The Lagrangian for Scalar Electrodynamics

The interaction Lagrangian must be gauge invariant in order to maintain the dynamical degrees of freedom when going from the free theory to the interacting theory.

To accomplish this, A^μ must be coupled to a complex scalar field transforming as

$$\phi(x) \longrightarrow e^{-i\alpha(x)}\phi(x)$$

under the gauge transformation

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x)$$

It is conventional to write the gauge transformation as

$$A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

Then under a gauge transformation

$$D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi \longrightarrow (\partial_\mu + ieA_\mu + i\partial_\mu\alpha)e^{-i\alpha}\phi$$

covariant derivative

$$= e^{-i\alpha}(\partial_\mu + ieA_\mu)\phi$$

$$= e^{-i\alpha} D_\mu \phi$$

transforms like ϕ - i.e. it is covariant under the gauge transformation

Then

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi$$

is gauge invariant.

This is the Lagrangian density for Scalar Electrodynamics.
The field couplings are contained within the second term. The Lagrangian density can be expanded as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

$$\left.\begin{aligned} &+ ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \\ &+ e^2 A_\mu A^\mu \phi^*\phi \end{aligned}\right\} \mathcal{L}_{INT}$$

This is of the form

$-e\, j^\mu A_\mu$

A gauge symmetry

e.g., $\alpha(x) = \alpha = \text{constant}$

$$\phi(x) \longrightarrow e^{-i\alpha(x)}\phi(x)$$

implies a global symmetry as well, for which we know ~~there~~

exists a Noether current, j^μ.

In this case

$$j^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta \phi^*}{\delta \alpha}$$

$$= \frac{\partial}{\partial(\partial_\mu \phi)} \left(\partial^\alpha \phi^* \partial_\alpha \phi - ie A_\alpha \phi^* \partial^\alpha \phi \right) \frac{\delta \phi}{\delta \alpha}$$

$$+ \frac{\partial}{\partial(\partial_\mu \phi^*)} \left(\partial_\alpha \phi^* \partial^\alpha \phi + ie A_\alpha \phi \partial^\alpha \phi^* \right) \frac{\delta \phi^*}{\delta \alpha}$$

$$= \left(\delta^\mu{}_\alpha \partial^\alpha \phi^* - ie A_\alpha \eta^{\alpha\rho} \phi^* \delta^\mu{}_\rho \right) \frac{\delta \phi}{\delta \alpha}$$

$$+ \left(\delta^\mu{}_\alpha \partial^\alpha \phi + ie A_\alpha \eta^{\alpha\rho} \phi \delta^\mu{}_\rho \right) \frac{\delta \phi^*}{\delta \alpha}$$

$$= \left(\partial^\mu \phi^* - ie A^\mu \phi^* \right) \frac{\delta \phi}{\delta \alpha} + \left(\partial^\mu \phi + ie A^\mu \phi \right) \frac{\delta \phi^*}{\delta \alpha}$$

$$\frac{\delta \phi}{\delta \alpha} = \frac{\delta}{\delta \alpha} \left(e^{-i\alpha} \phi \right) = -ie^{-i\alpha} \phi = -i\phi$$

$$\frac{\delta \phi^*}{\delta \alpha} = i\phi^*$$

Then

$$j^\mu = -i\phi \partial^\mu \phi^* + i\phi^* \partial^\mu \phi - eA^\mu \phi^* \phi - eA^\mu \phi \phi^*$$

$$= -i\left(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right) - 2eA^\mu\phi^*\phi$$