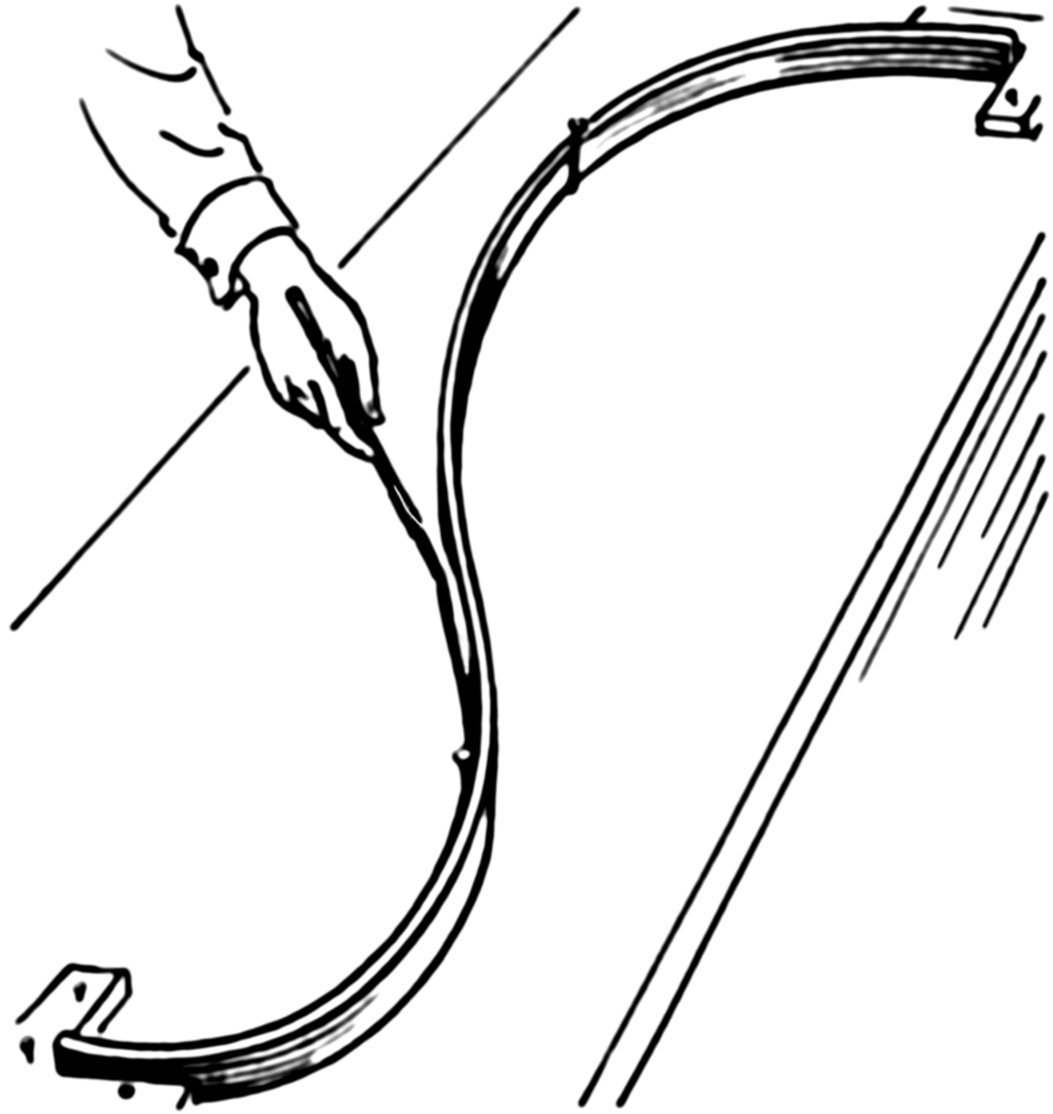
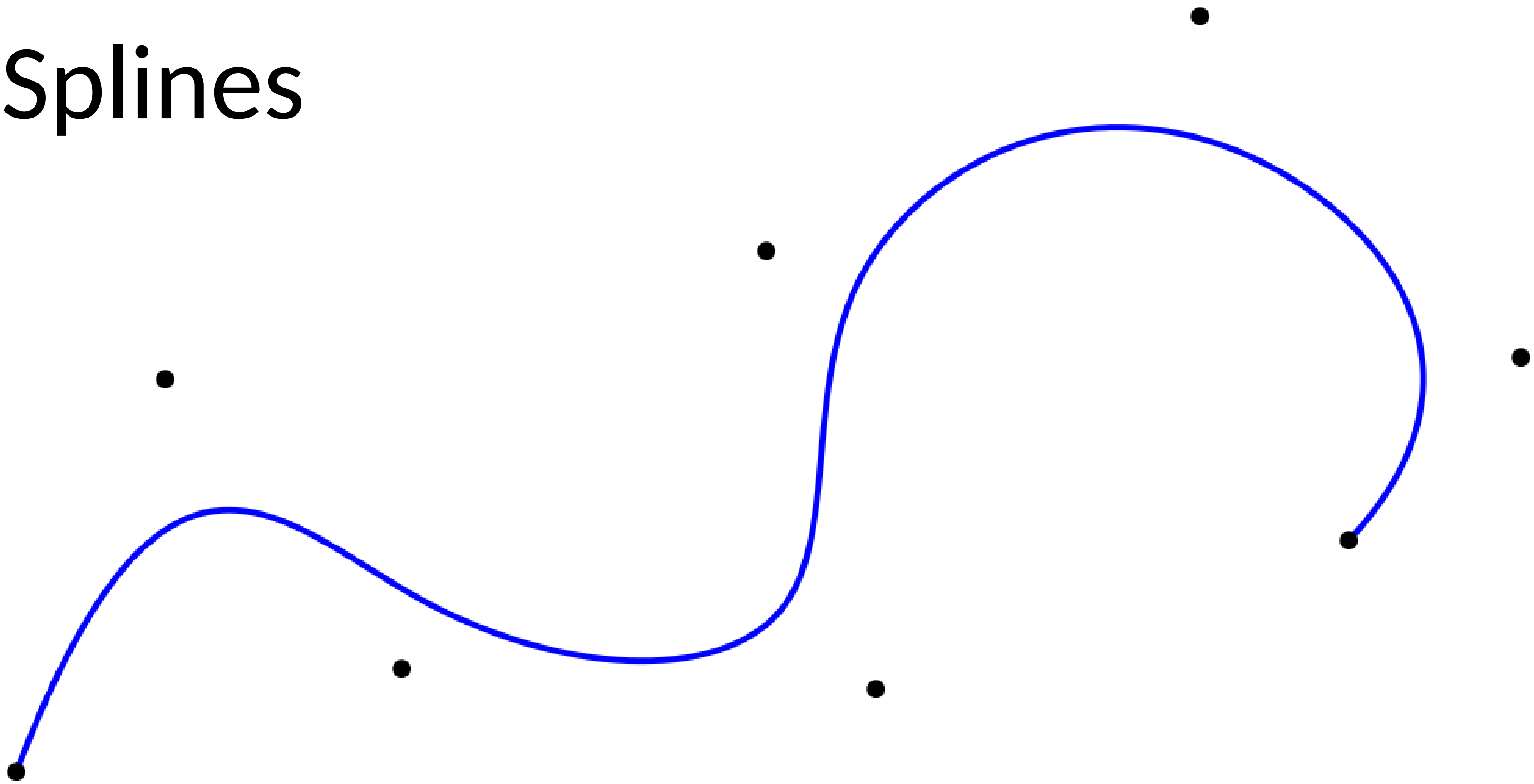
Lecture VIII – Splines

