Effect of Coriolis' Theorem on the Foucault Pendulum

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Abstract

In this report, Coriolis' Theorem is derived and its consequences discussed. The Foucault pendulum and its historical imprint are introduced and the mathematics of the pendulum is developed. The path the Pendulum traces due to being in a rotating frame of reference is found to be circular and how the period of this path varies based on the location of the pendulum on the Earth is discussed. Opportunities for further analysis beyond this report are also mentioned.

1 Coriolis' Theorem

Normally, in order to simplify the problems that we are considering in classical mechanics, we disregard the effect of the Earth rotating and consider any frame of reference that our problems occur in on Earth to be an **Inertial Frame of Reference** [1] - i.e. a frame of reference that is not accelerating.
However, it is known that the Earth is rotating about an axis through its line of symmetry, and hence any frame of reference taken on the Earth is accelerating and is non-inertial. We now want to formalise this notion mathematically.

If we consider a set of mutually orthogonal axes situated on the Earth $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ and do not require them to be constant (infact - we take them to be rotating with a constant velocity), we can express any arbitrary point particle in this frame as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.1}$$

Normally, taking the time derivative of Eq.1.1 would return an expression for the velocity of the particle at a time t. However, we have stated that the axes need not be invariant and as such, we must invoke the product rule when differentiating:

$$\frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) + \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}\right) \tag{1.2}$$

This gives us the true velocity of the particle inside the rotating reference frame. We see that the left-hand term of Eq.1.2 is the time derivative of the particle's position in an inertial reference frame. Thus, we will term it as $\frac{\delta\mathbf{r}}{\delta t}$, giving us:

$$\frac{d\mathbf{r}}{dt} = \frac{\delta\mathbf{r}}{\delta t} + \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}\right) \tag{1.3}$$

We now assume that an arbitrary vector $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ exists such that:

$$A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} = x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt} \tag{1.4}$$

As the scalar multiple of any unit vecotr with itself is 1 and the scalar multiple of any unit vector with another unit vector perpendicular to it is 0, we can obtain the following 3 equations for A_1, A_2 and A_3 from Eq.1.4 by taking the scalar multiple of each side with each unit vector of our axes:

$$\begin{aligned} A_1 &= \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}\right)\cdot\mathbf{i} \\ A_2 &= \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}\right)\cdot\mathbf{j} \\ A_3 &= \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}\right)\cdot\mathbf{k} \end{aligned} \tag{1.5}$$

Now, we notice that despite the fact the axes are rotating, the length of the axes will not vary and nor will the angle between any of the axed. Simple manipulation of derivatives and of vector laws allows us to see that

$$0 = \frac{d(\mathbf{i}\cdot\mathbf{i})}{dt} = \frac{d\mathbf{i}}{dt}\cdot\mathbf{i} + \mathbf{i}\cdot\frac{d\mathbf{i}}{dt} = 2\frac{d\mathbf{i}}{dt}\cdot\mathbf{i} \tag{1.6}$$

due to the commutativity of the scalar vector product product. The same line of thinking follows for the other two axes. This leads us to the conclusion that

$$\frac{d\mathbf{i}}{dt}\cdot\mathbf{i} = \frac{d\mathbf{j}}{dt}\cdot\mathbf{j} = \frac{d\mathbf{k}}{dt}\cdot\mathbf{k} = 0 \tag{1.7}$$

This leaves us with 6 other terms to consider. We can see that

$$0 = \frac{d}{dt}(0) = \frac{d}{dt}(\mathbf{i}\cdot\mathbf{j}) = \frac{d\mathbf{i}}{dt}\mathbf{j} + \mathbf{i}\frac{d\mathbf{j}}{dt} \tag{1.8}$$

The same logic following for the other two possible combinations of the derivative of the scalarproducts of the axes unit vectors. This gives the following three expressions:

$$\begin{aligned} \frac{d\mathbf{i}}{dt}\mathbf{j} &= -\mathbf{i}\frac{d\mathbf{j}}{dt} \\ \frac{d\mathbf{i}}{dt}\mathbf{k} &= -\mathbf{i}\frac{d\mathbf{k}}{dt} \\ \frac{d\mathbf{j}}{dt}\mathbf{k} &= -\mathbf{j}\frac{d\mathbf{k}}{dt} \end{aligned} \tag{1.9}$$

