A Review of the Canonical Formalism and Quantization Procedure for Particles

Anthony Mezzacappa

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Consider the one-dimensional motion of a single particle in a conservative force field. Let q be the generalized coordinate of the particle. Then

$$\dot{q} \equiv \frac{\mathrm{d}q}{\mathrm{d}t}$$

is the generalized velocity of the particle. Let $L(q, \dot{q})$ be the Lagrangian.

Hamilton's Principle

The physical path $q(t)$ that the particle takes in going from $q(t_1) \equiv q_1$ to $q(t_2) \equiv q_2$ is that along which the Action, S, is stationary.
The action is defined by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) \ \mathrm{d}t$$

Along the physical path, small variations in the path

$$q(t) \longrightarrow q(t) + \delta q(t)$$

leading to a variation of the action, δS, leave the action unchanged to first order in the variation, $\delta q(t)$, – i.e.,

$$\delta S = 0$$

Let's compute δS:

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L(q, \dot{q}) \ \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta(\dot{q}) \right] \ \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}}{\mathrm{d}t} (\delta q) \right] \ \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] \ \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \ \mathrm{d}t
\end{aligned}$$

The last equality arises because

$$\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\,\mathrm{d}t = \left.\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_2} - \left.\frac{\partial L}{\partial \dot{q}}\delta q\right|_{t_1} = 0$$

since $\delta q(t_1)$ and $\delta q(t_2)$ are by definition 0 given the definite starting point, $q(t_1) \equiv q_1$, and ending point, $q(t_2) \equiv q_2$.

Given a functional

$$F = F\left(x, y(x), y'(x)\right)$$

consider the variation

$$y(x) \longrightarrow y(x) + \epsilon\eta(x)$$

where $\epsilon \ll 1$.

$$\delta y(x) \equiv \epsilon\eta(x)$$

Then, at a fixed x,

$$\delta F = F\left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right) - F\left(x, y(x), y'(x)\right)$$

Expand in $\underline{\underline{\epsilon}}$:

$$\delta F = \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$$

For

$$\begin{aligned} F = y &\Longrightarrow \delta y = \epsilon\eta \\ F = y' &\Longrightarrow \delta y' = \epsilon\eta' = (\delta y)' \end{aligned}$$

δ and $\frac{\mathrm{d}}{\mathrm{d}x}$ commute!

$$\begin{aligned} \int_{t_1}^{t_2} [\quad]\,\delta q\ \mathrm{d}t = [\quad]_{t_i}\delta q(t_i)\Delta t + [\quad]_{t_{i+1}}\delta q(t_{i+1})\Delta t \\ + \cdots = 0 \quad \Longrightarrow \quad \text{all } [\quad]_{t_i} = 0 \end{aligned}$$

Since the variation $\delta q(t)$ of the function $q(t)$ is arbitrary, the physical path (for which $\delta S = 0$) is given by the solution of the Euler-Lagrange EOM

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

The momentum conjugate to q is defined by

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

The Hamiltonian, which is a function of (q, p) rather than $(q, \dot{q})$, is defined by

$$H \equiv p\dot{q} - L$$

The EOM are (using the Poisson brackets)

$$\dot{q} = -\{H, q\} = -\left(\frac{\partial H}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial q}{\partial q}\right) = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\{H, p\} = -\left(\frac{\partial H}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q}\right) = \frac{\partial H}{\partial q}$$

To quantize the system, q and p become Hermitian operators

$$q \longrightarrow \hat{q} \quad \text{whose action on } \psi(q,t) \text{ is multiplication by } q$$
$$p \longrightarrow \hat{p} \equiv -i\frac{\partial}{\partial q}$$

and the Hamiltonian becomes a Hermitian operator. The Schrödinger Equation is

$$\hat{H}\psi(q,t) = i\frac{\partial \psi(q,t)}{\partial t}$$

Considering the Hamiltonian as a function of (q, p), not explicitly t, we have

$$\mathrm{d}H(p,q) = \frac{\partial H}{\partial p}\mathrm{d}p + \frac{\partial H}{\partial q}\mathrm{d}q \tag{1}$$

