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Chaotic Features of a Class of Discrete Maps without Fixed Points

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Chaotic dynamical systems without fixed points are promising in the generation of pseudo-random sequences because orbits will not converge to a fixed point. In this class of systems there are no final fixed points, so the basin of attraction of a fixed point does not exist. The absence of fixed points makes it difficult to analyze the dynamics of the systems because a fixed point gives us a lot of information about the dynamics of the system. In addition, if there is an amplitude control parameter for the generated chaotic signals, the tolerance and adaptability are higher and better in the application process. In view of the aforementioned, in this paper, a novel class of discrete maps is presented and described by a kind of piecewise linear (PWL) maps. Necessary and sufficient conditions are given to guarantee that this class of discrete maps does not have any fixed point. Furthermore, we introduce families of these PWL discrete maps without fixed points that present positive Lyapunov exponents and have chaotic dynamics with amplitude control. From these families, we select one particular map, which is analyzed theoretically and proved to be chaotic in the sense of Devaney.

Keywords: Dynamical system; discrete map; map without fixed points; piecewise linear map; chaotic map; Lyapunov exponent; amplitude control.

1. Introduction

The design and behavior of chaotic dynamical systems have been extensively investigated since the late 19th century. The generation of chaotic systems, as well as their subsequent study, has allowed to discover important applications in different fields of science such as: engineering [Lorenz, 1963; Chen & Yu, 2003; Chua *et al.*, 1986], biology [May, 1976] and computer security [Cassal-Quiroga & Campos-Cantón, 2020; García-Martínez & Campos-Cantón, 2015; García-Martínez *et al.*, 2015]. In secure communications, cryptography plays a central role. In this area, chaotic dynamical systems have been employed specifically for data protection purposes, some of them have made use of chaotic PWL maps with fixed points [Chargé *et al.*, 2008; Hu *et al.*, 2014].

Equilibrium points give us important information about the behavior of the continuous dynamical system in the neighborhood of each equilibrium point. Some important theorems have been established based on equilibrium points, for example, one of the most important tools for analyzing the chaotic motion of a nonlinear dynamical system is the well-known Shil'nikov theorem. Shil'nikov proved that the chaotic behavior is exhibited in

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the neighborhoods of parameter space where certain homoclinic orbits appear, surrounding the saddle-focus equilibrium point. The chaotic behavior generated by the Shil'nikov method is called homoclinic chaos or heteroclinic chaos [Shil'nikov, 1965]. In the same vein, the chaos definition of Marotto is based on fixed points for discrete dynamical systems [Marotto, 1978].

Recently, both continuous and discrete dynamical systems without equilibrium points or fixed points, respectively, have been proposed and studied. Since there are no equilibrium points, then the methods mentioned in the previous paragraph cannot be applied to determine chaos, however, there is still the possibility of analyzing these systems using phase portraits, Lyapunov exponents, bifurcation diagrams, period doubling route to chaos, Poincaré maps and to verify chaotic behavior by means of Devaney's definition. The presence of chaos via Lyapunov exponents is wildly used between chaotic systems and hyperchaotic systems with or without equilibria [Chen & Yu, 2003; García-Martínez & Campos-Cantón, 2015; García-Martínez *et al.*, 2015]. No-equilibrium chaotic systems generate attractors in vector fields without equilibrium points that satisfy the definition of hidden attractors [Dudkowski *et al.*, 2016]. However, not all hidden attractors are generated in a vector field without equilibria. Usually, hidden attractors in vector fields with equilibria imply multistability where self-exited attractors and hidden attractors coexist. Hidden attractors are important in applications because they allow unexpected responses to disturbances in some structures like a bridge or an airplane wing [Jafari *et al.*, 2016a].

Examples of continuous dynamical system without equilibrium points were reported by Sprott [1994], this system was found by numerical examination of three-dimensional autonomous differential equations with quadratic nonlinearities, and the chaotic behavior was determined by means of Lyapunov exponents. On the other hand, Escalante-González and Campos-Cantón [2017, 2018] proposed the generation of chaotic dynamics without equilibrium points based on PWL systems, chaos was also determined through Lyapunov exponents. In the same vein of [Sprott, 1994], Jafari *et al.* [2013] proposed a work on chaotic flows without equilibrium points in which 17 three-dimensional dynamical systems show chaos (they use the Lyapunov exponents to verify this). Wang *et al.* [2012] introduced a hyperchaotic system by using four ordinary differential equations without equilibrium points. Dynamical systems of fractional order and without equilibrium points have been presented by Cafagna and Grassi [2014]. Other approaches to generate no-equilibrium chaotic system with interesting properties have been reported in [Pham *et al.*, 2014; Jafari *et al.*, 2016a; Pham *et al.*, 2017].

Compared to continuous dynamical systems, research on discrete dynamical systems without fixed points has been few to date. Within discrete dynamical systems, Jafari *et al.* [2016b] proposed a one-dimensional chaotic map which is an extended version of the logistic map, in which there are hidden chaotic attractors shown by bifurcations diagrams. A characteristic of the aforementioned class of maps without fixed points is that it introduces several discontinuities because the logistic map is defined in different intervals without crossing the diagonal. The use of Lyapunov exponents to demonstrate the presence of chaos has also been used in discrete dynamical systems; Lambić [2015] introduced a one-dimensional chaotic map based on composition of permutations. Jiang *et al.* [2016] proposed a two-dimensional chaotic map design without fixed points. Wang and Ding [2018] studied hidden attractors of a two-dimensional chaotic map class inspired by Henon's map.

The absence of fixed points makes it difficult to analyze the dynamics of the systems because a fixed point gives us a lot of information about the dynamics of the system. In chaos-based cryptography, fixed points are a weakness for cryptosystem because all initial conditions at a fixed point or at an eventually fixed point do not oscillate chaotically. Therefore, as Huynh *et al.* [2019] mentioned, maps without fixed points could have a very plausible application in encryption schemes or for the generation of pseudo-random sequences that are cryptographically secure. Recently, a pseudo-random number generator based on discrete map without fixed points was introduced by Lambić and Nikolić [2017], which produces long cycle lengths.

The amplitude control of a chaotic signal provides a lot of flexibility in applications of chaos such as chaotic encryption, secure communications and chaotic radar [Li & Wang, 2009a; Li *et al.*, 2009b; Li & Wang, 2015; Li *et al.*, 2017a]. The amplitude control of chaotic signals is a technique in which a control parameter of the amplitude of the signal is sought, taking into account the total or partial preservation of the Lyapunov exponents [Li & Wang, 2009a; Li *et al.*, 2012] or proportionally rescales the exponents without introducing new bifurcations [Li & Sprott, 2014a]. The amplitude control technique in some electronic applications allows better hardware performance [Li *et al.*, 2009b; Li & Wang, 2015]. In some systems the amplitude control can be implemented without any extra circuitry and also provide a new security encoding key [Li & Sprott, 2013a]. Li and Wang [2009a] have proposed amplitude control in piecewise linear systems with a single constant term. Under this scenario, the value of the constant is able to control the amplitude of all signals without modifying the Lyapunov exponent spectrum. Although there are works reported in the literature in which amplitude control in continuous dynamical systems is widely discussed [Li & Wang, 2009a; Li *et al.*, 2009b; Li & Sprott, 2014b; Li & Wang, 2015; Li *et al.*, 2017b], the research in discrete dynamical systems is incipient [Wang & Ding, 2020; Zhou *et al.*, 2021].

In this paper, a novel class of discrete PWL maps without fixed points and with families that have chaotic dynamics is proposed and their dynamics are analyzed. The presence of chaos for families of these PWL maps is analyzed through bifurcation diagrams and Lyapunov exponents. Additionally, for a specific PWL map, chaos is demonstrated through Devaney's definition. A characteristic of this proposed class of maps is that it has only one discontinuity. In addition, the dynamical system is provided with an amplitude control parameter, which keeps the Lyapunov exponent of the system's chaotic signals constant. These proposed PWL maps have computational advantages over other maps, such as the tent map or the doubling map. Chaotic dynamics in these last PWL maps is something that does not happen in numerical simulation, in contrast to the PWL maps exhibited here. For example, the dynamics of the tent map, $T : [0,1] \to [0,1]$, always converges to the fixed point independently of the initial condition $x_0 \in [0,1]$. In addition, since the proposed maps are defined by linear functions in sections, their implementation by software and hardware is fast and easy to perform. Given the remarkable application of discrete chaotic systems to cryptography, the amplitude-controlled, fixed-point-free PWL chaotic maps proposed here could jointly bring new approaches to applications such as the generation of cryptographically secure pseudorandom sequences.

2. A Class of Piecewise Linear Maps

In this section, we give the basis to design a class of PWL maps based on two lines L_1 and L_2, given by $y = m_1x + b_1$, and $y = m_2x + b_2$, respectively, with $m_1 \neq 0$, $m_2 \neq 0$, and $m_1 \neq m_2$.

Remark 2.1. We assume that the intersection point between the two lines L_1 and L_2 is at $x = -a < 0$, and $y = 0$.