Combining Eq.1.6 with these three expressions allows us to write

$$\begin{aligned} A_1 &= z\mathbf{i}\cdot\frac{d\mathbf{k}}{dt} - y\mathbf{j}\cdot\frac{d\mathbf{i}}{dt} \\ A_2 &= x\mathbf{j}\cdot\frac{d\mathbf{i}}{dt} - z\mathbf{k}\cdot\frac{d\mathbf{j}}{dt} \\ A_3 &= y\mathbf{k}\cdot\frac{d\mathbf{j}}{dt} - x\mathbf{i}\cdot\frac{d\mathbf{k}}{dt} \end{aligned} \tag{1.10}$$

These three equations can be factorised into the vector product of two vectors, giving us

$$\mathbf{A} = \left[\mathbf{k}\cdot\frac{d\mathbf{j}}{dt}, \mathbf{i}\cdot\frac{d\mathbf{k}}{dt}, \mathbf{j}\cdot\frac{d\mathbf{i}}{dt}\right]\mathbf{x}(x, y, z) \tag{1.11}$$

It follows that $\mathbf{A}$ must be a velocity for the dimensions of Eq.1.2 to hold. If we have a particle that is stationary with respect to the reference frame that it is in, it appears that it still has a velocity $\mathbf{A}$ governed by Eq.1.2. Thus, it follows that $\mathbf{A}$ is the velocity of the frame at the point $\mathbf{r}$.
Now, as we know for any rotating body with radial velocity $\boldsymbol{\omega}$ at a point $\mathbf{r}$ the tangential velocity at that point is given by:

$$\mathbf{v} = \boldsymbol{\omega}\mathbf{x}\mathbf{r} \tag{1.12}$$

This clearly follows the same format as Eq.1.11, and so we can define

$$\boldsymbol{\omega} = \left[\mathbf{k}\cdot\frac{d\mathbf{j}}{dt}, \mathbf{i}\cdot\frac{d\mathbf{k}}{dt}, \mathbf{j}\cdot\frac{d\mathbf{i}}{dt}\right] \tag{1.13}$$

As being the radial velocity of the frame. This allows us to rewrite Eq.1.2 as:

$$\frac{d\mathbf{r}}{dt} = \frac{\delta\mathbf{r}}{\delta t} + \boldsymbol{\omega}\mathbf{x}\mathbf{r} \tag{1.14}$$

Also notice that this is a differential operator acting on the vector $\mathbf{r}$. If we apply this operator again to $\frac{d\mathbf{r}}{dt}$ we obtain the acceleration of the particle in the rotating reference frame. This gives the final form of Coriolis' theorem:

$$m\frac{\delta^2\mathbf{r}}{dt^2} = \mathbf{F} - 2m\boldsymbol{\omega}\mathbf{x}\frac{\delta\mathbf{r}}{\delta t} - m\frac{\delta\boldsymbol{\omega}}{\delta t}\mathbf{x}\mathbf{r} - m\boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r}) \tag{1.15}$$

where $\mathbf{F}$ is the translational force (which is **not** a fictitious force) given by Newton's Second Law of Motion where

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} \tag{1.16}$$

We see that Newton's Law is modified and has extra terms in Eq.1.15. We term these as **Fictitious Forces** [2] and they arise due to the frame the particle being considered in accelerating relative to the particle, rather than the motion of the particle itself. This can cause interesting effects on the trajectory of objects in the rotating frame that would not otherwise occur.

We term the $-2m\boldsymbol{\omega}\mathbf{x}\frac{\delta\mathbf{r}}{\delta t}$ term as being the **Coriolis Force** [3] (which is responsible for the trajectory of the Foucault Pendulum that we will be analysing in the rest of this report).

The $-m\frac{\delta\boldsymbol{\omega}}{\delta t}\mathbf{x}\mathbf{r}$ term is known as the **azimuthal force** or the **Euler Force** [3]. For all situations we will be considering in this report, the rotating frame of reference we are considering is one on the Earth, which is rotating with a constant angular frequency $\boldsymbol{\omega}$, hence the azimuthal force will go to zero.

