Pion Decay Rate in Scalar Yukawa Theory

$$d\Gamma = \frac{1}{2E_1} |\mathcal{M}|^2 \, d\Pi_{LIPS}$$

$$= \frac{1}{2E_1} |-g|^2 (2\pi)^4 \delta^{(4)}(k^\mu - q^\mu - q'^\mu) \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \frac{1}{2E(\vec{q})} \frac{1}{2E(\vec{q}')}$$

In the rest frame of the pion ($\vec{k} = 0$)

$$E_1 = m$$

Then

$q^\mu = k^\mu - q'^\mu$

$$\Gamma = \frac{g^2}{32\pi^2 m} \int d^3q \, d^3q' \frac{1}{E(\vec{q})E(\vec{q}')} \delta^{(4)}(k^\mu - q^\mu - q'^\mu)$$

$$= \frac{g^2}{32\pi^2 m} \int d^3q' \frac{1}{E(\vec{q}')\underbrace{E(\vec{k}-\vec{q}')}_{E(-\vec{q}')=E(\vec{q}')}} \delta\big(\underbrace{E(\vec{k})}_{m} - \underbrace{E(\vec{k}-\vec{q}')}_{E(-\vec{q}')=E(\vec{q}')} - E(\vec{q}')\big)$$

$$= \frac{g^2}{32\pi^2 m} \int d\Omega_{q'} \int q'^2 dq' \frac{1}{E^2(\vec{q}')} \delta[m - 2E(\vec{q}')]$$

Transform variables

$$E(\vec{q}) = [q^2 + M^2]^{1/2}$$

← mass of the nucleon

Then

$$p^2 = E^2(\vec{p}) - M^2$$

and

$$2p\,dp = 2E(\vec{p})\,dE(\vec{p})$$

and

$$p^2\,dp = p\,E(\vec{p})\,dE(\vec{p})$$

$$= \left[E^2(\vec{p}) - M^2\right]^{1/2} E(\vec{p})\,dE(\vec{p})$$

Then

$$\Gamma = \frac{g^2}{32\pi^2 m}\int d\Omega_p \int \underbrace{dE(\vec{p})\left[E^2(\vec{p}) - M^2\right]^{1/2}\frac{1}{E(\vec{p})}\,\delta\left[m - 2E(\vec{p})\right]}_{\text{use change of variables}}$$

$E(\vec{p}) \rightarrow 2E(\vec{p})$

$\left[E^2(\vec{p}) - M^2\right]^{1/2} \rightarrow$

$\left[\frac{1}{4}\left(2E(\vec{p})\right)^2 - M^2\right]^{1/2}$

$\frac{1}{E(\vec{p})} \rightarrow \frac{1}{2E(\vec{p})}$

$$= \frac{g^2}{32\pi^2 m}\,4\pi\,\frac{1}{m}\left[\left(\frac{m}{2}\right)^2 - M^2\right]^{1/2}$$

$$= \frac{g^2}{8\pi m^2}\left[\left(\frac{m}{2}\right)^2 - M^2\right]^{1/2}$$

The half life, $\tau_{1/2}$, in this case would be

$$\tau_{1/2} = \frac{1}{\Gamma}$$

Two-Nucleon Differential Scattering Cross Section in Scalar Yukawa Theory

Let's consider the canonical case

$$p_1 + p_2 \longrightarrow p_3 + p_4$$

Then

$$d\sigma = \frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1-\vec{v}_2|}|\mathcal{M}|^2\, d\Pi_{LIPS}$$

$$= \frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1-\vec{v}_2|}|\mathcal{M}|^2 \frac{1}{4E_3E_4}\frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$$

$$= \frac{1}{4}\left[(p_1\cdot p_2)^2 - m_1^2 m_2^2\right]^{-1/2}|\mathcal{M}|^2\frac{1}{4E_3E_4}\frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$$

And let's consider the case when $m_1 = m_2 \equiv m$.

Since $d\sigma$ is Lorentz invariant, we can evaluate it in whatever frame makes the calculation simplest.

