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Correlations for the Graetz problem in convection – Part 1: For round pipes and parallel plates

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ABSTRACT

A general correlation is proposed for the laminar Graetz problem associated with convection in round pipes and parallel plate ducts. The general correlation for the average Nusselt number can be expressed under conditions of either constant wall temperature or constant wall heat flux. Additionally, closed form expressions for the local Nusselt number can be derived from the general correlation. New correlations are found to be within ±1.5% of exact solutions to the Graetz problem for round pipes and parallel plates, and are contrasted with existing correlations in the literature.

1. Introduction

Heat transfer into fully developed laminar flows is of great practical importance, and has been a topic of study for much of the last century. When an internal flow is already hydrodynamically developed, thermal boundary layers grow into a prescribed velocity distribution over the entry region of a heated duct. This is sometimes referred to as the Graetz problem because of his pioneering analysis of heat transfer in circular pipes [1]. A formulation of this problem can be found in Refs. [2,3] for flows in circular pipes and between parallel plates, and for wall conditions that are either constant temperature or constant heat flux. All correlations for the Graetz problem are basically interpolation functions designed to span the asymptotic behaviors of very short and very long ducts. Generalized correlations attempt to utilize one master equation that can be modified for the desired geometric and thermal conditions of the duct wall.

There has only been two previous attempts at a generalized correlation to the Graetz problem. Yilmaz and Cihan provided one correlation for the average Nusselt number in ducts of arbitrary geometry and constant wall temperature [4], and a second general correlation for the local Nusselt number in ducts of constant wall heat flux [5]. Yilmaz and Cihan assert a ±6.5% agreement with first principle solutions for ducts of constant wall temperature and −8.7% to +8.0% for ducts of constant wall heat flux. More recently, Muzychka and Yovanovich [6] provided one correlation for addressing both wall conditions and both local and average Nusselt numbers. This correlation was developed in the context of a limiting case ($Pr \to \infty$) of the more general combined entry problem, for which they assert ±15% agreement with first principle solutions. The reported uncertainty by both sets of authors are consistent with the level of accuracy of many of the early first principle calculations used in their studies.

Both previous attempts at generalizing a correlation for the Graetz problem have left significant room for improvement. Yilmaz and Cihan correlations [4,5] are incomplete from the stand point of not offering expressions for the local Nusselt number for ducts of constant wall temperature, or the average Nusselt number for ducts of constant wall heat flux. In general, there are four desired cases: local and average Nusselt numbers for constant temperature and constant heat flux wall conditions. The Muzychka and Yovanovich [6] correlation is able to express all four of these cases, but does not adhere to required relationships between local and average convection coefficients (except in the limiting case of very short or very long ducts). Additionally, Muzychka and Yovanovich used the same convection coefficient averaging rule for the constant wall heat flux condition as for the constant wall temperature condition. This leads to erroneous results when the conventional definition of average surface temperature is applied to the constant heat flux wall condition.

A significantly improved general correlation is developed in the current study by the following methodology. First, a set of trial functions was developed with forms informed by an extensive literature review. The fitting performance of these functions were contrasted using four bench mark calculations in order to select trial functions with the greatest potential. There is no evidence of this step in earlier works, and both previous attempts advanced different functions from each other and from the current work. The benchmark calculations used in this study were performed by Cotta and Özişik [2,3] for flows in circular tubes and between parallel plates that considers both constant temperature and constant heat flux wall conditions. The second step in the current

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Nomenclature

A	constants Lévêque solution, Eq. (40)
c_1, c_2, c_3	constants in trial functions, Eqs. (41), (42), (44) and (45)
D_h	hydraulic diameter (m)
f	Darcy friction factor
Gz_L^{-1}	inverse Graetz number, see Eq. (12)
h	convection coefficient (W/m^2/K)
k	thermal conductivity of the fluid (W/m/K)
L	length of heated duct (m)
Nu	Nusselt number, see Eqs. (4) and (7)
Nu_∞	Nusselt number for fully developed heat transfer
n	exponent in Eq. (46)
O^{Lev}	offset in extended Lévêque solution, see Eq. (46)
Pr	Prandtl number
R^2	statistical measure of data closeness to fitted line
Re_D	Reynolds number based on hydraulic diameter
x	distance from start of heated duct (m)
Subscripts	
L	duct length used with averaged values
Pipe	circular duct geometry
Plates	parallel plates duct geometry
x	local position used with local values
Superscripts	
Grtz	Graetz problem
H	constant wall heat flux
Lev	Lévêque solution
T	constant wall temperature

study was to determine whether the fitted constants in the trial functions could be correlated back to variables describing the asymptotic limits. This has not been done previously, and becomes the bases by which the best interpolation function is selected. As will be demonstrated in this paper, the generalized Graetz problem correlation resulting from this methodology offers a substantial quantitative improvement over the two previous attempts, and even as compared to most of the geometry specific correlations found in the literature.

2. Relations for local and average convection coefficients

Correlations to the Graetz problem are used to describe the local and average convection coefficients. The local convection coefficient is defined based on a local energy balance between the change in advected energy in the flow and the heat convected from the walls of the duct:

$$h_x = \frac{\dot{m}C_p(dT_{m,x}/dx)}{P_h \Delta T_x}. \tag{1}$$

In this definition, $\dot{m}C_p$ is the heat capacity transfer rate through the duct, P_h is the constant hydraulic perimeter of the duct, and $\Delta T_x = (T_{s,x} - T_{m,x})$ is the local temperature difference between the wall (surface) of the duct and the mean (bulk) temperature of the flow [1]. The average convection coefficient is defined by:

$$h_L = \frac{\int_0^L \dot{m}C_p(dT_{m,x}/dx)dx}{\int_0^L P_h \Delta T_x dx} = \frac{Q}{A_{conv}\Delta T_L}, \tag{2}$$

where the average temperature difference between the duct wall and the flow is defined by:

$$\Delta T_L = \frac{\int_0^L P_h \Delta T_x dx}{A_{conv}} = \frac{\int_0^L \Delta T_x dx}{L} \tag{3}$$

Notice that definitions for the average convection coefficient h_L, Eq. (2), and average temperature difference ΔT_L, Eq. (3), are true irrespective of the imposed thermal conditions on the wall. If the wall condition happens to be isothermal, then the average temperature difference is the same as the log-mean temperature difference [7].

Convection coefficients can be expressed in dimensionless form by the Nusselt number. For convection in ducts of constant wall temperature, the local and average value of the Nusselt numbers are defined by:

$$Nu_x^T = \frac{h_x^T D_h}{k} \text{ and } Nu_L^T = \frac{h_L^T D_h}{k}, \tag{4}$$

h_x^T is the local convection coefficient at some streamwise distance x from the entrance of the isothermal section of duct, and h_L^T is the corresponding average value over a section of length L. Notice that expressions denoted with the superscript T are referencing the constant wall temperature condition. For the case of constant wall temperature, the average Nusselt number is determined from the local value by:

$$Nu_L^T = \frac{1}{L}\int_0^L Nu_x^T dx \tag{5}$$

It is observed from Eq. (5) that for ducts of constant wall temperature, the local Nusselt number can be calculated from correlations for the average value using:

$$Nu_x^T = \frac{d}{dL}(Nu_L^T L)\bigg|_{L=x} \tag{6}$$

For convection in ducts of constant wall heat flux, Nusselt numbers for the local and average value are defined by:

$$Nu_x^H = \frac{h_x^H D_h}{k} \text{ and } Nu_L^H = \frac{h_L^H D_h}{k}, \tag{7}$$

where D_h is the hydraulic diameter.

