Interactions in Scalar Yukawa Theory

Yukawa Theory

- spin $\frac{1}{2}$ nucleons
- interactions mediated by spin 0 pseudoscalar pion field

Scalar Yukawa Theory

- nucleons / antinucleons represented by a complex scalar field
- interactions mediated by scalar pion field

$$\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - M^2 \psi^\dagger \psi - \frac{1}{2} m^2 \phi^2 \underbrace{- g \psi^\dagger \psi \phi}_{\mathcal{L}_{int} = -\mathcal{H}_{int}}$$

Weak coupling ($g \ll M, m$) is assumed.

We'll use

$$\hat{\psi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[e^{ik_\mu x^\mu} \hat{a}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu} \hat{b}(\vec{k}) \right]$$

$$\hat{\psi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[e^{-ik_\mu x^\mu} \hat{a}(\vec{k}) + e^{ik_\mu x^\mu} \hat{b}^\dagger(\vec{k}) \right]$$

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{k})}} \left[e^{ik_\mu x^\mu} \hat{c}^\dagger(\vec{k}) + e^{-ik_\mu x^\mu} \hat{c}(\vec{k}) \right]$$

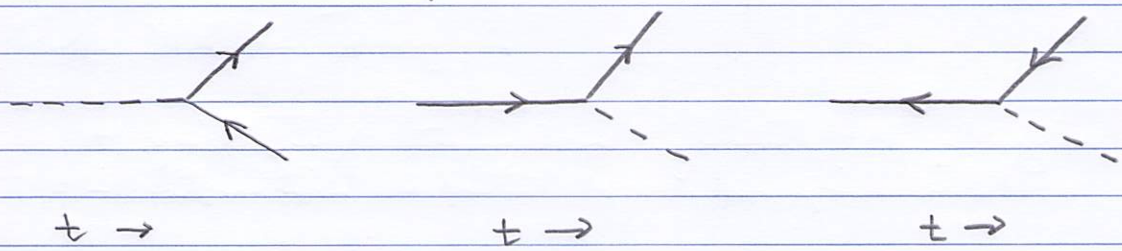
where

$\hat{a}^\dagger(\vec{k})$ creates a positively charged nucleon with momentum $\vec{k}$

$\hat{b}^\dagger(\vec{k})$ " " negatively " " " " "

$\hat{c}^\dagger(\vec{k})$ creates a neutral pion with momentum $\vec{k}$

In diagrams, to first order in perturbation theory we have



pion decay to nucleon/antinucleon pair — nucleon emission of a pion — antinucleon emission of a pion

Let's consider pion decay:

$$|i\rangle \equiv \sqrt{2E(\vec{k})}\,\hat{c}^\dagger(\vec{k})\,|0\rangle$$

$$|f\rangle \equiv \sqrt{2E(\vec{p})}\sqrt{2E(\vec{q})}\,\hat{a}^\dagger(\vec{p})\,\hat{b}^\dagger(\vec{q})\,|0\rangle$$

$$\langle f|\hat{S}|i\rangle = \langle f|T\{e^{i\int d^4x\,\mathcal{L}_I}\}|i\rangle \simeq 1 + i\langle f|\int d^4x\,\mathcal{L}_I|i\rangle$$

to 1st order in perturbation theory

The S-matrix amplitude for this process is

$$\langle f|\hat{S}|i\rangle \simeq -i\,\langle f|\int d^4x\, g\,\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\phi}(x)|i\rangle$$

