The Harmonic Oscillator

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September 1, 2020

$$
\begin{aligned}
H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \\
&= \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]
\end{aligned}
$$

This has the form $u^2 + v^2$, which can be written as

$$u^2 + v^2 = (iu + v)(-iu + v)$$

Classically, we are dealing with functions. Quantum mechanically we are dealing with operators, so we have to be careful about ordering.

The Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Define the operators

$$
\begin{aligned}
\hat{a} &\equiv \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} + i\hat{p}) \\
\hat{a}^\dagger &\equiv \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} - i\hat{p})
\end{aligned}
$$

Now consider the product

$$
\begin{aligned}
\hat{a}^\dagger\hat{a} &= \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) \\
&= \frac{1}{\sqrt{2m\omega}}\left\{m^2\omega^2\hat{x}^2 + im\omega[\hat{x},\hat{p}] + \hat{p}^2\right\}
\end{aligned}
$$

and the product

$$
\begin{aligned}
\hat{a}\hat{a}^{\dagger} &= \frac{1}{\sqrt{2m\omega}}(m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) \\
&= \frac{1}{\sqrt{2m\omega}}\left\{m^2\omega^2\hat{x}^2 - im\omega[\hat{x},\hat{p}] + \hat{p}^2\right\}
\end{aligned}
$$

The sum

$$
\begin{aligned}
\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger} &= m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{m\omega} \\
&= \tfrac{2}{\omega}\hat{H}
\end{aligned}
$$

Then

$$
\hat{H} = \tfrac{1}{2}\omega(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger})
$$

Let's consider the commutator of the operators $\hat{a}$ and $\hat{a}^{\dagger}$. ($\hbar = 1$)

$$
\begin{aligned}
[\hat{a},\hat{a}^{\dagger}] &= \frac{1}{2m\omega}\left\{(m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p}) - (m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p})\right\} \\
&= \frac{1}{2m\omega}\left\{m^2\omega^2\hat{x}^2 - im\omega[\hat{x},\hat{p}] + \hat{p}^2 - (m^2\omega^2\hat{x}^2 + im\omega[\hat{x},\hat{p}] + \hat{p}^2)\right\} \\
&= -i(i) \\
&= +1
\end{aligned}
$$

That is

$$
[\hat{a},\hat{a}^{\dagger}] = 1 \qquad \Rightarrow \qquad \hat{a}\hat{a}^{\dagger} = 1 + \hat{a}^{\dagger}\hat{a}
$$

and of course

$$
\begin{aligned}
[\hat{a},\hat{a}] &= 0 \\
[\hat{a}^{\dagger},\hat{a}^{\dagger}] &= 0
\end{aligned}
$$

Then

$$
\begin{aligned}
\hat{H} &= \tfrac{1}{2}\omega(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}) \\
&= \tfrac{1}{2}\omega(\hat{a}^{\dagger}\hat{a} + 1 + \hat{a}^{\dagger}\hat{a}) \\
&= \omega(\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2})
\end{aligned}
$$

Now look at

$$
\begin{aligned}
[\hat{H},\hat{a}] &= \omega\left\{(\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2})\hat{a} - \hat{a}(\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2})\right\} \\
&= \omega(\hat{a}^{\dagger}\hat{a}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{a}) \\
&= \omega[\hat{a}^{\dagger}\hat{a}\hat{a} - (1 + \hat{a}^{\dagger}\hat{a})\hat{a}] \\
&= -\omega\hat{a}
\end{aligned}
$$

and

$$
\begin{aligned}
[\hat{H}, \hat{a}^\dagger] &= \omega\left\{(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\hat{a}^\dagger - \hat{a}^\dagger(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\right\} \\
&= \omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a}) \\
&= \omega[\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)] \\
&= \omega\hat{a}^\dagger
\end{aligned}
$$