For the constant wall heat flux condition, the average Nusselt number is determined from the local value by:

$$\frac{L}{Nu_L^H} = \int_0^L \frac{dx}{Nu_x^H} \tag{8}$$

It should be noted that the average convection coefficient for the constant heat flux wall condition is not calculated from Eq. (5), as has been abundantly suggested elsewhere in the literature [8]. For the case of constant wall heat flux, the local heat flux can be expressed in terms of the average convection coefficient h_L^H and average temperature difference between the flow and the duct wall ΔT_L by the requirement that:

$$q_x^H = \frac{Q}{A_{conv}} = h_L^H \Delta T_L \tag{9}$$

Consequently, the validity of Eq. (8) can be demonstrated by:

$$\int_0^L \frac{dx}{Nu_x^H} = \frac{k}{D_h}\int_0^L \frac{\Delta T_x dx}{\Delta T_x h_x^H} = \frac{k}{D_h}\int_0^L \frac{\Delta T_x dx}{q_x^H} = \frac{k}{D_h}\frac{\Delta T_L L}{h_L^H \Delta T_L} = \frac{L}{Nu_L^H} \tag{10}$$

For ducts of constant wall heat flux, Eq. (8) can be used to demonstrate that the local Nusselt number is related to the average value by:

$$\frac{1}{Nu_x^H} = \frac{d}{dL}\left(\frac{L}{Nu_L^H}\right)\Big|_{L=x} \tag{11}$$

3. First principle solutions

Traditional solutions to the Graetz problem use a method of separation of variables that yields a series solution [9]. Historically, the task of evaluating the required eigenfunctions to the Graetz problem has been laborious, with the series converging slowly for short duct lengths. Traditional solutions to the Graetz problem must be carried out to 10 or more terms in order to achieve accurate solutions for $Gz_L^{-1} \geqslant 0.0005$, where the length of heated duct is measured using the inverse Graetz number:

$$Gz_L^{-1} = \frac{L}{D_h Re_D Pr} \tag{12}$$

where Re_D is the Reynolds number (based on hydraulic diameter) and Pr is the Prandtl number.

For shorter ducts, an asymptotic solution to the Graetz problem was proposed by Lévêque [10], which considers the thermal boundary layer to be thin compared to the hydraulic diameter. This condition limits convection to a linear region of the velocity profile near the wall, greatly simplifying the solution to the heat equation. The traditional Lévêque solution has been extended into a power series solution in $Gz_L^{-1/3}$. This series has been calculated to the first few terms in Ref. [11], and provides useful solutions for when $Gz_L^{-1} < 0.0005$.

Classical solutions to the Graetz problem are illustrated in Fig. 1 for flows between parallel plates and through circular pipes, for a constant wall temperature condition. For duct lengths approaching $Gz_L^{-1} \to 0.1$, the separation of variables solution to the Graetz problem was performed using the first ten (plates) or eleven (pipe) eigenvalue and eigenfunction terms in the series as calculated by [9], shown as solid lines in Fig. 1. For the limiting case of $Gz_L^{-1} \to 0$, the extended Lévêque solution for a pipe [11] (using the first two terms in the series) and the standard (one term) Lévêque solution for parallel plates are plotted as dashed lines in Fig. 1.

Computational techniques for solving Sturm-Liouville problems allow the series solution to be extended to large numbers of terms. By this method, Cotta and Özişik [2,3] determined "exact" solutions to the Graetz problem for flows through round pipes and between parallel plates for duct lengths as short as $Gz_L^{-1} \geqslant 10^{-6}$.

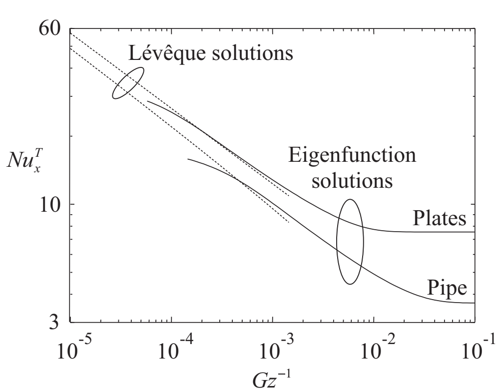

Fig. 1. Classical solutions to the Graetz problem for isothermal pipes and parallel plate ducts, following Refs. [9,11].

Cotta and Özişik's solution to the Graetz problem for an isothermal pipe are within 0.1% of the differencing method solution to the same problem by Jensen [12]. Average and local Nusselt numbers for both pipe and parallel plate geometries, and for wall conditions of constant temperature and constant heat flux, were reproduced from Cotta and Özişik and are presented in Table 1. The average Nusselt number for the constant wall heat flux case was calculated from reported local Nusselt numbers using the definition given by Eq. (8). The contents of Table 1 will be referred to as the "exact solution" for the purpose of evaluating the accuracy of correlations to the Graetz problem (see Table 2).

4. Review literature on correlations for the graetz problem

The complexity of analysis of the Graetz problem has motivated attempts to correlate solutions with simple functions. A number of correlations for the Graetz problem have been developed over the years, mostly for the pipe flow geometry. Hausen [13] provided one of the earliest correlations for an isothermal pipe:

$$Nu_{Pipe,L}^T = Nu_\infty^T + \frac{0.0668/Gz_L^{-1}}{1 + 0.04/Gz_L^{-2/3}} \tag{13}$$

where $Nu_\infty^T = 3.657$, is the fully developed value of the Nusselt number for pipes with isothermal walls. Hausen's correlation deviates from the exact solution provided in Table 1 by +0.4% to +13%, with the poorest comparison occurring in the $Gz_L^{-1} \to 0$ limit.

For the average Nusselt number corresponding to isothermal pipes, Shah [14] recommends:

$$Nu_{Pipe,L}^T = \begin{cases} 1.615/Gz_L^{-1/3} - 0.7 & Gz_L^{-1} \leqslant 0.005 \\ 1.615/Gz_L^{-1/3} - 0.2 & 0.005 < Gz_L^{-1} \leqslant 0.03 \\ 3.657 + 0.0499/Gz_L^{-1} & 0.03 < Gz_L^{-1} \end{cases} \tag{14}$$

This correlation deviates from the exact solution in Table 1 by −2.2 to +2.0%. For the local Nusselt number corresponding to isothermal pipes, Shah [14] recommends:

$$Nu_{Pipe,x}^T = \begin{cases} 1.077/Gz_x^{-1/3} - 0.7 & Gz_x^{-1} \leqslant 0.01 \\ 3.657 + 6.874\left(10^3 Gz_x^{-1}\right)^{-0.488} \exp\left(-57.2 Gz_x^{-1}\right) & 0.01 < Gz_x^{-1} \end{cases} \tag{15}$$

This correlation deviates from the exact solution by −13% to +1.0%, with the largest error occurring at the break between the two analytic functions at $Gz_x^{-1} = 0.01$.