Let's calculate $\langle f|\int d^4x\,\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\phi}(x)|i\rangle$:

$$\langle f | d^4x\, \hat{\phi}^\dagger \hat{\psi} \hat{\phi} | i \rangle$$

$$= \int d^4x \sqrt{2E(\vec{p})}\sqrt{2E(\vec{q})} \langle 0 | \hat{b}(\vec{q})\hat{a}(\vec{p})$$

$$\times \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{p}_1)}} \left[e^{-ip_{1\mu}x^\mu} \hat{a}(\vec{p}_1) + e^{ip_{1\mu}x^\mu} \hat{b}^\dagger(\vec{p}_1) \right]$$

$$\times \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{p}_2)}} \left[e^{ip_{2\mu}x^\mu} \hat{a}^\dagger(\vec{p}_2) + e^{-ip_{2\mu}x^\mu} \hat{b}(\vec{p}_2) \right]$$

$$\times \int \frac{d^3p_3}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{p}_3)}} \left[e^{ip_{3\mu}x^\mu} \hat{c}^\dagger(\vec{p}_3) + e^{-ip_{3\mu}x^\mu} \hat{c}(\vec{p}_3) \right] \sqrt{2E(\vec{k})}\, \hat{c}^\dagger(\vec{k}) | 0 \rangle$$

$$= \int d^4x \sqrt{2E(\vec{p})}\sqrt{2E(\vec{q})}\sqrt{2E(\vec{k})} \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{p}_1)}}\frac{1}{\sqrt{2E(\vec{p}_2)}}\frac{1}{\sqrt{2E(\vec{p}_3)}}$$

$$\times \langle 0 | \hat{b}(\vec{q})\hat{a}(\vec{p}) \left[e^{-ip_{1\mu}x^\mu} \hat{a}(\vec{p}_1) + e^{ip_{1\mu}x^\mu} \hat{b}^\dagger(\vec{p}_1) \right]$$

$$\times \left[e^{ip_{2\mu}x^\mu} \hat{a}^\dagger(\vec{p}_2) + e^{-ip_{2\mu}x^\mu} \hat{b}(\vec{p}_2) \right]$$

$$\times \left[e^{ip_{3\mu}x^\mu} \hat{c}^\dagger(\vec{p}_3) + e^{-ip_{3\mu}x^\mu} \hat{c}(\vec{p}_3) \right] \hat{c}^\dagger(\vec{k}) | 0 \rangle$$

Let's focus on $\langle 0 | \ldots | 0 \rangle$:

$$\langle 0 | \hat{b}(\vec{q})\hat{a}(\vec{p}) \left[e^{-i(p_{1\mu}-p_{2\mu})x^\mu} \hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2) + e^{-i(p_{1\mu}+p_{2\mu})x^\mu} \hat{a}(\vec{p}_1)\hat{b}(\vec{p}_2) \right.$$

$$\left. + e^{i(p_{1\mu}+p_{2\mu})x^\mu} \hat{b}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2) + e^{i(p_{1\mu}-p_{2\mu})x^\mu} \hat{b}^\dagger(\vec{p}_1)\hat{b}(\vec{p}_2) \right]$$

$$\times \left[e^{ip_{3\mu}x^\mu} \hat{c}^\dagger(\vec{p}_3) + e^{-ip_{3\mu}x^\mu} \hat{c}(\vec{p}_3) \right] \hat{c}^\dagger(\vec{k}) | 0 \rangle$$

$$= \ldots$$

$$\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\Big[e^{-i(p_{1\mu}-p_{2\mu}-p_{3\mu})x^\mu}\,\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)$$
$$+e^{-i(p_{1\mu}-p_{2\mu}+p_{3\mu})x^\mu}\,\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}(\vec{p}_3)$$
$$+e^{-i(p_{1\mu}+p_{2\mu}-p_{3\mu})x^\mu}\,\hat{a}(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)$$
$$+e^{-i(p_{1\mu}+p_{2\mu}+p_{3\mu})x^\mu}\,\hat{a}(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}(\vec{p}_3)$$
$$+e^{i(p_{1\mu}+p_{2\mu}+p_{3\mu})x^\mu}\,\hat{b}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)$$
$$+e^{i(p_{1\mu}+p_{2\mu}-p_{3\mu})x^\mu}\,\hat{b}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}(\vec{p}_3)$$
$$+e^{i(p_{1\mu}-p_{2\mu}+p_{3\mu})x^\mu}\,\hat{b}^\dagger(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)$$
$$+e^{i(p_{1\mu}-p_{2\mu}-p_{3\mu})x^\mu}\,\hat{b}^\dagger(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}(\vec{p}_3)\Big]c^\dagger(\vec{k})|0\rangle$$

