Expected value as operator:

Notation: $E[x]$, $E(x)$, Ex

Operator: $(a+b)^{10} \neq a^{10}+b^{10}$ $\quad (a+b)(a+b) \ldots (a+b)$

$$E[(x-\mu_x)(y-\mu_y)] \neq E[x-\mu_x]\,E[y-\mu_y]$$

Definition: $E[x] = \begin{cases} \sum_x x f(x) & \text{PMF} \\ \int_x x f(x)\,dx & \text{PDF} \end{cases}$

$$E[a+bx] = \int_x (a+bx) f(x)\,dx = \int_x a f(x)\,dx + \int_x bx f(x)\,dx$$

$$= a\underbrace{\int_x f(x)\,dx}_{1} + b\underbrace{\int_x x f(x)\,dx}_{E[x]} =$$

$$= a + bE[x] \equiv a + b\mu_x$$

$$var[a+bx] = E\left[\left((a+bx) - E[a+bx]\right)^2\right] =$$

$$= E\left[\left[a+bx-(a+b\mu_x)\right]^2\right] =$$

$$= E\left[(bx - b\mu_x)^2\right] = E\left[\left[b(x-\mu_x)\right]^2\right] =$$

$$= E\left[b^2(x-\mu_x)^2\right] = b^2 \underbrace{E\left[(x-\mu_x)^2\right]}_{var[x]} =$$

$$= b^2\,var[x] \equiv b^2\sigma_x^2$$

$$\boxed{var[x] \equiv \sigma_x^2 = E\left[(x-\mu_x)^2\right] \quad \text{Definition}}$$

$$= E\left[(x^2 - 2\mu_x x + \mu_x^2)\right]$$

$$= E[x^2] - 2\mu_x \underbrace{E[x]}_{\mu_x} + \mu_x^2$$

$$= E[x^2] - 2\mu_x^2 + \mu_x^2$$

$$= E[x^2] - \mu_x^2 \equiv E[x^2] - E[x]^2$$

$$\boxed{cov(x,y) = E\left[(x-\mu_x)(y-\mu_y)\right] \quad \text{Definition}}$$

$$= E[xy - \mu_x y - x\mu_y + \mu_x\mu_y]$$

$$= E[xy] - \mu_x \underbrace{E[y]}_{\mu_y} - \mu_y \underbrace{E[x]}_{\mu_x} + \mu_x\mu_y =$$

$$= E[xy] - \mu_x\mu_y \cancel{- \mu_x\mu_y + \mu_x\mu_y}$$

$$= E[xy] - \mu_x\mu_y = E[xy] - E[x]E[y]$$

---

pdf $\quad f(x) = \dfrac{1}{\sigma_x\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$

$$f(\mu_x) = \frac{1}{\sigma_x\sqrt{2\pi}}\, \cancel{e^{0}} \to 1 = \frac{1}{\sigma_x\sqrt{2\pi}}$$

As $\sigma_x \to 0 \Rightarrow f(\mu_x) \to \infty$

$$E[a + bx] = a + b\,E[x]$$

$$z = \frac{x - \mu_x}{\sigma_x} = \frac{x}{\sigma_x} - \frac{\mu_x}{\sigma_x} = \overset{a}{-\frac{\mu_x}{\sigma_x}} + \overset{bx}{\frac{1}{\sigma_x}x}$$

$$E(z) = E[a + bx] = -\frac{\mu_x}{\sigma_x} + \frac{1}{\sigma_x}\underbrace{E[x]}_{\mu_x} = -\frac{\mu_x}{\sigma_x} + \frac{\mu_x}{\sigma_x} = 0$$

$$Var[z] = \left(\frac{1}{\sigma_x}\right)^2 \underbrace{Var[x]}_{\sigma_x^2} = 1$$

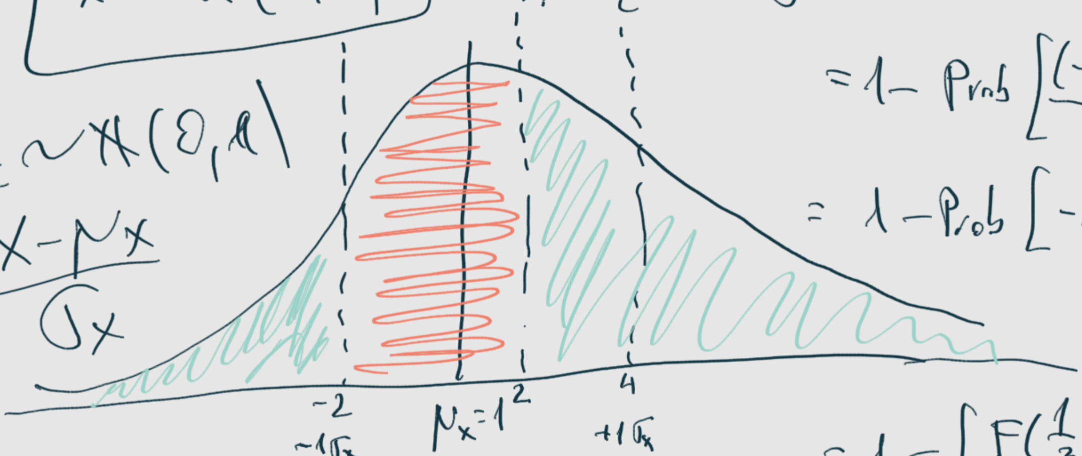
Q1. $\boxed{X \sim N(1,3)}$ $\quad Prob[|x| > 2] = 1 - Prob[-2 \le x \le 2] =$