Churchill and Ozoe [15] proposed the following isothermal pipe correlation for the local Nusselt number:

$$Nu_{Pipe,x}^T = 5.357\left[1 + \left(97\frac{4}{\pi}Gz_x^{-1}\right)^{-8/9}\right]^{3/8} - 1.7 \tag{16}$$

The authors note that the appropriateness of the offset (1.7) is suggested by the first term in the extended Lévêque solution, although the magnitude of this offset was decided empirically. They asserted agreement between this correlation and the Graetz problem solution to within 6% for all values of Gz_x^{-1} computed by Munakata [16]. Comparison with the more modern results provided in Table 1 suggests that this correlation is in error by −4.7% to +8.7% for $10^{-6} \leqslant Gz_x^{-1} \leqslant 1$. It should be noted that Churchill and Ozoe developed this correlation in the context of a limiting case ($Pr \to \infty$) to the more general consideration of the combined entry problem in which the state of the flow is hydrodynamically developing along with the state of heat transfer.

Churchill and Ozoe [17] also proposed the following correlation for the local Nusselt number associated with pipes having a constant heat flux wall condition:

Table 1
Exact Graetz problem solutions for local and average Nusselt numbers as taken or derived from Refs. [2,3].

	Round pipe				Parallel plates			
Gz_L^{-1}	Nu_x^T	Nu_L^T	Nu_x^H	Nu_L^H	Nu_x^T	Nu_L^T	Nu_x^H	Nu_L^H
0.000001	106.5	160.1	129.2	172.3	122.9	185.1	148.8	198.4
0.0000015	92.93	140.1	112.8	150.5	107.4	161.7	129.9	173.3
0.0000025	78.22	117.6	94.96	126.8	90.53	135.1	109.6	146.1
0.000005	61.88	93.59	75.19	100.5	71.83	107.6	86.95	116.0
0.00001	48.91	73.86	59.51	79.56	57.00	85.65	69.01	92.03
0.000015	42.61	64.46	51.89	69.39	49.79	74.67	60.29	80.38
0.000025	35.80	53.90	43.65	58.37	42.01	62.75	50.87	67.77
0.00005	28.25	42.83	34.51	46.19	33.38	50.05	40.42	53.84
0.0001	22.28	33.80	27.28	36.52	26.56	39.74	32.16	42.79
0.00015	19.38	29.44	23.77	31.83	23.26	34.74	28.16	37.43
0.00025	16.26	24.61	19.99	26.75	19.72	29.22	23.85	31.63
0.0005	12.82	19.50	15.81	21.16	15.83	23.41	19.11	25.27
0.001	10.13	15.38	12.54	16.75	12.82	18.75	15.43	20.27
0.0015	8.840	13.40	10.97	14.62	11.41	16.52	13.68	17.87
0.0025	7.474	11.21	9.299	12.34	9.964	14.08	11.87	15.32
0.005	6.002	8.943	7.494	9.869	8.517	11.62	9.988	12.63
0.01	4.916	7.155	6.148	7.973	7.741	9.825	8.803	10.68
0.015	4.441	6.321	5.547	7.093	7.583	9.097	8.439	9.879
0.025	4.017	5.422	4.984	6.196	7.545	8.366	8.275	9.189
0.05	3.710	4.641	4.514	5.335	7.541	8.010	8.236	8.690
0.10	3.658	4.156	4.375	4.833	7.541	7.776	8.235	8.457
0.15	3.657	3.990	4.365	4.667	7.541	7.697	8.235	8.382

$$Nu^H_{Pipe,x} = 5.364\left[1+\left(55\frac{4}{\pi}Gz_x^{-1}\right)^{-10/9}\right]^{3/10} - 1.0 \tag{17}$$

This correlation differs from the exact solution provided in Table 1 by −3.4% to +4.9%.

Shah and London [18] report the following correlation for local convection in circular ducts subject to a constant wall heat flux:

$$Nu^H_{Pipe,x} = \begin{cases} 1.302/Gz_x^{-1/3} - 1.0 & Gz_x^{-1} \leqslant 5\times10^{-5} \\ 1.302/Gz_x^{-1/3} - 0.5 & 5\times10^{-5} < Gz_x^{-1} \leqslant 0.0015 \\ 4.364 + 8.68\left(10^3 Gz_x^{-1}\right)^{-0.506}\exp\left(-41 Gz_x^{-1}\right) & 0.0015 < Gz_x^{-1} \end{cases} \tag{18}$$

This correlation agrees within −1.0% to +0.9 of the exact solution in Table 1. For the average convection coefficient in pipes with constant wall heat flux, Shah [14] recommends:

$$Nu^H_{Pipe,L} = \begin{cases} 1.953/Gz_L^{-1/3} - 1.0 & Gz_L^{-1} \leqslant 0.03 \\ 4.364 + 0.0722/Gz_L^{-1} & 0.03 < Gz_L^{-1} \end{cases} \tag{19}$$

However, this correlation is in error by −8.3% to +13%, and exhibits a 25% jump in the correlation value between the two analytic functions at Gz_x^{-1} =0.03.

Shome and Jensen [19] made use of their accurate numerical solution to the Graetz problem to propose the following average Nusselt number correlation for isothermal pipes:

$$Nu^T_{Pipe,L} = \begin{cases} -0.5632 + 1.571/Gz_L^{-0.3351} & 10^{-6} < Gz_L^{-1} \leqslant 10^{-3} \\ 0.9828 + 1.129/Gz_L^{-0.3686} & 10^{-3} < Gz_L^{-1} \leqslant 10^{-2} \\ 3.6568 + 0.1272/Gz_L^{-0.7373}\exp\left(-3.1563 Gz_L^{-1}\right) & 10^{-2} < Gz_L^{-1} \end{cases} \tag{20}$$

This correlation agrees within −0.3% to +0.5% of the exact solution in Table 1. For the constant wall heat flux condition, Shome and Jensen [19] recommend the following modification of the isothermal pipe correlation:

$$Nu^H_{Pipe,L} = Nu^T_{Pipe,L}\left[1 + 209.92\left(1 + Gz_L^{+0.20}\right)\right]^{0.03} \tag{21}$$

where $Nu^T_{Pipe,L}$ corresponding to the isothermal pipe is evaluated with Eq. (20). Unfortunately, Shome and Jensen calculated the average Nusselt number based on Eq. (5) and not Eq. (8) for the constant wall heat flux condition. Therefore, Eq. (21) deviates from the exact solution by +2.8% to +19%.

For the local convection coefficient, Shome and Jensen [19] proposed the following correlations for round pipes. For the constant temperature wall condition:

$$Nu^T_{Pipe,x} = \begin{cases} 1.022/Gz_x^{-0.3366} - 0.3856 & Gz_x^{-1} \leqslant 0.001 \\ 3.6568 + 0.2249/Gz_x^{-0.4956}\exp\left(-55.9857 Gz_x^{-1}\right) & 0.001 < Gz_x^{-1} \end{cases} \tag{22}$$

and for the constant wall heat flux condition:

$$Nu^H_{Pipe,x} = Nu^T_{Pipe,x}\left[1 + 98.42\left(1 + Gz_x^{+0.24}\right)\right]^{0.03} \tag{23}$$

in which $Nu^T_{Pipe,x}$ is evaluated with Eq. (22). For the constant wall temperature condition, correlation (22) is within −0.6% to +0.2% of the exact solution in Table 1. For the constant heat flux wall condition, correlation (23) is within −4.5% to +4.6% of the exact solution.

