Minkowski Spacetime, Its Metric, and Tensor Operations

proper length → $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ (assumes Cartesian Coordinates)

proper time → $= -c^2 d\tau^2$

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \xrightarrow[c=1]{} dt^2 - dx^2 - dy^2 - dz^2$$

$$d\tau^2 \equiv -\eta_{\mu\nu} dx^\mu dx^\nu \equiv \bar{\eta}_{\mu\nu} dx^\mu dx^\nu$$

$$\text{Minkowski metric} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\eta_{00} = -1$

$\eta_{ii} = 1$

$\eta_{\mu\nu} = 0 \quad \mu \neq \nu$

$dx^0 = dt$
$dx^1 = dx$
$dx^2 = dy$
$dx^3 = dz$

All tensors are built from contravariant vectors (vectors) and covariant vectors (covectors or 1-forms).

Contravariant Vector

$\vec{V}: T_pM \to \mathbb{R}$

$\vec{V} = V^\mu \hat{e}_\mu$ index on top (contravariant)

N.B. The Einstein index convention is at play here:

$$\vec{V} = \underbrace{V^\mu \hat{e}_\mu}_{\text{sum over repeated indices}} = V^0 \hat{e}_0 + V^1 \hat{e}_1 + V^2 \hat{e}_2 + V^3 \hat{e}_3$$

Covariant Vector

$$\tilde{V} : T_p M^* \to \mathbb{R}$$

$$\tilde{V} = V_\mu \tilde{dx}^\mu =$$

index down

1-form dual to $\hat{e}_\mu$ - i.e., $\tilde{dx}^\mu(\hat{e}_\nu) = \delta^\mu{}_\nu$

$$\tilde{V}(\vec{V}) = \tilde{V}(V^\alpha \hat{e}_\alpha) = V_\beta V^\alpha \tilde{dx}^\beta(\hat{e}_\alpha)$$

General Tensors are created via the Tensor Product. Let's look at the important case of the Metric Tensor

$$\eta \equiv \eta_{\mu\nu} \tilde{dx}^\mu \otimes \tilde{dx}^\nu$$

tensor product

Then

$$\vec{U} \cdot \vec{V} = \eta(\vec{U}, \vec{V}) = \eta_{\mu\nu} \tilde{dx}^\mu \otimes \tilde{dx}^\nu (\vec{U}, \vec{V})$$

The metric defines the inner product.

feeds on 2 vectors

$$= \eta_{\mu\nu} U^\alpha V^\beta \tilde{dx}^\mu(\hat{e}_\alpha) \tilde{dx}^\nu(\hat{e}_\beta)$$

$$= \eta_{\mu\nu} U^\alpha V^\beta \delta^\mu{}_\alpha \delta^\nu{}_\beta$$

$$= \eta_{\alpha\beta} U^\alpha V^\beta$$

Since this is just a scalar (a number), the following must be a 1-form

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$$\eta(\vec{U}, \; . \;) = \eta_{\mu\nu} \tilde{dx}^\mu \otimes \tilde{dx}^\nu (\vec{U}, \; . \;)$$

empty vector slot

$$= \eta_{\mu\nu} U^\alpha \tilde{dx}^\mu \otimes \tilde{dx}^\nu (\hat{e}_\alpha, \; . \;)$$

$$= \eta_{\mu\nu} U^\alpha \delta^\mu{}_\alpha \tilde{dx}^\nu (\; . \;)$$

$$= \eta_{\mu\nu} U^\mu \tilde{dx}^\nu (\; . \;)$$

$$\equiv U_\nu \tilde{dx}^\nu (\; . \;)$$

That is, the metric raises and lowers indices (the inverse, $\eta^{\mu\nu}$, would raise indices)

$$U_\nu = \eta_{\mu\nu} U^\mu$$

$$U^\mu = \eta^{\mu\nu} U_\nu$$

So the inner product

$$\vec{U} \cdot \vec{V} = \eta_{\alpha\beta} U^\alpha V^\beta$$

$$= U_\beta V^\beta$$

$$= U_\beta \tilde{dx}^\beta (V^\alpha \hat{e}_\alpha) = \tilde{U}(\vec{V})$$

$$= U_\beta V^\alpha \tilde{dx}^\beta (\hat{e}_\alpha)$$

$$= U_\beta V^\alpha \delta^\beta_\alpha$$

$$= U_\beta V^\beta$$

In Euclidean space in Cartesian coordinates, we write

$$\vec{U} \cdot \vec{V} \equiv U^1 V^1 + U^2 V^2 + U^3 V^3$$

there's a lot of differential geometry hidden here

which is really

$$U_1 V^1 + U_2 V^2 + U^3 V^3$$

But, given the metric

$$\eta_{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

we have

$$U_1 = \eta_{1i} U^i = \eta_{11} U^1 = (+1) U^1 = U^1$$

so it doesn't matter.

But in Minkowski space, it does matter.

Note that

$$\vec{U}\cdot\vec{V} = \eta(\vec{U},\vec{V}) = U_\alpha V^\alpha$$

is just a scalar. As a scalar, it is Lorentz invariant. That means that U_α and V^α must transform under a Lorentz transformation in an inverse manner.

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

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Then

$$V^{\mu'} V_{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu} V_{\mu}$$

$$= V^{\mu} V_{\mu}$$

Some important specific tensors:

The differential (a 1-form)

$$\tilde{d}f = \frac{\partial f}{\partial x^{\mu}} \tilde{d}x^{\mu} \equiv \partial_{\mu} f \, \tilde{d}x^{\mu}$$

The gradient (a contravariant vector) is defined by

$$\tilde{d}f(\hat{e}_\nu) = \eta(\vec{\nabla}f, \hat{e}_\nu)$$

$$= \eta((\vec{\nabla}f)^\mu \hat{e}_\mu, \hat{e}_\nu)$$

$$= (\vec{\nabla}f)^\mu \eta(\hat{e}_\mu, \hat{e}_\nu)$$

$$= \eta_{\mu\nu} (\vec{\nabla}f)^\mu$$

But

$$\tilde{d}f(\hat{e}_\nu) = \frac{\partial f}{\partial x^\nu}$$

Then

$$\frac{\partial f}{\partial x^\nu} = \eta_{\mu\nu} (\vec{\nabla}f)^\mu$$

$$\eta^{\alpha\nu}\eta_{\mu\nu}(\vec{\nabla}f)^\mu = \delta^\alpha_\mu (\vec{\nabla}f)^\mu$$

$$= (\vec{\nabla}f)^\alpha$$

$$\eta^{\alpha\nu}\frac{\partial f}{\partial x^\nu} =$$

or

$$(\vec{\nabla}f)^\mu = \eta^{\mu\nu}\frac{\partial f}{\partial x^\nu}$$

since

$$\frac{\partial f}{\partial x^\mu} \equiv \partial_\mu f$$

we must have

$$(\vec{\nabla} f)^{\mu} = \eta^{\mu\nu} \partial_{\nu} f$$

$$= \partial^{\mu} f$$

Since

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

we must have

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$$

The x_{μ} are the components of the 1-form dual to the position vector

$$\vec{X} = x^{\mu} \hat{e}_{\mu}$$

– i.e.,

$$\tilde{X} = x_{\mu} \widetilde{dx}^{\mu}$$