The Harmonic Oscillator

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$$
\begin{aligned}
H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \\
&= \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]
\end{aligned}
$$

This has the form $u^2 + v^2$, which can be written as

$$u^2 + v^2 = (iu + v)(-iu + v)$$

Classically, we are dealing with functions. Quantum mechanically we are dealing with operators, so we have to be careful about ordering.
The Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Define the operators

$$
\begin{aligned}
\hat{a} &\equiv \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} + i\hat{p}) \\
\hat{a}^\dagger &\equiv \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} - i\hat{p})
\end{aligned}
$$

Now consider the product

$$
\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) \\
&= \frac{1}{\sqrt{2m\omega}}\left\{m^2\omega^2\hat{x}^2 + im\omega[\hat{x},\hat{p}] + \hat{p}^2\right\}
\end{aligned}
$$

and the product

$$
\begin{aligned}
\hat{a}\hat{a}^\dagger &= \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) \\
&= \frac{1}{\sqrt{2m\omega}}\left\{m^2\omega^2\hat{x}^2 - im\omega[\hat{x},\hat{p}] + \hat{p}^2\right\}
\end{aligned}
$$

The sum

$$\begin{aligned}\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger &= m\omega^2\hat{x}^2+\frac{\hat{p}^2}{m\omega}\\ &= \tfrac{2}{\omega}\hat{H}\end{aligned}$$

Then

$$\hat{H}=\tfrac{1}{2}\omega(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)$$

Let's consider the commutator of the operators $\hat{a}$ and $\hat{a}^\dagger$. ($\hbar=1$)

$$\begin{aligned}[][\hat{a},\hat{a}^\dagger] &= \frac{1}{2m\omega}\left\{(m\omega\hat{x}+i\hat{p})(m\omega\hat{x}-i\hat{p})-(m\omega\hat{x}-i\hat{p})(m\omega\hat{x}+i\hat{p})\right\}\\ &= \frac{1}{2m\omega}\left\{m^2\omega^2\hat{x}^2-im\omega[\hat{x},\hat{p}]+\hat{p}^2-(m^2\omega^2\hat{x}^2+im\omega[\hat{x},\hat{p}]+\hat{p}^2)\right\}\\ &= -i(i)\\ &= +1\end{aligned}$$

That is

$$[\hat{a},\hat{a}^\dagger]=1 \qquad \Rightarrow \qquad \hat{a}\hat{a}^\dagger=1+\hat{a}^\dagger\hat{a}$$

and of course

$$\begin{aligned}[][\hat{a},\hat{a}] &= 0\\ [\hat{a}^\dagger,\hat{a}^\dagger] &= 0\end{aligned}$$

Then

$$\begin{aligned}\hat{H} &= \tfrac{1}{2}\omega(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)\\ &= \tfrac{1}{2}\omega(\hat{a}^\dagger\hat{a}+1+\hat{a}^\dagger\hat{a})\\ &= \omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\end{aligned}$$

Now look at

$$\begin{aligned}[][\hat{H},\hat{a}] &= \omega\left\{(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\hat{a}-\hat{a}(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\right\}\\ &= \omega(\hat{a}^\dagger\hat{a}\hat{a}-\hat{a}\hat{a}^\dagger\hat{a})\\ &= \omega[\hat{a}^\dagger\hat{a}\hat{a}-(1+\hat{a}^\dagger\hat{a})\hat{a}]\\ &= -\omega\hat{a}\end{aligned}$$

and

$$\begin{aligned}[][\hat{H},\hat{a}^\dagger] &= \omega\left\{(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\hat{a}^\dagger-\hat{a}^\dagger(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\right\}\\ &= \omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}^\dagger\hat{a})\\ &= \omega[\hat{a}^\dagger\hat{a}\hat{a}^\dagger-\hat{a}^\dagger(\hat{a}\hat{a}^\dagger-1)]\\ &= \omega\hat{a}^\dagger\end{aligned}$$