Stephan [20] proposed the following correlation for convection in an isothermal pipe:

$$Nu^T_{Pipe,L} = \frac{3.657}{\tanh\left[2.264 Gz_L^{-1/3} + 1.7 Gz_L^{-2/3}\right]} + \frac{0.0499}{Gz_L^{-1}}\tanh\left(Gz_L^{-1}\right) \tag{24}$$

This correlation agrees within −0.5% to +1.9% of the exact solution in Table 1. It has been asserted this equation approaches the two term eigenfunction series solution that is suitable for $Gz_L^{-1} \geqslant 0.05$ [21]:

$$Nu^T_{Pipe,L} = Nu^T_\infty + \frac{0.0499}{Gz_L^{-1}} + \cdots \tag{25}$$

However, it is noteworthy that $\tanh(Gz_L^{-1})$ does not approximate 1 until $Gz_L^{-1} > 1$. In fact, the $1.7Gz_L^{-2/3}$ term in the denominator of the preceding term of Eq. (24) is significant in reproducing the desired behavior of Eq. (25) as $Gz_L^{-1} \to 1$.

Gnielinski [7] proposed Graetz problem correlations for pipe flow. His correlation for the local Nusselt number for pipes of constant wall temperature is:

Table 2
Correlations compared against exact solutions presented in Table 1.

Geometry	Wall condition		Author	Year	Ref	Eq.	Maximum error	
							Lower	Upper
Pipe	(T)	Average	Hausen	1943	[13]	(13)	+0.4%	+13%
			Shah	1975	[14]	(14)	−2.2%	+2.0%
			Gnielinski	1993	[7]	(27)	+0.7%	+1.4%
			Shome and Jensen	1993	[19]	(20)	−0.3%	+0.5%
			Yilmaz and Cihan	1993	[4]	(34)	−0.6%	+4.7%
			Yilmaz and Cihan	1993	[4]	(35)	+0.4%	+4.7%
			Stephan	2001	[20]	(24)	−0.5%	+1.9%
			Muzychka and Yovanovich	2004	[6]	(39)	−3.6%	+1.7%
			Bennett			(51)	−0.4%	+1.4%
Pipe	(T)	Local	Churchill and Ozoe	1973	[15]	(16)	−4.7%	+8.7%
			Shah	1975	[14]	(15)	−13%	+1.0%
			Gnielinski	1993	[7]	(26)	+0.5%	+5.7%
			Shome and Jensen	1993	[19]	(22)	−0.6%	+0.2%
			Muzychka and Yovanovich	2004	[6]	(39)	−3.2%	+3.2%
			Bennett			(52)	−0.9%	+1.3%
Plates	(T)	Average	Shah	1975	[14]	(31)	−1.8%	+1.8%
			Yilmaz and Cihan	1993	[4]	(34)	−4.8%	+0.0%
			Yilmaz and Cihan	1993	[4]	(35)	−3.6%	+0.3%
			Muzychka and Yovanovich	2004	[6]	(39)	−7.6%	−1.7%
			Bennett			(55)	−0.5%	+1.4%
Plates	(T)	Local	Shah	1975	[14]	(30)	−0.7%	1.5%
			Muzychka and Yovanovich	2004	[6]	(39)	−6.2%	+1.4%
			Bennett			(56)	−0.8%	+0.8%
Pipe	(H)	Average	Shah	1975	[14]	(19)	−8.3%	+13%
			Gnielinski	1993	[7]	(29)	+3.2%	+14%
			Shome and Jensen	1993	[19]	(21)	+2.8%	+19%
			Muzychka and Yovanovich	2004	[6]	(39)	−0.5%	+13%
			Bennett			(53)	−1.0%	+1.0%
Pipe	(H)	Local	Churchill and Ozoe	1973	[17]	(17)	−3.4%	+4.9%
			Shah and London	1978	[18]	(18)	−1.0%	+0.9%
			Gnielinski	1993	[7]	(28)	−4.0%	+2.8%
			Shome and Jensen	1993	[19]	(23)	−4.5%	+4.6%
			Yilmaz and Cihan	1993	[5]	(37)	−0.7%	+3.8%
			Muzychka and Yovanovich	2004	[6]	(39)	−2.3%	+1.9%
			Bennett			(54)	−1.2%	+1.0%
Plates	(H)	Average	Shah	1975	[14]	(32)	−0.0%	+14%
			Muzychka and Yovanovich	2004	[6]	(39)	−1.9%	+9.4%
			Bennett			(57)	−0.5%	+0.6%
Plates	(H)	Local	Shah	1975	[14]	(33)	−6.1%	+0.6%
			Yilmaz and Cihan	1993	[5]	(37)	−3.9%	+2.4%
			Muzychka and Yovanovich	2004	[6]	(39)	−5.8%	+1.0%
			Bennett			(58)	−0.8%	+0.8%

$$Nu^{T}_{Pipe,x} = \left(\left\{1.077 Gz_x^{1/3} - 0.7\right\}^3 + 0.7^3 + \left(Nu^T_\infty\right)^3\right)^{1/3} \tag{26}$$

and correlation for the average Nusselt number is:

$$Nu^{T}_{Pipe,L} = \left(\left\{1.615 Gz_x^{1/3} - 0.7\right\}^3 + 0.7^3 + \left(Nu^T_\infty\right)^3\right)^{1/3} \tag{27}$$

For the constant wall temperature condition, correlation (26) is within +0.5% to +5.7% of the exact solution in Table 1, and correlation (27) is within −0.7% to +1.4%. Gnielinski's correlation for the local Nusselt number for pipes of constant wall heat flux is:

$$Nu^{H}_{Pipe,x} = \left(\left\{1.302 Gz_x^{1/3} - 1\right\}^3 + 1^3 + \left(Nu^H_\infty\right)^3\right)^{1/3} \tag{28}$$

and for the average Nusselt number is:

$$Nu^{H}_{Pipe,L} = \left(\left\{1.953 Gz_x^{1/3} - 0.6\right\}^3 + 0.6^3 + \left(Nu^H_\infty\right)^3\right)^{1/3} \tag{29}$$

where $Nu^H_\infty = 48/11$, is the fully developed value of the Nusselt number for pipes with walls of constant heat flux. For the constant wall heat flux condition, correlation (28) is within −4.0% to +2.8% of the exact solution in Table 1, and correlation (29) is within +3.2% to +14%.

For the local Nusselt number for flows between parallel plates with constant wall temperature, Shah [14] recommends:

$$Nu^{Grtz,T}_{Plates,x} = \begin{cases} 1.233/Gz_L^{-1/3} + 0.4 & Gz_L^{-1} \leqslant 0.001 \\ 7.541 + 6.874\left(10^3 Gz_L^{-1}\right)^{-0.488} \exp\left(-245 Gz_x^{-1}\right) & 0.001 < Gz_L^{-1} \end{cases} \tag{30}$$

This correlation agrees within −0.7% to +1.5% of the exact solution in Table 1.

For the average Nusselt number for flows between parallel plates with constant wall temperature, Shah [14] recommends:

$$Nu^{Grtz,T}_{Plates,L} = \begin{cases} 1.849 Gz_L^{1/3} & Gz_L^{-1} \leqslant 0.0005 \\ 1.849 Gz_L^{1/3} + 0.6 & 5\times10^{-4} < Gz_L^{-1} \leqslant 0.006 \\ 7.541 + 0.0235 Gz_L & 0.006 < Gz_L^{-1} \end{cases} \tag{31}$$

This correlation agrees within −1.8% to +1.8% of the exact solution.

For the average Nusselt number for flows between parallel plates with constant wall heat flux, Shah [14] recommends:

$$Nu^{Grtz,H}_{Plates,L} = \begin{cases} 2.236 Gz_L^{1/3} & Gz_L^{-1} \leqslant 10^{-3} \\ 2.236 Gz_L^{1/3} + 0.9 & 10^{-3} < Gz_L^{-1} \leqslant 10^{-2} \\ 8.235 + 0.0364 Gz_L & 10^{-2} < Gz_L^{-1} \end{cases} \tag{32}$$

However, this correlation over predicts the exact solution by as much as +14%.

