SPIN 1

Given that we chose

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi$$

for the spin 0 case, why not choose

$$\mathcal{L} = \frac{1}{2}\partial_\mu A^\nu \partial^\mu A_\nu - \frac{1}{2} m^2 A_\mu A^\mu$$

for the spin 1 case?

Let's look at the EOM:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}$$

$$= \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \frac{1}{2}\partial_\alpha A^\beta \partial^\alpha A_\beta - \frac{1}{2} m^2 A_\alpha A^\alpha \right\}$$

$$= \frac{1}{2} \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial_\alpha A^\beta \right\} \partial^\alpha A_\beta + \frac{1}{2} \partial_\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial^\alpha A_\beta \right\}$$

$$= \frac{1}{2} \delta^\mu{}_\alpha \delta^\beta{}_\nu \partial^\alpha A_\beta + \frac{1}{2} \eta^{\alpha\gamma} \eta_{\beta\rho} \partial_\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A^\nu)} \left\{ \partial_\gamma A^\rho \right\}$$

$$= \frac{1}{2} \partial^\mu A_\nu + \frac{1}{2} \partial^\gamma A_\rho \delta^\mu{}_\gamma \delta^\rho{}_\nu$$

$$= \frac{1}{2}\partial^\mu A_\nu + \frac{1}{2}\partial^\mu A_\nu$$

$$= \partial^\mu A_\nu$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = \frac{\partial}{\partial A^\nu}\left\{\frac{1}{2}\partial_\alpha A^\beta \partial^\alpha A_\beta - \frac{1}{2}m^2 A_\alpha A^\alpha\right\}$$

$$= -\frac{1}{2}m^2 \frac{\partial}{\partial A^\nu}\left\{A_\alpha A^\alpha\right\}$$

$$= -\frac{1}{2}m^2\left(\frac{\partial A^\alpha}{\partial A^\nu}\right)A_\alpha - \frac{1}{2}m^2 A^\alpha \frac{\partial}{\partial A^\nu}\left(\eta_{\alpha\beta} A^\beta\right)$$

$$= -\frac{1}{2}m^2 \delta^\alpha{}_\nu A_\alpha - \frac{1}{2}m^2 \eta_{\alpha\beta} A^\alpha\left(\frac{\partial A^\beta}{\partial A^\nu}\right)$$

$$= -\frac{1}{2}m^2 A_\nu - \frac{1}{2}m^2 A_\beta \delta^\beta{}_\nu$$

$$= -m^2 A_\nu$$

Then the EOM read

$$\partial_\mu\left(\partial^\mu A_\nu\right) + m^2 A_\nu = 0$$

or

$$\left(\Box + m^2\right)A_\nu = 0$$

which are the EOM's for 4 scalar fields, A_μ, not for a vector field.

The most general Lagrangian that is

1) quadratic in the fields
   - then the resultant EOM's are linear in the fields

2) has not more than two derivatives
   - then $E = \int d^3x\, \mathcal{E} > 0$

3) Lorentz invariant

4) excludes total (four) divergences
   - which do not contribute to the ACTION

$$S \equiv \int d^4x\, \mathcal{L}$$

can be constructed from

$$A_\mu A^\mu$$

and

$$\partial_\mu A^\nu \partial^\mu A_\nu \quad , \quad \partial_\mu A^\mu \partial^\nu A_\nu$$

Any other two-derivative term can be written in terms of these two terms and a total divergence.

For example,

$$\begin{aligned} A_\mu \Box A^\mu &= A_\mu \partial_\nu \partial^\nu A^\mu \\ &= \partial_\nu (A_\mu \partial^\nu A^\mu) - (\partial_\nu A_\mu) \partial^\nu A^\mu \\ &= -\partial_\nu A_\mu \partial^\nu A^\mu + \text{TOTAL DIVERGENCE} \end{aligned}$$

and

$$\partial_\mu (A^\mu \partial^\nu A_\nu) = (\partial_\mu A^\mu)(\partial^\nu A_\nu) + A^\mu \partial_\mu \partial^\nu A_\nu$$

Rearranging the last equation

$$A^\mu \partial_\mu \partial^\nu A_\nu = -(\partial_\mu A^\mu)(\partial^\nu A_\nu) + \text{TOTAL DIVERGENCE}$$

Then, equivalently, we can build our Lagrangian from

$$A_\mu A^\mu$$

and

$$A_\mu \Box A^\mu \quad , \quad A^\mu \partial_\mu \partial^\nu A_\nu$$

Consider the general Lagrangian

$$\mathcal{L} = \frac{a}{2} A_\mu \Box A^\mu + \frac{b}{2} A^\mu \partial_\mu \partial_\nu A^\nu + \frac{1}{2} m^2 A_\mu A^\mu$$

N.B. The presence of $\partial_\nu A^\nu$ and the demand of Lorentz invariance means that in this theory A^ν must transform as a 4-vector, not 4 scalars.