Finally, the $-m\boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r})$ is termed the **centrifugal force** which has interesting effects on the Earth - Namely causing the Earth to flatten at the poles and bulge at the equator, causing it to not be a true sphere [4]

2 The Foucault Pendulum

A way to unequivocally demonstrate the Coriolis effect and prove that the Earth is rotating was devised by French physicist Léon Foucault in 1851 [5]. It is a pendulum known as **Foucault's Pendulum**. It is a simple concept - if we displaced a pendulum in an inertial reference frame we would see oscillations in only one plane and the trajectory of the pendulum will trace out a circle. However, if we have a pendulum that is free to move in all directions in a non-inertial reference frame, the plane that the pendulum swings in will gradually precess around the vertical axis.

The set-up is shown in Fig.1 below. The system begins with many pins placed upright in a circle around the pendulum such that the maximum displacement of the pendulum is the same as the radius of the circle. The pendulum will be capable of oscillating in all possible directions, but will only have an initial displacement in one plane of motion (and also typically a motor will be applied to ensure that the pendulum executes forced oscillations and does not come to rest due to the dampening effect of the air surrounding it. But, this will not confine the pendulum to move in one plane.)

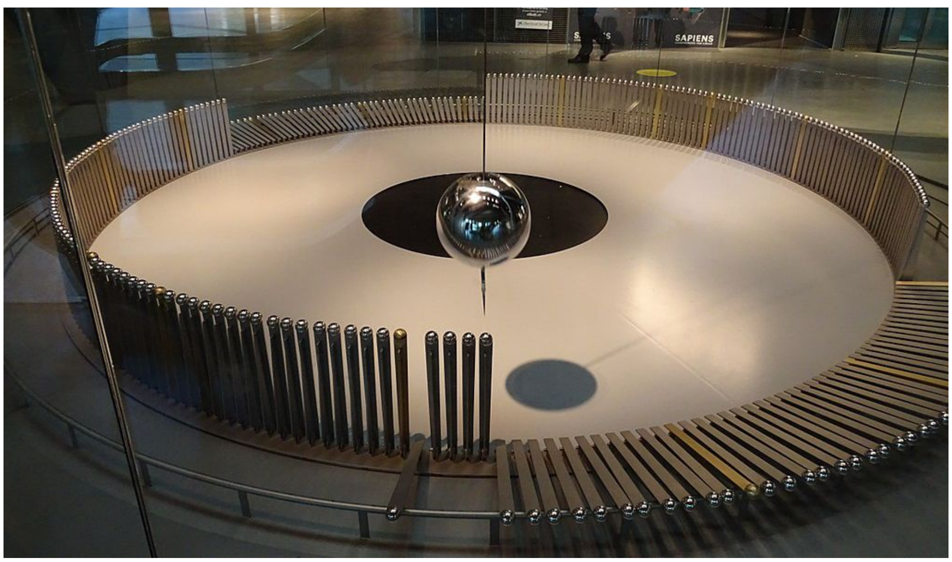

Figure 1: The Foucault Pendulum on display in the CosmoCaixa museum in Barcelona, Spain. The pendulum has executed some of the period of its rotation already - with some of the pins knocked down and the remaining still standing. [6]

If the pendulum was executing its oscillations in an inertial reference frame, clearly only 2 of the pins would be knocked down - independent of how long the pendulum oscillates. However, the plane of oscillation rotates, and all of the pins are eventually knocked down by the pendulum, suggesting the pendulum traces out a circle as the frame rotates. This effect can be explained mathematically by considering the Earth to be a rotating frame of reference and investigating the effect Coriolis' Theorem has on the oscillation of the pendulum.

One of the first Foucault Pendulums was set-up by Foucault himself in the Panthéon in Paris and is still an attraction there today. Foucault Pendulums have become a common attraction in many science museums due to the impressive set-up the require. The wire the bob oscillations on has to be very long (the pendulum in the Panthéon is 68m long for reference [5]) and the bob comparatively large and heavy in order for the circular path traced due to the Coriolis effect to be noticeable. We will derive the mathematics of this effect in the next section.

