# Noether's Theorem

If $\phi$ satisfies the EL EOM ($\delta S = 0$), for any variation we have

$$\delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$\delta \mathcal{L} = \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$\delta S = \int d^4x \, \delta\mathcal{L} = \int d^4x \, \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = 0$$

$\boxed{\text{REMINDER}}$

Assume the action also has the symmetry

$$\delta_\Delta S = 0$$

under the specific transformation

$$\phi \rightarrow \phi + \Delta$$

Given the action is invariant under the transformation even though the Lagrangian density may not be $\left[ \delta_\Delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \right) \neq 0 \right]$, (1)

(2) $$\delta_\Delta \mathcal{L} = \partial_\mu K^\mu \qquad \delta_\Delta S = \int d^4x \, \delta_\Delta \mathcal{L} = \int d^4x \, \partial_\mu K^\mu = 0$$

– i.e. $\delta_\Delta \mathcal{L}$ must be a total divergence. Equating (1) and (2), we get

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu \right) = 0$$

– i.e., we have a conserved current ($\partial_\mu j^\mu = 0$)

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

so, with a symmetry there is a conserved current

How do we find $K^\mu$?

Consider the translation invariance of the action under the transformation

A conserved current will be associated with a conserved "charge"

$$\partial_\mu j^\mu = \partial_0 j^0 + \partial_i j^i$$

Integrate over all of space

$$\int d^3x (\partial_0 j^0 + \partial_i j^i) = 0$$

Then

$$\partial_0 \int d^3x \, j^0 = - \int d^3x \, \partial_i j^i$$

$$= 0$$

and we identify

$$Q \equiv \int d^3x \, j^0$$

with the conserved "charge."

$$x^\mu \longrightarrow x'^\mu = x^\mu - a^\mu$$

Then

2a-b) $$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + \frac{\partial\phi}{\partial x^\mu} a^\mu$$

and

$$\delta_\Delta \phi = \frac{\partial\phi}{\partial x^\mu} a^\mu$$

Then

$$\delta_\Delta(\partial_\mu \phi) = \partial_\mu(\delta_\Delta \phi)$$

$$= \partial_\mu\left(\frac{\partial\phi}{\partial x^\alpha} a^\alpha\right)$$

$$= \frac{\partial^2\phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

Now we can compute

$$\delta_\Delta \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta_\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\Delta(\partial_\mu\phi)$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x^\alpha} a^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial^2\phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x^\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial}{\partial x^\alpha}(\partial_\mu\phi)\right) a^\alpha$$

$$= \frac{\partial\mathcal{L}}{\partial x^\alpha} a^\alpha$$

$$= \partial_\alpha(\mathcal{L} a^\alpha)$$

Then

Consider a coordinate transformation corresponding to a spacetime translation

$$x \longrightarrow x' = x - a$$

- i.e.,

$$x^\mu \longrightarrow x'^\mu = x^\mu - a^\mu$$

where

$$a^\mu = (a^0, a^1, a^2, a^3)$$

and $a^{0,1,2,3}$ are all constants (they can be different).

Consider three points P, Q, and R, and their coordinates before and after the transformation:

| P | Q | R | | P | Q | R |
|---|---|---|---|---|---|---|
| • | • | • | | • | • | • |
| $x-a$ | $x$ | $x+a$ | before | $\phi(x-a)$ | $\phi(x)$ | $\phi(x+a)$ |
| $x'-a$ <br> $= x-2a$ | $x'$ <br> $= x-a$ | $x'+a$ <br> $= x$ | after | $\phi'(x'-a)$ | $\phi'(x')$ | $\phi'(x'+a)$ <br> $= \phi'(x)$ |

$\longrightarrow$

WE HAVE SLID THE COORDINATES OVER

$\longleftarrow$

WE HAVE SLID THE FIELDS OVER SINCE THE NEW FIELD AT $x$, $\phi'(x)$, IS THE OLD FIELD AT $x+a$, $\phi(x+a)$

So, at $x$, we have replaced the field $\phi(x)$ by $\phi(x+a)$, and

$$\delta\phi(x) = \phi'(x) - \phi(x) = \phi(x+a) - \phi(x)$$

$$= \phi(x) + \partial_\mu\phi(x)\, a^\mu - \phi(x)$$

$$= \partial_\mu\phi(x)\, a^\mu$$

It is in this sense that we mean we have moved the fields rather than the coordinates, which is a viewpoint that emerges when we focus on a given $x$.

This is the ACTIVE viewpoint.

