Second and Higher Order in Perturbation Theory

In general, we need to compute matrix elements of the form

$$\langle f | T[\mathcal{H}_I(x_1) \ldots \mathcal{H}_I(x_n)] | i \rangle$$

We will make use of <u>Wick's</u> <u>Theorem</u>.

Wick's Theorem tells us how to convert time-ordered products of operators to normal ordered products of operators.

Consider our scalar field operator

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\phi^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot x} \hat{a}(\vec{k})$$

$$\phi^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x} \hat{a}^\dagger(\vec{k})$$

Let $x^0 > y^0$. Then

$$T[\phi(x)\phi(y)] = \phi(x)\phi(y)$$
$$= [\phi^{(+)}(x) + \phi^{(-)}(x)][\phi^{(+)}(y) + \phi^{(-)}(y)]$$

$$= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y)$$
$$+ \phi^{(-)}(x)\phi^{+}(y) + \phi^{(-)}(x)\phi^{(-)}(y)$$

$$= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y)$$
$$+ \phi^{(-)}(y)\phi^{(+)}(x)$$
$$- \phi^{(-)}(y)\phi^{(+)} x$$
$$+ \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y)$$

$$= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$
$$+ \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y)$$
$$+ [\phi^{(+)}(x), \phi^{(-)}(y)]$$

$$= :\phi(x)\phi(y): + [\phi^{(+)}(x), \phi^{(-)}(y)]$$

But

$$[\phi^{(+)}(x), \phi^{(-)}(y)] = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\frac{1}{\sqrt{2E(\vec{k}')}} e^{-ik\cdot x} e^{ik'\cdot y}$$
$$\times [\hat{a}(\vec{k}), \hat{a}^{\dagger}(\vec{k}')]$$
$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\frac{1}{\sqrt{2E(\vec{k}')}} e^{-ik\cdot x} e^{ik'\cdot y}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')$$
$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(\vec{k})} e^{-ik\cdot(x-y)}$$

$$= \Delta(x-y)$$

That is

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + \Delta(x-y)$$

For $y^0 > x^0$, we get

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + \Delta(y-x)$$

All of this can be succinctly written

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + \Delta_F(x-y)$$

Now let's generalize this to 4 fields (to see how the pattern goes).

$$T[\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)]$$

$$= :\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) + \overbrace{\phi_1(x_1)\phi_2(x_2)}\phi_3(x_3)\phi_4(x_4)$$

$$+ \overbrace{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)}\phi_4(x_4) + \overbrace{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)}$$

$$+ \phi_1(x_1)\overbrace{\phi_2(x_2)\phi_3(x_3)}\phi_4(x_4) + \ldots$$

$$+ \overbrace{\phi_1(x_1)\phi_2(x_2)}\overbrace{\phi_3(x_3)\phi_4(x_4)} + \overbrace{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)}\phi_4(x_4) + \ldots$$

$$= :\phi_1\phi_2\phi_3\phi_4: + D_F(x_1 - x_2) :\phi_3\phi_4: + \ldots$$

$$+ D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4)$$

$$+ D_F(x_1 - x_4) D_F(x_2 - x_3)$$

When taking

$$\langle 0 | T[\phi_1\phi_2\phi_3\phi_4] | 0 \rangle$$

only the fully contracted terms survive !

With this machinery, let's compute nucleon-nucleon scattering in scalar Yukawa theory.

