1 The State Vector Y

We define the complete state of the system as a single vector $\mathbf{Y}$:

$$\mathbf{Y} = [\mathbf{r}, \mathbf{v}, \mathbf{q}, \boldsymbol{\omega}, \mathbf{M}_{irr}]^T \tag{1}$$

Where:

- $\mathbf{r}$: Position vector in the **ECI** frame (m).

- $\mathbf{v}$: Velocity vector in the **ECI** frame (m/s).

- $\mathbf{q}$: Attitude quaternion (ECI $\rightarrow$ Body), defined as $[w, x, y, z]$.

- $\boldsymbol{\omega}$: Angular velocity vector in the **Body** frame (rad/s).

- $\mathbf{M}_{irr}$: Scalar irreversible magnetization for each hysteresis rod (A/m).

2 The Differential Equation $\frac{d\mathbf{Y}}{dt}$

The system of first-order differential equations governing the state evolution is:

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \\ \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{M}}_{irr} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{g}_{\text{total}}(\mathbf{r}, t) \\ \frac{1}{2}\mathbf{q} \otimes [0, \omega_x, \omega_y, \omega_z]^T \\ \mathbf{I}^{-1}\left(\boldsymbol{\tau}_{total} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right) \\ \mathbf{f}_{hyst}(\mathbf{B}_{body}, \dot{\mathbf{B}}_{body}, \mathbf{M}_{irr}) \end{bmatrix} \tag{2}$$

2.1 Gravitational Acceleration

The acceleration $\mathbf{g}_{\text{total}}$ is defined as the negative gradient of the geopotential V:

$$\mathbf{g}_{\text{total}} = -\nabla V(r, \theta, \lambda) \tag{3}$$

The potential V follows the EGM2008 model. `GeographicLib::GravityModel` returns the acceleration vector directly, which includes the central mass term and all spherical harmonic perturbations (zonal, tesseral, and sectoral).

2.2 Quaternion Normalization

To maintain the constraint $\|\mathbf{q}\| = 1$, the quaternion is explicitly normalized after **every successful integration step**:

$$\mathbf{q}_{next} = \frac{\mathbf{q} + \dot{\mathbf{q}}\Delta t}{\|\mathbf{q} + \dot{\mathbf{q}}\Delta t\|} \tag{4}$$

Frequent normalization prevents the accumulation of errors that could otherwise lead to unphysical rotations or "jumps" that destabilize adaptive-step solvers.

3 Simulation Workflow $f(t, \mathbf{Y})$

3.1 Time and Frame Setup

To synchronize with real-world observations (e.g., TLEs), we must account for the Earth's initial orientation:

1. **Calculate Rotation Angle:**
   $\theta_{rot} = \theta_{GMST,0} + \omega_{\oplus} \cdot t$, where $\theta_{GMST,0}$ is the Greenwich Mean Sidereal Time at $t = 0$.

2. **Rotation Matrix $\mathbf{R}_{ECEF}^{ECI}$:** Standard Z-axis rotation using θ_{rot}.

3. **Position Conversion:** $\mathbf{r}_{ECEF} = (\mathbf{R}_{ECEF}^{ECI})^T \mathbf{r}_{ECI}$.

3.2 GeographicLib and Environmental Vectors

The `GravityModel` and `MagneticModel` return values in the Local Tangent Plane (ENU). These are rotated back to ECI for the ODE:

$$\mathbf{g}_{ECI} = \mathbf{R}^{ECI}_{ECEF} \cdot \mathbf{R}^{ECEF}_{ENU} \cdot \mathbf{g}_{ENU} \tag{5}$$

$$\mathbf{B}_{ECI} = \mathbf{R}^{ECI}_{ECEF} \cdot \mathbf{R}^{ECEF}_{ENU} \cdot \mathbf{B}_{ENU} \tag{6}$$

3.3 Magnetic Field Derivative

The material derivative captures field changes due to orbital motion:

$$\frac{D\mathbf{B}_{ECI}}{Dt} \approx \frac{\mathbf{B}_{ECI}(\mathbf{r} + \mathbf{v}\Delta t, t + \Delta t) - \mathbf{B}_{ECI}(\mathbf{r}, t)}{\Delta t} \tag{7}$$

In the Body frame, including the transport term (apparent change due to spacecraft rotation):

$$\frac{d\mathbf{B}_{body}}{dt} = \mathbf{R}^{Body}_{ECI}(\mathbf{q})\frac{D\mathbf{B}_{ECI}}{Dt} - \boldsymbol{\omega} \times \mathbf{B}_{body} \tag{8}$$

3.4 Hysteresis Dynamics ($\dot{\mathbf{M}}_{irr}$)

For each rod i, we project the field H_i and its rate $\dot{H}_i$ onto the rod's axis $\mathbf{u}_i$. The Jiles-Atherton evolution is:

$$\frac{dM_{irr,i}}{dt} = \dot{H}_i \cdot \frac{M_{an,i} - M_{irr,i}}{k \cdot \text{sign}(\dot{H}_i)} = \frac{M_{an,i} - M_{irr,i}}{k}|\dot{H}_i| \tag{9}$$

Physicality and Stability Constraints:

1. **Directional Constraint:** Magnetization must always move towards the anhysteretic curve. If $(M_{an} - M_{irr})\dot{H}_i < 0$, we set $\dot{M}_{irr,i} = 0$ to prevent unphysical state excursions.

2. **Smoothing:** The $\text{sign}(\dot{H}_i)$ function is approximated by $\tanh(\beta\dot{H}_i)$ with $\beta \approx 10^6$ to maintain continuity for the ODE solver.

3. **Singularity:** If $|\dot{H}_i| < \epsilon$, then $\dot{M}_{irr,i} = 0$.

3.5 Torque Dynamics

Magnetic Torque: Includes permanent magnets and the total rod magnetization (reversible + irreversible components):

$$\boldsymbol{\tau}_{mag} = \left(\mathbf{m}_{perm} + \sum[V_{rod,i}((1-c)M_{irr,i} + cM_{an,i})\mathbf{u}_i]\right) \times \mathbf{B}_{body} \tag{10}$$

Gravity Gradient Torque:

$$\boldsymbol{\tau}_{grad} = \frac{3\mu}{\|\mathbf{r}_{ECI}\|^5}\left(\mathbf{r}_{body} \times (\mathbf{I} \cdot \mathbf{r}_{body})\right) \tag{11}$$