The Big Picture: In Equations

We began in classical field theory by constructing Lorentz invariant Lagrangian densities from which the classical field EOM could be derived.

For our real, scalar field (spin 0)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^2(x)$$

The Euler-Lagrange EOM

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

then gives

$$(\Box + m^2)\phi(x) = 0$$

whose solution is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\left[a(\vec{k})e^{-ik\cdot x} + a^*(\vec{k})e^{ik\cdot x}\right]$$

We demonstrated that quantum mechanical operators obey the classical EOM. Beginning with the Heisenberg EOM for a quantum mechanical operator $\hat{O}$

$$\partial_t \hat{O} = i[\hat{H}, \hat{O}]$$

we showed, for example, that

$$\partial_t \hat{q} = i[\hat{H}, \hat{q}] = \frac{\partial \hat{H}}{\partial p}$$

$$\partial_t \hat{p} = i[\hat{H}, \hat{p}] = -\frac{\partial \hat{H}}{\partial q}$$

which mirror the classical EOM for q and p.

Given this, our path to constructing quantum field theories is

1. Construct a classical Lorentz invariant Lagrangian density.
2. Derive the Euler-Lagrangian classical field equations.
3. Solve the field equations.
4. Quantize the classical field solution.

Define our quantized scalar field as

$$\hat{\phi}(x) \equiv \int \frac{d^3k}{(2\pi)^3} \left[\hat{a}(\vec{k}) e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k}) e^{ik\cdot x} \right]$$

where $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ are operators whose meaning will become clear later.

The classical canonical momentum is

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = -i \int \frac{d^3k}{(2\pi)^3} E(\vec{k}) \left[a(\vec{k}) e^{-ik\cdot x} - a^*(\vec{k}) e^{ik\cdot x} \right]$$

Then

$$\hat{\Pi}(x) \equiv -i \int \frac{d^3k}{(2\pi)^3} E(\vec{k}) \left[\hat{a}(\vec{k}) e^{-ik\cdot x} - \hat{a}^\dagger(\vec{k}) e^{ik\cdot x} \right]$$

We impose the canonical commutation relations

$$[\phi(\vec{x},t), \Pi(\vec{x}',t)] = (2\pi)^3 \delta^{(3)}(\vec{x}-\vec{x}')$$

which in turn demands that

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = 0$$

$$[\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = 0$$

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$$

which are the commutators for the creation and annihilation operators of the harmonic oscillator. In this case, each mode k is an oscillator.

Using these commutators and the Hamiltonian operator, we are able to prove that $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ are creation and annihilation operators for quanta in mode $\vec{k}$.

Causality demands

$$[\phi(x), \phi(y)] = 0 \qquad (x-y)^2 < 0$$

Our Hamiltonian is

FIELD VIEW

$$\hat{H} = \int d^3x\, \hat{\mathcal{H}}(x) = \int d^3x \left(\tfrac{1}{2}\hat{\pi}^2 + \tfrac{1}{2}|\vec{\nabla}\hat{\phi}|^2 + \tfrac{1}{2}m^2\hat{\phi}^2 \right)$$

PARTICLE VIEW

$$= \int d^3k\, E(\vec{k})\, \hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{k}) \leftarrow \text{actually } :H:$$

We make the connection to the field quanta (particles) by focusing on the modes of the field.

Our number operator is

$$\hat{N} = \int d^3k\, \hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{k})$$

Then

$$\hat{N}\,|n_k\rangle = n_k\,|n_k\rangle$$

$$\hat{H}\,|n_k\rangle = n_k\,\hbar\omega_k\,|n_k\rangle$$

Our field operator $\hat{\phi}(x)$ satisfies the Heisenberg EOM

$$\partial_t \hat{\phi} = i[\hat{H}, \hat{\phi}]$$

We proved that $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ are annihilation and creation operators, respectively, by using $\hat{H}$ together with $[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')]$ to show that

$$\hat{H}\,\hat{a}(\vec{k})\,|n_k\rangle = (n_k - 1)\hbar\omega_k\,|n_k - 1\rangle$$

$$\hat{H}\,\hat{a}^{\dagger}(\vec{k})\,|n_k\rangle = (n_k + 1)\hbar\omega_k\,|n_k + 1\rangle$$

That is, that

$$\hat{a}(\vec{k})\,|n_k\rangle \propto |n_k - 1\rangle$$

$$\hat{a}^{\dagger}(\vec{k})\,|n_k\rangle \propto |n_k + 1\rangle$$

So far, we have focused on the free scalar field $\hat{\phi}(x)$.