① $\psi\psi \to \psi\psi$

$$|i\rangle = \sqrt{2E(\vec{p}_1)}\sqrt{2E(\vec{p}_2)}\, \hat{a}^\dagger(\vec{p}_1)\, \hat{a}^\dagger(\vec{p}_2)\, |0\rangle$$

$$|f\rangle = \sqrt{2E(\vec{q}_1)}\sqrt{2E(\vec{q}_2)}\, \hat{a}^\dagger(\vec{q}_1)\, \hat{a}^\dagger(\vec{q}_2)\, |0\rangle$$

The first order that contributes to scattering is second order, and the matrix element is

$$\langle f | \frac{(-ig)^2}{2} \int d^4x\, d^4y\, T\left[\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y)\right] | i \rangle$$

Let's use Wick's Theorem to evaluate the time-ordered product.

$$T\left[\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y)\right]$$

$$= \,:\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y)$$

$$+ \overline{\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)}\phi(y)$$

$$+ \psi^\dagger(x)\overline{\psi(x)\phi(x)\psi^\dagger(y)}\psi(y)\phi(y)$$

$$+ \psi^\dagger(x)\psi(x)\overline{\phi(x)\psi^\dagger(y)\psi(y)\phi(y)}:$$

$$= \,:\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y):$$

$$+ \Delta_F^\psi(y-x)\,:\psi(x)\phi(x)\psi^\dagger(y)\phi(y):$$

$$+ \Delta_F^\psi(x-y)\,:\psi^\dagger(x)\phi(x)\psi(y)\phi(y):$$

all 0 because there is no pion in the initial or final state

$$+ \Delta_F^\phi(x-y)\,:\psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y):$$

So, we are left to compute

$$\frac{(-ig)^2}{2}\int d^4x\, d^4y\, \Delta_F^\phi(x-y)\langle f|:\psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y):|i\rangle$$

$$= \ldots$$

$$= \frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^{\phi}(x-y) \sqrt{2E(\vec{q}_1)} \sqrt{2E(\vec{q}_2)} \sqrt{2E(\vec{\gamma}_1)} \sqrt{2E(\vec{\gamma}_2)}$$

$$\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \frac{d^3k_4}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k}_1)}} \frac{1}{\sqrt{2E(\vec{k}_2)}} \frac{1}{\sqrt{2E(\vec{k}_3)}} \frac{1}{\sqrt{2E(\vec{k}_4)}}$$

$$\langle 0 | \hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2) : \left[e^{ik_1\cdot x} \hat{b}^\dagger(\vec{k}_1) + e^{-ik_1\cdot x} \hat{a}(\vec{k}_1) \right]$$

$$\times \left[e^{ik_2\cdot y} \hat{b}^\dagger(\vec{k}_2) + e^{-ik_2\cdot y} \hat{a}(\vec{k}_2) \right]$$

$$\times \left[e^{ik_3\cdot y} \hat{a}^\dagger(\vec{k}_3) + e^{-ik_3\cdot y} \hat{b}(\vec{k}_3) \right]$$

$$\times \left[e^{ik_4\cdot y} \hat{a}^\dagger(\vec{k}_4) + e^{-ik_4\cdot y} \hat{b}(\vec{k}_4) \right] : \hat{a}^\dagger(\vec{\gamma}_1)\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle$$

$\hat{b}^\dagger(\vec{k}_1)$ and $\hat{b}^\dagger(\vec{k}_2)$ will annihilate to the left. They commute with $\hat{a}(\vec{q}_1)$ and $\hat{a}(\vec{q}_2)$. Then

$$\langle 0 | \hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{b}^\dagger(\vec{k}_1) = 0$$

and in turn

$$\langle 0 | \hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{b}^\dagger(\vec{k}_2) = 0$$

Similarly $\hat{b}(\vec{k}_3)$ and $\hat{b}(\vec{k}_4)$ annihilate to the right and and commute with $\hat{a}^\dagger(\vec{\gamma}_1)$ and $\hat{a}^\dagger(\vec{\gamma}_2)$. Then

$$\hat{b}(\vec{k}_4)\hat{a}^\dagger(\vec{\gamma}_1)\hat{a}^\dagger(\vec{\gamma}_2)|0\rangle = 0$$

and in turn

$$\hat{b}(\vec{k}_3)\,\hat{a}^\dagger(\vec{r}_1)\,\hat{a}^\dagger(\vec{r}_2)\,|0\rangle = 0$$