3 Modelling the Pendulum

We want to model the pendulum and see how Eq.1.15 changes for these conditions. In doing this, we will attempt to obtain equations of motion to demonstrate how the pendulum does not just perform oscillations in one plane. Our goal is to be able to determine equations of motion of the pendulum and calculate the frequency of the rotation of the frame the pendulum is in. Consider the situation as shown below in Fig.2.

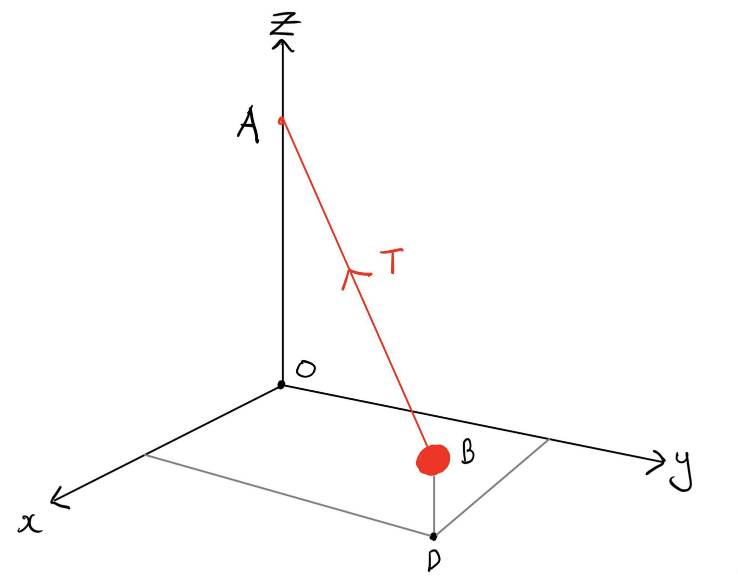

Figure 2: The model of the situation we will be considering. The pendulum is executing oscillations in the mutually perpendicular set of axis x, y, z which is rotating with a constant angular velocity $\boldsymbol{\omega}$

The fixed point of the pendulum is at A and the bob of the pendulum of mass m is free to move and is at point B. We will be considering the bob to be a particle with all of its mass concentrated at the point B. We also assume the string the bob is connected to is light and extensible. We have already established in Section 1 that the azimuthal force will be zero in our considerations of this frame. Let us investigate the centrifugal force. If we term the magnitude of the frequency of rotation of the Earth as Ω then we can clearly see that

$$\Omega = \frac{2\pi}{T} = \frac{2\pi}{24 \cdot 60 \cdot 60} \approx 7.272 \cdot 10^{-5} \text{ rad} \cdot s^{-1} \tag{3.1}$$

Where this is the same as the magnitude of the angular frequency of the frame that appears in Eq.1.15 i.e. $|\boldsymbol{\omega}| = \Omega$. Inspecting the centrifugal force we see that

$$|\boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r})| \leq \Omega^2|\mathbf{r}| \tag{3.2}$$

by considering the definition of the vector product. $\Omega^2 \approx 5 \times 10^{-9}$, which is a tiny factor compared to the translational force and the coriolois force. We will thus absorb it into the translational force and not consider it in our calculations.
We then have Coriolis' Theorem Eq.1.15 in the reduced form for this problem:

$$m\frac{\delta^2\mathbf{r}}{dt^2} = \mathbf{F} - 2m\boldsymbol{\omega}\mathbf{x}\frac{\delta\mathbf{r}}{\delta t} \tag{3.3}$$

If we consider $\mathbf{k}$ to be a unit vector point up from the surface of the Earth, $\mathbf{j}$ to be a unit vector pointing in the Northerly direction and $\mathbf{i}$ being a vector pointing in the Easterly direction we can clearly write the translational force on the bob as being

$$\mathbf{F} = \mathbf{T} - mg\mathbf{k} \tag{3.4}$$

We take these oscillations to be occurring at point O which we will say is at a latitude λ above the equator line. Now due to our previously established directional unit vectors, this means we can resolve the angular frequency into components relative to the $\mathbf{j}$ and $\mathbf{k}$ axes and express the angular frequency of the frame in the form

$$\boldsymbol{\omega} = \Omega(cos(\lambda)\mathbf{j} + sin(\lambda)\mathbf{k}) \tag{3.5}$$

