Clase 15 - Análisis Matemático 1 - LC: Derivadas

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Resumen

$F(x)$	$F'(x)$
$f+g$	$f'+g'$
$f.g$	$f'.g+f.g'$
$c.f$	$c.f'$
$\frac{f}{g}$	$\frac{f'.g-f.g'}{g^2}$
$f(g(x))$	$f'(g(x)).g'(x)$

	$f(x)$	$f'(x)$
constante	c	0
$r \in \mathbb{R}$	x^r	$r.x^{r-1}$
	$sen(x)$	$cos(x)$
	$cos(x)$	$-sen(x)$
	e^x	e^x
$a > 0$	a^x	$ln(a) \cdot a^x$
$x > 0$	$ln(x)$	$\frac{1}{x}$
$a > 0 \wedge x > 0$	$log_a(x)$	$\frac{1}{ln(a) \cdot x}$

Ejemplos

1. $f(x) = cos\left(\frac{\pi}{2}e^{x+1}\right)$

   $F(x) = cos(x)$, $G(x) = \frac{\pi}{2}e^x$ y $H(x) = x+1$

   $$f(x) = F(G(H(x))) \qquad f'(x) = F'(G(H(x))) \cdot G'(H(x)) \cdot H'(x)$$

   $F'(x) = -sen(x)$, $G'(x) = \frac{\pi}{2}e^x$ y $H'(x) = 1x^{(1-1)=1}$

   $$f'(x) = -sen\left(\frac{\pi}{2}e^{x+1}\right) \cdot \frac{\pi}{2}e^{x+1} \cdot (1) = \boxed{-\frac{\pi}{2}e^{x+1} sen\left(\frac{\pi}{2}e^{x+1}\right)}$$

2. $f(x) = tan(\sqrt{x})$ $\qquad F(x) = tan(x) = \frac{sen(x)}{cos(x)}$ y $G(x) = \sqrt{x} = x^{\frac{1}{2}}$

   $f(x) = F(G(x))$ y $f'(x) = F'(G(x)).G'(x)$

   $F'(x) = \frac{cos(x).cos(x) - sen(x).(-sen(x))}{cos^2(x)} = \frac{1}{cos^2(x)} = sec^2(x)$

   $$f'(x) = sec^2(\sqrt{x}) \cdot \frac{1}{2}x^{(\frac{1}{2}-1)} = \boxed{\frac{1}{2\sqrt{x}}sec^2(\sqrt{x})}$$

3. $f(x) = e^x sen(3-x)$ $\qquad F(x) = e^x$, $G(x) = sen(x)$, $H(x) = 3-x$

   $f(x) = F(x) \cdot G(H(x))$ $\qquad f'(x) = F'(x).G(H(x)) + F(x).[G'(H(x)).H'(x)]$

   $$f'(x) = e^x sen(3-x) + e^x\left[cos(3-x)\cdot(-1)\right] = \boxed{e^x\left[sen(3-x) - cos(3-x)\right]}$$

Derivada de la función inversa

$f(x) = y \Rightarrow f^{-1}(y) = x$

$$f^{-1}(x)\ /\ \boxed{f^{-1}\left(f(x)\right) = x = f\left(f^{-1}(x)\right)}$$

Ejemplos:

- $f(x) = x^3$ y $f^{-1}(x) = \sqrt[3]{x}$ $\quad f(f^{-1}(x)) = (f^{-1}(x))^3 = (\sqrt[3]{x})^3 = x$
- $f(x) = e^x$ y $f^{-1}(x) = ln(x)$ $\quad f(f^{-1}(x)) = e^{ln(x)} = x = ln(e^x)$
- a^x y $log_a(x)$
- $sen(x)$ y $arcsen(x)$
- $cos(x)$ y $arccos(x)$
- $tan(x)$ y $arctan(x)$
- ...