QFT was developed to treat particle decays and scattering, where the number of particles of a particular type can change and where particles interact.

Interacting particles implies interacting fields.

Here it is important to emphasize that we are always doing quantum mechanics.

In particle decays and scattering, we start with some initial state and we end up with some final state and we want to know the probability for such a transition to occur. To speak to this, we have to discuss how the initial state evolves.

In a given frame of reference, the evolution of a state vector in our Hilbert space in the Schroedinger picture is given by

$$i \frac{d}{dt} |\psi\rangle_S = \hat{H} |\psi\rangle_S \qquad \hat{H} = \hat{H}_0 + \hat{H}_{INT}$$

whose solution is

$$|\psi\rangle_S(t) = e^{-i\hat{H}(t-t_0)} |\psi\rangle_S$$

Since $\hat{H} = \hat{H}^\dagger$, this defines a unitary operator

$$U(t, t_0) = e^{-i\hat{H}(t-t_0)}$$

that transforms $|\psi\rangle_S(t_0)$ to $|\psi\rangle_S(t)$ in our Hilbert space.

The fact that

$$U^\dagger U = 1$$

preserves probabilities.

In the above Schroedinger picture, the state vector $|\psi\rangle_S(t)$ depends on time whereas the Hamiltonian operator does not.

In the Heisenberg representation, the reverse is true. The state vector and operator in the two pictures are related by

$$|\psi\rangle_S(t) = e^{-i\hat{H}t} |\psi\rangle_H$$

$|\psi\rangle_H$ has no time dependence.

And

$$\hat{O}_H = e^{i\hat{H}t}\,\hat{O}_S\,e^{-i\hat{H}t}$$

We will adopt the Interaction Picture. In this picture

$$|\psi\rangle_I(t) = e^{-i\hat{H}_{INT}t}\,|\psi\rangle_H$$

That is, the interaction picture state vectors evolve according to the interaction Hamiltonian. If $\hat{H}_{INT} = 0$, $|\psi\rangle_I = |\psi\rangle_H$.

We also have

$$\hat{O}_I = e^{i\hat{H}_0 t}\,\hat{O}_S\,e^{-i\hat{H}_0 t}$$

This is important. Operators – e.g., our field operators – evolve as they would in a free (interaction-free) theory.

The time evolution of our state vector in the interaction picture, $|\psi\rangle_I$, is given in our frame of reference by

$$i\frac{d}{dt}|\psi\rangle_I = \hat{H}_I\,|\psi\rangle_I$$

whose solution is

$$|\psi\rangle_I(t) = T\left\{e^{-i\int_{t_0}^{t}\hat{H}_I(t')\,dt'}\right\}|\psi\rangle_I(t_0) \equiv \hat{U}(t,t_0)\,|\psi\rangle_I(t_0)$$

The S-Matrix is defined as

$$S_{fi} \equiv {}_S\langle f, +\infty | i, -\infty \rangle_S$$

The S-Matrix Operator, $\hat{S}$, is defined as follows

$$S_{fi} = {}_S\langle f, +\infty | i, -\infty \rangle_S$$

$$= {}_H\langle f | \hat{U}(+\infty, -\infty) | i \rangle_H$$

$$\equiv {}_H\langle f | \hat{S} | i \rangle_H$$

All of our experiments will be set up such that

$$H_I(t) \xrightarrow[t \to \pm\infty]{} 0$$

At these early and late times, the interaction and Heisenberg pictures coincide.