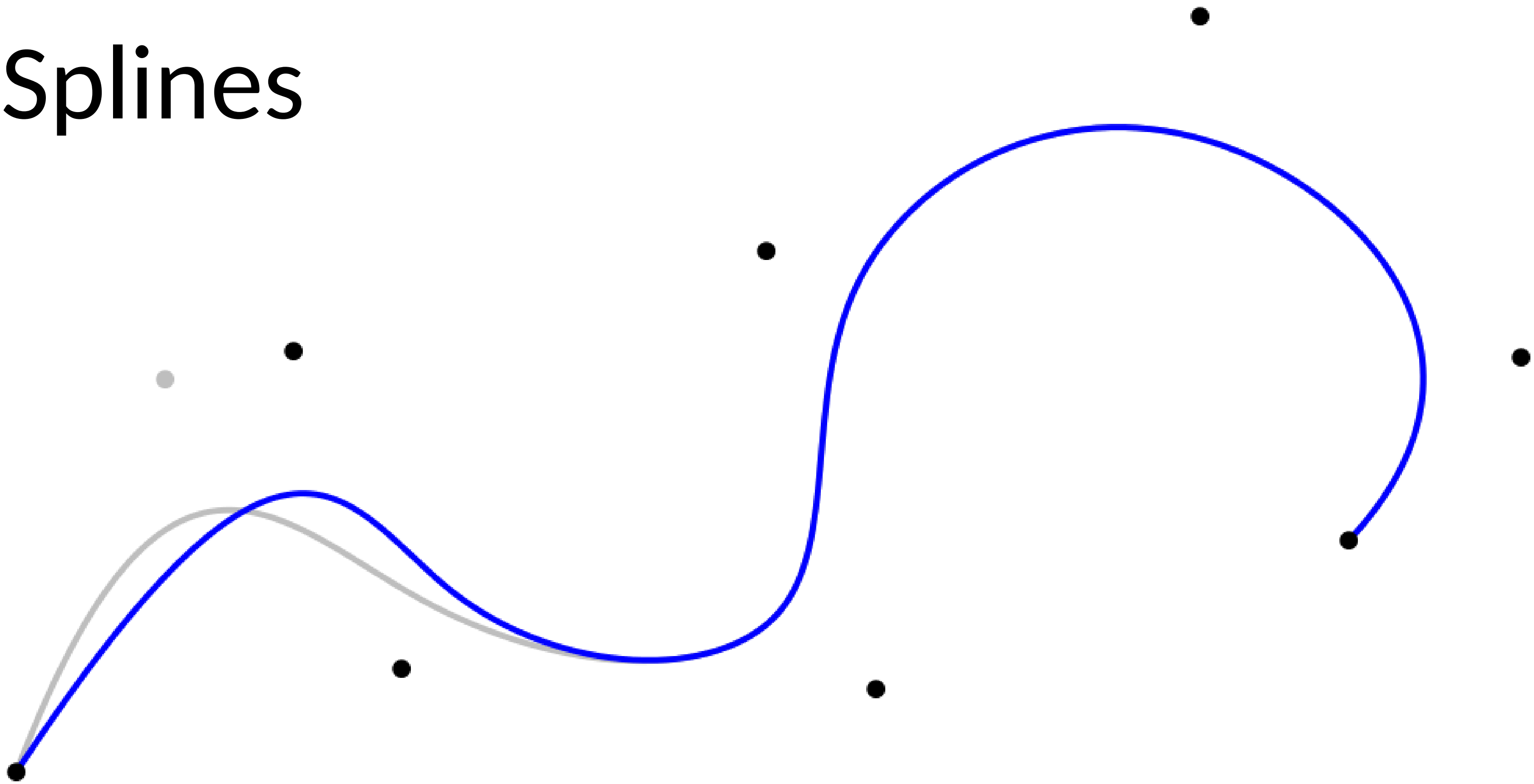
Splines



Splines

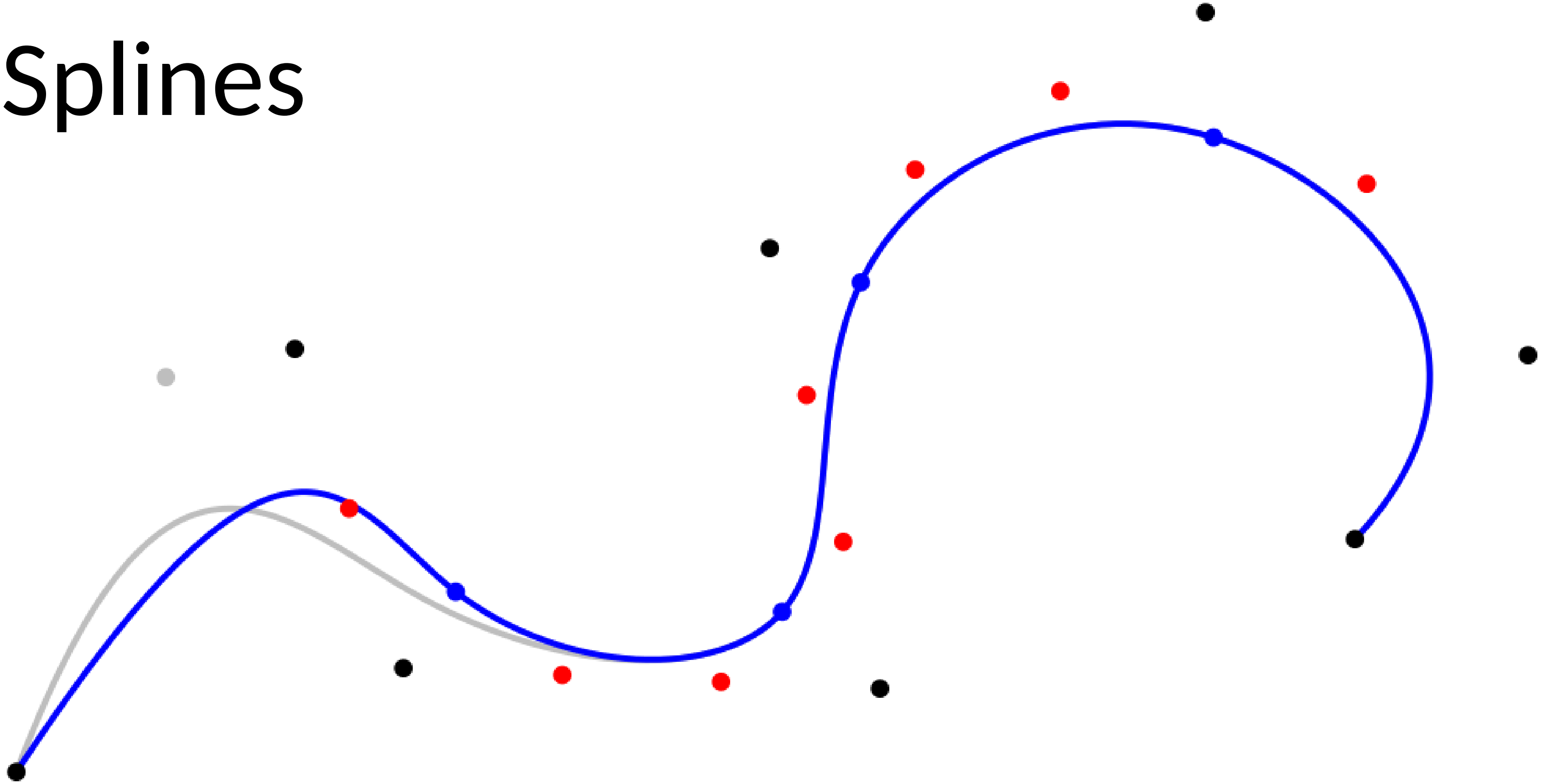


Splines



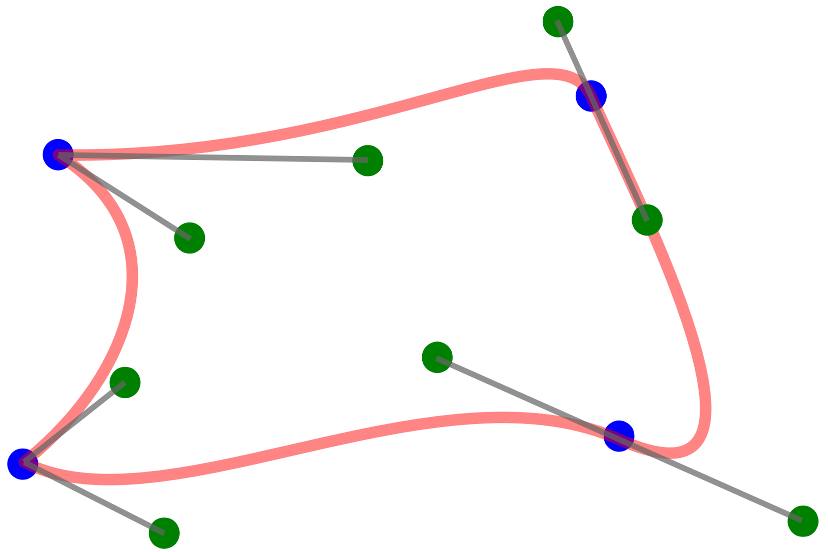
Arbitrarily long, smooth, curves, which only locally react to changes in control points.

