Causality

$$[\hat{O}_1(x), \hat{O}_2(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \qquad c^2(x_0-y_0)^2 - (x_1-y_1)^2 - \dots < 0$$

Measurement associated with operators $\hat{O}_{1,2}$ will commute for spacelike separations in spacetime.

Consider (for a real scalar field the fields are observable)

$$\Delta(x,y) \equiv [\hat{\phi}(x), \hat{\phi}(y)]$$

$$= \int \frac{d^3g}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{g})}}\frac{1}{\sqrt{2E(\vec{k})}}\left[e^{ik_\mu x^\mu}\hat{a}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu}\hat{a}(\vec{k}),\ e^{ig_\mu y^\mu}\hat{a}^\dagger(\vec{g}) + e^{-ig_\mu x^\mu}\hat{a}(\vec{g})\right]$$

$$= \int \frac{d^3g}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{g})}}\frac{1}{\sqrt{2E(\vec{k})}}\Big\{e^{i(k_\mu x^\mu + g_\mu y^\mu)}[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{g})] + e^{i(k_\mu x^\mu - g_\mu y^\mu)}[a^\dagger(\vec{k}), \hat{a}(\vec{g})]$$

$$+ e^{-i(k_\mu x^\mu - g_\mu y^\mu)}[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{g})] + e^{-i(k_\mu x^\mu + g_\mu x^\mu)}[\hat{a}(\vec{k}), \hat{a}(\vec{g})]\Big\}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(\vec{k})}\left\{-e^{ik_\mu(x^\mu - y^\mu)} + e^{-ik_\mu(x^\mu - y^\mu)}\right\}$$

$$\equiv D(x-y) - D(y-x)$$

where

$$D(x-y) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(\vec{k})}e^{-ik\cdot(x-y)}$$

is the propagator.

Now look at

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$$

$$= \langle 0|\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\left[e^{ik_\mu x^\mu}\hat{a}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu}\hat{a}(\vec{k})\right]$$

$$\times\int\frac{d^3g}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{g})}}\left[e^{ig_\mu y^\mu}\hat{a}^\dagger(\vec{g}) + e^{-ig_\mu y^\mu}\hat{a}(\vec{g})\right]|0\rangle$$

$$= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\int\frac{d^3\vec{g}}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{g})}}e^{-ik_\mu x^\mu}e^{-ig_\mu y^\mu}\underbrace{\langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{g})|0\rangle}_{(2\pi)^3\delta^{(3)}(\vec{k}-\vec{g})}$$

$$= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E(\vec{k})}e^{-ik_\mu(x^\mu-y^\mu)}$$

$$= D(x-y)$$

= AMPLITUDE FOR A PARTICLE TO BE CREATED AT y, PROPAGATE TO x, AND BE ANNIHILATED AT x

<u>Feynman</u> Propagator

Consider the time-ordered product

$$T\hat{\phi}(x)\hat{\phi}(y) \equiv \begin{cases}\hat{\phi}(x)\hat{\phi}(y) & x^0 > y^0\\ \hat{\phi}(y)\hat{\phi}(x) & x^0 < y^0\end{cases}$$

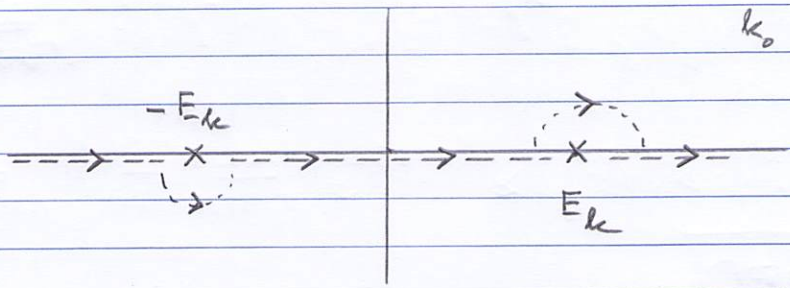
The Feynman propagator is defined as

$$D_F(x-y) = \langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle$$

$D_F(x-y)$ can be written as

$$D_F(x-y) = i \int_{C_F} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2} e^{-ik_\mu(x^\mu-y^\mu)}$$

where C_F is the k_0 contour shown below.