Therefore, the only term that survives in the normal-ordered product above is

$$\langle 0|\,\hat{a}(\vec{q}_1)\,\hat{a}(\vec{q}_2) : \hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{k}_3)\,\hat{a}^\dagger(\vec{k}_4) : \hat{a}^\dagger(\vec{r}_1)\,\hat{a}^\dagger(\vec{r}_2)\,|0\rangle$$

$$= \langle 0|\,\hat{a}(\vec{q}_1)\,\hat{a}(\vec{q}_2)\,\hat{a}^\dagger(\vec{k}_3)\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{r}_1)\,\hat{a}^\dagger(\vec{r}_2)\,|0\rangle$$

$$= \langle 0|\,\hat{a}(\vec{q}_1)\,\hat{a}^\dagger(\vec{k}_3)\,\hat{a}(\vec{q}_2)\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{r}_1)\,\hat{a}^\dagger(\vec{r}_2)\,|0\rangle$$

$$+ \langle 0|\,\hat{a}(\vec{q}_1)\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{r}_1)\,\hat{a}^\dagger(\vec{r}_2)\,|0\rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_2)$$

$$= \ldots$$

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The end result is:

$$(2\pi)^{12}\Big[\underbrace{\delta^{(3)}(\vec{q}_1 - \vec{k}_3)\,\delta^{(3)}(\vec{q}_2 - \vec{k}_4)}_{A} + \underbrace{\delta^{(3)}(\vec{q}_1 - \vec{k}_4)\,\delta^{(3)}(\vec{q}_2 - \vec{k}_3)}_{B}\Big]$$

$$\times \Big[\underbrace{\delta^{(3)}(\vec{r}_1 - \vec{k}_1)\,\delta^{(3)}(\vec{r}_2 - \vec{k}_2)}_{C} + \underbrace{\delta^{(3)}(\vec{r}_1 - \vec{k}_2)\,\delta^{(3)}(\vec{r}_2 - \vec{k}_1)}_{D}\Big]$$

Then, we have

$$\frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^\phi(x-y)\, \sqrt{2E(\vec{q}_1)}\,\sqrt{2E(\vec{q}_2)}\,\sqrt{2E(\vec{r}_1)}\,\sqrt{2E(\vec{r}_2)}$$

$$\times \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3}\,\frac{1}{\sqrt{2E(\vec{k}_1)}}\,\frac{1}{\sqrt{2E(\vec{k}_2)}}\,\frac{1}{\sqrt{2E(\vec{k}_3)}}\,\frac{1}{\sqrt{2E(\vec{k}_4)}}$$

$$\times\, e^{-ik_1\cdot x}\, e^{-ik_2\cdot y}\, e^{ik_3\cdot y}\, e^{ik_4\cdot y}$$

$$= \langle 0|\hat{a}^\dagger(\vec{k}_3)\,\hat{a}(\vec{q}_{\gamma_1})\,\hat{a}(\vec{q}_{\gamma_2})\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle$$

$$① \quad + \langle 0|\hat{a}(\vec{q}_{\gamma_2})\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})$$

$$+ \langle 0|\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{q}_{\gamma_1})\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_2})$$

$$② \quad + \langle 0|\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_1})$$

$$① \quad = \langle 0|\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{q}_{\gamma_2})\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})$$

$$+ \langle 0|\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})$$

$$= \langle 0|\hat{a}(\vec{k}_1)\,\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})$$

$$+ \langle 0|\hat{a}(\vec{k}_1)\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_1})$$

$$= \langle 0|\hat{a}^\dagger(\vec{q}_{\gamma_1})\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})$$

$$+ \langle 0|\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{q}_{\gamma_2})|0\rangle(2\pi)^3(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_1})$$

$$+ \langle 0|\hat{a}^\dagger(\vec{q}_{\gamma_2})\,\hat{a}(\vec{k}_1)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_1})$$

$$+ (2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_2})$$

$$= \langle 0|\hat{a}^\dagger(\vec{q}_{\gamma_2})\,\hat{a}(\vec{k}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_1})$$