Splines



Arbitrarily long, smooth, curves, which only locally react to changes in control points.

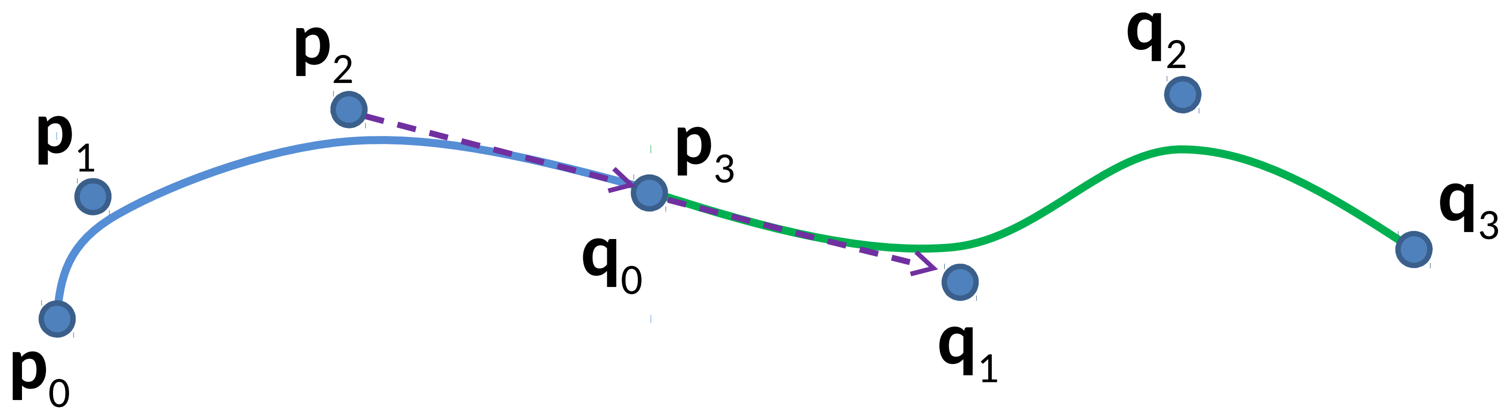
Splines from Bezier curves



- Bézier curves can be glued together
- Also called Béziergons or polyBézier
- Not automatically smooth!

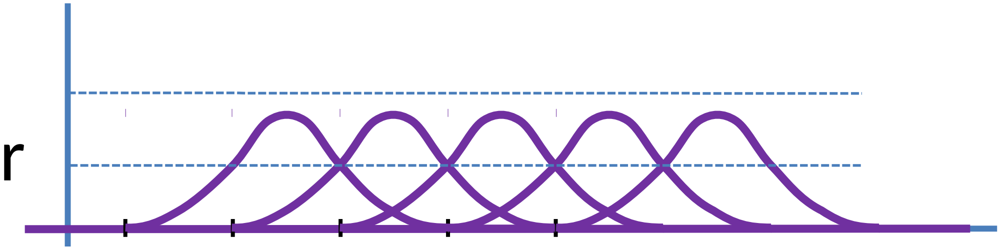
Splines from Bezier curves

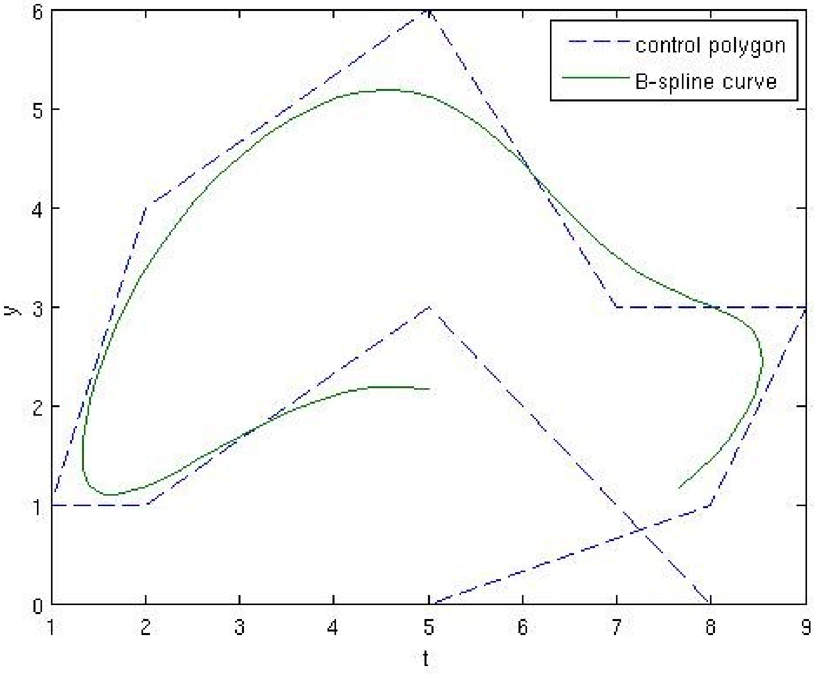
- To ensure continuity
  - C^0 : last control point of first piece must be same as first control point of second piece
  - G^1 : last two control points of first piece must align with the first two control points of the second piece
  - C^1 : distances must be the same as well



B-spline curves

- Idea: use copies of *the same* blend function for every control point
- Cannot sum to 1 everywhere
- Result: Approximates all control points, will *not* pass through first and last control point



