Interacting Fields : DYSON'S FORMULA

Recall the Schrodinger picture of quantum mechanics :

$$i \frac{d|\psi\rangle_S}{dt} = \hat{H}|\psi\rangle_S$$

States are time dependent, operators are not.

Recall the Heisenberg picture instead :

$$|\psi\rangle_H = e^{i\hat{H}t}|\psi\rangle_S$$ cancels out the time evolution of $|\psi\rangle_S$

$$|\psi\rangle_S = e^{-i\hat{H}t}|\psi\rangle_H$$

$$\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}$$

$$_S\langle\psi|\hat{O}_S|\psi\rangle_S = {}_H\langle\psi|\hat{O}_H|\psi\rangle_H$$

States do not evolve in time, operators do.

Now consider the "Interaction Picture" :

Let

$$\hat{H} = \hat{H}_0 + \hat{H}_{INT}$$

Then

$$|\psi\rangle_I \equiv e^{i\hat{H}_0 t}|\psi\rangle_S$$

cancels out the evolution in $|\psi\rangle_S$ from $\hat{H}_0$, leaving the time evolution associated with $\hat{H}_{INT}$ only $|\psi\rangle_I = e^{-i\hat{H}_{INT}t}|\psi\rangle_H$

and

$$\hat{O}_I(t) = e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}$$

time independent ($\hat{O}_S$)

evolves according to $\hat{H}_0$

$$_S\langle\psi|\hat{O}_S|\psi\rangle_S = \underbrace{{}_S\langle\psi|e^{-i\hat{H}_0 t}}_{{}_I\langle\psi|}\underbrace{e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}}_{\hat{O}_I}\underbrace{e^{i\hat{H}_0 t}|\psi\rangle_S}_{|\psi\rangle_I}$$

We can derive the Schroedinger-like equation for $|\psi\rangle_I$:

Begin with the Schroedinger equation

$$i \frac{d|\psi\rangle_s}{dt} = \hat{H}\,|\psi\rangle_s$$

Substitute

$$|\psi\rangle_s = e^{-i\hat{H}_0 t}\,|\psi\rangle_I$$

for $|\psi\rangle_s$, to obtain

$$i \frac{d}{dt}\left(e^{-i\hat{H}_0 t}\,|\psi\rangle_I\right) = (\hat{H}_0 + \hat{H}_{INT})\, e^{-i\hat{H}_0 t}\,|\psi\rangle_I$$

Expand out the left-hand side

$$\begin{aligned} i \frac{d}{dt}\left(e^{-i\hat{H}_0 t}\,|\psi\rangle_I\right) &= i e^{-i\hat{H}_0 t} \frac{d|\psi\rangle_I}{dt} + i(-i\hat{H}_0)\, e^{-i\hat{H}_0 t}\,|\psi\rangle_I \\ &= i e^{-i\hat{H}_0 t} \frac{d}{dt}|\psi\rangle_I + \hat{H}_0\, e^{-i\hat{H}_0 t}\,|\psi\rangle_I \\ &= i e^{-i\hat{H}_0 t} \frac{d|\psi\rangle_I}{dt} + \hat{H}_0\,|\psi\rangle_s \end{aligned}$$

Now consider the right-hand side

$$(\hat{H}_0 + \hat{H}_{INT})\, e^{-i\hat{H}_0 t}\,|\psi\rangle_I = \hat{H}_0\, e^{-i\hat{H}_0 t}\,|\psi\rangle_I + \hat{H}_{INT}\, e^{-i\hat{H}_0 t}\,|\psi\rangle_I$$

$$= \hat{H}_0 |\psi\rangle_S + \hat{H}_{INT} e^{-i\hat{H}_0 t} |\psi\rangle_I$$

Equating the two sides and ignoring the common term, $\hat{H}_0 |\psi\rangle_S$, we have

$$i e^{-i\hat{H}_0 t} \frac{d|\psi\rangle_I}{dt} = \hat{H}_{INT} e^{-i\hat{H}_0 t} |\psi\rangle_I$$

or

$$i \frac{d|\psi\rangle_I}{dt} = e^{i\hat{H}_0 t} \hat{H}_{INT} e^{-i\hat{H}_0 t} |\psi\rangle_I$$

$$\equiv \hat{H}_I(t) |\psi\rangle_I$$

where $H_I(t)$ is defined to be H_{INT} in the interaction picture – i.e.,

$$\hat{H}_I(t) \equiv e^{i\hat{H}_0 t} \hat{H}_{INT} e^{-i\hat{H}_0 t}$$

So, we have

$$\frac{d|\psi\rangle_I}{dt} = -i \hat{H}_I(t) |\psi\rangle_I$$

To obtain the corresponding integral equation, integrate both sides with respect to t, from some initial time t_0 to t, to obtain

$$\int_{t_0}^{t} \frac{d|\psi\rangle_I}{dt'} dt' = |\psi\rangle_I(t) - |\psi\rangle_I(t_0) = -i \int_{t_0}^{t} \hat{H}_I(t') |\psi\rangle_I(t') dt'$$

