1 The State Vector Y

We define the complete state of the system as a single vector $\mathbf{Y}$:

$$\mathbf{Y} = [\mathbf{r}, \mathbf{v}, \mathbf{q}, \boldsymbol{\omega}, \mathbf{M}_{irr}]^T \tag{1}$$

Where:

- $\mathbf{r}$: Position vector in the **ECI** (Inertial) frame (m).
- $\mathbf{v}$: Velocity vector in the **ECI** frame (m/s).
- $\mathbf{q}$: Attitude quaternion (ECI $\rightarrow$ Body frame), defined as $[w, x, y, z]$ with unit norm.
- $\boldsymbol{\omega}$: Angular velocity vector in the **Body** frame (rad/s).
- $\mathbf{M}_{irr}$: A vector containing the scalar irreversible magnetization for each hysteresis rod (A/m).

2 The Differential Equation $\frac{d\mathbf{Y}}{dt}$

The system of first-order differential equations governing the state evolution is:

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \\ \dot{\mathbf{q}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{M}}_{irr} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \mathbf{a}_{pert,ECI} \\ \frac{1}{2}\mathbf{q} \otimes [0, \omega_x, \omega_y, \omega_z]^T \\ \mathbf{I}^{-1}\left(\boldsymbol{\tau}_{total} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right) \\ \mathbf{f}_{hyst}(\mathbf{B}_{body}, \dot{\mathbf{B}}_{body}, \mathbf{M}_{irr}) \end{bmatrix} \tag{2}$$

Note: μ is the standard gravitational parameter, retrieved via `GeographicLib::GravityModel::GM()`. $\mathbf{a}_{pert,ECI}$ represents gravitational perturbations transformed into the inertial frame. The term $\mathbf{f}_{hyst}$ represents the complex nonlinear evolution of the hysteresis rods.

3 Simulation Workflow $f(t, \mathbf{Y})$

3.1 Time and Frame Setup

We must bridge the gap between the Inertial frame (ECI) and the Earth-Fixed frame (ECEF/Geodetic).

1. **Calculate GMST (Greenwich Mean Sidereal Time):**
   Compute the angle θ_{gst} based on the Julian Date derived from time t.
2. **Construct Rotation Matrix $\mathbf{R}_{ECEF}^{ECI}$:**
$$\mathbf{R}_{ECEF}^{ECI} = \begin{bmatrix} \cos\theta_{gst} & -\sin\theta_{gst} & 0 \\ \sin\theta_{gst} & \cos\theta_{gst} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$
3. **Position Conversion:**
$$\mathbf{r}_{ECEF} = (\mathbf{R}_{ECEF}^{ECI})^T \mathbf{r}_{ECI} \tag{4}$$

3.2 GeographicLib Coordinates

Use the `Geocentric` class to obtain Geodetic coordinates and the local frame rotation matrix.

```cpp
GeographicLib::Geocentric earth(Constants::WGS84_a(), Constants::WGS84_f());
double lat, lon, h;
// Computes lat/lon/h AND the rotation from ENU to ECEF (M)
earth.Reverse(r_ecef.x, r_ecef.y, r_ecef.z, lat, lon, h, M);
```

- **Input:** $\mathbf{r}_{ECEF}$.
- **Output:** Latitude (ϕ), Longitude (λ), Height (h).
- **Output:** $\mathbf{R}_{ENU}^{ECEF}$ (Returned by `Reverse` as a 3×3 matrix or computed via `LocalCartesian`).

3.3 Environmental Vectors

A. Gravity Perturbations (GravityModel)

1. Get disturbance in Local Tangent Plane (ENU).

```
grav.Disturbance(lat, lon, h, gx, gy, gz); // gx=East, gy=North, gz=Up
Vector3d a_pert_ENU(gx, gy, gz);
```

2. Transform to Inertial Frame (ECI) for the integrator:

$$\mathbf{a}_{pert,ECI} = \mathbf{R}_{ECEF}^{ECI} \cdot \mathbf{R}_{ENU}^{ECEF} \cdot \mathbf{a}_{pert,ENU} \tag{5}$$

B. Magnetic Field (MagneticModel)

1. Get field in ENU.

```
mag(year, lat, lon, h, Bx, By, Bz); // nT
Vector3d B_ENU(Bx, By, Bz);
B_ENU *= 1e-9; // Convert nT to Tesla
```

2. Transform to Body Frame (needed for Torque & Hysteresis):

   - First to ECI: $\mathbf{B}_{ECI} = \mathbf{R}_{ECEF}^{ECI} \cdot \mathbf{R}_{ENU}^{ECEF} \cdot \mathbf{B}_{ENU}$
   - Then to Body: $\mathbf{B}_{body} = \mathbf{R}_{ECI}^{Body}(\mathbf{q}) \cdot \mathbf{B}_{ECI}$