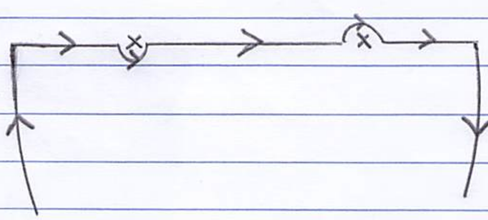
We can express

$$\frac{1}{k^2-m^2} = \frac{1}{(k^0)^2-(E_k)^2} = \frac{1}{(k^0-E_k)(k^0+E_k)}$$

so there are two poles at $\pm E_k$.

For $x^0 > y^0$, the contour must extend into the lower half plane since

$$\lim_{k_0 \to -i\infty} e^{-ik_0(x^0-y^0)} = 0$$



and we can ignore the contribution from the semicircle that closes the contour in the lower half plane and encloses the pole at E_k. Note too, this contour runs counterclockwise.

The Cauchy Integral Formula then gives

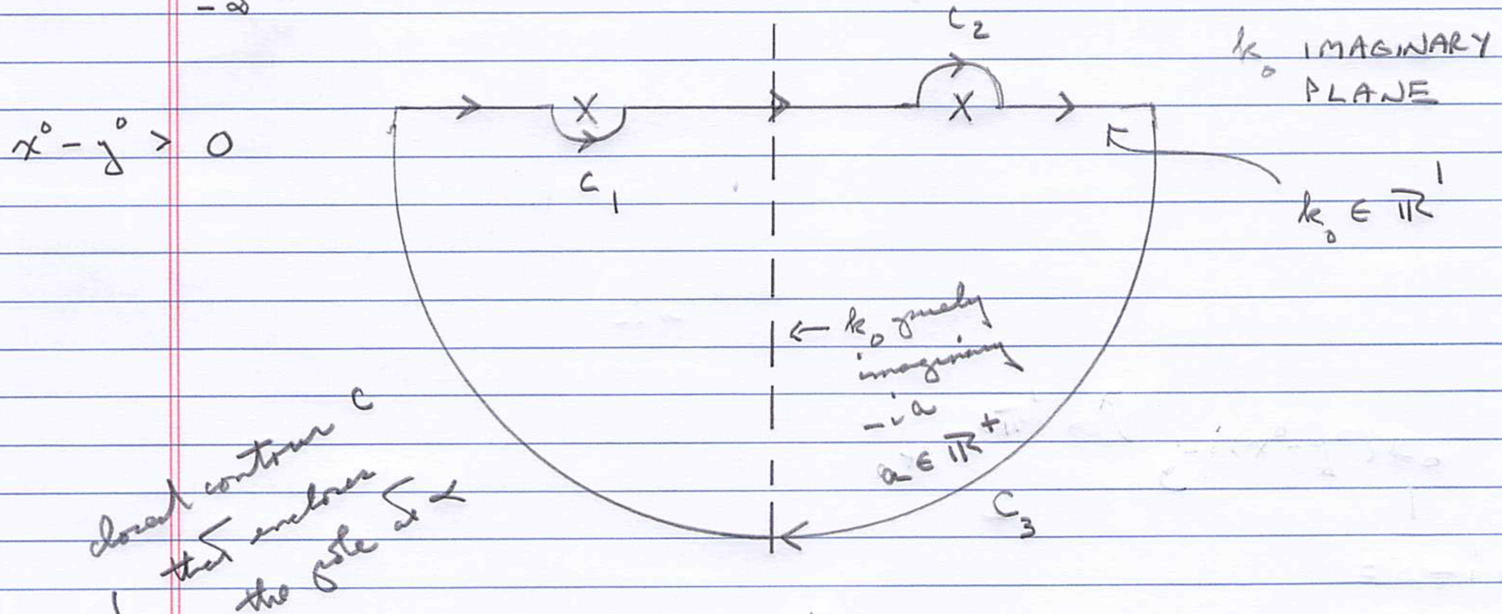
Cauchy's Integral Formula

This is an excellent approach to evaluating real integrals that have singularities in the integrand.

