Second and Higher Order in Perturbation Theory

In general, we need to compute matrix elements of the form

$$< f \,|\, T\left[\mathcal{H}_I(x_1) \ldots \mathcal{H}_I(x_n)\right] | i >$$

We will make use of Wick's Theorem.

Wick's Theorem tells us how to convert time-ordered products of operators to normal ordered products of operators.

Consider our scalar field operator

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\phi^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{-ik\cdot x}\, \hat{a}(\vec{k})$$

$$\phi^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{ik\cdot x}\, \hat{a}^\dagger(\vec{k})$$

Let $x^0 > y^0$. Then

$$T[\phi(x)\phi(y)] = \phi(x)\phi(y)$$
$$= \left[\phi^{(+)}(x) + \phi^{(-)}(x)\right]\left[\phi^{(+)}(y) + \phi^{(-)}(y)\right]$$

$$= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y)$$
$$+ \phi^{(-)}(x)\phi^{+}(y) + \phi^{(-)}(x)\phi^{(-)}(y)$$

$$= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y)$$
$$+ \phi^{(-)}(y)\phi^{(+)}(x)$$
$$- \phi^{(-)}(y)\phi^{(+)} x$$
$$+ \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y)$$

$$= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$
$$+ \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y)$$
$$+ [\phi^{(+)}(x), \phi^{(-)}(y)]$$

$$= :\phi(x)\phi(y): + [\phi^{(+)}(x), \phi^{(-)}(y)]$$

But

$$[\phi^{(+)}(x), \phi^{(-)}(y)] = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\frac{1}{\sqrt{2E(\vec{k}')}} e^{-ik\cdot x} e^{ik'\cdot y}$$
$$\times [\hat{a}(\vec{k}), \hat{a}^{\dagger}(\vec{k}')]$$
$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k})}}\frac{1}{\sqrt{2E(\vec{k}')}} e^{-ik\cdot x} e^{ik'\cdot y} (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')$$
$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E(\vec{k})} e^{-ik\cdot(x-y)}$$

$$= \Delta(x-y)$$

That is

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + \Delta(x-y)$$

For $y^0 > x^0$, we get

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + \Delta(y-x)$$

All of this can be succinctly written

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + \Delta_F(x-y)$$

Now let's generalize this to 4 fields (to see how the pattern goes).

$$T[\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)]$$

means $\Delta_F(x_1-x_2)$

$$= :\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) + \overbrace{\phi_1(x_1)\phi_2(x_2)}\phi_3(x_3)\phi_4(x_4)$$

means $\Delta_F(x_1-x_3)$

$$+ \overbrace{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)}\phi_4(x_4) + \overbrace{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)}$$

$$+ \phi_1(x_1)\overbrace{\phi_2(x_2)\phi_3(x_3)}\phi_4(x_4) + \ldots$$

$$+ \overbrace{\phi_1(x_1)\phi_2(x_2)}\overbrace{\phi_3(x_3)\phi_4(x_4)} + \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) + \ldots :$$

$$= :\phi_1\phi_2\phi_3\phi_4: + \Delta_F(x_1-x_2):\phi_3\phi_4: + \dots$$

$$+ \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4)$$

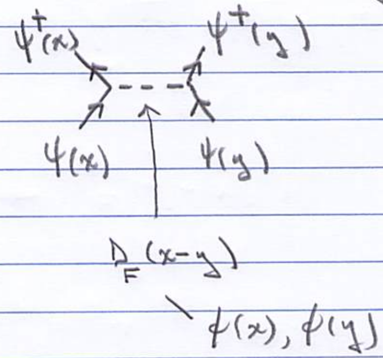
$$+ \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)$$

When taking

$$\langle 0|T[\phi_1\phi_2\phi_3\phi_4]|0\rangle$$

only the <u>fully contracted</u> terms survive!

With this machinery, let's compute nucleon-nucleon scattering in scalar Yukawa theory.