$$(f^{-1}(x))' = ??$$

Derivada de la función inversa

$$f\left(f^{-1}(x)\right) = x$$

$$\left(f\left(f^{-1}(x)\right)\right)' = (x)'$$

$$f'\left(f^{-1}(x)\right) \cdot (f^{-1}(x))' = 1$$

$$\boxed{(f^{-1}(x))' = \frac{1}{f'\left(f^{-1}(x)\right)}}$$

Ejemplo: $f(x) = e^x$ y $f^{-1}(x) = ln(x)$
$f'(x) = e^x$

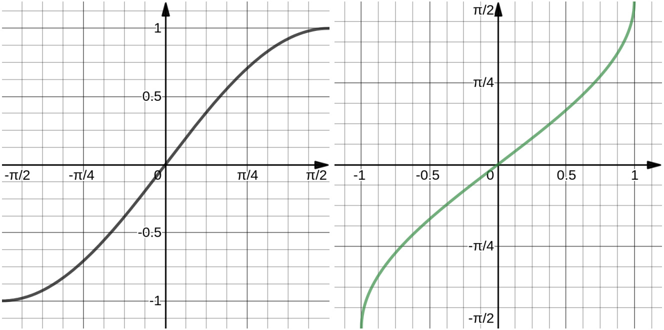
$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{e^{ln(x)}} = \frac{1}{x}$$

$$\boxed{(ln(x))' = \frac{1}{x}}$$

Derivadas de inversas de las trigonométricas

$$g(x) = arcsen(x)$$

$f(x) = sen(x)$ es inyectiva si $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$
$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\Rightarrow f^{-1}(x) = g(x)$



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$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow f^{-1}(x) = g(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{cos(g(x))} = \frac{1}{cos(arcsen(x))}$$

$$cos^2(\alpha) + sen^2(\alpha) = 1 \to cos^2(arcsen(x)) + sen^2(arcsen(x)) = 1$$
$$cos^2(arcsen(x)) + x^2 = 1 \Rightarrow cos^2(arcsen(x)) = 1 - x^2$$

$$cos(arcsen(x)) = \pm\sqrt{1 - x^2} \quad arcsen(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow cos(\alpha) \geq 0$$

$$\boxed{g'(x) = \frac{1}{\sqrt{1 - x^2}}}$$

Derivadas de inversas de las trigonométricas

$$g(x) = arccos(x)$$

$f(x) = cos(x)$ es inyectiva si $x \in [0, \pi]$
$f : [0, \pi] \rightarrow [-1, 1]$
$f^{-1} : [-1, 1] \rightarrow [0, \pi] \Rightarrow f^{-1}(x) = g(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{-sen(g(x))} = \frac{1}{-sen(arccos(x))}$$

$$cos^2(arccos(x)) + sen^2(arccos(x)) = 1$$

$$sen^2(arccos(x)) + x^2 = 1 \Rightarrow sen^2(arccos(x)) = 1 - x^2$$

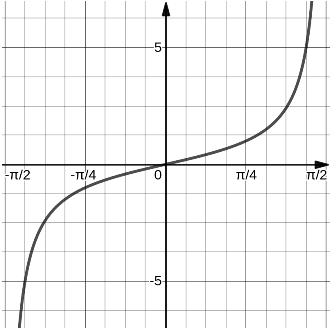
$$sen(arccos(x)) = \pm\sqrt{1 - x^2} \quad arccos(x) \in [0, \pi] \Rightarrow sen(\alpha) \geq 0$$

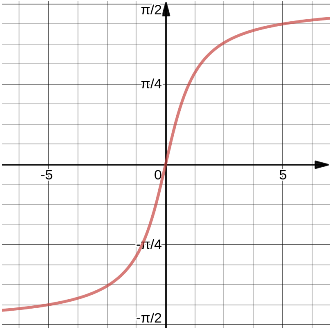
$$\boxed{g'(x) = -\frac{1}{\sqrt{1 - x^2}}}$$

Derivadas de inversas de las trigonométricas

$$g(x) = arctan(x)$$

$f(x) = tan(x)$ es inyectiva si $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to (-\infty, +\infty)$
$f^{-1} : (-\infty, +\infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow f^{-1}(x) = g(x)$





Derivadas de inversas de las trigonométricas

$$g(x) = arctan(x)$$

$f(x) = tan(x)$ es inyectiva si $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to (-\infty, +\infty)$
$f^{-1} : (-\infty, +\infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow f^{-1}(x) = g(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{sec^2(g(x))} = \frac{1}{sec^2(arctan(x))}$$

$$os^2(\alpha) + sen^2(\alpha) = 1 \to \frac{cos^2(\alpha)}{cos^2(\alpha)} + \frac{sen^2(\alpha)}{cos^2(\alpha)} = \frac{1}{cos^2(\alpha)} \Rightarrow 1 + tan^2(\alpha) = sec^2(\alpha)$$