Rearranging, we get

$$|\psi\rangle_I(t) = |\psi\rangle_I(t_0) - i\int_{t_0}^{t} \hat{H}_I(t')\,|\psi\rangle_I(t')\,dt'$$

Substituting the above equation for $|\psi\rangle_I(t')$ in the above integrand yields

$$|\psi\rangle_I(t) = |\psi\rangle_I(t_0)$$

$$- i\int_{t_0}^{t} \hat{H}_I(t')\left[\,|\psi\rangle_I(t_0) - i\int_{t_0}^{t'} \hat{H}_I(t'')\,|\psi_I\rangle(t'')\,dt''\right]dt'$$

Do this again

$$|\psi\rangle_I(t) = |\psi\rangle_I(t_0)$$

$$- i\int_{t_0}^{t} \hat{H}_I(t')\,dt' \;\; |\psi\rangle_I(t_0)$$

$$+(i)^2\int_{t_0}^{t} \hat{H}_I(t')\int_{t_0}^{t'} \hat{H}_I(t'')\left[\,|\psi\rangle_I(t_0) - i\int_{t_0}^{t''} \hat{H}_I(t''')\,|\psi\rangle_I(t''')\,dt'''\right]dt''\,dt'$$

$$= \Big\{ 1 - i\int_{t_0}^{t} dt'\,\hat{H}_I(t')$$

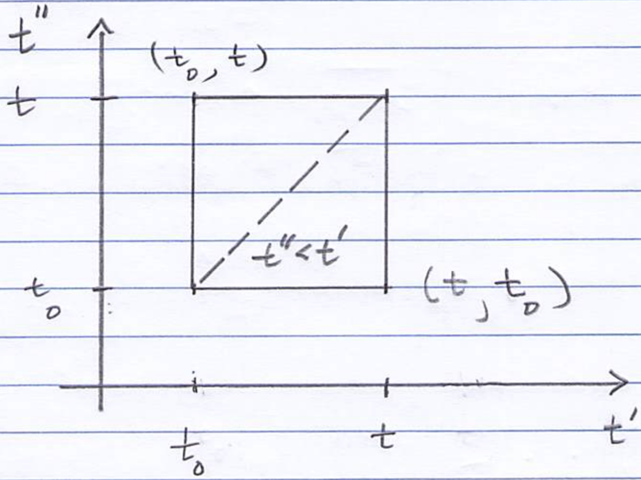
$$+ (i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\;\hat{H}_I(t')\,\hat{H}_I(t'')$$

$$- \ldots \Big\}\,|\psi\rangle_I(t_0)$$

Now consider

$$\int_{t_0}^{t}\int_{t_0}^{t'} dt'\, dt''\, \hat{H}_I(t')\, \hat{H}_I(t'')$$

and note that $t' > t''$. Consider the integration domain in the (t', t'') plane:



Now also consider

$$T\left\{\hat{H}_I(t')\,\hat{H}_I(t'')\right\} = \begin{cases} \hat{H}_I(t')\,\hat{H}_I(t'') & t'' < t' \\ \hat{H}_I(t'')\,\hat{H}_I(t') & t'' > t' \end{cases}$$

nothing changes if $t' \leftrightarrow t''$

But

$$T\left\{\hat{H}_I(t')\,\hat{H}_I(t'')\right\} = T\left\{\hat{H}_I(t'')\,\hat{H}_I(t')\right\}$$

Then

This is just the integral of $T\{\hat{H}_I(t')\hat{H}_I(t'')\}$ in the lower half triangle.

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, \hat{H}_I(t')\hat{H}_I(t'') = \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, T\{\hat{H}_I(t')\hat{H}_I(t'')\}$$

This integrand is equal to the integrand on the left in the lower triangle and is equal to itself in the upper triangle.

and

$$|\psi\rangle_I(t) = \Big\{ 1 - i\int_{t_0}^{t} dt' \, H_I(t')$$

$$+ (i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, T\{\hat{H}_I(t')\hat{H}_I(t'')\}$$

$$- \ldots \Big\} |\psi\rangle_I(t_0)$$

$$= T\Big\{ e^{-i\int_{t_0}^{t} dt' \hat{H}_I(t')} \Big\} |\psi\rangle_I(t_0)$$

DYSON'S FORMULA

$$\equiv \hat{U}(t, t_0)|\psi\rangle_I(t_0)$$

The S-Matrix Operator is defined in terms of $\hat{U}(t, t_0)$

$$\langle f|\hat{S}|i\rangle \equiv \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle f|\hat{U}(t_f, t_i)|i\rangle$$

Note that the states $|i\rangle$ and $|f\rangle$ are typically time-independent eigenstates of $\hat{H}_0(t)$.

