culture will be. The fact that some philosophers of science and epistemologists only value the useful part of logic and mathematics has even led some of them to say that we should cut all the “useless fat” of mathematics and throw it to a waste basket.[2] If Bolyai, Riemann, and Lobachevsky had followed that advice, most probably science would not have advanced the way it has.

Most philosophers forget that one of the cases used by Kant to exemplify a categorical imperative is our duty to promote culture according to our talents.[3] In a society like ours, it seems that only commercial culture is valued. So what can the proof of a theorem mean to someone who just wakes up in the morning, goes to work, returns

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1 Bunge, 1997, p. 21; Popper, 1979, p. 108.
2 Kitcher, 1988, pp. 315-316.
3 Kant, 1785/1999, AK 4:422-423.

home, watches TV and goes to sleep every day? Not a single opportunity arises to enrich him or herself with the legacy of the past. As Seneca put it beautifully in his *On the Brevity of Life*, one could prolong life in such a wonderful way. Philosophy is a way to open the doors to new possibilities and options, without losing *rationality* and *rigorous thinking* along the way. I would not be surprised if many of the ills of society were caused precisely by a lack of love for philosophy, mathematics, and logic. In other words, society could heal itself with more "love for Wisdom", an activity done for its own sake. Formal knowledge is as enriching as natural scientific knowledge. I agree with Peter Hilton when he said that the value of mathematics does not lie in the testing of the market, nor scientific usefulness, nor its economic value. It lies solely in itself and no more.[4]

As a daughter of philosophy, natural science enriches itself from mathematics and logic. It is not merely a tool, but a discipline with which scientists, in the long run, learn to think rigorously about the empirical world. However, unlike formal sciences, natural sciences are full of *a posteriori* suppositions, hypotheses, and theories. When we look at the world, our mind already has a theory on how to interpret what it perceives, and when natural sciences confront the world, they do so from an entire theoretical body. There are hypotheses that can be wrong, not exclusively because of themselves, but because of everything they suppose. At the beginning, we saw everything through the eyes of myth. Then philosophers began seeing the world a different way, developing whole theoretical bodies or paradigms throughout history.

The fact that we have many world-paradigms and theories, means that underdetermination exists. A phenomenon or event is not enough to state categorically that a scientific theory is true. There can be rival theories that can explain the same evidence. Since, as Husserl would say, we are talking about objects formally related to each other, and these are correlated with a formal network of theoretical statements, we have the problem of how much should we revise these theories ("fictions *cum fundamento in re*") in order to account for these states-of-affairs.[5] Very important questions on this matter arise from Duhem's and Quine's works: How far does this underdetermination go? How do changes in theoretical bodies happen? How many of these theoretical principles are rejected along the way? Why do scientists accept new ones? Why do they accept old ones again? Like philosophy, science is an endless

4 Gullberg, 1997, pp. xvii-xxii.
5 LI. Vol. I. §§23,62-63.

pursuit of a certain kind of truth: the truth about the physical world and the universe.

There is a sense that the subject of underdetermination is not irrelevant, it is very important because it covers all relevant subjects in philosophy of science: the process of conjectures and refutations, paradigm shifts, religion-science debate, the problem of demarcation, scientific research programs, among others.

This ambitious project, which is called *Underdetermination of Science* is divided in two parts. Part I was written to establish the importance of formal sciences (logic and mathematics) from a platonist standpoint, and see their *true* relationship with natural science. It will consist in mostly presenting a viable and coherent platonist doctrine based on Husserl's own philosophy of mathematics, and will refute many of the objections to platonism. Then, some challenges by Quine and Hilary Putnam will be addressed in order to see that scientific underdetermination does not reach the formal sciences.

Part II will examine practically all aspects that have to do with underdetermination: from an examination of how the mind constitutes experience, to the way scientific paradigms and research programs are formulated, validated and accepted within the scientific enterprise and its implications to both science and philosophy.

ABBREVIATIONS

I'll use these abbreviations to quote these specific works. In the rest of the cases, I will use mostly the APA style. With the exception of *The Foundations of Arithmetic*, all pages referred to regarding Frege's writings will be those of the original texts as presented by Michael Beaney in *The Frege Reader*. Regarding Husserl's writings, we will make reference to the sections where the passages are found. In the case of *Logical Investigations*, we will refer to the volume, investigation, and section.

BY GOTTLOB FREGE

FA. = *The Foundations of Arithmetic* = Frege 1888/1999

FC. = “Fuction and Concept” = Frege 1891/1998a

SR. = “On Sense and Referent” = Frege 1892/1998b

T. = “The Thought: a Logical Inquiry” = Frege 1919/1998c

BY EDMUND HUSSERL

LI. = *Logical Investigations* = Husserl 1913/2001.

I. = *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy* (Vol. 1) = Husserl 1913/1973.

FTL. = *Formal and Transcendental Logic* = Husserl 1929/1969.

EJ. = *Experience and Judgment* = Husserl 1939/1973.

INTRODUCTION

When W. V. O. Quine wrote “Two Dogmas of Empiricism”, he confronted philosophers with the nature of mathematics, logic and science. For him, all of these disciplines are inter-related forming a very big web of theoretical dependency on each other. There is no separation between analytic and synthetic judgments, there can only be a whole body of “knowledge”. Scientific theories and mathematics are posits like the early Greek gods of Homer. The so-called *a priori* disciplines can be changed for empirical reasons. A clear example of this is the way logic was revised by quantum phenomena, and mathematics was revised by the general theory of relativity. Of course, Quine says: “Posited objects can be real. As I also wrote elsewhere, to call a posit a posit is not to patronize it”.[6] But posits are posits regardless of whether they are real or not.

We should leave behind intensional notions, such as the platonist conception of meaning. “Meanings are what essences become, when they are detached from an object and wedded to the word”.[7] Science, on the other hand, is just extensional, it does not deal with intensions. Since physics is the north which guides philosophy in its research, we should be content with that. Logic and mathematics, though posits, are important *tools* for science; and with the exception of some problems in set theory, mathematics can be pretty reliable. In the long run, meanings cannot establish the difference between analytic and synthetic in any case. The only meanings we should pay attention to are the ones which can account for sensible experience.

In this book I shall attempt to challenge this aspect of “Two Dogmas of Empiricism” as a way to start dealing with the all-permeating problem of the underdetermination of science. Many still

6 Quine, 1953, p. viii.
7 Quine, 1953, p. 22.

stipulate and support the idea that logic and mathematics can be changed in order to account for raw sensory-data. One of them has been Hilary Putnam, who shows that the general theory of relativity refuted euclidean geometry when it adopted a non-euclidean view of space-time. This way of thinking seems to be supported by some epistemologists, and is generally accepted among philosophers of science. For example, in Donald Gillies' excellent demystification of the Duhem-Quine Thesis, he says that Duhem rejected non-euclidean geometry and accepted euclidean geometry as a common-sense way of providing physical foundations. Gillies points out correctly that Duhem's belief is not true, but Gillies *agrees* with Quine when he says that non-euclidean geometry superseded euclidean geometry, and that Duhem's common-sense foundation of physics is false.[8] He also quotes Duhem saying that Aristotle formulated logic in almost its final form, when we all know that Frege, Peano and Russell clearly superseded classic logic, and that we can also look at Brower's intuitionistic logic and quantum logic as proofs that logic itself can also be changed. From here he says that it "seems reasonable to extend the holistic thesis to include logic as well as to allow the possibility of altering logical laws as well as scientific laws to explain recalcitrant observations".[9] He later develops a new version of the Duhem-Quine Thesis which allows logical and mathematical revision in light of recalcitrant experience.[10]

Philosophers of science in general are not friends of supposing other entities besides physical entities. Perhaps they can, like Popper, accept a kind of semi-platonism, or a cultural realm, but not the independent existence of meanings, truth values, mathematical objects such as cardinal numbers, ordinal numbers, sets, and the true and false formal relationships between them. In fregean terms, almost no philosopher of science is a "third realm" lover. From a scientific point of view, it can even be seen as a posit of unnecessary entities (Occam's Razor). For example, like Mario Bunge, they even reject third realms either in platonist or popperian forms; it is for them

8 Gillies, 1993, pp. 114-115.
9 Gillies, 1993, p. 115.
10 Gillies, 1993, pp. 115-116.

a kind of medieval concept which is not needed.[11] For many philosophers of science, the more naturalistic the epistemology, the better! So, apparently, there is no space for platonism with respect to natural science.

I hope to show that one sort of platonism is the *only* way to understand science well and adequately. In fact, it is the *only* way to make epistemological sense on the *a priori* and its relationship with the *a posteriori*. A naturalistic view of logic and mathematics is not satisfactory to understand the real relation between formal science and natural science.

This book is intended to refute many antiplatonist claims that logic and mathematics can both be changed on the basis of a recalcitrant sensory experience. For that, it is necessary to refute Quine's claim that there can be no distinction between analytic and synthetic judgments. To make platonism a viable option for philosophers, a particular platonist philosophy of mathematics will be adopted, namely that of Edmund Husserl. His views on mathematics have been neglected by many analytic philosophers. His philosophy happens to be practically the *only* platonist proposal that provides an adequate epistemology of mathematics, one which is actually very close to twenty-first century mathematics. At the same time, some of the most important objections to platonism will be addressed.

Finally, once the nature of formal science and its epistemology are understood, there will be a thorough refutation of two philosophical statements presented by Quine and Putnam, which will be called, following Jerrold Katz, the "Quine-Putnam Theses". These refutations will provide an adequate basis to understand the real relationship between science, logic and mathematics.

11 Bunge, 1997, pp. 66, 76.

CHAPTER 1: ON THE NATURE OF FORMAL SCIENCE

To be able to provide an adequate doctrine on the relationship between formal science and natural science, we need to explore the nature of formal science itself, which is indispensable to understand its relationship with natural science. I wish to address one of the most important problems that arose in the twentieth century and still continues today, which is the division between analytic and synthetic propositions, and the problem between platonism and antiplatonism in philosophy of mathematics.

1.1 — QUINE'S REJECTION OF ANALYTIC AND SYNTHETIC DISTINCTION: A CHALLENGE FOR PLATONISTS

Willard van Orman Quine's position is the most popular philosophical doctrine that rejects the traditional analytic/synthetic distinction.[12] It must be clarified that Quine's criticism is only directed at Rudolf Carnap's way of making such a distinction in virtue of meanings. In "Two Dogmas of Empiricism", the two dogmas Carnap wished to attack were the analytic/synthetic distinction and the verifiability criterion of science. We will deal in this chapter only with the former, and we will deal with the latter in the second volume of *Underdetermination of Science*.

Quine made a distinction between two kinds of definitions of analyticity in philosophy. There are those who define analyticity as propositions that are *logically true*, like: "No unmarried man is married", and there are also those who define it *in virtue of*

12 Quine, 1953, pp. 20-46.

meanings, such as “No bachelor is married”. In the latter case, there is a synonymy of terms: we can replace “bachelor” with “unmarried man”. In this way, any analytic statement built on these notions would be a kind of a “second class” of analytic statements. Quine argues that this can only serve to reconstruct logical truth, but not analyticity as such.[13]

For Quine, to establish the definition on analyticity on the basis of synonymy of meanings is doomed to failure. It can be argued that there are judgments that are analytic in virtue of their *definitions*, for example, “bachelor” is defined as “unmarried man”. However, the way that the word “bachelor” is defined depends greatly on linguistic usage in people's daily lives. Another apparent criterion of analyticity is the interchangeability of terms *salva veritate*, as Leibniz suggested. Let us take this case:

“Bachelor has less than ten letters.”

In this sentence, the word “bachelor” cannot be substituted by “unmarried man”. Also, cases like “bachelor of arts” present us counter-instances of interchangeability of terms. In the former case we could say that we mean the word “bachelor,” while in the latter the word “bachelor” means something different from “unmarried man.” We must take into account *cognitive* synonymy. To claim that “bachelor” and “unmarried man” are synonymous means that the proposition “All and only bachelors are unmarried men” is analytic. This is to say that “*Necessarily* all and only bachelors are unmarried men.” Let us carry out the substitution and say: “Necessarily all and only bachelors are bachelors.” In both propositions, even though they interchanged terms “bachelor” and “unmarried men,” the cognitive information they offer is very different.[14]

We could try saving the argument by appealing to extensions. For example, two terms are interchangeable *salva veritate* if they have the same extension. However, extensions that fall under concepts depend greatly on accidental matters-of-fact. For example, the concepts of “creature with heart” and “creature with kidneys” have the same extension (presumably), but they are not

13 Quine, 1953, pp. 23-24.
14 Quine, 1953, pp. 27-31.

interchangeable *salva veritate*. It seems that the only way to assert the synonymy is by supposing that the identification of the terms "bachelor" and "unmarried man" is analytic. Hence, we see, not a circularity, but a "closed curve in space", because all of these concepts and theories of meanings are closely related.[15] It can be expressed simply in this way:

> *In order for us to distinguish between analytic and synthetic we must appeal to synonymy. At the same time, we should also understand synonymy as related to interchangeability* salva veritate. *However, such a condition to understand synonymy is not enough, so we not only argue that the terms should be interchangeable, but necessarily so. And to explain this logical necessity we must appeal to analyticity once again.*[16]

It could be that the failure to establish the interchangeability *salva veritate* is due to the vagueness of language. We could use instead, as Carnap certainly tried, an artificial language to avoid such vagueness and establish semantic rules to distinguish some propositions from others. Now, we have to explain why there are semantic rules that make a difference between analytic and synthetic propositions, and how they are different from other semantic rules. The answer is its adoption because they can pick up analytic propositions and distinguish them from synthetic ones. Here we find a circular reasoning: What distinguishes analytic propositions from synthetic ones is the supposition of semantic rules, which are themselves adopted because they can distinguish between analytic and synthetic propositions. In seeking the definition of analytic, we are led to presuppose the notion we wish to define in the first place.[17]

From this analysis, Quine presents this analytic/synthetic distinction as a kind of article of faith held by logical empiricists. For him, propositions that are logically true ("No unmarried man is married"), and those of science and experience are all posits, no different epistemologically to the Greek gods of antiquity which were also posited to explain what people perceived at that time. For Quine,

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15 Quine, 1953, p. 30; Katz, 2004, p. 19.
16 Quine, 1953, pp. 24-32.
17 Quine, 1953, pp. 32-39.

mathematics and logic are qualitatively no different from empirical statements, or scientific laws, and can be revised in light of recalcitrant experience. Therefore, the difference between posits of formal laws and posits of science is only of degree of abstraction, which means that there can be no analytic/synthetic distinction between propositions.[18]

1.2 — TAKING UP THE CHALLENGE

1.2.1 — WHO DOES QUINE CRITICIZE?

We must be aware that most of the criticism made by Quine (1953) is only directed to Rudolf Carnap's way of distinguishing analytic and synthetic judgments. For the late Carnap, we must focus on meaning, especially through cognitive synonymy, which determines which propositions are analytic or synthetic. Practically, from its refutation, Quine deduces that there is no qualitative distinction between judgments.

First, we have to see if this criticism applies to all of the versions of analytic and synthetic judgments. Before Rudolf Carnap, there were some well-known versions of these notions. The first was the one defined by Kant, who believed that synthetic *a priori* judgments are at the very heart of science. He gave two definitions of analytic judgments and, respectively, two definitions of synthetic judgments.

I. **Analytic**: For judgments that have a subject-predicate structure, if the concept of the predicate is included already in the concept of the subject, then the judgment is analytic.

   Synthetic: If the concept of the predicate is not included in the concept of the subject, then the judgment is synthetic.

II. **Analytic**: If a judgment is based on the principles of identity and no-contradiction, then the judgment is analytic.

   Synthetic: If the judgment is not based only on the principles of identity and no-contradiction, then it is synthetic.[19]

18 Quine, 1953, pp. 17-19, 42-46.
19 Kant, 1998, A7-10/B11-14.

Notice that at most Quine's criticism can be applied to version (II) of the analytic and synthetic distinction, but not definition (I).[20] Version (I) basically states that if I have a subject-predicate kind of proposition "*S is P*" in which *S* is a concept of the subject, and *P*, the concept of the predicate, is an essential or necessary property of *S*, then "*S* is *P*" is an analytic judgment. This is a qualitative difference from version (II) which says that a proposition "$a = b$" is an analytic proposition in virtue of the principle of identity. Not only are both of these definitions incompatible with each other, but they are also unsatisfactory for today's semantics. At least in Kant we have *one* instance where Quine's criticism does not apply, version (I) of these definitions.

Another version of analyticity can be found in Frege's philosophical masterpiece *The Foundations of Arithmetic*. Based on version (II) of Kant's definitions, he says:

> If in carrying out this process, we come only on general logical laws and on definitions, the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which admissibly of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general, since they contain assertions about particular objects. But if, on the contrary, its proof can be derived exclusively from general laws, which are themselves neither need nor admit of proof, then the truth is a priori.[21]

For Frege, logical and arithmetical propositions are all analytic *a priori*, and this fact points to the reducibility of arithmetic to logic. Geometry, on the other hand, is synthetic *a priori*. This distinction between analytic and synthetic does not fall under Quine's criticism.