Label the states by

$$|n\rangle \qquad E_n = (n + \tfrac{1}{2})\omega \qquad \hat{H}|n\rangle = (n + \tfrac{1}{2})\omega|n\rangle$$

and look at

$$\begin{aligned}
\hat{H}\hat{a}|n\rangle &= \omega(\hat{a}^\dagger \hat{a} + \tfrac{1}{2})\hat{a}|n\rangle \\
&= \omega(\hat{a}^\dagger \hat{a}\hat{a} + \tfrac{1}{2}\hat{a})|n\rangle \\
&= \omega\left[(\hat{a}\hat{a}^\dagger - 1)\hat{a} + \tfrac{1}{2}\hat{a}\right]|n\rangle \\
&= \omega\hat{a}(\hat{a}^\dagger\hat{a} + \tfrac{1}{2} - 1)|n\rangle \\
&= \hat{a}(\hat{H} - \omega)|n\rangle \\
&= \hat{a}\left[(\hat{n} + \tfrac{1}{2})\omega - \omega\right]|n\rangle \\
&= (n - \tfrac{1}{2})\omega\hat{a}|n\rangle \\
&\Rightarrow \quad \hat{a}|n\rangle \propto |n-1\rangle
\end{aligned}$$

What is the constant of proportionality?
From

$$\hat{H}|n\rangle = (\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\omega|n\rangle = (n + \tfrac{1}{2})\omega|n\rangle$$

we see that the number operator is

$$\hat{N} = \hat{a}^\dagger\hat{a}$$

Then

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n$$

But

$$\hat{a}|n\rangle = c|n-1\rangle$$

and

$$\langle n|\hat{a}^\dagger = c^*\langle n-1|$$

Then

$$n = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = |c|^2\langle n-1|n-1\rangle = |c|^2$$

and

$$c = \sqrt{n}$$

That is

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Now look at

$$\begin{aligned}
\hat{H}\hat{a}^\dagger|n\rangle &= \omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\hat{a}^\dagger|n\rangle \\
&= \omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\hat{a}^\dagger|n\rangle \\
&= \omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \tfrac{1}{2}\hat{a}^\dagger)|n\rangle \qquad\qquad \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \\
&= \omega\hat{a}^\dagger[(1 + \hat{a}^\dagger\hat{a}) + \tfrac{1}{2}]|n\rangle \\
&= \hat{a}^\dagger(\hat{H} + \omega)|n\rangle \\
&= \hat{a}^\dagger(n + \tfrac{1}{2} + 1)\omega|n\rangle \\
&= (n + \tfrac{1}{2} + 1)\omega\hat{a}^\dagger|n\rangle
\end{aligned}$$

$$\Rightarrow \quad \hat{a}^\dagger |n\rangle \propto |n+1\rangle$$

But

$$\langle n+1|\hat{a}^\dagger \hat{a}|n+1\rangle = n+1$$

and

$$n+1 = \langle n+1|\hat{a}^\dagger \hat{a}|n+1\rangle = |c|^2 \langle n+1|n+1\rangle = |c|^2$$

Then

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

One of the modt important aspects of the spectrum of states of the quantized harmonic oscillator is that its energy levels are separated by a uniform amount: $\hbar\omega$. The state $|n\rangle$ has energy $n\hbar\omega$ and the state $|n+1\rangle$ has energy $(n+1)\hbar\omega$. With

$$\hat{H}\,|n\rangle = \left(n + \tfrac{1}{2}\right)\hbar\omega\,|n\rangle$$

we have the vacuum state $|0\rangle$ with energy $\frac{1}{2}\hbar\omega$. Then we can think of the state $|n\rangle$ as a state of n quanta each of energy $\hbar\omega$ – i.e., we can think of it as a multiparticle state.
With this interpretation, the operator $\hat{a}^\dagger$ is a creation operator that creates a quantum of energy $\hbar\omega$, whereas $\hat{a}$ is a annihilation operator that annihilates a quantum of energy $\hbar\omega$.
Now we can generate all of the states beginning with the vacuum state defined by

$$\hat{a}\,|0\rangle = 0$$

Then

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n |0\rangle$$

$\dfrac{1}{\sqrt{n!}}$ cancels out the $\sqrt{n!}$ that appears when the $\hat{a}^\dagger$ act n times.
For example,

$$\begin{aligned}
|2\rangle &= \frac{1}{\sqrt{2}}\hat{a}^\dagger\left(\hat{a}^\dagger\,|0\rangle\right) \\
&= \frac{1}{\sqrt{2}}\hat{a}^\dagger\left(\sqrt{1}\,|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\sqrt{1+1}\,|2\rangle \\
&= |2\rangle
\end{aligned}$$