Complex Scalar Fields

Consider the following Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^*$$

EOM for ϕ

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*}$$

$$= \partial_\mu \left[\frac{\partial}{\partial(\partial_\mu \phi^*)} \left(\partial_\nu \phi \, \eta^{\nu\alpha} \partial_\alpha \phi^* \right) \right] + m^2 \phi$$

$$= \partial_\mu \left(\partial_\nu \phi \, \eta^{\nu\alpha} \delta^\mu_\alpha \right) + m^2 \phi$$

$$= \partial_\mu \partial^\mu \phi + m^2 \phi$$

Then

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

Similarly,

$$\partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0$$

The conjugate momenta are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

$$= \partial^0 \phi^* = \eta^{00} \partial_0 \phi^* = \partial_0 \phi^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)}$$

$$= \frac{\partial}{\partial(\partial_0 \phi^*)} \left[\partial_0 \phi \, \partial^0 \phi^* \right] = \frac{\partial}{\partial(\partial_0 \phi^*)} \left[\partial_0 \phi \, \eta^{00} \partial_0 \phi^* \right] = \eta^{00} \partial_0 \phi = \partial_0 \phi$$

We can now construct the Hamiltonian density

$$\mathcal{H} = \pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L}$$

$$= \pi \dot{\phi} + \pi^* \dot{\phi}^* - \left(\partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^* \right)$$

$$= \pi \dot{\phi} + \pi^* \dot{\phi}^* - \dot{\phi} \dot{\phi}^* - \partial_i \phi \, \eta^{ij} \partial_j \phi^* + m^2 \phi \phi^*$$

$$= \pi \pi^* + \pi^* \pi - \pi^* \pi + \vec{\nabla} \phi \cdot \vec{\nabla} \phi^* + m^2 \phi \phi^*$$

$$= \pi \pi^* + \vec{\nabla} \phi \cdot \vec{\nabla} \phi^* + m^2 \phi \phi^*$$

The Poisson Brackets are

$$\left\{ \phi(\vec{x}, t), \pi(\vec{x}', t) \right\}_{PB} = \delta^{(3)}(\vec{x} - \vec{x}')$$

$$\left\{ \phi^*(\vec{x}, t), \pi^*(\vec{x}', t) \right\}_{PB} = \delta^{(3)}(\vec{x} - \vec{x}')$$

with all other Poisson brackets equal to zero.

The Lagrangian density is invariant under the global transformation

$$\phi \to e^{i\alpha}\phi$$

where α is independent of the spacetime position $x = (\vec{x}, t)$.

$\Rightarrow$ By Noether's Theorem, there is a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\,\delta\phi_a$$

N.B. Since this is a symmetry of $\mathcal{L}$ – i.e., $\delta\mathcal{L} = 0$ – we have $K^\mu = 0$.

And the expression for j^μ assumes the ϕ_a satisfy the EOM.

Here,

$$\phi_a = \{\phi, \phi^*\}$$

and

$$\delta\phi_a = \{i\alpha\phi, -i\alpha\phi^*\}$$

$$\mathcal{L} = \partial_\alpha\phi\,\partial^\alpha\phi^* + \ldots$$

$$= \partial_\alpha\phi\,\eta^{\alpha\nu}\partial_\nu\phi^* + \ldots$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \partial_\alpha\phi\,\eta^{\alpha\nu}\delta^\mu{}_\nu$$

$$= \partial^\mu\phi$$

Then

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\,\delta\phi^*$$

$$= \partial^\mu\phi^*(i\alpha\phi) + \partial^\mu\phi(-i\alpha\phi^*)$$

$$= i\alpha \left(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi \right)$$

We'll assign physical meaning to the conserved charge j^0 when we quantize the theory.

Quantization of a Complex Scalar Field

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[e^{ik_\mu x^\mu} \hat{a}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu} \hat{b}(\vec{k}) \right]$$

doesn't have to be $\hat{a}(\vec{k})$ since $\hat{\phi}^\dagger(x) \neq \hat{\phi}(x)$

$$\hat{\phi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[e^{-ik_\mu x^\mu} \hat{a}(\vec{k}) + e^{ik_\mu x^\mu} \hat{b}^\dagger(\vec{k}) \right]$$

with

$$[\hat{\phi}(\vec{x},t), \hat{\Pi}(\vec{x}',t)] = i\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$[\hat{\phi}^\dagger(\vec{x},t), \hat{\Pi}^\dagger(\vec{x}',t)] = i\,\delta^{(3)}(\vec{x}-\vec{x}')$$

and all other commutators set to zero.

The above imply

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$$

$$[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$$

with all other commutators equal to zero.