Example

$$\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{1}{(k_0 - E_k)(k_0 + E_k)} e^{-ik_0(x^0 - y^0)}$$

$x^0 - y^0 > 0$



$$\oint \frac{f(z)}{z - \alpha} dz = \pm 2\pi i\, f(\alpha)$$

$+$ counterclockwise around C

$-$ clockwise around C

$$\Rightarrow \underbrace{\int_{-\infty}^{+\infty} \frac{f(x)}{x - \alpha} dx}_{\text{as the } C_1 \text{ and } C_2 \text{ contours shrink to zero radius}} + \underbrace{\int_{C_3} \frac{f(z)}{z - \alpha} dz}_{\to 0 \text{ as the } C_3 \text{ contour is extended to } \infty} = \pm 2\pi i\, f(\alpha)$$

$$D_F(x-y) = i\int_{C_F} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}e^{-ik_\mu(x^\mu-y^\mu)}$$

$$= i\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\pi}\underbrace{\frac{-2\pi i}{2E_k}e^{-iE_k(x^0-y^0)}}_{-2\pi i\, f(E_k)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}e^{-ik\cdot(x-y)}$$

$$= D(x-y)$$

Similarly, for $y^0 > x^0$, we close the contour in the upper half plane, enclosing the pole at $-E_k$ and

$$D_F(x-y) = D(y-x)$$

Now consider

$$(\Box + m^2)D_F(x-y)$$

$$= i(\Box+m^2)\int_{C_F}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}e^{-ik\cdot(x-y)}$$

$$= i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}(\partial_\mu\partial^\mu+m^2)e^{-ik_\mu(x^\mu-y^\mu)}$$

$$= i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}(\partial_0\partial^0-\partial_i\partial^i+m^2)e^{-ik_0(x^0-y^0)}e^{-ik_i(x^i-y^i)}$$

$$= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\left[\eta^{00}(-ik_0)^2 - \eta^{ii}(-ik_i)^2 + m^2\right]e^{-ik\cdot(x-y)}$$

$$= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\left[-\underbrace{(k_0^2 + k_i^2)}_{k_0k^0 - k_ik^i = k_\mu k^\mu = k^2} + m^2\right]e^{-ik\cdot(x-y)}$$

$$= i\int \frac{d^4k}{(2\pi)^4}\frac{-k^2+m^2}{k^2-m^2}e^{-ik\cdot(x-y)}$$

$$= -i\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}$$

$$= -i\,\delta^{(4)}(x-y)$$

That is

$$(\Box + m^2)\,\Delta_F(x-y) = -i\,\delta^{(4)}(x-y)$$

so $\Delta_F(x-y)$ is the <u>Green's</u> <u>Function</u> for the Klein-Gordon equation.

$$c^2\underbrace{\Delta t}_{x^0-y^0}{}^2 - \underbrace{\Delta x}_{x^1-y^1}{}^2 - \Delta y^2 - \Delta z^2 < 0 \quad (1)$$

spatial distance is larger than $c\Delta t$

Proof that $\Delta(x-y) = 0$ for $(x-y)^2 < 0$

$$\Delta(x-y) = D(x-y) - D(y-x)$$

Let's compute $D(x-y)$ for $(x-y)^2 < 0$ – specifically for equal time (i.e., $x^0 = y^0$):

$$(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$$

Recall

$$D(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(\vec{k})} e^{-ik_\mu(x^\mu - y^\mu)}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(\vec{k})} e^{i\vec{k}\cdot\vec{r}}$$