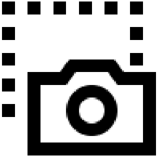
$$1 + tan^2(arctan(x)) = sec^2(arctan(x)) \Rightarrow sec^2(arctan(x)) = x^2 + 1$$

$$\boxed{g'(x) = \frac{1}{x^2+1}}$$

Reglas de derivacion y derivadas de funciones elementales

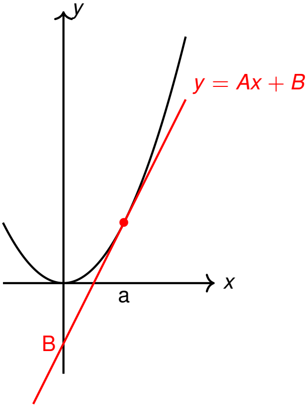
$F(x)$	$F'(x)$
$f+g$	$f'+g'$
$f.g$	$f'.g+f.g'$
$c.f$	$c.f'$
$\frac{f}{g}$	$\frac{f'.g-f.g'}{g^2}$
$f(g(x))$	$f'(g(x)).g'(x)$
$f^{-1}(x)$	$\frac{1}{f'(f^{-1}(x))}$

	$f(x)$	$f'(x)$
constante	c	0
$r \in \mathbb{R}$	x^r	$r.x^{r-1}$
	$sen(x)$	$cos(x)$
	$cos(x)$	$-sen(x)$
	e^x	e^x
$a>0$	a^x	$ln(a) \cdot a^x$
$x>0$	$ln(x)$	$\frac{1}{x}$
$a>0 \wedge x>0$	$log_a(x)$	$\frac{1}{ln(a)\cdot x}$
$-1 \leq x \leq 1$	$arcsen(x)$	$\frac{1}{\sqrt{1-x^2}}$
$-1 \leq x \leq 1$	$arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
	$arctan(x)$	$\frac{1}{1+x^2}$



Ecuación de la recta tangente

"La derivada de una función en un punto es la pendiente de la recta que es tangente a la función en ese punto"



$$y = A\,x + B$$

$$A = f'(a)$$

$$B??? \; x = a \Rightarrow y = f(a) \Rightarrow f(a) = A \cdot a + B$$

$$B = f(a) - A \cdot a = f(a) - f'(a) \cdot a$$

$$y = \overbrace{f'(a)}^{A}\, x + \overbrace{f(a) - a\,f'(a)}^{B} = \boxed{f'(a)(x-a) + f(a) = y}$$

Ejemplos

$$f(x) = cos\left(\frac{\pi}{2}e^{x+1}\right)$$

Dar la ecuación de la recta tangente al gráfico de $f(x)$ en el punto $(-1, 0)$

$$f(-1) = cos\left(\frac{\pi}{2}e^{-1+1}\right) = cos\left(\frac{\pi}{2}\right) = 0$$

$$y = A\,x + B$$

$$A = f'(-1) \qquad f'(x) = -sen\left(\frac{\pi}{2}e^{x+1}\right)\cdot\frac{\pi}{2}e^{x+1} \Rightarrow f'(-1) = -sen\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{2} = -1\cdot\frac{\pi}{2}$$

$$\boxed{A = -\frac{\pi}{2}} \qquad y = -\frac{\pi}{2}x + B$$

$$(-1, 0) \Rightarrow 0 = -\frac{\pi}{2}\cdot(-1) + B \qquad \Rightarrow \boxed{B = -\frac{\pi}{2}}$$

$$\boxed{y = -\frac{\pi}{2}\,x - \frac{\pi}{2}} = -\frac{\pi}{2}(x+1)$$

Ejemplos

$$f(x) = e^x$$

Dar la ecuación de la recta tangente al gráfico de $f(x)$ en el punto $(0,1)$

$$y = A\,x + B$$

$$A = f'(0) \qquad\qquad f'(x) = e^x \Rightarrow f'(0) = e^0 = \boxed{1 = A}$$

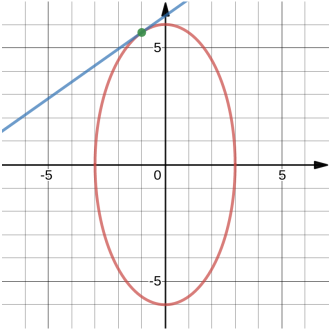
$$x = 0 \rightarrow y = 1 \Rightarrow 1 = A.0 + B \qquad\qquad \boxed{B = 1}$$

$$y = \overbrace{1}^{A}\, x + \overbrace{1}^{B}$$

Ejemplos

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

Dar la ecuación de la recta tangente al gráfico de la elipse en el punto $(-1, 4\sqrt{2})$