For the local Nusselt number for flows between parallel plates with constant wall heat flux, Shah [14] recommends:

$$Nu^{Grtz,H}_{Plates,x} = \begin{cases} 1.409/Gz_x^{-1/3} & Gz_L^{-1} \leqslant 2\times10^{-4} \\ 1.409/Gz_x^{-1/3} + 0.4 & 2\times10^{-4} < Gz_L^{-1} \leqslant 10^{-3} \\ 8.235 + 8.68\left(10^3 Gz_x^{-1}\right)^{-0.506}\exp\left(-164 Gz_x^{-1}\right) & 10^{-3} < Gz_L^{-1} \end{cases} \tag{33}$$

This correlation agree to within −6.1% to +0.6% of the parallel plate data in Table 1. However, the constant offset in the second function of Eq. (33) has been changed from −0.4 (as originally published) to +0.4 for better agreement with the exact solution.

Yilmaz and Cihan [4] proposed two alternate average correlations for the Graetz problem in ducts of arbitrary geometry and with walls of constant temperature:

$$\frac{Nu_L^{Grtz,T}}{Nu_\infty^T} = 1 + \frac{1.615X^{-1/3}}{\left(1+1.88X^{1/3}+3.93X^{4/3}\right)^{1/2}} \tag{34}$$

$$\text{and} \quad \frac{Nu_L^{Grtz,T}}{Nu_\infty^T} = \left[1 - \frac{0.8}{X^{2/3}} + \frac{4.212}{X}\right]^{1/3}, \tag{35}$$

where in both correlations

$$X = \frac{64\left(Nu_\infty^T\right)^3}{\Phi^3 (fReGz_L)} \tag{36}$$

Yilmaz and Cihan [4] suggest further correlations for estimating fRe, Nu_∞^T and Φ, where fRe is the product of the Darcy friction factor f and Reynolds number Re_D (based on hydraulic diameter). The correlation for Φ evaluates to $\Phi = 1$ for both pipes and parallel plates, but returns $\Phi \neq 1$ for other duct geometries. It is noted that only when $\Phi = 1$ will Eqs. (34) and (35) return the traditional Lévêque solution as $X \to 0$. Yilmaz and Cihan did not recommend one equation over the other. For an isothermal pipe, where $fRe_D = 64$, correlation (34) agrees within −0.6% to +4.7% of the exact solution in Table 1, while correlation (35) agrees within +0.4% to +4.7%. For isothermal parallel plates, where $fRe_D = .96$ and $Nu_\infty^T = 7.541$, correlation (34) agrees within −4.8% to 0.0% of the exact solution, while correlation (35) agrees within −3.6% to +0.3% of data in Table 1.

For ducts of arbitrary geometry and walls of constant heat flux, Yilmaz and Cihan [5] proposed the following local correlation for the Graetz problem:

$$\frac{Nu_x^{Grtz,H}}{Nu_\infty^T} = \left[1 - \frac{0.76}{X^{2/3}} + \frac{2.212}{X}\right]^{1/3} \tag{37}$$

where

$$X = \frac{64\left(Nu_\infty^H\right)^3}{\Phi^3 (fReGz_x)} \tag{38}$$

Again, for both pipes and parallel plates $\Phi = 1$. For pipes with walls of constant heat flux, the Yilmaz and Cihan correlation (37) agrees within −0.7% to +3.8% of the exact solution. For parallel plates with constant wall heat flux, where $Nu_\infty^H = 8.235$, correlation (37) agrees within −3.9% to +2.4% of the exact solution in Table 1.

Muzychka and Yovanovich [6] proposed the following correlation for the Graetz problem in ducts of arbitrary geometry:

$$Nu^{Grtz} = \left(\left\{C_2 C_3 \left(\frac{fRe_D}{4Gz_L^{-1}}\right)^{1/3}\right\}^5 + \left(Nu_\infty\right)^5\right)^{1/5} \tag{39a}$$

where the constants are selected based on whether a local or average correlation (Nu_x or Nu_L) is desired and on whether the wall condition is constant temperature or constant heat flux (Nu^T or Nu^H):

$$C_2 = \begin{cases} 1 & (Nu_x) \\ 3/2 & (Nu_L) \end{cases} \text{ and } C_3 = \begin{cases} 0.409 & (Nu^T) \\ 0.501 & (Nu^H) \end{cases} \tag{39b}$$

For example, to construct a correlation for $Nu_L^{Grtz,T}$ (the average Nusselt number for constant wall temperature conditions), $C_2 = 3/2$ and $C_3 = 0.409$. Similar to the work of Churchill and Ozoe [15], the Muzychka and Yovanovich correlation was developed in the context of a limiting case ($Pr \to \infty$) of the more general combined entry problem. In the original correlation, the authors assume that Nu_∞ in Eq. (39) is unknown, and offer a separate correlation for its estimation.

For an isothermal pipe, the Muzychka and Yovanovich correlation (39) agrees within −3.6% to +1.7% of the exact solution in Table 1 for the average Nusselt number and –3.2% to +3.2% for the local Nusselt number. For a flow between isothermal parallel plates, the average Nusselt number calculated with the Muzychka and Yovanovich correlation (39) under predicts the exact solution by −7.6% to −1.7% and agrees within −6.2% to +1.4% of the exact solution in Table 1 for the local Nusselt number.

For flows through pipes with walls of constant heat flux, the Muzychka and Yovanovich correlation (39) agrees within −0.5% to +13% of the exact solution in Table 1 for the average Nusselt number and –2.3% to +1.9% for the local Nusselt number. For a flow between parallel plates with walls of constant heat flux, the average Nusselt number calculated with the Muzychka and Yovanovich correlation (39) agrees within −1.9% to +9.4% of the exact solution and agrees within −5.8% to +1.0% for the local Nusselt number.

The principle limitation of the Muzychka and Yovanovich correlation results from the mistake of assuming Eq. (5) applies to ducts of constant wall heat flux. This throws the average convection coefficient off by as much as +13% for ducts of constant wall heat flux. Additionally, despite the correlation being designed to express both local and average Nusselt numbers, it should be noted that the required relationships given by Eqs. (6) and (11) cannot be realized by the Muzychka and Yovanovich correlation. In contrast, the principle limitation of the Yilmaz and Cihan correlations are their inability to explicitly calculate local Nusselt numbers for ducts with isothermal walls and calculate average Nusselt numbers for ducts with constant wall heat flux.

5. Trial expressions for a generalized graetz problem correlation

Based on the review of preceding work, several asymptotic methods were considered for creating an improved general Graetz problem correlation. Such correlations are designed to return the Lévêque solution in the limit of $Gz_L^{-1} \to 0$ and return Nu_∞ in the limit of large Gz_L^{-1} for fully developed heat transfer. It is assumed that the value of Nu_∞ can be determined from first principles for the appropriate wall condition and duct geometry. Furthermore, it is assumed that the generalized Lévêque solution is given by:

$$Nu_L^{Lev} = A(fRe_D Gz_L)^{1/3} \tag{40a}$$

where the product of the Darcy friction factor f and Reynolds number Re_D (based on hydraulic diameter) is a constant for fully developed flows. It is assumed that the value of fRe_D can be determined from first principles for each duct geometry. The numeric value of the coefficient A in the Lévêque solution is selected based on the wall condition [11]:

$$A = \begin{cases} 0.40377 & (Nu^T) \\ 0.43399 & (Nu^H) \end{cases} \tag{40b}$$

Each of the following trial functions has two or three fitted coefficients that are a function of duct geometry and wall condition. Trial expressions are separately fitted to exact solutions with the goal of testing which type of function is best suited for developing accurate correlations.

