Noether's Theorem

If ϕ satisfies the EL EOM ($\delta S = 0$), for any variation we have

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \leftarrow \delta\mathcal{L} = \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right]}_{=0} \delta\phi$$

$$+ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$= \partial_\mu (\quad)$$

Assume the action also has the symmetry

$$\delta_\Delta S = 0$$

$$\delta S = \int d^4x \, \delta\mathcal{L}$$

under the specific transformation

$$= \int d^4x \, \partial_\mu (\quad)$$

$$\phi \to \phi + \Delta$$

$$= 0$$

Given the action is invariant under the transformation even though the Lagrangian density may not be $\left[\delta_\Delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \right) \neq 0 \right]$, (1)

$$\delta_\Delta \mathcal{L} = \partial_\mu K^\mu \qquad \delta_\Delta S = \int d^4x \, \delta_\Delta \mathcal{L} = \int d^4x \, \partial_\mu K^\mu = 0$$

(2)

– i.e. $\delta_\Delta \mathcal{L}$ must be a total divergence. Equating (1) and (2), we get

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu \right) = 0$$

– i.e., we have a conserved current ($\partial_\mu j^\mu = 0$)

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

so, with a symmetry there is a conserved current

How do we find K^μ?

Consider the translation invariance of the action under the transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu$$

Then

$$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + \frac{\partial\phi}{\partial x^\mu} a^\mu$$

and

$$\delta_\Delta \phi = \frac{\partial\phi}{\partial x^\mu} a^\mu$$

Then

$$\delta_\Delta(\partial_\mu \phi) = \partial_\mu(\delta_\Delta \phi)$$

$$= \partial_\mu\left(\frac{\partial\phi}{\partial x^\alpha} a^\alpha\right)$$

$$= \frac{\partial^2\phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

Now we can compute

$$\delta_\Delta \mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi} \delta_\Delta \phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta_\Delta(\partial_\mu \phi)$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x^\alpha} a^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial^2\phi}{\partial x^\mu \partial x^\alpha} a^\alpha$$

$$= \left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x^\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial}{\partial x^\alpha}(\partial_\mu \phi)\right) a^\alpha$$

$$= \frac{\partial\mathcal{L}}{\partial x^\alpha} a^\alpha$$

$$= \partial_\alpha(\mathcal{L} a^\alpha)$$

Then

$$K^\mu = \mathcal{L} a^\mu$$

and

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - K^\mu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} a^\alpha - \mathcal{L} a^\mu$$

$$= \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) a^\alpha$$

Then

$$\partial_\mu j^\mu = a^\alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) = 0$$

Since a^α is arbitrary, for the above to be true always, each "coefficient" of a^α must be zero:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\alpha} - \mathcal{L} \delta^\mu{}_\alpha \right) = 0$$

Contract with $\eta^{\nu\alpha}$ $\qquad \leftarrow \eta^{\nu\alpha} \partial_\mu (\quad) = 0$

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi}{\partial x_\nu} - \mathcal{L} \eta^{\mu\nu} \right)}_{= T^{\mu\nu}} = 0 \qquad \Rightarrow \partial_\mu [\eta^{\nu\alpha} (\quad)] = 0$$

$= T^{\mu\nu}$ The energy-momentum tensor for the scalar field ϕ

What about Lorentz invariance?

$$x^\beta \to \Lambda^\beta{}_\nu x^\nu = x^\beta + \underbrace{\varepsilon^\beta{}_\nu x^\nu}_{\equiv \delta x^\beta}$$

Then

$$\phi(x) \to \phi(\Lambda x) = \phi(x) + \frac{\partial \phi}{\partial x^\alpha} \delta x^\alpha$$

and

$$\delta_\Delta \phi = \frac{\partial \phi}{\partial x^\alpha} \delta x^\alpha$$
$$= \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu$$

Then

$$\delta_\Delta (\partial_\mu \phi) = \partial_\mu (\delta_\Delta \phi)$$
$$= \partial_\mu (\partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu)$$
$$= (\partial_\mu \partial_\alpha \phi) \varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu \delta^\nu{}_\mu$$
$$= (\partial_\mu \partial_\alpha \phi) \varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