Let's derive the EOM:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} :$$

$$\frac{\partial}{\partial(\partial_\mu A^\nu)}\left(A_\alpha \Box A^\alpha\right)$$

$$= \frac{\partial}{\partial(\partial_\mu A^\nu)}\left(A_\alpha \partial_\beta \partial^\beta A^\alpha\right)$$

$$= \frac{\partial}{\partial(\partial_\mu A^\nu)}\left(A_\alpha \partial^\beta \partial_\beta A^\alpha\right)$$

$$= A_\alpha \frac{\partial}{\partial(\partial_\mu A^\nu)}\left(\partial^\beta \partial_\beta A^\alpha\right)$$

$$= A_\alpha \partial^\beta\left(\frac{\partial}{\partial(\partial_\mu A^\nu)} \partial_\beta A^\alpha\right)$$

$$= A_\alpha \partial^\beta\left(\delta^\mu{}_\beta \, \delta^\alpha{}_\nu\right)$$

$$= 0$$

$$\frac{\partial}{\partial(\partial_\mu A^\nu)}\left(A^\alpha \partial_\alpha \partial_\beta A^\beta\right)$$

$$= 0$$

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = \frac{a}{2}\Box A_\mu + \frac{b}{2}\partial_\mu \partial_\nu A^\nu + \frac{1}{2}m^2 A_\mu$$

The EOM are

$$-\frac{a}{2}\Box A_\mu - \frac{b}{2}\partial_\mu \partial_\nu A^\nu - \frac{1}{2}m^2 A_\mu = 0$$

or

$$a \Box A_\mu + b \partial_\mu \partial^\nu A_\nu + m^2 A_\mu = 0$$

Take the derivative of this equation w.r.t. ∂^μ

$$a \Box (\partial^\mu A_\mu) + b \partial^\mu \partial_\mu (\partial^\nu A_\nu) + m^2 \partial^\mu A_\mu = 0$$

which can be rewritten as

$$[(a+b)\Box + m^2] \partial^\mu A_\mu = 0$$

For $a = -b$ and $m \neq 0$

$$\partial_\mu A^\mu = 0$$

which removes 1 degree of freedom from the field A^μ, leaving 3.

For $a = 1$ and $b = -1$, and with $\partial_\mu A^\mu = 0$, the EOM read

$$(\Box + m^2) A_\mu = 0$$

That is, the components of A^μ still satisfy the Klein-Gordon equation, but A^μ is a 4-vector.

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} A_\mu \Box A^\mu - \frac{1}{2} A^\mu \partial_\mu \partial_\nu A^\nu + \frac{1}{2} m^2 A_\mu A^\mu$$

Now define

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

Then

$$F^{\mu\nu} F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\overset{①}{=} \partial^\mu A^\nu \partial_\mu A_\nu - \overset{②}{\partial^\mu A^\nu \partial_\nu A_\mu}$$

$$\underset{③}{- \partial^\nu A^\mu \partial_\mu A_\nu} + \underset{④}{\partial^\nu A^\mu \partial_\nu A_\mu}$$

Look at the term ②

$$\partial^\mu A^\nu \partial_\nu A_\mu = \partial^\mu (A^\nu \partial_\nu A_\mu) - A^\nu \partial^\mu \partial_\nu A_\mu$$

That is, up to a total 4-divergence

$$- \partial^\mu A^\nu \partial_\nu A_\mu = A^\nu \partial_\mu \partial_\nu A^\mu$$

Look at ③

$$\partial^\nu A^\mu \partial_\mu A_\nu = \partial^\nu (A^\mu \partial_\mu A_\nu) - A^\mu \partial^\nu \partial_\mu A_\nu$$

or

$$- \partial^\nu A^\mu \partial_\mu A_\nu = A^\mu \partial_\nu \partial_\mu A^\nu$$

Now look at 1

$$\partial^\mu A^\nu \partial_\mu A_\nu = \partial^\mu (A^\nu \partial_\mu A_\nu) - A^\nu \partial^\mu \partial_\mu A_\nu$$

or

$$\partial^\mu A^\nu \partial_\mu A_\nu = - A^\mu \Box A_\mu$$

Similarly, for 4

$$\partial^\nu A^\mu \partial_\nu A_\mu = \partial^\nu (A^\mu \partial_\nu A_\mu) - A^\mu \partial^\nu \partial_\nu A_\mu$$

or

$$\partial^\nu A^\mu \partial_\nu A_\mu = - A^\mu \Box A_\mu$$

Then

$$F^{\mu\nu} F_{\mu\nu} = - A^\mu \Box A_\mu + A^\nu \partial_\mu \partial_\nu A^\mu$$
$$+ A^\mu \partial_\nu \partial_\mu A^\nu - A^\mu \Box A_\mu$$
$$= -2 A^\mu \Box A_\mu + 2 A^\mu \partial_\mu \partial_\nu A^\nu$$

or

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}A^{\mu}\Box A_{\mu} - \frac{1}{2}A^{\mu}\partial_{\mu}\partial_{\nu}A^{\nu}$$

Then, the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_{\mu}A^{\mu}$$

which is the PROCA LAGRANGIAN.