20 Notice that Quine (1953) believes his criticism reaches Kant's original proposal (p. 21).

21 FA. p. 4.

Finally, there is another less studied criterion to distinguish analytic and synthetic propositions, the husserlian criterion. Husserl did not base it on Kant's but was inspired by Bernard Bolzano, who, at that time, was a relatively unknown philosopher and mathematician. In the second volume of *Logical Investigations*, the Third Investigation, Edmund Husserl establishes some semantic distinctions to develop his mereological doctrine. He distinguishes concepts and propositions that are completely free from all "matter that contains the thing" (or free from all empirical content), and those which are not free from them. He introduces the notion of *logical categories* which include concepts like: "something", "one", "object", "property", "relation", "plurality", "number", "order", "whole", "parts", "magnitude", and so on. These logical categories are qualitatively different from material concepts such as "tree", "house", "color", "sound", "space", "sensation", and "feeling", which have empirical content.[22]

For Husserl, analytic propositions are those which express a truth exclusively in virtue of forms, devoid of all material concepts, while synthetic judgments do contain them. Also, Husserl's distinction allows for the existence of synthetic *a priori* judgments. To understand the difference between analyticity and syntheticity, Husserl makes a distinction between "analytic laws" and "analytic necessity".[23] He describes these analytic laws this way:

> Analytic Laws are unconditionally universal propositions, which are accordingly free from all explicit or implicit assertions of individual existence; they include none but formal concepts, and if we go back to such as are primitive, they contain only formal categories. Analytic Laws stand opposed to their *specifications*, which arise when we introduce concepts with content, and thoughts perhaps positing individual existence, e.g. *this, the Kaiser*. The specification of laws always yields necessary connections: specifications of analytic laws therefore yield

22 LI. Vol. II. Inv. III. §11.

23 LI. Vol. II. Inv. III. §§11-12. Rosado (2008a, 2008b) states that Quine (1970) and Chateaubriand (2005) arrived separately to almost the same notion of Husserl's "analytic laws" with their respective notions of "logical truth". (see Chateaubriand, 2005, pp. 254-255, 262)

> *analytically necessary connections*. When they imply existential assertions (e.g. *If this house is red, the redness pertains to this house*) such analytic necessity relates to that content of the proposition in virtue of which it empirically specifies the analytic law, not to its empirical assertion of existence.
>
> We may define *analytically necessary propositions* whose truth is completely independent of the peculiar content of their objects (whether thought of with definite or indefinite universality) and of any possible existential assertions. They are propositions which permit of a *complete* '*formalization*' and can be regarded as special cases or empirical applications of the formal, analytic laws whose validity appears in such formalization. In an analytic proposition it must be possible, without altering the proposition's logical form, to replace all material which has content, with an empty formal Something, and to eliminate every assertion of existence by giving all one's judgments the form of universal, unconditional laws.[24]

Analytic laws are purely formal, while an analytic necessity is an instantiation of an analytic law. Here he gives an example:

> It is, e.g., an analytic proposition that *the existence of this house includes that of its roof, its walls and its other parts*. For the *analytic* formula holds that the existence of the whole *W*(*A*, *B*, *C* . . .) generally includes that of its parts *A*, *B*, *C* . . . This law contains no meaning which gives expression to material Genus and Species. The assertion of individual existence, implied by the *this* of our illustration, is seen to fall away by our passage into the pure law. This is an analytic law: it is built up exclusively out of formal-logical categories and categorial forms.[25]

Husserl distinguishes analytic law and analytic necessity from synthetic *a priori* law and synthetic *a priori* necessity. If a proposition is necessary, but not formalizable *salva veritate*, it is a synthetic *a priori* proposition. Its laws are founded on material concepts or essences. Synthetic necessities are particularizations of synthetic laws, such as the one expressed in the proposition “this red

24 LI. Vol. II. Inv. III. §12.
25 LI. Vol. II. Inv. III. §12.

is different from this green".[26] Propositions like "anything colored must be extended" are necessary, but are synthetic, because we are not establishing direct analytic correlatives such as in the case of "There are no fathers without children". Here, the concepts of "father" and "child" are analytically founded on each other, and the analytic necessity of that proposition relies in its forms (the way they are formally related). On the other hand, in the case of color we are not appealing to a direct correlative. It is true that a color cannot be exist without a colored extension, but the existence of such an extension is not founded analytically on the concept of color.[27]

Husserl includes any geometry based on material categories (such as spatial form, geometrical congruence, among others) as a synthetic *a priori* discipline.[28] Notice that under these husserlian criteria, "No bachelor is married" would not be an analytic judgment nor a synthetic *a priori* proposition, so Quine's arguments do not apply to this case.[29]

26 LI. Vol. II. Inv. III. §12.

27 LI. Vol. II. Inv. III. §11.

28 LI. Vol. II. Inv. III. §12; Rosado, 2008b, p. 109.

29 It should be mentioned that, although Husserl's criteria seem to satisfy our intuition of what should be analytic or synthetic, it is not perfect. Rosado (2008a) noticed that, under husserlian criteria, some mathematical statements that should be considered analytic become synthetic *a priori*. This is the case of the mathematical truth "$1^3+2^3+3^3+4^3=100$" which formalized will give us "$x^3+y^3+z^3+w^3=100$". The problem with both of these mathematical truths is that the latter can *only* be instantiated with the former (Rosado, 2008a, p. 134). The same could be said regarding the analyticity of certain metalogical statements such as the Löwenheim-Skolem Theorems, or Gödel's Completeness Theorem for first-order logic, or Gödel's Incompleteness Theorems, and so on. (Rosado, 2008a, pp. 134-135). Rosado proposed a model-theoretic definition: *A statement S is analytic if it satisfies the following two conditions: (i) it is true in at least one structure, and (ii) if it is true in a structure M, then it is true in at least any structure M* isomorphic to M.* (Rosado, 2008a, p. 137; Rosado, 2008b, p. 110) Under such a definition, all mathematical (excluding usual geometric statements that include what Husserl would consider "material concepts") and metalogical truths are considered analytic. This definition could also allow the existence of some sort of synthetic *a priori* definition: *A statement is synthetic* a

As a result of this analysis we find in Quine's argument a *non sequitur*. The fact that Carnap's criterion for analyticity and syntheticity fails semantically, does *not* mean that there is no criterion to distinguish analytic and synthetic propositions, and much less that there is no qualitative difference between them.

1.2.2. — A Reply to Quine's Criticism

Even if the analytic and synthetic distinction appears arbitrary, from a *pragmatic* point of view it is logically legitimate to make a qualitative difference between kinds of propositions. Let us assume, for the sake of the argument that analytic propositions are posits like scientific theories. However, Quine is one of those philosophers who refuse to believe that logical laws and mathematical propositions are abstracted from experience. Posits about the world are not abstracted from experience either, they only serve to explain the “sense-data” we receive from our bodily senses. Nothing prevents anyone from making a difference among posits, just as we can make a difference between imaginary objects we cannot represent mentally (the round square or Hegel's universal spirit) and objects we can represent mentally (a table, a unicorn, Pegasus). Some of these objects do actually exist in the physical world, while others do not (or at least no one has good reason to believe they exist, especially when they have no scientific use). This is not merely artificial, this distinction can be made very clearly and legitimately.

We can say the same concerning posits that have one characteristic and posits that have other ones. For example, we can follow Husserl and identify posits like these as true:

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priori *if: (i) it is true in at least one physical world, and (ii) if true in a physical world W, it is true in any possible physical world*. (Rosado, 2008a, p. 139; Rosado, 2008b, p. 110) This definition would distinguish between empirical statements on space-time which can be applicable to this world, while statements such as “anything coloured must have extension” would apply to any world and include what Husserl would consider “material concepts” (Rosado, 2008a, p. 139). These definitions have not been challenged yet, and Quine's criticisms do not apply to them.

$$((\forall x)(F(x) \to G(x)) \wedge (\forall x)(G(x) \to H(x))) \to ((\forall x)(F(x) \to H(x)))$$

$$x^2 - y^2 = (x+y)(x-y)$$

These have nothing to do with material objects or concepts. They only express formal truths, whose variables can represent any propositions in the first case, or, any number in the second. The way they are shown to be true according to logical rules and mathematical axioms does not depend on empirical objects or temporal events.

These logical and mathematical statements differ from propositions like

$$F_g = G\frac{m_1 m_2}{d^2}$$

which is a newtonian formula to find the gravitational force between two masses. The variables *F*, *G*, *m*, *d* represent material concepts that can only be applied to the phenomenal or temporal world. These are not pure formal relations like our previous logical and mathematical formulas. Why can't this be taken into account when establishing legitimate criteria to distinguish between analytic and synthetic propositions? It can be argued that this statement is circular in a way, like in Carnap's case, because we make the difference between these kinds of propositions in order to make a difference between analytic and synthetic. However, the undisputed fact is that a difference *can* be made legitimately, and, furthermore, it is scientifically acceptable to make such a difference.

In the following sections and chapters I will discuss in depth the important differences between formal truths as analytic and natural-scientific propositions as synthetic.

1.3 — FORMAL SCIENCE: A NEGLECTED PLATONIST PERSPECTIVE ON *MATHESIS UNIVERSALIS*

One author who has been ignored by philosophy of mathematics is Edmund Husserl. Only recently we have rediscovered his influence on analytic philosophy, especially regarding his philosophy of language, of logic, of mathematics, and his influence over eminent figures in logical empiricism such as Rudolf Carnap.[30] Some authors have reexamined his philosophy of logic and mathematics in order to solve contemporary problems in these fields.[31] In this section, I am not going to say anything new about Husserl's philosophy of mathematics, but I think we should look at this relatively unknown aspect of Husserl's philosophy.

In 1890, when Husserl began his anti-psychologist turn, he was heavily influenced by G. W. von Leibniz, Bernard Bolzano, Hermann Lotze and David Hume. He realized that meanings (logical contents) and mathematical objects were clearly different from entities and events that are subject to changes in time. It is difficult to base the necessary and universal validity of logical and mathematical truths on temporal psychological acts or material abstractions from experience. One thing is the temporal *act* of recognizing a truth, and another the objective *validity of truth* itself; the *act* of counting and the *number itself*; the *act* of collecting, and *collection* (*set*) *itself*. Hence, we recognize, on the one hand, the *temporal* psychological acts and events in the world, and on the other the *atemporal* meanings and abstract mathematical entities. He says that temporality is the sphere of the real (*reell*) and the other the realm of the ideal (*ideel*).[32] Husserl retains this platonist doctrine throughout his life, even to the point of extending it to phenomenology.

Some have tried to argue against the view that Husserl held some kind of platonist doctrine. He pointed out that he rejects a certain form of platonism: that which considers abstract entities as existing

30 See Beaney, 2004; Dummett, 1993; Friedman, 1999, 2000; Hill, 1991; Hill & Rosado, 2000; Hintikka, 1995; Mayer, 1991, 1992; Mohanty, 1974, 1982a, 1982b; and Rosado, 2008b.

31 Many of these problems are addressed in Hill, 1991; Hill & Rosado, 2000; and Hintikka, 1995.

32 The term “real” in this context should not be identified with the German words “*Realität*”, or “*Wirklichkeit*”. “*Reell*” in this context only refers to the sphere of the temporal. As we shall see, the realm of the atemporal is also real in the sense of the word “*Wirklchkeit*”, that which forms part of everything that indeed exists.

as real (*reell*) outside of thought. He regards this as a mistake he called “metaphysical hypostatization”.[33] He clarifies:

> [. . .] what is 'in' consciousness counts as real just as much as what is 'outside' of it. What is real is the individual with all its constituents: it is something here and now. For us temporality is a sufficient mark of reality. Real being and temporal being may not be identical notions, but they coincide in extension. [. . .] Should we wish, however, to keep all metaphysics out, we may simply define 'reality' in terms of temporality. For the only point of importance to oppose it to the timeless 'being' of the ideal.[34]

> *Ideal objects* [. . .] *exist genuinely*. Evidently there is not merely a good sense in speaking of such objects (e.g. Of the number 2, the quality of redness, of the principle of [non-]contradiction etc.) and in conceiving them as predicates: we also have insight into certain categorial truths that relate to such ideal objects. *If these truths hold, everything presupposed as an object by their holding must have being*. If I see the truth that 4 is an even number, that the predicate of my assertion actually pertains to the ideal number 4, then this object cannot be a mere fiction, a mere *façon de parler*, a mere nothing in reality.[35]

In other words, ideality, an atemporal realm, *does exist* for Husserl. The fact that he was a platonist cannot be questioned.

Inspired by Leibniz, Husserl held an idea of *mathesis universalis*, where pure logic and mathematics come together to form the most universal mathematics of all. To understand this, we should examine the way Husserl conceived both formal disciplines, especially in his official position which appears in Chapter 11 of the first volume of his *Logical Investigations*.[36]

For Husserl, there is formal knowledge and material knowledge. This is made clear when we take into account the differences made by

33 LI. Vol. II. Inv. II. §7.
34 LI. Vol. II. Inv. II. §8.
35 LI. Vol. II. Inv. II. §8, my italics.
36 Rosado (2006) makes a more detailed and thorough exposition of Chapter 11 of the first volume of *Logical Investigations*.

Leibniz between "truths-of-reason" and "truths-of-fact", or by Hume between "relations-of-ideas" and "matters-of-fact".[37] However, as we have seen, his view on analyticity and syntheticity is more complex. From his criticism to psychologism, it is evident that in its theoretical standpoint, logic and mathematics are qualitatively different from any discipline that deals with matters-of-fact. They express truths that are *necessary*, and, because of that, they *prescribe* all forms-of-truth and all forms-of-being respectively.[38]

This is also true for knowledge. Science, in the widest sense (*Wissenschaft*), is an activity that has an anthropological origin, that is, it originates in the acts and dispositions of our thinking. This anthropological unity of sciences has as its correlate an ideal and objective unity of propositions *and* of objects. We must realize that all sciences refer to objects *in a specific manner*. So we see, in the first place, formal interconnections of objects (objectualities) or, as Husserl called them "*states-of-affairs*" (*Sachverhalte*) to which scientific truths refer to. On the other hand, science itself consists in the interconnections among truths which refer to those states-of-affairs. Truths are propositions that are fulfilled in states-of-affairs.[39] In the realm of truth, the interconnection of objects and the interconnection of propositions that refer to them are given *a priori* together and are inseparable. Because true propositions tell us that these states-of-affairs are indeed the case, these objectualities are necessarily correlated to truths about them. In other words, truth-in-itself is a necessary correlate of being-in-itself.[40]

If we abstract all subtracts (low level substrates of sensible objects) of a given state-of-affairs, we will end up with two different sorts of formal unities: the objectual unity and the unity of truth. In the former, we discover the ideal laws of all kinds of objectual relations, and in the latter we discover the ideal laws of truth. These legalities are supposed *a priori* for every science in order to provide knowledge. It is up to mathematics to explore the *a priori* laws of the

37 LI. Vol. I. §51.

38 LI. Vol. I. §§41, 46, 51.

39 See a more detailed account of this doctrine of sense and referent in Appendix A, Section A.2.

40 LI. Vol. I. §62.

ideal unity of objects and objectualities, while it is up to logic to explore the *a priori* unity of truths. In other words, logic is a *formal theory of judgment* or *formal apophantics*, and mathematics is a *formal theory of object* or *formal ontology*. For Husserl, mathematics is logic's ontological correlate, or in his words, “logic's adult sister”.[41]

He develops an epistemology of mathematics which we discuss in detail in Chapter 2. However, we must point out that formal interconnections of objects are given in intuition along with the objects themselves. In other words, our consciousness constitutes at once the objects *and their formal relationship* in a state-of-affairs, and not through a process of psychological reflection as many phenomenalists and psychologists thought at the time.[42]

Since formal components are intuitively given in a state-of-affairs, and once abstracted from all empirical content these reveal *a priori* legality, we must explore three groups of problems regarding the laws of pure logic and their relationship with mathematics. These will reveal three logical strata and correlatively three mathematical strata:

- **First Problem**: We should look for primitive concepts that make such legal interconnections possible in an objective and theoretical sense. On one hand, all the ideal interconnections of truths in science form a complete deductive unity. If we substitute all propositions with indeterminates (variables), we are able to discover their elementary forms of deductive combinations. Husserl calls these elementary forms “*meaning categories*”, which include: the copulative, disjunctive and hypothetical combinations of propositions in new propositions. So, what we today would call “connectives”, would form part of these meaning categories: conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), equivalence ($\leftrightarrow$), among others. However, meaning categories also include forms of combination between concepts themselves so that meaningful propositions can be expressed. This means that these categories include subject-predicate forms, forms of copulative or disjunctive combinations,

41 LI. Vol. I. §§46, 63, 66.
42 LI. Vol. II. Inv. IV. §§43, 45.

forms of plural, and so on.[43] In other words, we are talking about a *morphology of meanings*, or a *pure universal grammar*, and Husserl calls the ideal laws that govern this sphere "laws to prevent non-sense [*Unsinn*]". Thanks to Carnap, today we call them "*formation rules*".[44] In this logical stratum, we can construct an infinity of possible forms of meaningful propositions. For example, in the case of subject-predicate forms, if I say "*S* is *p*", we can also use this proposition to form another proposition "*S*(*p*) is *q*", and at the same time use this judgment to form a new one "*S*(*p*,*q*) is *r*", and we could continue indefinitely.[45]

On the other hand, correlated with the morphology of meanings, we find a *morphology of intuitions,* which includes all the laws of adequate meaning fulfillment in objectualities.[46] According to Husserl, once we substitute the sensible or material objects with indeterminates in any state-of-affairs, we find in this stratum the elementary formal components of every objectuality which he called "*formal-objectual categories*" or, more properly, "*formal-ontological categories*". These include concepts of object, state-of-affairs, unity, plurality, cardinal number, ordinal number, sets, part and whole, among others. This morphology of intuitions is nothing more than a morphology of formal-ontological categories or of formal-objectual categories.[47]

- **Second Problem**: We have to deal with the ideal laws of the objectual validity of formal-ontological categories and the truth and falsity of propositions or judgments on the basis of meaning categories. On the side of logic, we find a stratum Husserl called "logic of consequence", which is concerned with theories of forms of deductions or theories of inference which preserve truth. He would call the laws of this sphere "*laws to prevent counter-sense* [*Windersinn*]", which include syllogistics among other

43 LI. Vol. I. §67.

44 LI. Vol. II. Inv. IV. §§10-14; Hill & Rosado, 2000, p. 203.

45 LI. Vol. II. Inv. IV. §§10-14; FTL. §§12-13.

46 LI. Vol. II. Inv. VI. §59. See Section A-2, in Appendix A for more details.

47 LI. Vol. I. §67; Bernet, Kern & Marbach, 1999, p. 48.

forms of deductions. Today, thanks to Carnap, we call them "transformation rules".[48] If we have a logical proposition like "$(\alpha\rightarrow\beta)\leftrightarrow\neg(\neg\alpha\vee\beta)$", according to the first logical stratum it would be meaningful, but according to this second stratum, it would be contradictory.[49]

Distinct from the logic of consequence we find the "logic of truth", where the concepts of "truth", "falsity" and other related concepts are being considered. Any deductive relationship between judgments can be turned into a deductive interconnection of truths if these judgments are determined to be true on the basis of states-of-affairs.[50] If the judgments themselves are formally contradictory, this *a priori* excludes them from being true.

Logic's second stratum has its mathematical correlate. It deals with the being and non-being of "objects in general" and of "states-of-affairs in general". Each theory in this level of formal ontology (mathematics) is founded on formal-ontological categories. We consider many sorts of theories of plurality founded on the category of plurality, arithmetic founded on the category of number, set theory founded on the category of sets, and so on.[51] Husserl holds the ideal completeness of these categorial theories on ideal grounds for any true and valid judgment as well as any valid or true combination or arrangement of objects in any way. It makes possible a science of the conditions of possibility of any theory in general.[52]

- **Third Problem**: The discovery of a morphology of meanings and forms of inference point to a still higher logical level, which deals *a priori* with all essential forms or sorts of theories and their corresponding laws. In other words, it points to a *theory of all possible forms of theories* or a *theory of deductive systems*. This logical stratum

48 Hill & Rosado, 2000, p. 203.
49 LI. Vol. I. §68; LI. Vol. II. Inv. IV. §12; FTL. §§14-22.
50 FTL. §§15, 19. See Section A-2, Appendix A for more details on truth fulfillment.
51 LI. Vol. I. §68.
52 LI. Vol. I. §§65, 68.