Let's choose the Center of Momentum (COM) Frame. In this frame

$$\vec{p}_1 + \vec{p}_2 = 0$$

Then

$$\vec{p}_3 + \vec{p}_4 = 0$$

We have overall 4-momentum (not just 3-momentum) conservation

$$p_1 \cdot p_2 = p_1^{\mu} p_{2\mu}$$

In the rest frame of particle 1,

$$p_1^{\mu} = (E_1, \vec{0}) = (m_1, \vec{0})$$

and

$$p_1^{\mu} p_{2\mu} = E_1 p_{20} = E_1 \eta_{00} p_2^0 = E_1 E_2 = m_1 E_2$$

Then

$$E_2 = \frac{1}{m_1} p_1 \cdot p_2$$

The relative velocity in this frame of reference is

$$|\vec{v}_2|$$

$$= \left| \frac{\vec{p}_2}{E_2} \right| \qquad \text{recall } \vec{p}_2 = \gamma_2 m_2 \vec{v}_2 \text{ and } E_2 = \gamma_2 m_2$$

$$\gamma_2 \equiv 1/\sqrt{1-v_2^2}$$

But

$$\left| \frac{\vec{p}_2}{E_2} \right| = \frac{1}{E_2} |\vec{p}_2| = \frac{1}{E_2} \left[E_2^2 - m_2^2 \right]^{1/2}$$

All Lorentz invariants.

$$= \frac{1}{E_2} \left[\frac{(p_1 \cdot p_2)^2}{m_1^2} - m_2^2 \right]^{1/2}$$

$E_1^2 = m_1^2$

$E_2^2 = |\vec{p}_2|^2 + m_2^2$

$$= \frac{1}{E_2 m_1} \left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]^{1/2}$$

$\Rightarrow E_1 E_2 |\vec{v}_2|$

$= \left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]^{1/2}$

$$= \frac{1}{E_1 E_2} \left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]^{1/2}$$

$E_1^2 = |\vec{p}_1|^2 + m_1^2$

$= E_1 E_2 |\vec{v}_1 - \vec{v}_2|$

$E_2^2 = |\vec{p}_2|^2 + m_2^2$

$$p_1 + p_2 = p_3 + p_4$$

Then

$$s^2 \equiv (p_1 + p_2)^2$$

$$= (p_3 + p_4)^2$$

Compute

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2\, p_1 \cdot p_2 = 2m^2 + 2\, p_1 \cdot p_2$$

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2\, p_3 \cdot p_4 = 2m^2 + 2\, p_3 \cdot p_4$$

Then

$$p_1 \cdot p_2 = p_3 \cdot p_4$$

and

$$\left[(p_1 \cdot p_2)^2 - m^4 \right]^{1/2} = \left[(p_3 \cdot p_4)^2 - m^4 \right]^{1/2}$$

$$= \left[\left(E_3 E_4 - \vec{p}_3 \cdot \vec{p}_4 \right)^2 - \left(E_3^2 - |\vec{p}_3|^2 \right)\left(E_4^2 - |\vec{p}_4|^2 \right) \right]^{1/2}$$

$$= \left[\left(E_3 E_4 + |\vec{p}_3|^2 \right)^2 - \left(E_3^2 - |\vec{p}_3|^2 \right)\left(E_4^2 - |\vec{p}_3|^2 \right) \right]^{1/2}$$

$$= \Big[\cancel{E_3^2 E_4^2} + \cancel{|\vec{p}_3|^4} + 2 E_3 E_4 |\vec{p}_3|^2$$

$$- \cancel{E_3^2 E_4^2} + E_3^2 |\vec{p}_3|^2 + E_4^2 |\vec{p}_3|^2 - \cancel{|\vec{p}_3|^4} \Big]^{1/2}$$

$$= \left[(E_3 + E_4)^2 \, |\vec{p}_3|^2 \right]^{1/2}$$

But

$$s^2 = (p_3 + p_4)^2 = \eta_{\mu\nu}(p_3 + p_4)^\mu (p_3 + p_4)^\nu$$

$$= (E_3 + E_4)^2 - |\vec{p}_3 + \vec{p}_4|^2$$

$$= (E_3 + E_4)^2$$

Then

$$[(p_1 \cdot p_2)^2 - m^4]^{1/2} = \sqrt{s}\,|\vec{p}_3|$$

and

$$d\sigma = \frac{1}{4}\frac{1}{\sqrt{s}}\frac{1}{|\vec{p}_3|}|\mathcal{M}|^2 \frac{1}{4E_3E_4}\frac{d^3p_3}{(2\pi)^3}\frac{d^3p_4}{(2\pi)^3}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