The first trial function is a generalization of Hausen's correlation, Eq. (13), which is proposed to have the form:

$$Nu_{Trial1}^{Grtz} = Nu_{\infty} + \frac{1}{\frac{c_3}{Nu_L^{Lev}} + c_1 \left(Gz_L^{-1}\right)^{c_2}} \quad (41)$$

This trial function requires $c_3 \approx 1$ to return the limiting value of Nu^{Lev} for $Gz_L^{-1} \to 0$. However, the best least-square fitting generally yields $c_3 \neq 1$. The $c_1 (Gz_L^{-1})^{c_2}$ term allows the bridging behavior between the asymptotic limits to be fitted. To obtain a function of the form of Hausen's isothermal pipe correlation requires: $c_1 = 1/0.0668$, $c_2 = 1$ and $c_3 = 0.242$. Exact solutions in Table 1 are fitted individually with Eq. (41) for the correlation constants. The best fit values of the correlation constants are reported in Table 3 for each case, along with the minimum and maximum correlation error. For example, when the constant wall temperature pipe data in Table 1 is fitted, the best fit values of $c_1 = 15.7$, $c_2 = 1.03$, $c_3 = 1.16$ are found to yield an error of −2.2% to +3.0% when Eq. (41) is compared against the exact solution. For the parallel plate geometry, values of $c_1 = 46.9$, $c_2 = 1.10$, $c_3 = 1.20$, yields an error of −2.1% to +1.8% for the isothermal wall condition. For walls of constant heat flux, the best fit results shown in Table 3 yield errors of −1.9% to +2.5% for the pipe geometry and −1.8% to +2.0% for the parallel plate geometry.

The second trial function is a generalization of Stephan's correlation (24), which is proposed to have the form:

$$Nu_{Trial2}^{Grtz} = \frac{Nu_{\infty}}{\tanh\left[Nu_{\infty}/Nu_L^{Lev} + c_1 \left(Gz_L^{-1}\right)^{c_2}\right]} + \frac{\tanh\left(Gz_L^{-1}\right)}{c_3 Gz_L^{-1}} \quad (42)$$

The term $c_1 (Gz_L^{-1})^{c_2}$, in combination with the added term $\tanh(Gz_L^{-1})/(c_3 Gz_L^{-1})$, allows for the desired bridging behavior between the asymptotic limits. To obtain a function of the form of Stephan's isothermal pipe correlation requires: $c_1 = 1.7$, $c_2 = 2/3$ and $c_3 = 1/0.0499$. As reported in Table 3, the correlation yields an error of −0.7% to +1.0% with best fit values of these constant when compared against the exact solution for the isothermal pipe. For the parallel plate geometry with isothermal walls, best fit results yields errors from −2.1% to +1.9% when compared against the exact solution. For walls of constant heat flux, the best fit results yields errors of −0.4% to +0.6% for the pipe geometry and −1.4% to +1.5% for the parallel plate geometry.

The third trial function is based on the desire to have the form:

$$Nu = \left(\left(Nu_L^{Lev} + c_1\right)^{c_2} + \left(Nu_{\infty}\right)^{c_2}\right)^{1/c_2} \quad (43)$$

This function uses an extended Lévêque solution, in which the second term in the expansion is the constant c_1. The exponential c_2 is used to affect the behavior of the function between the asymptotic limits of the Lévêque solution and Nu_{∞}. This is the same form as the Muzychka and Yovanovich correlation (39) when $c_1 = 0$. Unfortunately, as $Nu_L^{Lev} \to 0$ the quantity $(Nu_L^{Lev} + c_1)^{c_2}$ can become a small complex number when $c_1 < 0$. To avoid this situation, it is preferred to subtract c_1 from the result for Nu and from both asymptotic limits. In this way, the third trial function becomes:

$$Nu_{Trial3}^{Grtz} = \left(\left(Nu_L^{Lev}\right)^{c_2} + \left(Nu_{\infty} - c_1\right)^{c_2}\right)^{1/c_2} + c_1 \quad (44)$$

This trial function has the same form as Churchill and Ozoe's local pipe correlation (16). For the pipe geometry with isothermal walls, the best fit values of $c_1 = -0.708$ and $c_2 = 3.40$, yields an error of −1.1% to +1.3%. For parallel plates with isothermal walls, the values of $c_1 = 0.091$ and $c_2 = 3.67$ yields an error of −0.8% to +0.8%. For walls of constant heat flux, the best fit results yields errors of −1.2% to +1.1% for the pipe geometry and −0.8% to +0.8% for the parallel plate geometry, as reported in Table 3.

The last trial function is a variant on the correlation by Yilmaz and Cihan [4], which is proposed to have the form:

$$Nu_{Trial4}^{Grtz} = \left[\left(Nu_L^{Lev}\right)^3 + c_1 \left(Nu_L^{Lev}\right)^{3/c_2} + \left(Nu_{\infty}\right)^3\right]^{1/3} \quad (45)$$

To obtain a correlation of the form of Eq. (35) requires $c_2 = 1.5$. For the pipe geometry with isothermal walls, the best fit values of $c_1 = -1.11$ and $c_2 = 1.32$, yields an error of −1.2% to +1.3%. For the parallel plates with isothermal walls, the values of $c_1 = -4.44$ and $c_2 = 2.07$ yields an error of −1.3% to +1.8%. For walls of constant heat flux, the best fit results yields errors of −0.8% to +1.1% for the pipe geometry and −1.1% to +1.6% for the parallel plate geometry, as reported in Table 3.

It can be seen from the results in Table 3 that trial functions 2, 3 and 4 are able to fit the first principle solutions of Cotta and Özişik [2,3] within ±2% for both pipe and parallel plate geometries, while

Table 3
Fitted coefficients to the trial functions and associated errors to exact solution in Table 1.

			$10^{-6} \leqslant Gz^{-1} \leqslant 1$			% error	
			c_1	c_2	c_3	min	max
Eq. (41)	Pipe	Nu^T	15.7	1.03	1.16	−2.2	+3.0
		Nu^H	16.2	1.03	1.14	−1.9	+2.5
	Plates	Nu^T	46.9	1.10	1.20	−2.1	+1.8
		Nu^H	42.6	1.08	1.17	−1.8	+2.0
Eq. (42)	Pipe	Nu^T	2.43	0.711	7.37	−0.7	+1.0
		Nu^H	2.20	0.704	9.02	−0.4	+0.6
	Plates	Nu^T	7.40	0.832	6.27	−2.1	+1.9
		Nu^H	7.44	0.847	7.18	−1.4	+1.5
Eq. (44)	Pipe	Nu^T	−0.708	3.40	–	−1.1	+1.3
		Nu^H	−0.556	3.43	–	−1.2	+1.1
	Plates	Nu^T	0.091	3.67	–	−0.8	+0.8
		Nu^H	0.196	3.70	–	−0.8	+0.8
Eq. (45)	Pipe	Nu^T	−1.11	1.32	–	−1.2	+1.3
		Nu^H	−1.20	1.39	–	−0.8	+1.1
	Plates	Nu^T	−4.44	2.07	–	−1.3	+1.8
		Nu^H	−5.96	2.30	–	−1.1	+1.6