Now we need a means of expressing $\mathbf{T}$ in the unit vectors of our frame. From the set-up of Fig.2, we can see that clearly $\mathbf{T}$ acts parallel to $\vec{BA}$ and so we can say

$$\begin{aligned}\vec{BA} &= \vec{BD} + \vec{DO} + \vec{OA}\\ &= -z\mathbf{k} - x\mathbf{i} - y\mathbf{j} + a\mathbf{k}\\ \implies \frac{\vec{BA}}{a} &= -\frac{x}{a}\mathbf{i} - \frac{y}{a}\mathbf{j} + \left(1 - \frac{z}{a}\right)\mathbf{k}\end{aligned} \tag{3.6}$$

So we now if we say that $\mathbf{T} = T\hat{\mathbf{T}}$ then we can express $\mathbf{T}$ as

$$\mathbf{T} = \left(-\frac{x}{a}T\right)\mathbf{i} - \left(\frac{y}{a}T\right)\mathbf{j} + \left(1 - \frac{z}{a}\right)T\mathbf{k} \tag{3.7}$$

and finally the position of B is expressed as the position vector $\mathbf{r}$ where

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\\ \frac{\delta\mathbf{r}}{\delta t} &= \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}\\ \frac{\delta^2\mathbf{r}}{\delta t^2} &= \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}\end{aligned} \tag{3.8}$$

Now we can re-express Eq.3.4 as a system of 3 equations by considering the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components separately:

$$m\ddot{x} = -\frac{xT}{a} - 2m\Omega(\dot{z}cos(\lambda) - \dot{y}sin(\lambda)) \tag{3.9}$$

$$m\ddot{y} = -\frac{yT}{a} - 2m\Omega(\dot{x}sin(\lambda)) \tag{3.10}$$

$$m\ddot{z} = \left(1 - \frac{z}{a}\right)T - mg + 2m\Omega\dot{x}cos(\lambda) \tag{3.11}$$

And we obtain our equations of motion that describe the trajectory of the Pendulum. We observe that these equations of motion are coupled, each equation is dependent on more than one space variable or their derivatives and we will need all three equations to obtain a solution for x, y, z at any time t.

4 Path of the Pendulum

The current equations of motion we have of the pendulum are impossible to solve analytically [7], so we must make some assumptions to simplify the system. We have established that the pendulum will trace a sphere centred at $A(0, 0, a)$, hence we can write the constraint that

$$x^2 + y^2 + (a - z)^2 = a^2 \tag{4.1}$$

hence rearranging this constraint and expanding via the binomial theorem to 2nd order allows us to see:

$$(a - z) = (a^2 - x^2 - y^2)^{\frac{1}{2}}$$
$$(a - z) = a - \frac{1}{2a^2}(x^2 + y^2) + ...$$
$$\frac{z}{a} = \frac{1}{2a^2}(x^2 + y^2) + ...$$

but if we assume that the pendulum is only executing small oscillations, ie

$$\frac{x}{a}, \frac{y}{a} << 1$$

then we can approximate from the previous expression

$$z, \dot{z} \approx 0$$

so we can now simplify Eq.3.11 by this approximation and say that

$$T - mg = 2m\Omega\dot{x}cos(\lambda) \tag{4.2}$$

but we have established already that $\Omega^2 << \Omega << 1$ and claimed that small oscillations are occurring, thus all the terms on the right-hand side of Eq.4.2 are negligible compared to the terms on the left-hand side, and so we can approximate that

$$T \approx mg \tag{4.3}$$

combining these approximations with Eq.3.9 and Eq.3.10 allows us to obtain linear differential equations of the form

$$\ddot{x} = -g\frac{x}{a} + 2\Omega\dot{y}sin(\lambda) \tag{4.4}$$

$$\ddot{y} = -g\frac{x}{a} - 2\Omega\dot{x}sin(\lambda) \tag{4.5}$$

All of the above terms are similar in magnitude, so we may make no further approximations. These equations of motion are not easily decoupled - we must introduce complex numbers to achieve a solution. We allow