① $\psi\psi \to \psi\psi$

$$|i\rangle = \sqrt{2E(\vec{p}_1)}\sqrt{2E(\vec{p}_2)}\,\hat{a}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)|0\rangle$$

$$|f\rangle = \sqrt{2E(\vec{q}_1)}\sqrt{2E(\vec{q}_2)}\,\hat{a}^\dagger(\vec{q}_1)\hat{a}^\dagger(\vec{q}_2)|0\rangle$$

The first order that contributes to scattering is second order, and the matrix element is

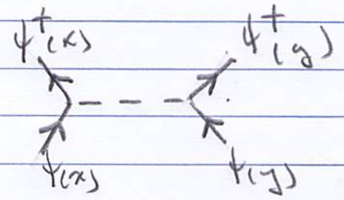
$$\langle f|\frac{(-ig)^2}{2}\int d^4x\,d^4y\,T\left[\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y)\right]|i\rangle$$

Let's use Wick's Theorem to evaluate the time-ordered product.

$$T\left[\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y)\right]$$

$$= :\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y)$$

$$+ \overline{\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)}\phi(y)$$



$$+ \psi^\dagger(x)\overline{\psi(x)\phi(x)\psi^\dagger(y)}\psi(y)\phi(y)$$

$$+ \psi^\dagger(x)\psi(x)\overline{\phi(x)\psi^\dagger(y)\psi(y)\phi(y)}$$

$$+ \ldots :$$

terms with 2 or 3 contractions left out here because they cannot connect a 2-particle initial state with a 2-particle final state

5a

$$= :\psi^\dagger(x)\psi(x)\phi(x)\psi^\dagger(y)\psi(y)\phi(y):$$

$$+ \Delta_F^\psi(y-x)\; :\psi(x)\phi(x)\psi^\dagger(y)\phi(y):$$

$$+ \Delta_F^\psi(x-y)\; :\psi^\dagger(x)\phi(x)\psi(y)\phi(y):$$

all 0 because there is no pion in the initial or final state

$$+ \Delta_F^\phi(x-y)\; :\psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y):$$

So, we are left to compute

$$\frac{(-ig)^2}{2}\int d^4x\, d^4y\, \Delta_F^\phi(x-y)\langle f|:\psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y):|i\rangle$$

$$= \ldots$$

...

$$= :\psi^\dagger(x)\,\psi(x)\,\phi(x)\,\psi^\dagger(y)\,\psi(y)\,\phi(y)$$

$$+ \psi^\dagger(x)\,\psi(x)\,\phi(x)\,\psi^\dagger(y)\,\psi(y)\,\phi(y)$$

$$+ \psi^\dagger(x)\,\psi(x)\,\phi(x)\,\psi^\dagger(y)\,\psi(y)\,\phi(y)$$

$$+ \psi^\dagger(x)\,\psi(x)\,\phi(x)\,\psi^\dagger(y)\,\psi(y)\,\phi(y) \; :$$

← this would be the only term that would contribute if the initial and final states are $|0\rangle$

$$= \frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^{\phi}(x-y) \sqrt{2E(\vec{q}_1)} \sqrt{2E(\vec{q}_2)} \sqrt{2E(\vec{\gamma}_1)} \sqrt{2E(\vec{\gamma}_2)}$$

$$\times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \frac{d^3k_4}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k}_1)}} \frac{1}{\sqrt{2E(\vec{k}_2)}} \frac{1}{\sqrt{2E(\vec{k}_3)}} \frac{1}{\sqrt{2E(\vec{k}_4)}}$$

$$\langle 0| \hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2) : \left[e^{ik_1\cdot x} \hat{b}^\dagger(\vec{k}_1) + e^{-ik_1\cdot x} \hat{a}(\vec{k}_1) \right]$$

$$\times \left[e^{ik_2\cdot y} \hat{b}^\dagger(\vec{k}_2) + e^{-ik_2\cdot y} \hat{a}(\vec{k}_2) \right]$$

$$\times \left[e^{ik_3\cdot y} \hat{a}^\dagger(\vec{k}_3) + e^{-ik_3\cdot y} \hat{b}(\vec{k}_3) \right]$$

$$\times \left[e^{ik_4\cdot y} \hat{a}^\dagger(\vec{k}_4) + e^{-ik_4\cdot y} \hat{b}(\vec{k}_4) \right] : \hat{a}^\dagger(\vec{\gamma}_1)\hat{a}^\dagger(\vec{\gamma}_2)|0\rangle$$

$\hat{b}^\dagger(\vec{k}_1)$ and $\hat{b}^\dagger(\vec{k}_2)$ will annihilate to the left. They commute with $\hat{a}(\vec{q}_1)$ and $\hat{a}(\vec{q}_2)$. Then

$$\langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{b}^\dagger(\vec{k}_1) = 0$$

and in turn

$$\langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{b}^\dagger(\vec{k}_2) = 0$$

