Massless Spin 1

Easiest to start by postulating a Lagrangian for the massless case rather than taking the $m \to 0$ limit of the massive case.

Let's Try

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This Lagrangian is gauge invariant in that the Lagrangian is unchanged ~~under the transformation~~

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$$

where $\alpha(x)$ is a (scalar) function. Consider

$$F'^{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

$$= \partial_\mu (A_\nu + \partial_\nu \alpha) - \partial_\nu (A_\mu + \partial_\mu \alpha)$$

$$= \partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha - \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= F_{\mu\nu}$$

Let's derive the EOM:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} = \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ -\frac{1}{4}\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \right\}$$

$$= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial^\alpha A^\beta \partial_\alpha A_\beta - \partial^\alpha A^\beta \partial_\beta A_\alpha - \partial^\beta A^\alpha \partial_\alpha A_\beta + \partial^\beta A^\alpha \partial_\beta A_\alpha \right\}$$

The term in brackets is symmetric under the interchange of $\alpha$ and $\beta$, so we only need to compute the derivative of the first two terms.

$$\frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial^\alpha A^\beta \partial_\alpha A_\beta \right\}$$

$$= \partial^\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial_\alpha A_\beta \right\} + \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial^\alpha A^\beta \right\} \partial_\alpha A_\beta$$

$$= \eta_{\gamma\beta}\, \partial^\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial_\alpha A^\gamma \right\} + \eta^{\alpha\gamma} \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial_\gamma A^\beta \right\} \partial_\alpha A_\beta$$

$$= \eta_{\gamma\beta}\, \partial^\alpha A^\beta \delta^\mu{}_\alpha \delta^\gamma{}_\nu + \eta^{\alpha\gamma} \delta^\mu{}_\gamma \delta^\beta{}_\nu \partial_\alpha A_\beta$$

$$= \eta_{\nu\beta}\, \partial^\mu A^\beta + \eta^{\alpha\mu} \partial_\alpha A_\nu$$

$$= \partial^\mu A_\nu + \partial^\mu A_\nu$$

$$= 2\partial^\mu A_\nu$$

$$\frac{\partial}{\partial(\partial_\mu A^\nu)}\left\{\partial^\alpha A^\beta \partial_\beta A_\alpha\right\}$$

$$= \partial^\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A^\nu)}\left\{\partial_\beta A_\alpha\right\} + \frac{\partial}{\partial(\partial_\mu A^\nu)}\left\{\partial^\alpha A^\beta\right\}\partial_\beta A_\alpha$$

$$= \eta_{\alpha\gamma}\,\partial^\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A^\nu)}\left\{\partial_\beta A^\gamma\right\} + \eta^{\alpha\gamma}\frac{\partial}{\partial(\partial_\mu A^\nu)}\left\{\partial_\gamma A^\beta\right\}\partial_\beta A_\alpha$$

$$= \eta_{\alpha\gamma}\,\partial^\alpha A^\beta \delta^\mu{}_\beta\, \delta^\gamma{}_\nu + \eta^{\alpha\gamma}\delta^\mu{}_\gamma\, \delta^\beta{}_\nu\, \partial_\beta A_\alpha$$

$$= \eta_{\alpha\nu}\,\partial^\alpha A^\mu + \eta^{\alpha\mu}\partial_\nu A_\alpha$$

$$= \partial_\nu A^\mu + \partial_\nu A^\mu$$

$$= 2\partial_\nu A^\mu$$

Then

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)} = -\frac{1}{4}\left\{2\partial^\mu A_\nu - 2\partial_\nu A^\mu - 2\partial_\nu A^\mu + 2\partial^\mu A_\nu\right\}$$

$$= \partial_\nu A^\mu - \partial^\mu A_\nu$$

We can immediately write down

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = 0$$

Then, the EOM are

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) = \partial_\mu \left( \partial_\nu A^\mu - \partial^\mu A_\nu \right)$$

$$= \partial_\nu \partial_\mu A^\mu - \partial_\mu \partial^\mu A_\nu$$

$$= \partial_\nu \partial_\mu A^\mu - \Box A_\nu$$

$$= 0$$

or

$$\Box A_\nu - \partial_\nu (\partial_\mu A^\mu) = 0$$

As before, let's separate the 0 and i components:

