

A Review of the Canonical Formalism and Quantization Procedure for Particles

Anthony Mezzacappa

August 27, 2020

Consider the one-dimensional motion of a single particle in a conservative force field. Let q be the generalized coordinate of the particle. Then

$$\dot{q} \equiv \frac{dq}{dt}$$

is the generalized velocity of the particle. Let $L(q, \dot{q})$ be the Lagrangian.

Hamilton's Principle

The physical path $q(t)$ that the particle takes in going from $q(t_1) \equiv q_1$ to $q(t_2) \equiv q_2$ is that along which the Action, S , is stationary.

The action is defined by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt$$

Along the physical path, small variations in the path

$$q(t) \longrightarrow q(t) + \delta q(t)$$

leading to a variation of the action, δS , leave the action unchanged to first order in the variation, $\delta q(t)$, – i.e.,

$$\delta S = 0$$

Let's compute δS :

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L(q, \dot{q}) \, dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta(\dot{q}) \right] \, dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right] \, dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] \, dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \, dt
\end{aligned}$$

The last equality arises because

$$\int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \, dt = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_2} - \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1} = 0$$

since $\delta q(t_1)$ and $\delta q(t_2)$ are by definition 0 given the definite starting point, $q(t_1) \equiv q_1$, and ending point, $q(t_2) \equiv q_2$.

Given a functional

$$F = F(x, y(x), y'(x))$$

consider the variation

$$y(x) \longrightarrow y(x) + \epsilon \eta(x)$$

where $\epsilon \ll 1$.

$$\delta y(x) \equiv \epsilon \eta(x)$$

Then, at a fixed x ,

$$\delta F = F(x, y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x)) - F(x, y(x), y'(x))$$

Expand in $\underline{\epsilon}$:

$$\delta F = \frac{\partial F}{\partial y} \epsilon \eta + \frac{\partial F}{\partial y'} \epsilon \eta'$$

For

$$\begin{aligned}
F = y &\implies \delta y = \epsilon \eta \\
F = y' &\implies \delta y' = \epsilon \eta' = (\delta y)'
\end{aligned}$$

δ and $\frac{d}{dx}$ commute!

$$\int_{t_1}^{t_2} [\] \delta q \, dt = [\]_{t_i} \delta q(t_i) \Delta t + [\]_{t_{i+1}} \delta q(t_{i+1}) \Delta t \\ + \dots = 0 \implies \text{all } [\]_{t_i} = 0$$

Since the variation $\delta q(t)$ of the function $q(t)$ is arbitrary, the physical path (for which $\delta S = 0$) is given by the solution of the Euler-Lagrange EOM

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

The momentum conjugate to q is defined by

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

The Hamiltonian, which is a function of (q, p) rather than (q, \dot{q}) , is defined by

$$H \equiv p\dot{q} - L$$

The EOM are (using the Poisson brackets)

$$\dot{q} = -\{H, q\} = -\left(\frac{\partial H}{\partial q} \frac{\partial q}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial q}{\partial q} \right) = \frac{\partial H}{\partial p} \\ \dot{p} = -\{H, p\} = -\left(\frac{\partial H}{\partial q} \frac{\partial p}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial p}{\partial q} \right) = \frac{\partial H}{\partial q}$$

To quantize the system, q and p become Hermitian operators

$$q \longrightarrow \hat{q} \quad \text{whose action on } \psi(q, t) \text{ is multiplication by } q \\ p \longrightarrow \hat{p} \equiv -i \frac{\partial}{\partial q}$$

and the Hamiltonian becomes a Hermitian operator. The Schrödinger Equation is

$$\hat{H}\psi(q, t) = i \frac{\partial \psi(q, t)}{\partial t}$$

Considering the Hamiltonian as a function of (q, p) , not explicitly t , we have

$$dH(p, q) = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq \quad (1)$$

Considering the definition of the Hamiltonian

$$H(p, q) \equiv p\dot{q} - L(q, \dot{q}) \quad (2)$$

$$dH = p d\dot{q} + \dot{q} dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} d\dot{q} \quad (3)$$

where we have assumed that L is not an explicit function of time (this would break Lorentz invariance of the Lagrangian, which is a scalar quantity, when we build our relativistic QFT).

From the Euler-Lagrange EOM

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0 \quad (4)$$

But

$$p \equiv \frac{\partial L}{\partial \dot{q}} \quad (5)$$

The Euler-Lagrange EOM then tell us

$$\frac{d}{dt}(p) = \dot{p} = \frac{\partial L}{\partial q} \quad (6)$$

Then, insertion of (5) and (6) in (3) gives

$$\begin{aligned} dH &\equiv \left(p - \frac{\partial L}{\partial \dot{q}} \right) d\dot{q} + \dot{q} dp - \frac{\partial L}{\partial q} dq && \longleftarrow \text{just a rewrite of (3)} \\ &= \dot{q} dp - \dot{p} dq \end{aligned}$$

But, from (1), equating coefficients

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

These are Hamilton's EOM, which can be expressed in terms of the so-called Poisson Brackets as

$$\begin{aligned} \dot{q} &= -\{H, q\} = -\left(\frac{\partial H}{\partial q} \frac{\partial q}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial q}{\partial q} \right) = \frac{\partial H}{\partial p} \\ \dot{p} &= -\{H, p\} = -\left(\frac{\partial H}{\partial q} \frac{\partial p}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial p}{\partial q} \right) = \frac{\partial H}{\partial q} \end{aligned}$$

