Noether's Theorem

If ϕ satisfies the EL EOM ($\delta S = 0$), for any variation we have

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$\delta \mathcal{L} = \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$= \partial_\mu (\quad)$$

Assume the action also has the symmetry

$$\delta_\Delta S = 0$$

under the specific transformation

$$\phi \to \phi + \Delta$$

$$\delta S = \int d^4x \, \delta \mathcal{L}$$

$$= \int d^4x \, \partial_\mu (\quad)$$

$$= 0$$

Given the action is invariant under the transformation even though the Lagrangian density may not be [$\delta_\Delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \right) \neq 0$],

$$\delta_\Delta \mathcal{L} = \partial_\mu K^\mu \quad (2)$$

$$\delta_\Delta S = \int d^4x \, \delta_\Delta \mathcal{L} = \int d^4x \, \partial_\mu K^\mu = 0 \quad (1)$$

- i.e. $\delta_\Delta \mathcal{L}$ must be a total divergence. Equating (1) and (2), we get

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu \right) = 0$$

- i.e., we have a conserved current ($\partial_\mu j^\mu = 0$)

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

so, with a symmetry there is a conserved current

How do we find K^μ?

Consider the translation invariance of the action under the transformation

A conserved current will be associated with a conserved "charge"

$$\partial_\mu j^\mu = \partial_0 j^0 + \partial_i j^i$$

Integrate over all space

$$\int d^3x \left(\partial_0 j^0 + \partial_i j^i \right) = 0$$

Then

$$\partial_0 \int d^3x\, j^0 = - \int d^3x\, \partial_i j^i$$

$$= 0$$

and we identify

$$Q \equiv \int d^3x\, j^0$$

with the conserved "charge."

$$x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu$$

Then

$$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + \frac{\partial \phi}{\partial x^\mu} a^\mu$$

and

$$\delta_\Delta \phi = \frac{\partial \phi}{\partial x^\mu} a^\mu$$

Then

$$\delta_\Delta(\partial_\mu \phi) = \partial_\mu(\delta_\Delta \phi)$$

$$= \partial_\mu \left(\frac{\partial \phi}{\partial x^\alpha} a^\alpha \right)$$

$$= \frac{\partial^2 \phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

Now we can compute

$$\delta_\Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta_\Delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\Delta(\partial_\mu \phi)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial \phi}{\partial x^\alpha} a^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial \phi}{\partial x^\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial}{\partial x^\alpha}(\partial_\mu \phi) \right) a^\alpha$$

$$= \frac{\partial \mathcal{L}}{\partial x^\alpha} a^\alpha$$

$$= \partial_\alpha (\mathcal{L} a^\alpha)$$

Then

$$K^\mu = \mathcal{L} a^\mu$$

and

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} a^\alpha - \mathcal{L} a^\mu$$

$$= \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) a^\alpha$$

Then

$$\partial_\mu j^\mu = a^\alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) = 0$$

Since a^α is arbitrary, for the above to be true always, each "coefficient" of a^α must be zero:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) = 0$$

Contract with $\eta^{\nu\alpha}$ $\qquad \leftarrow \eta^{\nu\alpha} \partial_\mu (\ ) = 0$

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x_\nu} - \mathcal{L} \eta^{\mu\nu} \right)}_{} = 0 \qquad \Rightarrow \partial_\mu [\eta^{\nu\alpha} (\ )] = 0$$

$$= T^{\mu\nu}$$

The energy-momentum tensor for the scalar field ϕ

What about Lorentz invariance?

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0$$

$\nu = 0$

$$\partial_0 T^{00} + \partial_i T^{i0} = 0$$ conservation of energy

$\nu = j$

$$\partial_0 T^{0j} + \partial_i T^{ij} = 0$$ conservation of momentum

$\nu = 0$

$$\partial_0 \int d^3x\, T^{00} = \partial_0 \int d^3x\, \mathcal{H} = -\int d^3x\, \partial_i T^{i0}$$

$\Rightarrow H = \int d^3x\, \mathcal{H}$ is the conserved "charge"

$\nu = j$

$$\partial_0 \int d^3x\, T^{0j} = -\int d^3x\, \partial_i \underbrace{T^{ij}}_{\text{stress tensor}} = 0$$