$$= 1 - Prob\left[\frac{(-2-1)}{3} \le z \le \frac{2-1}{3}\right] =$$

$Z \sim N(0,1)$

$\dfrac{X - \mu_x}{\sigma_x}$

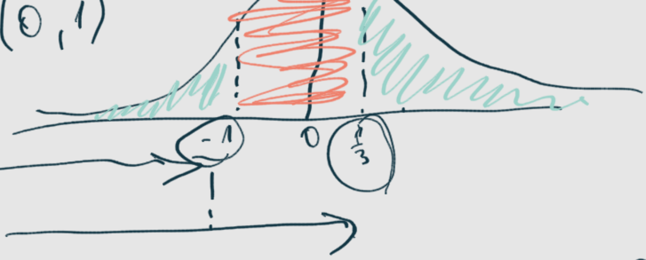
$$= 1 - Prob\left[-1 \le z \le \frac{1}{3}\right] =$$



$$= 1 - \left[F\left(\tfrac{1}{3}\right) - F(-1)\right] =$$

$Z \sim N(0,1)$

CDF $-\infty$



$\approx 1 - (.6293 - .15866) =$

$\approx 1 - .47 \approx .53$

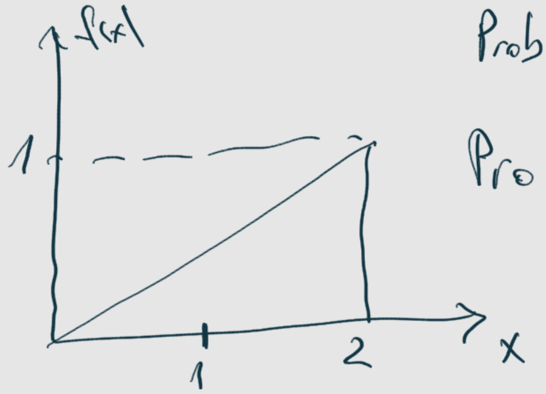
Notation: PDF $= f(x)$ CDF $= F(x)$

Q2. $f(x) = \frac{1}{2}x$

a) $\int_x f(x)dx = 1$

$$\int_0^2 \frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_0^2 = \frac{4}{4} - 0 = 1 \checkmark$$

b)

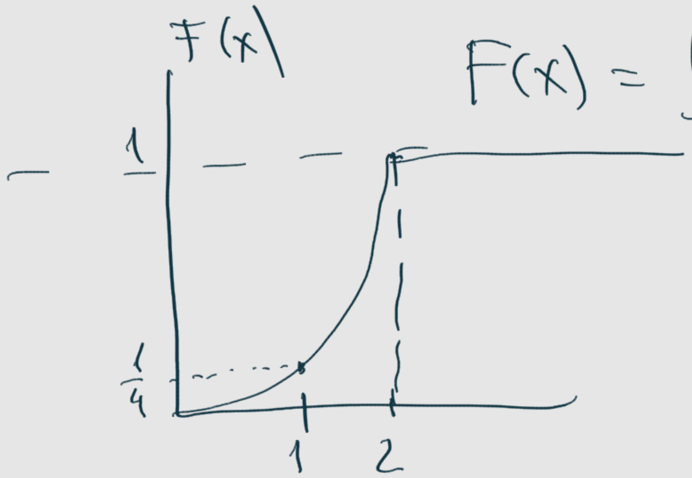


$$\text{Prob}(x<1) = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_0^1 = \frac{1}{4}$$

$$\text{Prob}(x>1) = 1 - \frac{1}{4} = \frac{3}{4}$$

c) CDF $= \int$ PDF $= \frac{1}{4}x^2,\ 0 \le x \le 2$

$$F(x) = \int f(x)\,dx$$



$$\mu_x = E[x] = \int_x x f(x)\,dx$$

$$= \int_0^2 x \frac{1}{2}x\,dx = \int_0^2 \frac{1}{2}x^2\,dx$$

$1 \quad 3\Big|^2 \quad 8 \quad 4$

$$= \frac{1}{6} x^3 \Big|_0 = \frac{}{6} - \emptyset = \frac{1}{3}$$

$$Var[X] \equiv \sigma_x^2 = E\left[(X-\mu_x)^2\right] = \int_x \underbrace{(x-\mu_x)^2}\underbrace{f(x)}_{PDF} dx$$

$$= \int_0^2 \left(x - \frac{4}{3}\right)^2 \frac{1}{2} x \, dx =$$

$$= \int_0^2 \left(x^2 - \frac{8}{6}x + \frac{16}{9}\right) \frac{1}{2} x \, dx \quad \ldots\ldots = \frac{4}{18}$$