Under assumption given in the Remark 2.1, the PWL map turns out as $x_{n+1} = 0$ if $x_n = -a$. Hence from L_1 and L_2, we have

$$-a =: x = \frac{-b_1}{m_1}, \tag{1}$$

$$-a =: x = \frac{-b_2}{m_2}, \tag{2}$$

respectively. From these equalities, the following statement can be deduced:

Proposition 1. *Let L_1 and L_2 be two lines that fulfill the assumption given in Remark 2.1. If $m_i > 0$, then $b_i > 0$, on the other hand, if $m_i < 0$, then $b_i < 0$, with $i = 1, 2$.*

Proof. From Eq. (1), we have $b_1 = am_1$. As $-a < 0$, it follows that $a > 0$; if $m_1 > 0$, then b_1 is the product of two positive numbers, that is, $b_1 > 0$. If $m_1 < 0$, then b_1 is the product of one positive number for another negative, therefore, $b_1 < 0$.

If the values of m_1, b_1 and m_2 are set, then by equating Eqs. (1) and (2), we get:

$$b_2 = \frac{m_2 b_1}{m_1}. \tag{3}$$

Similarly, as proved above, the quotient $\frac{b_1}{m_1}$ is always positive, as a consequence:
if $m_2 > 0$, then, $m_2\frac{b_1}{m_1}$ is a positive number, therefore, $b_2 > 0$. And if $m_2 < 0$, then, $m_2\frac{b_1}{m_1}$ is a negative number, and it follows that, $b_2 < 0$. ■

Under the aforementioned scenario, it is possible to define the following class of PWL maps.

Definition 2.1. Let L_1 and L_2 be two lines that fulfill the assumption given in Remark 2.1. A class of PWL maps is defined by three given parameters m_1, m_2, and b_1 as follows

$$f(x) = \begin{cases} m_1 x + b_1 & \text{if } x \le -a < 0, \\ m_2 x + b_2 & \text{if } -a < x < 0, \\ m_2 x - b_2 & \text{if } 0 \le x < a, \\ m_1 x - b_1 & \text{if } x \ge a, \end{cases} \tag{4}$$

where $m_1 \neq 0$, $m_2 \neq 0$ and $b_1 \neq 0$, in addition, the parameters satisfy Proposition 1. The parameter values of a and b_2 are adjusted according to Eqs. (1) and (3) as follows: $a = \frac{b_1}{m_1}$ and $b_2 = \frac{m_2 b_1}{m_1}$, respectively.

It is important to highlight the following observations.

Remark 2.2. $f(-a) = f(-\frac{b_1}{m_1}) = m_1(-\frac{b_1}{m_1}) + b_1 = 0$, and $f(a) = f(\frac{b_1}{m_1}) = m_1(\frac{b_1}{m_1}) - b_1 = 0$.

Remark 2.3. The map is continuous in $\mathbb{R}\backslash\{0\}$. To see why this is true, first note that since the map is linear in parts, then it is continuous in parts. So, it is enough to see that the map is continuous at $x = -a$ and at $x = a$.

$$\lim_{x \to -a^-} f(x) = 0 = f(-a);$$

$$\begin{aligned} \lim_{x \to -a^+} f(x) &= -m_2 \left(\frac{b_1}{m_1} \right) + b_2 \\ &= -b_2 \left(\frac{m_1}{b_1} \right) \left(\frac{b_1}{m_1} \right) + b_2 \\ &= 0 = f(-a); \end{aligned}$$

As

$$\lim_{x \to -a^-} f(x) = \lim_{x \to -a^+} f(x) = f(-a),$$

then, the map is continuous at $x = -a$. In an analog way, it can be verified that f is continuous at $x = a$.

Example 2.1. Consider the map $f : \mathbb{R} \mapsto \mathbb{R}$ given by (4) and the following three parameter values $m_1 = 0.5$, $m_2 = 10$, $b_1 = 3$.

$$f(x) = \begin{cases} 0.5x + 3 & \text{if } x \le -6, \\ 10x + 60 & \text{if } -6 < x < 0, \\ 10x - 60 & \text{if } 0 \le x < 6, \\ 0.5x - 3 & \text{if } x \ge 6. \end{cases} \tag{5}$$

Then, f defines a PWL map, discontinuous only at zero. Its graph can be seen in Fig. 1.

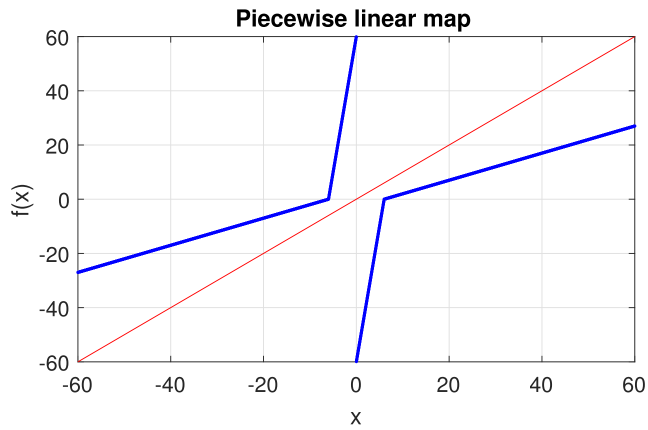


Fig. 1. The PWL map given by (5) with a discontinuity at zero.

2.1. *Fixed points analysis*

In this Sec. 2.1, we analyze the two totally free parameters $m_1, m_2 \in \mathbb{R}\backslash\{0\}$ of the class of PWL maps defined by (4). The interest is to know what conditions each parameter must meet to guarantee the absence or existence of fixed points. To accomplish this task, we analyze each of the intervals where such maps are defined.

2.1.1. *Analysis in the interval $x \le -a$*

Lemma 1. *If $m_1 \le 1$, then the class of PWL maps defined by (4) has no fixed points in the interval $(-\infty, -a]$, otherwise, this class of maps has fixed points.*

Proof. Assume that the maps do have fixed points in this region and that $0 < m_1 < 1$, therefore the following equality should be satisfied:

$$f(x) = m_1 x + b_1 = x.$$

By equaling to zero the above equation, we have

$$m_1 x + b_1 - x = x(m_1 - 1) + b_1 = 0. \tag{6}$$

Then, the point of intersection would be

$$x_i =: x = \frac{-b_1}{m_1 - 1}. \tag{7}$$

As $m_1 > 0$, by Proposition 1, thus $b_1 > 0$, consequently $-b_1 < 0$; furthermore, $0 < m_1 < 1$, therefore $m_1 - 1 < 0$. Thus the value of x_i is a quotient between two negative numbers, hence $x_i \in \mathbb{R}^+$, but this leads to a contradiction with the fact that, $x \in (-\infty, -a]$, where $-a < 0$.

If, we have the case $m_1 < 0$, then the fixed point x_i is given by $f(x_i) = m_1 x_i + b_1 = x_i$, therefore

$x_i = \frac{-b_1}{m_1-1}$. Since, we have $m_1 - 1 < m_1 < 0$, then $\frac{1}{m_1-1} > \frac{1}{m_1}$. From Proposition 1, we know that $b_1 < 0$, then $-a = \frac{-b_1}{m_1} < \frac{-b_1}{m_1-1} = x_i$, but this leads to a contradiction. In both cases, the class of PWL maps defined by (4) has no fixed points in the interval $(-\infty, -a]$.

If $m_1 = 1$, then the map defined by (4) and the line $y = x$ have the same slope, therefore there is no point of intersection between them and there is no fixed point. ■

2.1.2. *Analysis in the interval*: $-a < x < 0$

Lemma 2. *If* $m_2 > 0$, *then the class of PWL maps defined by Eq. (4) has no fixed points in the interval* $(-a, 0)$. *Otherwise, there are fixed points.*

Proof. By Proposition 1, if $m_2 > 0$, then $b_2 > 0$. As well:

$f(x) = m_2x + b_2 \geq 0 \Leftrightarrow m_2x \geq -b_2 = -m_2a \Leftrightarrow m_2x + m_2a \geq 0 \Leftrightarrow m_2(x + a) \geq 0$. Since $m_2 > 0$, then $x + a \geq 0 \Leftrightarrow x \geq -a \Leftrightarrow x \in [-a, \infty)$, in particular, if $x \in (-a, 0)$. As $f(-a) = 0$, then if $x > -a$, it is satisfied that $f(x) > 0$. Since the map $y_{id} = x$, takes only negative values in the interval $(-a,\ 0)$, then, $f(x) \neq x$, consequently in this case, there are no fixed points.

If $m_2 < 0$, then $b_2 < 0$. If there was a fixed point that verified for some $x \in (-a, 0)$, then

$$f(x) = m_2x + b_2 = x, \tag{8}$$

this equality will be true, if and only if $x(m_2 - 1) = -b_2 \Leftrightarrow x = \frac{-b_2}{m_2-1} \Rightarrow x < 0$. On the other hand, we have $-a = \frac{-b_2}{m_2} < \frac{-b_2}{m_2-1} = x \Rightarrow -a < x$. Then the fixed point $x \in (-a, 0)$. ■

2.1.3. *Analysis in the intervals*: $0 \leq x < a$ *and* $x \geq a$

We first analyze the interval $x \geq a$, we have that the class of PWL maps defined by (4) has or does not have a fixed point according to the following statements:

Lemma 3

(1) *If* $0 < m_1 \leq 1$, *then the map has no fixed points.*
(2) *If* $m_1 > 1$, *then the map has a fixed point.*
(3) *If* $m_1 < 0$, *then the map has no fixed points.*

In the other interval $0 \leq x < a$, the class of PWL maps fulfills one of the following statements according to the m_2 parameter:

Lemma 4

(1) *If* $m_2 > 0$, *then the map does not have fixed points.*
(2) *If* $m_2 < 0$, *then the map has a fixed point.*

Proof. The proofs for the intervals: $0 \leq x < a$ and $x \geq a$, are analogous to those for the previous two intervals. Since the map is odd, just take $-f$ instead of f at the first two intervals to get the dynamics in the intervals $0 \leq x < a$ and $x \geq a$, in this way when we make the changes in the inequalities used in the previous proof, we demonstrate in the same way the enunciated cases. ■

Therefore, the only thing left to do is to eliminate the cases where there are fixed points, so that the conditions that determine families of PWL maps without fixed points can finally be established.

