Classical Field Theory

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We can extend the Lagrangian formulation of classical dynamics for point particles to fields.

The correspondence is

$$
\begin{aligned}
q(t) &\longrightarrow \phi(\mathbf{x}) \qquad \mathbf{x} = (\vec{x}, t) \\
\dot{q}(t) &\longrightarrow \partial_\mu \phi(\mathbf{x})
\end{aligned}
$$

We define a Lagrangian density related to the Lagrangian and Action by

$$
S = \int \mathrm{d}t\ L = \int \mathrm{d}^4\mathbf{x}\ \underbrace{\mathscr{L}(\phi, \partial_\mu \phi)}_{\text{LAGRANGIAN DENSITY}}
$$

Compute δS:

$$
\begin{aligned}
\delta S &= \int \mathrm{d}^4\mathbf{x}\ \left[\frac{\partial \mathscr{L}}{\partial \phi}\ \delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\ \delta(\partial_\mu \phi) \right] \\
&= \int \mathrm{d}^4\mathbf{x}\ \left[\frac{\partial \mathscr{L}}{\partial \phi}\ \delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\ \partial_\mu(\delta \phi) \right] \\
&= \left\{ \int \mathrm{d}^4\mathbf{x}\ \frac{\partial \mathscr{L}}{\partial \phi}\ \delta\phi + \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\ \delta\phi \Big|_{-\infty}^{+\infty} \right) - \int \mathrm{d}^4\mathbf{x}\ \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right\} \\
&= \int \mathrm{d}^4\mathbf{x} \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi
\end{aligned}
$$

Now instead of using $\delta q(t_i) = \delta q(t_f) = 0$, we simply assume that our fields are well-behaved functions of spacetime – i.e., that they go to zero at $x = \pm\infty$.

The solutions render the action on extremum ($\delta S = 0$) and satisfy the EOM

$$
\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = 0
$$

One can now define the momentum conjugate to ϕ

$$
\pi(\mathbf{x}) \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi(\mathbf{x}))}
$$

and the Hamiltonian density

$$\mathscr{H}(\mathbf{x}) \equiv \pi(\mathbf{x})\ \partial_0\phi(\mathbf{x}) - \mathscr{L}$$

which is related to the Hamiltonian by

$$H = \int \mathrm{d}^3\mathbf{x}\ \mathscr{H}$$

As an example, consider the following Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2$$

Then

$$\begin{aligned}
\frac{\partial\mathscr{L}}{\partial\phi} &= -m^2\phi \\
\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} &= \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi)}\left[\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)\right] \\
&= \frac{1}{2}\eta^{\alpha\beta}\frac{\partial}{\partial(\partial_\mu\phi)}\left[(\partial_\alpha\phi)(\partial_\beta\phi)\right] \\
&= \frac{1}{2}\eta^{\alpha\beta}\left[\frac{\partial}{\partial(\partial_\mu\phi)}(\partial_\alpha\phi)\right](\partial_\beta\phi) + \frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)\left[\frac{\partial}{\partial(\partial_\mu\phi)}(\partial_\beta\phi)\right] \\
&= \frac{1}{2}\eta^{\alpha\beta}\delta^\mu{}_\alpha\ \partial_\beta\phi + \frac{1}{2}\eta^{\alpha\beta}\ \partial_\alpha\phi\ \delta^\mu{}_\beta \\
&= \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi \\
&= \partial^\mu\phi
\end{aligned}$$

and

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right) &= \partial_\mu(\partial^\mu\phi) \\
&= \Box\ \phi
\end{aligned}$$

The EOM then read

$$(\Box + m^2)\ \phi = 0$$

which is the Klein-Gordon equation.