$$+ (2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_1})$$

$$+ (2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_2})$$

$$Ⓐ \times Ⓒ = (2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_1})$$

$$Ⓐ \times Ⓓ \; + (2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_{\gamma_2})(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_{\gamma_1})(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_{\gamma_2})$$

$$② \quad \langle 0|\hat{a}(\vec{k}_1)\hat{a}^{\dagger}(\vec{q}_1)\hat{a}(\vec{k}_2)\hat{a}^{\dagger}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ \langle 0|\hat{a}(\vec{k}_1)\hat{a}^{\dagger}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$=\ \langle 0|\hat{a}^{\dagger}(\vec{q}_1)\hat{a}(\vec{k}_1)\hat{a}(\vec{k}_2)\hat{a}^{\dagger}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ \langle 0|\hat{a}(\vec{k}_2)\hat{a}^{\dagger}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ \langle 0|\hat{a}^{\dagger}(\vec{q}_2)\hat{a}(\vec{k}_1)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$+\ (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$=\ \langle 0|\hat{a}^{\dagger}(\vec{q}_2)\hat{a}(\vec{k}_2)|0)(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ (2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(2)}(\vec{k}_4-\vec{q}_1)$$

$$+\ (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$Ⓑ \times Ⓒ = (2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$Ⓑ \times Ⓓ + (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$\times (2\pi)^{12} \left[\delta^{(3)}(\vec{q}_1 - \vec{k}_3)\delta^{(3)}(\vec{q}_2 - \vec{k}_4) + \delta^{(3)}(\vec{q}_1 - \vec{k}_4)\delta^{(3)}(\vec{q}_2 - \vec{k}_3) \right]$$

$$\times \left[\delta^{(3)}(\vec{p}_1 - \vec{k}_1)\delta^{(3)}(\vec{p}_2 - \vec{k}_2) + \delta^{(3)}(\vec{p}_1 - \vec{k}_2)\delta^{(3)}(\vec{p}_2 - \vec{k}_1) \right]$$

$$= \frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^\phi(x-y)$$

$$\times \left\{ e^{-ip_1\cdot x} e^{-ip_2\cdot y} e^{iq_1\cdot x} e^{iq_2\cdot y} \right.$$

$$+ e^{-ip_2\cdot x} e^{-ip_1\cdot y} e^{iq_1\cdot x} e^{iq_2\cdot y}$$

$$+ e^{-ip_1\cdot x} e^{-ip_2\cdot y} e^{iq_2\cdot x} e^{iq_1\cdot y}$$

$$\left. + e^{-ip_2\cdot x} e^{-ip_1\cdot y} e^{iq_2\cdot x} e^{iq_1\cdot y} \right\}$$

$$= \frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^\phi(x-y)$$

$$\times \left\{ e^{-i(p_1-q_1)\cdot x} e^{-i(p_2-q_2)\cdot y} \right.$$

$$+ e^{-i(p_2-q_1)\cdot x} e^{-i(p_1-q_2)\cdot y}$$

$$+ e^{-i(p_1-q_2)\cdot x} e^{-i(p_2-q_1)\cdot y}$$

$$\left. + e^{-i(p_2-q_2)\cdot x} e^{-i(p_1-q_1)\cdot y} \right\}$$

$$= (-ig)^2 \int d^4x\, d^4y\, \Delta_F^\phi(x-y) \left\{ e^{-i(p_1-q_1)\cdot x} e^{-i(p_2-q_2)\cdot y} \right.$$

$$\left. + e^{-i(p_2-q_1)\cdot x} e^{-i(p_1-q_2)\cdot y} \right\}$$

The last equality is due to the fact that

$$D_F^{\phi}(x-y) = D_F^{\phi}(y-x)$$

Inserting

$$D_F^{\phi}(x-y) = i\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$$

we now have

$$(-ig)^2 \, i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon}$$

$$\times \int d^4x \, d^4y \, e^{-ik\cdot(x-y)} \left\{ e^{-i(p_1-q_1)\cdot x} e^{-i(p_2-q_2)\cdot y} + e^{-i(p_2-q_1)\cdot x} e^{-i(p_1-q_2)\cdot y} \right\}$$

