Minkowski Spacetime, Its Metric, and Tensor Operations

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$$
\begin{aligned}
\text{proper length}\longrightarrow \mathrm{d}s^2 &= -c^2\ \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \qquad \text{(assumes Cartesian coordinates)} \\
\text{proper time}\longrightarrow \quad &= -c^2\ \mathrm{d}\tau^2 \\
c^2\ \mathrm{d}\tau^2 &= c^2\ \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \xrightarrow[c=1]{} \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \\
\mathrm{d}\tau^2 &= -\eta_{\mu\nu}\ \mathrm{d}x^\mu\ \mathrm{d}x^\nu \equiv \bar{\eta}_{\mu\nu}\ \mathrm{d}x^\mu\ \mathrm{d}x^\nu
\end{aligned}
$$

$$
\text{Minkowski metric} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{aligned}
\eta_{00} &= -1 \\
\eta_{ii} &= 1 \\
\eta_{\mu\nu} &= 0 \qquad \mu \neq \nu
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}x^0 &= \mathrm{d}t \\
\mathrm{d}x^1 &= \mathrm{d}x \\
\mathrm{d}x^2 &= \mathrm{d}y \\
\mathrm{d}x^3 &= \mathrm{d}z
\end{aligned}
$$

All tensors are built from contravariant vectors (vectors) and covariant vectors (covectors or 1-forms).

Contravariant Vector

$\vec{V} : T_P M \to \mathbb{R}$

$$
\vec{V} = V^\mu \hat{\mathbf{e}}_\mu \qquad \text{index on top (contravariant)}
$$

N.B. The Einstein index convention is at play here:

$$
\vec{V} = \underbrace{V^\mu \hat{\mathbf{e}}_\mu}_{\text{sum over repeated indices}} = V^0 \hat{\mathbf{e}}_0 + V^1 \hat{\mathbf{e}}_1 + V^2 \hat{\mathbf{e}}_2 + V^3 \hat{\mathbf{e}}_3
$$

Covariant Vector

$\tilde{V} : T_P M^* \to \mathbb{R}$

$$\tilde{V} = V_\mu \, \tilde{\mathrm{dx}}^\mu$$

index down — 1-form dual to $\hat{\mathbf{e}}_\mu$ – i.e., $\tilde{\mathrm{dx}}^\mu(\hat{\mathbf{e}}_\nu) = \delta^\mu{}_\nu$

$$\tilde{V}(\vec{V}) = \tilde{V}(V^\alpha \hat{\mathbf{e}}_\alpha) = V_\beta \, V^\alpha \tilde{\mathrm{dx}}^\beta(\hat{\mathbf{e}}_\alpha)$$

General Tensors are created via the Tensor Product. Let's look at the important case of the Metric Tensor

$$\eta \equiv \eta_{\mu\nu} \tilde{\mathbf{dx}}^\mu \otimes \tilde{\mathbf{dx}}^\nu$$

tensor product

Then

$$\begin{aligned}
\vec{u}\cdot\vec{v} \equiv \eta(\vec{u},\vec{v}) &= \eta_{\mu\nu} \tilde{\mathbf{dx}}^\mu \otimes \tilde{\mathbf{dx}}^\nu (\vec{u},\vec{v}) \\
&= \eta_{\mu\nu} u^\alpha v^\beta \underbrace{\tilde{\mathbf{dx}}^\mu(\hat{\mathbf{e}}_\alpha)}\underbrace{\tilde{\mathbf{dx}}^\nu(\hat{\mathbf{e}}_\beta)} \\
&= \eta_{\mu\nu} u^\alpha v^\beta \delta^\mu{}_\alpha \delta^\nu{}_\beta \\
&= \eta_{\mu\nu} u^\alpha v^\beta
\end{aligned}$$

Since this is just a scalar (a number), the following must be a 1-form

$$\begin{aligned}
\eta(\vec{u}, \,.\,) &= \eta_{\mu\nu} \tilde{\mathbf{dx}}^\mu \otimes \tilde{\mathbf{dx}}^\nu (\vec{u}, \,.\,) \\
&= \eta_{\mu\nu} u^\alpha \tilde{\mathbf{dx}}^\mu \otimes \tilde{\mathbf{dx}}^\nu (\hat{\mathbf{e}}_\alpha, \,.\,) \\
&= \eta_{\mu\nu} u^\alpha \delta^\mu{}_\alpha \tilde{\mathbf{dx}}^\nu(\,.\,) \\
&= \eta_{\mu\nu} u^\mu \tilde{\mathbf{dx}}^\nu(\,.\,) \\
&= u_\nu \tilde{\mathbf{dx}}^\nu(\,.\,)
\end{aligned}$$

