A Review of the Canonical Formalism and Quantization Procedure for Particles

Consider the one-dimensional motion of a single particle in a conservative force field.

Let q be the generalized coordinate of the particle.

Then

$$\dot{q} \equiv \frac{dq}{dt}$$

is the generalized velocity of the particle.

Let $L(q, \dot{q})$ be the Lagrangian.

Hamilton's Principle

The physical path, $q(t)$, that the particle takes in going from $q(t_1) \equiv q_1$ to $q(t_2) \equiv q_2$ is that along which the Action, S, is stationary.

The action is defined by

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

Along the physical path, small variations in the path

$$q(t) \longrightarrow q(t) + \delta q(t)$$

leading to a variation of the action, δS, leave the action unchanged to first order in the variation, $\delta q(t)$ — i.e.,

$$\delta S = 0$$

Let's compute δS:

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta(\dot{q}) \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\, dt$$

The last equality arises because

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) dt = \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_2} - \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1} = 0$$

since $\delta q(t_1)$ and $\delta q(t_2)$ are by definition 0 given the definite starting point, $q(t_1) \equiv q_1$, and ending point, $q(t_2) \equiv q_2$.

Given a functional

$$F = F(x, y(x), y'(x))$$

Consider the variation

$$y(x) \longrightarrow y(x) + \epsilon\,\eta(x)$$

where $\epsilon << 1$.

$$\delta y(x) \equiv \epsilon\,\eta(x)$$

Then, at a *fixed* x,

$$\delta F = F(x, y(x) + \epsilon\,\eta(x), y'(x) + \epsilon\,\eta'(x)) - F(x, y(x), y'(x))$$

Expand in $\underline{\underline{\epsilon}}$:

$$\delta F = \frac{\partial F}{\partial y}\epsilon\,\eta + \frac{\partial F}{\partial y'}\epsilon\,\eta'$$

For

$$F = y \Rightarrow \delta y = \epsilon\,\eta$$

$$F = y' \Rightarrow \delta y' = \epsilon\,\eta' = (\delta y)'$$ δ and $\frac{d}{dx}$ commute!

$$\int_{t_i}^{t_{i+1}} [\ ]\,\delta q\, dt = [\ ]_{t_i}\,\delta q(t_i)\,\Delta t + [\ ]_{t_{i+1}}\,\delta q(t_{i+1})\,\Delta t + \dots = 0 \Rightarrow \text{all } [\ ]_{t_i} = 0$$

Since the variation $\delta q(t)$ of the function $q(t)$ is arbitrary, the physical path (for which $\delta S = 0$) is given by the solution of the Euler-Lagrange EOM

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

The momentum conjugate to q is defined by

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

The Hamiltonian, which is a function of (q, p) rather than $(q, \dot{q})$, is defined by

$$H \equiv p\dot{q} - L$$

The EOM are (using the Poisson Brackets)

$$\dot{q} = -\{H, q\} = -\left(\frac{\partial H}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial q}{\partial q}\right) = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\{H, p\} = -\left(\frac{\partial H}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q}\right) = -\frac{\partial H}{\partial q}$$

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To quantize this system, q and p become Hermitian operators

$q \longrightarrow \hat{q}$ whose action on $\Psi(q, t)$ is multiplication by q

$p \longrightarrow \hat{p} \equiv -i\frac{\partial}{\partial q}$

Considering the Hamiltonian as a function of (q, p), not explicitly t, we have

$$dH(p, q) = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq \qquad (1)$$

Considering the definition of the Hamiltonian

$$H(p, q) \equiv p\dot{q} - L(q, \dot{q}) \qquad (2)$$

$$dH = p\, d\dot{q} + \dot{q}\, dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} d\dot{q} \qquad (3)$$

where we have assumed that L is not an explicit function of time (this would break Lorentz invariance of the Lagrangian which is a scalar quantity, when we build our relativistic QFT)

From the Euler-Lagrange EOM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad (4)$$

But

$$p \equiv \frac{\partial L}{\partial \dot{q}} \qquad (5)$$

The Euler-Lagrange EOM then tell us

$$\frac{d}{dt}(p) = \dot{p} = \frac{\partial L}{\partial q} \qquad (6)$$

Then, insertion of (5) and (6) in (3) gives

$$dH = \left(p - \frac{\partial L}{\partial \dot{q}}\right) d\dot{q} + \dot{q}\, dp - \frac{\partial L}{\partial q}\, dq$$ ← just a rewrite of (3)

$$= \dot{q}\, dp - \dot{p}\, dq$$

But, from (1), equating coefficients

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

These are Hamilton's EOM, which can be expressed in terms of the so-called Poisson Brackets as

$$\dot{q} = -\{H, q\} \equiv -\left(\frac{\partial H}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial q}{\partial q}\right) = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\{H, p\} \equiv -\left(\frac{\partial H}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q}\right) = -\frac{\partial H}{\partial q}$$

and the Hamiltonian becomes a Hermitian operator. The Schroedinger Equation is

$$\hat{H}\psi(q,t) = i\frac{\partial\psi(q,t)}{\partial t}$$

In the Heisenberg Representation, it's the operators that depend on time, not the states.

