The Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$= \frac{1}{2m}\left[p^2 + (m\omega x)^2 \right]$$

This has the form $u^2 + v^2$, which can be written as

$$u^2 + v^2 = (iu + v)(-iu + v)$$

Classically, we are dealing with functions. Quantum mechanically we are dealing with operators, so we have to be careful about ordering.

The Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

Define the operators

$$\hat{a} \equiv \frac{1}{\sqrt{2m\omega}} \left(m\omega \hat{x} + i \hat{p} \right)$$

$$\hat{a}^{+} \equiv \frac{1}{\sqrt{2m\omega}} \left(m\omega \hat{x} - i \hat{p} \right)$$

Now consider the product

$$\hat{a}^{+}\hat{a} = \frac{1}{2m\omega}(m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p})$$

$$= \frac{1}{2m\omega}\left\{m^2\omega^2\hat{x}^2 + im\omega[\hat{x},\hat{p}] + \hat{p}^2\right\}$$

and the product

$$\hat{a}\hat{a}^{+} = \frac{1}{2m\omega}(m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p})$$

$$= \frac{1}{2m\omega}\left\{m^2\omega^2\hat{x}^2 - im\omega[\hat{x},\hat{p}] + \hat{p}^2\right\}$$

The sum

$$\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+} = m\omega\hat{x}^2 + \frac{\hat{p}^2}{m\omega}$$

$$= \frac{2}{\omega}\hat{H}$$

Then

$$\hat{H} = \frac{1}{2}\omega(\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+})$$

Let's consider the commutator of the operators $\hat{a}$ and $\hat{a}^{+}$.

$$[\hat{a},\hat{a}^{+}]$$

$$= \frac{1}{2m\omega}\left\{(m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p}) - (m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p})\right\}$$

$$= \frac{1}{2m\hbar}\Big\{ m^2\omega^2\hat{x}^2 - im\omega[\hat{x},\hat{p}] + \hat{p}^2$$

$$- \left(m^2\omega^2\hat{x}^2 + im\omega[\hat{x},\hat{p}] + \hat{p}^2 \right) \Big\}$$

$$= -i[\hat{x},\hat{p}]$$

$\hbar = 1$ $$= -i(i)$$

$$= +1$$

That is

$$[\hat{a},\hat{a}^\dagger] = 1 \quad \Rightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$$

and of course

$$[\hat{a},\hat{a}] = 0$$

$$[\hat{a}^\dagger,\hat{a}^\dagger] = 0$$

Then

$$H = \frac{1}{2}\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$$

$$= \frac{1}{2}\omega\left(\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}\right)$$

$$= \omega\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right)$$

Now look at

$$[\hat{H}, \hat{a}]$$

$$= \omega\left\{\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right)\hat{a} - \hat{a}\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right)\right\}$$

$$= \omega\left(\hat{a}^{+}\hat{a}\,\hat{a} - \hat{a}\hat{a}^{+}\hat{a}\right)$$

$$= \omega\left[\hat{a}^{+}\hat{a}\,\hat{a} - \left(1 + \hat{a}^{+}\hat{a}\right)\hat{a}\right]$$

$$= -\omega\hat{a}$$

and

$$[H, \hat{a}^{+}]$$

$$= \omega\left\{\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right)\hat{a}^{+} - \hat{a}^{+}\left(\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right)\right\}$$

$$= \omega\left(\hat{a}^{+}\hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}^{+}\hat{a}\right)$$

$$= \omega\left[\hat{a}^{+}\hat{a}\hat{a}^{+} - \hat{a}^{+}\left(\hat{a}\hat{a}^{+} - 1\right)\right]$$

$$= \omega\hat{a}^{+}$$

Label the states by

$|n\rangle \qquad E_n = (n+\tfrac{1}{2})\omega \qquad H|n\rangle = (n+\tfrac{1}{2})\omega|n\rangle$

and look at

$\hat{H}\hat{a}|n\rangle$

$= \omega(\hat{a}^{+}\hat{a} + \tfrac{1}{2})\hat{a}|n\rangle$

$= \omega(\hat{a}^{+}\hat{a}\hat{a} + \tfrac{1}{2}\hat{a})|n\rangle \qquad \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1$

