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Minkowski Spacetime, Its Metric, and Tensor Operations

proper length $\rightarrow ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ (assuming Cartesian Coordinates)

proper time $\rightarrow = -c^2 d\tau^2$

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \xrightarrow{c=1} dt^2 - dx^2 - dy^2 - dz^2$$

$$d\tau^2 \equiv -\eta_{\mu\nu} dx^\mu dx^\nu \equiv \bar{\eta}_{\mu\nu} dx^\mu dx^\nu$$

↑

$$\text{Minkowski metric} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta_{00} = -1$$

$$\eta_{ii} = 1$$

$$\eta_{\mu\nu} = 0 \quad \mu \neq \nu$$

$$dx^0 = dt$$

$$dx^1 = dx$$

$$dx^2 = dy$$

$$dx^3 = dz$$

All tensors are built from contravariant vectors (vectors) and covariant vectors (covectors or 1-forms).

Contravariant Vectors

$$\vec{V}: T_P M \rightarrow \mathbb{R}$$

$$\vec{V} = V^\mu \hat{e}_\mu \quad \text{index on Type (contravariant)}$$

N.B. The Einstein index convention is at play here:

$$\vec{V} = V^\mu \hat{e}_\mu = V^0 \hat{e}_0 + V^1 \hat{e}_1 + V^2 \hat{e}_2 + V^3 \hat{e}_3$$

sum over repeated indices

Covariant Vectors

$\tilde{V}: T_P^*M \rightarrow \mathbb{R}$

$$\tilde{V} = V \tilde{dx}^\mu =$$

index down \uparrow 1-form dual to \hat{e}_μ - i.e., $\tilde{dx}^\mu(\hat{e}_\nu) = \delta^\mu_\nu$

$$\tilde{V}(\vec{V}) = \tilde{V}(V^\alpha \hat{e}_\alpha) = V_\mu V^\alpha \tilde{dx}^\mu(\hat{e}_\alpha)$$

General Tensors are created via the Tensor Product. Let's look at the important case of the Metric Tensor

$$\eta \equiv \eta_{\mu\nu} \tilde{dx}^\mu \otimes \tilde{dx}^\nu$$

↑
tensor product

Then

$$\begin{aligned} \vec{U} \cdot \vec{V} &\equiv \eta(\vec{U}, \vec{V}) = \eta_{\mu\nu} \tilde{dx}^\mu \otimes \tilde{dx}^\nu(\vec{U}, \vec{V}) \\ &= \eta_{\mu\nu} U^\alpha V^\beta \tilde{dx}^\mu(\hat{e}_\alpha) \tilde{dx}^\nu(\hat{e}_\beta) \\ &= \eta_{\mu\nu} U^\alpha V^\beta \delta^\mu_\alpha \delta^\nu_\beta \end{aligned}$$

↑
The metric η is the inner product.

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feeds on 2 vectors

$$= \eta_{\alpha\beta} U^\alpha V^\beta$$

Since this is just a scalar (a number), the following must be a 1-form

must be a 1-form \rightarrow

$$\begin{aligned} \eta(\vec{U}, \cdot) &= \eta_{\mu\nu} \tilde{dx}^\mu \otimes \tilde{dx}^\nu (\vec{U}, \cdot) \\ &= \eta_{\mu\nu} U^\alpha \tilde{dx}^\mu \otimes \tilde{dx}^\nu (\hat{e}_\alpha, \cdot) \\ &= \eta_{\mu\nu} U^\alpha \delta^\mu_\alpha \tilde{dx}^\nu (\cdot) \\ &= \eta_{\mu\nu} U^\mu \tilde{dx}^\nu (\cdot) \\ &\equiv U_\nu \tilde{dx}^\nu (\cdot) \end{aligned}$$

empty vector slot

This is the metric raises and lowers indices (the inverse, $\eta^{\mu\nu}$, would raise indices)

$$U_\nu = \eta_{\mu\nu} U^\mu$$

$$U^\mu = \eta^{\mu\nu} U_\nu$$

is the inner product

$$\vec{U} \cdot \vec{V} = \eta_{\alpha\beta} U^\alpha V^\beta$$

$$= U_{\mu} V^{\mu}$$

$$= U_{\mu} \tilde{dx}^{\mu} (V^{\alpha} \hat{e}_{\alpha}) = \tilde{U}(\vec{V})$$

$$= U_{\mu} V^{\alpha} \tilde{dx}^{\mu} (\hat{e}_{\alpha})$$

$$= U_{\mu} V^{\alpha} \delta^{\mu}_{\alpha}$$

$$= U_{\mu} V^{\mu}$$

In Euclidean space, in Cartesian coordinates, we write

$$\vec{U} \cdot \vec{V} = U^1 V^1 + U^2 V^2 + U^3 V^3 \leftarrow$$

which is really

there's a bit of differential geometry hidden here

$$U_1 V^1 + U_2 V^2 + U_3 V^3$$

But, given the metric

$$\eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we have

$$U_i = \eta_{ij} U^j = \eta_{ii} U^i = (+1) U^i = U^i$$

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or \vec{v} doesn't matter.

But in Minkowski space, it does matter.

Note that

$$\vec{U} \cdot \vec{V} = \eta(\vec{U}, \vec{V}) = U_\alpha V^\alpha$$

is just a scalar. As a scalar, it is Lorentz invariant. That means that U and V must transform under a Lorentz transformation in an inverse manner.

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

$$V_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} V_\mu$$

Then

$$\begin{aligned} V^{\mu'} V_{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^{\mu'}} V^\mu V_\mu \\ &= V^\mu V_\mu \end{aligned}$$