1 The State Vector $\mathbf{Y}$

We define the complete state of the system as a single vector $\mathbf{Y}$:

$$\mathbf{Y} = [\mathbf{r}, \mathbf{v}, \mathbf{q}, \boldsymbol{\omega}, \mathbf{M}_{irr}]^T \tag{1}$$

Where:

- $\mathbf{r}$: Position vector in the **ECI** (Inertial) frame (m).
- $\mathbf{v}$: Velocity vector in the **ECI** frame (m/s).
- $\mathbf{q}$: Attitude quaternion (ECI $\rightarrow$ Body frame), defined as $[w, x, y, z]$ with unit norm.
- $\boldsymbol{\omega}$: Angular velocity vector in the **Body** frame (rad/s).
- $\mathbf{M}_{irr}$: A vector containing the scalar irreversible magnetization for each hysteresis rod (A/m).

2 The Differential Equation $\frac{d\mathbf{Y}}{dt}$

The system of first-order differential equations governing the state evolution is:

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \\ \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{M}}_{irr} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g}_{\text{total}}(\mathbf{r}, t) \\ \frac{1}{2}\mathbf{q} \otimes [0, \omega_x, \omega_y, \omega_z]^T \\ \mathbf{I}^{-1}\left(\boldsymbol{\tau}_{total} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right) \\ \mathbf{f}_{hyst}(\mathbf{B}_{body}, \dot{\mathbf{B}}_{body}, \mathbf{M}_{irr}) \end{bmatrix} \tag{2}$$

2.1 Gravitational Acceleration

The term $\mathbf{g}_{\text{total}}(\mathbf{r}, t)$ represents the total gravitational acceleration vector in the ECI frame. It is computed via the `GeographicLib::GravityModel`, which evaluates the gradient of the Earth's geopotential V:

$$\mathbf{g}_{\text{total}} = \nabla V(r, \theta, \lambda) \tag{3}$$

The potential V is defined by a spherical harmonic expansion (EGM2008 model):

$$V(r, \theta, \lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{N}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\cos\theta)\right] \tag{4}$$

This naturally includes the central Newtonian term $(n = 0)$, the J_2 oblateness perturbation $(n = 2, m = 0)$, and higher-order zonal and tesseral harmonics up to degree N.
Note: The term $\mathbf{f}_{hyst}$ represents the complex nonlinear evolution of the hysteresis rods, detailed in Section 3.5.

2.2 Quaternion Normalization

The quaternion derivative $\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \otimes [0, \omega_x, \omega_y, \omega_z]^T$ is exact, but numerical integration causes quaternions to drift from unit norm due to accumulated roundoff errors. To maintain the constraint $\|\mathbf{q}\| = 1$, the quaternion is explicitly normalized at regular intervals during integration:

$$\mathbf{q}_{\text{normalized}} = \frac{\mathbf{q}}{\|\mathbf{q}\|} \tag{5}$$

This is performed every N integration steps (typically $N = 10$) to prevent numerical drift while minimizing computational overhead.

3 Simulation Workflow $f(t, \mathbf{Y})$

3.1 Time and Frame Setup

We must bridge the gap between the Inertial frame (ECI) and the Earth-Fixed frame (ECEF/Geodetic).

1. **Calculate Earth Rotation Angle:**
   Compute the rotation angle $\theta_{rot} = \omega_{\oplus} \cdot t$, where $\omega_{\oplus} = 7.2921159 \times 10^{-5}$ rad/s is Earth's rotation rate.

2. **Construct Rotation Matrix $\mathbf{R}_{ECEF}^{ECI}$:**

$$\mathbf{R}_{ECEF}^{ECI} = \begin{bmatrix} \cos\theta_{rot} & -\sin\theta_{rot} & 0 \\ \sin\theta_{rot} & \cos\theta_{rot} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

3. **Position Conversion:**

$$\mathbf{r}_{ECEF} = (\mathbf{R}_{ECEF}^{ECI})^T \mathbf{r}_{ECI} \tag{7}$$

3.2 GeographicLib Coordinates

Use the `Geocentric` class to obtain Geodetic coordinates and the local frame rotation matrix.

```
GeographicLib::Geocentric earth(Constants::WGS84_a(), Constants::WGS84_f());
double lat, lon, h;
std::vector<double> M;
// Computes lat/lon/h AND the rotation from ENU to ECEF
earth.Reverse(r_ecef.x, r_ecef.y, r_ecef.z, lat, lon, h, M);
```

- **Input:** $\mathbf{r}_{ECEF}$.
- **Output:** Latitude (ϕ), Longitude (λ), Height (h).
- **Output:** $\mathbf{R}_{ENU}^{ECEF}$ as a 3×3 matrix.