> deals a priori with the essential sorts (forms) of theories and the relevant laws of relation. The Idea [of science of the conditions of the possibility of theory in general] arises, all of this being taken together, of a more comprehensive science of theory in general. In its fundamental part, the essential concepts and laws which pertain constitutively to the Idea of Theory will be investigated. It will then go over to differentiating this Idea, and investigating possible theories in a priori fashion, rather than the possibility of theory in general.[53]

We could establish theories of possible relations between pure forms of theories, investigate these logical relations and the deductions from such legal interconnections. The logician is perfectly free to see the extension of this deductive, theoretical sphere of pure logic.[54]

This logical level is correlated with what Husserl, inspired by Bernard Riemann, called a "*theory of manifolds*", which he describes as the "supreme flower of modern mathematics".[55] This theory of manifolds is a free investigation where a mathematician can assign several meanings to several symbols, and all their possible valid deductions in a general and indeterminate manner. Through the posit of several indeterminate objects as well as any combination of mathematical axioms, mathematicians can explore the apodeictic interconnections between them, just as long as consistency is preserved. Husserl says:

> The most general Idea of a Theory of Manifolds is to be a science which definitely works out the form of the essential types of possible theories or fields of theory, and investigates their legal relations with one another. All actual theories are then specializations or singularizations of corresponding forms of theory, just as all theoretically worked-over fields of knowledge are individual manifolds. If the formal theory in question is actually worked out in the theory of manifolds,

53 LI. Vol. I. §69.
54 LI. Vol. I. §69; FTL. §28.
55 LI. Vol. I. §70.

> then all deductive theoretical work in constructing all actual theories of the same form has been done.[56]

Both, the theories of all possible forms of theories (pure logic) and the theory of manifolds (mathematics in its highest expression) together form a *mathesis universalis*, the most universal mathematics of all. He gives an example of the partial realizations of this *mathesis universalis* or theory of manifolds:

> When I spoke above of theories of manifolds which arose out of generalizations of geometric theory, I was of course referring to the theory of *n*-dimensional manifolds, whether euclidean and non-euclidean, to Grassmann's theory of extensions, and, among others, to the related theories of W. Rowan Hamilton, which can be readily purged of anything geometric. Lie's theory of transformation-groups and G. Cantor's investigations into numbers and manifolds also belong here.[57]

Rosado (in Hill & Rosado, 2000) also points out that Husserl's view of logic and mathematics is amazingly very contemporary. General topology, universal algebra, category theory and other mathematical disciplines can be seen as “important partial realizations of Husserl's view of mathematics as ultimately the theory of all forms of possible multiplicities or, to use a more frequent term nowadays, forms of possible structures”.[58]

56 LI. Vol. I. §70.
57 LI. Vol. I. §70.
58 p. 205.

CHAPTER 2: REPLY TO USUAL OBJECTIONS TO PLATONISM

There are many philosophers who reject mathematical platonism on a basis very similar to that expressed by Michael Dummett:

> If mathematics is not about some particular realm of empirical reality, what, then *is* it about? Some have wished to maintain that it is indeed a science like any other, or rather, differing from others only in that its subject-matter is super-empirical realm of abstract entities, to which we have access by means of an intellectual faculty of intuition analogous to those sensory faculties by means of which we are aware of physical realm. Whereas the empiricist view tied mathematics too closely to certain of its applications, this view generally labeled "platonists," separates it too widely from them: it leaves unintelligible how the denizens of this atemporal, supra-sensible realm could have any connection with, or bearing upon, conditions in the temporal, sensible realm that we inhabit.
>
> Like the empiricist view, the platonist one fails to do justice to the role of proof in mathematics. For presumably, the supra-sensible realm is as much God's creation as is the sensible one; if so, conditions in it must be as contingent as in the latter.[59]

In other words, here we find three objections to platonism in general. The first one is of epistemological nature, if abstract objects cannot be perceived or have any causal relationship with us, how do we know them? The second has to do with the question about which intellectual, mental or psychological ability do we use to know them. Finally, are these abstract objects contingent (God's creations)?

59 Dummett, 2002, p. 20.

This chapter will address these usual objections to platonism. It will not be an exhaustive refutation of every argument against platonism presented in philosophy. However, it will provide enough answers to cover the necessary bases to explore the extent of the relationship between formal and natural sciences, and see if *a posteriori* matters-of-fact can revise logic and mathematics.

Before we begin, we must point out that Benacerraf (1983) challenged philosophers of mathematics in general to formulate a philosophical doctrine of mathematics that fulfills the following requirements:

(1) First, any proposed good philosophy of mathematics should reach the objectivity and high level of consistency of mathematics, thus providing an adequate account of mathematical truth. This has been accomplished by platonism.[60]

(2) Second, it should provide an adequate epistemology of mathematics, which supposedly platonism is not able to offer.[61]

We will certainly address the problem presented in the second requirement that, according to Benacerraf, platonism is not able to satisfy.

2.1 — CAUSAL KNOWLEDGE FOR LEGITIMATE EPISTEMOLOGY?

Platonism as such is not positing objects “floating in the middle of the air”, or “creatures of God”, or anything similar. We are talking about an abstract reality that is absolutely necessary, and is *the formal condition of possibility of any object or any state-of-affairs whatsoever*. Therefore, we are referring to an ideal realm autonomous from human psychology and the physical world, and in the specific case of pure logical and mathematical truths, completely independent. We have to add, as Husserl pointed out, that we are not talking about numbers or other ideal objects and concepts as forming

60 p. 408.
61 p. 409.

part of the real (*reell*) world, which would lead to a “metaphysical hypostatization”.[62]

This abstract nature of logical and mathematical entities makes it impossible to establish a causal relation between such ideal realm on the one hand, and the psychological acts and physical world on the other. This goes against certain philosophers' doctrines that ask for causality as a condition for all acceptable epistemology. The requirement of causality for an adequate epistemology of mathematics does not seem convincing for many reasons. First, it is not self-evident. Our view of mathematical epistemology is non-causal, the abstract entities themselves do not cause our knowing them, and yet, our cognitive processes are perfectly natural. Whichever causal mechanisms our brain or mind uses to constitute objectualities and essences do not depend causally on numbers or ideal entities themselves.

Second, in natural science, the criterion of causality is not applied to all cases. For example, in quantum physics we cannot establish certain causal relations among subatomic particles, and yet we have legitimate scientific knowledge about them. The following example shows one case:

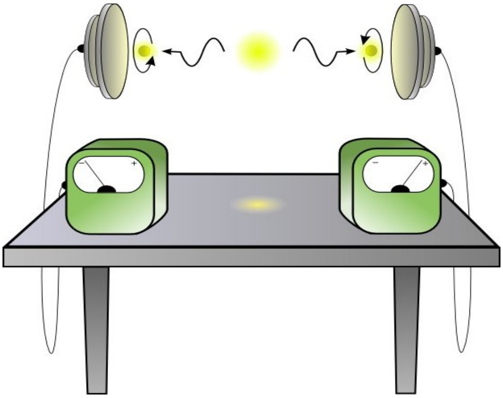

Illustration 1

62 LI. Vol. II. Inv. II. §7. See p. 16 of this book.

Illustration 1 shows an EPR Machine[63]. Let us suppose we have a positronium[64] that annihilates and two photons are released in opposite directions. According to experiments carried out by scientists, one photon will spin in one direction and the other will spin in the opposite direction. Apparently, there are no exceptions to this phenomenon. We could conjecture why this happens, but it seems we cannot formulate a theory that can causally link both spins. We could say that this happens because one photon seems to affect the other. Since photons travel at the speed of light, the effect of one on the other should be *faster* than the speed of light, which goes against special relativity's assertion that nothing can travel faster than this speed. We could perhaps think that what affects their spin is at the very source of the photons at the moment of annihilation, but quantum physics states that if such a thing were true, the outcome would be completely different. Therefore, *we can know non-causally* the spin of one photon by knowing the spin of the other one. The causal requirement for legitimate knowledge is refuted.[65]

63 "EPR" refers to the Einstein-Podolsky-Rosen Paradox ("EPR Paradox" for short). Albert Einstein, Boris Podolsky, and Nathan Rosen tried to refute the claim made by quantum physicists that somehow the observers have influence over quanta. They made their case the following way: suppose there is a positronium and that it releases two photons which travel to two opposite sides of the galaxy. Quantum theory suggests that the photons would literally be anywhere within a probabilistic path. However, there are two detectors on both sides of the galaxy, and despite the fuzzy path of the photons, those two detectors will detect both photons at exactly the right locations. Given that faster-than-light communication among both photons is impossible, the photons would "magically" know where and when each other is being detected to locate each other at exactly the right positions. For them, this shows that the Copenhagen interpretation of quantum phenomena is wrong, since it leaves unexplained how does this happen. In other words, quantum physics as formulated is not complete. The EPR machine operates in a slightly different manner. Based on suggestions by physicist David Bohm, this machine does not measure position but spin, making and EPR-like experiment possible (Popper, 2000, pp. 14-25).

64 A positronium is an atom consisting of an electron and a positron.

65 Brown, 1999, pp. 16-17.

2.2 — SEMANTIC FICTIONALISM

Some philosophers of science like Mario Bunge are not completely satisfied with platonist proposals, not even Popper's semi-platonism. Bunge chooses fictionalism with respect to propositions, logic, and mathematics as an alternative. He says that there are no "propositions-in-themselves" as Bernard Bolzano, Gottlob Frege, Edmund Husserl and others seem to believe, and says that we have to make believe that they exist. He says that unlike Mickey Mouse or Superman,[66] we can establish formal and semantic exactness using these fictitious notions.[67]

Since his conception of meanings and propositions is fictionalist, the truth value for these propositions is linked to human psychology and to scientists' and mathematicians' "conventional knowledge". He gives the following examples to show that not all propositions are true or false, and that there are other propositions with no truth value: the trivial propositions "the trillionth decimal character of π is 7", "in the center of the Earth there is a piece of iron", and the non-trivial proposition "The value of function *f*, representative of a *P* property for an individual *x*, is *y*" where *f* is an attribute of the figure in a scientific theory. Bunge says that in these and other cases a proposition lacks truth value, not because one has not been assigned to it, but because it is impossible to decide if it is true or false.[68]

In light of platonism, Bunge's real problem is that he supposes that a proposition is true or false only when we *know* such truth value on the basis of facts. For platonists, it would be as absurd as claiming that the proposition "the Sun is in the center of the Solar System" was not true one time because human beings did not know it. The proposition just expressed has always been (and is) objectively true, even when throughout the centuries humanity believed the false proposition "the Earth is the center of the universe". That humanity did believe that "the Earth is the center of the universe" is true did not

66 Bunge (1997) originally used Minerva and Mafalda, but it is the same idea anyway (p. 65).
67 Bunge, 1997, p. 65.
68 Bunge, 1997, p. 72.

make such a proposition true. Bunge confuses holding a proposition as true with it *being* true.

For a platonist, it is possible for a proposition to have a truth value, even if we do not know it, or if we do not know the proposition itself. Let us take the example of “the trillionth decimal character of π is 7”. This proposition will depend ontologically on the number π itself. According to platonists, there *is* a trillionth decimal character, but we *do not know* which one it is. Knowing and being are two very different things. Unfortunately, Bunge's own conception of numbers as being only products of human imagination leads him to confuse knowledge of the number with its being, and he imagines that this is enough to refute platonism. Epistemological limitation is not the same as lack of truth value of a proposition.

With respect to “the value of function *f*, representative of a *P* property for an individual *x*, is *y*”, this itself is a function, not a proposition. Its truth value will depend on the values to be assigned to, *P*, *x*, and *y*. Once the assignment is made, *then* a proposition is expressed. At least in logic we have made a very clear difference between a propositional *function* such as “$F(x)$” where x is a variable, from singular logical *statements* such as “$F(a)$” (where a is a constant), or as propositions where a quantifier is prefixed such as:

$$((\forall x)(F(x)\rightarrow G(x))\wedge(\forall x)(G(x)\rightarrow H(x)))\rightarrow(\forall x)(F(x)\rightarrow H(x))$$

As Bunge clearly stated, truth value can only be applied to propositions. If this is correct, they are not applied to functions.

Bunge develops an epistemological doctrine about propositions which does not let us talk about *all* propositions as being true or false. His proposal overcomes some difficulties with Quine's statements about the problem of identity of meanings.[69] For him, it is possible to say that there are propositions, just like mathematics can state that there are numbers, but without stating that there are “propositions-in-themselves” or “numbers-in-themselves”.

In light of this epistemological doctrine, he establishes the unequivocal ontological existence of physical organisms capable of thinking and forming judgments. Possibly there are *no identical*

69 Quine, 1953, pp. 20-46.

judgments, nor an identical process in two or more brains, or even in the same brain. However, there are *similar judgments*. If that similarity is pointed out enough, then we can conclude that two brains are thinking the same proposition. This leads him to think that concepts, propositions, and numbers lack autonomous existence. At most, we can define their existence psychologically: "*x* exists conceptually if there is at least one brain that thinks *x*".[70]

Apparently, Bunge is not aware that platonists already have replied to such statements. For example, in *The Foundations of Arithmetic*, Gottlob Frege addressed precisely this kind of psychological view of propositions and numbers. Such view states that there are no identical propositions or notions, since they are subject to psychological change, and subjective notions cannot be shared. Let us suppose, for the sake of the argument, that mathematical objects are figments of our brains' neurons and nothing else beyond that. If there is no abstract ideal realm, nor a cultural realm that is essentially abstract, it is difficult to account for our agreement about propositions and mathematical entities.[71] Throughout history and in every society, humans have represented numbers in very different ways (think about Greek numbers, Roman numbers, Hebrew numbers, Arabic numbers, and so on), and yet, they all have the same *meaning*, and refer to exactly the same objects without there being any difference between the properties of the Arabic "1", the Hebrew "א" or the Roman "I". We can all agree that the number five is a prime number, the result of adding 2 and 3, the result of adding 1 and 4, the number whose square is 25, and whose cube is 125, and so on.[72] Bunge does not solve the problem of defining numbers as subjective representations. Each representation of the number one (as similar as it can be to other number ones in

70 Bunge, 1997, pp. 75-76.

71 Bunge's position is paradoxic. On one hand, he accepts the existence of conventions, which do belong to the cultural realm. On the other hand, he says that no cultural abstract products exist beyond the brain. Then conventions cannot exist in any real sense, because they are abstract. The cultural realm is rejected also by Bunge, especially as a form of rejecting Popper's World 3.

72 FA. pp. 33-38.

other people's brains) *will always be different* with different subjective characteristics. So we cannot talk about *the* number one. Instead we would be talking about 1, 1', 1'', 1''', and so on.[73] It would be a miracle if rational beings can understand *one* proposition independently of non-shared mental representations.

2.3 — Non-Twilight Zone View of Formal Knowledge: Platonist Epistemological Theories

Platonist epistemology is often accused of being mysterious and mystical. As Katz (1998) has pointed out, the fact that something is mysterious does not make it illegitimate. Philosophy is replete with mysteries, and it thrives in solving those mysteries rationally.[74] It has been accused of mysticism, because it would require a kind of an extra-eye to "see" those mysterious entities floating in the air waiting for someone to grasp them and know them. Mysticism proposes that we can obtain knowledge beyond our cognitive faculties.[75] In this section, we will see that this is not so. First, we will see three platonist theories regarding formal knowledge, choose one of them, and then look at empirical evidence for this in animals and humans.

2.3.1 — The Mind's Eye

Several platonist philosophers of mathematics like James Robert Brown posit the existence of "the mind's eye" which, in many ways, is analogous to the physical eye. We must understand the term "mind's eye" only as a metaphor to describe that aspect of the mind that lets us access the mathematical realm.[76]

For example, says Brown, we ordinarily perceive a cup of coffee on the table, and we are compelled to believe in the actual existence of the cup. In the same way, we are compelled to believe in the actual

73 FA. pp. 33-38, 47-51.
74 p. 33.
75 p. 33.
76 Brown, 1999, p. 13.

existence of mathematical objects and truths, such as the necessity of the proposition "2 + 2 = 4" and the contingency of the judgment "the bishops move diagonally".[77]

An objection may be raised that we know more or less the physical processes by which our physical eyes perceive objects. On the other hand, the way this mind's eye perceives needs to be explained.[78] Brown argues against this objection saying that this mystery of the "mechanics" of the mind's eye does not undermine this platonist claim that we perceive abstract mathematical objects. *It is a fact*. As we have seen, the criterion of causation has been refuted, so opponents cannot use it against platonism.[79]

Finally, Brown argues in favor of the concept of the mind's eye by recognizing the fallibility of formal *a priori* knowledge. As Descartes argued very well, perceptions can deceive us. What we perceive does not necessarily correspond to the actual facts. What seems to be a lake from afar may be, in reality, a hallucination, an illusion, or a mirage. The same thing happens to the mind's eye. When we formulate a mathematical conjecture that is not self-evident, we may be subject to error. He calls this philosophical standpoint "fallibilist platonism".[80]

The epistemological proposal of the mind's eye has one advantage, and it is precisely fallibilist platonism. As I will show later in this chapter, *a priori* knowledge can be fallible.

However, the "mind's eye" doctrine has several serious disadvantages. All that this doctrine says is that we perceive abstract objects, but it does not begin to explain *how* we perceive them. Also, it seems that the analogy with the physical eye is flawed. In the realm of all possible worlds, we can see as possible that the world as we experience and perceive it is an illusion, and that our compelling belief about the existence of a cup that we perceive may not be true. But in the case of formal knowledge there seems to be no possible scenario where the principle of no-contradiction is false, or that the

77 Brown, 1999, pp. 13, 15.
78 Brown, 1999, p. 15.
79 Brown, 1999, pp. 15-18.
80 Brown, 1999, p. 14.

square root of two is not an irrational number. In this case, *a priori* knowledge seems to be far more compelling than an *a posteriori* knowledge of the existence of a cup based on physical perception.

We also need to recognize another aspect where the analogy with the physical eye fails. Let us assume, for the sake of the argument, that it is still a mystery today how our brain works regarding the way physical perception operates. We can still argue that despite this mystery, something seems to be *given* to us. Even if the world were an illusion, it would still be reasonable to suppose its existence. So, even if we don't know how the brain works, we *know* that there must be a natural mechanism for perception. The problem with abstract mathematical objects is that they are not perceived sensibly, which leads to several hypotheses that would seem *prima facie* viable regarding our mathematical knowledge: numbers are conventions, or fictions, or arise from sensible experience, and so on. These explanations do seem plausible in relation to natural processes, and the hypothesis of the "eye" that "grasps" abstract mathematical objects is not itself powerful enough to be an explanation.