$= \omega[(\hat{a}\hat{a}^{+} - 1)\hat{a} + \tfrac{1}{2}\hat{a}]|n\rangle$

$= \omega\hat{a}(\hat{a}^{+}\hat{a} + \tfrac{1}{2} - 1)|n\rangle$

$= \hat{a}(\hat{H} - \omega)|n\rangle$

$= \hat{a}[(\hat{n} + \tfrac{1}{2})\omega - \omega]|n\rangle$

$= (n - \tfrac{1}{2})\omega\,\hat{a}|n\rangle$

$\Rightarrow \hat{a}|n\rangle \propto |n-1\rangle$

What is the constant of proportionality?

From

$$\hat{H}|n\rangle = (\hat{a}^\dagger \hat{a} + \tfrac{1}{2})\omega|n\rangle = (n+\tfrac{1}{2})\omega|n\rangle$$

we see that the number operator is

$$N = \hat{a}^\dagger \hat{a}$$

Then

$$\langle n|\hat{a}^\dagger \hat{a}|n\rangle = n$$

But

$$\hat{a}|n\rangle = c\,|n-1\rangle$$

and

$$\langle n|\hat{a}^\dagger = c^* \langle n-1|$$

Then

$$n = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = |c|^2 \langle n-1|n-1\rangle = |c|^2$$

and

$c = \sqrt{n}$

That is

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

Now look at

$\hat{H}\hat{a}^{+}|n\rangle$

$= \omega(\hat{a}^{+}\hat{a} + \frac{1}{2})\hat{a}^{+}|n\rangle$

$= \omega(\hat{a}^{+}\hat{a}\hat{a}^{+} + \frac{1}{2}\hat{a}^{+})|n\rangle$

$= \omega\hat{a}^{+}(\hat{a}\hat{a}^{+} + \frac{1}{2})|n\rangle$ $\qquad \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1$

$= \omega\hat{a}^{+}[(1 + \hat{a}^{+}\hat{a}) + \frac{1}{2}]|n\rangle$

$= \hat{a}^{+}(H + \omega)|n\rangle$

$= \hat{a}^{+}(n + \frac{1}{2} + 1)\omega|n\rangle$

$= (n + \frac{1}{2} + 1)\omega\hat{a}^{+}|n\rangle$

$\Rightarrow \hat{a}^{+}|n\rangle \propto |n+1\rangle$

But

$$\langle n+1 | \hat{a}^\dagger \hat{a} | n+1 \rangle = n+1$$

and

$$n+1 = \langle n+1 | \hat{a}^\dagger \hat{a} | n+1 \rangle = |c|^2 \langle n+1 | n+1 \rangle = |c|^2$$

Then

$$\hat{a}^\dagger | n \rangle = \sqrt{n+1}\, | n+1 \rangle$$

One of the most important aspects of the spectrum of states of the quantized harmonic oscillator is that its energy levels are separated by a uniform amount: $\hbar\omega$. The state $|n\rangle$ has energy $n\hbar\omega$ and the state $|n+1\rangle$ has energy $(n+1)\hbar\omega$.

With

$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega\, |n\rangle$$

we have the vacuum state $|0\rangle$ with energy $\frac{1}{2}\hbar\omega$. Then we can think of the state $|n\rangle$ as a state of n quanta each of energy $\hbar\omega$ - i.e., we can think of it as a <u>multiparticle</u> state.

With this interpretation, the operator $a^\dagger$ is a <u>creation</u> <u>operator</u> that creates a quantum of energy $\hbar\omega$, whereas a is an <u>annihilation</u> <u>operator</u> that annihilates a quantum of energy $\hbar\omega$.

Now we can generate all of the states beginning with the vacuum state defined by

$$\hat{a}|0\rangle = 0$$

Then

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0\rangle$$

cancels out the $\sqrt{n!}$ that appears when the a^{+} acts n times

For example,

$$|2\rangle = \frac{1}{\sqrt{2}}\hat{a}^{+}(\hat{a}^{+}|0\rangle)$$

$$= \frac{1}{\sqrt{2}}\hat{a}^{+}(\sqrt{1}\,|1\rangle)$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+1}\,|2\rangle$$

$$= |2\rangle$$