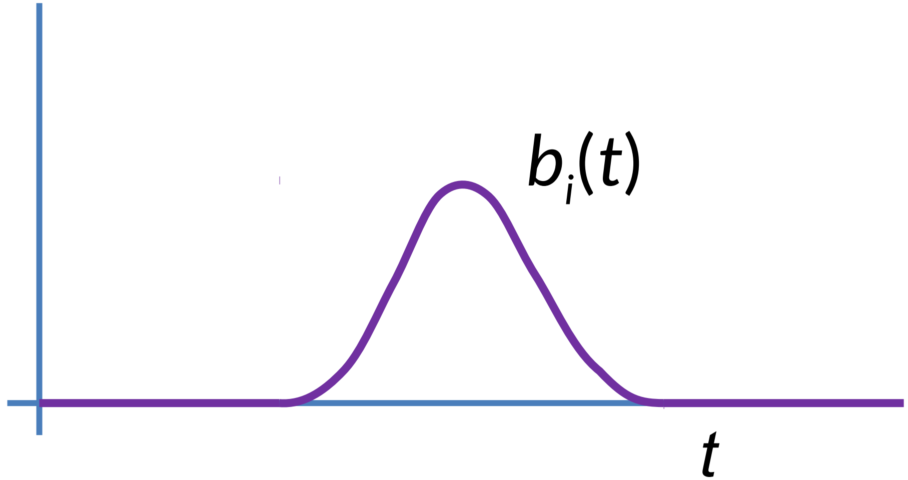


B-spline curves

- A linear combination of the control points

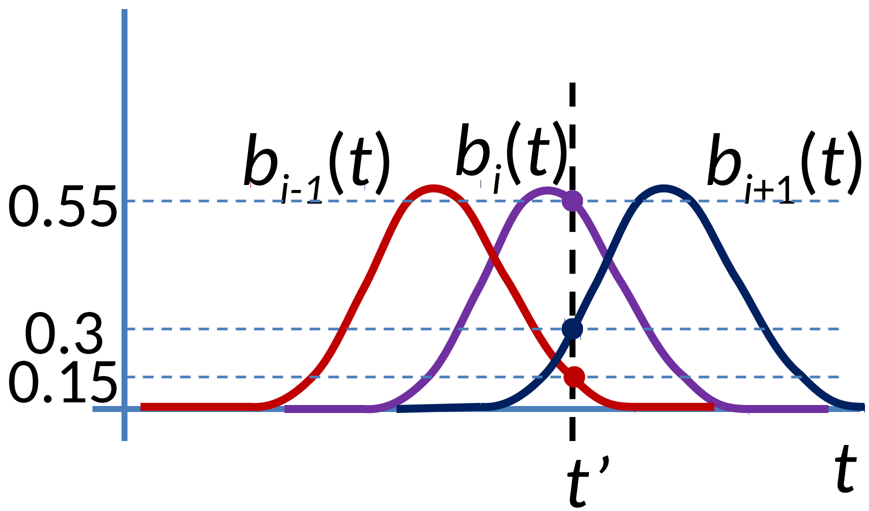
$$\mathbf{f}(t) = \sum_{i=1}^{n} \mathbf{p}_i b_i(t)$$

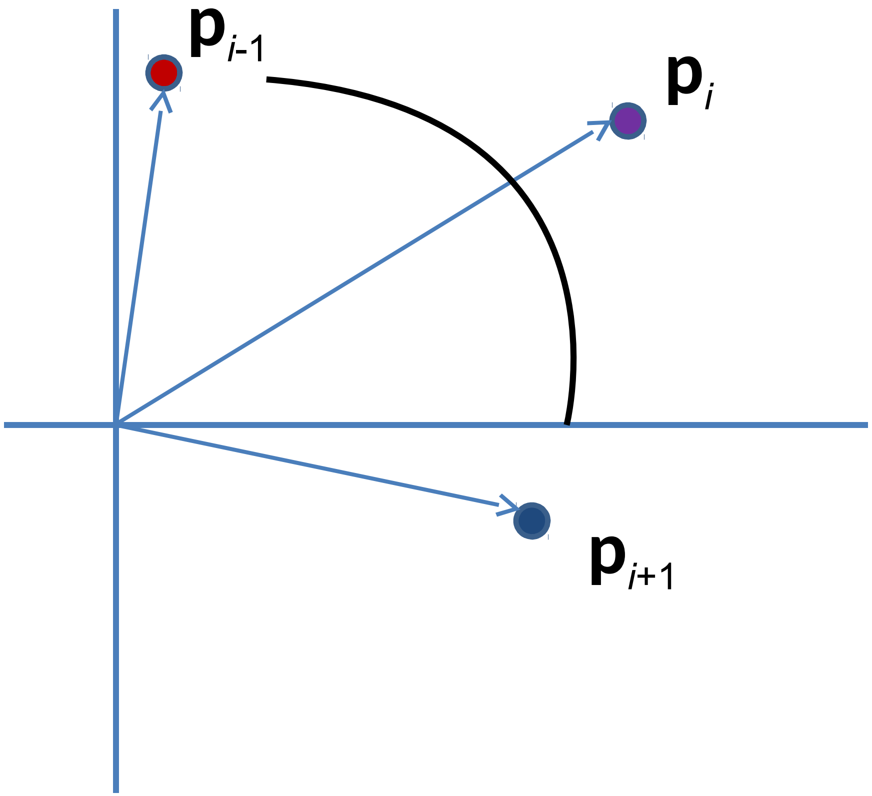
- The $b_i(t)$ are the *basis functions* (blend functions) and show how to blend the points
- To confuse people, the basis functions are often also called are B-splines themselves



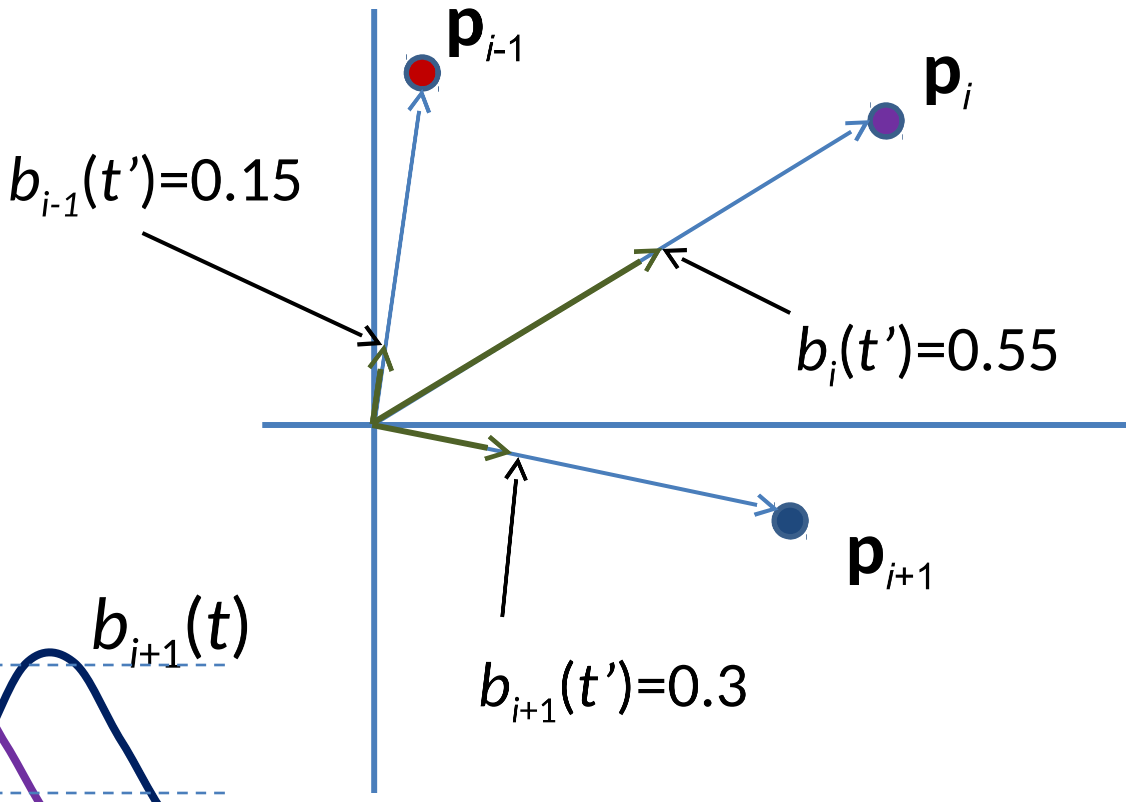
the contribution of point $\mathbf{p}_i$ to the curve depending on the parameter value t

