Continuum Limit of the Linear Chain

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September 8, 2020

$$\left.\begin{array}{l} a \to \mathrm{d}x \longrightarrow a \to 0 \\ \qquad\qquad\; N \to 0 \end{array}\right\} \; l = Na = \text{length of the chain} = \text{constant}$$

$$m \to \mathrm{d}m \longrightarrow m \to 0$$

$$\rho = \frac{m}{a} = \text{constant}$$

displacement of infinitesimal mass at x

$$q_n(t) \to q(x,t) \longrightarrow \underline{\underline{\text{ a scalar field! }}}$$

$$q_{n+1}(t) - q_n(t) \to dq = a\frac{\partial q}{\partial x}$$

$$\sum_{n=1}^{N} \to \frac{1}{a}\int_0^l \mathrm{d}x \qquad (\text{this just says } N = \frac{l}{a})$$

$$\sigma = \kappa a = \text{constant} = \text{string tension}$$

Then

$$\begin{aligned} L(q_n, \dot{q}_n) &= \sum_n \left[\frac{m}{2}(\dot{q}_n)^2 - \frac{\kappa}{2}(q_{n+1} - q_n)^2\right] \\ &\to \frac{1}{a}\int_0^l \mathrm{d}x \; \frac{1}{2}\left[\rho a\left(\frac{\partial q}{\partial t}\right)^2 - \frac{\sigma}{a}a^2\left(\frac{\partial q}{\partial x}\right)^2\right] \\ &= \frac{1}{2}\int_0^l \mathrm{d}x \; \left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \sigma\left(\frac{\partial q}{\partial x}\right)^2\right] \end{aligned}$$

and

"Lagrangian density" $$\mathscr{L}(\dot{q}, q') = \frac{1}{2}\left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \sigma\left(\frac{\partial q}{\partial x}\right)^2\right] \qquad \boxed{q' \equiv \frac{\partial q}{\partial x}}$$

and

$$S = \int_{t_i}^{t_f} L \; \mathrm{d}t = \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \mathscr{L}$$

The EOM are determined by extremizing the action.

$$
\begin{aligned}
\delta S &= \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \delta\mathscr{L} \\
&= \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathscr{L}}{\partial q'} \delta q' \right) \\
&= \int_0^l \mathrm{d}x \int_{t_i}^{t_f} \left[\frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \delta q \right] \mathrm{d}t \\
&\quad + \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \left[\frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \right) \delta q \right] \; \mathrm{d}x \\
&= -\int_0^l \mathrm{d}x \int_{t_i}^{t_f} \mathrm{d}t \; \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \delta q \\
&\quad - \int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \right) \delta q \\
&= -\int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \left[\frac{\partial}{\partial t} \left(\rho \frac{\partial q}{\partial t} \right) + \frac{\partial}{\partial x} \left(-\sigma \frac{\partial q}{\partial x} \right) \right] \delta q \\
&= -\int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \left(\rho \frac{\partial^2 q}{\partial t^2} - \sigma \frac{\partial^2 q}{\partial x^2} \right) \delta q \\
&= 0
\end{aligned}
$$

now $\delta q = \delta q(x,t)$
before $\delta q_n = \delta q_n(t)$
$\Rightarrow (\quad)$ must be 0

Then the EOM for $q(x,t)$ is

$$\frac{\partial^2 q}{\partial t^2} - \frac{\sigma}{\rho} \frac{\partial^2 q}{\partial x^2} = 0$$

which is a wave equation, with wave speed

$$c = \sqrt{\frac{\sigma}{\rho}}$$

$$\int_0^l \mathrm{d}x \int_{t_i}^{t_f} \mathrm{d}t \; \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right) = \int_0^l \mathrm{d}x \left[\left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right)_{t_f} - \left(\frac{\partial \mathscr{L}}{\partial \dot{q}} \delta q \right)_{t_i} \right] = 0$$

since

$$\delta q(x, t_f) = 0 \text{ for all } x \in [0, l]$$
$$\delta q(x, t_i) = 0 \text{ for all } x \in [0, l]$$

$$\int_{t_i}^{t_f} \mathrm{d}t \int_0^l \mathrm{d}x \; \frac{\partial}{\partial x} \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right) = \int_{t_i}^{t_f} \mathrm{d}t \left[\left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_l - \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_0 \right] = 0$$

since

$$\left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_l = \left(\frac{\partial \mathscr{L}}{\partial q'} \delta q \right)_0$$

by periodicity.