$$= (-ig)^2 \, i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon}$$

$$\times (2\pi)^8 \left[\delta^{(4)}(k+p_1-q_1)\, \delta^{(4)}(k-p_2+q_2) \right.$$

$$\left. + \delta^{(4)}(k+p_2-q_1)\, \delta^{(4)}(k-p_1+q_2) \right]$$

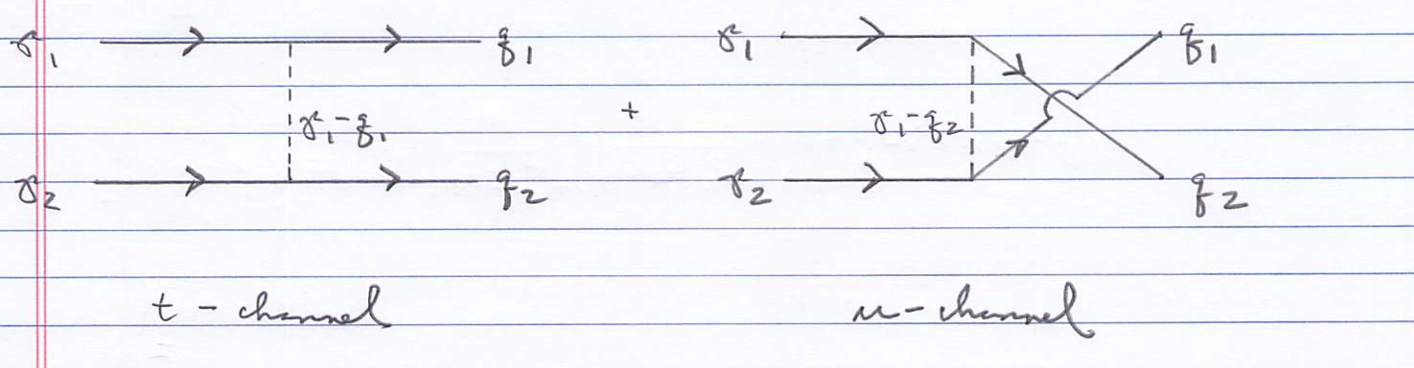
$$= (-ig)^2 i (2\pi)^4 \left\{ \frac{1}{(p_1-q_1)^2 - m^2 + i\epsilon} \delta^{(4)}\left[(q_1-p_1) - p_2 + q_2\right] \right.$$

$$\left. + \frac{1}{(p_2-q_1)^2 - m^2 + i\epsilon} \delta^{(4)}\left[(q_1-p_2) - p_1 + q_2\right] \right\}$$

$$= \ldots$$

$$i(-ig)^2 \left\{ \frac{1}{(p_1-q_1)^2 - m^2} + \frac{1}{(p_1-q_2)^2 - m^2} \right\} (2\pi)^4 \delta^{(4)}(p_1+p_2-q_1-q_2)$$

We can write down 2 Feynman Diagrams associated with each term above.



While Wick's Theorem helped in reducing the number of terms in our calculation of S-matrix elements, the number of terms remained daunting (imagine even higher orders in perturbation theory!).

Feynman Diagrams eliminate all of this intermediate work. What are the rules?

From our calculation of pion decay and nucleon-nucleon scattering in scalar Yukawa theory, we see that the final result for the S-matrix element can be obtained from the Feynman diagram(s) for the interaction using the following rules:

1. For each vertex, assign a factor of $-ig$.
2. For each internal line, assign a factor of $\frac{1}{k^2-m^2}$.
3. Multiply by $2\pi\,\delta^{(4)}(\dots)$, corresponding to conservation of energy.