Why Quantum Field Theory

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1 To treat physics that cannot be treated using non-relativistic quantum mechanics.

The fundamental shortcoming of non-relativistic quantum mechanics is its inability to treat systems where the number of particles changes.

e.g., $\gamma\gamma \leftrightarrow e^+e^-$

Under relativistic conditions, we can <u>expect this</u> to happen.

Consider a particle in a box of size L. The uncertainty in momentum is

$$\Delta p \geq \frac{\hbar}{L}$$

For a relativistic particle

$$E \simeq pc$$

Then

$$\Delta p \geq \frac{\hbar}{L} \qquad \Rightarrow \qquad \Delta E \geq \frac{\hbar c}{L}$$

When

$$\Delta E = 2mc^2$$

where m is the mass of the particle, the uncertainty in energy is above the threshold for particle-antiparticle production.

At what L will this happen? When

$$\Delta E = 2mc^2 = \frac{\hbar c}{L}$$

Solving for L

$$\frac{\hbar c}{L} = 2mc^2$$
$$L = \frac{\hbar c}{2mc^2} = \frac{\hbar}{2mc}$$

The quantity

$$\lambda_{\text{Compton}} \equiv \frac{\hbar}{mc}$$

is the Compton wavelength.

$\Rightarrow$ When $L \sim \lambda_{\text{Compton}}$, we should expect to see a swarm of particle-antiparticle pairs surrounding the original particle (virtual particles, whose existence is limited by the uncertainty principle).

Note that

$$\lambda_{\text{de Broglie}} \equiv \frac{\hbar}{p} > \lambda_{\text{Compton}} = \frac{\hbar}{mc}$$

since $p < mc$ ($p = \gamma m_0 v = mv$).

$\lambda_{\text{de Broglie}}$ – the distance at which the wavelike nature of the particle becomes apparent

λ_{Compton} – the distance at which the concept of a single particle breaks down

$\lambda \ll \lambda_{\text{Compton}}$	$\lambda \sim \lambda_{\text{Compton}}$	$\lambda \sim \lambda_{\text{de Broglie}}$	$\lambda \gg \lambda_{\text{de Broglie}}$
	RQM a.k.a. QFT	NRQM	Classical

2 To treat physics that can be treated using quantum mechanics but more easily (naturally)

The treatment of systems of N identical bosons or fermions ($N \gg 1$) in quantum mechanics is truly cumbersome (if not impossible for N sufficiently large).

We work with symmetrical (antisymmetrical) sums of products of single-particle wave functions. The symmetry (antisymmetry) is put in by hand.

There is a better way, known as the Number Representation, which bundles N-particle systems in a much easier and more natural way. This representation is used in conjunction with quantum field operators, and symmetry (antisymmetry) is built into the theory!

Other Reasons:

3 Causality

Consider the amplitude in quantum mechanics for a free particle to propagate from $\vec{x} = \vec{x}_0$ at $t = 0$ to $\vec{x}$ at t:

$$\begin{aligned}
&\langle \vec{x} | e^{-i\hat{H}t} | \vec{x}_0 \rangle \qquad \text{position eigenstate} \\
&= \langle \vec{x} | e^{-i[\hat{p}^2 + m^2]^{\frac{1}{2}} t} | \vec{x}_0 \rangle \\
&= \int \mathrm{d}^3\vec{p} \ \langle \vec{x} | e^{-i[\hat{p}^2 + m^2]^{\frac{1}{2}} t} | \vec{p} \rangle \langle \vec{p} | \vec{x}_0 \rangle
\end{aligned}$$

$$\int \mathrm{d}^3\vec{p} \ |\vec{p}\rangle \langle \vec{p}| = 1?$$

$$\begin{aligned}
&\langle \vec{x}' | \left(\int \mathrm{d}^3\vec{p} \ |\vec{p}\rangle \langle \vec{p}| \right) |\vec{x}\rangle \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\vec{p} \ \langle \vec{x}' | \vec{p} \rangle \langle \vec{p} | \vec{x} \rangle \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\vec{p} \ e^{+i\vec{p}\cdot\vec{x}'} e^{-i\vec{p}\cdot\vec{x}} \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\vec{p} \ e^{-i\vec{p}\cdot(\vec{x} - \vec{x}^prime)} \\
&= \delta^{(3)}(\vec{x} - \vec{x}') \\
&= \langle \vec{x}' | \vec{x} \rangle
\end{aligned}$$

$$\Rightarrow \quad \int \mathrm{d}^3\vec{p} \ |\vec{p}\rangle \langle \vec{p}| = 1$$

For a free particle $[\hat{H}, \hat{p}] = 0$.

$$\begin{aligned}
\langle \vec{x}|e^{-i\hat{H}t}|\vec{x}_0\rangle &= \int \mathrm{d}^3\vec{p}\ e^{-i[p^2+m^2]^{\frac{1}{2}}t} \underbrace{\langle \vec{x}|\vec{p}\rangle}_{\frac{1}{(2\pi)^{\frac{3}{2}}}e^{-i\vec{p}\cdot\vec{x}}} \underbrace{\langle \vec{p}|\vec{x}_0\rangle}_{\frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\vec{p}\cdot\vec{x}_0}} \\
&= \int \mathrm{d}^3\vec{p}\ e^{-i[p^2+m^2]^{\frac{1}{2}}t} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \\
&= \int \mathrm{d}^3\vec{p}\ e^{i\{-[p^2+m^2]^{\frac{1}{2}}t-\vec{p}\cdot(\vec{x}-\vec{x}_0)\}}
\end{aligned}$$