Solutions to

$$(\Box + m^2) A_\mu = 0$$

for massive, spin-1 fields A_μ:

$$A^\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \underset{\text{sum over } i}{a_i(\vec{p})} \underbrace{\epsilon_i^\mu(p)}_{\text{3 polarization basis 4-vectors } (i=1,2,3)} e^{ip\cdot x} \qquad p_0 = \omega_p = [|\vec{p}|^2 + m^2]^{1/2}$$

since

$$\Box e^{ip\cdot x} = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) e^{i(p_0 t - p_x x - p_y y - p_z z)}$$

$$= (-p_0^2 + p_x^2 + p_y^2 + p_z^2)\, e^{i(p_0 t - p_x x - p_y y - p_z z)}$$

$$= -m^2 e^{ip\cdot x}$$

Then

$$(\Box + m^2) e^{ip\cdot x} = 0$$

We also know that

$$\partial_\mu A^\mu = 0$$

Let's compute

$$p^\mu \equiv (p^0, p^x, p^y, p^z)$$

$$p_\mu = (p_0, p_x, p_y, p_z) = (p^0, -p^x, -p^y, -p^z)$$

$$\partial_\mu e^{ip\cdot x} = \left(\frac{\partial}{\partial t} e^{ip\cdot x}, \frac{\partial}{\partial x} e^{ip\cdot x}, \ldots\right)$$

$$= (ip_0, ip_x, ip_y, ip_z)\, e^{ip\cdot x}$$

$$= i(p_0, p_x, p_y, p_z)\, e^{ip\cdot x}$$

$$= i p_\mu\, e^{ip\cdot x}$$

$$p\cdot x = p_\mu x^\mu = p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3 = p^0 x^0 - p^1 x^1 - p^2 x^2 - p^3 x^3$$

Then

$$\partial_\mu A^\mu = \int \frac{d^3\vec{p}}{(2\pi)^3}\, a_i(\vec{p})\, i p_\mu\, \epsilon^\mu_i(p)\, e^{ip\cdot x} = 0$$

Then, we must have

$$p_\mu \epsilon^\mu_i(p) = 0$$

for all p. Since

$$p^\mu p_\mu = m^2$$

there are 3 independent (4-vector) solutions : $\epsilon^\mu_i(p)$, $i=1,2,3$.

The polarization basis elements are normalized to

$$\eta(\vec{\epsilon}_i, \vec{\epsilon}_i) = -1$$

That is

$$\eta_{\mu\nu}\, \epsilon^{\mu}_{i}(p)\, \epsilon^{\nu}_{i}(p) = -1$$

If

$$p^{\mu} = (E, 0, 0, p^{z})$$

with

$$E^{2} - (p^{z})^{2} = m^{2}$$

then

$$\epsilon^{\mu}_{1} = (0, 1, 0, 0)$$

$$\epsilon^{\mu}_{2} = (0, 0, 1, 0)$$

satisfy

$$p_{\mu}\, \epsilon^{\mu}_{i} = 0$$

and

$$\eta_{\mu\nu}\, \epsilon^{\mu}_{i}\, \epsilon^{\nu}_{i} = -1$$

These are both <u>transverse</u> polarization vectors.

We also have a longitudinal polarization vector:

$$\epsilon^{\mu}_{L} = \left(\frac{p^z}{m}, 0, 0, \frac{E}{m} \right)$$

Here

$$p_\mu \epsilon^{\mu}_{L} = p_0 \epsilon^{0}_{L} + p_z \epsilon^{z}_{L}$$

$$= E \frac{p^z}{m} - p^z \frac{E}{m}$$

$$= 0$$

and

$$\epsilon^{\mu}_{L} \epsilon_{\mu L} = \eta_{\mu\nu} \epsilon^{\mu}_{L} \epsilon^{\nu}_{L}$$

$$= \epsilon^{0}_{L} \epsilon^{0}_{L} - \epsilon^{z}_{L} \epsilon^{z}_{L}$$

$$= \left(\frac{p^z}{m} \right)^2 - \left(\frac{E}{m} \right)^2$$

$$= -1$$