Table 4
R^2 values corresponding to linear regression between model coefficients and constants of fully developed transport.

	c_1		c_2		c_3	
	fRe_D	Nu_∞	fRe_D	Nu_∞	fRe_D	Nu_∞
Eq. (41)	0.989	0.935	0.947	0.863	0.653	0.480
Eq. (42)	0.999	0.961	0.992	0.971	0.549	0.377
Eq. (44)	0.972	0.999	0.989	0.995	–	–
Eq. (45)	0.933	0.971	0.960	0.991	–	–

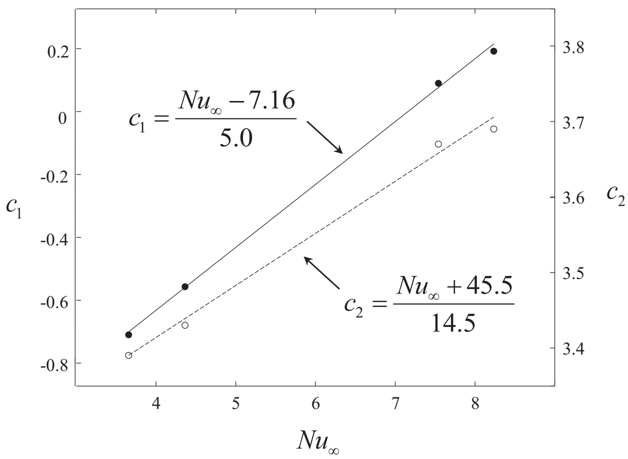

Fig. 2. Fitted coefficients for the third trial function with least square fits to a linear function of Nu_∞.

trial function 1 performs almost as well. To generalize the four trial functions, it is desired that the coefficients appearing in each correlation relate back to known parameters used to characterize convection, i.e., fRe_D and Nu_∞. To explore this possibility, linear fits of the coefficients reported in Table 3 can be performed. For example, trial function 3 given by Eq. (44) has best fit values for c_1 of: –0.708 for isothermal pipes, –0.556 for pipes of constant heat flux, +0.091 for isothermal plates and +0.196 for plates of constant heat flux (as reported in Table 3). If these four values of c_1 are linearly fitted against fRe_D, the fit has an R^2 =0.972 value, as reported in Table 4. However, if these values of c_1 are linearly fitted against Nu_∞, the fit has an R^2 =0.999 value. Table 4 shows how well the coefficients of the four trial functions correlate to a linear dependence on either fRe_D or Nu_∞.

Defining $R^2 > 0.98$ as a strong correlation, the first and forth trial functions only exhibit strong correlations for one coefficient: c_1 has a strong correlation with fRe_D in Eq. (41), andc_2 has a strong correlation with Nu_∞ in Eq. (45). The second trial function (42) exhibits a strong correlation for c_1 and c_2 with fRe_D, but none for c_3. However, both coefficients in the third trial function (44) have strong correlations with Nu_∞. Fig. 2 plots the suggested values of c_1 and c_2 for the third trial function along with the best linear fits. Based on these findings, the third trial function is selected as the best general purpose correlation for the Graetz problem.

6. General correlation for graetz problem

Based on the preceding analysis the following general correlation is proposed for the Graetz problem based on trial function 3:

$$Nu_L^{Grtz} = \left(\left(A\left(fRe_D/Gz_L^{-1}\right)^{1/3}\right)^n + \left(Nu_\infty - O^{Lev}\right)^n\right)^{1/n} + O^{Lev} \tag{46a}$$

The coefficient A is defined by Eq. (40b), and the constants c_1 and c_2 in the third trial function have been renamed O^{Lev} and n, respectively. In the proceeding section, the constants were fitted to exact Graetz problem solutions of Cotta and Özişik [2,3], as summarized in Table 1 for ducts having constant temperature and constant wall heat flux, and geometries corresponding to round pipes and parallel plates. The best fit values of these constants were reported in Table 3 as $O^{Lev} = c_1$ and $n = c_2$. As shown in Fig. 2, these constants were found to correlate well with linear relationships to Nu_∞, as determined to be:

$$O^{Lev} = \frac{Nu_\infty - 7.16}{5.0} \text{ and } n = \frac{Nu_\infty + 45.5}{14.5} \quad (3 < Nu_\infty < 9) \tag{46b}$$

The recommended range $3 < Nu_\infty < 9$ reflects the conditions over which fitting was performed.

Eq. (46) provides the general Graetz problem correlation for the average Nusselt number. Using Eq. (6), the local Nusselt number for ducts of constant wall temperature can be determined from Eq. (46) as:

$$\begin{aligned} Nu_x^{Grtz,T} &= \frac{d}{dL}\left(Nu_L^{Grtz,T}L\right)\Big|_{L=x} \\ &= Nu_{L=x}^{Grtz,T} - \frac{1}{3}\frac{\left(Nu_{L=x}^{Lev,T}\right)^n}{\left(Nu_{L=x}^{Grtz,T} - O^{Lev}\right)^{(n-1)}} \end{aligned} \tag{47}$$

$$\text{or,}\quad \frac{Nu_x^{Grtz,T}}{Nu_{L=x}^{Grtz,T}} = 1 - \frac{\left(\frac{Gz_x^{+n/3}}{3}\right)/\left(Gz_x^{+n/3} + \left(\frac{Nu_\infty - O^{Lev}}{A(fRe_D)^{1/3}}\right)^n\right)}{1 + \frac{O^{Lev}}{A(fRe_D)^{1/3}}/\left(Gz_x^{+n/3} + \left(\frac{Nu_\infty - O^{Lev}}{A(fRe_D)^{1/3}}\right)^n\right)^{1/n}} \tag{48}$$

Similarly, using Eq. (11) the local Nusselt number for ducts of constant wall heat flux can be determined from Eq. (46) as:

$$\begin{aligned} \frac{1}{Nu_x^{Grtz,H}} &= \frac{d}{dL}\left(\frac{L}{Nu_L^{Grtz,H}}\right)\Bigg|_{L=x} \\ &= \frac{1}{Nu_{L=x}^{Grtz,H}} + \frac{1}{3}\frac{\left(Nu_{L=x}^{Lev,H}\right)^n/\left(Nu_{L=x}^{Grtz,H}\right)^2}{\left(Nu_{L=x}^{Grtz,H} - O^{Lev}\right)^{(n-1)}} \end{aligned} \tag{49}$$

$$\text{or,}\quad \frac{Nu_{L=x}^{Grtz,H}}{Nu_x^{Grtz,H}} = 1 + \frac{\left(\frac{Gz_x^{+n/3}}{3}\right)/\left(Gz_x^{+n/3} + \left(\frac{Nu_\infty - O^{Lev}}{A(fRe_D)^{1/3}}\right)^n\right)}{1 + \frac{O^{Lev}}{A(fRe_D)^{1/3}}/\left(Gz_x^{+n/3} + \left(\frac{Nu_\infty - O^{Lev}}{A(fRe_D)^{1/3}}\right)^n\right)^{1/n}} \tag{50}$$

In both Eqs. (48) and (50), the average Nusselt number $Nu_{L=x}^{Grtz}$ is evaluated using Eq. (46) for a duct of length $L = x$.