Let's integrate over $\vec{p}_4$ $(= -\vec{p}_3)$

$$d\sigma = \frac{1}{4}\frac{1}{\sqrt{s}}\frac{1}{|\vec{p}_3|}|\mathcal{M}|^2 \frac{1}{2E_3}\frac{d^3p_3}{(2\pi)^3}\int \frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}(2\pi)^4\delta^4\underbrace{(p_1 + p_2 - p_3 - p_4)}$$

$$\Rightarrow \vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3$$
$$= -\vec{p}_3$$

$$= \frac{1}{4}\frac{1}{\sqrt{s}}\frac{1}{|\vec{p}_3|}|\mathcal{M}|^2\frac{1}{4E_3^2}\frac{d^3p_3}{(2\pi)^3}\,2\pi\,\delta(E_1 + E_2 - 2E_3)$$

$$= \frac{1}{4}\frac{1}{\sqrt{s}}\frac{1}{|\vec{p}_3|}|\mathcal{M}|^2\frac{1}{4E_3^2}\frac{1}{(2\pi)^2}\,d\Omega_{\vec{p}_3}\,|\vec{p}_3|^2\,d|\vec{p}_3|\,\delta(E_1 + E_2 - 2E_3)$$

$$= \frac{1}{64\pi^2} \frac{1}{\sqrt{s}} |\mathcal{M}|^2 \frac{1}{E_3} dE_3 \, \delta(E_1 + E_2 - 2E_3) \, d\Omega_{\vec{p}_3}$$

Then

$$\frac{d\sigma}{d\Omega_{\vec{p}_3}} = \frac{1}{64\pi^2} \frac{1}{\sqrt{s}} \int |\mathcal{M}|^2 \underbrace{\frac{1}{E_3} dE_3}_{\frac{1}{2E_3} d(2E_3)} \delta(E_1 + E_2 - 2E_3) \quad 2E_3 = E_1 + E_2$$

$$= \frac{1}{64\pi^2} \frac{1}{\sqrt{s}} |\mathcal{M}|^2 \frac{1}{E_1 + E_2}$$

$$= \frac{1}{64\pi^2} \frac{1}{s} |\mathcal{M}|^2$$

$$\mathcal{M} = (-ig)^2 \left\{ \frac{1}{(p_1 - q_1)^2 - m^2} + \frac{1}{(p_1 - q_2)^2 - m^2} \right\}$$

Then

$$\frac{d\sigma}{d\Omega_{q_1}} = \frac{1}{64\pi^2} \frac{1}{s} g^4 \left| \frac{1}{(p_1 - q_1)^2 - m^2} + \frac{1}{(p_1 - q_2)^2 - m^2} \right|^2$$

Choose

$$p_1 = (E, |\vec{p}|, 0, 0)$$

$$p_2 = (E, -|\vec{p}|, 0, 0)$$

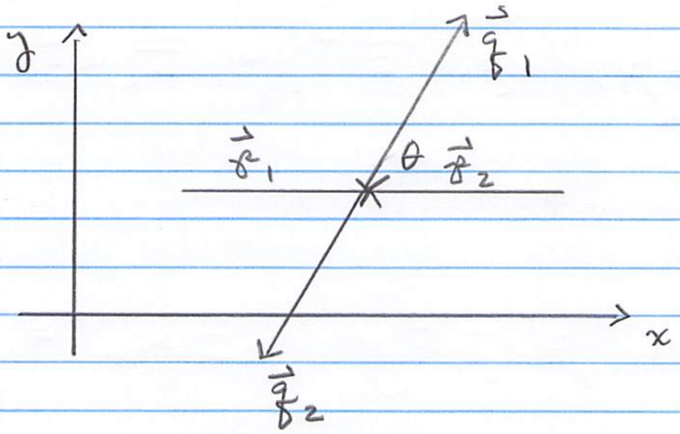
$$q_1 = (E, |\vec{p}|\cos\theta, |\vec{p}|\sin\theta, 0)$$