2.3.2 — REALISTIC RATIONALISM

Jerrold Katz is one of the most recent philosophers of mathematics who tried to meet Benacerraf's challenge, and had tried to account for a platonist epistemology of mathematics without the need to establish a certain kind of supernatural faculty that grasps abstract entities. He calls his philosophical position "realistic rationalism". "Realism" in this context means that mathematical truths refer to abstract objects: entities no located in space and time.[81] "Rationalism" means that reason alone is the source of all formal *a priori* knowledge.[82] Therefore, his epistemological theory can be rightfully called "rationalist epistemology".[83]

Katz establishes several conditions of an adequate rationalist epistemology:

81 Katz, 1998, p. 1.
82 Katz, 1998, p. 24.
83 Katz, 1998, pp. xxxii, 38.

1. There should be no causal relationship between the knower and the abstract objects to be known.[84]

2. The truth of a mathematical proposition depends exclusively on the nature of the proposition itself. For example, the truth of propositions about physical objects depends on the facts of the physical world. The knowledge of mathematical truths depends ultimately on their *a priori* nature and their epistemological foundation in reason.[85]

3. That the truths discovered through reason should be justified according to the objects they talk about, mathematical entities and their properties.[86]

Contrary to the objects of the external world, whose nature is contingent, all abstract mathematical entities have necessary properties. The fact that the square root of two is an irrational number, and that two is the only even prime number are examples of this. Both statements about number two are examples of apodeictic conclusions at which we arrive using *Reductio ad Absurdum*.[87]

Realistic rationalism recognizes the existence of a kind of *mathematical intuition* which is necessary to find these mathematical truths. This is not a mystical intuition, but rather one that can be explained naturally. He uses this example:

> It is also crucial to the notion of intuition in our sense that intuitions are apprehensions of structure that can reveal the limits of possibility with respect to the abstract objects having the structure. Intuitions are of structure, and the structure cannot be certain ways. [. . .] Consider the pigeon-hole principle. Even mathematically naive people immediately see that if *m* things are put into *n* pigeon-holes, then, when *m* is greater than *n*, some hole must contain more than one thing. We can eliminate prior acquaintance with the proof of the pigeon-hole principle, instantaneous discovery of the proof, lucky guesses, and so on as

84 Katz, 1998, pp. 34-35.
85 Katz, 1998, p. 36.
86 Katz, 1998, pp. 36-41.
87 Katz, 1998, pp. 39-40.

"impossibilities." The only remaining explanation for the immediate knowledge of the principle is intuition.[88]

In this way, Katz refutes antirealists who accuse realists of not providing an alternative to this mathematical intuition that they reject as mystical.

It seems by the context of this epistemological doctrine that this notion of "mathematical entities" is conceptualized as structures, not as objects in the fregean sense.[89] For this reason, we find Katz as closer to Nicolas Bourbaki and Edmund Husserl than to Frege.

Katz also tells us that he favors a kind of fallibilist platonism, because mathematical intuition itself is also subject to error. The revision of logic and mathematics is the result of our cognitive limitations of mathematics, which are eventually superseded through reason and careful analysis.[90]

Katz concludes his presentation of his rationalist epistemology, showing that there is no begging a question within reason. For him, to beg the question within rationalist epistemology would be like falling into a kind of Cartesian skepticism, which questions even the most elementary mathematical principles. If this is the game empiricists wish to play against realism, then we should also remember that they can fall into a humean skepticism if we ask on what logical basis can we say that we have one single pencil in our hand if all the flow of experience never remains the same, and that the fact that we perceive the pencil does not mean necessarily that it exists or that the pencil is the source of our sensations. Therefore, if the empiricist is willing to accept within reason (without falling into radical skepticism) that what we have in our hand has the traits of a pencil, and that such phenomenon can mean that the pencil does indeed exist, the empiricist must grant that through *Reductio ad Absurdum* we should show the mathematical truth that, for example, the number 2 is the only even prime number.[91]

88 Katz, 1998, p. 45.

89 For Frege, objects are saturated entities, and numbers are seen as logical objects in his philosophy of mathematics (FC. pp. 17-18).

90 Katz, 1998, pp. 48-51.

91 Katz, 1998, pp. 51-58.

Also, he pointed out that not holding up some basic logical principles that are necessary for any rational reasoning would lead immediately into paradoxes, such as the *paradox of revisability*. The paradox goes like this:

> Since the constitutive principles are premises of every argument for belief revision, it is impossible for an argument for belief revision to revise any of them because revising any one of them saws off the limb on which the argument rests. Any argument for changing the truth value of one of the constitutive principles must make a conclusion that contradicts the premise of the argument, and hence must be an unsound argument for revising the constitutive principle.[92]

The principle of no-contradiction is an example. If all logic were revisable, the principle of no-contradictions should also be revisable. If it is revisable, it is because certain "events" seem to contradict the principle of no-contradiction. Since the principle of no-contradiction should be supposed as a premise necessary to find a contradiction, then the reason to question this principle is also questioned, and there would be no base at all for the revision. Therefore, we reach the conclusion that the principle of no-contradiction is revisable and it is not revisable.[93] He states that realistic rationalism meets Benacerraf's challenge, because it explains how we can grasp mathematical entities and truths non-causally.[94]

Katz's advantage over the mind's eye proposal is significant. He shows the mechanisms through which we can achieve a logical or mathematical truth without going into metaphors like the "mind's eye". He also embraces "fallibilist platonism" and gives us a glimpse of how we are able to grasp truths about mathematical entities.

However, there is an aspect of Katz's epistemological doctrine that is not completely satisfied. For example, he implicitly recognizes that we are able to perceive some mathematical structures in a given experience. In fact, much later in his book, as we shall see more thoroughly with Husserl, he says that we are able to see these

92 Katz, 1998, p. 73.
93 Katz, 1998, pp. 73-74.
94 Katz, 1998, pp. 26-28, 54, 55.

structures along with sensible objects. We are able to recognize these abstract objects once we "purify" them from the sensible components.[95] However, he does not talk much about the way different kinds of mathematical objects can be found in experience, he just assumes that they are there and that we can perceive them rationally. More to the point, his notion of mathematical intuition is not very clear, since he seems to confuse the "perception" of structures with the intuition that lets us find their necessary properties or relationships. As we shall see, his portrayal of mathematical intuition with the pigeon-hole example would seem a confusion of categorial intuition and eidetic intuition in husserlian terms.

Finally, Katz lacks a theory explaining how from these elementary structures we are able to reach the level of mathematical complexity we know today (various theories of numbers such as complex numbers, category theory, universal algebra, general topology, among others)

I do not wish to finish this section without making a comment on a significant point made by Katz regarding the revisability paradox. As we shall see, this paradox is a very important stumbling block to Quine's assertion that in principle all logical truths can be revised.

2.3.3 – HUSSERLIAN EPISTEMOLOGY OF MATHEMATICS

To address Husserl's epistemological proposal, we have to realize that it is a result of his epistemological interest in mathematical knowledge. After his antipsychological turn, in a section called "Sensibility and Understanding", in the sixth investigation of his philosophical work, *Logical Investigations*, he explained in full detail his phenomenological approach to mathematics and addresses the problem of mathematical knowledge. To understand this, we must look at the way judgments or propositions are fulfilled by states-of-affairs.

According to phenomenology, intentionality is an essential property or consciousness, and it consists in directing ourselves to

95 For instance, "the set of all my children" is an example of "impure sets" (Katz, 1998, p. 133).

objects. An intentional act is an act of thinking, a *cogito* in the cartesian sense. Every *cogito* has a *cogitatum*, every act of thinking has an object that is being thought of.[96]

As we have seen in Chapter 1, Husserl says that an objectual act, which is also an intentional act, constitutes an objectuality, a state-of-affairs. As we have seen, a state-of-affairs is composed of sensible objects interconnected by categorial forms. For him, objectualities can be correlated with judgments that are fulfilled in them (true judgments or propositions). If we say that "Mary is taller than John", such a proposition is fulfilled precisely on the objects Mary and John, *and* the way they are related by categorial forms. If we say, "John is shorter than Mary", this judgment refers to *another* state-of-affairs, another objectuality. The *situation-of-affairs* (*Sachlage*), the sensible objects that serve as a passive basis for both states-of-affairs, remains the same. If we have sensible objects a and b, the propositions "$a < b$" and "$b > a$" would both refer to two different states-of-affairs, but they are based on the same situation-of-affairs.[97]

This is the way propositions about the temporal world are fulfilled. Our words can refer to sensible objects: "house", "book", "bike", "Aristotle", and many other concepts. However, we cannot disregard *formal words*, which are words with no sensible correlates, like "is", "the", "three", "below", "over", "under", "then", "and", or "first". These refer to categorial forms.[98] Given that they have no sensible correlates, how do we know about them?

An intentional analysis can reveal how states-of-affairs are constituted. The origin of all our knowledge begins with two sorts of intuitions: *sensible intuition* and *categorial intuition*. Sensible intuition is what lets us grasp sensible objects directly. We can identify two kinds of sensible intuition: *sensible perception* and *sensible imagination*. The latter lets us constitute sensible objects in our fantasy or imagination, while the former lets us grasp sensible objects that are present "in-the-flesh" so-to-speak. Through sensible

96 I. §28.
97 LI. Vol. II. Inv. VI. §48; EJ. §§58-60.
98 LI. Vol. II. Inv. VI. §40.

perception, sensible objects are given immediately in a low-level objectual act. They appear as "external" objects before us.[99]

On the other hand, categorial intuition lets us constitute categorial forms *on the basis of* sensible objects. In this sense, they are higher-level formal concepts that relate low-level sensible objects.[100] However, these categorial forms are not given in the same manner than sensible objects, because they require the intervention of understanding through categorial acts. In the act of constituting a state-of-affairs through a categorial act, objects are given in a *specific manner*. We can either talk about a *set* of pencils, or about *five* pencils or the *total* of pencils, and other kinds of states-of-affairs based on one situation-of-affairs.

In all of these cases, we treat sensible objects as a *unity of experience*. This lets us use these states-of-affairs as basis for other categorial or objectual acts, there can still be higher objectual levels.[101] This can be clearly seen in the case of sets. He gives us the following example:

> In the domain of receptivity there is already an act of plural contemplation in the act of collectively taking things together; it is not the mere apprehension of one object after the other but retaining-in-grasp of the one in the apprehension of the next, and so forth [. . .]. But this unity of taking-together, of collection, does not yet have *one* object: the pair, the collection, more generally, the set of the two objects. It is limited consciousness, we are turned toward one object in particular, then toward another in particular, and nothing beyond this. We can then, while we hold on the apprehension, again, carry out a new act of taking together [of, let us say] the inkwell and a noise that we have just heard, or we retain the first two objects in apprehension and look at a third object as separate from others. The connection of the first two is not loosened thereby. It is another thing to the combination or to take a new object into consideration in addition to the two objects already in special combination. And then we

99 LI. Vol. II. Inv. VI. §45.
100 LI. Vol. II. Inv. VI. §46.
101 LI. Vol. II. Inv. VI. §46.

> have a unity of apprehension in the form of {{A,B},C}: likewise {{A,B},{C,D}}, etc. It is necessary to say again here that each apprehension of a complex form has as objects A B C . . . and not, for example {A,B} as *one* object, and so on.[102]

In other words, the example of sets shows us that there can be a hierarchy of objectualities. If we wish to find the sensible subtracts of all of these objectual acts, we can trace them down to the sensible components.[103]

Of course, in states-of-affairs we find sensible components *and* formal components, and we are able to distinguish matter and form. They are essentially the result of *mixed categorial acts* where we constitute a state-of-affairs which include empirical content and formal content. As we have seen, every categorial form given in categorial intuition rests on sensible intuition.[104]

This has to be distinguished from *pure categorial acts* where *pure categorial concepts* are constituted without any sort of sensible components. We know they are constituted, because all of the *mathesis universalis* is empty of all sensible content.[105] How can it be constituted?

To reach the level of *pure* categorial forms, an act of *categorial abstraction* is needed. With this act of understanding, we purge a state-of-affairs of all of its sensible contents, and stay with the categorial forms. In this process, all sensible concepts are substituted with indeterminates (variables), and the only thing that matters in mathematics and logic is the essential relationship between these categorial forms.[106] In the case of mathematics we explore the necessary or essential relationship between formal-ontological categories: unity, plurality, relation, cardinal numbers, ordinal numbers, part and whole, and so on. What lets us constitute these essences (necessities or possibilities) is what Husserl called "*general intuition*", "*essential intuition*" or "*eidetic intuition*". Through eidetic

102 EJ. §61.
103 LI. Vol. II. Inv. VI. §60.
104 LI. Vol. II. Inv. VI. §60.
105 LI. Vol. II. Inv. VI. §60.
106 LI. Vol. II. Inv. VI. §60.

intuition, essences are constituted, and we are able to discover the necessary relationships between concepts, be them material or formal.[107]

Also, Husserl explains that to understand logic, we must understand that the acts by which we constitute a state-of-affairs, while we are able to formulate judgments or propositions through meaning acts. The most basic of these meaning acts is expressed by the concept of “being”. Once we constitute a state-of- affairs, through meaning acts we establish that such a state-of-affairs *is* the case. If we find Mary taller than John as a state-of-affairs, through a meaning act we can express that “Mary *is* taller than John”. “Is”, in this case, is a meaning category constituted by a meaning act. Through more meaning acts we can formulate judgments such as: “Mary *is* taller than John *and* John *is* shorter than Mary” where we establish a conjunction among elementary propositions. We can abstract a pure logical truth once the material concepts in propositions are substituted by indeterminates or variables (categorial abstraction). We can obtain from such abstraction true subject-predicate forms, disjunction, conjunction, and so forth: “A *is* B”, “A *and* B”, “A *or* B”, and so on.[108]

Since every objectual act can have a meaning act as correlate, and given the fact that there are unlimited ways to form complex propositions about objectualities, we can also form a hierarchy of meanings which can be traced down and be founded on sensible components, and their complication also has to follow laws, namely, the “laws to avoid non-sense”, which are the laws of pure universal grammar. The essential relationship between meaning categories and between any sort of propositions can be discovered through categorial abstraction, as we explained above. Analytic laws are discovered when these judgments, purified of all material concepts, follow the laws to avoid non-sense, and the laws to avoid counter-sense in order to preserve truth.[109]

107 LI. Vol. II. Inv. VI. §§52, 60; I. §§3, 9-10.
108 LI. Vol. I. §67; LI. Vol. II. Inv. VI. §§42-44, 55, 61, 63-64.
109 LI. Vol. II. Inv. VI. §63.

For Husserl, we are able to reach the *mathesis universalis* from these pure categorial acts. From a transcendental standpoint we can distinguish between what he called “*formal logic*” and “*transcendental logic*”. For Husserl, the world is already given in a determined and determinable way to transcendental consciousness. It is to this experience that we must turn to, so we can understand the relationship between the subjective forms of reasoning (transcendental logic) and the objective logic (formal logic) in its supreme form as *mathesis universalis*. As we will show, he argued that transcendental logic turns to objective formal logic, while formal logic needs transcendental logic to be constituted.[110]

As we have seen in Chapter 1, logic consists in three different strata. From a transcendental point of view, each one of these strata is further removed from psychology until it constitutes formal logic. The first stratum on the side of logic is the morphology of meanings or the universal pure grammar, which follows the laws to prevent non-sense. On the side of mathematics we find the *morphology of intuitions* or *morphology of formal-ontological categories*.[111]

Logic's second stratum builds on the first one, and deals with forms of deductions and demonstrations, and its laws, called by Husserl “laws to prevent counter-sense”. This stratum is still syntactic. However, he makes a distinction between the “logic of consequence” and the “logic of truth”. The latter includes the concept of truth among other related concepts. In other words, there must be a difference between syntax and semantics. On the side of mathematics we also find a second stratum that deals with theories based on formal-ontological categories completely purified from any empirical content, such as set theory, theory of plurality, theory of numbers (arithmetic), and so on. It simply consists in theories of possible objects and states-of-affairs based formal-ontological categories, in other words, based on mathematics' first stratum.[112]

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110 FTL. §§7, 9; EJ. §§3, 11.

111 LI. Vol. I. §67; LI. Vol. II. Inv. IV. §§10-14; LI. Vol. II. Inv. VI. §59; FTL. §§12-13.

112 LI. Vol. I. §68; LI. Vol. II. Inv. IV. §12; FTL. §§14-22.

Finally there is the third stratum, where logic becomes a theory of deductive systems or a theory of all possible forms of theories. On the side of mathematics, its supreme form is a theory of manifolds. Both of these correlated strata become a *mathesis universalis* where we only reason completely on a level of pure forms, and purified from all empirical content, whose truth cannot be reduced to psychological activities. This *mathesis universalis*, from an epistemological standpoint, is the *a priori* basis of any and every science in the broadest sense.[113]

In the next page, we illustrate a summary of Husserl's doctrine regarding these three strata of formal logic and how we reach this *mathesis universalis*.

The advantage of Husserl's epistemology over the other two are great. First, it can tell us *how* from our experience of states-of-affairs we are able to abstract mathematical objects, or formal-ontological categories. We constitute them through objectual acts, purify them from all empirical content, and then reach the theory of manifolds in the husserlian sense. It does not state just that there is a "mind's eye", but Husserl explains that due to the fact that we constitute states-of-affairs through mixed categorial acts, we can perceive sensibly the objects, but *we can also perceive the formal relationship between them*. Katz is close to this notion when he talks about mathematical relationships that need to be purified of all empirical concepts or objects.

However, there is a difference between what Katz calls "mathematical intuition" and what Husserl conceives as mathematical intuition. For Katz, mathematical intuition is closer to a mix of categorial intuition and eidetic intuition, with which we are able to "perceive" the *necessary* and *possible* relationships among formal structures. On the other hand, for Husserl, mathematical intuition is a pure categorial act: categorial intuition and categorial abstraction. That is, we constitute a state-of-affairs and then, through an act of formalization, we obtain the mathematical objects themselves purified from all empirical content. Once these formal-ontological categories

113 LI. Vol. I. §§69-70; FTL. §28-36.

	Formal Theory of Judgment (Formal Apophantics: Pure Logic)	***Formal Theory of Object (Formal Ontology: Mathematics)***
	Mathesis Universalis	
Third Stratum	Theory of All Possible Forms of Theories	Pure Theory of Manifolds
Second Stratum	Logic of Consequence	Being and Non-Being of “Objects” and “States-of-Affairs” in General
First Stratum	Morphology of Meanings	Morphology of Formal-Ontological Categories
Formal Constituents	Meaning Categories	Formal-Ontological Categories
Acts of Understanding	Categorial Intuition and Categorial Abstraction	

Illustration 2

are constituted in their pure form, we develop theories based on them and on the way they are *essentially* related to each other. Furthermore, in a metamathematical level, we are able to posit new mathematical entities (negative roots, fractions, and more) and formulate axioms based on them, or alter axioms in order to explore mathematical consistency (as when the so-called "axiom of the parallels" was not used and mathematicians could discover consistent non-euclidean spaces). Such activity can be carried on as long as logical consistency is preserved.

Hence, we have reached with Husserl a satisfactory platonist epistemology of mathematics.

2.4 — SCIENTIFIC EVIDENCE FOR MATHEMATICAL INTUITION

The evidence for mathematical intuition, meaning categorial intuition and categorial abstraction, is evident everywhere in the world, in every civilization. It is not a mystical intuition, but the result of acts carried out by our mind in order to constitute objects and states-of-affairs. George Ifrah shows how categorial intuition manifests in animals:

> Some animal species possess some kind of notion of number. At a rudimentary level, they can distinguish concrete quantities (an ability that must be differentiated from the ability to count numbers in abstract). For what of a better term we will call animals' basic number-recognition the *sense of number*. [. . .]
>
> Domesticated animals (for instance, dogs, cats, monkeys, elephants) notice straight away if one item is missing from a small set of familiar objects. In some species, mothers show by their behaviour that they know if they are missing one or more than one of their litter. A sense of number is marginally present in such reactions. The animal possesses a natural disposition to recognise that a small set seen for a second time has undergone a numerical change.