$$(1)\quad = e^{-i(p_{1\mu}-p_{2\mu}-p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)c^\dagger(\vec{k})|0\rangle$$
$$(2)\quad + e^{-i(p_{1\mu}-p_{2\mu}+p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}(\vec{p}_3)c^\dagger(\vec{k})|0\rangle$$
$$(3)\quad + e^{-i(p_{1\mu}+p_{2\mu}-p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{a}(\vec{p}_1)\hat{b}(\vec{p}_2)c^\dagger(\vec{p}_3)c^\dagger(\vec{k})|0\rangle$$
$$(4)\quad + e^{-i(p_{1\mu}+p_{2\mu}+p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{a}(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}(\vec{p}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$
$$(5)\quad + e^{i(p_{1\mu}+p_{2\mu}+p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{b}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$
$$(6)\quad + e^{i(p_{1\mu}+p_{2\mu}-p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{b}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}(\vec{p}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$
$$(7)\quad + e^{i(p_{1\mu}-p_{2\mu}+p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{b}^\dagger(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$
$$(8)\quad + e^{i(p_{1\mu}-p_{2\mu}-p_{3\mu})x^\mu}\,\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{b}^\dagger(\vec{p}_1)\hat{b}(\vec{p}_2)\hat{c}(\vec{p}_3)c^\dagger(\vec{k})|0\rangle$$

(1):

$$\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{p})\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{c}^\dagger(\vec{p}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$

Since all of the commutators vanish except those for $[\hat{a}, \hat{a}^\dagger]$, $[\hat{b}, \hat{b}^\dagger]$, and $[\hat{c}, \hat{c}^\dagger]$, I can

i) slide $\hat{b}(q)$ to the right to $|0\rangle$, which gives 0

or

ii) slide $\hat{c}^\dagger(\vec{y}_3)$ to the left to $\langle 0|$, which gives 0

(2):

$$\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{y})\hat{a}(\vec{y}_1)\hat{a}^\dagger(\vec{y}_2)\hat{c}(\vec{y}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$

$= 0$ for reason (i) above

(3):

$$\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{y})\hat{a}(\vec{y}_1)\hat{b}(\vec{y}_2)\hat{c}^\dagger(\vec{y}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$

$= 0$

since I can slide $\hat{b}(\vec{y}_2)$ to $|0\rangle$, which gives 0.

(4):

$$\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{y})\hat{a}(\vec{y}_1)\hat{b}(\vec{y}_2)\hat{c}(\vec{y}_3)\hat{c}^\dagger(\vec{k})|0\rangle$$

$= 0$

for same reason as in case (3).

5 :

$$\langle 0 | \hat{b}(\vec{q}) \hat{a}(\vec{r}) \hat{b}^\dagger(\vec{r}_1) \hat{a}^\dagger(\vec{r}_2) \hat{c}^\dagger(\vec{r}_3) \hat{c}(\vec{k}) | 0 \rangle$$

$$= 0$$

since I can slide $\hat{c}^\dagger(\vec{r}_3)$ to the left to $\langle 0|$, which gives 0.