Classical Field Theory

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September 8 and 10, 2020

We can extend the Lagrangian formulation of classical dynamics for point particles to fields.

The correspondence is

$$
\begin{aligned}
q(t) &\longrightarrow \phi(\mathbf{x}) \qquad \mathbf{x} = (\vec{x}, t) \\
\dot{q}(t) &\longrightarrow \partial_\mu \phi(\mathbf{x})
\end{aligned}
$$

We define a Lagrangian density related to the Lagrangian and Action by

$$
S = \int \mathrm{d}t \; L = \int \mathrm{d}^4\mathbf{x} \; \underbrace{\mathscr{L}(\phi, \partial_\mu \phi)}_{\text{LAGRANGIAN DENSITY}}
$$

Compute δS:

$$
\begin{aligned}
\delta S &= \int \mathrm{d}^4\mathbf{x} \; \left[\frac{\partial \mathscr{L}}{\partial \phi} \; \delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \; \delta(\partial_\mu \phi) \right] \\
&= \int \mathrm{d}^4\mathbf{x} \; \left[\frac{\partial \mathscr{L}}{\partial \phi} \; \delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \; \partial_\mu(\delta\phi) \right] \\
&= \left\{ \int \mathrm{d}^4\mathbf{x} \; \frac{\partial \mathscr{L}}{\partial \phi} \; \delta\phi + \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \; \delta\phi \Big|_{-\infty}^{+\infty} \right) - \int \mathrm{d}^4\mathbf{x} \; \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right\} \\
&= \int \mathrm{d}^4\mathbf{x} \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi
\end{aligned}
$$

Now instead of using $\delta q(t_i) = \delta q(t_f) = 0$, we simply assume that our fields are well-behaved functions of spacetime – i.e., that they go to zero at $x = \pm\infty$.

The solutions render the action on extremum ($\delta S = 0$) and satisfy the EOM

$$
\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = 0
$$

One can now define the momentum conjugate to ϕ

$$
\pi(\mathbf{x}) \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi(\mathbf{x}))}
$$

and the Hamiltonian density

$$\mathscr{H}(\mathbf{x}) \equiv \pi(\mathbf{x})\ \partial_0 \phi(\mathbf{x}) - \mathscr{L}$$

which is related to the Hamiltonian by

$$H = \int \mathrm{d}^3\mathbf{x}\ \mathscr{H}$$

As an example, consider the following Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2} m^2 \phi^2$$

Then

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \phi} &= -m^2 \phi \\
\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} &= \frac{1}{2} \frac{\partial}{\partial(\partial_\mu \phi)} \left[\eta^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi)\right] \\
&= \frac{1}{2}\eta^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu \phi)} \left[(\partial_\alpha \phi)(\partial_\beta \phi)\right] \\
&= \frac{1}{2}\eta^{\alpha\beta} \left[\frac{\partial}{\partial(\partial_\mu \phi)}(\partial_\alpha \phi)\right](\partial_\beta \phi) + \frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha \phi)\left[\frac{\partial}{\partial(\partial_\mu \phi)}(\partial_\beta \phi)\right] \\
&= \frac{1}{2}\eta^{\alpha\beta}\delta^\mu{}_\alpha\ \partial_\beta \phi + \frac{1}{2}\eta^{\alpha\beta}\ \partial_\alpha \phi\ \delta^\mu{}_\beta \\
&= \frac{1}{2}\partial^\mu \phi + \frac{1}{2}\partial^\mu \phi \\
&= \partial^\mu \phi
\end{aligned}$$

and

$$\begin{aligned}
\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right) &= \partial_\mu (\partial^\mu \phi) \\
&= \Box\ \phi
\end{aligned}$$

The EOM then read

$$(\Box + m^2)\ \phi = 0$$

which is the Klein-Gordon equation.