> Some birds have shown that they can be trained to recognise more precise quantities. Goldfinches, when trained to choose between two different piles of seed, usually manage to distinguish successfully between three and one, three and two, four and two, four and three, and six and three.
>
> Even more striking is the untutored ability of nightingales, magpies, and crows to distinguish between concrete sets ranging from one to three or four.
>
> [. . .]
>
> What we see in domesticated animals is the rudimentary perception of equivalence and non-equivalence between sets, but only in respect of numerically small sets. In goldfinches, there is something more than just a perception of equivalence — there seems to be a sense of "more than" and "less then". Once trained, these birds seem to have a perception or intensity, halfway, between a perception of quantity (which requires an ability to numerate beyond a certain point) and a perception of quality. However, it only works for goldfinches when the "moreness" or "lessness" is quite large; the bird will almost always confuse five and four, seven and five, eight and six, ten and six. In other words, goldfinches can recognise differences of intensity if they are large enough, but not otherwise.
>
> Crows have rather greater abilities: they can recognise equivalence and non-equivalence, they have considerable powers of memory, and they can perceive the relative magnitudes of two sets of the same kind separated in time and space. Obviously, crows do not count in the sense that we do, since in the absence of any generalising or abstracting capacity they cannot conceive any "absolute quantity". But they do manage to distinguish concrete quantities. They do therefore seem to have a basic number sense.[114]

From a husserlian standpoint, the term "number sense" is equivocal, since we can see categorial objectualities that are not necessarily numerical. There are different categorial forms being

114 Ifrah, 2000, pp. 3-4.

considered, like “more than”, “less than”, and “sets”.. We should remember that no categorial form is reducible to the other, and the notion of sets is not reducible to the notion of number, and vice-versa.[115] This “number sense” only describes the ability animals have of conceiving objects as groups (sets). The data provided by Ifrah only makes sense if we take into consideration that animals perceive categorially because these categorial forms are founded on the sensible objects in the states-of-affairs constituted by their mind.

However, we must recognize that depending on the animals' mental faculties, its perception is limited by its ability to grasp certain quantities of objects. This is not different with humans. Ifrah carries out an experiment which I essentially reproduce in Illustration 3 next page. Try to know how many objects there are in each frame in Illustration 3 *without counting them*, just by glancing at them.

The reader may have noticed that he or she can grasp the number of objects just by glancing at one, two, three or four elements in a frame. But if there are five or more objects it is almost impossible to know right away the right amount without mentally grouping or counting them. Since animals do not know how to count, they have a very limited capacity for numbering. This not only applies to animals, but in some human civilizations around the world, some people are not able to count beyond three or four. For example,the Murray islanders use numbers “one”, “two”, “three”, or , “four” but above that they call “a crowd of . . .”. The same thing happens with Torres Straits islanders, among other isolated civilizations.[116]

Even babies have this mental ability of constituting states-of-affairs through mixed categorial acts. For instance, Karen Wynn has experimented with five-month-old babies and found that they can perform elementary forms of mental arithmetic. Steven Pinker describes this process:

> In Wynn's experiment, the babies were shown a rubber Mickey Mouse doll on a stage until their little eyes wandered. Then a screen came up, and a prancing hand visibly reached out from

115 Husserl, 1994/2004, pp. 12-19.
116 Ifrah, 2000, p. 6.

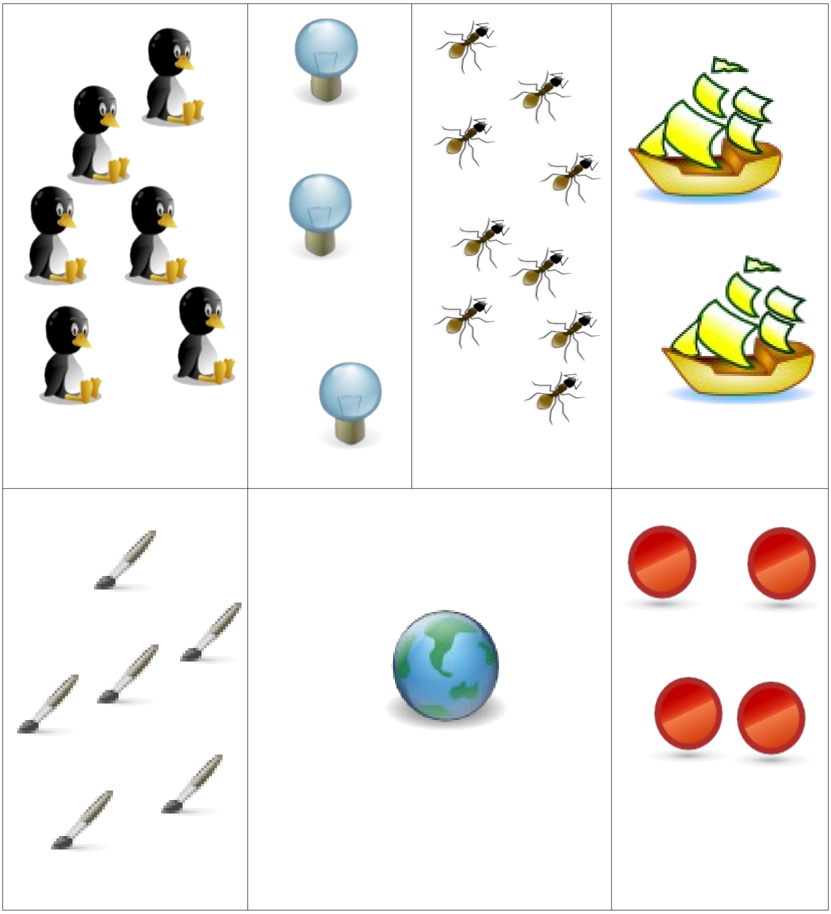


Illustration 3

> behind a curtain and placed a second Mickey Mouse behind the screen. When some screen was removed, if there were two Mickey Mouses visible (something the babies had never actually seen), the babies looked for only a few moments. But if there was only one doll, the babies were captivated—even though this was exactly the scene that had bored them before the screen was put into place. Wynn also tested a second group of babies, and this time, after the screen came up to obscure a *pair* of dolls, a hand visibly reached behind the screen and removed one of them. If the screen fell to reveal a single Mickey, the babies looked briefly; if it revealed the old scene with two, the babies had more trouble tearing themselves away. The babies must have been keeping track of how many dolls were behind the screen, updating their counts as dolls were added or subtracted. If the number inexplicably departed from what they expected, they scrutinized the scene, as if searching for some explanation.[117]

Even a similar methodology to this one has shown that five-days-old babies are "sensitive to numbers".[118] Husserl would say "Of course, they are sensitive to numbers! Their consciousness is constituting states-of-affairs, because that *is* what is originally given. They can see the sensible objects, and simultaneously the categorial forms founded on them."

In our case, we have reached a level of categorial abstraction, which makes us separate sensible experience from formal judgments constituted by formal categories. We do not need to perceive 99,999+1 sensible objects (which would be impossible in light of our recent experiment) to know that it *must* equal 100,000. Here we see our eidetic intuition in action when we recognize the *necessary* relationship between three different numbers (99,999 + 1 = 100,000). In fact, we are able to see that we can even conceive all kinds of mathematical entities, many of them not grasped on sensible basis like irrational numbers, imaginary numbers (negative roots), among others. These are constituted in light of mathematics' third stratum, where the mathematician is free to posit any mathematical objects

117 Pinker, 1994, p. 59. See Wynn, 1992.
118 Pinker, 1994, p. 59.

whatsoever, formulate axioms, and explore them as long as consistency is preserved.

Hence, categorial intuition, categorial abstraction, and eidetic intuition are perfectly natural activities of the mind.

2.5 — PLATONISM AND THE FALLIBILITY OF KNOWLEDGE

One of the least understood aspects of platonism is that of the fallibility of mathematical knowledge. For many antiplatonists, the infallibility of truths and all the logical and objectual relationships of abstract *entities* of all kinds implies necessarily infallibility of logical and mathematical *knowledge*. Also they make the mistake of equating *a priori* knowledge with the *infallibility* of *a priori* knowledge. This constitutes a *non sequitur*, it somehow implies that for an infallible mathematical or logical proposition to exist, it follows that our knowledge about it *has* to be infallible. It would be like saying that Uranus never existed for millenia because humans never knew it. If we take it to the extreme suggested by antiplatonists, Uranus still would not exist because our knowledge of it is limited and not infallible.

In fact, for platonists, logic and mathematics *are* science in the original sense of the word "*episteme*" in Greek. They provide *formal* knowledge. The way we grasp it is completely different from that of natural science. For example, in the case of natural science it is not enough to formulate consistent theories, *we need to find the empirical correlates* that might indicate that our theories are indeed correct. But logic and mathematics do not proceed in such a way. Let us see two very simple, but clear, examples, one of logic and one of mathematics.

THEOREM. *Modus Ponens preserves truth; so if* α *and* $\alpha\rightarrow\beta$ *are true in an interpretation I, then* β *has to be true in I.*

PROOF. Let us suppose that α and $\alpha\rightarrow\beta$ are true in *I*, but that β is false in *I*. If α is true and β false in *I*, then that would mean that $\alpha\rightarrow\beta$ should be false according to the definition of implication.

This, however, would contradict the theorem's position that $\alpha \rightarrow \beta$ is true. Therefore β should be true in *I*. ❑

Notice that in the case of logic we do not have to appeal to experience in any way, its truth relies solely in its own definitions, rules, and axioms.

Let us see now an example in mathematics:

THEOREM. *There is no rational number c which satisfies $c^2=2$, so* $\sqrt{2}$ *is an irrational number*.

PROOF. An irrational number is that which cannot be expressed as a fraction (a ratio). Suppose that there is a fraction p/q reduced to its lowest terms, in such a way that

$$\left(\frac{p}{q}\right)^2 = 2$$

$$p^2=2q^2$$

This means that regardless of *q*'s numerical value, *p* has to be an even number. Therefore, the value of *p* is 2*a*. If we substitute *p* for 2*a* then we will obtain the following equation:

$$(2a)^2=2q^2$$

Therefore:

$$2q^2=4a^2$$

$$q^2=2a^2$$

This means that *q* is an even number. If *p* is even and *q* is even, then that means that they can both be divisible by two, so the fraction p/q is *not* reduced to the lowest terms. This would contradict the hypothesis of the theorem that states that the fraction is reduced to the lowest terms. Hence, there is *no* rational number *c* which satisfies $c^2=2$. ❑

We can see here that there is no appeal at all to sensible experience in order to prove that $\sqrt{2}$ is an irrational number.

For platonists, both of these theorems are true and have always been timelessly true. What we have done is not an “invention” or “construction” of the truth of the theorems but a *discovery* of their absolute truth. The very notion of discovery implies these truths have existed, but were not previously known, and they constitute mathematical knowledge. If platonists argue the *discovery* of such objects, then the antiplatonists' view that platonism posits infallible knowledge is invalid.

By the way, given that logic and mathematics as analytic disciplines carry out their proofs in a completely different way than natural-scientific theories as a body of synthetic judgments, *we can see a qualitative difference between analytic and synthetic propositions*. If this is the case, Quine is wrong in denying the analytic/synthetic dichotomy on the basis that there is no qualitative difference between both kinds of statements. There *is* a difference in kind, and not merely in degree of abstraction.

Now, we have to confront a usual objection against platonists concerning mathematical knowledge: the difficulty of proving certain theorems because their proof is too long and sometimes some important factors are missing. Philip Kitcher, for instance, presents this argument:

> We suppose, with the apriorist, that when we follow a proof we begin by undergoing a process which is an a priori warrant belief in an axiom. The process serves as a warrant for the belief so long as it is present to mind. As we proceed with the proof, there comes a stage when we can no longer keep the process and the subsequent reasoning present to mind: we cannot attend to everything at once. In continuing beyond this stage, we no longer believe the axiom on the basis of the original warrant, but rather because we recall having apprehended its truth in the approximate way. However, this new process of recollection although it normally warrants belief in the axiom, does not provide an a priori warrant for the belief. So, when we follow long proofs we lose our a priori warrants for their beginnings.[119]

119 Kitcher, 1984, pp. 44-45.

But this argument does not refute the fact that logic and mathematics considered in themselves are *a priori*, nor does it refute the statement that we have some *a priori* knowledge. All this argument shows is that our knowledge and psychological processes are limited and that we are not fully able to grasp the truth of certain logical or mathematical propositions. But through a revision of the deductive process using mathematical truths known by all mathematicians, the flaws of such a "long proof" can be shown, then see what went wrong, and, sometimes, how to make it right. Sometimes there *is* no proof because the mathematical conjecture is wrong, but *we do not know* yet that it *is* wrong. James R. Brown shows an example of Euler's conjecture, which is a generalization of Fermat's Last Theorem. It states the following:

> [. . .] if $n \geq 3$, then fewer than n nth powers cannot sum to an nth power. As a special case, this means that there are no solutions $\{w,x,y,z\}$ to equation $w^4+x^4+y^4=z^4$. The conjecture was well tested by examples, and for about two centuries was widely believed as [Fermat's Last Theorem]. However, counter-examples have been found recently, for example, $2{,}682{,}440^4 + 15{,}365{,}639^4 + 18{,}796{,}760^4 = 20{,}615{,}673^4$.[120]

So, Euler's conjecture was always false, even when everyone thought it was true. The point platonists wish to make is not that our knowledge of the logical-mathematical realm is infallible, but that sometimes many of these *a priori* truths can be discovered either by proving a certain conjecture or providing its refutation. But our beliefs in them being true or false have nothing to do with their objective truth value.

2.6 — Where and Why Do Mistakes Occur?

Most of the mistakes made in mathematics are due to three main reasons:[121]

(1) We could formulate false mathematical conjectures, but not know that they are actually false. This is the case of Euler's

120 Brown, 1999, p. 166.
121 Brown, 1999, pp. 18-23.

generalization of Fermat's Last Theorem which we have shown above, or David Hilbert's (and Husserl's) belief in the completeness of mathematics. In both cases, it was the apodeictic certainty of mathematics that led to the discovery of these conjectures' falsity. These refutations do not mean that mathematics is doomed to be uncertain as fallible inventions of humans, nor does it mean that because this sole aspect has a similarity in conjecturing and refuting like in science, it belongs to natural science.[122] It is an exaggerated claim to say that because some of these conjectures were false there are "disasters" in mathematics.[123] On the contrary, due to the refutation of these conjectures we have more *certain* knowledge of mathematics: our knowledge about Euler's conjecture is now more certain than before we knew it was false.

(2) The use of wrong and naïve concepts is also a source of mathematical mistakes. For example, some of the ancient Greeks associated numbers with geometrical objects, and adhered to these concepts the notion of perfection (circles, squares, or equilateral triangles as perfect shapes). Of course, the number 3 cannot be identified with the equilateral triangle, nor is 4 identified to the square, etc. Nor are these numbers expressing perfection of any kind. Also, the misconception of numbers as distances prevented many philosophers in history to adopt negative numbers and negative roots, and regarded them as contradictory to certain axioms of mathematics. Simultaneously, this conception of numbers prevented many ancient mathematicians from developing a theory of complex numbers.

(3) Incorrect application of accepted principles can be a source of mistakes. This phenomenon can be shown with a simple example. Let us suppose that $x = 1$. Let us follow the rules of algebra and multiply both sides of the equation with the same variable x:

122 Kline, 1985, pp. 391-427.
123 Kline, 1985, pp. 3-7.

$$x^2 = x$$

$$x^2 - 1 = x - 1$$

$$(x - 1)(x + 1) = x - 1$$

If we divide both sides by $x - 1$ the result will be:

$$x + 1 = 1$$

$$x + 1 - 1 = 1 - 1$$

$$x = 0$$

> Therefore, $x = 1$ means $x = 0$. This is a mathematical impossibility because x cannot be 1 and 0 simultaneously. Although this demonstration seems to follow correctly all algebraic laws, in reality it does not. If the premise is that $x = 1$, then $x - 1$ would be zero, and division by zero is forbidden in algebra. The mistake was to divide both sides of the equation by $x - 1$.[124]

As we have seen, we must be careful not to confuse platonism with certainty of knowledge. *Our* knowledge of abstract objects is indeed affected by our limits and fallibility, but that does not mean that there is no *a priori* knowledge of logical and mathematical objects. Mathematical conjectures, to attain absolute certainty, must be proved or refuted in some apodeictic way; if they are not, then they are just that: conjectures.

124 Bunch, 1982, p. 13.

CHAPTER 3:
FORMAL SCIENCE AND NATURAL SCIENCE

Having established natural science as dealing with the *physical* world, and formal science as dealing with *ideal* categorial forms and their relationships between them, we proceed to explore how any theory of natural science can imply a revision in the formal science.

3.1 — THE QUINE-PUTNAM THESES

One of the most controversial subjects in philosophy of science has to do with the underdetermination of natural science. Most people refer to the Duhem-Quine Thesis as one of its foundations. It should be mentioned here that *there is no such thing as the Duhem-Quine Thesis*. As famous as this "thesis" may be, Pierre Duhem and W. V. O. Quine held different points of view concerning natural science and how its theories affect other branches of knowledge.

This should be corrected because this term has been widely used in the fields of epistemology and philosophy of science. Donald Gillies made an excellent exposition about the similarities and differences regarding Duhem's thesis and Quine's thesis. Pierre Duhem said that *in the case of physics* an experiment can never condemn a hypothesis but a whole theoretical group. That what is really put to the test is not merely a hypothesis, but a whole bunch of hypotheses, laws and theories that the tested hypothesis supposes.[125] This aspect of physics *does not extend* to other branches of science such as medicine and physiology, and definitely does not extend to formal science.[126] Evidence of this is the fact that he refused to think

125 Duhem, 1991, pp. 183-188; Gillies, 1993, pp. 98-99.
126 Duhem, 1991, pp. 180-183.

that the general theory of relativity was legitimate, because its use of non-euclidean geometry goes against our intuition that space is euclidean.[127]

Quine holds a very different point of view in "Two Dogmas of Empiricism". He denied the distinction between analytic and synthetic propositions, and, as a result, he concluded that there is no actual distinction between formal science and natural science except in degrees of abstraction. Since there is no distinction among both sciences, all of the formal and natural posits are nothing more than convenient fictions. Quine says the following in "Two Dogmas":

> If this view is right, it is misleading to speak of the empirical content of an individual statement—especially if it is a statement at all remote from the experiential periphery of the field. Furthermore it becomes folly to seek a boundary between synthetic statements, which holds contingently on experience, and analytic statements, which hold come what may. Any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system. Even a statement very close to the periphery can be held true in the face of recalcitrant experience by pleading hallucination or by amending certain statements of the kind called logical laws. Conversely, by the same token, no statement is immune to revision. Revision even of the logical law of the excluded middle has been proposed as a means of simplifying quantum mechanics; and what difference is there in principle between such a shift and the shift whereby Kepler superseded Ptolemy, or Einstein Newton, or Darwin Aristotle? [. . .]
>
> As an empiricist I continue to think of the conceptual scheme of science as a tool, ultimately, for predicting future experience in light of past experience. Physical objects are conceptually imported into the situation as convenient intermediaries—not by definition in terms of experience, but simply as irreducible posits comparable, epistemologically, to the gods of Homer. For my part I do, qua lay physicist, believe in physical objects and not Homer's gods; and I consider it a scientific error to believe

127 Curd & Cover, 1998, p. 377; Gillies, 1993, p. 105.

> otherwise. But in point of epistemological footing the physical objects and the gods differ only in degree and not in kind. Both sorts of entities enter our conception only as cultural posits. [. . .]
>
> The over-all algebra of rational and irrational numbers is underdetermined by the algebra of rational numbers, but is smoother and more convenient; and it includes the algebra of rational numbers as jagged on gerry-mandered part. Total science, mathematical and natural and human, is similarly but more extremely under-determined by experience. The edge of the system must be kept squared with experience; the rest with all its elaborate myths or fictions, has as its objective the simplicity of laws.[128]

As we can see, Quine's proposal is much more radical than Duhem's, and it does extend to logic and mathematics, specifically he presents the case of quantum logic.