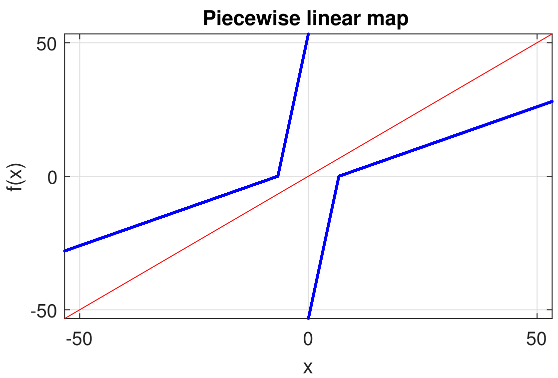
Theorem 1. *Let* f *be a class of PWL maps defined by (4), with* $m_1 \in (-\infty, 0) \cup (0, 1]$, *and* $m_2 \in (0, \infty)$, *then the class of PWL maps does not have fixed points in* $\mathbb{R}$.

Proof. The proof is the one given in each of the cases presented above. ■

Figure 2 shows two particular examples of PWL maps without fixed points, illustrating the two cases: (a) $m_1 > 0$ and $m_2 > 0$ (this case is analyzed in Sec. 2.2) and (b) $m_1 < 0$ and $m_2 > 0$.

2.2. *Bifurcation diagram and Lyapunov exponent of monoparametric families of PWL maps without fixed point*

There are several definitions that capture the essence of chaos, for instance, computationally speaking, orbits are accepted as chaotic if they are bounded, generated by a deterministic system, and present a positive Lyapunov exponent [Alligood *et al.*, 1997]. The advantage of Lyapunov exponents is that it is easy and fast to find intervals of parameter values where the system exhibits chaotic behavior. In this Sec. 2.2, we show the existence of chaotic behavior by analyzing bifurcation diagrams and Lyapunov exponents for two families of PWL maps defined by (4). These monoparametric families of PWL maps, $f_{m_1}(x)$ and $f_{m_2}(x)$, are analyzed



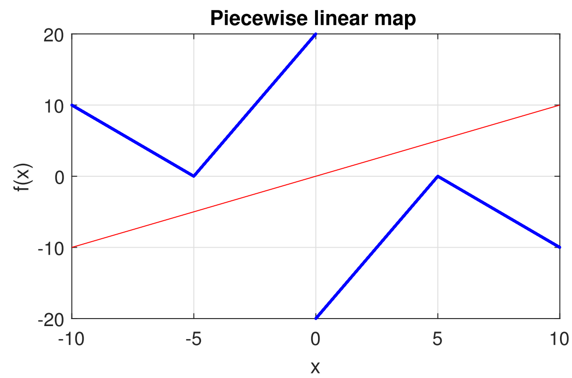


Fig. 2. Examples of PWL maps $f(x)$ without fixed points. (a) $m_1 = 0.6$, $m_2 = 8$, $b_1 = 4$, $b_2 = 53.3$ and (b) $m_1 = -2$, $m_2 = 4$, $b_1 = -10$, $b_2 = 20$.

by varying the parameters m_1 and m_2, respectively. The calculation of the Lyapunov exponent is carried out as in [Elaydi, 2016], i.e. for an initial condition x_0, the Lyapunov exponent is

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|f'(x(k))|. \tag{9}$$

The first family of PWL maps $f_{m_1}(x)$ considers the parameter m_1 in the range $0 < m_1 < 1$; the other parameters are set at $m_2 = 5$ and $b_1 = 4$. Figure 3(a) shows the bifurcation diagram that is generated for $m_1 \in (0, 1)$ and taking the initial condition x_0 randomly in the interval $(0, 1)$. In general, it is possible to observe two types of behavior: periodic and chaotic. The periodic behavior appears in two windows, in the first one, it shows an orbit of period four when $0 < m_1 < 0.2$. In the second one, it shows an orbit of period six when $0.25 < m_1 < 0.45$. Furthermore, in the bifurcation diagram there are two windows of chaotic behavior: $0.2 < m_1 < 0.25$ and $0.45 < m_1 < 1$. Figure 3(b) shows a zoom of the bifurcation diagram for $0.45 < m_1 < 1$. Figure 3(c) shows the Lyapunov exponent against parameter m_1. The Lyapunov exponent is negative for the two windows given by $0 < m_1 < 0.2$ and $0.25 < m_1 < 0.45$, and it is positive for the intervals: $0.20 < m_1 < 0.25$ and $0.45 < m_1 < 1$. The chaotic behavior is verified because all the orbits in these intervals are bounded, there is a positive Lyapunov exponent, and deterministic systems given by the family of maps $f_{m_1}(x)$ in the intervals $0.20 < m_1 < 0.25$ and $0.45 < m_1 < 1$, with $m_2 = 5$ and $b_1 = 4$.

The second monoparametric family of PWL maps, $f_{m_2}(x)$, is analyzed by varying the parameter $m_2 \in (0, 10]$ and setting the parameter values $m_1 = 0.8$ and $b_1 = 4$. The bifurcation diagram is shown in Fig. 4(a). Again, the initial condition x_0 is taken randomly in the interval $(0, 1)$. In general, it is possible to observe two behaviors: periodic for $0 < m_2 < 1.2$, and chaotic for $1.2 < m_2 < 10$. Figure 4(b) shows a zoom of the bifurcation diagram for $1 < m_2 < 5$. Figure 4(c) shows the Lyapunov exponent. In contrast to the first case that is shown in Fig. 3, the Lyapunov exponent remains positive once it is greater than zero, see Fig. 4(d), where the parameter m_2 varies from 0 to 100. The Lyapunov exponent is greater than zero when $1.2 < m_2 < 100$, the orbits are bounded and the system is deterministic then there exists chaotic behavior for $1.2 < m_2 < 100$.

In conclusion, through this numerical analysis, we state the existence of two monoparametric families that present chaotic behavior, which are:

(1) **The family of PWL maps** $f_{m_1}(x)$, with the parameter m_1 in the range $(0.20, 0.25) \cup (0.45, 1)$. The other parameters are set at $m_2 = 5$ and $b_1 = 4$.
(2) **The family of PWL maps,** $f_{m_2}(x)$, with the parameter $m_2 \in (1.2, 100)$ and setting the value of the parameters $m_1 = 0.8$, and $b_1 = 4$.

2.3. *Amplitude control*

In this section, we discuss a method to analyze the existence of amplitude control in the PWL maps (4). The analysis of the PWL maps (4) shows that the Lyapunov exponent given an initial condition remains constant when the amplitude control is implemented.