So,

$$\hat{S} = T\left\{ e^{-i\int_{-\infty}^{\infty} H_I(t')\,dt'} \right\}$$

$$= T\left\{ e^{-i\int d^4x\, \mathcal{H}(x)} \right\}$$

$$= T\left\{ e^{i\int d^4x\, \mathcal{L}_I(x)} \right\}$$

The last equality expresses the S-Matrix Operator in a manifestly Lorentz invariant way.

1. Our Lagrangian density $\mathcal{L}_I(x)$ is constructed to be Lorentz invariant.

2. For $(x-y)^2 > 0$, time ordering is Lorentz invariant.

3. For $(x-y)^2 < 0$, time ordering is not Lorentz invariant, but

$$[\mathcal{L}_I(x), \mathcal{L}_I(y)] = 0$$

Now let's talk about Lorentz invariance with respect to our actual experiments - i.e., w.r.t. our probabilities.

$|i\rangle$ and $|f\rangle$ are defined in a particular frame of reference.

For example, I can experimentally set

$|i\rangle = |\vec{k}\rangle$

- i.e., I create a particle with momentum $\vec{k}$ <u>in</u> <u>this</u> frame of reference.

Suppose another observer is observing the experiment. In the second observer's frame of reference

$|i\rangle' = |\vec{k}'\rangle = |\Lambda\vec{k}\rangle$

where $\Lambda \vec{k}$ is the Lorentz transformed $\vec{k}$.

We require that there be a unitary operator, $\hat{U}(\Lambda, a)$, such that

$$|i\rangle' = \hat{U}(\Lambda, a)\,|i\rangle$$

$$|f\rangle' = \hat{U}(\Lambda, a)\,|f\rangle$$

so that

$${}_S\langle f, \infty | i, -\infty\rangle_S = {}_S'\langle f, \infty | \underbrace{\hat{U}^\dagger(\Lambda, a)\,\hat{U}(\Lambda, a)}_{= \hat{1} \text{ by unitarity}} | i, -\infty\rangle'$$

$$= {}_S'\langle f, \infty | i, -\infty\rangle_S'$$

That is, the probability amplitudes must agree.

This also tells us that the S-Matrix Operator must be Lorentz invariant:

$$S_{fi} \equiv {}_S\langle f, \infty | i, -\infty\rangle_S$$

$$= {}_H\langle f | \hat{S} | i\rangle_H$$

$$= {}_H'\langle f | \hat{U}^\dagger(\Lambda, a)\,\hat{S}\,\hat{U}(\Lambda, a) | i\rangle_H'$$

$$= {}_H'\langle f | \hat{S} | i \rangle'_H$$

$$= {}_S'\langle f, \infty | i, -\infty \rangle'_S$$

$$= S_{f'i'}$$

where we have used the fact that, by construction,

$$\hat{U}^\dagger(\Lambda, a)\, \hat{S}\, \hat{U}(\Lambda, a) = \hat{S}$$

For $|f\rangle \neq |i\rangle$,

$${}_H\langle f | \hat{S} | i \rangle_H \equiv i \mathcal{M} (2\pi)^4 \delta^{(4)}(\Sigma p_i^\mu - p_f^\mu)$$

This is a <u>definition</u> of the Lorentz invariant <u>amplitude</u> $\mathcal{M}$.

Once $\mathcal{M}$ is known, we can compute particle decay rates and differential scattering cross sections as

$$d\Gamma = \frac{1}{2E} |\mathcal{M}|^2 \, d\Pi_{LIPS}$$

for

$$p \rightarrow \{p_j\}$$

where

Γ is defined to be the decay width and the particle's half life is defined as

$$\tilde{\tau}_{1/2} \equiv \frac{1}{\Gamma}$$

and

$$d\sigma = \frac{1}{4}\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{-1/2} |\mathcal{M}|^2 \, d\Pi_{L,PS}$$

for

$$p_1 + p_2 \rightarrow \{ p_j \}$$

It is customary to express $d\sigma$ in Lorentz invariant form.

We can evaluate $d\sigma$ in the COM frame. We can then integrate out all of the final state dependence except, for example, the scattering angle for one of the particles, to obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{COM}$$

This last quantity is frame dependent even though we started with the Lorentz invariant $d\sigma$.