$\Rightarrow P^i = \int d^3x\, \underbrace{T^{0i}}_{\text{momentum density}}$ is the conserved charge

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = \underbrace{x^\mu + \varepsilon^\mu{}_\nu x^\nu}_{\equiv x'^\mu}$$

Then

$$\phi(x) \to \phi(\Lambda x) = \phi(x) + \varepsilon^\rho{}_\mu x^\mu \frac{\partial \phi}{\partial x^\rho}$$

and

$$\delta^\nabla_\rho \phi = \varepsilon^\rho{}_\mu x^\mu \frac{\partial \phi}{\partial x^\rho}$$

$$= \varepsilon^\lambda{}_\mu x^\mu \partial_\lambda \phi^\nu \epsilon_\nu$$

Then

$$(\delta^\nabla_\rho \phi^\nu)\epsilon_\nu = \delta^\nabla_\rho(\phi^\nu \epsilon_\nu)$$

$$= (\varepsilon^\lambda{}_\mu x^\mu \partial_\lambda \phi^\nu)\epsilon_\nu =$$

$$= \varepsilon^\lambda{}_\mu \partial_\lambda \phi^\nu \epsilon_\nu + \varepsilon^\lambda{}_\mu x^\mu \partial_\lambda (\phi^\nu \epsilon_\nu) =$$

$$= \varepsilon^\lambda{}_\mu \partial_\lambda \phi^\nu \epsilon_\nu + \varepsilon^\lambda{}_\mu x^\mu \partial_\lambda (\phi^\nu \epsilon_\nu)$$

Now

$$\delta^\nabla_\rho \chi^\nabla_\sigma = (\phi^\nu \epsilon_\nu)^\nabla_\sigma \left(\frac{\partial^{(1)}\chi^\nu_\sigma}{\partial x^\rho \epsilon_\nu}\right) + \phi^\nabla_\sigma \frac{\partial \epsilon_\nu}{\partial x^\rho}$$

$$= \left[\varepsilon^\nu{}_\mu \phi^\nu \epsilon_\nu + \varepsilon^\lambda{}_\mu x^\mu \partial_\lambda (\phi^\nu \epsilon_\nu)\right]\left(\frac{\partial^{(1)}\chi^\nu_\sigma}{\partial x^\rho}\right) + \varepsilon^\lambda{}_\mu x^\mu \partial_\lambda \phi^\nu \epsilon_\nu \cdot \frac{\partial \epsilon_\nu}{\partial x^\rho}$$

$$= \varepsilon^\nu{}_\mu \phi^\nu \epsilon_\nu \left(\frac{\partial^{(1)}\chi^\nu_\sigma}{\partial x^\rho}\right) + \varepsilon^\lambda{}_\mu x^\mu \partial_\lambda \left(\phi^\nu \epsilon_\nu \frac{\partial^{(1)}\chi^\nu_\sigma}{\partial x^\rho}\right)$$

We can write

$$\frac{\partial x^\rho}{\partial \epsilon} \varepsilon^\alpha{}_\nu x^\nu = {}_\lambda(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho \zeta^\nu{}_\lambda \chi - {}_\lambda(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho =$$

$$= {}_\lambda(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho - \cancel{{}_\lambda \varepsilon^\alpha{}_\nu \chi}^{0 \text{ since } \varepsilon^\mu{}_\nu = -\varepsilon^\nu{}_\mu} =$$

So

$$\Delta^\zeta{}_\chi = {}_\lambda(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho + \frac{\partial (\phi^\mu)^\rho}{\partial \epsilon} {}_\mu \varepsilon^\lambda{}_\rho \phi^\rho{}_\lambda =$$

$$= {}_\alpha(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho + \left[\phi^\mu{}_\rho \frac{\partial (\phi^\lambda)^\rho}{\partial \epsilon} - \phi^\lambda{}_\rho \frac{\partial (\phi^\mu)^\rho}{\partial \epsilon} \right] \frac{1}{2} {}_\mu \varepsilon^\lambda{}_\rho =$$

$$= {}_\alpha(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho + \frac{1}{2} \left[{}^\mu{}_\rho \zeta^\lambda{}_\nu \chi + {}^\lambda{}_\rho L^\mu{}_\nu - {}^\mu{}_\rho L^\lambda{}_\nu \right] {}_\mu \varepsilon^\lambda{}_\rho =$$