Hilary Putnam is a bit more careful than Quine. He says that propositions such as “2 + 2 = 4” without a doubt are true and not subject to revision. However, he holds the belief that there are mathematical propositions that are “quasi-empirical”, which may be revised. Therefore there is no such thing as *a priori* knowledge.[129] He seems to equate revisability with empirical experience, and in his mind empiricism implies revision of supposed *a priori* knowledge. We have argued in the previous chapter against Kitcher that *a prioricity* is not incompatible with revisability.[130] Also Putnam shows the example of quantum logic as an instance of revisability in classic logic. He says:

> [in rejecting] the traditional philosophical distinction between statements necessary in some eternal sense and statements contingent in some eternal sense [. . .] could some of the 'necessary truths' of logic ever turn out to be false *for empirical*

128 Quine, 1953, pp. 43-45.

129 Putnam, 1975, pp. 124-126.

130 See Hale's (1987) comments on Putnam's view on revisability of *a priori* disciplines (p. 143).

> *reasons*? I shall argue that the answer to this question is affirmative.[131]
>
> I am inclined to think that the situation is not substantially different in logic and mathematics. I believe that if I had the time I could describe for you a case in which we could have a choice between accepting a physical theory based upon non-standard logic, on the one hand, and retaining standard logic and postulating hidden variables on the other. In this case, too, the decision to retain the old logic is not merely the decision to keep the meaning of certain words unchanged, for it has physical and perhaps metaphysical consequences. In quantum mechanics, for example, the customary interpretation says that an electron does not have a definite position measurement; the position measurement causes the electron to take on suddenly the property that we call its 'position' (this is the so-called 'quantum jump'). Attempts to work out a theory of quantum jumps and of measurement in quantum mechanics have been notoriously unsuccessful to date. It has been pointed out that it is entirely unnecessary to postulate the absence of sharp values prior to measurement and the occurrence of quantum jumps, if we are willing to regard quantum mechanics as a theory formalized within a certain non-standard logic, the modular logic proposed in 1935 by Birkhoff and von Newmann, for precisely the purpose of formalizing quantum mechanics.[132]

Therefore, we have here two philosophers who seem to argue that natural science can indeed revise formal science. Their statement can be summarized in two theses, which shall be called here the *Quine-Putnam Theses*[133]:

(1) *First Quine-Putnam Thesis*: Mathematics and logic can be revised in light of recalcitrant experience as well as changes in scientific theories.

131 Putnam, 1975, p. 174.

132 Putnam, 1975, p. 248.

133 This is a phrase used by Katz (1998) to refer to Quine's and Putnam's statement on the revisability of formal sciences in light of experience (p. 50).

(2) *Second Quine-Putnam Thesis*: Mathematics exists because it is indispensable to science. This is the so-called *indispensability argument*.

Here I shall discuss both of these theses in light of the case they present and the refutation of such claims.

3.2 — QUANTUM LOGIC

One of the most cited cases we see in Quine's and Putnam's arguments about the possibility of revising classic logic is the famous quantum logic which has as its basis the empirical data on quantum behavior. According to classic logic, this well-formed-formula, one distributive law, is a tautology:

$$(\alpha\wedge(\beta\vee\gamma))\leftrightarrow((\alpha\wedge\beta)\vee(\alpha\wedge\gamma))$$

If this is a tautology, then it is always true, regardless of the truth value assigned to the propositional variables. However, this logical truth would not seem to hold in quantum physics. Martin Curd and J. A. Cover give us an example of how this is so. Let us look at Illustration 4:

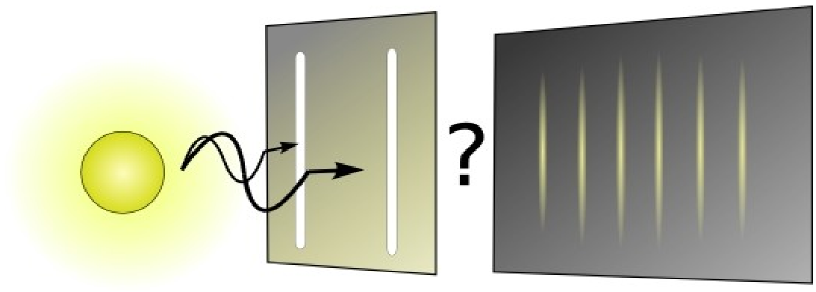


Illustration 4

It presents us the famous double-slit experiment. If we have a light source on one side of a panel with two slits, we will be able to see an interference pattern in a screen at the other end. If we conceive light as a wave, we will be able to account for the interference pattern in the screen. If we conceive light as made up of photons (light particles), we are not unable to explain that phenomenon. At least according to our experience with particles, the interference pattern should not appear, we should see instead a set of two light bands corresponding to the two slits in the panel. If we cover one of the slits, the interference pattern is not shown anymore; if the two slits are open, then it appears once again. How does the photon "know" that the other is going to pass through the other slit to then form an interference pattern? It might seem that the *same* photon can pass through both slits to show the interference pattern.

Now, let us suppose that p is the proposition "The photon is in region R of the screen", q_1 is the proposition "The photon went through slit 1", and q_2 stands for the proposition "The photon went through slit 2". If the photon goes through slit 1 or slit 2, we do not see the interference pattern formed in the screen. However, if they go through *both* slits simultaneously, then we can see the pattern. Therefore, this means that:

$$p \wedge (q_1 \vee q_2)$$

should not interderive with:

$$(p \wedge q_1) \vee (p \wedge q_2)$$

This is true in the quantum world, because a photon is assumed to pass through *both* slits, not one.[134] Is this a refutation of classic logic on empirical grounds, just as the Quine-Putnam Theses suggest?

3.3 — REFUTATION OF THE FIRST QUINE-PUTNAM THESIS

3.3.1 — ABOUT THE PROBLEM OF QUANTUM LOGIC

One of the ironies of philosophy is that one of the philosophers who proposed the First Quine-Putnam Thesis was precisely the one who refuted it: W. V. O. Quine. In his *Philosophy of Logic*, he

134 Curd & Cover, 1998, p. 380.

discusses many cases of the so-called “deviant logics”, one of them being quantum logic.[135] Many of those who favor and those who oppose Quine pay no attention to this very important retraction from statements he made in “Two Dogmas”.

For Quine, to deny a law of logic or redefine logical connectives according to quantum phenomena would be “changing the subject”. In classic logic, the meaning of logical connectives is defined by their truth values. Quantum logic is not truth functional and we cannot determine through truth tables whether conjunction, disjunction, implication mean the same as in classic logic. Therefore, quantum logic can hardly be considered a refutation of classic logic. At most, it is an alternative logic, but not a refutation of the older one.[136] He further says from his pragmatic view:

> [. . .] I would cite again the maxim of minimum mutilation as a deterring consideration. [. . .] let us not underestimate the price of a deviant logic. There is a serious loss of simplicity, especially when the new [quantum] logic is not even a many-valued truth-functional logic. And there is a loss, still more serious, on the score of familiarity. Consider again the case [. . .] of begging the question in an attempt to defend classical negation. This only begins to illustrate the handicap of having to think within a deviant logic. The price is perhaps not quite prohibitive, but returns had better be good.
>
> [. . .] Now the present objection from quantum mechanics is in a way reminiscent of this, though without the confusion. [. . .] Certainly a scientist admits as significant many sentences that are not linked in a distinctive way to any possible observations. He admits them so as to round out the theory and make it simpler, just as the arithmetician admits the irrational numbers so as to round out arithmetic and simplify computation; just, also as the grammarian admits such sentences as Carnap's 'This stone is thinking about Vienna' and Russell's 'Quadruplicity drinks

135 The other “deviant logics” Quine (1970) mentions are intuitionistic logic (which rejects the principle of excluded middle) and various multi-valued logics.

136 Quine, 1970, pp. 83-86.

> procrastination' so as to round out and simplify the grammar. Other things being equal, the less such fat the better; but when one begins to consider complicating logic to cut fat from quantum physics, I can believe that other things are far from equal. The fat must have been admirably serving its purpose of rounding out a smooth theory, and it is rather to be excused than excited.[137]

To add to this quinean rejection, many other philosophers and scientists have objected to quantum logic for not helping us understand what happens in the quantum world. For many of them, such a logic only shifts the mystery from quantum physics to logic.[138] It is for this reason that many philosophers of science as well as many scientists themselves have completely rejected the Copenhagen interpretation.

Also, from an epistemological standpoint, this is one attempt to translate quantum phenomena given in experience (*a posteriori*), and turn it into some sort of *a priori* logical laws of quanta. The problem relies in the fact that, ideally speaking, there can be numerous possible explanations for that phenomenon, some which may not have been formulated yet. Those who favor quantum logic only rely on *one particular interpretation* of quantum phenomena, and turn what apparently is a contradiction in light of classic logic into a new kind of logic. The fallacy of this procedure can be seen clearly once we realize that this logic is only valid within a particular *a posteriori* natural-scientific theory. If in the future such a theory is refuted or abandoned, this new logic will cease to be valid. However, historically, classic logical laws remain true regardless of which *a posteriori* natural-scientific theories are adopted.

Finally, we must not forget that the disjunction in the distributive laws is an inclusive disjunction: or one, or the other, *or both*.

The norm in science is never to change or revise the axioms and theorems of formal sciences on empirical grounds, but *to change the natural-scientific theory so it is consistent with logico-mathematical laws and truths*. There is a very simple but good example given by Carl G. Hempel about this:

137 Quine, 1970, p. 86.
138 Curd & Cover, 1998, p. 380.

> [. . .] consider now a simple "hypothesis" from arithmetic 3+2=5. If this is actually an empirical generalization of the past experiences, then it must be possible to state what kind of evidence would oblige us to concede the hypothesis was not generally true after all. If any disconfirming evidence for the given proposition can be thought of, the following illustration might well be typical of it: We place some microbes on a slide, putting down first three of them and then another two. Afterward we count all the microbes to test whether in this instance 3 and 2 actually added up to 5. Suppose now what we counted 6 microbes altogether. Would we consider this an empirical disconfirmation of the given proposition, or at least as a proof that it does not apply to microbes? Clearly not; rather, we would assume we had made a mistake in counting or that one of the microbes had split in two between the first and second count. *But under no circumstances could the phenomenon just described invalidate the arithmetical proposition in question*.[139]

This illustrates very well the relation between formal sciences and natural sciences. Some events in natural sciences seem to revise mathematics, when in reality they do not. Let us see two more cases where this only *seems* to happen.

3.3.2 — General Theory of Relativity and Non-Euclidean Geometry

One of the most cited arguments in favor of revision of mathematics is the general theory of relativity and its adoption of non-euclidean geometry. It can be said, for instance, that Einstein's discovery of physical space-time being non-euclidean refuted euclidean geometry.[140] However, we must look carefully at these claims to understand well what Einstein *really* did in the case of non-euclidean geometry and his theories of relativity.

To be able to understand well what happens in the relationship between geometry in general and the general theory of relativity, we

139 Hempel, 2001, p. 4, my italics.
140 Putnam, 1975, pp. xv-xvi.

should examine the reason why non-euclidean geometry was developed. One of the most significant axioms in euclidean geometry was the so-called "axiom of the parallels" which states that given a line and a non-collinear point, there is one and only one line that goes through that point which is parallel to the given line. For many mathematicians, this axiom was self-evident, but for others it was not. The same assumptions were necessary for the proof of the theorem that stated that the sum of the angles of a triangle is 180°. The more mathematics became an abstract and rigorous discipline, the more the supposed self-evidence of the axiom and the validity of the theorem were questioned. In the eighteenth and nineteenth centuries, there was a conviction among some mathematicians that to deny such an "axiom" would *not* lead to any contradictions.

The Jesuit priest Gerolamo Saccheri (1667-1733), after trying to prove the axiom of the parallels, discovered accidentally that consistent non-euclidean geometry is possible. He could not show through the method of *Reductio ad Absurdum* that the negation of the axiom of the parallels was false. So, without knowing it, he showed that non-euclidean geometry could be consistent and that it could be perfectly possible to conceive the angles of a triangle being less than 180°. He rejected this conclusion on intuitive grounds, but later Carl Friedrich Gauss (1777-1855) realized that non-euclidean geometries are as valid as the euclidean one.

It was not until János Bolyai (1802-1860) and Nikolai Lobachevsky (1793-1856) that a variant of non-euclidean geometry called "hyperbolic geometry" was developed, which was ignored and rejected by most of the other mathematicians at the time for being counterintuitive. This hyperbolic geometry denied the axiom of the parallels and assumed, not a flat kind of space, but pseudo-spherical. In it, the sum of the angles of the triangle is less than 180°, it was possible to "draw" more lines going through non-collinear points parallel to a given line.

Another mathematician, Bernard Riemann (1826-1866), developed another kind of non-euclidean geometry called "elliptic geometry", where the sum of the angles of a triangle is greater than 180°, and where the shortest distance between two points lies in a great circle (the line which divides the sphere in two halves). It also makes possible for more than one line to pass through two points.

As we can see, these non-euclidean geometries *were not grasped by experience in any way*, they came out as a direct result of centuries of reflections by mathematicians, especially those in the eighteenth and nineteenth centuries. Such theories did revise mathematics *but did not refute euclidean geometry*. In fact, euclidean space, under this conception, became *one* of an infinity of possible mathematical spaces. It did not refute at all that in euclidean space the sum of the angles of the triangle is 180°, or that the Pythagorean Theorem is true. What it *did* refute was the conception that the only valid geometry is euclidean geometry. *Natural science had nothing to do with this revision*.

Then what did Einstein do? He was acquainted with the philosophy of the famous mathematician Henri Poincaré, who is considered today one of the fathers of the general theory of relativity. Poincaré accepted the mathematical validity of non-euclidean geometry.[141] For him, a non-euclidean world is perfectly possible and is very different from euclidean space,[142] but he further states the following:

> It is seen that experiment plays a considerable rôle in the genesis of geometry; but it would be a mistake to conclude from that geometry, is, even part, an experimental science [. . .] [Geometry] is not concerned with natural solids: its object is certain ideal solids, absolutely invariable, which are put a greatly simplified and very remote image of them. The concept of these ideal bodies is entirely mental, and experiment is but the opportunity which enables us to reach the idea. The object of geometry is the study of a particular "group"; but the general concept of group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our sensitiveness, but as a form of our understanding; *only, from among all possible groups, we must choose one that will be the* standard*, so to speak, to which we shall refer natural phenomena*.

141 Poincaré, 1952, p. 50.
142 Poincaré, 1952, pp. 64-68.

> Experiment guides us in this choice, which it does not impose on us. It tells us not what is the truest, *but what is the most convenient geometry*. It will be noticed that my description of these [non-euclidean] worlds has required no language other than that of ordinary geometry. Then, were we transported to those worlds, there would be no need to change that language. Beings educated there would no doubt find it more convenient to create a geometry different from ours, and better adapted to their impressions; but as for us, in the presence of the same impressions, it is certain that we should not find it more convenient to make a change.[143]

Poincaré's statement regarding non-euclidean geometry is that it is as valid as euclidean geometry, but it would not serve well at all to adopt non-euclidean geometry as a convention in this world. It is perfectly conceivable that it would make sense *to adopt non-euclidean geometry as a way to make the theories about the world simpler*, even if non-euclidean geometry itself is not as simple as euclidean geometry. But due to the fact that our world seems to be in euclidean space, he rejects the possibility of the eventual adoption of non-euclidean geometry to make scientific theories simpler.

This is where Einstein comes in. One of the mathematical consequences of the Lorentz Transformations and the adoption of the independence of the constancy of light's speed with respect to all inertial reference frames is that nothing can travel faster than the speed of light. This left a significant problem: according to newtonian mechanics, the effect of gravity among massive objects is instantaneous, and there is no account for Lorentz's spatial contraction. Poincaré influenced Einstein by letting him see that it was possible to adopt a more complicated mathematical model *in order to simplify a scientific theory*. Due to the special theory of relativity we cannot talk about rigid bodies, i.e. bodies in which the relative length, time frame, and mass are not affected due to its speed with respect to other inertial reference frames. Euclidean geometry would be an inappropriate model to build a theory using special relativity. So, he had two options:

143 Poincaré, 1952, pp. 70-71, my italics.

(1) To retain euclidean geometry as the mathematical model for simplicity, which would mean sacrificing the simplicity of the scientific theory about space-time behavior.

(2) To adopt a more complicated non-euclidean geometry as a mathematical model, but having the benefit of a simpler scientific theory.[144]

By choosing the latter, Einstein not only formulated a very consistent general theory of relativity, but also was able to predict and include a series of phenomena which were not accounted for in classic newtonian mechanics: the Second Twin Paradox, the way light deviates near massive objects, the motion of Mercury's Perihelion, and the Doppler Effect.[145]

So, what we see here in the case of general theory of relativity is not that it refuted euclidean geometry. Euclidean geometry itself does not contradict non-euclidean geometry, because an euclidean space is one of an infinity of possible spaces. Non-euclidean geometry came to be because of *internal problem solving processes within mathematics* itself, and its historical origin has nothing to do with its adoption or rejection within natural science.

Therefore, *the general theory of relativity did not revise mathematics* at all. Quite the contrary. Einstein chose another mathematical model of space which was available thanks to mathematicians' development of non-euclidean geometry the previous century. He formulated the simplest theory when he considered the mathematical implications of the special theory of relativity as well as other phenomena. Instead of natural science revising mathematics, *it was mathematics the field that revised natural science*.