3.3 Environmental Vectors

A. Gravitational Acceleration (GravityModel) The gravity vector is obtained in the Local Tangent Plane (ENU) and includes all perturbations:

```
// Returns total acceleration g (including J2, higher harmonics)
grav.Gravity(lat, lon, h, gx, gy, gz);
Vector3d g_ENU(gx, gy, gz);
```

Transform to Inertial Frame (ECI):

$$\mathbf{g}_{\text{total}}(\mathbf{r}, t) = \mathbf{R}_{ECEF}^{ECI} \cdot \mathbf{R}_{ENU}^{ECEF} \cdot \mathbf{g}_{ENU} \tag{8}$$

B. Magnetic Field (MagneticModel) The magnetic field vector is obtained in ENU coordinates:

```
mag(year, lat, lon, h, Bx, By, Bz); // Output in nT
Vector3d B_ENU(Bx * 1e-9, By * 1e-9, Bz * 1e-9); // Convert to Tesla
```

Transform to ECI frame:

$$\mathbf{B}_{ECI} = \mathbf{R}_{ECEF}^{ECI} \cdot \mathbf{R}_{ENU}^{ECEF} \cdot \mathbf{B}_{ENU} \tag{9}$$

3.4 Magnetic Field Derivative ($\dot{\mathbf{B}}_{ECI}$)

The time derivative of the magnetic field in the ECI frame is computed using a numerical spatial gradient approach. This captures the **material derivative** (also called the total derivative) following the satellite's trajectory:

$$\frac{D\mathbf{B}_{ECI}}{Dt} = \frac{\partial \mathbf{B}_{ECI}}{\partial t} + (\mathbf{v}_{ECI} \cdot \nabla)\mathbf{B}_{ECI} \tag{10}$$

where the first term represents secular variation (negligible on orbital timescales, $\sim 10^{-9}$ T/s) and the second term represents spatial variation due to orbital motion (dominant, $\sim 10^{-6}$ T/s).

Numerical Implementation: Using a finite difference approximation with a fixed timestep $\Delta t = 0.01$ s:

$$\frac{D\mathbf{B}_{ECI}}{Dt} \approx \frac{\mathbf{B}_{ECI}(\mathbf{r} + \mathbf{v}\Delta t, t + \Delta t) - \mathbf{B}_{ECI}(\mathbf{r}, t)}{\Delta t} \tag{11}$$

This automatically captures the spatial gradient term, which is the dominant contribution. The fixed timestep Δt is chosen to balance numerical accuracy (avoiding cancellation errors) with approximation quality (capturing local field curvature). For LEO velocities (~ 7.5 km/s), this corresponds to a spatial displacement of ~ 75 m.

Important Notes:

- The timestep Δt is a *numerical differentiation parameter*, independent of the ODE integrator's adaptive timestep.
- This approach works correctly with variable-step integrators (RK45, Dormand-Prince).
- The secular variation $\partial \mathbf{B}/\partial t$ from the WMM model ($\sim$nT/year) is negligible compared to the orbital motion term and is not explicitly computed.

3.5 Magnetic Field Rate in Body Frame

For hysteresis rod dynamics, we need the time derivative of $\mathbf{B}$ as measured in the *body-fixed* frame. This has two contributions:

$$\frac{d\mathbf{B}_{body}}{dt} = \underbrace{\mathbf{R}_{ECI}^{Body}(\mathbf{q})\frac{D\mathbf{B}_{ECI}}{Dt}}_{\text{Orbital Motion (Dominant)}} - \underbrace{\boldsymbol{\omega} \times \mathbf{B}_{body}}_{\text{Body Rotation (Secondary)}} \tag{12}$$

where:

- The first term transforms the material derivative from ECI to the body frame, representing how the field changes as the satellite moves through Earth's magnetic field ($\sim 10^{-6}$ T/s).
- The second term arises from the transport theorem (time derivative in a rotating reference frame), representing the apparent field change due to spacecraft rotation ($\sim 10^{-7}$ T/s for typical tumbling rates).
- $\mathbf{R}_{ECI}^{Body}(\mathbf{q})$ is the rotation matrix derived from the attitude quaternion $\mathbf{q}$.