where $\vec{r} = \vec{x} - \vec{y}$. Choosing spherical k coordinates

$$D(x-y) = \frac{1}{(2\pi)^3} \int k^2 dk \sin\theta_k \, d\theta_k \, d\phi_k \frac{1}{2E(\vec{k})} e^{ikr\cos\theta_k}$$

$$= \frac{1}{2}\frac{1}{(2\pi)^2} \int_0^\infty \int_{-1}^{+1} k^2 dk \, d\cos\theta_k \frac{1}{E(\vec{k})} e^{ikr\cos\theta_k}$$

$$= \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_0^\infty dk \frac{k^2}{[k^2+m^2]^{1/2}} \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$= \frac{-i}{2(2\pi)^2 r} \left\{ \int_0^\infty dk \frac{k}{[k^2+m^2]^{1/2}} e^{ikr} - \int_0^{\infty} dk \frac{k}{[k^2+m^2]^{1/2}} e^{-ikr} \right\}$$

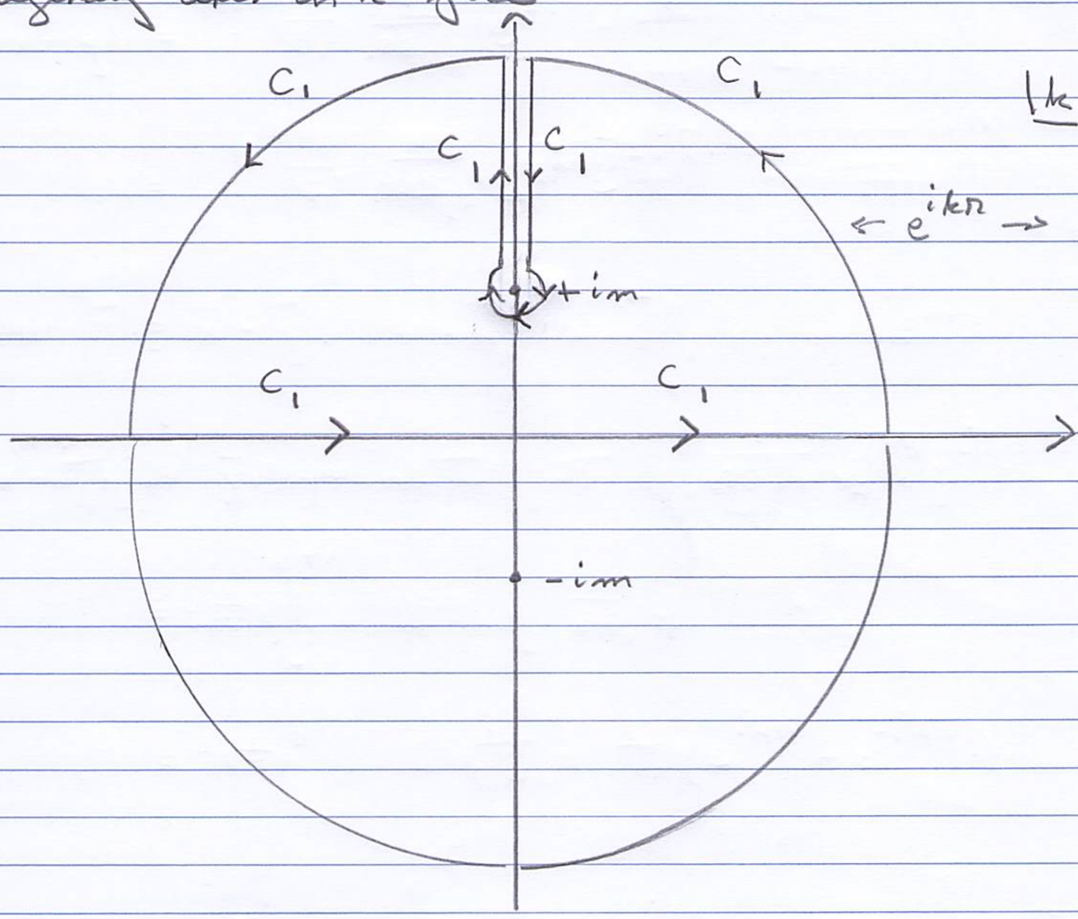
$$\int_0^\infty dk \frac{k}{k^2+m^2} e^{-ikr} = \int_0^{-\infty} d(-k) \frac{-k}{[(-k)^2+m^2]} e^{i(-k)r}$$