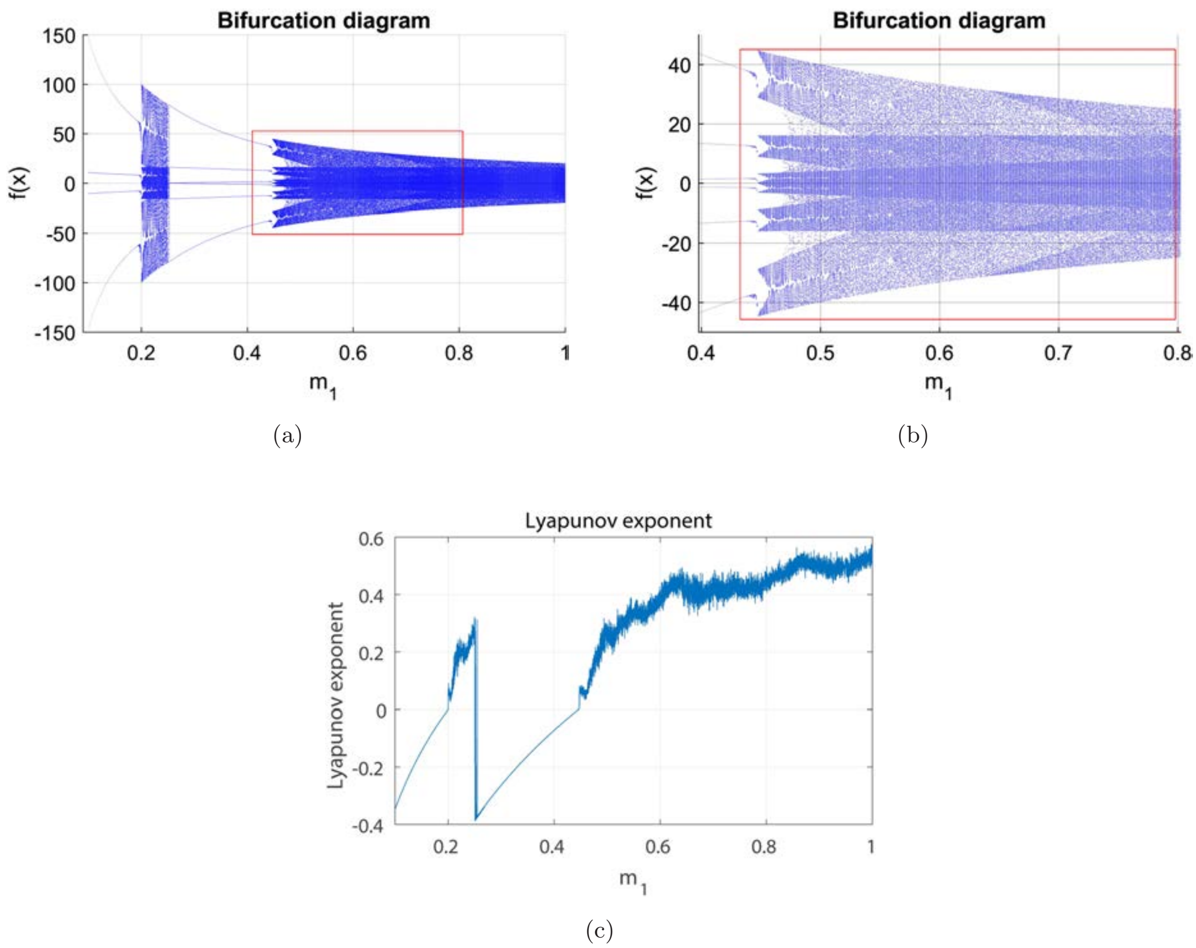


Fig. 3. Bifurcation diagram and Lyapunov exponent of f_{m_1}, $0 < m_1 < 1$, $m_2 = 5$, and $b_1 = 4$. $x_0 = 1$. (a) Bifurcation diagram of the monoparametric family of PWL maps, (b) close-up of the previous bifurcation diagram and (c) the Lyapunov exponent of a monoparametric family of PWL maps.

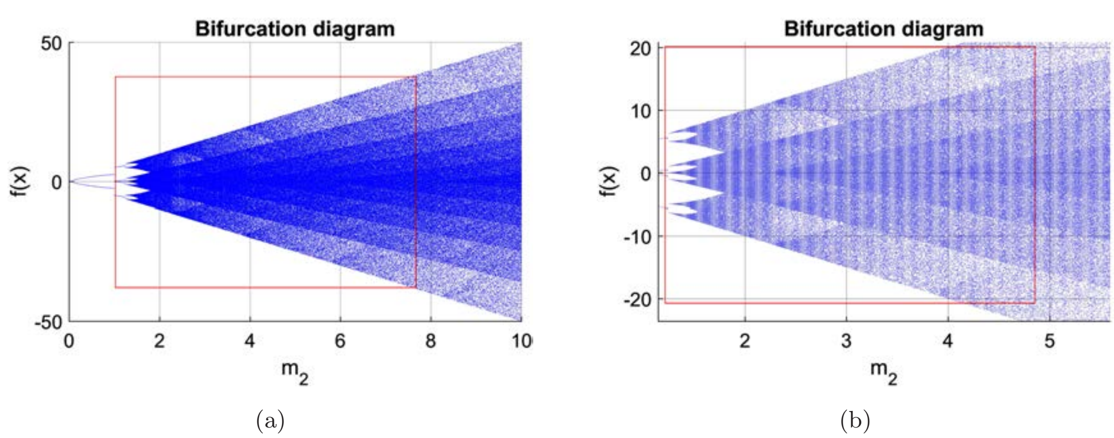
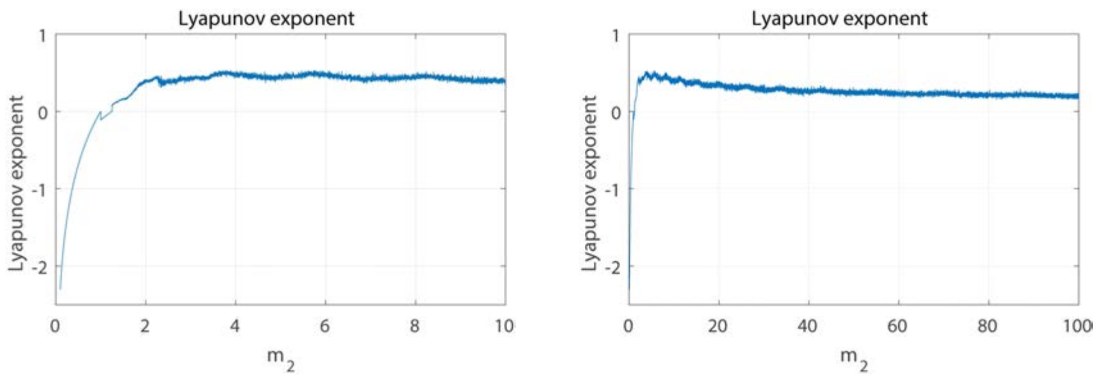


Fig. 4. Bifurcation diagram and Lyapunov exponent of f_{m_2}, $1 < m_2 < 100$, $m_1 = 0.8$, and $b_1 = 4$. $x_0 = 1$. (a) Bifurcation diagram of the monoparametric family of PWL maps, (b) close-up of the previous bifurcation diagram, (c) first Lyapunov exponent of another monoparametric family of PWL maps and (d) second Lyapunov exponent of the same monoparametric family of PWL maps.



(c) (d)

Fig. 4. (*Continued*)

There are several methods for designing an amplitude control parameter for a chaotic dynamical system. In some systems where the terms are monomials, control falls on the coefficient of a parameter whose degree is different from the rest of the terms present in the system and in which the degree of these is the same [Li & Sprott, 2013b; Li & Wang, 2015]. A particular case of this method is when there is a PWL continuous chaotic dynamical system with a single constant value, which by varying it allows to control the amplitude of the signals present in the system without modifying the Lyapunov exponent [Li & Wang, 2009a].

The method used in this work to verify the existence of a parameter that controls the amplitude and preserves the Lyapunov exponent is similar to the one used in the PWL systems mentioned in the previous paragraph. The PWL maps (4) has a constant term and a linear term in each of the intervals where it is defined. The linear term x_n can be accompanied by the coefficient m_1 or m_2, depending on the region in which x_n is located. Recall from the previous section that m_1 and m_2 are bifurcation parameters. Since $b_2 = b_1 \frac{m_2}{m_1}$, then the only constant term in each of the intervals is b_1, thus the variation of it could control the amplitude of the signal x_n, while the exponent holds constant. Proposition 2 formally states what we have discussed in this paragraph.

Proposition 2. *The parameter b_1 is an amplitude control parameter of the PWL map (4), and the magnitude of each element of an orbit given always the same initial condition x_0 increases directly proportional to b_1, while its Lyapunov exponent remains constant.*

Proof. By considering the following change of variable $x_n = kx_n^\star$ and $k \in \mathbb{R}^+$, the system (4) is given as follows.

$$x_{n+1}^\star = \begin{cases} m_1 x_n^\star + \dfrac{b_1}{k}, & \text{if } x_n^\star \le -\dfrac{a}{k} < 0; \\ m_2 x_n^\star + \dfrac{b_2}{k}, & \text{if } -\dfrac{a}{k} < x_n^\star < 0; \\ m_2 x_n^\star - \dfrac{b_2}{k}, & \text{if } 0 \le x_n^\star < \dfrac{a}{k}; \\ m_1 x_n^\star - \dfrac{b_1}{k}, & \text{if } x_n^\star \ge \dfrac{a}{k}. \end{cases} \tag{10}$$

From (3) and (10), we have:

$$x_{n+1}^\star = \begin{cases} m_1 x_n^\star + \dfrac{b_1}{k}, & \text{if } x_n^\star \le -\dfrac{a}{k} < 0; \\ m_2 x_n^\star + \dfrac{b_1 \dfrac{m_2}{m_1}}{k}, & \text{if } -\dfrac{a}{k} < x_n^\star < 0; \\ m_2 x_n^\star - \dfrac{b_1 \dfrac{m_2}{m_1}}{k}, & \text{if } 0 \le x_n^\star < \dfrac{a}{k}; \\ m_1 x_n^\star - \dfrac{b_1}{k}, & \text{if } x_n^\star \ge \dfrac{a}{k}. \end{cases} \tag{11}$$

The above linear transformation yields $\frac{b_1}{k}$ on the right-hand side of the system (11). This means that the transformation $x_n = kx_n^\star$ has the same effect on the signal x_n, as that produced by the change of the constant b_1. Therefore, the parameter b_1 rescales linearly the sequence x_n, and is an amplitude control parameter of the PWL map (4). In particular, when $k = 1$, the PWL (10) is the same as the PWL (4).