Ejemplos

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

Dar la ecuación de la recta tangente al gráfico de la elipse en el punto $(-1, 4\sqrt{2})$

$$y^2 = 36\left(1 - \frac{x^2}{9}\right) \qquad y = \pm\sqrt{36 - \frac{36}{9}x^2}$$

(y toma valores $\pm$ porque NO es una función sino la curva de la elipse)

$$(-1, \overbrace{4\sqrt{2}}^{>0}) \to \boxed{f(x) = \sqrt{36 - 4x^2}} \qquad f(-1) = \sqrt{36-4} = \sqrt{32} = \sqrt{16.2} = 4\sqrt{2}$$

$$r = A\,x + B \qquad A = f'(-1) \qquad f'(x) = \frac{1}{2}\cdot(36 - 4x^2)^{-\frac{1}{2}}\cdot(-8x)$$

$$f'(-1) = \frac{1}{2\sqrt{36-4}}\cdot(-8)\cdot(-1) = \frac{8}{2.4.\sqrt{2}} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2} = A}$$

$$x = 1 \to r = f(-1) = 4\sqrt{2} \qquad \Rightarrow 4\sqrt{2} = A(-1) + B \to 4\sqrt{2} = -\frac{\sqrt{2}}{2} + B$$

$$\to B = 4\sqrt{2} + \frac{\sqrt{2}}{2} = \sqrt{2}\left(4 + \frac{1}{2}\right) = \boxed{\frac{9}{2}\sqrt{2} = B} \qquad \boxed{r = \underbrace{\frac{\sqrt{2}}{2}}_{A}x + \underbrace{\frac{9}{2}\sqrt{2}}_{B}}$$

Derivadas de orden superior

La derivada de una función f es una función f' que a su vez puede ser derivable en algunos puntos. La derivada de la derivada se llama **derivada segunda** y se denota f'' o $f^{(2)}$

Si continuamos el proceso podemos definir la $n-$ésima derivada o derivada de orden n: $f^{(n)}$. Otra notación: $\dfrac{d^n}{dx^n}f(x) = f^{(n)}$

Ejemplos:

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

$$f^{(3)} = 24x$$

$$f^{(4)} = 24$$

$$f^{(5)} = 0$$

Si $n \geq 5$, $f^{(n)} = 0$

Diferenciación logarítmica

$$f(x) = \frac{x^{\frac{5}{4}}\sqrt{x^2+4}}{(4x-2)^2}$$

1. Tomar logaritmo de ambos lados

$$ln(f(x)) = ln\left(\frac{x^{\frac{5}{4}}\sqrt{x^2+4}}{(4x-2)^2}\right) = ln\left(x^{\frac{5}{4}}\right) + ln(\sqrt{x^2+4}) - ln\left((4x-2)^2\right)$$

$$ln(f(x)) = \frac{5}{4}ln(x) + \frac{1}{2}ln(x^2+4) - 2ln(4x-2)$$

2. Derivar de ambos lados

$$(ln(f(x)))' = \left(\frac{5}{4}ln(x)\right)' + \left(\frac{1}{2}ln(x^2+4)\right)' - (2ln(4x-2))'$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{5}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2+4} \cdot 2x - 2 \cdot \frac{1}{4x-2} \cdot 4$$

3. Despejar $f'(x)$

$$f'(x) = f(x) \cdot \left(\frac{5}{4x} + \frac{x}{(x^2+4)} - \frac{8}{4x-2}\right) = \boxed{\frac{x^{\frac{5}{4}}\sqrt{x^2+4}}{(4x-2)^2} \cdot \left(\frac{5}{4x} + \frac{x}{(x^2+4)} - \frac{8}{4x-2}\right)}$$

FIN