To make the math easy, let's assume we are in one spatial dimension, x, with $x \gg t$ (i.e., outside the light cone) and $x_0 = 0$. Then

$$\langle x|e^{-i\hat{H}t}|0\rangle = \int \frac{\mathrm{d}p}{2\pi} \underbrace{\underbrace{e^{-i[p^2+m^2]^{\frac{1}{2}}t}e^{-ipx}}_{e^{i\left\{-p-[p^2+m^2]^{\frac{1}{2}}\left(\frac{x}{t}\right)^{-1}\right\}x}}}_{\equiv e^{ig(p)x}} \quad \Rightarrow \quad g'(p) = -1 - p[p^2+m^2]^{-\frac{1}{2}}\left(\frac{x}{t}\right)^{-1}$$

$$\begin{aligned}
g'(p) &= 0 \text{ when} \\
-1 - p[p^2+m^2]^{-\frac{1}{2}}\left(\frac{x}{t}\right)^{-1} &= 0 \\
-p[p^2+m^2]^{-\frac{1}{2}} &= \frac{x}{t} \\
\frac{p^2}{p^2+m^2} &= \left(\frac{x}{t}\right)^2 \\
p^2 &= \left(\frac{x}{t}\right)^2\left(p^2+m^2\right) \\
\left[1-\left(\frac{x}{t}\right)^2\right]p^2 &= \left(\frac{xm}{t}\right)^2 \\
p &= \pm\frac{\frac{xm}{t}}{\sqrt{1-\left(\frac{x}{t}\right)^2}} = \pm\frac{xm}{\sqrt{x^2-t^2}} \equiv p_{\mathrm{s}}
\end{aligned}$$

The Method of Stationary Phase tells us that

$$\begin{aligned}
I(x) &= \int_a^b e^{ixg(p)}\mathrm{d}p \qquad\qquad x \gg 1,\ g(p) \in \mathbb{R} \text{ and smooth} \\
&\simeq e^{ixg(p_{\mathrm{s}})}\int_{-\infty}^{\infty} e^{ixg''(p_{\mathrm{s}})p^2}\mathrm{d}p \\
&= e^{ixg(p_{\mathrm{s}})}\left(\frac{2\pi i}{xg''(p_{\mathrm{s}})}\right)^{\frac{1}{2}}
\end{aligned}$$

where

$$g'(p_{\mathrm{s}}) = 0$$

So, for us,

$$I(x) \simeq \frac{1}{2\pi} e^{ixg(p_{\mathrm{s}})} \left(\frac{2\pi i}{x g''(p_{\mathrm{s}})} \right)^{\frac{1}{2}}$$

where

$$g(p_{\mathrm{s}}) = \left\{ \pm \frac{xm}{\sqrt{x^2 - t^2}} + \left[\left(\pm \frac{xm}{\sqrt{x^2 - t^2}} \right)^2 + m^2 \right]^{\frac{1}{2}} \left(\frac{x}{t} \right)^{-1} \right\}$$

The important point is

$$\left| \langle x | e^{-i\hat{H}t} | 0 \rangle \right|^2 = \frac{1}{2\pi x |g''(p_{\mathrm{s}})|} \neq 0 \; !$$

– i.e., causality is violated.

We need to double check that $|g''(p_{\mathrm{s}}|$ is not ∞.

$$\begin{aligned} g''(p) &= \frac{\mathrm{d}}{\mathrm{d}p} \left\{ -1 - p[p^2 + m^2]^{-\frac{1}{2}} \left(\frac{x}{t} \right)^{-1} \right\} \\ &= -p \left(\frac{x}{t} \right)^{-1} \left(-\tfrac{1}{2} \right) 2p[p^2 + m^2]^{-\frac{3}{2}} - \left(\frac{x}{t} \right)^{-1} [p^2 + m^2]^{-\frac{1}{2}} \\ &\overset{?}{=} \infty \text{ for } p = p_{\mathrm{s}} \\ &= \left. \frac{p^2 - (p^2 + m^2)}{(p^2 + m^2)^{\frac{3}{2}}} \right|_{p = p_{\mathrm{s}}} \left(\frac{x}{t} \right)^{-1} \\ &= \left. \frac{-m^2}{(p^2 + m^2)^{\frac{3}{2}}} \right|_{p = p_{\mathrm{s}}} \left(\frac{x}{t} \right)^{-1} \end{aligned}$$

$$\begin{aligned} p_{\mathrm{s}}^2 + m^2 &= \frac{x^2 m^2}{x^2 - t^2} + m^2 \qquad (x \gg t) \\ &\simeq 2m^2 \end{aligned}$$

$$\begin{aligned} g''(p_{\mathrm{s}}) &\simeq -\frac{1}{(2)^{\frac{3}{2}}} \frac{1}{m} \frac{t}{x} \\ |g''(p_{\mathrm{s}})|^2 &\simeq \frac{\sqrt{2}m}{\pi t} \longrightarrow 0 \text{ as } t \longrightarrow \infty \end{aligned}$$

No.

4 Locality (related to causality)

Instantaneous action at a distance is incompatible with relativity.

Classically, we've solved this problem for Coulomb's Law and Newton's Law of Gravitation by introducing fields, whose solution is governed by Maxwell's Equations and Einstein's Equations, respectively.

Why wouldn't all interactions in Nature (gravitational, electromagnetic, weak, and strong) be describable through fields?

When we consider quantized fields, we will see that interactions are mediated by particles of a particular spacetime point – i.e., they are <u>local</u>.