In addition, since the derivative of $x_{n+1} = f(x_n)$ is

$$x'_{n+1} = \begin{cases} m_1, & \text{if } x_n \le -a < 0; \\ m_2, & \text{if } -a < x_n < 0; \\ m_2, & \text{if } 0 < x_n < a; \\ m_1, & \text{if } x_n \ge a. \end{cases} \tag{12}$$

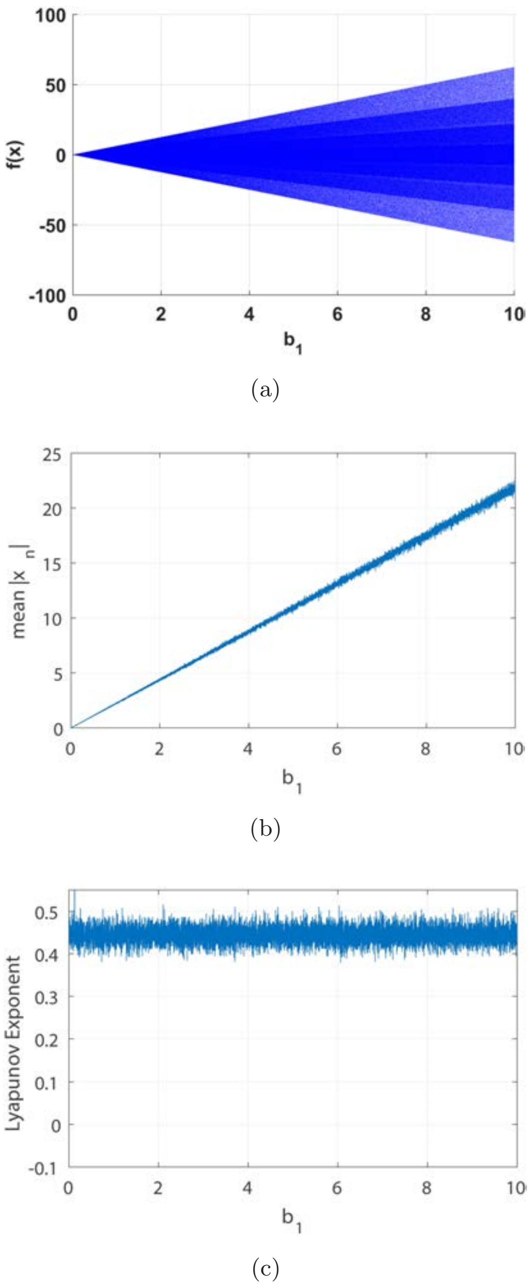


Fig. 5. Amplitude control of the PWL (4) map $x_{n+1} = f(x_n)$, with $m_1 = 0.8$, $m_2 = 5$, $x_0 = 1$ and $b_1 \in (0, 10]$. (a) Amplitude diagram, (b) mean of the absolute value of the signal x_n and (c) Lyapunov exponents.

Then, the parameter b_1 is not required for the calculation of the Lyapunov exponent. Furthermore, as b_1 has the property of amplitude control, then, the Lyapunov exponent remains constant when b_1 is varied and the other parameters are fixed. ■

For example, when $m_1 = 0.8$, $m_2 = 5$, $b_1 = 1$ and $x_0 = 1$, the PWL map is given as follows

$$x_{n+1} = \begin{cases} 0.8x_n + 1 & \text{if } x_n \le -1.25, \\ 5x_n + 6.25 & \text{if } -1.25 < x_n < 0, \\ 5x_n - 6.25 & \text{if } 0 \le x_n < 1.25, \\ 0.8x_n - 1 & \text{if } x_n \ge 1.25, \end{cases} \tag{13}$$

the system (13) exhibits chaos because it has a positive Lyapunov exponent, $\text{Le} = 0.4439$.

The amplitude control that the parameter b_1 exerts on the system (13) is illustrated in Fig. 5. Figure 5(a) shows the behavior of the system when the parameter $b_1 \in (0, 10]$, then the parameter b_1 is an amplitude control parameter and not a bifurcation parameter. Figure 5(b) shows the mean of the absolute values of the orbit $x_0 = 1$. Note that the value increases linearly with respect to the parameter b_1, meanwhile the Lyapunov exponent remains almost constant, varying with slight oscillations around the value $\text{Le} = 0.4439$, see Fig. 5(c).

3. Chaos in the Sense of Devaney's Definition

The chaotic behavior in the system (4) was numerically verified by computing the Lyapunov exponents. However, this way of showing chaos in a system is not a mathematical demonstration of chaotic behavior in the system (4). An analytical way to demonstrate chaos is through Devaney's definition of chaos [Afraĭmovich & Hsu, 2000]. The downside of using Devaney's definition is that it is for a particular set of parameter values, rather than detecting ranges of parameter values.

In this section, we demonstrate that the dynamical system (4) is chaotic in Devaney's sense for a specific set of parameter values $\{m_1, m_2, b_1\}$. Devaney's definition is given by assuming the existence of an invariant set. In this context, it is confirmed that chaos is present in the invariant set due to the fulfillment of the three properties given in Devaney's definition.

3.1. *Invariant set of a PWL map and its basin of attraction*

Proposition 3. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a PWL map given by (4) with parameters $m_1 = 0.8$, $m_2 = 5$, $b_1 = 4$. Then, the closed interval $\Lambda = [-25, 25]$ is an invariant set for the map f.*

Proof. According to the given parameter values, the PWL map (4) is defined as follows:

$$f(x) = \begin{cases} 0.8x + 4, & \text{if } x \le -5; \\ 5x + 25, & \text{if } -5 < x < 0; \\ 5x - 25, & \text{if } 0 \le x < 5; \\ 0.8x - 4, & \text{if } x \ge 5. \end{cases} \tag{14}$$

Now, we need to prove that the closed interval $\Lambda = [-25, 25]$ is an invariant set for the map f.

We consider the subintervals:

$$\Lambda_1 = [-25, -19.0625], \quad \Lambda_2 = (-19.0625, -11.25],$$

$$\Lambda_3 = (-11.25, -5], \quad \Lambda_4 = (-5, 0), \quad \Lambda_5 = [0, 5],$$

$$\Lambda_6 = (5, 11.25], \quad \Lambda_7 = (11.25, 19.0625],$$

$$\Lambda_8 = (19.0625, 25].$$

Each $\Lambda_i \subset \Lambda$, and $\Lambda = \bigcup \Lambda_i$, with $i = 1, \ldots, 8$. Then we have

$$f(\Lambda_1) \subset \Lambda_2, \quad f(\Lambda_2) \subseteq \Lambda_3,$$

$$f(\Lambda_3) \subseteq \Lambda_4 \quad \text{and} \quad f(\Lambda_4) \subseteq \bigcup_{i=5}^{8} \Lambda_i.$$

And we also have

$$f(\Lambda_8) \subset \Lambda_7, \quad f(\Lambda_7) \subseteq \Lambda_6,$$

$$f(\Lambda_6) \subseteq \Lambda_5 \quad \text{and} \quad f(\Lambda_5) \subseteq \bigcup_{i=1}^{4} \Lambda_i.$$

Therefore, we can conclude that $\Lambda = [-25, 25]$ is an invariant set under the map f. ■

Proposition 4. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a PWL map (4) with parameters $m_1 = 0.8$, $m_2 = 5$, $b_1 = 4$. Then there exists $n \in \mathbb{N}$ such that $f^n(x) \in \Lambda$, for all $x \in \mathbb{R}$.*

Proof. We have that $\mathbb{R} = (-\infty, -25) \cup \Lambda \cup (25, \infty)$. By Proposition 3, Λ is an invariant set. Then we have to prove only two cases, when x belongs to $(-\infty, -25)$ or $(25, \infty)$.

For the first case, we consider the backward orbit of $\overline{x}_0 = -5$, i.e.

$$\overline{x}_0 = -5, \quad \overline{x}_{-1} = -11.25,$$

$$\overline{x}_{-2} = -19.0625, \quad \overline{x}_{-3} = -28.8282,$$

$$\overline{x}_{-4} = -41.0352, \quad \ldots.$$

Note that $\overline{x}_n \to -\infty$ for $n \to -\infty$. Then, we have $(-\infty, -25) \subset \bigcup_{i=0}^{\infty} (\overline{x}_{-i-1}, \overline{x}_{-i}]$. Thus for all $x \in (-\infty, -25)$, x belongs to an interval $(\overline{x}_{-i-1}, \overline{x}_{-i}]$ and since in its region, f is a contractive affine map then $f : (\overline{x}_{-i-1}, \overline{x}_{-i}] \to (\overline{x}_{-i}, \overline{x}_{-i+1}]$.

Then there exists a $n \in \mathbb{N}$ that defines the interval $(\overline{x}_{-n-1}, \overline{x}_{-n}] \ni x$, such that $f^n : (\overline{x}_{-n-1}, \overline{x}_{-n}] \to (\overline{x}_{-1}, \overline{x}_0]$, and $(\overline{x}_{-1}, \overline{x}_0] \subset \Lambda$. Therefore, there exists $n \in \mathbb{N}$ such that $f^n(x) \in \Lambda$, for all $x \in (-\infty, -25)$.

In a similar way, we can generate the backward orbit of $\overline{x}_0 = 5$ and the intervals $[\overline{x}_{-i}, \overline{x}_{-i-1})$ such that $(25, \infty) \subset \bigcup_{i=0}^{\infty} [\overline{x}_{-i}, \overline{x}_{-i-1})$. Then, there exists a $n \in \mathbb{N}$ that defines the interval $[\overline{x}_{-n}, \overline{x}_{-n-1}) \ni x$, such that $f^n : [\overline{x}_{-n}, \overline{x}_{-n-1}) \to [\overline{x}_0, \overline{x}_{-1})$, and $[\overline{x}_0, \overline{x}_{-1}) \subset \Lambda$. Therefore, there exists $n \in \mathbb{N}$ such that $f^n(x) \in \Lambda$, for all $x \in (25, \infty)$.