$$\psi_s(q,t) = e^{-i\hat{H}t}\psi_s(q,0) \equiv e^{-i\hat{H}t}\psi_H$$

The time-independent operators in the Schroedinger picture are replaced in the Heisenberg picture by

$$\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}_s e^{-i\hat{H}t}$$

and the time development of the operators is given by

$$\frac{d\hat{O}_H}{dt} = i[\hat{H}, \hat{O}_H]$$

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Let's look at $\hat{O}_H = \hat{p}(t)$. Using an arbitrary function, $f(p,q)$, as a placeholder,

$$\frac{d\hat{p}(t)}{dt}f = i[\hat{H},\hat{p}]f$$

$$= i\left[\hat{H}\left(-i\frac{\partial}{\partial q}\right) - \left(-i\frac{\partial}{\partial q}\right)\hat{H}\right]f$$

$$= \hat{H}\left(\frac{\partial f}{\partial q}\right) - \frac{\partial}{\partial q}(\hat{H}f)$$

Consider the time derivative of the expectation value of the operator, $\hat{O}_S$, in the Schrödinger picture in the state $|\psi_S\rangle$:

$$\frac{d}{dt}\langle \psi_S | \hat{O}_S | \psi_S \rangle \qquad (1)$$

Here the state $|\psi_S\rangle$ is time-dependent but the operator $\hat{O}_S$ is not.

We can reexpress the matrix element in terms of $|\psi_H\rangle$ and $\hat{O}_H$ as

$$\frac{d}{dt}\langle \psi_H | e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t} |\psi_H\rangle \qquad (2)$$

Now all of the time dependence sits in the exponentials $e^{\pm i\hat{H}t}$.

Taking the derivative

$$\langle \psi_H | (i\hat{H} e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t} - i e^{i\hat{H}t}\hat{O}_S \hat{H} e^{-i\hat{H}t}) | \psi_H \rangle$$

$$= \langle \psi_H | i (\hat{H} e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t} - e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}\hat{H}) | \psi_H \rangle$$

$$= \langle \psi_H | i (\hat{H}\hat{O}_H - \hat{O}_H\hat{H}) | \psi_H \rangle$$

$$= \langle \psi_H | i [\hat{H}, \hat{O}_H] | \psi_H \rangle \qquad (3)$$

Then, from (2) and (3)

$$\frac{d}{dt} \langle \psi_H | \hat{O}_H | \psi_H \rangle$$

$$= \langle \psi_H | \frac{d}{dt} \hat{O}_H | \psi_H \rangle$$

$$= \langle \psi_H | i [\hat{H}, \hat{O}_H] | \psi_H \rangle$$

We arrive at the operator relation

$$\frac{d\hat{O}_H}{dt} = i [\hat{H}, \hat{O}_H]$$

which is the Heisenberg EOM for the time-dependent operator $\hat{O}_H$.

$$= \hat{H}\left(\frac{\partial f}{\partial q}\right) - \frac{\partial \hat{H}}{\partial q}(f) - \hat{H}\left(\frac{\partial f}{\partial q}\right)$$

$$= -\frac{\partial \hat{H}}{\partial q}(f)$$

Then

$$\frac{d\hat{p}(t)}{dt} = -\frac{\partial \hat{H}}{\partial q}$$

Now, let's look at $\hat{O}_H = \hat{q}(t)$.

$$\frac{d\hat{q}(t)}{dt}(f) = i\,[\hat{H}, \hat{q}]\,f$$

$$= i\left[\hat{H}\left(i\frac{\partial}{\partial p}\right) - \left(i\frac{\partial}{\partial p}\right)\hat{H}\right] f$$

$$= -\hat{H}\left(\frac{\partial f}{\partial p}\right) + \frac{\partial}{\partial p}(\hat{H} f)$$

$$= -\hat{H}\left(\frac{\partial f}{\partial p}\right) + \frac{\partial \hat{H}}{\partial p}(f) + \hat{H}\left(\frac{\partial f}{\partial p}\right)$$

$$= \frac{\partial \hat{H}}{\partial p}(f)$$

Then

$$\frac{d\hat{q}(t)}{dt} = \frac{\partial \hat{H}}{\partial p}$$

But these are just the <u>classical</u> <u>equations</u> <u>of</u> <u>motion</u>.

$\Rightarrow$ In the Heisenberg representation, the relevant generators evolve according to the classical EOM.

This is very important in QFT and motivates the approach we will take to build our theories:

1) CONSTRUCT LAGRANGIANS FOR CLASSICAL FIELDS

2) DERIVE THE EOM FOR THOSE FIELDS

3) FIND THE CLASSICAL SOLUTIONS TO THESE EOM

4) QUANTIZE THESE CLASSICAL FIELD SOLUTIONS

- ELEVATE THEM TO OPERATOR STATUS
- INTRODUCE CREATION AND ANNIHILATION OPERATORS
- MAKE THE CONNECTION TO PARTICLES