Physical Interpretation: Even when the spacecraft is not rotating ($\boldsymbol{\omega} = 0$), the rods experience a time-varying magnetic field due to orbital motion at ~ 7.5 km/s through Earth's spatially-varying field. This is the primary mechanism for hysteresis damping.

3.6 Hysteresis Dynamics ($\dot{\mathbf{M}}_{irr}$)

The function $\mathbf{f}_{hyst}$ mentioned in Section 2 is computed here. For each rod i aligned with the body axis unit vector $\mathbf{u}_i$:

1. **Project Field and Rate onto Rod:**

$$H_i = \frac{1}{\mu_0}(\mathbf{B}_{body} \cdot \mathbf{u}_i) \tag{13}$$

$$\dot{H}_i = \frac{1}{\mu_0}\left(\frac{d\mathbf{B}_{body}}{dt} \cdot \mathbf{u}_i\right) \tag{14}$$

2. **Calculate Theoretical Anhysteretic Magnetization (M_{an}):**

$$H_{eff} = H_i + \alpha M_{irr,i} \tag{15}$$

$$M_{an} = M_s \left(\coth\left(\frac{H_{eff}}{a}\right) - \frac{a}{H_{eff}} \right) \tag{16}$$

3. **Calculate Derivative (The Hysteresis Differential Equation):**
The classic Jiles-Atherton model gives dM/dH. We require dM/dt for state integration:

$$\frac{dM_{irr,i}}{dH} = \frac{M_{an} - M_{irr,i}}{k\delta} \quad \text{where } \delta = \text{sign}(\dot{H}_i) \tag{17}$$

Applying the chain rule:

$$\frac{dM_{irr,i}}{dt} = \left(\frac{M_{an} - M_{irr,i}}{k \cdot \text{sign}(\dot{H}_i)} \right) \cdot \dot{H}_i \tag{18}$$

Singularity Handling: When $\dot{H}_i \to 0$, the sign function creates a discontinuity. To prevent integrator instability:

- Use a smooth approximation: $\text{sign}(x) \approx \tanh(\beta x)$ with large β.
- Or implement a dead zone: if $|\dot{H}_i| < \epsilon$, set $dM_{irr,i}/dt = 0$.

3.7 Torque Dynamics ($\boldsymbol{\tau}_{total}$)

The total external torque acting on the spacecraft is the sum of magnetic and gravity gradient contributions:

$$\boldsymbol{\tau}_{total} = \boldsymbol{\tau}_{mag} + \boldsymbol{\tau}_{grad} \tag{19}$$

3.7.1 Magnetic Torque ($\boldsymbol{\tau}_{mag}$)

This is the interaction between the spacecraft's total dipole moment and the Earth's magnetic field. The dipole moment consists of the permanent magnet ($\mathbf{m}_{perm}$) and the hysteresis rods ($\mathbf{m}_{rods}$).

$$\mathbf{m}_{rods} = \sum_{i=1}^{N} (V_{rod} \cdot M_{total,i} \cdot \mathbf{u}_i) \tag{20}$$

where the total magnetization M_{total} includes both the irreversible state variable and the reversible (elastic) component, scaled by the coefficient c:

$$M_{total,i} = (1-c)M_{irr,i} + cM_{an}(H_{eff,i}) \tag{21}$$

The total magnetic torque is then:

$$\boldsymbol{\tau}_{mag} = (\mathbf{m}_{perm} + \mathbf{m}_{rods}) \times \mathbf{B}_{body} \tag{22}$$

3.7.2 Gravity Gradient Torque ($\boldsymbol{\tau}_{grad}$)

In LEO, the Earth's gravitational field varies across the body of the spacecraft, creating a torque that tries to align the minimum moment of inertia with the nadir vector.

$$\boldsymbol{\tau}_{grad} = \frac{3\mu}{\|\mathbf{r}_{ECI}\|^5} (\mathbf{r}_{body} \times (\mathbf{I} \cdot \mathbf{r}_{body})) \tag{23}$$

where:

- $\mathbf{r}_{body} = \mathbf{R}_{ECI}^{Body}(\mathbf{q}) \cdot \mathbf{r}_{ECI}$ is the position vector in the Body frame.
- $\mathbf{I}$ is the spacecraft inertia tensor (diagonal for principal axes).
- $\mu = 3.986004418 \times 10^{14}$ m^3/s^2 is Earth's standard gravitational parameter.