Continuum Limit of the Linear Chain

$a \to dx$ $\quad a \to 0$, $N \to \infty$ $\Big\}$ $\ell = Na$ = length of the chain = constant

$m \to dm$ $\quad m \to 0$

$\rho = m/a$ = constant

displacement of infinitesimal mass at x

$q_n(t) \longrightarrow q(x,t)$ $\qquad$ a scalar field!

$q_{n+1}(t) - q_n(t) \longrightarrow dq = a\frac{\partial q}{\partial x}$

$\sum_{n=1}^{N} \longrightarrow \frac{1}{a}\int_0^\ell dx$ $\qquad$ (this just says $N = \frac{\ell}{a}$)

$\sigma \equiv Ka$ = constant = string tension

$m\ddot{x} + Kx = 0$
as $m \to 0$, $\ddot{x} \to \infty$
$\Rightarrow$ restorative force $\to \infty$
$\Rightarrow x \to 0$ w/ $K \to \infty$
$\not\Rightarrow m\ddot{x} + Kx = 0$

Then

$$L(q_n, \dot{q}_n) = \sum_n \left[\frac{m}{2}(\dot{q}_n)^2 - \frac{K}{2}(q_{n+1} - q_n)^2\right]$$

$$\to \frac{1}{a}\int_0^\ell dx\, \frac{1}{2}\left[\rho a\left(\frac{\partial q}{\partial t}\right)^2 - \frac{\sigma}{a}\,a^2\left(\frac{\partial q}{\partial x}\right)^2\right]$$

$$= \frac{1}{2}\int_0^\ell dx \left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \sigma\left(\frac{\partial q}{\partial x}\right)^2\right]$$

and

a "Lagrangian density"

$$\mathcal{L}(\dot{q}, q') = \frac{1}{2}\left[\rho\left(\frac{\partial q}{\partial t}\right)^2 - \sigma\left(\frac{\partial q}{\partial x}\right)^2\right] \qquad q' \equiv \frac{\partial q}{\partial x}$$

and

$$S = \int_{t_i}^{t_f} L\, dt = \int_{t_i}^{t_f} dt \int_0^{\ell} dx\, \mathcal{L}$$

The EOM are determined by extremizing the action.

$$\delta S = \int_{t_i}^{t_f} dt \int_0^{\ell} dx\, \delta\mathcal{L}$$

$$= \int_{t_i}^{t_f} dt \int_0^{\ell} dx \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathcal{L}}{\partial q'} \delta q' \right)$$

$$= \int_0^{\ell} dx \int_{t_i}^{t_f} \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \right] dt$$

$$+ \int_{t_i}^{t_f} \int_0^{\ell} dx \left[\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial q'} \delta q \right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial q'} \right) \delta q \right] dx$$

$$= - \int_{t_i}^{t_f} dt \int_0^{\ell} dx\, \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q$$

$\delta q(t_i) = 0$

$\delta q(t_f) = 0$

$$- \int_{t_i}^{t_f} dt \int_0^{\ell} dx\, \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial q'} \right) \delta q$$

$$= - \int_{t_i}^{t_f} dt \int_0^{\ell} dx \left[\frac{\partial}{\partial t}\left(\rho \frac{\partial q}{\partial t} \right) + \frac{\partial}{\partial x}\left(-\sigma \frac{\partial q}{\partial x} \right) \right] \delta q$$

$$= -\int_{t_i}^{t_f} dt \int_0^{\ell} dx \left(\rho \frac{\partial^2 q}{\partial t^2} - \sigma \frac{\partial^2 q}{\partial x^2} \right) \delta q$$

now $\delta q = \delta q(x,t)$

before $\delta q_n = \delta q_n(t)$

$\Rightarrow$ ( ) must be 0

$$= 0$$

Then the EOM for $q(x,t)$ is

$$\frac{\partial^2 q}{\partial t^2} - \frac{\sigma}{\rho} \frac{\partial^2 q}{\partial x^2} = 0$$

which is a wave equation, with wave speed

$$c = \sqrt{\frac{\sigma}{\rho}}$$

Classical Field Theory

We can extend the Lagrangian formulation of classical dynamics for point particles to fields.

The correspondence is

$$q(t) \longrightarrow \phi(x) \qquad x = (\vec{x}, t)$$

$$\dot{q}(t) \longrightarrow \partial_\mu \phi(x)$$

We define a Lagrangian density related to the Lagrangian and Action by

$$S = \int dt\, L = \int d^4x \underbrace{\mathcal{L}(\phi, \partial_\mu \phi)}_{\text{LAGRANGIAN DENSITY}}$$

Compute δS:

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi) \right]$$

$$= \left\{ \int d^4x \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \left. \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right|_{-\infty}^{+\infty} - \int d^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right\}$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi$$

Now instead of using $\delta q(t_i) = \delta q(t_f) = 0$, we simply assume that our fields are well-behaved functions of spacetime - i.e., that they go to zero at $x = \pm\infty$.

The solutions render the action an extremum ($\delta S = 0$) and satisfy the EOM

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

One can now define the momentum conjugate to ϕ

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(x))}$$

the Hamiltonian density

$$\mathcal{H}(x) \equiv \pi(x)\, \partial_0 \phi(x) - \mathcal{L}$$

which is related to the Hamiltonian by

$$H = \int d^3x\, \mathcal{H}$$

As an example, consider the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

Then

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{1}{2} \frac{\partial}{\partial(\partial_\mu \phi)} \left[\eta^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \right]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu \phi)} \left[(\partial_\alpha \phi)(\partial_\beta \phi) \right]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \left[\frac{\partial}{\partial(\partial_\mu \phi)} (\partial_\alpha \phi) \right] \partial_\beta \phi$$

$$+ \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \left[\frac{\partial}{\partial(\partial_\mu \phi)} (\partial_\beta \phi) \right]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \delta^\mu{}_\alpha \partial_\beta \phi$$

$$+ \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \, \delta^\mu{}_\beta$$

$$= \frac{1}{2} \partial^\mu \phi$$

$$+ \frac{1}{2} \partial^\mu \phi$$

$$= \partial^\mu \phi$$

and

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \partial_\mu(\partial^\mu \phi)$$

$$= \Box \phi$$

The EOM then read

$$(\Box + m^2)\phi = 0$$

which is the Klein-Gordon equation.