We will assume that as $t \to \pm\infty$, $\hat{H}_I(t) \to 0$.

When $\hat{H}_I(t) = 0$,

CLAIM

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, \hat{H}_I(t') \hat{H}_I(t'')$$

$$= \frac{1}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, T\left\{ \hat{H}_I(t') \hat{H}_I(t'') \right\}$$

PROOF

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, \hat{H}_I(t') \hat{H}_I(t'')$$

The integrands are equal for $t' > t''$ - i.e., in the lower triangle.

$$= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, T\left\{ \hat{H}_I(t') \hat{H}_I(t'') \right\}$$

The integrand is equal to itself in the upper triangle.

$$= \frac{1}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, T\left\{ \hat{H}_I(t') \hat{H}_I(t'') \right\}$$

$$|\psi\rangle_H = e^{i\hat{H}t}|\psi\rangle_s$$ ← INVERSE OF $|\psi_s\rangle = e^{-i\hat{H}t}|\psi\rangle_H$

t-independent

$$= e^{i\hat{H}_I t} e^{i\hat{H}_0 t}|\psi\rangle_s$$

$$= e^{i\hat{H}_I t}|\psi\rangle_I$$

$$= |\psi\rangle_I$$

Also, when $\hat{H}_I(t) = 0$,

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_I(t)$$

$$= \hat{H}_0(t)$$

and

$$|\psi\rangle_s = e^{-i\hat{H}_0 t}|\psi\rangle_H$$

$$= e^{-i\hat{H}_0 t}|\psi\rangle_I$$

So

$$\langle f|\hat{S}|i\rangle \equiv \langle f|T\left\{e^{-i\int_{-\infty}^{\infty} dt\, \hat{H}_I(t)}\right\}|i\rangle$$

$$= \langle f|T\left\{e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\right\}|i\rangle$$

where we have expressed the interaction Hamiltonian in terms of the interaction Hamiltonian density

One last, but very important, thing:

What fields are used in $\hat{\mathcal{H}}_I(x)$?

Recall that for a general operator $\hat{O}(t)$

$$\hat{O}_H(t) = e^{i\hat{H}t}\,\hat{O}_S\,e^{-i\hat{H}t}$$

$$\hat{O}_I(t) = e^{i\hat{H}_0 t}\,\hat{O}_S\,e^{-i\hat{H}_0 t}$$

This is true for $\hat{\phi}(x)$. That is

$$\hat{\phi}_H(\vec{x},t) = e^{i\hat{H}t}\,\hat{\phi}(\vec{x},0)\,e^{-i\hat{H}t} \qquad \hat{\phi}_S(\vec{x}) \equiv \hat{\phi}(\vec{x},0)$$

Technically just $\hat{\phi}(\vec{x},t)$.

$$\hat{\phi}_I(\vec{x},t) = e^{i\hat{H}_0 t}\,\hat{\phi}(\vec{x},0)\,e^{-i\hat{H}_0 t}$$

Then, at $t=0$

$$\hat{\phi}_H(\vec{x},0) = \hat{\phi}_I(\vec{x},0) = \hat{\phi}_S(\vec{x})$$ all three pictures agree

Since

$$\hat{\mathcal{H}}_I(x) = \hat{\mathcal{H}}_I[\hat{\phi}_H(x)]$$

$$\hat{\mathcal{H}}_I(\vec{x},0) = \hat{\mathcal{H}}[\hat{\phi}_H(\vec{x},0)]$$

$$= \hat{\mathcal{H}}[\hat{\phi}_I(\vec{x},0)]$$

But

$$\hat{\mathcal{H}}_I(x) = e^{i\hat{H}_0 t}\,\hat{\mathcal{H}}_I(\vec{x},0)\,e^{-i\hat{H}_0 t}$$

$$= e^{i\hat{H}_0 t}\,\hat{\mathcal{H}}_I[\hat{\phi}_I(\vec{x},0)]\,e^{-i\hat{H}_0 t}$$

$$= \hat{\mathcal{H}}_I[\hat{\phi}_I(x)]$$

That is, $\hat{\mathcal{H}}_I$ is a function of the field operators in the interaction representation.

But $\hat{\phi}_I(x)$ evolve according to $\hat{H}_0$!

These are nothing other than the free fields!

So, finally

$$\langle f|\hat{S}|i\rangle = \langle f|\,T\left\{e^{-i\int d^4x\,\hat{\mathcal{H}}_I[\hat{\phi}_I(x)]}\right\}|i\rangle$$