We can conclude that there exists $n \in \mathbb{N}$ such that $f^n(x) \in \Lambda$, for all $x \in \mathbb{R}$. ■

3.2. *Chaos in the sense of Devaney*

The main result of this work is to exhibit that the PWL map according to Proposition 3 has chaotic dynamics in the Devaney's sense. To carry this out, first, some lemmas are proposed for the map defined by (14). To facilitate the structure of the proofs, let us consider and observe the following:

(1) Given the points $x_{0_l} = -19.0625$ and $x_{0_r} = 19.0625$, the orbits of these points are defined as X_l and X_r, respectively, i.e.

$$X_l : \ x_{0_l} = -19.0625, \quad x_{1_l} = -11.25,$$

$$x_{2_l} = -5, \quad x_{3_l} = 0, \quad \ldots$$

$$X_r : \ x_{0_r} = 19.0625, \quad x_{1_r} = 11.25,$$

$$x_{2_r} = 5, \quad x_{3_r} = 0, \quad \ldots$$

(2) From the above, we define the sets:

$$X_l^\star = \{x_{0_l} = -19.0625, x_{1_l} = -11.25,$$

$$x_{2_l} = -5, x_{3_l} = 0\},$$

$$X_r^\star = \{x_{0_r} = 19.0625, x_{1_r} = 11.25,$$

$$x_{2_r} = 5, x_{3_r} = 0\}.$$

(3) Let us consider again the subintervals Λ_i, with $i = 1, \ldots, 8$. Recall that $\Lambda = \bigcup_{i=1}^{8} \Lambda_i$, where:

$$\Lambda_1 = [-25, x_{0_l}], \quad \Lambda_2 = (x_{0_l}, x_{1_l}],$$

$$\Lambda_3 = (x_{1_l}, x_{2_l}], \quad \Lambda_4 = (x_{2_l}, x_{3_l}],$$

$$\Lambda_5 = (x_{3_r}, x_{2_r}], \quad \Lambda_6 = (x_{2_r}, x_{1_r}],$$

$$\Lambda_7 = (x_{1_r}, x_{0_r}], \quad \Lambda_8 = (x_{0_r}, 25].$$

Remark 3.1. From 1 and 2, we can note that any element $x \in \Lambda$ which satisfies that $f^{k_1}(x) \in X_l^\star \cup X_r^\star$ for some $k_1 \in \mathbb{N}$, then there is a $k_2 \geq k_1$ such that $f^{k_2}(x) = 0$. At this point, the map f presents a discontinuity. In this way, any open interval (a, b), that contains the element $f^{k_2}(x)$, will be split into two subintervals in the next iteration:

$$I_1 = (f(a), 25),$$

$$I_2 = [-25, f(b)),$$

such that $f(-25) = -16 \leq f(a) < 25$ and $-25 < f(b) \leq f(25) = 16$.

Lemma 5. *Let $(a, b) \subset \Lambda$ be an open interval such that $(a, b) \subset \Lambda_i$, for some $i = 1, 2, 3, \ldots, 8$.*

(1) *If $\Lambda_i \subset [-25, 0)$, then there exists a $j \in \{0, 1, 2, 3\}$, such that the interval $(f^j(a), f^j(b)) \subset \Lambda_4$.*
(2) *If $\Lambda_i \subset [0, 25]$, then there exists a $j \in \{0, 1, 2, 3\}$, such that the interval $(f^j(a), f^j(b)) \subset \Lambda_5$.*

Proof. Let us first consider the case $\Lambda_i \subset [-25, 0)$. If $(a, b) \subset \Lambda_4$, then $j = 0$, and the lemma is true.

Otherwise, $(a, b) \subset \Lambda_i$, for some $i = 1, 2, 3$. In these regions, f is a contractive map such that $f(\Lambda_1) \subset \Lambda_2$, $f(\Lambda_2) \subseteq \Lambda_3$, and $f(\Lambda_3) \subseteq \Lambda_4$.

Therefore, there exists an $j \in \{1, 2, 3\}$ such that for $(a, b) \subset \Lambda_j$ then $f^{4-j}(a, b) \subset \Lambda_4$.

The case for $\Lambda_i \subset [0, 25]$ is proved analogously to the previous case. ■

Lemma 6. *Let (a, b) be an open interval in Λ, then there are $\bar{a}, \bar{b} \in [a, b]$, with $\bar{a} < \bar{b}$, such that the open interval $(\bar{a}, \bar{b})$ is contained in one of the intervals Λ_i, $(i = 1, 2, \ldots, 8)$.*

Proof. If (a, b) is contained in an interval Λ_i, then we take $\bar{a} = a$ and $\bar{b} = b$ and $(\bar{a}, \bar{b}) \subset \Lambda_i$.

If the interval (a, b) is not entirely contained in any of the intervals Λ_i, then (a, b) is contained in at least two contiguous intervals Λ_i and Λ_{i+1}, so there is an element $c \in (a, b)$ such that $c \in X_l^\star \cup X_r^\star$, as a result of this, there are two subintervals: $(a, c]$ and $[c, b)$, for which it is fulfilled that $(a, c] \subset \Lambda_i$ and $[c, b) \cap \Lambda_{i+1} \neq \emptyset$. Hence, if we take $\bar{a} = a$ and $\bar{b} = c$ (or $\bar{a} = c$ and $\bar{b} = b$, if $(c, b) \subset \Lambda_{i+1}$), it is satisfied that $(\bar{a}, \bar{b})$ is contained in some interval Λ_i. ■

Lemma 7. *Let $(\bar{a}, \bar{b}) \subseteq \Lambda_i$ and $L_0 = |\bar{b} - \bar{a}|$, be an open interval and its length respectively, then for some $k \in \{1, 2, 3, 4\}$, there exists an open interval $(f^k(\bar{a}), f^k(\bar{b}))$, such that its length $L_k = |f^k(\bar{b}) - f^k(\bar{a})|$ fulfills that $L_k \geq 2.56 L_0$.*

Proof. If $(\bar{a}, \bar{b}) \subset \Lambda_4$ or $(\bar{a}, \bar{b}) \subset \Lambda_5$, then the interval evolves with one of the following functions:

$$f(x) = 5x - 25, \quad \text{or}$$

$$f(x) = 5x + 25, \text{ respectively.}$$

Since in this region the map is expansive with a factor of expansion equal to five, then for $k = 1$ we have that $L_1 = |f^1(\bar{b}) - f^1(\bar{a})| = 5L_0 > 2.56 L_0$.

If $(\bar{a}, \bar{b}) \subset \Lambda_i$, with $i = 1, 2, 3$, then the interval belongs to a region where the map f is contractive and evolves according to the function $f(x) = 0.8x + 4$. Due to Lemma 5, for some $k \in \{2, 3, 4\}$, the $(k-1)$th iteration of the interval $(\bar{a}, \bar{b})$ is finally contained in Λ_4. Because the maximum number of iterations for this to happen is three, then the maximum contraction is $0.8^3 = 0.512$. Therefore, at the kth iteration it will have an expansion such that $L_k = |f^k(\bar{b}) - f^k(\bar{a})| \geq 0.8^3 \cdot 5L_0 = 2.56 L_0$.

The cases when $(\bar{a}, \bar{b}) \subset \Lambda_i$, with $i = 5, 6, 7, 8$, are proved in a similar way that for $i = 1, 2, 3, 4$. ■

Lemma 8. *For any open interval $(a, b) \subset \Lambda$, there is an integer n_1, such that at the n_1th iteration, there are two intervals*:

(1) $(f^{n_1}(a), 0)$.
(2) $[0, f^{n_1}(b))$.

Proof. There are three cases and the first occurs as follows:

Let $(a, b) \subset \Lambda$ and $L_0 = |b - a|$ be an open interval and its initial length, respectively. If there is a $c \in (a, b)$ such that $c = 0$, then for $n_1 = 0$ the statement is true and the test ends.

The other two cases occur when $0 = c \notin (a, b)$. In the second case, the open interval $(a, b) \subset [-25, 0)$ or $(a, b) \subset [0, 25]$. Without loss of generality, we assume that $(a, b) \subset [-25, 0)$. If any element

$x_l^\star$ of the set $X_l^\star$ satisfies that $x_l^\star \in (a,b)$, then as we saw previously, there exists a $k \in \{1,2,3\}$ such that $f^k(x_l^\star) = 0$, therefore we have:

(a) $(f^k(a), 0)$.
(b) $[0, f^k(b))$.

In this way, for $n_1 = k$, the desired result is fulfilled for any open interval $(a,b) \subset \Lambda_i \subset [-25,0)$, with $i \in \{1,2,3,4\}$. Similarly, it can be demonstrated when $(a,b) \subset [0,25)$.

The last case occurs when the elements $x_l^\star$ of the set $X_l^\star$ satisfy that $x_l^\star \notin (a,b) \subset [-25,0)$. According to Lemma 7, there is a $k_1 \in \{1,2,3,4\}$ such that:

$$L_{k_1} = |f^{k_1}(b) - f^{k_1}(a)| \geq 2.56 L_0 \quad \text{and}$$
$$(f^{k_1}(a), f^{k_1}(b)) \subset [0,25].$$

Therefore, one and only one of the following two situations can occur:

(1) There exists a $x_r^\star \in (f^{k_1}(a), f^{k_1}(b))$, such that $x_r^\star \in X_r^\star$, and for the above comment, there is a $n_1 \geq k_1$ such that we have the intervals: $(f^{n_1}(a), 0)$ and $[0, f^{n_1}(b))$ and the proof is done.