7. Graetz problem correlation for round pipes and parallel plates

To illustrate the use of correlation (46), consider the case of fully developed flow through a circular pipe with constant wall temperature. In this case, Nu_L^{Lev} is evaluated with Eq. (40) using A =0.40377 and fRe_D =64 for the result $Nu_L^{Lev} = 1.615Gz_L^{1/3}$. Furthermore, with Nu_∞^T =3.6568 Eq. (46b) yields the values O^{Lev} =–0.701 and n =3.39. Therefore, substituting these values into Eq. (46a) yields:

$$Nu_{Pipe,L}^{Grtz,T} = \left(5.079Gz_L^{+1.13} + 146.9\right)^{0.295} - 0.701 \tag{51}$$

Using Eq. (48), the correlation for the local convection coefficient is found to be:

$$\frac{Nu_{Pipe,x}^{Grtz,T}}{Nu_{Pipe,L=x}^{Grtz,T}} = 1 - \frac{\left(Gz_x^{+1.13}/3\right)/\left(Gz_x^{+1.13} + 28.92\right)}{1 - 0.4338/\left(Gz_x^{+1.13} + 28.92\right)^{0.295}} \tag{52}$$

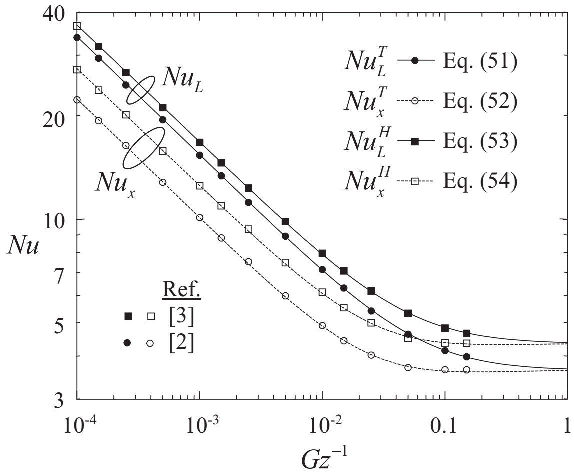

Fig. 3. New pipe flow Graetz problem correlations for average and local Nusselt numbers.

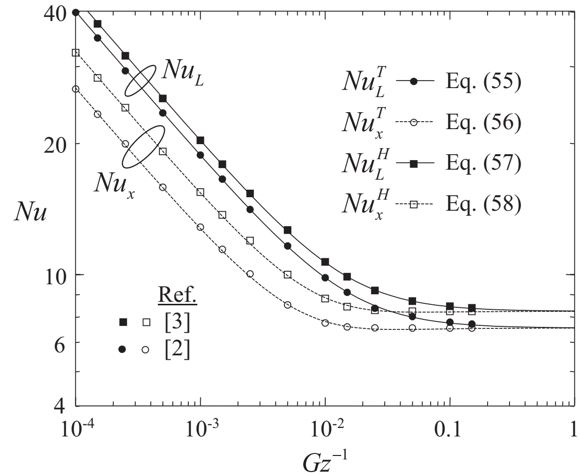

Fig. 4. New parallel plate Graetz problem correlations for average and local Nusselt numbers.

where the average Nusselt number $Nu^{Grtz,T}_{Pipe,L=x}$ is evaluated with Eq. (51). These isothermal pipe correlations agree within –0.9% to +1.4% of the exact solutions provided in Table 1.

Similarly, for pipes with walls of constant heat flux, where Nu^H_∞ =48/11, Eq. (46) yields the following correlation for the average Nusselt number:

$$Nu^{Grtz,H}_{Pipe,L} = \left(6.664Gz_L^{+1.146} + 240.1\right)^{0.2908} - 0.559 \tag{53}$$

Using Eq. (50), the correlation for the local convection coefficient is found to be:

$$\frac{Nu^{Grtz,H}_{Pipe,L=x}}{Nu^{Grtz,H}_{Pipe,x}} = 1 + \frac{\left(Gz_x^{+1.146}/3\right)/\left(Gz_x^{+1.146} + 36.03\right)}{1 - 0.3222/\left(Gz_x^{+1.146} + 36.03\right)^{0.2908}} \tag{54}$$

where the average Nusselt number $Nu^{Grtz,H}_{Pipe,L=x}$ is evaluated with Eq. (53). These correlations for pipes with constant wall heat flux agree within –1.2% to +1.0% of the exact solutions provided in Table 1.

Correlations (51) through (54) for pipe flow are plotted in Fig. 3, along with the exact results from Table 1. In all cases, the new correlations reproduce local and average Nusselt numbers that are within ±1.5% of the exact solution.

For flows between parallel plates of constant temperature, the constants needed for the general correlation are Nu^T_∞ =7.541 and fRe_D =96. Evaluating Eq. (46) yields the following correlation for the average Nusselt number:

$$Nu^{Grtz,T}_{Plates,L} = \left(9.469Gz_L^{+1.219} + 1561\right)^{0.2734} + 0.076 \tag{55}$$

Using Eq. (48), the correlation for the local Nusselt number is found to be:

$$\frac{Nu^{Grtz,T}_{Plates,x}}{Nu^{Grtz,T}_{Plates,L=x}} = 1 - \frac{\left(Gz_x^{+1.219}/3\right)/\left(Gz_x^{+1.219} + 164.9\right)}{1 + 0.04122/\left(Gz_x^{+1.219} + 164.9\right)^{0.2734}} \tag{56}$$

where the average Nusselt number $Nu^{Grtz,T}_{Plates,L=x}$ is evaluated with Eq. (55). These correlations for isothermal plates agree within –0.8% to +1.4% of the exact solutions in Table 1.

Similarly, for constant wall heat flux, for which Nu^H_∞ =8.235, the average Nusselt number for flow between parallel plates is found from Eq. (46) to have the correlation:

$$Nu^{Grtz,H}_{Plates,L} = \left(12.74Gz_L^{+1.235} + 2243\right)^{0.2698} + 0.215 \tag{57}$$

Using Eq. (50), the correlation for the local Nusselt number is found to be:

$$\frac{Nu^{Grtz,H}_{Plates,L=x}}{Nu^{Grtz,H}_{Plates,x}} = 1 + \frac{\left(Gz_x^{+1.235}/3\right)/\left(Gz_x^{+1.235} + 176\right)}{1 + 0.1082/\left(Gz_x^{+1.235} + 176\right)^{0.2698}} \tag{58}$$

where the average Nusselt number $Nu^{Grtz,H}_{Plates,L=x}$ is evaluated with Eq. (57). These correlations for plates with constant wall heat flux agree within –0.8% to +0.8% of the exact solutions in Table 1.

Correlations (55) through (58) are plotted in Fig. 4 along with exact results from Table 1. In all cases, the new correlations reproduce local and average Nusselt numbers that are within ±1.5% of the exact solution.

8. Conclusion

This work offers a general correlation for the laminar Graetz problem for round pipes and parallel plates that relies only on knowledge of the fully developed values of Nusselt number and friction factor. Eq. (46) is the main result, suitable for calculating the average convection coefficient for ducts of either constant temperature or constant heat flux wall condition. Two auxiliary equations are provided for calculating the local convection coefficients. Eq. (48) is for the constant temperature wall condition, and Eq. (50) for the constant wall heat flux condition. The new correlations for pipe and parallel plate duct geometries are within ±1.5% of exact solutions for both constant temperature and constant heat flux wall conditions, and for both local and average Nusselt numbers. Implicit to this work is the possibility that the general correlation can be extended to ducts of arbitrary geometry. This is the subject of the follow-up paper to this work.

Declaration of interest

None.

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