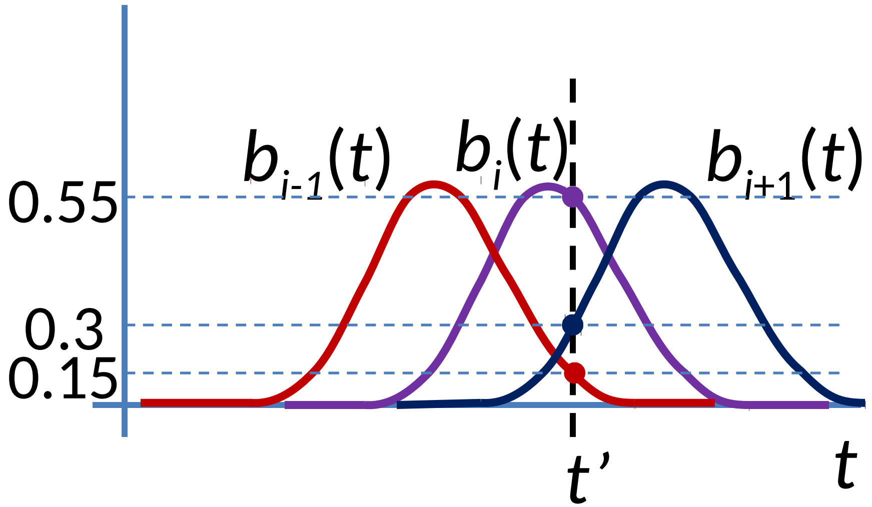
B-splines and B-spline curves



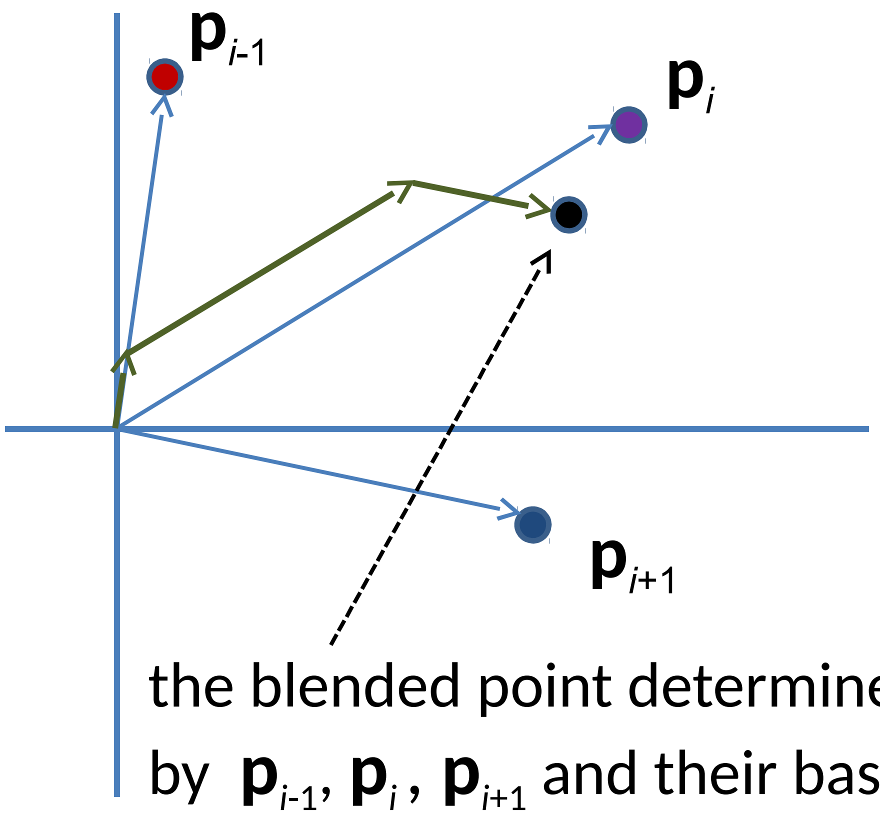


B-splines and B-spline curves

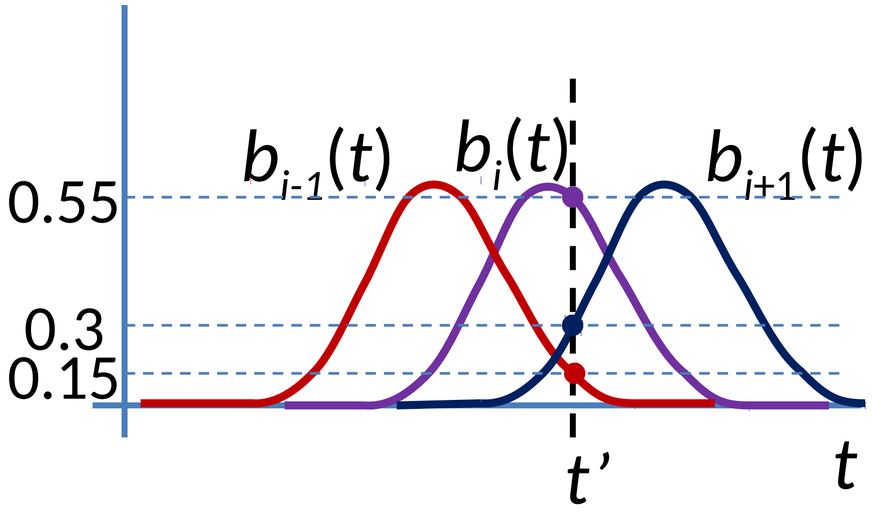




B-splines and B-spline curves

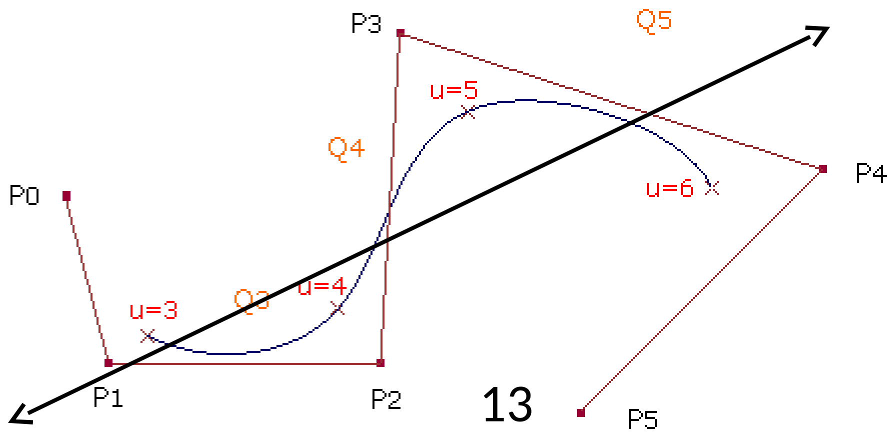


the blended point determined by $\mathbf{p}_{i-1}$, $\mathbf{p}_i$, $\mathbf{p}_{i+1}$ and their basis functions



B-spline curves

- A B-spline curve of n points and parameter value k
  - is C^{k-2} continuous
  - is made of polynomials of degree k – 1
  - has local control: any location on the curve is determined by only k control points
  - is bounded by the convex hull of the control points
  - has the variation diminishing property: any line intersects the B-spline curve at most as often as that line intersects the control polygon



Types of B-splines

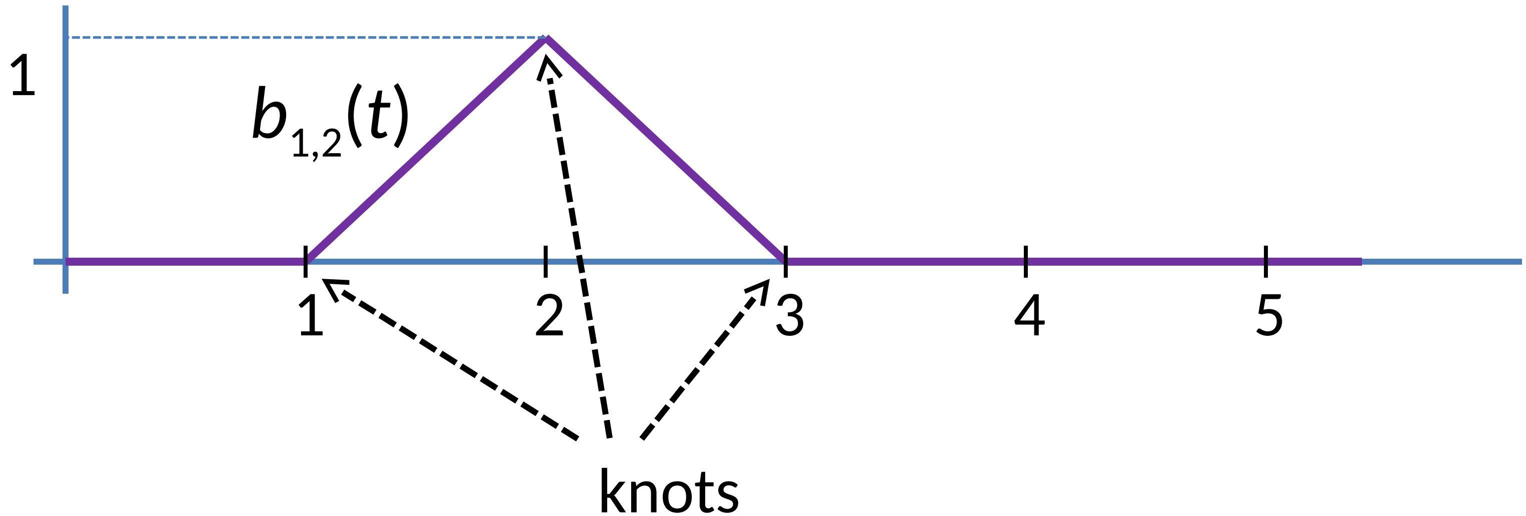
- Uniform linear B-splines
- Uniform quadratic B-splines
- Uniform cubic B-splines
- Non-uniform B-splines
- NURBS (non-uniform rational B-splines)