$$\begin{gathered} u = x + iy \\ \text{and so} \\ \dot{u} = \dot{x} + i\dot{y} \\ \ddot{u} = \ddot{x} + i\ddot{y} \end{gathered} \tag{4.6}$$

this allows us to combine Eq.4.4 and Eq.4.5 to say that

$$\begin{aligned} \ddot{u} &= -\frac{g}{a}(x + iy) + 2\Omega sin(\lambda)(\dot{y} - i\dot{x}) \\ \ddot{u} &= -\frac{g}{a}u - 2i\Omega sin(\lambda)\dot{u} \end{aligned}$$

or, alternatively

$$\ddot{u} + 2i\Omega sin(\lambda)\dot{u} + \frac{g}{a}u = 0 \tag{4.7}$$

This is a second order homogeneous differential equation that can be solved easily enough to give the solution

$$u = A\exp\left[(-i\omega sin(\lambda) + ip)t\right] + B\exp\left[(-i\omega sin(\lambda) - ip)t\right] \tag{4.8}$$

where we have that $p^2 = \Omega^2 sin^2\lambda + \frac{g}{a}$ and A and B are arbitrary real constants. If we make use of Euler's Formula [8] we can convert this into a trigonometric expression with real and imaginary components:

$$u = [cos(\Omega sin(\lambda)t) - isin(\Omega sin(\lambda)t)](Acos(pt) + isin(pt)) \tag{4.9}$$

Which forms our equations of motion for the trajectory of the pendulum as a function of time. Comparing real and imaginary coefficients at any time allows us to obtain the x and y co-ordinates of the pendulum at any time t.

5 Elliptical Path

If we consider the case with $\Omega = 0$ - i.e. an inertial reference frame - as u_0, we should obtain the equations of a simple compound pendulum with period of oscillation of $\sqrt{\frac{g}{a}}$:

$$u_0 = Acos(pt) + iBsin(pt) \tag{5.1}$$

comparing the real and imaginary components gives us

$$
\begin{aligned}
x &= Acos(pt) \\
y &= Bsin(pt)
\end{aligned}
\tag{5.2}
$$

This is the parametric equation of an ellipse and describes an ellipse with semi-major axis A, semi-minor axis B and angular frequency of p, as clearly if the time to complete one revolution is T then we have that

$$pT = 2\pi \tag{5.3}$$

And we have that $\Omega = 0$, so clearly from our definition for p we have that

$$p = \sqrt{\frac{g}{a}} \tag{5.4}$$

Which is the result we would expect for the angular frequency of a simple pendulum oscillating in an inertial reference frame. This then allows us to write that:

$$u = [cos(\Omega sin(\lambda)t) - isin(\Omega sin(\lambda)t)]u_0 \tag{5.5}$$

So clearly, the left term of Eq.5.5 is what determines the precession of the otherwise standard simple pendulum around the vertical axis. Investigating this equation via the same analysis before leads us to

$$
\begin{aligned}
x &= u_0 cos(\Omega sin(\lambda)t) \\
y &= -u_0 sin(\Omega sin(\lambda)t)
\end{aligned}
\tag{5.6}
$$

via the same analysis as before, this clearly describes the elliptical path the pendulum follows in the $x - y$ plane while precessing about the z-axis. We see the pendulum will actually precess in a circular path, due to having the same major and minor axis of u_0.

A plot of the path an arbitrary pendulum that is not located at the equator would trace out after completing one revolution is shown below in Fig.3. We see that the path traced by any Foucault Pendulum would indeed form a circle will precessing about the vertical axis due to the Coriolis Force.

