Quantization of a Free Scalar Field

Consider the free scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

that we considered before. ϕ satisfies the Klein-Gordon equation

$$(\Box + m^2)\phi = 0 \quad \leftarrow \text{ This has solutions } e^{\pm i k \cdot x}$$

$k \cdot x \equiv k_\mu x^\mu$

$k^\mu = (E(\vec{k}), \vec{k})$

$k_\mu = \eta_{\mu\nu} k^\nu = (E(\vec{k}), -\vec{k})$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$$

The general solution of the Klein-Gordon equation is

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[a(\vec{k}) e^{-ik\cdot x} + a^*(\vec{k}) e^{ik\cdot x} \right]$$

Then

$$\pi(x) = -\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[i E(\vec{k}) \right] \left[a(\vec{k}) e^{-ik\cdot x} - a^*(\vec{k}) e^{ik\cdot x} \right]$$

To quantize the field, we follow the usual canonical procedure:

$$\{\phi, \pi\} \longrightarrow i[\hat{\phi}, \hat{\pi}]$$

$$a(\vec{k}) \longrightarrow \hat{a}(k)$$

$$a^*(\vec{k}) \longrightarrow \hat{a}^\dagger(k)$$

In particular, we impose

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)] = i\delta^{(3)}(\vec{x}-\vec{x}')$$

What does this imply about $[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')]$?

To answer this question, compute

$$\hat{\phi}(\vec{x},t)\,\hat{\pi}(\vec{x}',t)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[\hat{a}(\vec{k}) e^{-iE(\vec{k})t} e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(\vec{k}) e^{iE(\vec{k})t} e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$\times(-i)\int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{E(\vec{k}')}{\sqrt{2E(\vec{k}')}} \left[\hat{a}(\vec{k}') e^{-iE(\vec{k}')t} e^{i\vec{k}'\cdot\vec{x}'} - \hat{a}^\dagger(\vec{k}') e^{iE(\vec{k}')t} e^{-i\vec{k}'\cdot\vec{x}'}\right]$$

$$= -i \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{E(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times \left[\hat{a}(\vec{k})\hat{a}(\vec{k}') e^{-i[E(\vec{k})+E(\vec{k}')]t} e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'}\right.$$

$$- \hat{a}(\hat{k})\, a^\dagger(\vec{k}') e^{-i[E(\vec{k})-E(\vec{k}')]t} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'}$$

$$+ \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}') e^{i[E(\vec{k})-E(\vec{k}')]t} e^{-i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'}$$

$$- \hat{a}^\dagger(\vec{k})\, \hat{a}^\dagger(\vec{k}')\, e^{i[E(\vec{k}) + E(\vec{k}')]t}\, e^{-i\vec{k}\cdot\vec{x}}\, e^{-\vec{k}'\cdot\vec{x}'} \Big]$$

Now compute

$$\hat{\pi}(\vec{x}', t)\, \hat{\phi}(\vec{x}, t)$$

$$= (-i) \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{E(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[\hat{a}(\vec{k}')\, \hat{a}(\vec{k})\, e^{-i[E(\vec{k}') + E(\vec{k})]t}\, e^{i\vec{k}'\cdot\vec{x}'}\, e^{i\vec{k}\cdot\vec{x}}$$

$$+ \hat{a}(\vec{k}')\, \hat{a}^\dagger(\vec{k})\, e^{-i[E(\vec{k}') - E(\vec{k})]t}\, e^{i\vec{k}'\cdot\vec{x}'}\, e^{-i\vec{k}\cdot\vec{x}}$$

$$- \hat{a}^\dagger(\vec{k}')\, \hat{a}(\vec{k})\, e^{i[E(\vec{k}') - E(\vec{k})]t}\, e^{-i\vec{k}'\cdot\vec{x}'}\, e^{i\vec{k}\cdot\vec{x}}$$

$$- \hat{a}^\dagger(\vec{k}')\, \hat{a}^\dagger(\vec{k})\, e^{i[E(\vec{k}') + E(\vec{k})]t}\, e^{-i\vec{k}'\cdot\vec{x}'}\, e^{-i\vec{k}\cdot\vec{x}} \Big]$$

Then

$$\hat{\phi}(\vec{x}, t)\, \hat{\pi}(\vec{x}', t) - \hat{\pi}(\vec{x}', t)\, \hat{\phi}(\vec{x}, t)$$

$$= -i \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{E(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[- \hat{a}(\vec{k})\, \hat{a}^\dagger(\vec{k}')\, e^{-i[E(\vec{k}) - E(\vec{k}')]t}\, e^{i\vec{k}\cdot\vec{x}}\, e^{-i\vec{k}'\cdot\vec{x}'} \qquad (1)$$

$$+ \hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{k}')\, e^{i[E(\vec{k}) - E(\vec{k}')]t}\, e^{-i\vec{k}\cdot\vec{x}}\, e^{i\vec{k}'\cdot\vec{x}'} \qquad (2)$$

$$- \hat{a}(\vec{k}')\hat{a}^{\dagger}(\vec{k})\, e^{-i[E(\vec{k}')-E(\vec{k})]t}\, e^{i\vec{k}'\cdot\vec{x}'}\, e^{-i\vec{k}\cdot\vec{x}} \qquad ③$$

$$+ \hat{a}^{\dagger}(\vec{k}')\hat{a}(\vec{k})\, e^{i[E(\vec{k}')-E(\vec{k})]t}\, e^{-i\vec{k}'\cdot\vec{x}'}\, e^{i\vec{k}\cdot\vec{x}} \Big] \qquad ④$$

If

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = 0$$

$$[\hat{a}^{\dagger}(\vec{k}), \hat{a}^{\dagger}(\vec{k}')] = 0$$

Further, if

$$[\hat{a}(\vec{k}), \hat{a}^{\dagger}(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$$

then

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]$$

$$= i\int \frac{d^3\vec{k}}{(2\pi)^3}\int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k}')}} \frac{E(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')\, e^{-i[E(\vec{k})-E(\vec{k}')]t}\, e^{i\vec{k}\cdot\vec{x}}\, e^{-i\vec{k}'\cdot\vec{x}'} \qquad ①+④$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k}'-\vec{k})\, e^{i[E(\vec{k})-E(\vec{k}')]t}\, e^{-i\vec{k}\cdot\vec{x}}\, e^{i\vec{k}'\cdot\vec{x}'} \Big] \qquad ③+②$$

$$= i\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2}\Big[e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \Big] = i\,\delta^{(3)}(\vec{x}-\vec{x}')$$