To define the B-splines is to define the B-spline curve

Uniform linear B-splines

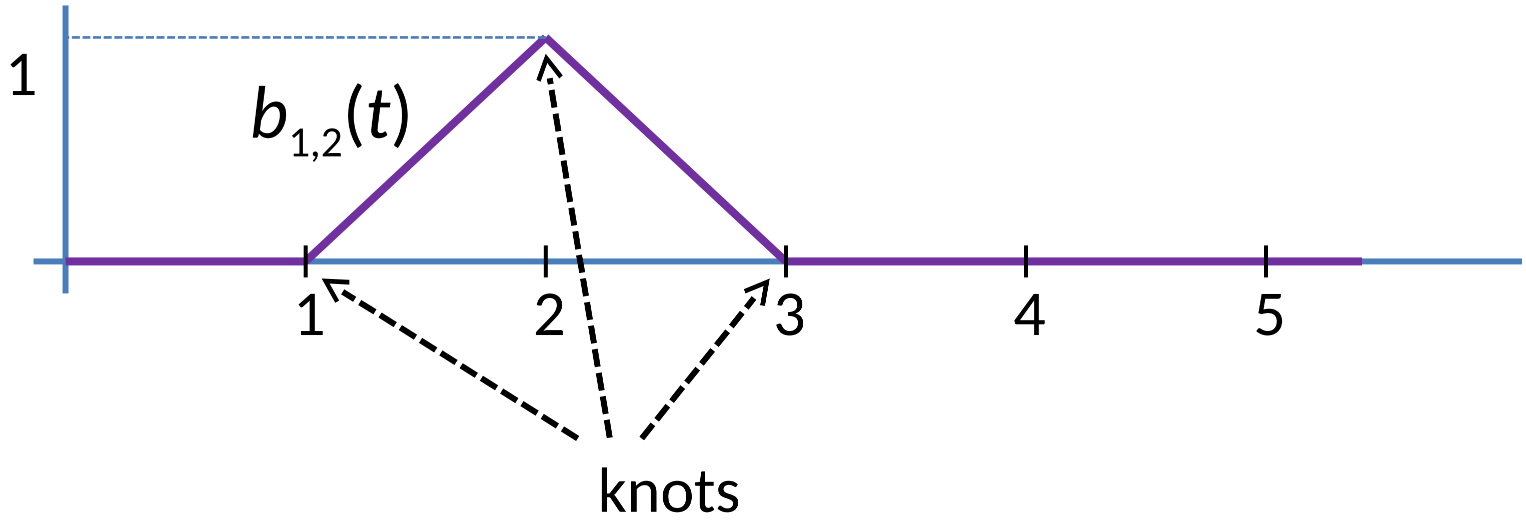
- Basis functions are piecewise linear, for example:

$$b_{i,2}(t) = \begin{cases} t - i & i \le t \le i+1 \\ 2 - t + i & i+1 \le t \le i+2 \\ 0 & \text{otherwise} \end{cases}$$

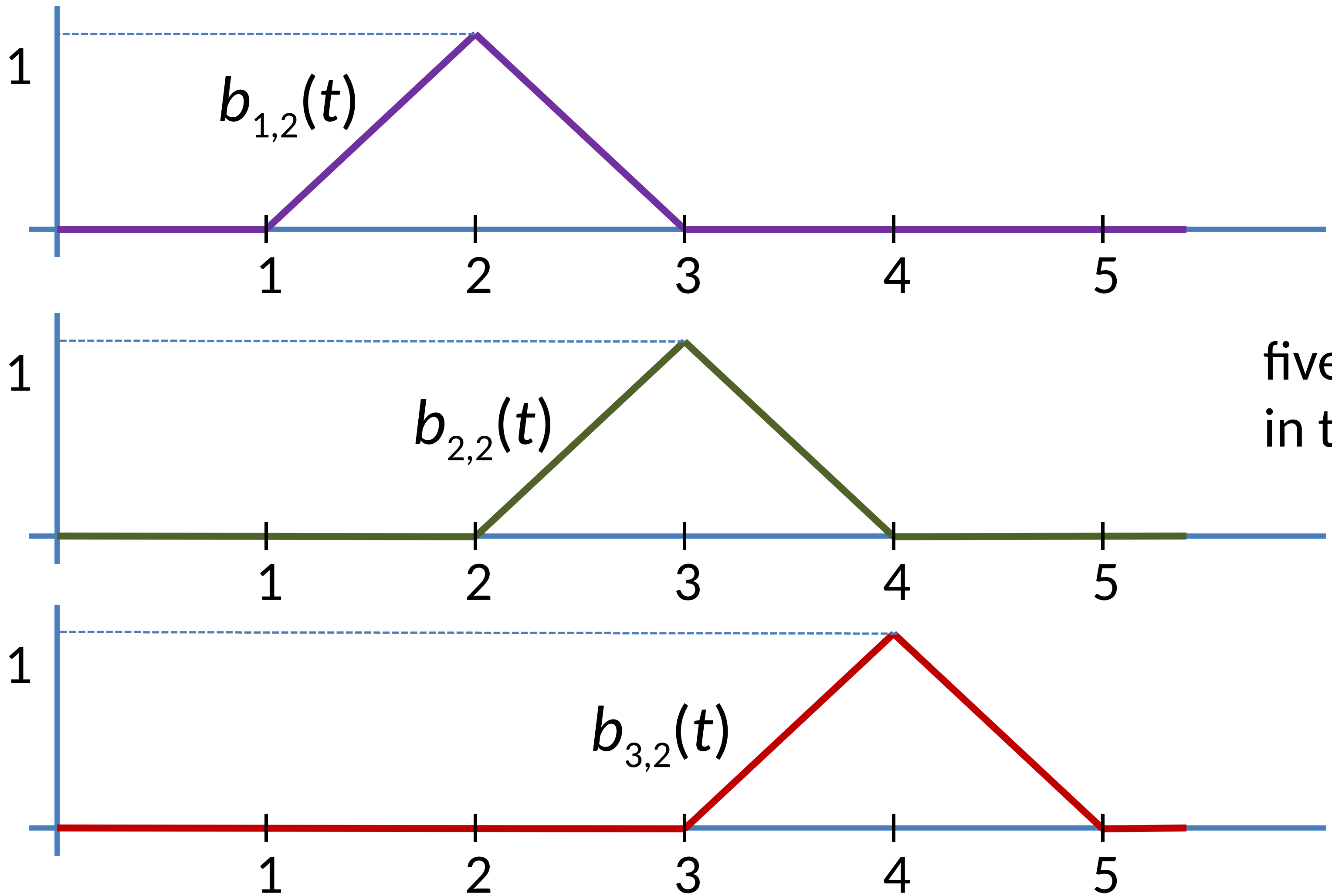


Uniform linear B-splines, knots

- A *knot* (for a uniform linear B-spline, or any type of B-spline) is a parameter value where the definition of the function of some B-spline changes
- Also: the corresponding point on the curve