That is, the metric raises and lowers indices (the inverse, $\eta^{\mu\nu}$, would raise indices)

$$\begin{aligned}
u_\nu &= \eta^{\mu\nu} u^\mu \\
u_\mu &= \eta^{\mu\nu} u_\nu
\end{aligned}$$

So the inner product

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= \eta_{\alpha\beta} u^\alpha v^\beta \\
&= u_\beta \tilde{\mathbf{dx}}^\beta (v^\alpha \hat{\mathbf{e}}_\alpha) = \tilde{u}(\vec{v}) \\
&= u_\beta v^\beta \\
&= u_\beta v^\alpha \tilde{\mathbf{dx}}^\beta (\hat{\mathbf{e}}_\alpha) \\
&= u_\beta v^\alpha \delta^\beta{}_\alpha \\
&= u_\beta v^\beta
\end{aligned}$$

In Euclidean space in Cartesian coordinates, we write

$$\vec{u}\cdot\vec{v} \equiv u^1 v^1 + u^2 v^2 + u^3 v^3$$

there's a lot of differential geometry hidden here

which is really

$$u_1 v^1 + u_2 v^2 + u_3 v^3$$

But, given the metric

$$\eta_{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

we have

$$u_1 = \eta_{1i} u^i = \eta_{11} u^1 = (+1) u^1 = u^1$$

so it doesn't matter.

But in Minkowski space, it <u>does</u> matter.

Note that

$$\vec{u}\cdot\vec{v} = \eta(\vec{u},\vec{v}) = u_\alpha v^\alpha$$

is just a number. As a scalar, it is Lorentz invariant. That means that u_α and v^α must transform under a Lorentz transformation in an inverse manner.

$$\begin{aligned} V^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} \\ V_{\mu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} V_{\mu} \end{aligned}$$

Then

$$\begin{aligned} V^{\mu'} V_{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu} V_{\mu} \\ &= V^{\mu} V_{\mu} \end{aligned}$$

Some important specific tensors:

- The differential (a 1-form)

$$\tilde{\mathrm{d}} f = \frac{\partial f}{\partial x^{\mu}}\ \tilde{\mathrm{d}} x^{\mu} \equiv \partial_\mu f\ \tilde{\mathrm{d}} x^{\mu}$$

- The gradient (a contravariant vector) is defined by

$$\begin{aligned} \tilde{\mathrm{d}} f(\hat{\mathbf{e}}_\nu) &= \eta(\vec{\nabla} f, \hat{\mathbf{e}}_\nu) \\ &= \eta\left((\vec{\nabla} f)^{\mu}\ \hat{\mathbf{e}}_\mu, \hat{\mathbf{e}}_\nu\right) \\ &= (\vec{\nabla} f)^{\mu}\ \eta(\hat{\mathbf{e}}_\mu, \hat{\mathbf{e}}_\nu) \\ &= \eta_{\mu\nu}\ (\vec{\nabla} f)^{\mu} \end{aligned}$$

But

$$\tilde{\mathrm{d}}f(\hat{\mathbf{e}}_\nu) = \frac{\partial f}{\partial x^\nu}$$

Then

$$\frac{\partial f}{\partial x^\nu} = \eta_{\mu\nu}\ (\vec{\nabla} f)^\mu$$

or

$$(\vec{\nabla} f)^\mu = \eta^{\mu\nu} \frac{\partial f}{\partial x^\nu}$$

$$\eta^{\alpha\nu}\ \eta_{\mu\nu}\ (\vec{\nabla} f)^\mu = \delta^\alpha{}_\mu\ (\vec{\nabla} f)^\mu$$
$$\eta^{\alpha\nu}\ \frac{\partial f}{\partial x^\nu} = (\vec{\nabla} f)^\alpha$$

Since

$$\frac{\partial f}{\partial x^\mu} = \partial_\mu f$$

we must have

$$\begin{aligned}(\vec{\nabla} f)^\mu &= \eta^{\mu\nu}\ \partial_\nu f \\ &= \partial^\nu f\end{aligned}$$

- Since

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

  we must have

$$\partial^\mu = \frac{\partial}{\partial x_\mu}$$

- The x_μ are the components of the 1-form dual to the position vector

$$\vec{x} = x^\mu \hat{\mathbf{e}}_\mu$$

  – i.e.,

$$\tilde{x} = x_\mu \tilde{\mathrm{d}x}^\mu$$