Label the states by

$$
|n\rangle \qquad E_n = (n + \tfrac{1}{2})\omega \qquad \hat{H}\,|n\rangle = (n + \tfrac{1}{2})\omega\,|n\rangle
$$

and look at

$$
\begin{aligned}
\hat{H}\hat{a}\,|n\rangle &= \omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\hat{a}\,|n\rangle \\
&= \omega(\hat{a}^\dagger\hat{a}\hat{a} + \tfrac{1}{2}\hat{a})\,|n\rangle \\
&= \omega\left[(\hat{a}\hat{a}^\dagger - 1)\hat{a} + \tfrac{1}{2}\hat{a}\right]|n\rangle \\
&= \omega\hat{a}(\hat{a}^\dagger\hat{a} + \tfrac{1}{2} - 1)\,|n\rangle \\
&= \hat{a}(\hat{H} - \omega)\,|n\rangle \\
&= \hat{a}\left[(\hat{n} + \tfrac{1}{2})\omega - \omega\right]|n\rangle \\
&= (n - \tfrac{1}{2})\omega\hat{a}\,|n\rangle
\end{aligned}
$$

$$
\Rightarrow \quad \hat{a}\,|n\rangle \propto |n-1\rangle
$$

What is the constant of proportionality?

From

$$
\hat{H}\,|n\rangle = (\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\omega\,|n\rangle = (n + \tfrac{1}{2})\omega\,|n\rangle
$$

we see that the number operator is

$$
\hat{N} = \hat{a}^\dagger\hat{a}
$$

Then

$$
\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n
$$

But

$$
\hat{a}\,|n\rangle = c\,|n-1\rangle
$$

and

$$
\langle n|\,\hat{a}^\dagger = c^*\,\langle n-1|
$$

Then

$$
n = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = |c|^2\,\langle n-1|n-1\rangle = |c|^2
$$

and

$$c = \sqrt{n}$$

That is

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

Now look at

$$\begin{aligned}
\hat{H}\hat{a}^\dagger |n\rangle &= \omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\hat{a}^\dagger |n\rangle \\
&= \omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\hat{a}^\dagger |n\rangle \\
&= \omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \tfrac{1}{2}\hat{a}^\dagger) |n\rangle \qquad\qquad \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \\
&= \omega\hat{a}^\dagger[(1 + \hat{a}^\dagger\hat{a}) + \tfrac{1}{2}] |n\rangle \\
&= \hat{a}^\dagger(\hat{H} + \omega) |n\rangle \\
&= \hat{a}^\dagger(n + \tfrac{1}{2} + 1)\omega |n\rangle \\
&= (n + \tfrac{1}{2} + 1)\omega\hat{a}^\dagger |n\rangle
\end{aligned}$$

$$\Rightarrow \quad \hat{a}^\dagger |n\rangle \propto |n+1\rangle$$

But

$$\langle n+1|\hat{a}^\dagger\hat{a}|n+1\rangle = n+1$$

and

$$n+1 = \langle n+1|\hat{a}^\dagger\hat{a}|n+1\rangle = |c|^2 \langle n+1|n+1\rangle = |c|^2$$

Then

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

One of the modt important aspects of the spectrum of states of the quantized harmonic oscillator is that its energy levels are separated by a uniform amount: $\hbar\omega$. The state $|n\rangle$ has energy $n\hbar\omega$ and the state $|n+1\rangle$ has energy $(n+1)\hbar\omega$.

With

$$\hat{H} |n\rangle = \left(n + \tfrac{1}{2}\right) \hbar\omega |n\rangle$$

we have the vacuum state $|0\rangle$ with energy $\frac{1}{2}\hbar\omega$. Then we can think of the state $|n\rangle$ as a state of n quanta each of energy $\hbar\omega$ – i.e., we can think of it as a <u>multiparticle</u> state.

With this interpretation, the operator $\hat{a}^\dagger$ is a <u>creation operator</u> that creates a quantum of energy $\hbar\omega$, whereas $\hat{a}$ is a <u>annihilation operator</u> that annihilates a quantum of energy $\hbar\omega$.

Now we can generate all of the states beginning with the vacuum state defined by

$$\hat{a} |0\rangle = 0$$

Then

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle$$

$\frac{1}{\sqrt{n!}}$ cancels out the $\sqrt{n!}$ that appears when the $\hat{a}^\dagger$ act n times.

For example,

$$\begin{aligned}
|2\rangle &= \frac{1}{\sqrt{2}}\hat{a}^\dagger \left(\hat{a}^\dagger |0\rangle\right) \\
&= \frac{1}{\sqrt{2}}\hat{a}^\dagger \left(\sqrt{1}\,|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\sqrt{1+1}\,|2\rangle \\
&= |2\rangle
\end{aligned}$$