Uniform linear B-splines, knots



five knots in total

Uniform linear B-splines

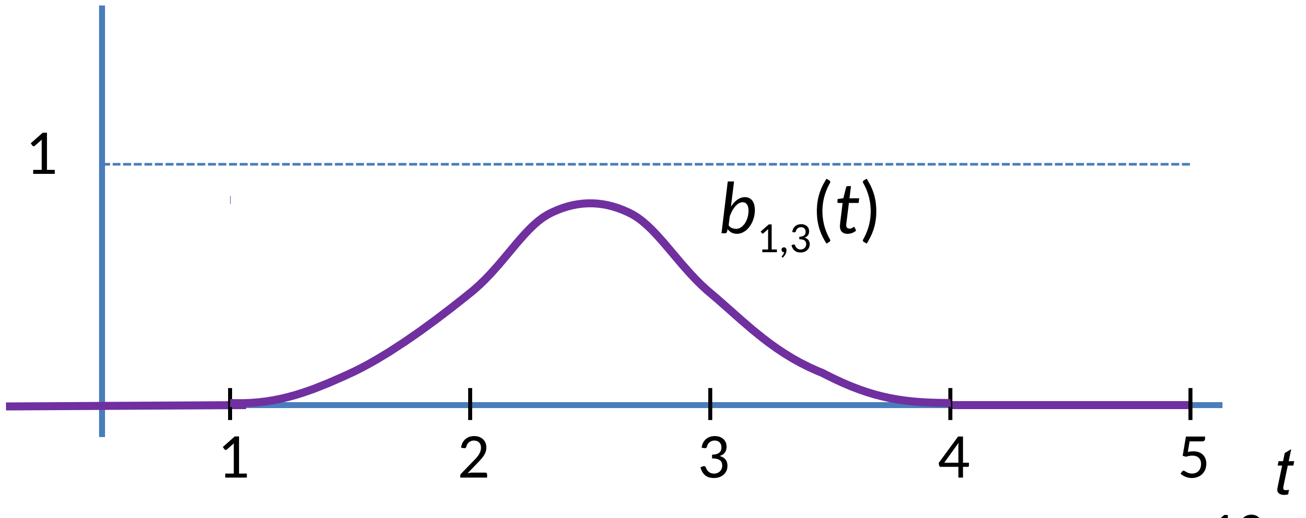
- Basis functions are piecewise linear, for example:

$$b_{i,2}(t) = \begin{cases} t - i & i \le t \le i+1 \\ 2 - t + i & i+1 \le t \le i+2 \\ 0 & \text{otherwise} \end{cases}$$

- The B-spline curve can be used for parameter values $t \in [2, n+1]$ (the t where the $b_{i,2}(t)$ sum up to 1), given points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$
- The B-spline curve is just the polygonal line through the control points; only two B-splines are non-zero for any t

Uniform quadratic B-splines

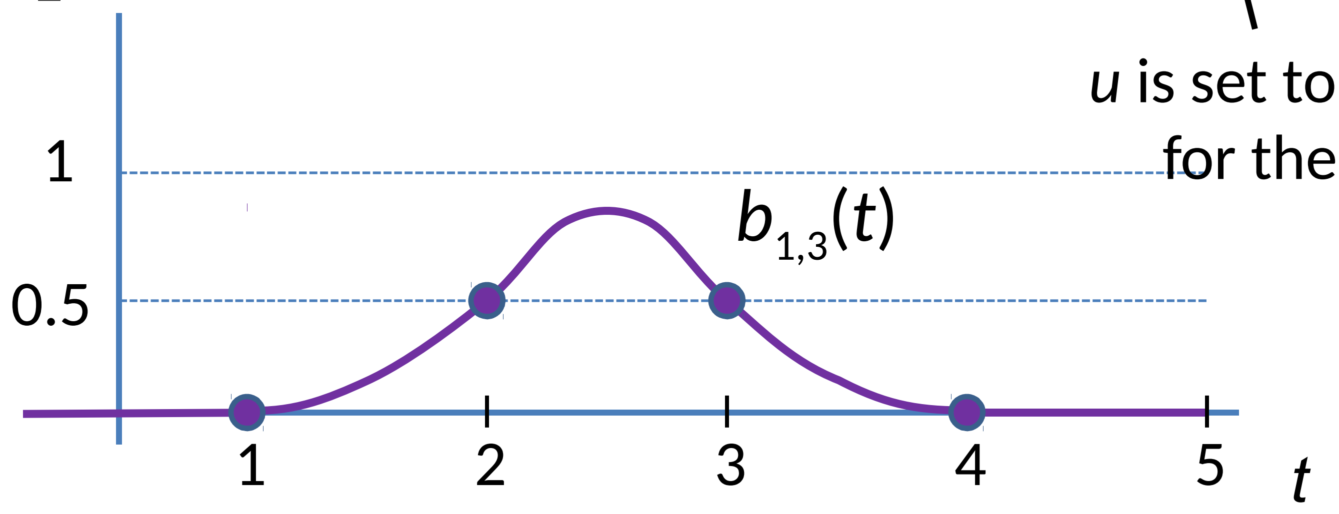
- Uniform quadratic B-splines $b_{i,3}(t)$ have knots at i, $i+1$, $i+2$, and $i+3$
- They are shifted copies of each other



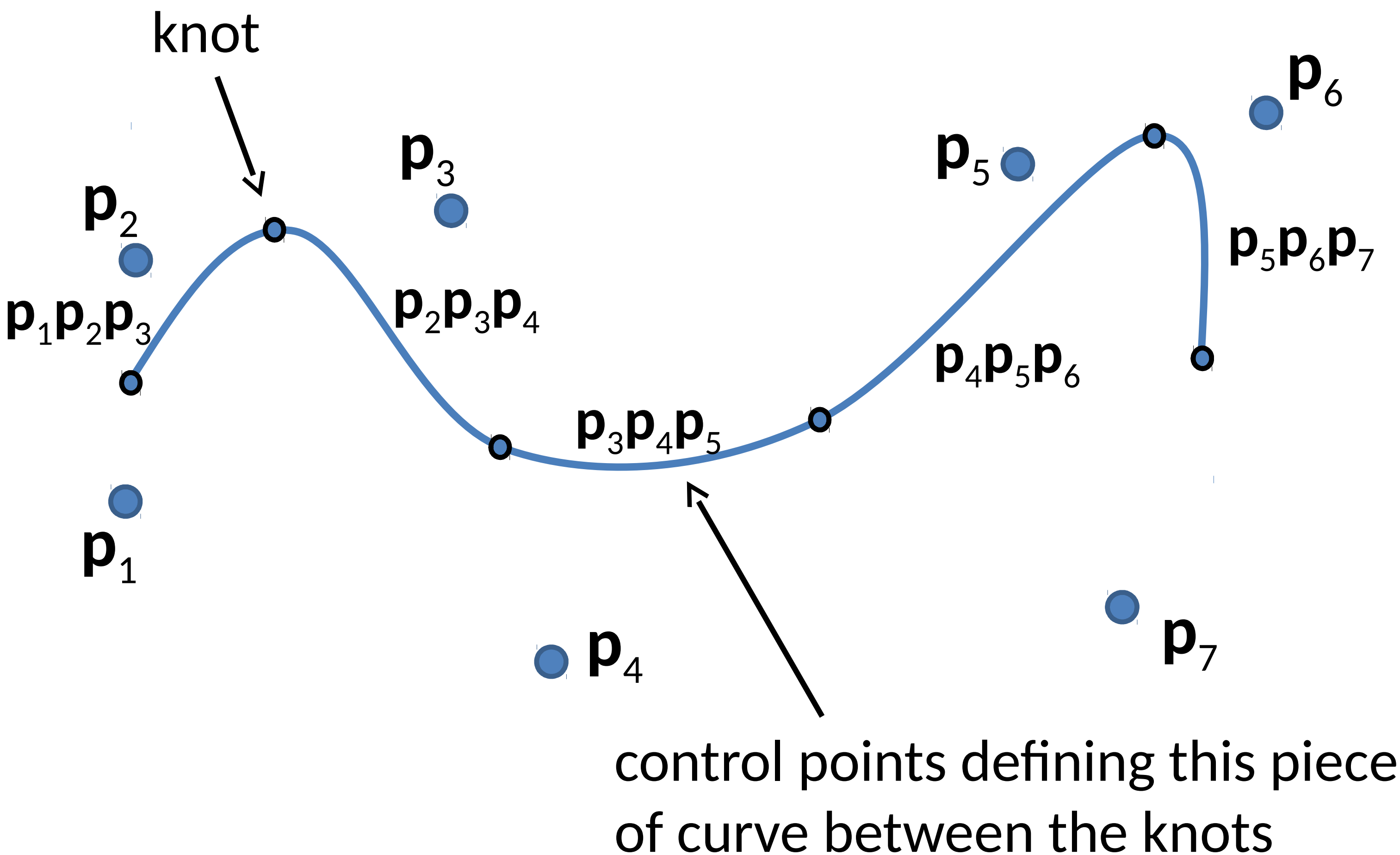
Uniform quadratic B-splines

$$b_{i,3} = \begin{cases} \frac{1}{2}u^2 & \text{if } t \in [i, i+1] \quad u = t - i \\ -u^2 + u + \frac{1}{2} & \text{if } t \in [i+1, i+2] \quad u = t - (i+1) \\ \frac{1}{2}(1-u)^2 & \text{if } t \in [i+2, i+3] \quad u = t - (i+2) \\ 0 & \text{otherwise} \end{cases}$$