0: $$\partial_0 \partial^0 A_0 - \partial_i \partial^i A_0 - \partial_0 (\partial_0 A^0 + \partial_i A^i) = 0$$

i: $$\partial_0 \partial^0 A_i - \partial_j \partial^j A_i - \partial_i (\partial_0 A^0 + \partial_i A^i) = 0$$

The 0 equation gives

$$-\partial_i \partial^i A_0 - \partial_0 \partial_i A^i = 0$$

To count the physical degrees of freedom of the theory, use the gauge invariance of the theory:

For

$$A^\mu(x) \longrightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x)$$

we have

$$\partial_i A^i(x) \longrightarrow \partial_i A'^i(x) = \partial_i A^i(x) + \partial_i \partial^i \alpha(x)$$

Choose $\alpha(x) \ni$

$$\partial_i A'^i = 0$$

This is the <u>Coulomb</u> Gauge. Then, from the EOM

In this gauge, it is easy to see how we get rid of $A_0$.

$$-\partial_i \partial^i (A'_0 - \partial_0 \alpha) - \partial_0 \partial_i (A'^i - \partial^i \alpha) = 0$$

Under the above gauge transformation, we also have

$$A^0 \longrightarrow A'^0 = A^0 + \partial^0 \alpha$$

Choose $\alpha(x) \ni$

$$A'_0 = 0 \quad \leftarrow \text{specifies the time dependence of } \alpha(x)$$

The EOM then read

$$\partial_i \partial^i \partial_0 \alpha + \partial_0 \partial_i \partial^i \alpha = 0$$

That is, if $\partial_i \partial^i \alpha = 0$, $A'^\mu$ still satisfies the EOM. We can choose $\alpha$ ∋

$$\partial_i \partial^i \alpha = 0 \quad \leftarrow \text{specifies the spatial dependence of } \alpha(x)$$

The EOM now read

$$\Box A'_i = 0$$

Let

$$A^\mu(x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{a}_j(p) \epsilon^\mu_j(p) e^{ip\cdot x}$$

Then

EOM $$\Box A^\mu = \int \frac{d^4 p}{(2\pi)^4} \tilde{a}_j(p) \epsilon^\mu_j(p) p^2 e^{ip\cdot x} = 0 \quad \Rightarrow p^2 = 0$$

Gauge $$\partial_i A^i = \int \frac{d^4 p}{(2\pi)^4} \tilde{a}_j(p) p_i \epsilon^i_j(p) e^{ip\cdot x} = 0 \quad \Rightarrow p_i \epsilon^i(p) = 0$$

$A_0 = 0$ $$\Rightarrow \epsilon^0_j(p) = 0$$

Let (i.e., choose a frame)

$$p^\mu = (E, 0, 0, E)$$

Then

$$\epsilon_1^\mu = (0, 1, 0, 0)$$

$$\epsilon_2^\mu = (0, 0, 1, 0)$$

In the Lorentz Gauge, as before in the massive spin 1 case, we would have had

$$p^2 = m^2$$

$$p_\mu \epsilon^\mu(p) = 0$$

and for the above choice of $p^\mu$ we would have wound up with 2 physical (transverse) polarizations

$$\epsilon_1^\mu = (0, 1, 0, 0)$$

$$\epsilon_2^\mu = (0, 0, 1, 0)$$

and 1 unphysical (longitudinal) polarization

$$\epsilon^{\mu}_{3} = (1, 0, 0, 1)$$

In the last case,

$$\gamma_{\mu} \epsilon^{\mu}_{3} = \gamma_0 \epsilon^{0}_{3} + \gamma_3 \epsilon^{3}_{3}$$

$$= (E)(1) + (-E)(1)$$

$$\gamma_3 = \eta_{3\mu} \gamma^{\mu} = -\gamma^{3}$$

$$= 0$$

but

$$\epsilon^{\mu}_{3} \, \epsilon_{3\mu} = \epsilon^{0}_{3} \, \epsilon_{30} + \epsilon^{3}_{3} \, \epsilon_{33}$$

$$= (1)(1) + (1)(-1)$$

$$= 0$$

$$\neq -1$$