The PASSIVE viewpoint emerges when we focus on what happens at a specific point (e.g., $Q$):

@ $Q$ : $\phi'(x') = \phi(x)$ $\qquad\qquad$ $\phi'(Q) = \phi(Q)$

@ $R$ : $\phi'(x'+a) = \phi(x+a)$ $\qquad$ $\phi'(R) = \phi(R)$

– i.e., the value of the scalar field at a point does not change as the coordinates of the point change

$$\delta\phi(Q) = \phi'(x') - \phi(x) = 0$$

$$K^\mu = \mathcal{L}\, a^\mu$$

and

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} a^\alpha - \mathcal{L}\, a^\mu$$

$$= \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L}\, \delta^\mu{}_\alpha \right) a^\alpha$$

Then

$$\partial_\mu j^\mu = a^\alpha \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L}\, \delta^\mu{}_\alpha \right) = 0$$

Since $a^\alpha$ is arbitrary, for the above to be true always, each "coefficient" of $a^\alpha$ must be zero:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L}\, \delta^\mu{}_\alpha \right) = 0$$

Contract with $\eta^{\nu\alpha}$ $\leftarrow \eta^{\nu\alpha} \partial_\mu (\quad) = 0$

$$\partial_\mu \underbrace{\left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x_\nu} - \mathcal{L}\, \eta^{\mu\nu} \right)}_{= T^{\mu\nu}} = 0 \qquad \Rightarrow \partial_\mu [\eta^{\nu\alpha}(\quad)] = 0$$

$= T^{\mu\nu}$ The energy-momentum tensor for the scalar field $\phi$

What about Lorentz invariance?

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$\underline{\nu = 0}$

$$\partial_0 T^{00} + \partial_i T^{i0} = 0$$ conservation of energy

$\underline{\nu = j}$

$$\partial_0 T^{0j} + \partial_i T^{ij} = 0$$ conservation of momentum

$\nu = 0$ $$\partial_0 \int d^3x\, T^{00} = \partial_0 \int d^3x\, \mathcal{H} = -\int d^3x\, \partial_i T^{i0}$$

$\Rightarrow H = \int d^3x\, \mathcal{H}$ is the conserved "charge"

$\nu = j$ $$\partial_0 \int d^3x\, T^{0j} = -\int d^3x\, \partial_i \underbrace{T^{ij}}_{\text{stress tensor}} = 0$$

$\Rightarrow P^i = \int d^3x\, \underbrace{T^{0i}}_{\text{momentum density}}$ is the conserved charge

$$x^\beta \to \Lambda^\beta{}_\nu x^\nu = x^\beta - \underbrace{\varepsilon^\beta{}_\nu x^\nu}_{\equiv \delta x^\beta}$$

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Then

$$\phi(x) \to \phi(\Lambda^{-1}x) = \phi(x) + \frac{\partial\phi}{\partial x^\alpha}\delta x^\alpha$$

and

$$\delta_\Delta \phi = \frac{\partial\phi}{\partial x^\alpha}\delta x^\alpha$$

$$= \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu$$

Then

$$\delta_\Delta(\partial_\mu \phi) = \partial_\mu(\delta_\Delta \phi)$$

$$= \partial_\mu(\partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu)$$

$$= (\partial_\mu \partial_\alpha \phi)\varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu \delta^\nu{}_\mu$$

$$= (\partial_\mu \partial_\alpha \phi)\varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

Now

$$\delta_\Delta \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta_\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\Delta(\partial_\mu\phi)$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\partial_\alpha\phi\,\varepsilon^\alpha{}_\nu x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left[(\partial_\mu\partial_\alpha\phi)\varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha\phi\,\varepsilon^\alpha{}_\mu\right]$$

$$= \frac{\partial\mathcal{L}}{\partial x^\alpha}\varepsilon^\alpha{}_\nu x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi\,\varepsilon^\alpha{}_\mu$$

For the case where there is relative motion in the x-direction only we know that the $(t, x)$ and $(t', x')$ coordinates are related by

$$t' = \gamma(t - \beta x)$$

$$x' = \gamma(x - \beta t)$$

where

$$\gamma = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}$$

$$\beta = \frac{v}{c}$$

and $v$ is the relative velocity.