$$= -\int_{-\infty}^0 dk \frac{k}{k^2+m^2} e^{ikr}$$

Then

$$D(x-y) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dk \frac{k}{[k^2+m^2]^{1/2}} e^{ikr}$$

This integral possesses two cuts along the upper and lower imaginary axis in k-space



$\underline{Ik}$

$\leftarrow e^{ikr} \rightarrow e^{\mu r}$

$\mu \equiv ik < 0$

Because of the sign of the exponential (e^{ikr}), we use the upper contour to evaluate the integral (the lower contour will be

Branch Points / Branch Cuts

We are working in the complex plane and we are working with the function $\sqrt{z}$.

At $z = 0$, $\sqrt{z}$ is singular (it does not have a derivative).

$\Rightarrow$ $\sqrt{z}$ will be discontinuous across contours that encircle $z=0$.

Consider the unit circle around $z = 0$ in the complex z plane.

In general, z can be written as

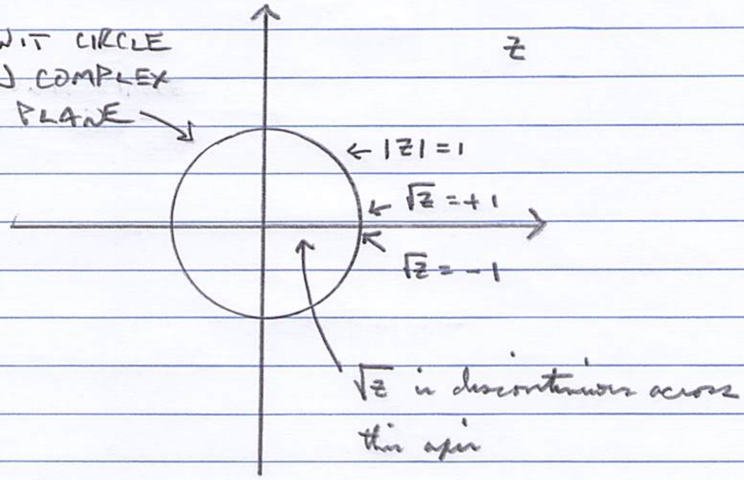
$$z = a + i b$$

or

$$= A e^{i\theta} \qquad a = A\cos\theta \quad b = A\sin\theta$$

On the unit circle, $A = 1$ and

$$\sqrt{z} = e^{i\theta/2}$$



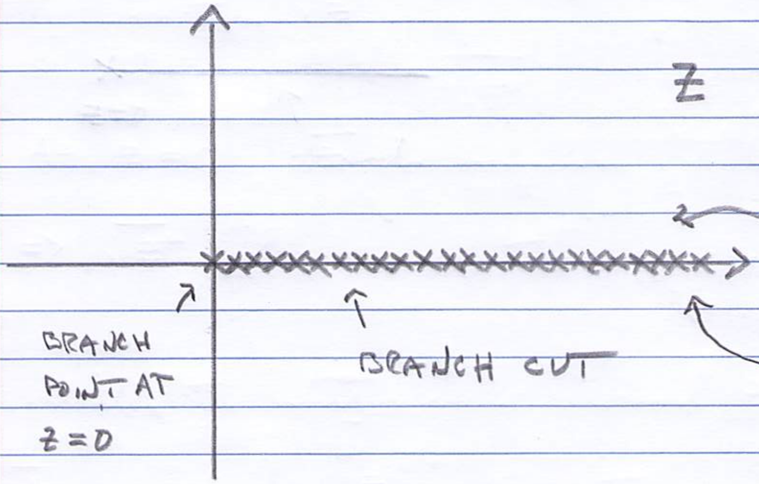
At $\theta = 0$,

$$\sqrt{z} = +1$$

At $\theta = 2\pi$

$$\sqrt{z} = -1$$

So, $\sqrt{z}$ is discontinuous across the $x > 0$ line. $z = 0$ is known as a <u>branch point</u>, and we introduce a <u>branch cut</u> to prevent our contour from crossing it



The values of z differ by $e^{i\pi} = -1$ on either side of the branch cut!

$$\sqrt{z}\Big|_{\theta = 2\pi} = -\sqrt{z}\Big|_{\theta = 0}$$

used for $D(y-x)$.

Cauchy's Theorem

If $f(z)$ is analytic on and inside C

$$\oint_C f(z)\, dz = 0$$

Applying Cauchy's Theorem to our case, this tells us that the integral along the real k axis must be opposite of the integral along the branch cut (the integral along the semicircle vanishes by virtue of the exponential).

The integral along the branch cut is

$$\int_{i\infty}^{im} dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr} - \int_{im}^{i\infty} dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr}$$

when passing from the right side of the branch cut to the left side of the branch cut, we pick up a factor of $e^{i\pi} = -1$