3.3.3 — Chaos Theory and Mathematics

Recently there has been an enthusiasm about chaos theory, even to the point of the absurd, mostly as a result of propaganda within the

144 Einstein, 1983, pp. 33-35, 39.
145 See also Carnap's comments in Reichenbach, 1958, p. v.

academy.[146] Some have presented chaos theory as a refutation of mathematical method in general. What is chaos theory? Alan Sokal and Jean Bricmont explain this very well:

> What is chaos theory about? There are many physical phenomena governed by deterministic laws, and therefore, predictable in principle, which are nevertheless unpredictable in practice because of their "sensitivity to initial conditions". This means that two systems obeying the same laws may, at some moment in time, be in very similar (but not identical) states and yet, after a brief lapse of time, find themselves in very different states. This phenomenon is expressed figuratively by saying that a butterfly flapping its wings today in Madagascar could provoke a hurricane three weeks from now in Florida. Of course, the butterfly by itself doesn't do much. But if one compares the two systems constituted by Earth's atmosphere with and without the flap of the butterfly's wings, the result three weeks from now may be very different (a hurricane or not). One consequence of this is that we do not expect to be able to predict the weather more than a few weeks ahead. Indeed, one would have to take into account such a vast quantity of data, and with such precision, that even the largest conceivable computers could not begin to cope.[147]

This is illustrated best with what occurred to one of the fathers of chaos theory, Edward Lorenz. He discovered what would later be known as the "butterfly effect", especially in relation to the weather. He used attractors in order to describe the behavior of certain systems. Chaotic behavior, in chaos theory, does not mean just pure disorder or pure randomness. A system is chaotic if it depends

146 I need not emphasize the huge problems of certain "thinkers" who use chaos theory to support the latest nonsense that comes to their mind. Alan Sokal and Jean Bricmont illustrate very well the overwhelming confusions concerning so-called "thinkers" like Jean-François Lyotard, Jean Baudrillard, Gilles Deleuze, and Félix Guattari (Sokal & Bricmont, 1999, pp. 147-168). For a very sober research on chaos theory, Sokal and Bricmont have suggested the following readings: Kadanoff, 1986; Matheson & Kirchoff, 1997; Ruelle, 1991; and Van Peer, 1998.

147 Sokal & Bricmont, 1998, p. 138.

greatly in the sensitivity of initial conditions. Lorenz, studying the weather and picking up data, eventually showed with an attractor, the behavior of a chaotic system. Illustration 5 is a display of what has come to be known as the *Lorenz attractor*.

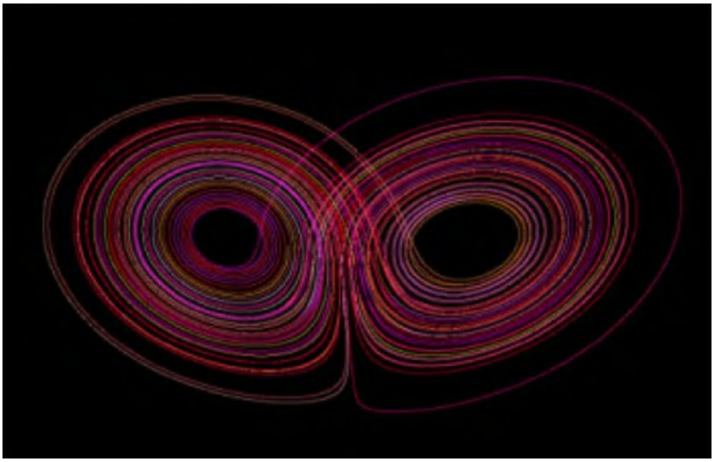


Illustration 5

The attractor shows that even in what appears to be disorderly behavior, there is an inherent structure within the stream of data, even when the trajectories never intersect, and the system never repeats itself. Such a view of chaotic patterns has been very useful in explaining, for instance, Jupiter's Red Spot, which is essentially a self-organizing system within a chaotic system. This seems to apply, not only to nature, but also to economy, population growth, and so on.[148]

Chaos theory also includes the idea of the fractal aspect of such behavior. Among many scientists and mathematicians, it was Benoit Mandelbrot who discovered accidentally that a diagram of income distribution can be correlated with the diagram of eight years of cotton prices. Gleick tells more of this story:

> [. . .] when Mandelbrot sifted the cotton-price data through IBM's computers, he found the astonishing results he was seeking. The numbers that produced aberrations from the point of view of

148 Gleick, 1987, p. 55.

> normal distribution produced symmetry from the point of view of scaling. Each particular price change was random and unpredictable. But the sequence of changes was independent of scale: curves for daily price changes and monthly price changes matched perfectly. Incredibly, analyzed Mandelbrot's way, the degree of variation had remained constant over a tumultuous sixty-year period that saw two World Wars and a depression.[149]

So, we see not only that there is an order within the pattern as discovered by Lorenz, but also, in a sense, there was a correlation between the whole and the parts of a chaotic system. Mandelbrot could calculate the fractional dimensions of real objects according to shape or any other irregular patterns. And it does not matter which dimensional fraction we reduce it to, we are able to see that irregular pattern again and again. He created the word "fractal" to refer to these fractional dimensions and "fractal geometry", to create the discipline which has fractals as its objects of study.

This non-conventional way of looking at the world and the creation of such a mathematical discipline was a very important step in understanding the behavior of the physical world. Up to now we have seen chaotic systems and fractals and their relationship with the physical world. What about the mathematical realm? Many of these views apparently also apply to pure mathematical objects, such as the very well known *Mandelbrot Set*. Gullberg explains in full detail what the Mandelbrot Set is:

> The fractal behavior in the complex number plane is demonstrated by iterating a nonlinear function whose variables *include its own result*. If a set of an infinite sequence $f(z)$, $f(f(z))$, $f(f(f(z)))$..., where z is a complex number, is plotted on a graph, the sequence of iterates may
>
> 1. be unbounded; or
> 2. jump around within a bounded region
>
> If (2) holds, we say that z lies in the "filled-in Julia set for f" [. . .]

149 Gleick, 1987, p. 86.

> The Mandelbrot set is related to the Julia set, but for it the defining variable is the c in $f(z)=z^2+c$, where z and c are complex numbers. Starting with $z=0+0i$, we look for the complex numbers c, such that 0, $f(0)$, $f[f(0)]$, ... remain bounded.
>
> If we let $z=0+0i$ in $f(z)=z^2+c$, then
>
> $$f(z) = f(0+0i) = (0+0i)^2 + c = c$$
>
> $$f(f(z)) = f(c) = c^2 + c$$
>
> $$f(f(f(z))) = f(c^2 + c) = (c^2 + c)^2 + c$$
>
> and the process may be iterated *ad infinitum*. [. . .]
>
> We are now in a position to define the **Mandelbrot set** as the set of complex numbers c for which the iterated $f(z)=z^2+c$ remains bounded. The initial value of z is $0+0i$ and each subsequent value of z is used to find the next one.[150]

A more formal definition of the Mandelbrot set is the following:

> DEFINITION. The *Mandelbrot Set* M is the set of complex numbers c such that the sequence $\langle f_c^n(0) \rangle$ does not approach ∞ as n gets larger.

Through the process of iteration in a computer program, we can actually produce Illustration 6, which is the graphical representation of the Mandelbrot Set. The image, as complex and irregular as it may appear, has also in it an intrinsic pattern. Using a certain sequence of finding fractions on the Mandelbrot Set, we are able to find again a fraction that contains once more its original image. In the page 73 we see an example with Illustration 7. The white rectangles show how we can "zoom in" the Mandelbrot Set, and find gradually the same image pattern of the Mandelbrot Set. If we repeat that process, we will find another image of the Mandelbrot Set, and so on.

So, it seems that once again we are apparently faced with a possibility that empirical sciences have revised mathematics. The reasoning we apply in empirical experience is the same we apply to

150 Gullberg, 1997, p. 633.

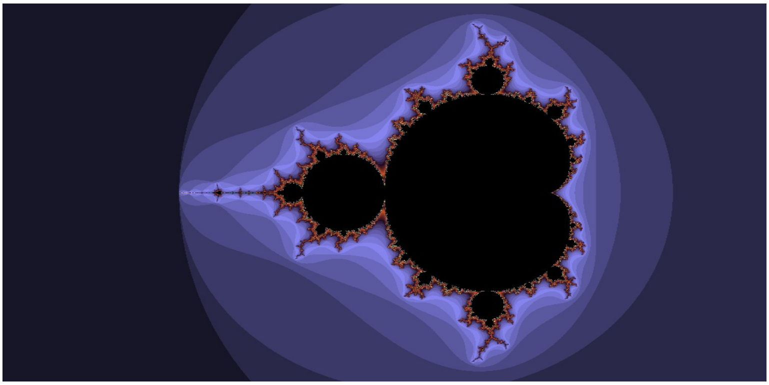
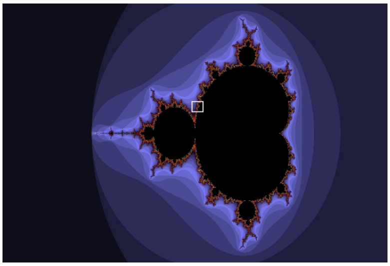


Illustration 6

mathematical objects, because fractal reasoning applies to objects of sensible experience and also to certain mathematical objects. This reasoning itself is based on experience. Therefore, on the basis of sensible experience, mathematical reasoning has been revised.

To be able to understand this well, we have to distinguish two very important sides of chaos theory, one that has to do with natural and empirical sciences, and another solely having to do with pure mathematics. Some people have alleged that chaos theory adds uncertainty to mathematics, especially on the basis of experience. First, I wish to ask: if mathematics is uncertain on the basis of experience, then what guarantees the certainty of chaos theory in the first place? If chaos theory in a sense refutes mathematical certainty, then why does it use mathematically *certain* rules to be able to understand this with *certainty*?

Alan Sokal and Jean Bricmont have pointed out the confusion of those who argue this way. They confuse determinism with predictability, but this aspect applies only to empirical sciences, not to pure mathematics. Chaos theory does not refute *determinism* at all, all chaotic systems in the world are determined according to physical laws. They just have the peculiar aspect of its determination depending on the sensitivity of initial conditions. However, due to the fact that we are not able to account for each and every single



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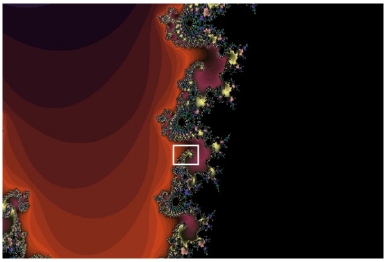

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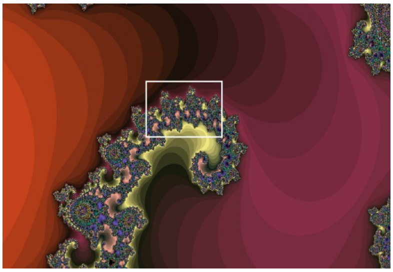

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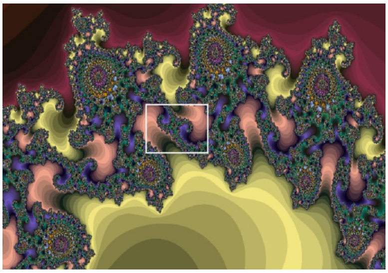

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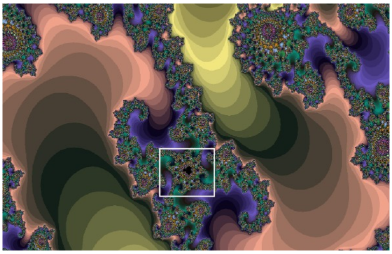

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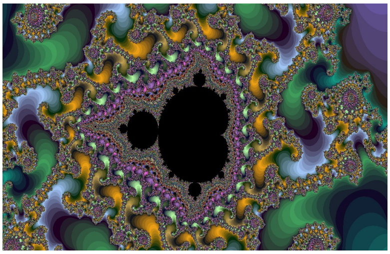

6

Illustration 7

variable that intervenes in a chaotic system, we are not able to *predict* with 100% accuracy certain phenomena. That is why Edward Lorenz discovered that we are able to predict the weather only in short term within a statistical model, and such a prediction loses its certainty as time goes by.[151]

But none of this revises mathematics at all, on the contrary, as I will show, the application of certain mathematical chaotic notions such as strange attractors or fractals, is no different from the application of non-euclidean geometry to physics. So, for this discussion we will have to look at the part of chaos theory that deals with mathematics.

In general, there are some misuses of the notion of fractals that should be mentioned here. Notice that many of the advocates for fractal geometry say that fractals are a more accurate representation of reality. As Gleick said about Mandelbrot:

> Clouds are not spheres, Mandelbrot is fond of saying. Mountains are not cones. Lightning does not travel in a straight line. The new [fractal] geometry mirrors a universe that is rough, not rounded, scabrous, not smooth. It is a geometry of the pitted, pocked, and broken up, the twisted, tangled, intertwined.[152]

The classic way in which Mandelbrot confronts a certain problem concerning the usual way we do geometry, is to ask: "How long is the Coast of Britain?" Mandelbrot found out that Lewis F. Richardson saw discrepancies about 20% in the estimated lengths of the coasts of Spain, Portugal, Belgium, and the Netherlands. Mandelbrot wanted to take another approach: the fractal approach. He arrived to the conclusion that a coastline is infinitely long. In theory, if we continue "zooming in" the coastline, not only will everyone discover how long it is, but also at one point, we will see the same irregular shapes as the original (as the famous fractal shore illustrates).[153] But is this true?

151 Sokal & Bricmont, 1998, pp. 140-146.
152 Gleick, 1987, p. 94.
153 Gleick, 1987, pp. 94-96.

Gullberg comments about this:

> The universe is replete with shapes that repeat themselves on different scales within the same object. In Mandelbrot's terminology, such objects are said to be **self-similar**.
>
> In the idealized world of mathematics, there are several well defined figures that are self-similar and an infinite number of such figures may be generated through iteration of functions [. . .] The word fractal — coined by Mandelbrot — was intended to describe a dimension that could not be expressed as an integer, today, "fractal" is generally understood to mean a set that is self-similar under magnification.
>
> Unlike mathematical fractals, no object in nature can be magnified an infinite number of times and still present the same shape of every detail in successive magnifications – one reason being the finite size of molecules and atoms. Yet fractal models may provide useful approximations of reality over a finite range of scales.
>
> Mandelbrot and others have applied fractals *as explanatory models of natural phenomena* involving irregularities on different size scales. This technique is used in graphical analysis in such diverse fields as fluid mechanics, economics, and linguistics and the study of crystal formation, vascular networks in biological tissue, and population growth.[154]

The word "models" here is key to understand exactly what is going on. As we can see, fractal geometry is as much an approximation to reality as euclidean and non-euclidean geometries are. What chaos theorists do essentially is use fractal geometry and many other mathematical notions and apply them to experience, the very same way non-euclidean geometry was applied to simplify scientific theory and explain the physical world more accurately. In this way, fractal geometry helps us to understand better the chaotic systems in the world. None of this refutes the validity of euclidean geometry at all nor any other mathematical axiom or theorem. The choice of the mathematical models depend greatly on which kind of natural-

154 Gullberg, 1997, p. 626, my italics.

scientific theory we choose, how that model is pertinent to it, and if it really simplifies the theory in such a way that actually makes a better understanding of the world possible.

So, attempts to make chaos theory a way to revise formal science basing ourselves on empirical experience is also doomed to failure.

3.3.4 — THE PARADOX OF REVISABILITY

As we said in Chapter 2, we cannot forget the paradox of revisability pointed out by Jerrold Katz. This is a big headache to all of those who believe in the First Quine-Putnam Thesis, because this paradox discovers two facts: the absolute necessity of some basic logical truths such as the principle of no-contradiction, and it also discovers the inviability of the potential revision of *all* formal science.

3.4 — REPLY TO SECOND QUINE-PUTNAM THESIS

The Second Quine-Putnam Thesis has been known as the "indispensability argument", which states that mathematics is meaningful because of the fact that it is indispensable to science. This is related to the First Quine-Putnam Thesis, that somehow formal sciences are revised in light of recalcitrant experience. I wish to point out the indispensability argument as a strange claim.

Rosado Haddock says that if mathematics is subordinated to physics, it is strange that mathematics does not refer at all to physical entities or theories of any kind. In fact, it seems that contrary, to physical theories, many mathematical truths are self-evident and true in every possible world.

> Now, although applicable to the physical (and other) sciences, mathematical theorems seem to be true even if all actually accepted physical theories were false and, thus, the claim that only after the advent of modern physical science can we argue that mathematical theorems are true seems really amazing, to say the least. It is also extremely unreasonable to think that before the advent of modern physical science there was no way to establish the existence of mathematical entities, thus, e.g., that

> there exists an immediate successor of 3 in the natural number series. Moreover, it is perfectly conceivable that there exists a world in which all mathematical theorems known to present-day mathematicians are true (supposing that current mathematics is consistent), and that mathematicians know as much mathematics as they actually know, but in which none of the physical laws accepted as true nowadays were known to humanity. What is not possible is a world in which physical science were as developed as it actually is, but in which our present mathematical theories (especially those applicable to present-day physical science) were not valid, or, at least, were not considered to be valid.[155]

Katz also made his criticism along this line, stating that we can establish the existence of these mathematical entities even without empirical science.[156] So, can we remain with a straight face when we state that the validity of mathematics depends on the validity of scientific theories? It seems the other way around. We admit that these replies to the Second Quine-Putnam Thesis are not a refutation *per-se* of the claims, but they show how unlikely this thesis seems to be.

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155 Hill & Rosado, 2000, p. 269.
156 Katz, 1998, pp. 50-51.

CONCLUSION

This book advocates for a platonist view of logic and mathematics, in order to truly understand the formal sciences' relationship with natural sciences. Philosophers like Mario Bunge, Philip Kitcher, Imre Lakatos, Karl Popper, Hilary Putnam, W. V. O. Quine among many others have expressed deep dissatisfaction with the view that there can exist an ideal world or a fregean "third realm".

Throughout our analysis we have seen that many of the arguments presented against platonism are completely false, products of misunderstandings, and sometimes *non sequiturs*. Behind all of these arguments, there is only one prejudice: there is no possibility for an abstract logico-mathematical reality. If there is no ideal realm, mathematics and logic cannot provide true knowledge like science provides knowledge. In the best cases, some philosophers can accept a certain kind of formal knowledge. In the worst of cases, there is essentially no difference at all between the formal and the natural, all our theories are posits.

Against the former, we recognize that it is possible that we make up concepts which may have foreseen or unforeseen logical consequences. But we can question if before the concept was invented everything that existed did not conform to the objective laws of logic and mathematics. Besides, the physical world and all of its laws did exist before us, and they are all subject to the same objectual relations. These logical-mathematical truths would be the condition of possibility of all beings and propositions.

Against those who hold that there is no difference between posits of formal science and the posits of natural science, a qualitative difference between such posits *must* be recognized. To distinguish between logic and mathematics on one hand and natural science on the other, we can observe that in these two fields *the epistemological foundations and knowing activities are very different*. In that sense, it is *useful* to distinguish between them. It is also *useful* to distinguish between qualitative aspects of logic and mathematics such as axioms, theorems, and proofs which require no sensory input whatsoever, from the qualitative aspects of natural science, which use the laws of formal sciences to formulate theories, that are confirmed or refuted in light of experience.

