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# Continuous Limit of the Linear Chain

$$\left. \begin{array}{l} a \rightarrow dx \\ a \rightarrow 0 \\ N \rightarrow \infty \end{array} \right\} l = Na = \text{length of the chain} = \text{constant}$$

$$\left. \begin{array}{l} m \rightarrow dm \\ m \rightarrow 0 \end{array} \right\} \rho = m/a = \text{constant}$$

displacement of infinitesimal mass at  $x$

$$f_n(t) \rightarrow q(x, t) \quad \text{a scalar field!}$$

$$f_{n+1}(t) - f_n(t) \rightarrow dq = a \frac{\partial q}{\partial x}$$

$$\sum_{n=1}^N \rightarrow \frac{1}{a} \int_0^l dx \quad (\text{this just says } N = \frac{l}{a})$$

$$\sigma \equiv Ka = \text{constant} = \text{string tension}$$

$$m\ddot{x} + Kx = 0$$

$$\text{as } m \rightarrow 0, \ddot{x} \rightarrow \infty$$

$$\Rightarrow \text{restorative force} \rightarrow \infty$$

$$\Rightarrow x \rightarrow 0 \text{ w/ } K \rightarrow \infty$$

$$\Rightarrow m\ddot{x} + Kx = 0$$

Then

$$L(q_n, \dot{q}_n) = \sum_n \left[ \frac{m}{2} (\dot{q}_n)^2 - \frac{K}{2} (q_{n+1} - q_n)^2 \right]$$

$$\rightarrow \frac{1}{a} \int_0^l dx \frac{1}{2} \left[ \rho a \left( \frac{\partial q}{\partial t} \right)^2 - \frac{\sigma}{a} a^2 \left( \frac{\partial q}{\partial x} \right)^2 \right]$$

$$= \frac{1}{2} \int_0^l dx \left[ \rho \left( \frac{\partial q}{\partial t} \right)^2 - \sigma \left( \frac{\partial q}{\partial x} \right)^2 \right]$$

and

"Lagrangian density"

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \left[ \rho \left( \frac{\partial q}{\partial t} \right)^2 - \sigma \left( \frac{\partial q}{\partial x} \right)^2 \right]$$

$$\dot{q} \equiv \frac{\partial q}{\partial t}$$

and

$$S = \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} dt \int_0^l dx \mathcal{L}$$

The EOM are determined by extremizing the action.

$$\delta S = \int_{t_i}^{t_f} dt \int_0^l dx \delta \mathcal{L}$$

$$= \int_{t_i}^{t_f} dt \int_0^l dx \left( \frac{\delta \mathcal{L}}{\delta \dot{y}} \delta \dot{y} + \frac{\delta \mathcal{L}}{\delta y'} \delta y' \right)$$

$$= \int_0^l dx \int_{t_i}^{t_f} \left[ \frac{\delta}{\delta t} \left( \frac{\delta \mathcal{L}}{\delta \dot{y}} \delta y \right) - \frac{\delta}{\delta t} \left( \frac{\delta \mathcal{L}}{\delta \dot{y}} \right) \delta y \right] dt$$

$$+ \int_{t_i}^{t_f} dt \int_0^l dx \left[ \frac{\delta}{\delta x} \left( \frac{\delta \mathcal{L}}{\delta y'} \delta y \right) - \frac{\delta}{\delta x} \left( \frac{\delta \mathcal{L}}{\delta y'} \right) \delta y \right]$$

$$= - \int_{t_i}^{t_f} dt \int_0^l dx \frac{\delta}{\delta t} \left( \frac{\delta \mathcal{L}}{\delta \dot{y}} \right) \delta y$$

$$- \int_{t_i}^{t_f} dt \int_0^l dx \frac{\delta}{\delta x} \left( \frac{\delta \mathcal{L}}{\delta y'} \right) \delta y$$

$$= - \int_{t_i}^{t_f} dt \int_0^l dx \left[ \frac{\delta}{\delta t} \left( \rho \frac{\delta \mathcal{L}}{\delta \dot{y}} \right) + \frac{\delta}{\delta x} \left( -\sigma \frac{\delta \mathcal{L}}{\delta x} \right) \right] \delta y$$