Transform coordinates

$$\mu \equiv i k$$

$$\int_{i\infty}^{im} dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr} \longrightarrow -i \int_{-\infty}^{-m} d\mu \frac{-i\mu}{\sqrt{-\mu^2+m^2}} e^{\mu r}$$

$$= -i \int_{-\infty}^{-m} d\mu \frac{\mu}{\sqrt{\mu^2-m^2}} e^{\mu r}$$

Transform coordinates again

$$y \equiv -\mu$$

Then

$$i\int_{-\infty}^{-m} d\mu \, \frac{\mu}{\sqrt{\mu^2-m^2}} \, e^{\mu r} \underset{y=-\mu}{=} -i\int_{\infty}^{m} dy \, \frac{-y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

$$= -i\int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

This leaves us with

$$\int_{i\infty}^{im} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} = -i\int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

Similarly

$$\int_{im}^{i\infty} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} \underset{\mu \equiv ik}{=} -i\int_{-m}^{-\infty} d\mu \, \frac{-i\mu}{\sqrt{-\mu^2+m^2}} \, e^{\mu r}$$

$$= +i\int_{-m}^{-\infty} d\mu \, \frac{\mu}{\sqrt{\mu^2-m^2}} \, e^{\mu r}$$

$$\underset{y=-\mu}{=} +i\int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

Pulling it all together, we have

$$D(x-y) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr}$$

$$= \frac{-i}{2(2\pi)^2 r} (-1)$$

$$\times \left\{ \int_{i\infty}^{im} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} - \int_{im}^{i\infty} dk \, \frac{k}{\sqrt{k^2+m^2}} \, e^{ikr} \right\}$$

$$= \frac{-i}{2(2\pi)^2 r} (-1)$$

$$\times \left\{ -i \int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr} - i \int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr} \right\}$$

$$= \frac{1}{4\pi^2 r} \int_{m}^{\infty} dy \, \frac{y}{\sqrt{y^2-m^2}} \, e^{-yr}$$

$$= \frac{1}{4\pi^2} \frac{m}{r} K_1(mr)$$

modified Bessel Function of the first kind

$$\underset{r\to\infty}{\longrightarrow} \frac{m}{r} e^{-mr}$$

HOMEWORK Performing the calculation for $D(y-x)$ yields the same answer, so

$$\Delta(x,y) = D(x-y) - D(y-x) = 0$$

for this case - i.e., equal times, $x^0 = y^0$. Lorentz invariance generalizes this to all $(x, y) \ni (x-y)^2 < 0$.