$$q_2 = (E, -|\vec{p}|\cos\theta, -|\vec{p}|\sin\theta, 0)$$

$$\vec{p}_1 + \vec{p}_2 = 0 = \vec{q}_1 + \vec{q}_2$$

$$\vec{p}_1 = -\vec{p}_2 \qquad \vec{q}_1 = -\vec{q}_2$$

$$E(\vec{p}_1) = E(\vec{p}_2) = E = E(\vec{q}_1) = E(\vec{q}_2)$$



We need

$$(p_1 - q_{\gamma 1})^2 - m^2$$

$$= p_1^2 + q_{\gamma 1}^2 - 2\, p_1 \cdot q_{\gamma 1} - m^2$$

$$= E^2 - |\vec{p}|^2 + E^2 - |\vec{p}|^2 \cos^2\theta - |\vec{p}|^2 \sin^2\theta$$
$$\quad - 2(E^2 - |\vec{p}|^2 \cos\theta) - m^2$$

$$= 2E^2 - 2|\vec{p}|^2 - 2E^2 + 2|\vec{p}|^2 \cos\theta - m^2$$

$$= -2|\vec{p}|^2 (1 - \cos\theta) - m^2$$

and

$$(p_1 - q_{\gamma 2})^2 - m^2$$

$$= p_1^2 + q_{\gamma 2}^2 - 2\, p_1 \cdot q_{\gamma 2} - m^2$$

$$= E^2 - |\vec{p}|^2 + E^2 - |\vec{p}|^2 - 2(E^2 + |\vec{p}|^2 \cos\theta) - m^2$$

$$= -2|\vec{p}|^2(1+\cos\theta) - m^2$$

Then

$$\frac{d\sigma}{d\Omega}_{\gamma_1} = \frac{1}{64\pi^2}\frac{1}{s}g^4 \left| \frac{1}{-2|\vec{p}|^2(1-\cos\theta)-m^2} + \frac{1}{-2|\vec{p}|^2(1+\cos\theta)-m^2} \right|^2$$

Denominator: $\left(2|\vec{p}|^2(1-\cos\theta)+m^2\right)\left(2|\vec{p}|^2(1+\cos\theta)+m^2\right)$

$$= 4|\vec{p}|^4(1-\cos\theta)(1+\cos\theta) + 2|\vec{p}|^2(1-\cos\theta)m^2 + m^2 2|\vec{p}|^2(1+\cos\theta) + m^4$$

$$= 4|\vec{p}|^4\sin^2\theta + 4|\vec{p}|^2m^2 + m^4$$

$$= (2|\vec{p}|^2 + m^2)^2 - 4|\vec{p}|^4\cos^2\theta$$

Numerator: $2|\vec{p}|^2(1+\cos\theta) + m^2 + 2|\vec{p}|^2(1-\cos\theta) + m^2$

$$= 4|\vec{p}|^2 + 2m^2$$

Then

$$\frac{d\sigma}{d\Omega}_{\gamma_1} = \underbrace{\frac{g^4}{64\pi^2 s}}_{\frac{g^4}{128\pi^2 E}} \left| \frac{4|\vec{p}|^2 + 2m^2}{(2|\vec{p}|^2+m^2)^2 - 4|\vec{p}|^4\cos^2\theta} \right|^2$$

since $s = E_1 + E_2 = 2E$

The variable

$$s = (p_1 + p_2)^2$$

is one of three <u>Mandelstam</u> <u>Variables</u>

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \qquad s\text{-channel}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \qquad t\text{-channel}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \qquad u\text{-channel}$$