Considering the definition of the Hamiltonian

$$H(p,q) \equiv p\dot{q} - L(q,\dot{q}) \tag{2}$$

$$\mathrm{d}H = p\,\mathrm{d}\dot{q} + \dot{q}\,\mathrm{d}p - \frac{\partial L}{\partial q}\mathrm{d}q - \frac{\partial L}{\partial \dot{q}}\mathrm{d}\dot{q} \tag{3}$$

where we have assumed that L is not an explicit function of time (this would break Lorentz invariance of the Lagrangian, which is a scalar quantity, when we build our relativistic QFT).
From the Euler-Lagrange EOM

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{4}$$

But

$$p \equiv \frac{\partial L}{\partial \dot{q}} \tag{5}$$

The Euler-Lagrange EOM then tell us

$$\frac{\mathrm{d}}{\mathrm{d}t}(p) = \dot{p} = \frac{\partial L}{\partial q} \tag{6}$$

Then, insertion of (5) and (6) in (3) gives

$$\begin{aligned} \mathrm{d}H \equiv &= \left(p - \frac{\partial L}{\partial \dot{q}}\right) \mathrm{d}\dot{q} + \dot{q}\,\mathrm{d}p - \frac{\partial L}{\partial q}\mathrm{d}q \qquad \longleftarrow \text{ just a rewrite of (3)} \\ &= \dot{q}\mathrm{d}p - \dot{p}\mathrm{d}q \end{aligned}$$

But, from (1), equating coefficients

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

These are Hamilton's EOM, which can be expressed in terms of the so-called Poisson Brackets as

$$\begin{aligned} \dot{q} &= -\{H, q\} = -\left(\frac{\partial H}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial q}{\partial q}\right) = \frac{\partial H}{\partial p} \\ \dot{p} &= -\{H, p\} = -\left(\frac{\partial H}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial q}\right) = \frac{\partial H}{\partial q} \end{aligned}$$

In the Heisenberg Representation, it's the operators that depend on time, not the states.

$$\psi_{\mathrm{S}}(q,t) = e^{-i\hat{H}t}\psi_{\mathrm{S}}(q,0) \equiv e^{-i\hat{H}t}\psi_{\mathrm{H}}$$

The time-independent operators in the Schrödinger picture are replaced in the Heisenberg picture by

$$\hat{O}_{\mathrm{H}}(t) = e^{i\hat{H}t}\,\hat{O}_{\mathrm{S}}\,e^{-i\hat{H}t}$$

and the time development of the operators is given by

$$\frac{\mathrm{d}\hat{O}_{\mathrm{H}}}{\mathrm{d}t} = i[\hat{H}, \hat{O}_{\mathrm{H}}]$$

Consider the time derivative of the expectation value of the operator, $\hat{O}_{\mathrm{S}}$, in the Schrödinger picture in the state $|\psi_{\mathrm{S}}\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\psi_{\mathrm{S}}|\hat{O}_{\mathrm{S}}|\psi_{\mathrm{S}}\rangle \tag{7}$$

Here the state $|\psi_{\mathrm{S}}\rangle$ is time-dependent but the operator $\hat{O}_{\mathrm{S}}$ is not.
We can re-express the matrix element in terms of $|\psi_{\mathrm{H}}\rangle$ and $\hat{O}_{\mathrm{H}}$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\psi_{\mathrm{H}}|e^{i\hat{H}t}\hat{O}_{\mathrm{S}}e^{-i\hat{H}t}|\psi_{\mathrm{H}}\rangle \tag{8}$$

Now <u>all</u> of the time dependence sits in the exponentials $e^{\pm i\hat{H}t}$.

Taking the derivative

$$\begin{aligned}
&\langle\psi_{\mathrm{H}}|\left(i\hat{H}e^{i\hat{H}t}\hat{O}_{\mathrm{S}}e^{-i\hat{H}t} - ie^{i\hat{H}t}\hat{O}_{\mathrm{S}}\hat{H}e^{-i\hat{H}t}\right)|\psi_{\mathrm{H}}\rangle \\
&= \langle\psi_{\mathrm{H}}|i\left(\hat{H}e^{i\hat{H}t}\hat{O}_{\mathrm{S}}e^{-i\hat{H}t} - e^{i\hat{H}t}\hat{O}_{\mathrm{S}}e^{-i\hat{H}t}\hat{H}\right)|\psi_{\mathrm{H}}\rangle \\
&= \langle\psi_{\mathrm{H}}|i\left(\hat{H}\hat{O}_{\mathrm{H}} - \hat{O}_{\mathrm{H}}\hat{H}\right)|\psi_{\mathrm{H}}\rangle \\
&= \langle\psi_{\mathrm{H}}|i[\hat{H},\hat{O}_{\mathrm{H}}]|\psi_{\mathrm{H}}\rangle
\end{aligned} \tag{9}$$