The conserved charge becomes

$$\hat{Q} = i\int d^3x \left[\hat{\pi}(x)\hat{\phi}(x) - \hat{\phi}^{\dagger}(x)\hat{\pi}^{\dagger}(x)\right]$$

$$j^{\mu} = i[\underbrace{(\partial^{\mu}\phi^{\dagger})}_{\pi}\phi - \phi^{\dagger}\underbrace{(\partial^{\mu}\phi)}_{\pi^*}]$$

where

$$\hat{\pi} = \dot{\hat{\phi}}^{\dagger}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{-iE(\vec{k})}{\sqrt{2E(\vec{k})}} \left[e^{-ik_{\mu}x^{\mu}}\hat{a}(\vec{k}) - e^{ik_{\mu}x^{\mu}}\hat{b}^{\dagger}(\vec{k})\right]$$

$$\hat{\pi}^{\dagger} = \dot{\hat{\phi}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{iE(\vec{k})}{\sqrt{2E(\vec{k})}} \left[e^{ik_{\mu}x^{\mu}}\hat{a}^{\dagger}(\vec{k}) - e^{-ik_{\mu}x^{\mu}}\hat{b}(\vec{k})\right]$$

Then

$$\hat{Q} = i\int d^3x \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} \frac{-iE(\vec{k})}{\sqrt{2E(\vec{k})}} \frac{1}{\sqrt{2E(\vec{k}')}}$$

$$\times\left[e^{-ik_{\mu}x^{\mu}}\hat{a}(\vec{k}) - e^{ik_{\mu}x^{\mu}}\hat{b}^{\dagger}(\vec{k})\right]$$

$$\times\left[e^{ik'_{\mu}x^{\mu}}\hat{a}^{\dagger}(\vec{k}') + e^{-ik'_{\mu}x^{\mu}}\hat{b}(\vec{k}')\right]$$

$$- i\int d^3x \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{iE(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times\left[e^{-ik_{\mu}x^{\mu}}\hat{a}(\vec{k}) + e^{ik_{\mu}x^{\mu}}\hat{b}^{\dagger}(\vec{k})\right]$$

$$\times\left[e^{ik'_{\mu}x^{\mu}}\hat{a}^{\dagger}(\vec{k}') - e^{-ik'_{\mu}x^{\mu}}\hat{b}(\vec{k}')\right]$$

$$= -i \int d^3x \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{-iE(\vec{k})}{\sqrt{2E(\vec{k})}} \frac{1}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[e^{-i(k_\mu - k'_\mu)x^\mu} \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}') + e^{-i(k_\mu + k'_\mu)x^\mu} \hat{a}(\vec{k}) \hat{b}(\vec{k}')$$

$$- e^{i(k_\mu + k'_\mu)x^\mu} \hat{b}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}') - e^{i(k_\mu - k'_\mu)x^\mu} \hat{b}^\dagger(\vec{k}) \hat{b}(\vec{k}') \Big]$$

$$- i \int d^3x \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{iE(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[e^{-i(k_\mu - k'_\mu)x^\mu} \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}') - e^{-i(k_\mu + k'_\mu)x^\mu} \hat{a}(\vec{k}) \hat{b}(\vec{k}')$$

$$+ e^{i(k_\mu + k'_\mu)x^\mu} \hat{b}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}') - e^{i(k_\mu - k'_\mu)x^\mu} \hat{b}^\dagger(\vec{k}) \hat{b}(\vec{k}') \Big]$$

$$= -i \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{-iE(\vec{k})}{\sqrt{2E(\vec{k})}} \frac{1}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}')$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') e^{-2iE(\vec{k})t} \hat{a}(\vec{k}) \hat{b}(\vec{k}')$$

$$- (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') e^{2iE(\vec{k})t} \hat{b}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}')$$

$$- (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \hat{b}^\dagger(\vec{k}) \hat{b}(\vec{k}') \Big]$$

$$- i \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \frac{iE(\vec{k}')}{\sqrt{2E(\vec{k}')}}$$

$$\times \Big[(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}')$$

$$- (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') e^{-2iE(\vec{k})t} \hat{a}(\vec{k}) \hat{b}(\vec{k}')$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') e^{2iE(\vec{k})t} \hat{b}^\dagger(\vec{k}) \hat{a}^\dagger(\vec{k}')$$

$$- (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \hat{b}^\dagger(\vec{k}) \hat{b}(\vec{k}') \Big]$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right]$$

$$+ \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right]$$

$$= \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right]$$

Then

$$:\hat{Q}: = \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) - \underbrace{\hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})}\right]$$

normal ordering gives $:Q:|0\rangle = 0$ – i.e., the charge of the vacuum is 0 (a requirement)

charge (+1)

number operator for particles with opposite charge (−1) but same mass (m) and momentum ($\vec{k}$)