Similarly $\hat{b}(\vec{k}_3)$ and $\hat{b}(\vec{k}_4)$ annihilate to the right and commute with $\hat{a}^\dagger(\vec{\gamma}_1)$ and $\hat{a}^\dagger(\vec{\gamma}_2)$. Then

$$\hat{b}(\vec{k}_4)\hat{a}^\dagger(\vec{\gamma}_1)\hat{a}^\dagger(\vec{\gamma}_2)|0\rangle = 0$$

and in turn

$$\hat{b}(\vec{k}_3)\,\hat{a}^\dagger(\vec{q}_1)\,\hat{a}^\dagger(\vec{q}_2)\,|0\rangle = 0$$

Therefore, the only term that survives in the normal-ordered product above is:

$$\langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2) : \hat{a}(\vec{k}_1)\hat{a}(\vec{k}_2)\hat{a}^\dagger(\vec{k}_3)\hat{a}^\dagger(\vec{k}_4) : \hat{a}^\dagger(\vec{q}_1)\hat{a}^\dagger(\vec{q}_2)|0\rangle$$

$$= \langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{a}^\dagger(\vec{k}_3)\hat{a}^\dagger(\vec{k}_4)\hat{a}(\vec{k}_1)\hat{a}(\vec{k}_2)\hat{a}^\dagger(\vec{q}_1)\hat{a}^\dagger(\vec{q}_2)|0\rangle$$

$$= \langle 0|\hat{a}(\vec{q}_1)\hat{a}^\dagger(\vec{k}_3)\hat{a}(\vec{q}_2)\hat{a}^\dagger(\vec{k}_4)\hat{a}(\vec{k}_1)\hat{a}(\vec{k}_2)\hat{a}^\dagger(\vec{q}_1)\hat{a}^\dagger(\vec{q}_2)|0\rangle$$

$$+ \langle 0|\hat{a}(\vec{q}_1)\hat{a}^\dagger(\vec{k}_4)\hat{a}(\vec{k}_1)\hat{a}(\vec{k}_2)\hat{a}^\dagger(\vec{q}_1)\hat{a}^\dagger(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)$$

$$= \ldots$$

7a-b >

The end result is:

$$(2\pi)^{12}\Big[\underbrace{\delta^{(3)}(\vec{q}_1-\vec{k}_3)\delta^{(3)}(\vec{q}_2-\vec{k}_4)}_{A} + \underbrace{\delta^{(3)}(\vec{q}_1-\vec{k}_4)\delta^{(3)}(\vec{q}_2-\vec{k}_3)}_{B}\Big]$$

$$\times\Big[\underbrace{\delta^{(3)}(\vec{q}_1-\vec{k}_1)\delta^{(3)}(\vec{q}_2-\vec{k}_2)}_{C} + \underbrace{\delta^{(3)}(\vec{q}_1-\vec{k}_2)\delta^{(3)}(\vec{q}_2-\vec{k}_1)}_{D}\Big]$$

Then, we have

$$\frac{(-ig)^2}{2}\int d^4x\,d^4y\,\Delta_F^\phi(x-y)\sqrt{2E(\vec{q}_1)}\sqrt{2E(\vec{q}_2)}\sqrt{2E(\vec{q}_1)}\sqrt{2E(\vec{q}_2)}$$

$$\times\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{k}_1)}}\frac{1}{\sqrt{2E(\vec{k}_2)}}\frac{1}{\sqrt{2E(\vec{k}_3)}}\frac{1}{\sqrt{2E(\vec{k}_4)}}$$

$$\times\ e^{-ik_1\cdot x}e^{-ik_2\cdot y}e^{ik_3\cdot y}e^{ik_4\cdot y}$$

$$= \langle 0 | \hat{a}^\dagger(\vec{k}_3)\,\hat{a}(\vec{q}_{\gamma 1})\,\hat{a}(\vec{q}_{\gamma 2})\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle$$

$$① \quad + \langle 0 | \hat{a}(\vec{q}_{\gamma 2})\,\hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1})$$

$$+ \langle 0 | \hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{q}_{\gamma 1})\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 2})$$

$$② \quad + \langle 0 | \hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 1})$$