Then, from (8) and (9)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_{\mathrm{H}}|\hat{O}_{\mathrm{H}}|\psi_{\mathrm{H}}\rangle \\
&= \langle\psi_{\mathrm{H}}|\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_{\mathrm{H}}|\psi_{\mathrm{H}}\rangle \\
&= \langle\psi_{\mathrm{H}}|i[\hat{H},\hat{O}_{\mathrm{H}}]|\psi_{\mathrm{H}}\rangle
\end{aligned}$$

We arrive at the operator relation

$$\frac{\mathrm{d}\hat{O}_{\mathrm{H}}}{\mathrm{d}t} = i[\hat{H},\hat{O}_{\mathrm{H}}]$$

which is the Heisenberg EOM for the time-dependent operator $\hat{O}_{\mathrm{H}}$.

Let's look at $\hat{O}_{\mathrm{H}} = \hat{p}(t)$. Using an arbitrary function, $f(p,q)$, as a placeholder,

$$\begin{aligned}
\frac{\mathrm{d}\hat{p}(t)}{\mathrm{d}t}f &= i[\hat{H},\hat{p}]f \\
&= i\left[\hat{H}\left(-i\frac{\partial}{\partial q}\right) - \left(-i\frac{\partial}{\partial q}\right)\hat{H}\right]f \\
&= \hat{H}\left(\frac{\partial f}{\partial q}\right) - \frac{\partial}{\partial q}\left(\hat{H}f\right) \\
&= \hat{H}\left(\frac{\partial f}{\partial q}\right) - \frac{\partial\hat{H}}{\partial q}(f) - \hat{H}\frac{\partial f}{\partial q} \\
&= -\frac{\partial\hat{H}}{\partial q}(f)
\end{aligned}$$

Then

$$\frac{\mathrm{d}\hat{p}(t)}{\mathrm{d}t} = -\frac{\partial\hat{H}}{\partial q}$$

Now let's look at $\hat{O}_{\mathrm{H}} = \hat{q}(t)$.

$$\begin{aligned}
\frac{\mathrm{d}\hat{q}(t)}{\mathrm{d}t} f &= i[\hat{H}, \hat{q}] f \\
&= i \left[\hat{H} \left(i \frac{\partial}{\partial p} \right) - \left(i \frac{\partial}{\partial p} \right) \hat{H} \right] f \\
&= -\hat{H} \left(\frac{\partial f}{\partial p} \right) + \frac{\partial}{\partial p} \left(\hat{H} f \right) \\
&= -\hat{H} \left(\frac{\partial f}{\partial p} \right) + \frac{\partial \hat{H}}{\partial p} (f) + \hat{H} \frac{\partial f}{\partial p} \\
&= \frac{\partial \hat{H}}{\partial p} (f)
\end{aligned}$$

Then

$$\frac{\mathrm{d}\hat{q}(t)}{\mathrm{d}t} = \frac{\partial \hat{H}}{\partial p}$$

But these are just the <u>classical equations of motion</u>.
$\Longrightarrow$ In the Heisenberg representation, the relevant operators evolve according to the classical EOM.
This is very important in QFT and motivates the approach we will take to build our theories:

1. CONSTRUCT LAGRANGIANS FOR CLASSICAL FIELDS
2. DERIVE THE EOM FOR THOSE FIELDS
3. FIND THE CLASSICAL SOLUTIONS TO THESE EOM
4. QUANTIZE THESE CLASSICAL FIELD SOLUTIONS
    - ELEVATE THEM TO OPERATOR STATUS
        - INTRODUCE CREATION AND ANNIHILATION OPERATORS
            - MAKE THE CONNECTION TO PARTICLES