(2) If the previous point is not satisfied, then $(f^{k_1}(a), f^{k_1}(b)) \subset \Lambda_i$, for some $i \in \{5,6,7,8\}$. For this interval and for Lemma 7, there is a $k_2 \in \{1,2,3,4\}$ such that:

$$L_{k_2} = |f^{k_2}(b) - f^{k_2}(a)| \geq 2.56 L_{k_1} \geq 2.56^2 L_0 \quad \text{and}$$
$$(f^{k_2}(a), f^{k_2}(b)) \subset [-25,0].$$

If for some $k_i \geq k_2$, with $i \geq 2$, we have that $x_l^\star \in (f^{k_i}(a), f^{k_i}(b))$ or $x_r^\star \in (f^{k_i}(a), f^{k_i}(b))$, then for some $n_1 \geq k_i$ the first case will occur and the proof is done.

If the above does not happen, and as will be seen below, we can continue iterating the open interval (a,b) up to a k_n, such that there exist a $x_l^\star \in (f^{k_n}(a), f^{k_n}(b))$ or a $x_r^\star \in (f^{k_n}(a), f^{k_n}(b))$.

The proof of this is as follows. Since Λ is an interval with finite length and due to Lemma 7, the sequence $\{k_1, k_2, k_3, \ldots, k_i, \ldots, k_{n-1}, k_n, \ldots\}$ generates longer interval $L_{k_n} \geq L_{k_{n-1}} \geq \cdots \geq L_{k_i} \geq \cdots \geq L_{k_3} \geq L_{k_2} \geq L_{k_1}$, thus for Archimedean property, we get that:

There exists a $k_n \in \mathbb{N}$ such that $L_{k_n} = |f^{k_n}(b) - f^{k_n}(a)| = 2.56^{k_n} \cdot L_0 \geq L_{\Lambda_i}$ for some $\Lambda_i \subset \Lambda$. Then, we have an initial condition $x_0 \in (a,b)$ such that $f^{k_n}(x_0) = x_l^\star$ or $f^{k_n}(x_0) = x_r^\star$, that belongs to the open interval $(f^{k_n}(a), f^{k_n}(b))$. Therefore, as we saw previously, there exists $n_1 \geq k_n$ such that $f^{n_1}(x_l^\star) = 0$. In this way the desired result is fulfilled. ■

Lemma 9. *Let* $(\bar{a}, \bar{b}) \subset \Lambda_i$ *be an open interval, where* $i \in \{1, \ldots, 8\}$, *then there exists an* $n_2 \in \mathbb{N}$ *such that one of the following cases holds*:

- $(f^{n_2}(\bar{a}), f^{n_2}(\bar{b})) \supset [-5,-4]$.
- $(f^{n_2}(\bar{a}), f^{n_2}(\bar{b})) \supset [4,5]$.

Proof. Let $(\bar{a}, \bar{b}) \subseteq \Lambda_i$ be an open interval and let $L_0 = |\bar{b} - \bar{a}|$ be its initial length. For Lemma 8, we can take the first $k_1 \in \mathbb{N}$, such that we have the intervals:

$$I_{1_{k_1}} = (f^{k_1}(\bar{a}), 0) \quad \text{and} \quad I_{2_{k_1}} = [0, f^{k_1}(\bar{b})).$$

Then, one and only one of the following two situations can occur:

(1) If $f^{k_1}(\bar{a}) \leq -5$ or $f^{k_1}(\bar{b}) \geq 5$, then $[-5,-4] \subset (f^{k_1}(\bar{a}), 0)$, or $[4,5] \subset (0, f^{k_1}(\bar{b}))$. In this situation, for $n_2 = k_1$, the statement is true and the proof is done.

(2) Otherwise, we have that $I_{k_1} = (f^{k_1}(\bar{a}), f^{k_1}(\bar{b})) \subseteq (-5,5)$. As $0 \in I_{k_1}$, we know for Remark 3.1, that in the $(k+1)$th iteration we have the two intervals:

$$I_{1_{k_1+1}} = (f^{k_1+1}(\bar{a}) = \epsilon_1, 25) \quad \text{and}$$
$$I_{2_{k_1+1}} = [-25, f^{k_1+1}(\bar{b}) = \delta_1),$$

such that, $0 < \epsilon_1 < 25$ and $-25 < \delta_1 < 0$. Therefore, the intervals $I_{1_{k_1+1}}$ and $I_{2_{k_1+1}}$ are disjoint, as a consequence of this:

$$I_{k_1+1} = I_{1_{k_1+1}} \sqcup I_{2_{k_1+1}},$$
$$L_{k_1+1} = L_{1_{k_1+1}} + L_{2_{k_1+1}}$$
$$= |25 - \epsilon_1| + |\delta_1 - (-25)|.$$

Although, for Lemma 7, we know that this length fulfills that:

$$L_{k_1+1} \geq 2.56 L_0. \tag{15}$$

Now, we can choose the interval with greater length or one of them if they have the same length, i.e. we can pick the interval $I_{1_{k_1+1}}$, if:

$$|25 - f^{k_1+1}(\bar{a})| \geq \frac{1}{2} L_{k_1+1} \tag{16}$$

or the interval $I_{2_{k_1+1}}$, if:

$$|f^{k_1+1}(\bar{b}) - (-25)| \geq \frac{1}{2} L_{k_1+1}. \tag{17}$$

In any case, as a consequence of (15):

$$|25 - f^{k_1+1}(\bar{a})| \geq 1.28L_0, \tag{18}$$

or

$$|f^{k_1+1}(\bar{b}) - (-25)| \geq 1.28L_0. \tag{19}$$

Therefore, the interval $\overline{I}_{k_1+1}$ is given as

$$\overline{I}_{k_1+1} = \begin{cases} I_{1_{k+1}} & \text{if } L_{1_{k+1}} \geq L_{2_{k+1}}, \\ I_{2_{k+1}} & \text{othercase.} \end{cases}$$

In such a way that $\overline{L}_{k_1+1} \geq 1.28$. If we follow the same argument from point two, we can construct a sequence of intervals.

$$S_I = \{\overline{I}_{k_1+1}, \overline{I}_{k_2+1}, \overline{I}_{k_3+1}, \ldots, \overline{I}_{k_n+1}, \ldots\},$$

such that

$$\overline{L}_{k_n+1} \geq \cdots \geq \overline{L}_{k_3+1} \geq \overline{L}_{k_2+1} \geq \overline{L}_{k_1+1}.$$

Since $[-5, -4]$ is an interval with finite length and for the Archimedean property, there exists a $k_i \in \mathbb{N}$. Such that:

$$\overline{L}_{k_i} \geq 1.28^{k_i} \cdot L_0 \geq L_{[-5,-4]} = |-4 - (-5)| = 1,$$

or

$$\overline{L}_{k_i} \geq 1.28^{k_i} \cdot L_0 \geq L_{[4,5]} = |5 - 4| = 1.$$

Then, we can take $n_2 \geq k_i$ such that $\overline{L}_{n_2} \geq \overline{L}_{k_i}$, and $\overline{I}_{n_2} \supset [-5, -4]$, or $\overline{I}_{n_2} \supset [4, 5]$. Therefore the proof is done. ■

With the above propositions, we are now ready to demonstrate the three properties that the map defined by Proposition 3 must satisfy to present chaos in Devaney's sense.

Proposition 5 [Transitivity and Dense Periodic Orbits]. *Let $f : [-25, 25] \mapsto [-25, 25)$ be a PWL map given by (4) with parameters $m_1 = 0.8$, $m_2 = 5$, $b_1 = 4$, defined as follows*:

$$f(x) = \begin{cases} 0.8x + 4, & \text{if } x \leq -5; \\ 5x + 25, & \text{if } -5 < x < 0; \\ 5x - 25, & \text{if } 0 \leq x < 5; \\ 0.8x - 4, & \text{if } x \geq 5; \end{cases} \tag{20}$$

for any open interval $(a, b) \subseteq [-25, 25]$ and a $k \in \mathbb{N}$. If f fulfills one of the following cases:

(a) $[-5, -4] \subseteq (f^k(a), f^k(b))$,
(b) $[4, 5] \subseteq (f^k(a), f^k(b))$,

then, the PWL map f is transitive and has dense periodic orbits.

Proof. If the hypothesis is fulfilled in any case, at the $(k+1)$th iteration, f^{k+1}, there is a subinterval:

$$(c, d) \subset (f^{k+1}(a), f^{k+1}(b)).$$

such that $(c, d) = [0, 5]$, (or $(-d, c) = [-5, 0]$, in the other case).

If the first case is true, at $(k+2)$th iteration, f^{k+2}, the interval $[0, 5]$ is mapped to the interval $[-25, 0]$. If $(a, b) \subset [-25, 0]$, there is an intersection in one subset of (a, b), between the line that is defined by f^{k+2} and the line $y = x$, then there is a periodic point p belonging to (a, b).

