1) $$E[a+bx] = \int_x (a+bx)\, f(x)\, dx =$$

$$= \int_x \left[a f(x) + b x f(x)\right] dx$$

$$= \int a f(x)\, dx + \int b x f(x)\, dx$$

$$= a \overbrace{\int f(x)\, dx}^{1} + b \underbrace{\int x f(x)\, dx}_{E[x]} = a + b E[x]$$

$$Var[x] = E\left[(x-\mu_x)^2\right]$$

$$Var[a+bx] = E\left[\left[(a+bx) - \underbrace{E[a+bx]}\right]^2\right] =$$

$$= E\left[\left[a+bx-(a+b\mu_x)\right]^2\right] = E\left[\left[b(x-\mu_x)\right]^2\right] =$$

$$= E\left[b^2(x-\mu_x)^2\right] = b^2 \underbrace{E\left[(x-\mu_x)^2\right]}_{Var\, x} = b^2\, Var[x]$$

$$Var[x] = E\left[(x-\mu_x)^2\right] =$$

$$= E\left[x^2 - 2\mu_x x + \mu_x^2\right]$$

$$= E[x^2] - 2\mu_x \underbrace{E[x]}_{\mu_x} + \mu_x^2 = E[x^2] - 2\mu_x^2 + \mu_x^2 =$$

$$= E[x^2] - \mu_x^2$$

$$\equiv E[x^2] - E[x]^2$$

$$Cov[x,y] = E\left[(x-\mu_x)(y-\mu_y)\right]$$

$$= E\left[xy - x\mu_y - y\mu_x + \mu_x\mu_y\right]$$

$$= E[xy] - \mu_x\mu_y \cancel{-\mu_x\mu_y} + \cancel{\mu_x}\mu_y =$$

$$= E[xy] - \mu_x\mu_y$$

$$X \sim N(\mu_x, \sigma_x^2)$$

$$\boxed{z = \frac{x-\mu_x}{\sigma_x}} \qquad E[z] = ? \quad Var[z] = ?$$

$$z = \frac{X}{\sigma_x} - \frac{\mu_x}{\sigma_x} = \underbrace{-\frac{\mu_x}{\sigma_x}}_{a} + \underbrace{\frac{1}{\sigma_x}}_{b} \underbrace{X}_{x}$$

$$E[a+bx] = a + b\,E[x]$$

$$= -\frac{\mu_x}{\sigma_x} + \frac{1}{\sigma_x} E[x] = -\frac{\mu_x}{\sigma_x} + \frac{\mu_x}{\sigma_x} = \emptyset$$

$$Var[z] = Var[a+bx] = b^2\,Var[x]$$

$$= \frac{1}{\sigma_x^2}\,\sigma_x^2 = \underline{1}$$

$b \equiv \hat{\beta}$ $\quad e \equiv \hat{\varepsilon} \equiv \upsilon$

2)

Population model

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

Sample model

$$y = xb \boxed{+e}$$

RSS	VCM
scalar $\to e^2$	ee'
matrix $\to e'e$	$n\times 1 \times 1\times n$
$1\times n \times n\times 1$	$n\times n$
1×1	

OLS Problem :

$$\min_b \; RSS = e'e$$

$$= (y-xb)'(y-xb)$$

$$= y'y - (xb)'y - y'xb + (xb)'(xb)$$

$$= y'y - \underbrace{(xb)'y - (xb)'y} + b'x'xb =$$

$$= y'y - 2(xb)'y + b'x'xb =$$

$$= \underbrace{y'y} - \underbrace{2b'x'y} + \underbrace{b'x'xb} \to \text{Quadratic form} \quad \frac{\partial x^2}{\partial x} = 2x$$

(scalar)' = scalar

$$\underline{(AB)' = B'A'}$$

$$\frac{\partial x'M}{\partial x} = M$$

$$\frac{\partial RSS}{\partial b} = \emptyset - 2x'y + 2x'xb = \emptyset$$

$$x'xb = x'y \qquad |(x'x)^{-1} \text{ premultiply}$$

$$(x'x)^{-1}(x'x)b = (x'x)^{-1}x'y$$

$$\boxed{b = (x'x)^{-1}x'y}$$

$$\boxed{\hat{\beta} = \frac{\Sigma xy}{\Sigma x^2}}$$

Univariate

Unbiased? $E[b] \overset{?}{=} \beta \Rightarrow$ need to link b and β

$$b = (X'X)^{-1}X'Y = (X'X)^{-1}X'(X\beta + \varepsilon) =$$
$$= \underbrace{(X'X)^{-1}X'X}_{I}\beta + (X'X)^{-1}X'\varepsilon$$
$$= \beta + (X'X)^{-1}X'\varepsilon$$

$$E[b|X] = \beta + E[\underbrace{(X'X)^{-1}X'}\varepsilon|X] =$$
$$= \beta + \underbrace{(X'X)^{-1}X'E[\varepsilon|X]}_{0} = \beta$$

assumption A3 (Greene T2.1 or T4.1)

$$E[b] = E_X\{E[b|X]\} = E_X[\beta] = \beta \Rightarrow \text{unbiased} \checkmark$$