$$= {}_\alpha(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho + \frac{1}{2} \left({}^\mu{}_\rho L^\lambda{}_\nu - {}^\lambda{}_\rho L^\mu{}_\nu \right) =$$

$$= {}_\alpha(\chi \varepsilon^\alpha{}_\nu x^\nu)^\lambda{}_\rho \quad \text{since } L^\mu{}_\alpha = L^\lambda{}_\nu \text{ for the scalar field}$$

$$= \chi^\lambda{}_\rho = \quad \text{since } \chi \varepsilon^\lambda{}_\nu x^\nu{}_\rho = \chi \varepsilon^\nu{}_\nu = 0$$

Now

$$\Delta^\zeta{}_\chi = {}_\alpha(\chi \varepsilon^\alpha{}_\nu x^\nu)^\mu{}_\rho = \left(\nabla \frac{\partial (\phi^\mu)^\rho}{\partial \epsilon} \right)$$

$$= \left(x^\nu \varepsilon^\lambda{}_\nu \phi^\rho{}_\lambda \frac{\partial (\phi^\mu)^\rho}{\partial \epsilon} \right)^\mu{}_\rho =$$

So

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\sigma)} \partial_\sigma \phi_\tau \varepsilon^\tau{}_\nu x^\nu - \mathcal{L} \varepsilon^\mu{}_\nu x^\nu \right) = 0$$

□

$$\varepsilon^\tau{}_\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\sigma)} \partial_\sigma \phi_\tau x^\nu - \mathcal{L} \delta^\mu{}_\tau x^\nu \right) = 0$$

Rewriting

$$\varepsilon^\tau{}_\nu \partial_\mu \Big[\underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\sigma)} \partial_\tau \phi - \mathcal{L} \delta^\mu{}_\tau \right)}_{\overline{T}^\mu{}_\tau} x^\nu \Big] = 0$$

Since $\varepsilon^\tau{}_\nu$ is antisymmetric, we can conclude that the antisymmetric component of the term in [ ] must vanish – i.e.,

$$\partial_\mu \left[\overline{T}^\mu{}_\tau x^\nu - \overline{T}^\mu{}_\nu x^\tau \right] = 0$$

So

$$M^\mu{}_\tau{}^\nu \equiv \overline{T}^\mu{}_\tau x^\nu - \overline{T}^\mu{}_\nu x^\tau$$

is the conserved current.

For $\tau, \nu = 1, 2, 3$, the Lorentz transformation corresponds to a

spatial rotation. In this case

$$\partial_0 \left[\bar{T}^0_{i} x^j - \bar{T}^0_{j} x^i \right] + \partial_k \left[\bar{T}^k_{i} x^j - \bar{T}^k_{j} x^i \right] = 0$$

and the conserved charge is

$$\int d^3x \left(\bar{T}^0_{i} x^j - \bar{T}^0_{j} x^i \right)$$

which is the total angular momentum tensor $Q_i^{\ j}$ with $Q_i^{\ i} = 0$ and $Q_i^{\ j} = -Q_j^{\ i}$ - i.e., $Q^i_{\ j}$ has 3 independent components.

Commutation of ∂_μ and δ

$F = F(x, y(x), y'(x))$ a functional

Change $y(x)$ in the following manner

$$y(x) \longrightarrow y(x) + \epsilon \eta(x)$$

The variation δy is defined to be

$$\delta y \equiv \epsilon \eta(x) \qquad (1)$$

The variation is a change in a function.

At fixed x,

$$F(x, y, y') \longrightarrow F(x, y + \epsilon\eta, y' + \epsilon\eta')$$

$$= \frac{\partial F}{\partial y} \epsilon\eta + \frac{\partial F}{\partial y'} \epsilon\eta'$$

$$\equiv \delta F$$

If we let $F = y'$, we can see that generally

$$\delta F = \delta y' = \epsilon \eta' \qquad (2)$$

So, at fixed x, from (1) and (2)

$$\delta y' = (\delta y)'$$

- i.e., the derivative w.r.t. the independent variable x and the variation of a function of x commute.