Cross Sections and Decay Rates

Recall that the quantum mechanical differential scattering cross section is

$$d\sigma = \underbrace{\frac{1}{T}\frac{1}{\Phi}}_{\text{time} \times \frac{1}{\text{time}}} \overbrace{dP}^{\text{probability / particle}}$$

total probability per particle in cm^2

where

$T \equiv$ time for the experiment

$\Phi \equiv$ particle flux normalized so there is one particle per cm^2 per s

$dP \equiv$ quantum mechanical probability of scattering per particle

Then

$$[d\sigma] = cm^2 = \frac{1}{[T][\Phi]}[dP] = \frac{1}{s \cdot \frac{\#}{cm^2 s}} \frac{1}{\#} = cm^2$$

In a collider, the differential number of scattering events, dN, is

$$dN = \overbrace{L}^{\#\text{ particles}/cm^2} \underbrace{d\sigma}_{\text{probability in } cm^2} \qquad [L] = \#/cm^2 \leftarrow \int_T dt \frac{\#}{cm^2 s}$$

e.g.) total # particles in $d\Omega$: $\frac{dN}{d\Omega} = L \frac{d\sigma}{d\Omega}$

where

$L \equiv$ integrated luminosity (defined by the above equation)

Specializing to the case of scattering of 2 particles

$$p_1 + p_2 \longrightarrow \{p_j\}$$

In the rest frame of one of the particles

$\Phi = |\vec{v}|/V$ $\qquad [\Phi] = \frac{cm}{s}\frac{1}{cm^3} = \frac{1}{cm^2 s}$ — i.e., a flux of one particle per cm^2 per s

where

V = volume of the experiment

In the CM frame

$\Phi = |\vec{v}_1 - \vec{v}_2|/V$

and

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|} dP$$

We relate dP to the S-matrix as follows

$$dP = \underbrace{\frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}} d\Pi$$

probability per particle per unit phase-space volume for $|i\rangle \to |f\rangle$

where $d\Pi$ is the phase-space volume

$$d\Pi = V\prod_j \frac{d^3 p_j}{(2\pi)^3}$$

Given

$$\hat{a}^{+}_{p}|0\rangle = \frac{1}{\sqrt{2\omega_p}}|p\rangle$$

$$[\hat{a}_{p}, \hat{a}^{+}_{q}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

we have (with $\langle 0|0\rangle = 1$)

$$\langle p|p\rangle = (2\pi)^3\, 2\omega_p\, \delta^{(3)}(0)$$

$$\langle q|p\rangle$$

$$= \sqrt{2\omega_q}\sqrt{2\omega_p}\,\langle 0|\hat{a}_q \hat{a}^\dagger_p|0\rangle$$

$$= \sqrt{2\omega_q}\sqrt{2\omega_p}\,\langle 0|\hat{a}^\dagger_p \hat{a}_q|0\rangle$$

$$+ \sqrt{2\omega_q}\sqrt{\omega_p}\,(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,\langle 0|0\rangle$$

$$= 2\omega_p (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p})$$

Recall that

$$(2\pi)^3\,\delta^{(3)}(\vec{p}) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}$$

then

$$(2\pi)^3\,\delta^{(3)}(0) = \int d^3x = V$$

In the 4-dimensional case (we'll need this later)

$$(2\pi)^4\,\delta^{(4)}(0) = TV$$

Therefore

$$\langle p|p\rangle = 2\omega_p V = 2E_p V$$

$$\langle i|i\rangle = (2E_1 V)(2E_2 V)$$

$$\langle f|f\rangle = \prod_j (2E_j V)$$

Now recall that

$S = 1 + i T$

where S is the scattering matrix and T is the Transfer Matrix.

Then

$$\langle f | S - 1 | i \rangle = i (2\pi)^4 \delta^{(4)}\left(\Sigma p_i^\mu - \Sigma p_f^\mu\right) \mathcal{M}$$

For $|f\rangle \neq |i\rangle$,

$$|\langle f | S | i \rangle|^2 = (2\pi)^8 \delta^{(4)}(0)\, \delta^{(4)}\left(\Sigma p_i^\mu - \Sigma p_f^\mu\right) |\mathcal{M}|^2$$

$$= (2\pi)^4\, T V\, \delta^{(4)}\left(\Sigma p_i^\mu - \Sigma p_f^\mu\right) |\mathcal{M}|^2$$

and

$$dP = \frac{1}{2E_1 V} \frac{1}{2E_2 V} \frac{1}{\prod_j 2E_j V} (2\pi)^4 \delta^{(4)}\left(\Sigma p_i^\mu - p_f^\mu\right) T V |\mathcal{M}|^2 \prod_j V \frac{d^3 p_j}{(2\pi)^3}$$

$$= \frac{T}{V} \frac{1}{2E_1} \frac{1}{2E_2} |\mathcal{M}|^2\, d\Pi_{LIPS}$$

Lorentz Invariant Phase Space

where

$$d\Pi_{LIPS} = (2\pi)^4 \delta^{(4)}\left(\Sigma p_i^\mu - \Sigma p_f^\mu\right) \prod_j \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_{p_j}}$$

Then

$$d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP$$

$$= \frac{1}{2E_1} \frac{1}{2E_2} \frac{1}{|\vec{v}_1 - \vec{v}_2|} |\mathcal{M}|^2 d\Pi_{LIPS}$$

and we can consider

$$V \longrightarrow \infty$$

$$T \longrightarrow \infty$$

A decay can be treated as a $1 \to n$ scattering, and the differential decay rate, $d\Gamma$, is given by

$$d\Gamma = \frac{1}{2E_1} |\mathcal{M}|^2 d\Pi_{LIPS}$$