Then

$$x'^{\mu} = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \equiv \Lambda^{\mu}{}_{\nu} x^{\nu}$$

We can write

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} \varepsilon^\alpha{}_\nu x^\nu = \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) - \mathcal{L} \varepsilon^\alpha{}_\nu \delta^\nu{}_\alpha$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) - \mathcal{L} \cancel{\varepsilon^\alpha{}_\alpha}^{\,0} \quad \text{since } \varepsilon^\mu{}_\nu = -\varepsilon^\nu{}_\mu$$

So

$$\delta_\Delta \mathcal{L} = \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \partial_\mu \phi\right] \varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left[T^\mu{}_\alpha + \mathcal{L}\delta^\mu{}_\alpha - T^\alpha{}_\mu - \mathcal{L}\delta^\alpha{}_\mu\right] \varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left(T^\mu{}_\alpha - T^\alpha{}_\mu\right)$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) \quad \text{since } T^\mu{}_\alpha = T^\alpha{}_\mu \text{ for the scalar field}$$

$$= \partial_\alpha \mathcal{L} \quad \text{since } \mathcal{L} \varepsilon^\alpha{}_\nu \partial_\alpha x^\nu = \mathcal{L} \varepsilon^\nu{}_\nu = 0$$

But

$$\delta_\Delta \mathcal{L} = \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \right)$$

$$= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu \right)$$

So

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi\,\varepsilon^\alpha{}_\nu x^\nu - \mathcal{L}\,\varepsilon^\mu{}_\nu x^\nu\right) = 0$$

or

$$\varepsilon^\alpha{}_\nu\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi\,x^\nu - \mathcal{L}\,\delta^\mu{}_\alpha x^\nu\right) = 0$$

Rewriting

$$\varepsilon^\alpha{}_\nu\,\partial_\mu\Big[\underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi - \mathcal{L}\,\delta^\mu{}_\alpha\right)}_{T^\mu{}_\alpha} x^\nu\Big] = 0$$

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Since $\varepsilon^\alpha{}_\nu$ is antisymmetric, we can conclude that the anti-symmetric component of the term in [ ] must vanish – i.e.,

$$\partial_\mu\left[T^\mu{}_\alpha x^\nu - T^\mu{}_\nu x^\alpha\right] = 0$$

So

$$M^\mu{}_\alpha{}^\nu \equiv T^\mu{}_\alpha x^\nu - T^\mu{}_\nu x^\alpha$$

is the conserved current.

For $\alpha, \nu = 1, 2, 3$, the Lorentz transformation corresponds to a

We can decompose the tensor

$$T^{\mu}{}_{\alpha} x^{\nu}$$

into its symmetric and antisymmetric components

$$T^{\mu}{}_{\alpha} x^{\nu} = \frac{1}{2}\left[T^{\mu}{}_{\alpha} x^{\nu} + T^{\mu}{}_{\nu} x^{\alpha} + T^{\mu}{}_{\alpha} x^{\nu} - T^{\mu}{}_{\nu} x^{\alpha}\right]$$

$$= \frac{1}{2}\left[T^{\mu}{}_{\alpha} x^{\nu} + T^{\mu}{}_{\nu} x^{\alpha}\right] \leftarrow \text{SYMMETRIC}$$

$$+ \frac{1}{2}\left[T^{\mu}{}_{\alpha} x^{\nu} - T^{\mu}{}_{\nu} x^{\alpha}\right] \leftarrow \text{ANTISYMMETRIC}$$

spatial rotation. In this case

$$\partial_0 \left[ T^0{}_i x^j - T^0{}_j x^i \right] + \partial_k \left[ T^k{}_i x^j - T^k{}_j x^i \right] = 0$$

and the conserved charge is

$$\int d^3x \left( T^0{}_i x^j - T^0{}_j x^i \right)$$

which is the total angular momentum tensor $Q_i{}^j$ with $Q_i{}^i = 0$ and $Q_i{}^j = -Q_j{}^i$ – i.e., $Q^i{}_j$ has 3 independent components.

Commutation of $\partial_\mu$ and $\delta$

$F = F(x, y(x), y'(x))$ a functional

Change $y(x)$ in the following manner

$$y(x) \longrightarrow y(x) + \epsilon\eta(x)$$

The variation $\delta y$ is defined to be

$$\delta y \equiv \epsilon\eta(x) \qquad (1)$$

The variation is a change in a function.

At fixed $x$,

$$F(x, y, y') \longrightarrow F(x, y+\epsilon\eta, y'+\epsilon\eta')$$

$$= \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$$

$$\equiv \delta F$$

If we let $F = y'$, we can see that generally

$$\delta F = \delta y' = \epsilon\eta' \qquad (2)$$

So, at fixed $x$, from (1) and (2)

$$\delta y' = (\delta y)'$$

– i.e., the derivative w.r.t. the independent variable $x$ and the variation of a function of $x$ commute.