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$$\int_0^l dx \int_{t_i}^{t_f} dt \frac{\partial}{\partial t} \left( \frac{\partial x}{\partial t} s_g \right)$$

$$= \int_0^l dx \left[ \left( \frac{\partial x}{\partial t} s_g \right)_{t_f} - \left( \frac{\partial x}{\partial t} s_g \right)_{t_i} \right]$$

$$= 0$$

since

$$s_g(x, t_f) = 0 \quad \text{for all } x \in [0, l]$$

$$s_g(x, t_i) = 0 \quad " \quad " \quad "$$

$$\int_{t_i}^{t_f} dt \int_0^l dx \frac{\partial}{\partial x} \left( \frac{\partial x}{\partial t} s_g \right)$$

$$= \int_{t_i}^{t_f} dt \left[ \left( \frac{\partial x}{\partial t} s_g \right)_l - \left( \frac{\partial x}{\partial t} s_g \right)_0 \right]$$

$$= 0$$

since

$$\left( \frac{\partial x}{\partial t} s_g \right)_l = \left( \frac{\partial x}{\partial t} s_g \right)_0$$

by periodicity

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$$= - \int_{t_i}^{t_f} dt \int_0^l dx \left( \rho \frac{\partial^2 \delta f}{\partial t^2} - \sigma \frac{\partial^2 \delta f}{\partial x^2} \right) \delta f$$

$$= 0$$

now  $\delta f = \delta f(x, t)$

before  $\delta f_m = \delta f_m(t)$

$\Rightarrow ( )$  must be 0

Then the EOM for  $g(x, t)$  is

$$\frac{\partial^2 \delta f}{\partial t^2} - \frac{\sigma}{\rho} \frac{\partial^2 \delta f}{\partial x^2} = 0$$

which is a wave equation, with wave speed

$$c = \sqrt{\frac{\sigma}{\rho}}$$



# Classical Field Theory

We can extend the Lagrangian formulation of classical dynamics for point particles to fields.

The correspondence is

$$f(t) \longrightarrow \phi(x) \quad x = (\vec{x}, t)$$

$$\dot{f}(t) \longrightarrow \partial_\mu \phi(x)$$

We define a Lagrangian density related to the Lagrangian or Action by

$$S = \int dt L = \int d^4x \underbrace{\mathcal{L}(\phi, \partial_\mu \phi)}_{\text{LAGRANGIAN DENSITY}}$$

Compute  $\delta S$ :

$$\delta S = \int d^4x \left[ \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right]$$

$$= \int d^4x \left[ \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \partial_\mu (\delta \phi) \right]$$

$$= \left\{ \int d^4x \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi + \int_{-\infty}^{+\infty} \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \delta \phi - \int d^4x \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \partial_\mu (\delta \phi) \right\}$$

$$= \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi$$

Now instead of using  $S_g(t_i) = S_g(t_f) = 0$ , we simply assume that our fields are well-behaved functions of spacetime - i.e., that they go to zero as  $x \rightarrow \pm \infty$ .

The solutions render the action an extremum ( $\delta S = 0$ ) and satisfy the EOM

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0$$

One can now define the momentum conjugate to  $\phi$

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi(x))}$$

the Hamiltonian density

$$\mathcal{H}(x) \equiv \pi(x) \partial_0 \phi(x) - \mathcal{L}$$

which is related to the Hamiltonian by

$$H = \int d^3x \mathcal{H}$$

As an example, consider the following Lagrangian density



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$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

Then

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \frac{1}{2} \frac{\partial}{\partial (\partial_\mu \phi)} [\eta^{\alpha\beta} (\partial_\alpha \phi) (\partial_\beta \phi)]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \frac{\partial}{\partial (\partial_\mu \phi)} [(\partial_\alpha \phi) (\partial_\beta \phi)]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \left[ \frac{\partial}{\partial (\partial_\mu \phi)} (\partial_\alpha \phi) \right] \partial_\beta \phi$$

$$+ \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \left[ \frac{\partial}{\partial (\partial_\mu \phi)} (\partial_\beta \phi) \right]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \delta^\mu_\alpha \partial_\beta \phi$$

$$+ \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \delta^\mu_\beta$$

$$= \frac{1}{2} \partial^\mu \phi$$

$$+ \frac{1}{2} \partial^\mu \phi$$

$$= \partial^\mu \phi$$

and

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$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = \partial_\mu (\partial^\mu \phi)$$

$$= \square \phi$$

The EOM then read

$$(\square + m^2) \phi = 0$$

which is the Klein-Gordon equation. The KG equation gives