Hence, the outcome of this analysis with respect to the issue of the underdetermination of science is perfectly clear: science is unable to revise logic and mathematics. Of course, it is possible to argue that *a posteriori* phenomena can stimulate mathematical works, like in Newton's and Leibniz's cases. However, all that this shows is that there is a dialogue between formal science and natural science, but none can be reduced to the other, and one cannot be the servant of the other. Let each field have its own investigations, its own knowledge, and its own discovery of the truth.

APPENDIX A: DEMYSTIFYING THE FREGEAN AND PSYCHOLOGICAL HUSSERL

A.1 — BIOGRAPHICAL MISCONCEPTIONS

At the very beginning of writing the book, I did not wish to include this appendix, but I was very surprised to see how very well rooted are the myths against Edmund Husserl, and how deep the rejection of his doctrine in the analytic tradition has been.[157]

My use of Husserl's philosophy of logic and mathematics as a starting point to clarify the relation between formal sciences and natural sciences would surprise many philosophers. Most analytic philosophers completely ignore Husserl's philosophy of mathematics, even if it is, perhaps, the one that best describes mathematics in the twenty-first century. Even in the case of mathematicians, they regard Gottlob Frege as being far superior to Edmund Husserl and much more worthy of recognition.

I want to explain briefly why this is so in the analytic world. First, there is an unfounded myth that began with Dagfinn Føllesdal's master's thesis: *Husserl and Frege: A Contribution to Elucidating the Origins of Phenomenological Philosophy* and his article "Husserl's Notion of Noema", where he says that at first Husserl began favoring psychologism, and from that perspective he wrote his *Philosophy of Arithmetic* (1891). Later, he changed his mind due to Frege's review against this work (Frege, 1894/1972). The myth

157 For a more thorough study on this subject, and a full demystification of wrong conceptions about Husserl see: Hill, 1991; Hill & Rosado, 2000; Bernet, Kern & Marbach, 1999, pp. 13-57; Mohanty, 1974, 1982b; Hintikka, 1995, pp. 78-105; and Rosado, 2008b.

further states that Husserl became a kind of fregean semanticist and that his phenomenological notions (like the notions of *noema* and object (*Gegenstand*)) are nothing more than an extension of the fregean distinction between sense (*Sinn*) and referent (*Bedeutung*). And as if that were not enough, many other authors accuse Husserl of falling into the claws of psychologism once again.

None of this is true. As it turns out, apparently Frege's review of Husserl's *Philosophy of Arithmetic* had nothing to do with him changing his mind. As some studies have shown, some of Frege's criticisms were valid, but others were not. He exaggerated Husserl's position to the point of caricaturing it. Many authors consider Frege's criticism against Husserl's notion of abstraction as accurate, when in reality Husserl did not favor the silly assertions Frege accused him of saying.[158] It is possible that Frege's attack was not directed exclusively to Husserl, but to Georg Cantor, who was Husserl's close friend, colleague, and mentor, who, like Husserl, was also a disciple of the renowned mathematician Karl Weierstrass, whom Frege opposed.[159] Frege would charge Cantor exactly with the same errors he charged Husserl with.[160]

The myth that Husserl adopted fregean semantics after the famous fregean review (1894) is also false. Husserl already had his own theory of sense (meaning) and referent (objectuality) by 1890. We can accept that *Philosophy of Arithmetic* was published in 1891, but that only means it was published that year. He finished writing it in 1890. So, *Philosophy of Arithmetic* represents his thinking on mathematics up to 1890. J. N. Mohanty also offers this information:

> the basic change in Husserl's mode of thinking which by itself could have led to the *Prolegomena* conception of pure logic had already taken place by 1891. [. . .] If pure logic is defined in the *Prolegomena* in terms of the concept of ideal objective meanings, then already the 1891 review of Schröder's work contains this concept. If the major burden of Frege's 1894 review of

158 Coffa, 1998, pp. 68-69; Frege, 1894/1972, pp. 323-330; Hill, 1991, pp. 14-16, 58-62, 66-70, 71-86.

159 Hill & Rosado, 2000, pp. 96-97.

160 Hill & Rosado, 2000, p. 98.

> [*Philosophy of Arithmetic*] is the lack of distinction, in that work, between the subjective and the objective, between *Vorstellung* and *Begriff*, then Husserl already had come to distinguish between *Vorstellung* meaning and object in his 1891 review.[161]

It was Frege, in a letter addressed to Husserl (May 24, 1891), who recognized that he made a distinction between sense and referent, and he (Frege) compares correctly both theories of sense and referent of concept words. Husserl himself recognized that by the time *Philosophy of Arithmetic* was published, he already disagreed with its content. He says that he began having doubts about psychologism from the very beginning. He attributes his change from psychologism to his reading of Gottfried Wilhelm von Leibniz, Bernard Bolzano, Rudolf Hermann Lotze, and David Hume. He makes no mention of Frege as being decisive for the change.[162] In fact, in his *Logical Investigations* Husserl mentions Frege only twice: the first one in a footnote to point out that he retracted three pages of his criticism of Frege's *The Foundations of Arithmetic,*[163] and the other one to question his use of the word "*Bedeutung*" to denote referent rather than meaning (sense).[164]

Finally, contrary to what many people think, Husserl did *not* fall again into psychologism. He maintained basically the core of the criticism against psychologism made in the "Prolegomena" throughout his life, and we can find some of them in works as late as *Formal and Transcendental Logic* (1929). That accusation usually is the product of a misunderstanding between husserlian semantics and his phenomenological doctrine. But even the ideal realm plays a very important role in phenomenology, *noemata* are the irreal (ideal) necessary correlates of real *noetic* acts of consciousness.

161 Mohanty, 1974, pp. 22-23.

162 Hill, 1991, pp. 16-17.

163 Notice that it is only *three* pages, not eight as the typographical mistake of the old English edition seems to imply. This typographical mistake, unfortunately has fueled the myth about Husserl being dramatically changed by Frege. In reality, Husserl leaves most of his criticism to Frege intact *after* his turn from psychologism (see LI. Vol I. §46; see also Hill & Rosado, 2000, pp. 4-5).

164 LI. Vol. II. Inv. I. §15.

Another reason why some philosophers do not know about Husserl's doctrine of formal sciences has to do with the fact that many phenomenologists have not paid attention at all to it, either because they do not know it, or because they are not acquainted enough with logic and mathematics to notice it. They mostly focus on Husserl's phenomenological doctrine in general as well as his doctrine of the crisis of the European sciences.

There are also those husserlian scholars who make some distorted expositions of Husserl's philosophy of mathematics. For example, some authorities in this area give the impression that Husserl said that using eidetic intuition and eidetic variation we are able to abstract numbers, sets, etc. This is false. For Husserl, as we have seen in Chapter 2, numbers and sets are constituted by consciousness through categorial intuition. I do agree with Rosado (1997) that not to mention categorial intuition in Husserl's philosophy of mathematics would be like trying to explain newtonian mechanics without mentioning the three laws of motion.[165] Others confuse categorial intuition and eidetic intuition, leading to confusing notions of state-of-affairs and situation-of-affairs, therefore, unable to understand mathematical intuition as being categorial intuition and categorial abstraction.[166]

Up to now, we have presented most of the reasons why people have dismissed Husserl's philosophy of logic and mathematics. So, to finish exorcising some philosophers' image of Husserl, I will present his background in mathematics and philosophy of mathematics.

1. While Husserl was a student at the University of Berlin, he studied with great mathematicians such as Leopold Kronecker

165 Tieszen (1995) seems to fall in this kind of error. See also Rosado, 1997, pp. 387-395.

166 Moran (2000) makes an excellent summary of Husserl's phenomenological doctrine, but his exposition has many flaws, among them the confusion between categorial intuition and eidetic intuition (see pp. 119-120), and he also confuses the concepts of state-of-affairs and situation-of-affairs (p. 112). In *Logical Investigations* eidetic intuition appears as just *one* among other kinds of categorial intuitions. (LI. Vol. II. Inv. VI. §52) However, in his later works, there is a qualitative difference between both intuitions (I. §§3-4).

and Karl Weierstrass (1878-1881). Later, Husserl became Weierstrass' assistant (1883-1884).[167]

2. After being Franz Brentano's disciple (1884-1886), he went to the University of Halle, where he was under the supervision of Carl Stumpf, who was also Brentano's disciple and to whom Husserl later dedicated his *Logical Investigations*. Claire Ortiz Hill points out that it was through Carl Stumpf that he increasingly became interested in platonic ideas and led him away from Brentano's philosophy.[168] Stumpf was the one who convinced Gottlob Frege to elaborate a philosophy to clear up the purpose of his recently created conceptual notation (*Begriffsschrift*), which led Frege to write his philosophical masterpiece *The Foundations of Arithmetic*.[169]

3. It was during his years in Halle where he befriended Georg Cantor, the father of set theory, who would become his mentor (1886-1901).[170]

4. Husserl's first philosophical works were precisely about mathematics. In Halle, he presented his professorship doctoral dissertation *On the Concept of Number* about the psychological origins of sets and numbers, and later in 1891 he published his *Philosophy of Arithmetic: Psychological and Logical Investigations*. We also have to take into account the first volume of *Logical Investigations* titled “Prolegomena of Pure Logic” where Husserl exposes his definitive doctrine on logic and mathematics. In all of these works he was concerned about logic and mathematics, and he wanted to develop a proper epistemology.

5. He revealed Franz Brentano in 1892 that he had already concluded that non-euclidean geometry was as legitimate as euclidean geometry. For him, space does not need to be euclidean space to be consistent, and that there could be many other *a priori* possible spaces. This happened years before the

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167 Hill & Rosado, 2000, p. xi; Verlade, 2000, p. 3.
168 Hill, 1991, p. 17.
169 Hill & Rosado, 2000, p. 3.
170 Hill & Rosado, 2000, p. xi.

general theory of relativity legitimized the use of non-euclidean geometry in natural science.[171]

6. Later, when Husserl went to the University of Götingen, he was a colleague of David Hilbert, the great mathematician who was looking for a definitive proof of the completeness of mathematics. He also believed in such completeness, and formed part of Hilbert's Circle (1901-1916).[172]

7. Husserl was deeply interested in the paradoxes of set theory. Ernst Zermelo, a mathematician known for his works on set theory, was Husserl's friend. Zermelo found a paradox and in 1902 communicated it to Husserl and Hilbert. The reason for this is that Husserl had discussed a similar paradox in a review he wrote in 1891. Bertrand Russell also found this very same paradox and told Frege that his logical foundation of arithmetic as presented in *Basic Laws of Arithmetic* made this paradox possible. This is the reason why the world knows it as the "Russell Paradox", even though Zermelo found it first. Here we will call it the "Zermelo-Russell Paradox". Many of Husserl's works on set paradoxes (which are hundreds of pages) still remain unpublished.[173]

8. Even in later works such as *Formal and Transcendental Logic* (1929) and *Experience and Judgment* (1938), Husserl elaborated his semantic theory and his philosophy of mathematics in its final form.

9. Apparently Husserl contributed to the field of logic more than analytic philosophers realize. For example, some suspect that the difference between the "formation rules" and the "transformation rules" proposed by Rudolf Carnap, was originally proposed by Husserl, but with different names ("laws to prevent non-sense" and "laws to prevent counter-sense").[174] We must remember that Carnap, in his *Der Raum* includes much of Husserl's thinking in his philosophy of space, and that between 1924 and 1925 he

171 Rosado, 2008b, p. 31.
172 Hill & Rosado, 2000, p. xi.
173 Hill, 1991, pp. 2-3.
174 Hill & Rosado, 2000, p. 203.

assisted advanced seminars given by Husserl for three semesters.[175] We also have to remember that Husserl influenced some Polish thinkers such as Alfred Tarski and Stanislaw Leśniewski, the founder of mereology. Leśniewski's studies of Husserl's *Logical Investigations* (specifically the Third Investigation, often disregarded by Husserl's scholars and analytic philosophers) made him interested in developing a theory of parts and wholes.

A.2 — Frege's Semantic Theory Compared to Husserl's

We talked in Chapter 1 and 2 about Husserl's conceptions of propositions, states-of-affairs, and situation-of-affairs. This subject has been explored thoroughly by J. N. Mohanty, Claire Ortiz Hill, and especially Rosado Haddock, who has written extensive articles about this specific subject. Here I will not present anything new, just a scheme to compare both semantic doctrines. To illustrate their differences, I am going to use Table 10, 11 and 12.

Table 10 — Sense and Referent of Proper Names

Semantic Notion	***Gottlob Frege***	***Edmund Husserl***
Sign	Proper Name	Proper Name
Sense	Sense of Proper Name	Sense of Proper Name
Referent	Object	Object

As we can see, there is practically no difference between Frege and Husserl with respect to proper names. Frege uses the example of "the morning star" and "the evening star".[176] Husserl uses the example of

175 Friedman, 1999, pp. 46-48; Hill & Rosado, 2000, pp. 202-203.
176 SR. p. 32.

"the victor at Jenna" and "the vanquished at Waterloo", and of the "equilateral triangle" and the "equiangular triangle".[177] Each pair expresses a sense and refers to one object.

It is very important to notice that Husserl in *Logical Investigations* makes a difference between a whole set of meanings (senses) which can refer only to one object, and those meanings which by themselves can refer to a whole set of objects. He called *universal names* to those which have only one meaning and many referents.[178]

Table 11 — Sense and Referent of Concept-Words

Semantic Notions	***Gottlob Frege***	***Edmund Husserl***
Sign	Concept-Word	Universal Name
Sense	Sense of Concept	Concept
Referent	Concept	Extension of Concept
	Extension of Concept	

This leads us to discuss Table 11. Notice this difference: For Frege it takes one more step from concept-words to objects which fall under concept. He makes the concept the referent of a concept-word. The reason for this is that he thinks about sentences, whose structure are in subject-predicate form. The subject is a proper name and refers to an object, the predicate is a concept word and it refers to a concept. A concept for Frege is a function of one argument, which can be filled by an object. So, under this scheme, the notion of concept as referent of a concept-word works well. However, there is a very

177 LI. Vol. II. Inv. I. §12. It is very interesting that in a letter to P. Linke, Frege borrowed Husserl's example and changes it a bit so he can explain his notion of the sense of proper names: "the loser of Waterloo" and "the victor of Austerlitz" (Hill, 1991, pp. 93-94).

178 LI. Vol. II. Inv. I. §12.

important gap in his theory: he never explains what the sense of a concept-word is.

In Husserl's case, he conceives a universal name as a name having one meaning, and referring to many objects. This can solve many problems of Frege's semantic theory; Husserl leaves no semantic gaps like Frege does.

The subject of sentences leads us to discuss Table 12.

Table 12 — Sense and Referent of Assertive Sentences

Semantic Notions	***Gottlob Frege***	***Edmund Husserl***
Sign	Assertive Sentence	Assertive Sentence
Sense	Thought	Proposition
Referent	Truth Value	State-of-Affairs
Referent Base		Situation-of-Affairs
		Truth Value

For Frege, the assertive sentence has a thought as its sense, and its referent is always a truth value. Husserl, on the other hand, says that the sense of a sentence is a proposition and its referent is a state-of-affairs. As we explained in Chapter 2, a state-of-affairs is the result of an objectual categorial act with a situation-of-affairs as a referent base. The truth value of a proposition will depend on the correlation between propositions and states-of-affairs founded on situations of affairs.

Here we can see the advantage of Edmund Husserl's semantics over Frege's. Frege apparently held two very different notions of sense.[179] One which is expressed in "On Sense and Referent" where "$a > b$" and "$b < a$" express two different thoughts, and the one of "The Thought" in which both of those relations express the same

179 See Rosado's article "On Frege's Two Notions of Sense" in Hill & Rosado, 2000, pp. 53-66.

thought. So, "On Sense and Referent" he can see that his notion of thought is similar to Husserl's propositions. In "The Thought", his notion is similar to that of Husserl's situation-of-affairs.

It is obvious for many that a thought should refer to a fact. However, in "The Thought", Frege does not conceive facts as referents, but ironically *as senses*. For him, a fact *is a thought* that is true.[180] It is semantically acceptable that "the morning star" and "the evening star" express senses which refer to the same object: Planet Venus. But then it would be semantically strange to say that "the sun is in the Milky Way" and "7 + 5 = 12" are both senses which denote the same object.

So, with respect to semantics of sense and referent, Husserl is clearly superior to Frege.

A.3 — FREGE'S AND HUSSERL'S PHILOSOPHIES OF MATHEMATICS

Another frequent point of dispute is that many believe that Frege's view of logic and mathematics is far superior to Husserl's. If we remember Frege's original project, he was solely interested in showing that arithmetic could be reduced to logic. We all know the outcome of his work, the Zermelo-Russell Paradox practically destroyed all the logical building which could derive arithmetic from logic. Even other projects like that of A. N. Whitehead and Bertrand Russell, who tried to show in *Principia Mathematica* that all of mathematics could be reduced to logic, failed miserably, especially after Kurt Gödel's proof of mathematics' incompleteness.

Husserl, on the other hand, did not hold this point of view. As we have seen in Chapter 1, he did see mathematics as logic's ontological correlate, but the former is not reducible to the latter. He even held that some formal-ontological categories cannot be reduced to other formal-ontological categories. For example, he did not accept the reducibility of arithmetic to set theory. Every effort to try to provide

180 T. p. 74.

a foundation of mathematics on logic has failed. It seems that Husserl was right after all.

An interesting point of comparison of both of their doctrines has to do with non-euclidean geometry and other mathematical notions. Frege, did not accept non-euclidean geometry as a legitimate mathematical enterprise. Husserl was more careful than that. His idea of a *mathesis universalis* as theory of manifolds did allow the mathematical truths of non-euclidean geometries as well as the scientificity of what he called then “imaginary numbers” (negative roots, fractions, irrational numbers, decimals, among many others).[181]

Another point of advantage of Husserl's views with respect to Frege's is that his philosophy of mathematics and mathematical epistemology are able to overcome the paradoxes of naïve set theory. Frege's logicism collapsed because of the Zermelo-Russell Paradox. However, in *Experience and Judgment*, Husserl uses sets as examples of objectual categorial acts, and, as we have seen in Chapter 2, in principle a new set can always be constituted on top of other ones. Rosado (in Hill & Rosado, 2000) has pointed out the way this hierarchy of objectualities practically blocks the paradoxes of naïve set theory. For example, the fact that in principle every set can be constituted on another set and that there is a hierarchy blocks the possibility of a set of all sets. This prevents the Cantor Paradox from happening. The hierarchical aspect of objectual acts also blocks the Zermelo-Russell Paradox, since we cannot form sets which are elements of themselves.[182]

And last, but not least, Husserl's view of mathematics has provided perhaps the only platonist epistemology that can be explained naturally. Unfortunately, Frege could not provide such an epistemology, he was content only with grasping thoughts and senses, but nothing more than that.

Frege made many contributions to the fields of logic and mathematics, but the truth is that Husserl not only contributed to them, but also his view of them is extremely close to the way they are developed in the twenty-first centuries.

181 Hill & Rosado, 2000, pp. 161-178.
182 pp. 235-236.

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Ezra SIL Hebrew Unicode Font (http://scripts.sil.org/EzraSIL_Home)