If $(a, b) \subsetneq [-25, 0]$, then it can happen that $c = 0 \in (a, b)$ such that $(a, c) \subset [-25, 0]$ and $(c, b) \in (0, 25)$. Thus, if we take the subinterval $(a, c) \subset [-25, 0]$, for the previous lemma, one of the hypotheses (a) or (b) will be fulfilled and for the analysis made in this test, we will have a periodic point.

If $(a, b) \subset [0, 25)$, then in the iteration f^{k+3}, the subinterval $(-5, 0) \subset [-25, 0]$ will be mapped to the interval $(0, 25)$. Thus we will have a periodic point p that belongs to (a, b).

In any case it is fulfilled that any arbitrary interval (a, b) has a periodic point, therefore the interval Λ has a dense set of periodic orbits.

In addition, it is fulfilled that for any open $U = (a, b) \subset \Lambda$, we will get that:

$$f^{k+2}(U) \cap (-25, 0) \neq \varnothing \quad \text{and}$$

$$f^{k+3}(U) \cap (0, 25) \neq \varnothing.$$

Therefore, for any open $V \in [-25, 25]$, there will be a positive integer n such that:

$$f^n(U) \cap V \neq \varnothing.$$

If the other case occurs, the proof is analogous. In this way, we can conclude that the map is transitive in Λ. ■

Proposition 6 [Sensitivity to Initial Conditions]. *The map f defined in (20), has sensitivity to the initial conditions.*

Proof. Let x be a point of the interval $[-25, 25]$ and let $B_\epsilon(x)$ be an open neighborhood of x, i.e. an interval of the form:

$$(x - \epsilon, x + \epsilon) = (a, b),$$

such that $\epsilon > 0$.

Choose the smallest integer m for which Lemma 7 is valid, that is, there are two intervals:

(1) $(f^m(a), f^m(y_1) = 0)$.
(2) $[f^m(y_1) = 0, f^m(b))$.

For some $y_1 \in (a, b)$.

Therefore, $f^m(x)$ belongs to one of the intervals: $(f^m(a), 0)$ or $[0, f^m(b))$.

If $f^m(x) \in (f^m(a), 0)$, in the following iteration we get:

$$f^{m+1}(x) \in (f^{m+1}(a), f(0)) \subset [-16, 25)$$

and

$$[f(0), f^{m+1}(b)) \subset [-25, 16].$$

As $f^{m+1}(y_1) = f(0) = -25$, then $|f^{m+1}(x) - f^{m+1}(y_1)| \geq 9$.

If $f^m(x) \notin (f^m(a), 0)$, therefore, $f^m(x) \in [0, f^m(b))$. Since $(f^m(a), 0) \in [-25, 0)$, exists $y_2 \in (f^m(a), 0)$ such that $y_2 \geq -\frac{1}{5}$, as $f(-\frac{1}{5}) = 24$, then:

$$|f^{m+1}(x) - f^{m+1}(y_2)| \geq 8.$$

In any case, it is satisfied that, there is a $\beta = 8 > 0$, such that for any x and for all $\epsilon > 0$, there is a point $y \in B_\epsilon(x)$, as well as a positive integer $n = m+1$, such that it is satisfied that the distance between $f^n(x)$ and $f^n(y)$ is at least β. That is:

$$|f^n(x) - f^n(y)| \geq 8.$$

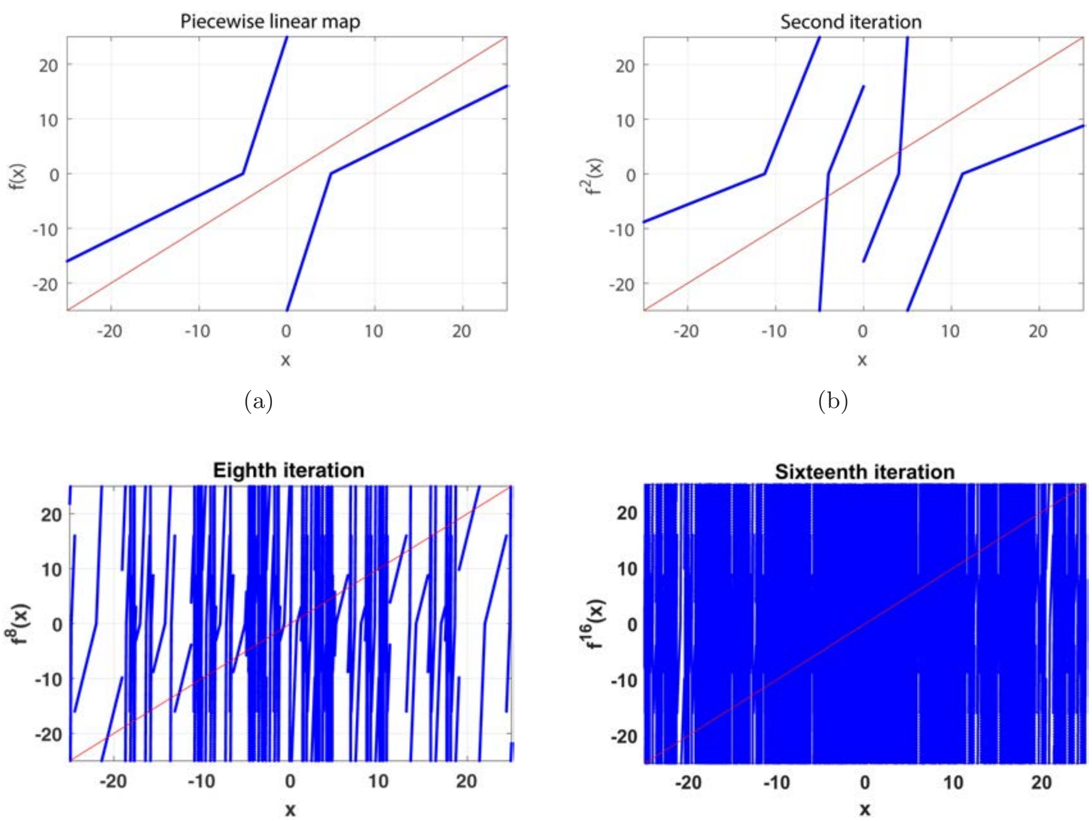
Consequently, the map f has sensitive dependence on the initial conditions. ■

Finally, due to the following Theorem 2, it is possible to accomplish the last of the objectives stated in this paper.

Theorem 2. *The map* $f : [-25, 25] \mapsto [-25, 25)$, *with parameters* $m_1 = 0.8$, $m_2 = 5$, $b_1 = 4$, *and defined as*

$$f(x) = \begin{cases} 0.8x + 4, & \text{if } x \leq -5; \\ 5x + 25, & \text{if } -5 < x < 0; \\ 5x - 25, & \text{if } 0 \leq x < 5; \\ 0.8x - 4, & \text{if } x \geq 5, \end{cases} \tag{21}$$

is chaotic in Devaney's sense.



(a) (b) (c) (d)

Fig. 6. Iteration of the PWL map; $m_1 = 0.8$, $m_2 = 5$, $b_1 = 4$. (a) Map without fixed point, (b) periodic points of period two, (c) periodic points of period eight and (d) periodic points of period sixteen.

Proof. By Proposition 5, in $\Lambda = [-25, 25]$ the transitivity property is satisfied, as well as the set of periodic orbits having the density property. Due to Proposition 6, in the interval $\Lambda = [-25, 25]$, the map f presents sensitivity to the initial conditions.

Thus, once we have verified the three properties of Devaney's definition, we conclude that the map f is chaotic in this sense. ■

In Fig. 6, it can be noted that as the map f is iterated, the three characteristics that define a chaotic map can be verified if the following observations are made:

(1) The periodic orbits begin to multiply in each new iteration and start to distribute for the whole interval $[-25, 25]$. In such a way that for any other interval contained in this, the existence of a periodic point will be given in some iteration. In other words, the periodic orbits will form a dense set within the interval $[-25, 25]$.

(2) From the previous point, it is visualized that as the iterations advance, in each region of the interval $[-25, 25]$, f maps to the intervals $[-25, 16]$ and $(-16, 25)$. Which, on the one hand, leads to the separation of two points that initially started close together, on the other hand, it is feasible to map to the whole interval $[-25, 25]$ and as a consequence to any open interval contained in this interval. In other words, we have the properties of sensitive dependence on initial conditions and transitivity.

4. Conclusions

In this work, a novel class of discrete maps with chaotic behavior has been presented and described by a kind of piecewise linear maps. Necessary and sufficient conditions have been given in order that this class of discrete maps does not have fixed points. Chaotic behavior was explored in this class of PWL maps by using the Lyapunov exponent and bifurcation diagrams for a large range of parameter values. Furthermore, we have formally proved for a particular set of parameter values that this PWL map without fixed points is chaotic by using the Devaney's definition of chaos. In addition, the dynamical system is provided with an amplitude control parameter, which preserves the Lyapunov exponent constant.

This new class of PWL maps without a fixed point is a good alternative to numerically simulate chaotic orbits that are not possible to achieve with PWL maps such as the doubling map or the tent map because their trajectories converge to the fixed point, regardless of whether these maps have been mathematically proven to be chaotic. We consider that this class of PWL maps without a fixed point and with amplitude control is an excellent candidate to be used in cryptography applications. Also its electronic implementation will be easy because it is a PWL with amplitude control mathematical model.

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