$$① \quad = \langle 0 | \hat{a}^\dagger(\vec{k}_4)\,\hat{a}(\vec{q}_{\gamma 2})\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1})$$

$$+ \langle 0 | \hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2})$$

$$= \langle 0 | \hat{a}(\vec{k}_1)\,\hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2})$$

$$+ \langle 0 | \hat{a}(\vec{k}_1)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_1)$$

$$= \langle 0 | \hat{a}^\dagger(\vec{\gamma}_1)\,\hat{a}(\vec{k}_1)\,\hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2})$$

$$+ \langle 0 | \hat{a}(\vec{k}_2)\,\hat{a}^\dagger(\vec{\gamma}_2) | 0 \rangle (2\pi)^3 (\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_1)$$

$$+ \langle 0 | \hat{a}^\dagger(\vec{\gamma}_2)\,\hat{a}(\vec{k}_1) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_1)$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_1) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_2)$$

$$= \langle 0 | \hat{a}^\dagger(\vec{\gamma}_2)\,\hat{a}(\vec{k}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_1)$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_2) (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_1)$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_1) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_2)$$

$$Ⓐ \times Ⓒ = (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_2) (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_1)$$

$$Ⓐ \times Ⓓ \quad + (2\pi)^3 \delta^{(3)}(\vec{k}_3 - \vec{q}_{\gamma 1}) (2\pi)^3 \delta^{(3)}(\vec{k}_4 - \vec{q}_{\gamma 2}) (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{\gamma}_1) (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{\gamma}_2)$$

$$(2)\quad \langle 0|\hat{a}(\vec{k}_1)\hat{a}^{+}(\vec{q}_1)\hat{a}(\vec{k}_2)\hat{a}^{+}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ \langle 0|\hat{a}(\vec{k}_1)\hat{a}^{+}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$=\ \langle 0|\hat{a}^{+}(\vec{q}_1)\hat{a}(\vec{k}_1)\hat{a}(\vec{k}_2)\hat{a}^{+}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ \langle 0|\hat{a}(\vec{k}_2)\hat{a}^{+}(\vec{q}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ \langle 0|\hat{a}^{+}(\vec{q}_2)\hat{a}(\vec{k}_1)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$+\ (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$=\ \langle 0|\hat{a}^{+}(\vec{q}_2)\hat{a}(\vec{k}_2)|0\rangle(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ (2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$+\ (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$(B)\times(C) =\ (2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)$$

$$(B)\times(D)\ +\ (2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_3-\vec{q}_2)(2\pi)^3\delta^{(3)}(\vec{k}_4-\vec{q}_1)(2\pi)^3\delta^{(3)}(\vec{k}_2-\vec{q}_1)$$

$$\times (2\pi)^{12} \left[\delta^{(3)}(\vec{q}_1 - \vec{k}_3)\, \delta^{(3)}(\vec{q}_2 - \vec{k}_4) + \delta^{(3)}(\vec{q}_1 - \vec{k}_4)\, \delta^{(3)}(\vec{q}_2 - \vec{k}_3) \right]$$

$$\times \left[\delta^{(3)}(\vec{p}_1 - \vec{k}_1)\, \delta^{(3)}(\vec{p}_2 - \vec{k}_2) + \delta^{(3)}(\vec{p}_1 - \vec{k}_2)\, \delta^{(3)}(\vec{p}_2 - \vec{k}_1) \right]$$

$$= \frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^{\phi}(x-y)$$

$$\times \Big\{ e^{-ip_1\cdot x}\, e^{-ip_2\cdot y}\, e^{iq_1\cdot x}\, e^{iq_2\cdot y}$$

$$+ e^{-ip_2\cdot x}\, e^{-ip_1\cdot y}\, e^{iq_1\cdot x}\, e^{iq_2\cdot y}$$

$$+ e^{-ip_1\cdot x}\, e^{-ip_2\cdot y}\, e^{iq_2\cdot x}\, e^{iq_1\cdot y}$$

$$+ e^{-ip_2\cdot x}\, e^{-ip_1\cdot y}\, e^{iq_2\cdot x}\, e^{iq_1\cdot y} \Big\}$$

