Continuum Limit of the Linear Chain

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September 8, 2020

$$a \to \mathrm{d}x \longrightarrow \left.\begin{matrix} a & \to 0 \\ N & \to 0 \end{matrix}\right\} \; l = Na = \text{length of the chain} = \text{constant}$$

$$m \to \mathrm{d}m \longrightarrow m \to 0$$

$$\rho = \frac{m}{a} = \text{constant}$$

displacement of infinitesimal mass at x

$$q_n(t) \to q(x,t) \longrightarrow \underline{\underline{\text{ a scalar field!}}}$$

$$q_{n+1}(t) - q_n(t) \to dq = a\frac{\partial q}{\partial x}$$

$$\sum_{n=1}^{N} \to \frac{1}{a}\int_0^l \mathrm{d}x \qquad (\text{this just says } N = \frac{l}{a})$$

$$\sigma = \kappa a = \text{constant} = \text{string tension}$$

Then

$$\begin{aligned} L(q_n, \dot{q}_n) &= \sum_n \left[\frac{m}{2}(\dot{q}_n)^2 - \frac{\kappa}{2}(q_{n+1} - q_n)^2\right] \\ &\to \frac{1}{a}\int_0^l \mathrm{d}x \; \frac{1}{2}\left[\rho a\left(\frac{\partial q}{\partial t}\right)^2 - \frac{\sigma}{a}a^2\left(\frac{\partial q}{\partial x}\right)^2\right] \\ &= \frac{1}{2}\int_0^l \mathrm{d}x \; \left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \sigma\left(\frac{\partial q}{\partial x}\right)^2\right] \end{aligned}$$

and

"Lagrangian density" $$\mathscr{L}(\dot{q}, q') = \frac{1}{2}\left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \sigma\left(\frac{\partial q}{\partial x}\right)^2\right] \qquad \boxed{q' \equiv \frac{\partial q}{\partial x}}$$

and

$$S = \int_{t_i}^{t_f} L \; \mathrm{d}t = \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \mathscr{L}$$

The EOM are determined by extremizing the action.

$$
\begin{aligned}
\delta S &= \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \delta \mathscr{L} \\
&= \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathscr{L}}{\partial q'} \delta q' \right) \\
&= \int_0^l \mathrm{d}x \int_{t_i}^{t_f} \left[\frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \delta q \right] \mathrm{d}t \\
&\quad + \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \left[\frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \right) \delta q \right] \; \mathrm{d}x \\
&= - \int_0^l \mathrm{d}x \int_{t_i}^{t_f} \mathrm{d}t \; \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \delta q \\
&\quad - \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \right) \delta q \\
&= - \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \left[\frac{\partial}{\partial t} \left(\rho \frac{\partial q}{\partial t} \right) + \frac{\partial}{\partial x} \left(-\sigma \frac{\partial q}{\partial x} \right) \right] \delta q \\
&= - \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \left(\rho \frac{\partial^2 q}{\partial t^2} - \sigma \frac{\partial^2 q}{\partial x^2} \right) \delta q \\
&= 0
\end{aligned}
$$

now $\delta q = \delta q(x,t)$
before $\delta q_n = \delta q_n(t)$
$\Rightarrow$ ( ) must be 0

Then the EOM for $q(x,t)$ is

$$\frac{\partial^2 q}{\partial t^2} - \frac{\sigma}{\rho} \frac{\partial^2 q}{\partial x^2} = 0$$

which is a wave equation, with wave speed

$$c = \sqrt{\frac{\sigma}{\rho}}$$

$$\int_0^l \mathrm{d}x \int_{t_i}^{t_f} \mathrm{d}t \; \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right) = \int_0^l \mathrm{d}x \left[\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right)_{t_f} - \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right)_{t_i} \right] = 0$$

since

$$
\begin{gathered}
\delta q(x, t_f) = 0 \text{ for all } x \in [0, l] \\
\delta q(x, t_i) = 0 \text{ for all } x \in [0, l]
\end{gathered}
$$

$$\int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right) = \int_{t_i}^{t_f} \mathrm{d}t \left[\left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_l - \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_0 \right] = 0$$

since

$$\left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_l = \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_0$$

by periodicity.