In the Heisenberg Representation, it's the operators that depend on time, not the states.

$$\psi_S(q, t) = e^{-i\hat{H}t}\psi_S(q, 0) \equiv e^{-i\hat{H}t}\psi_H$$

The time-independent operators in the Schrödinger picture are replaced in the Heisenberg picture by

$$\hat{O}_H(t) = e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t}$$

and the time development of the operators is given by

$$\frac{d\hat{O}_H}{dt} = i[\hat{H}, \hat{O}_H]$$

Consider the time derivative of the expectation value of the operator, \hat{O}_S , in the Schrödinger picture in the state $|\psi_S\rangle$:

$$\frac{d}{dt} \langle \psi_S | \hat{O}_S | \psi_S \rangle \quad (7)$$

Here the state $|\psi_S\rangle$ is time-dependent but the operator \hat{O}_S is not.

We can re-express the matrix element in terms of $|\psi_H\rangle$ and \hat{O}_H as

$$\frac{d}{dt} \langle \psi_H | e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t} | \psi_H \rangle \quad (8)$$

Now all of the time dependence sits in the exponentials $e^{\pm i\hat{H}t}$.

Taking the derivative

$$\begin{aligned} & \langle \psi_H | \left(i\hat{H}e^{i\hat{H}t}\hat{O}_Se^{-i\hat{H}t} - ie^{i\hat{H}t}\hat{O}_S\hat{H}e^{-i\hat{H}t} \right) | \psi_H \rangle \\ &= \langle \psi_H | i \left(\hat{H}e^{i\hat{H}t}\hat{O}_Se^{-i\hat{H}t} - e^{i\hat{H}t}\hat{O}_Se^{-i\hat{H}t}\hat{H} \right) | \psi_H \rangle \\ &= \langle \psi_H | i \left(\hat{H}\hat{O}_H - \hat{O}_H\hat{H} \right) | \psi_H \rangle \\ &= \langle \psi_H | i[\hat{H}, \hat{O}_H] | \psi_H \rangle \end{aligned} \quad (9)$$

Then, from (8) and (9)

$$\begin{aligned} & \frac{d}{dt} \langle \psi_H | \hat{O}_H | \psi_H \rangle \\ &= \langle \psi_H | \frac{d}{dt} \hat{O}_H | \psi_H \rangle \\ &= \langle \psi_H | i[\hat{H}, \hat{O}_H] | \psi_H \rangle \end{aligned}$$

We arrive at the operator relation

$$\frac{d\hat{O}_H}{dt} = i[\hat{H}, \hat{O}_H]$$

which is the Heisenberg EOM for the time-dependent operator \hat{O}_H .

Let's look at $\hat{O}_H = \hat{p}(t)$. Using an arbitrary function, $f(p, q)$, as a placeholder,

$$\begin{aligned} \frac{d\hat{p}(t)}{dt} f &= i[\hat{H}, \hat{p}] f \\ &= i \left[\hat{H} \left(-i \frac{\partial}{\partial q} \right) - \left(-i \frac{\partial}{\partial q} \right) \hat{H} \right] f \\ &= \hat{H} \left(\frac{\partial f}{\partial q} \right) - \frac{\partial}{\partial q} (\hat{H} f) \\ &= \hat{H} \left(\frac{\partial f}{\partial q} \right) - \frac{\partial \hat{H}}{\partial q} (f) - \hat{H} \frac{\partial f}{\partial q} \\ &= -\frac{\partial \hat{H}}{\partial q} (f) \end{aligned}$$

Then

$$\frac{d\hat{p}(t)}{dt} = -\frac{\partial \hat{H}}{\partial q}$$

Now let's look at $\hat{O}_H = \hat{q}(t)$.

$$\begin{aligned} \frac{d\hat{q}(t)}{dt} f &= i[\hat{H}, \hat{q}] f \\ &= i \left[\hat{H} \left(i \frac{\partial}{\partial p} \right) - \left(i \frac{\partial}{\partial p} \right) \hat{H} \right] f \\ &= -\hat{H} \left(\frac{\partial f}{\partial p} \right) + \frac{\partial}{\partial p} (\hat{H} f) \\ &= -\hat{H} \left(\frac{\partial f}{\partial p} \right) + \frac{\partial \hat{H}}{\partial p} (f) + \hat{H} \frac{\partial f}{\partial p} \\ &= \frac{\partial \hat{H}}{\partial p} (f) \end{aligned}$$

Then

$$\frac{d\hat{q}(t)}{dt} = \frac{\partial \hat{H}}{\partial p}$$

But these are just the classical equations of motion.

\implies In the Heisenberg representation, the relevant operators evolve according to the classical EOM.

This is very important in QFT and motivates the approach we will take to build our theories:

1. CONSTRUCT LAGRANGIANS FOR CLASSICAL FIELDS
2. DERIVE THE EOM FOR THOSE FIELDS
3. FIND THE CLASSICAL SOLUTIONS TO THESE EOM
4. QUANTIZE THESE CLASSICAL FIELD SOLUTIONS
 - ELEVATE THEM TO OPERATOR STATUS
 - INTRODUCE CREATION AND ANNIHILATION OPERATORS
 - MAKE THE CONNECTION TO PARTICLES