$$= \frac{(-ig)^2}{2} \int d^4x\, d^4y\, \Delta_F^{\phi}(x-y)$$

$$\times \Big\{ e^{-i(p_1 - q_1)\cdot x}\, e^{-i(p_2 - q_2)\cdot y}$$

$$+ e^{-i(p_2 - q_1)\cdot x}\, e^{-i(p_1 - q_2)\cdot y}$$

$$+ e^{-i(p_1 - q_2)\cdot x}\, e^{-i(p_2 - q_1)\cdot y}$$

$$+ e^{-i(p_2 - q_2)\cdot x}\, e^{-i(p_1 - q_1)\cdot y} \Big\}$$

$$= (-ig)^2 \int d^4x\, d^4y\, \Delta_F^{\phi}(x-y) \Big\{ e^{-i(p_1 - q_1)\cdot x}\, e^{-i(p_2 - q_2)\cdot y}$$

$$+ e^{-i(p_2 - q_1)\cdot x}\, e^{-i(p_1 - q_2)\cdot y} \Big\}$$

The last equality is due to the fact that

$$D_F^\phi(x-y) = D_F^\phi(y-x)$$

Inserting

$$D_F^\phi(x-y) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2+i\epsilon}\,e^{-ik\cdot(x-y)}$$

we now have

$$(-ig)^2\, i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2+i\epsilon}$$

$$\times \int d^4x\,d^4y\,e^{-ik\cdot(x-y)}\left\{e^{-i(p_1-q_1)\cdot x}\,e^{-i(p_2-q_2)\cdot y}+e^{-i(p_2-q_1)\cdot x}\,e^{-i(p_1-q_2)\cdot y}\right\}$$

$$= (-ig)^2\, i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2+i\epsilon}$$

$$\times (2\pi)^8\left[\delta^{(4)}(k+p_1-q_1)\,\delta^{(4)}(k-p_2+q_2)\right.$$

$$\left.+\,\delta^{(4)}(k+p_2-q_1)\,\delta^{(4)}(k-p_1+q_2)\right]$$

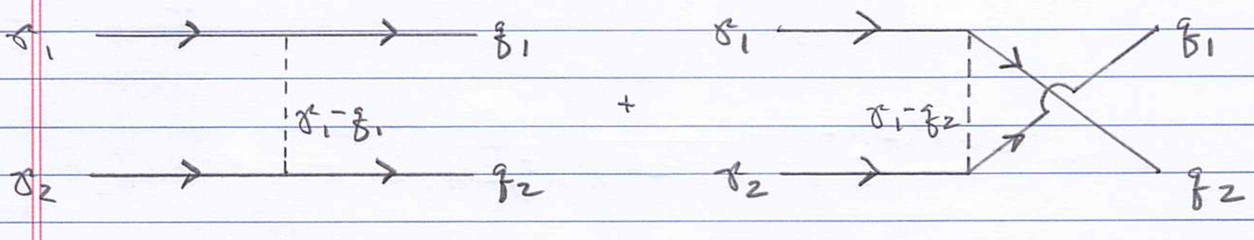
$$= (-ig)^2\, i\,(2\pi)^4\left\{\frac{1}{(p_1-q_1)^2-m^2+i\epsilon}\,\delta^{(4)}\left[(q_1-p_1)-p_2+q_2\right]\right.$$

$$\left.+\,\frac{1}{(p_2-q_1)^2-m^2+i\epsilon}\,\delta^{(4)}\left[(q_1-p_2)-p_1+q_2\right]\right\}$$

$$= \ldots$$

$$i(-ig)^2 \left\{ \frac{1}{(p_1-q_1)^2 - m^2} + \frac{1}{(p_1-q_2)^2 - m^2} \right\} (2\pi)^4 \delta^{(4)}(p_1+p_2-q_1-q_2)$$

We can write down 2 Feynman Diagrams associated with each term above.



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While Wick's Theorem helped in reducing the number of terms in our calculation of S-matrix elements, the number of terms remained sizeable (imagine even higher orders in perturbation theory!).

Feynman Diagrams eliminate all of this intermediate work. What are the rules?

From our calculation of pion decay and nucleon-nucleon scattering in scalar Yukawa theory, we see that the final result for the S-matrix element can be obtained from the Feynman diagram(s) for the interaction using the following rules:

1. For each vertex, assign a factor of $-ig$.
2. For each internal line, assign a factor of $\frac{i}{k^2-m^2}$.
3. Sum over the graphs.
4. Multiply by an overall factor of $(2\pi)^4 \delta^{(4)}\left(\sum_i p_i^\mu - \sum_i p_f^\mu\right)$.