3.4 Magnetic Field Derivative ($\dot{\mathbf{B}}$)

Hysteresis depends on how fast the field changes *inside the rod*. This has two components: Orbit movement and Satellite tumbling.

$$\frac{d\mathbf{B}_{body}}{dt} = \underbrace{\mathbf{R}_{ECI}^{Body}(\mathbf{q})\frac{\mathbf{B}_{ECI}(t) - \mathbf{B}_{ECI}(t-\Delta t)}{\Delta t}}_{\text{Orbital Change}} - \underbrace{\boldsymbol{\omega} \times \mathbf{B}_{body}}_{\text{Rotational Change}} \tag{6}$$

Logic Implementation:

- If $t = 0$, set $\dot{\mathbf{B}}_{body} = 0$ (or assume a small initial rate).
- Store $\mathbf{B}_{ECI}(t)$ to use as $\mathbf{B}_{ECI}(t - \Delta t)$ in the *next* integration step.
- The term $-\boldsymbol{\omega} \times \mathbf{B}$ arises from the transport theorem (derivative of a vector in a rotating frame).

3.5 Hysteresis Dynamics ($\dot{\mathbf{M}}_{irr}$)

The function $\mathbf{f}_{hyst}$ mentioned in Section 2 is computed here. For each rod i aligned with the body axis unit vector $\mathbf{u}_i$:

1. **Project Field and Rate onto Rod:**

$$H_i = \frac{1}{\mu_0}(\mathbf{B}_{body} \cdot \mathbf{u}_i) \tag{7}$$

$$\dot{H}_i = \frac{1}{\mu_0}\left(\frac{d\mathbf{B}_{body}}{dt} \cdot \mathbf{u}_i\right) \tag{8}$$

2. **Calculate Theoretical Anhysteretic Magnetization (M_{an}):**

$$H_{eff} = H_i + \alpha M_{irr,i} \tag{9}$$

$$M_{an} = M_s\left(\coth\left(\frac{H_{eff}}{a}\right) - \frac{a}{H_{eff}}\right) \tag{10}$$

3. **Calculate Derivative (The Hysteresis Differential Equation):**
Note: The classic equation gives dM/dH. We require dM/dt for the state integration.

$$\frac{dM_{irr,i}}{dH} = \frac{M_{an} - M_{irr,i}}{k\delta} \quad \text{where } \delta = \text{sign}(\dot{H}_i) \tag{11}$$

Applying the chain rule:

$$\frac{dM_{irr,i}}{dt} = \left(\frac{M_{an} - M_{irr,i}}{k \cdot \text{sign}(\dot{H}_i)}\right) \cdot \dot{H}_i \tag{12}$$

3.6 Torque Dynamics ($\boldsymbol{\tau}_{total}$)

The total external torque acting on the spacecraft is the sum of two components:

$$\boldsymbol{\tau}_{total} = \boldsymbol{\tau}_{mag} + \boldsymbol{\tau}_{grad} \tag{13}$$

3.6.1 Magnetic Torque ($\boldsymbol{\tau}_{mag}$)

This is the interaction between the spacecraft's total dipole moment and the Earth's magnetic field. The dipole moment consists of the permanent magnet ($\mathbf{m}_{perm}$) and the hysteresis rods ($\mathbf{m}_{rods}$).

$$\mathbf{m}_{rods} = \sum_{i=1}^{N} \left(V_{rod} \cdot (M_{irr,i} + \chi_r H_i) \cdot \mathbf{u}_i\right) \tag{14}$$

Note: $\chi_r H_i$ adds the reversible/linear component if explicit in the model; otherwise, M_{total} is used.

$$\boldsymbol{\tau}_{mag} = (\mathbf{m}_{perm} + \mathbf{m}_{rods}) \times \mathbf{B}_{body} \tag{15}$$

3.6.2 Gravity Gradient Torque ($\boldsymbol{\tau}_{grad}$)

In LEO, the Earth's gravitational field varies across the body of the spacecraft, creating a torque that tries to align the minimum moment of inertia with the nadir vector.

$$\boldsymbol{\tau}_{grad} = \frac{3\mu}{\|\mathbf{r}\|^5} \left(\mathbf{r}_{body} \times (\mathbf{I} \cdot \mathbf{r}_{body})\right) \tag{16}$$

Where:

- $\mathbf{r}_{body} = \mathbf{R}_{ECI}^{Body} \cdot \mathbf{r}_{ECI}$ is the position vector in the Body frame.
- $\mathbf{I}$ is the spacecraft inertia tensor.
- μ is the Earth's standard gravitational parameter.