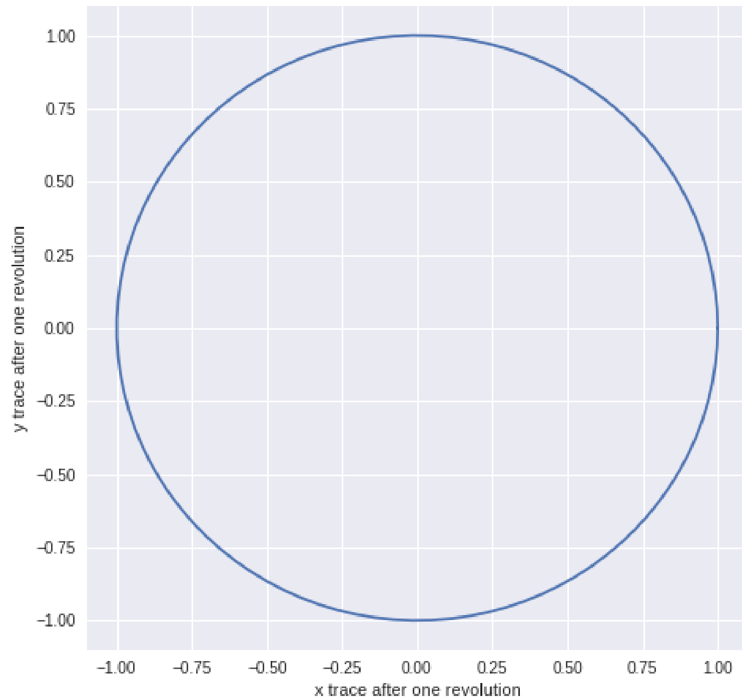

Figure 3: A birds-eye perspective of the path the pendulum would trace out in one period

6 Variation of Period of Pendulum Path with Latitude

We have analysed the precession of the pendulum about the vertical axis due to the rotation of the frame it is present in and shown that it clearly traces out an ellipse as it completes this precession. We now want to investigate the properties of this precession and see how they vary due to the location of the pendulum on the Earth. By Eq. 5.6. Via the same analysis as in Eq.5.3, we can clearly see that the angular frequency of this precession is given by

$$\omega_p = \Omega sin(\lambda) \tag{6.1}$$

And so the period is given by

$$T = \frac{2\pi}{\Omega|sin(\lambda)|} = \frac{T_{Earth}}{|sin(\lambda)|} \tag{6.2}$$

where T_{Earth} is the period of the rotation of Earth (i.e. 24hr).

Analysing what Eq.6.2 infers shows that the time taken for the pendulum to complete one full revolution about the vertical axis is inversely proportional to the sine of the latitude of the pendulum on the Earth. The mathematics here makes intuitive sense as at the North Pole where $\lambda = 90°$, the period of the precession is at its quickest, corresponding to where the Coriolis Effect is strongest on the Earth [9] due to a frame of reference at the pole rotating with the fastest possible angular velocity, that of Earth's.
We also see that the period tends to infinity as we approach the equator where $\lambda = 0°, 180°$.

This also makes sense as the Coriolis Force goes to zero at the equator [9], so it would take infinitely long for the pendulum to precess about the vertical axis, as it is not precessing.

We need to look at a subtlety here that has been thus ignored. The orientation of the precession of the pendulum. From Eq.5.6, the negative sign in the y term infers a clockwise rotation of the pendulum about the vertical axis in the Northern Hemisphere. This is because in the Northern Hemisphere, $sin(\lambda)$ is a positive value, and thus y is negative. Physically, this corresponds to a clockwise rotation as we have considered anti-clockwise to be positive in this frame.
In the Southern Hemisphere, however, $sin(\lambda)$ will return a negative value and hence we will observe the pendulum rotating in a counter-clockwise orientation. These effects are indeed observed in pendulums located across the world. This is also what calls for the modulus present in Eq.6.2 as otherwise, we would obtain negative periods for pendulums situated in the Southern hemisphere*.

A plot of this variation of the period of precession of the Pendulum against its latitude is shown below in Fig.4:

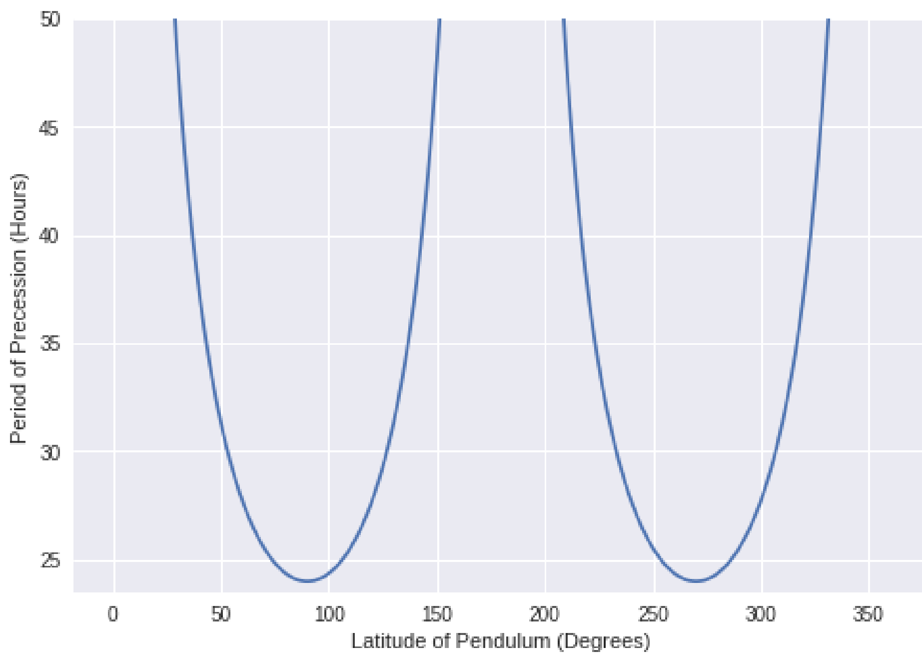

Figure 4: A plot of variation of period against latitude. We obtain the inverse nature as Eq.5.6 would suggest and an asymptotic nature is seen at the areas where the sine of the latitude goes to zero

The characteristic features being the pendulum taking only 24 hours to complete a revolution about its axis at the North and South poles as would be expected at $\lambda = 90^\circ$ and

*It is worth noting that if this modulus was discounted, the negative period obtained in the Southern Hemisphere would simply be an indication of the direction of the pendulum. But for simplicity's sake it has been disregarded.

$\lambda = 270°$. We also notice the sharp increase in the period as we move away from the minima points of the graph. Hence, the period is clearly very sensitive to the latitude. We obtain a reflection of what occurs in the Northern Hemisphere as what occurs in the Southern Hemisphere as our analysis lead us to conclude.

Due to these characteristics, it is rare for a Foucault Pendulum to be seen in areas of the world that are near the Equator. The most famous Foucault Pendulum in the Panthéon in **Paris** is at a latitude $\lambda = 48.87$ [10] and will complete one full revolution and knock down all the pins surrounding it in **31.86 hours**, making it a worthwhile attraction. Note that it will execute a clockwise path due to being in the Northern Hemisphere.
A pendulum located in a more exotic such as **Nairobi** in Kenya is at a latitude $\lambda = -1.2833$ [10] and would take **44.6 Days** to complete one full revolution. Clearly, this would not make as interesting a public attraction, but it would be interesting to see that after this time period all the pins do indeed end up knocked over despite the length of time it took, reinforcing the theory. Note that the pendulum in this situation would be in the Southern Hemisphere (hence the negative latitude) and so would rotate in the counter-clockwise direction.

Finally, **Nueva Loja** in **Ecuador** - the Country named after how close it is to the Equator is at a latitude of $\lambda = 0.090573$ [10] and would take a whopping **1.732 Years** to complete one revolution. But, the effect is indeed still present.

7 Conclusions

We have derived Coriolis' Theorem and applied it to the case of a simple pendulum to show how it affects the path of the pendulum. We discussed how this known as the Foucault Pendulum and looked at the historical and cultural aspects briefly. Following the mathematics lead us to see that the pendulum will in fact trace out a circle and time taken for this circle to be completed depends heavily on how far away the pendulum is from the equator.
It is worth noting that a number of assumptions had to be made to derive a set of equations for the position of the pendulum at a time t, showing how considering the Coriolis Effect makes systems that were comparably simple much more complicated and giving us perspective as to why this effect is disregarded in a large number of areas of Theoretical Mechanics. It would be interesting to see if any information can be gleaned from the equations of motion being solved numerically instead of analytically through assumptions. Applying the theory derived in this report of Coriolis' Theorem to other mechanical systems and analysing how it effects them would also be a logical continuation of this report.

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