$ee' \equiv VCM$

$$\boxed{(AB)' = B'A'}$$

$$Var[b|X] = E[(b - \underbrace{E[b]}_{\beta})(b - E[b])'|X]$$
$$= E[(b-\beta)(b-\beta)'|X]$$
$$= E[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}|X]$$
$$= (X'X)^{-1}X'\underbrace{E[\varepsilon\varepsilon'|X]}X(X'X)^{-1}$$
$$= (X'X)^{-1}X'\sigma_\varepsilon^2 I X(X'X)^{-1} =$$
$$= \sigma_\varepsilon^2 \underbrace{(X'X)^{-1}X'X}(X'X)^{-1} =$$
$$\boxed{= \sigma_\varepsilon^2 (X'X)^{-1}}$$

$\sigma^2 \rightarrow s^2 = RSS = e'e$

Unbiased?

$$E[e'e|X] \overset{?}{=} \sigma_\varepsilon^2$$

Need to link e and $\varepsilon \Rightarrow \begin{cases}\text{projection mtrx}\\ \text{res. maker mtrx}\end{cases}$

$$\hat{Y} = Xb = \underbrace{X(X'X)^{-1}X'}_{P}Y$$

$$M = I - P \qquad X - PX = X - X = 0$$

$$e = My = M(X\beta + \varepsilon) = M\varepsilon$$

$$P' = P = P^2$$

$$M' = M = M^2$$

symetric & idempotent

$$E[e'e|X] = E[\varepsilon' M' M \varepsilon | X] =$$

$$= E[\underbrace{\varepsilon' M \varepsilon}_{\text{trace}(\varepsilon' M \varepsilon)}|X]$$

$$= E[tr(\varepsilon' M \varepsilon)|X] =$$

$$= tr(E[\varepsilon' M \varepsilon|X]) = tr(M\underbrace{E[\varepsilon\varepsilon'|X]}_{\sigma^2})$$

$$= tr\, M \sigma_\varepsilon^2 I$$

$$= \sigma^2 tr(M)$$

$$= \sigma^2 tr(I_n - X(X'X)^{-1}X') =$$

$$= \sigma^2 tr(I_n) - tr(\underbrace{(X'X)^{-1}X'X}_{I_k})$$

$$tr(ABC) = tr(BCA)$$

$$= \sigma^2(n-k)$$

$$s^2 = \frac{e'e}{n-k}$$

$$E[s^2] = E\left[\frac{e'e}{n-k}\right] = \frac{(n-k)\sigma^2}{n-k} = \sigma^2 \Rightarrow \text{unbiased now}$$

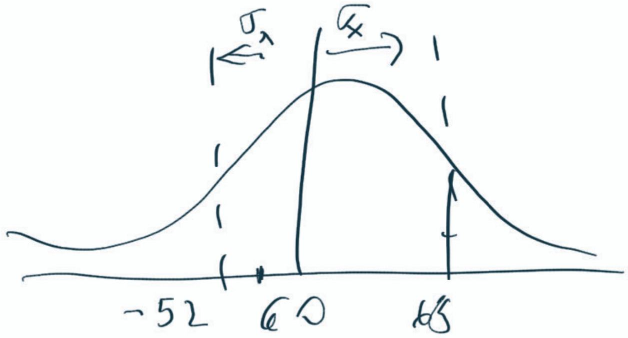
$$\Rightarrow \text{Est. } Var[b|X] = s^2(X'X)^{-1} \Rightarrow$$ standard error of b_k is square root of kth diagonal element

$$\Rightarrow S.E[b_k] = \left\{[s^2(X'X)^{-1}]_{kk}\right\}^{1/2}$$

3) X $\quad \mu_x = 60 \quad , \sigma_x^2 = 64 \qquad X \sim (60, 8)$

$$\bar{x} \sim N\left(\mu_x, \sqrt{\frac{\sigma_x^2}{n}}\right) = N\left(60, \sqrt{\frac{64}{100}}\right) = N(\underset{\mu_{\bar{x}}}{60}, \underset{\sigma_{\bar{x}}}{.8})$$

Pop $\rightarrow$



$$\bar{x} \Rightarrow Z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{58 - 60}{.8}$$

$$\text{Prob}(\bar{x} \leq 58) = F_z\left(\frac{-2}{.8}\right) = F_z(-2.5) = 1 - .9938 = 0.0062$$

Sample mean

$\bar{X} \rightarrow$