Now

$$\delta_\Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta_\Delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\Delta (\partial_\mu \phi)$$
$$= \frac{\partial \mathcal{L}}{\partial \phi} \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left[(\partial_\mu \partial_\alpha \phi) \varepsilon^\alpha{}_\nu x^\nu + \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu \right]$$
$$= \frac{\partial \mathcal{L}}{\partial x^\alpha} \varepsilon^\alpha{}_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

We can write

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} \varepsilon^\alpha{}_\nu x^\nu = \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) - \mathcal{L} \varepsilon^\alpha{}_\nu \delta^\nu{}_\alpha$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) - \mathcal{L} \cancel{\varepsilon^\alpha{}_\alpha}^{\,0} \quad \text{since } \varepsilon^\mu{}_\nu = -\varepsilon^\nu{}_\mu$$

So

$$\delta_\Delta \mathcal{L} = \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \partial_\mu \phi\right] \varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left[T^\mu{}_\alpha + \mathcal{L}\delta^\mu{}_\alpha - T^\alpha{}_\mu - \mathcal{L}\delta^\alpha{}_\mu\right] \varepsilon^\alpha{}_\mu$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) + \frac{1}{2}\left(T^\mu{}_\alpha - T^\alpha{}_\mu\right)$$

$$= \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) \quad \text{since } T^\mu{}_\alpha = T^\alpha{}_\mu \text{ for the scalar field}$$

$$= \partial_\alpha \mathcal{L} \quad \text{since } \mathcal{L} \varepsilon^\alpha{}_\nu \partial_\alpha x^\nu = \mathcal{L} \varepsilon^\nu{}_\nu = 0$$

But

$$\delta_\Delta \mathcal{L} = \partial_\alpha (\mathcal{L} \varepsilon^\alpha{}_\nu x^\nu) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\right)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu\right)$$

So

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, \varepsilon^\alpha{}_\nu x^\nu - \mathcal{L} \varepsilon^\mu{}_\nu x^\nu \right) = 0$$

or

$$\varepsilon^\alpha{}_\nu \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi \, x^\nu - \mathcal{L} \delta^\mu{}_\alpha x^\nu \right) = 0$$

Rewriting

$$\varepsilon^\alpha{}_\nu \, \partial_\mu \Big[\underbrace{\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \mathcal{L} \delta^\mu{}_\alpha \Big)}_{T^\mu{}_\alpha} x^\nu \Big] = 0$$

Since $\varepsilon^\alpha{}_\nu$ is antisymmetric, we can conclude that the anti-symmetric component of the term in [ ] must vanish – i.e.,

$$\partial_\mu \left[T^\mu{}_\alpha x^\nu - T^\mu{}_\nu x^\alpha \right] = 0$$

So

$$M^\mu{}_\alpha{}^\nu \equiv T^\mu{}_\alpha x^\nu - T^\mu{}_\nu x^\alpha$$

is the conserved current.

For $\alpha, \nu = 1, 2, 3$, the Lorentz transformation corresponds to a spatial rotation. In this case

$$\partial_0 \left[T^0{}_i x^j - T^0{}_j x^i \right] + \partial_k \left[T^k{}_i x^j - T^k{}_j x^i \right] = 0$$

and the conserved charge is

$$\int d^3x \left(T^0{}_i x^j - T^0{}_j x^i \right)$$

which is the total angular momentum tensor $Q_i{}^j$ with $Q_i{}^i = 0$ and $Q_i{}^j = -Q_j{}^i$ – i.e., $Q^i{}_j$ has 3 independent components.

Commutation of ∂_μ and δ

$F = F(x, y(x), y'(x))$ a functional

Change $y(x)$ in the following manner

$$y(x) \longrightarrow y(x) + \epsilon\,\eta(x)$$

The variation δy is defined to be

$$\delta y \equiv \epsilon\,\eta(x) \qquad (1)$$

The variation is a change in a function.

At fixed x,

$$F(x, y, y') \longrightarrow F(x, y + \epsilon\eta, y' + \epsilon\eta')$$

$$= \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$$

$$\equiv \delta F$$

If we let $F = y'$, we can see that generally

$$\delta F = \delta y' = \epsilon\,\eta' \qquad (2)$$

So, at fixed x, from (1) and (2)

$$\delta y' = (\delta y)'$$

- i.e., the derivative w.r.t. the independent variable x and the variation of a function of x commute.