6 :

$$\langle 0 | \hat{b}(\vec{q}) \hat{a}(\vec{r}) \hat{b}^\dagger(\vec{r}_1) \hat{a}^\dagger(\vec{r}_2) \hat{c}(\vec{r}_3) \hat{c}^\dagger(\vec{k}) | 0 \rangle$$

$$= \langle 0 | \hat{b}(\vec{q}) \hat{a}(\vec{r}) \hat{b}^\dagger(\vec{r}_1) \hat{a}^\dagger(\vec{r}_2) \hat{c}^\dagger(\vec{k}) \hat{c}(\vec{r}_3) | 0 \rangle$$

$$+ \langle 0 | \hat{b}(\vec{q}) \hat{a}(\vec{r}) \hat{b}^\dagger(\vec{r}_1) \hat{a}^\dagger(\vec{r}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{r}_3)$$

$$= \langle 0 | \hat{b}(\vec{q}) \hat{b}^\dagger(\vec{r}_1) \hat{a}(\vec{r}) \hat{a}^\dagger(\vec{r}_2) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{r}_3)$$

$$= \langle 0 | \hat{b}(\vec{q}) \hat{b}^\dagger(\vec{r}_1) \hat{a}^\dagger(\vec{r}_2) \hat{a}(\vec{r}) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{r}_3)$$

$$+ \langle 0 | \hat{b}(\vec{q}) \hat{b}^\dagger(\vec{r}_1) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{r} - \vec{r}_2) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{r}_3)$$

$$= \langle 0 | \hat{b}^\dagger(\vec{r}_1) \hat{b}(\vec{q}) | 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{r} - \vec{r}_2) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{r}_3)$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{r}_1) (2\pi)^3 \delta^{(3)}(\vec{r} - \vec{r}_2) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{r}_3)$$

7 :

$$\langle 0 | \hat{b}(\vec{q}) \hat{a}(\vec{r}) \hat{b}^\dagger(\vec{r}_1) \hat{b}(\vec{r}_2) \hat{c}^\dagger(\vec{r}_3) \hat{c}^\dagger(\vec{k}) | 0 \rangle$$

$$= 0$$

since $\hat{b}(\vec{r}_2)$ can slide to the right to $|0\rangle$, or $\hat{c}^\dagger(\vec{r}_3)$ can slide to the left to $\langle 0|$, or $\hat{a}(\vec{r})$ can slide to the right to $|0\rangle$, all of which give 0.

8:

$$\langle 0|\hat{b}(\vec{q})\hat{a}(\vec{r})\hat{b}^{\dagger}(\vec{r}_1)\hat{b}(\vec{r}_2)\hat{c}(\vec{r}_3)\hat{c}^{\dagger}(\vec{k})|0\rangle$$

$$= 0$$

since q can slide $\hat{b}(\vec{r}_2)$ to the right to $|0\rangle$, or push $\hat{a}(\vec{r})$ to the right to $|0\rangle$, both of which give 0.

Then

$$\langle f|\int d^4x\, \psi^{\dagger}\psi\phi\,|i\rangle$$

$$= \int d^4x \sqrt{2E(\vec{r})}\sqrt{2E(\vec{q})}\sqrt{2E(\vec{k})}\int \frac{d^3r_1}{(2\pi)^3}\frac{d^3r_2}{(2\pi)^3}\frac{d^3r_3}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{r}_1)}}\frac{1}{\sqrt{2E(\vec{r}_2)}}\frac{1}{\sqrt{2E(\vec{r}_3)}}$$

$$\times e^{i(r_{1\mu}+r_{2\mu}-r_{3\mu})x^{\mu}}(2\pi)^9\delta^{(3)}(\vec{k}-\vec{r}_3)\delta^{(3)}(\vec{r}-\vec{r}_2)\delta^{(3)}(\vec{q}-\vec{r}_1)$$

$$= \int d^4x\, e^{i(q_\mu+r_\mu-k_\mu)x^{\mu}}$$

$$= (2\pi)^4\delta^{(4)}(k_\mu-q_\mu-r_\mu)$$

and

$$\langle f|\hat{S}|i\rangle \simeq -ig(2\pi)^4\delta^{(4)}(k_\mu-q_\mu-r_\mu)$$

$$= i(2\pi)^4 \delta^{(4)}\left(\sum p_i^{\mu} - \sum p_f^{\mu}\right) \mathcal{M}$$

Then in this case,

$$\mathcal{M} = -g$$