u is set to be in [0,1] for the *t*-range

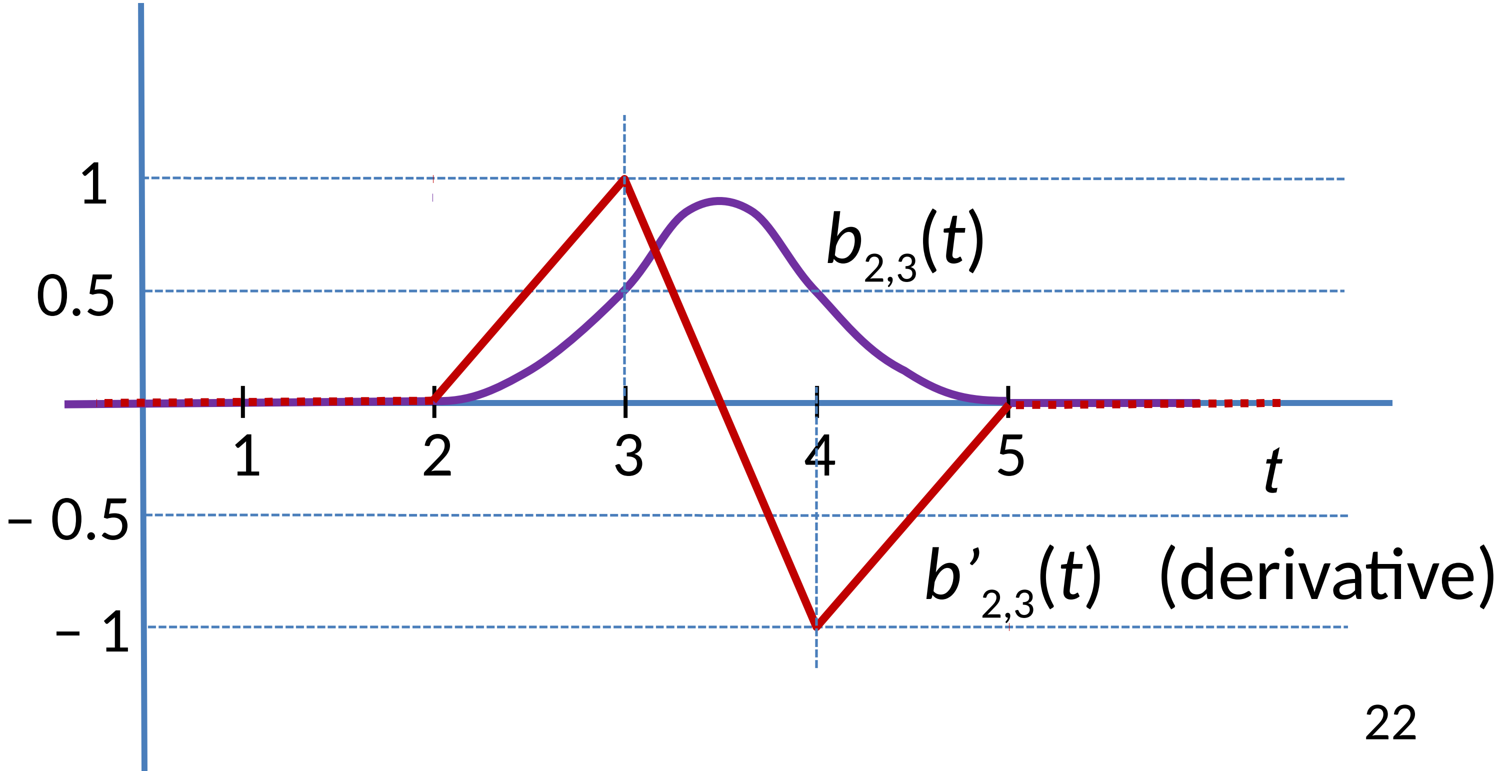


Quadratic B-spline curve



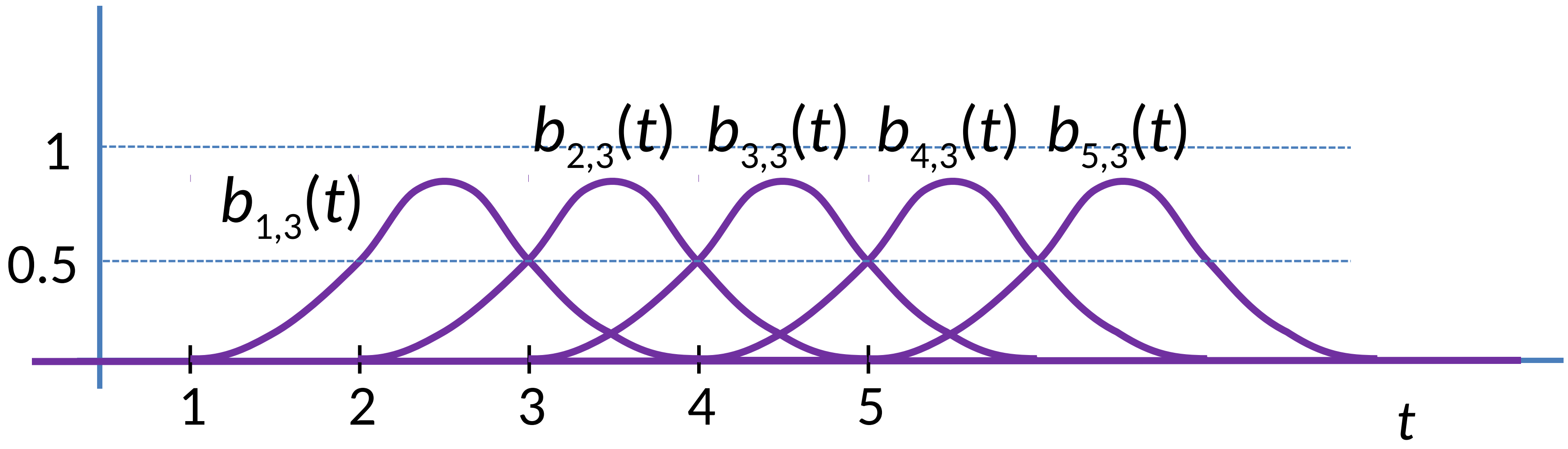
Uniform quadratic B-splines

- At the 4 knots, the left and right derivatives are equal
  $\rightarrow$ C^1 continuous B-splines
  $\rightarrow$ C^1 continuous B-spline curve



Uniform quadratic B-splines

- At the 4 knots, the left and right derivatives are equal
  - → C^1 continuous B-splines
  - → C^1 continuous B-spline curve
- Starting at t = 3, the B-splines sum up to exactly 1



Uniform quadratic B-splines

- Suppose n control points
- Then we have n B-splines and $n+3$ knots
- The B-spline curve has $n - 2$ quadratic pieces, starting at the 3rd knot and ending at the $n+1$st knot

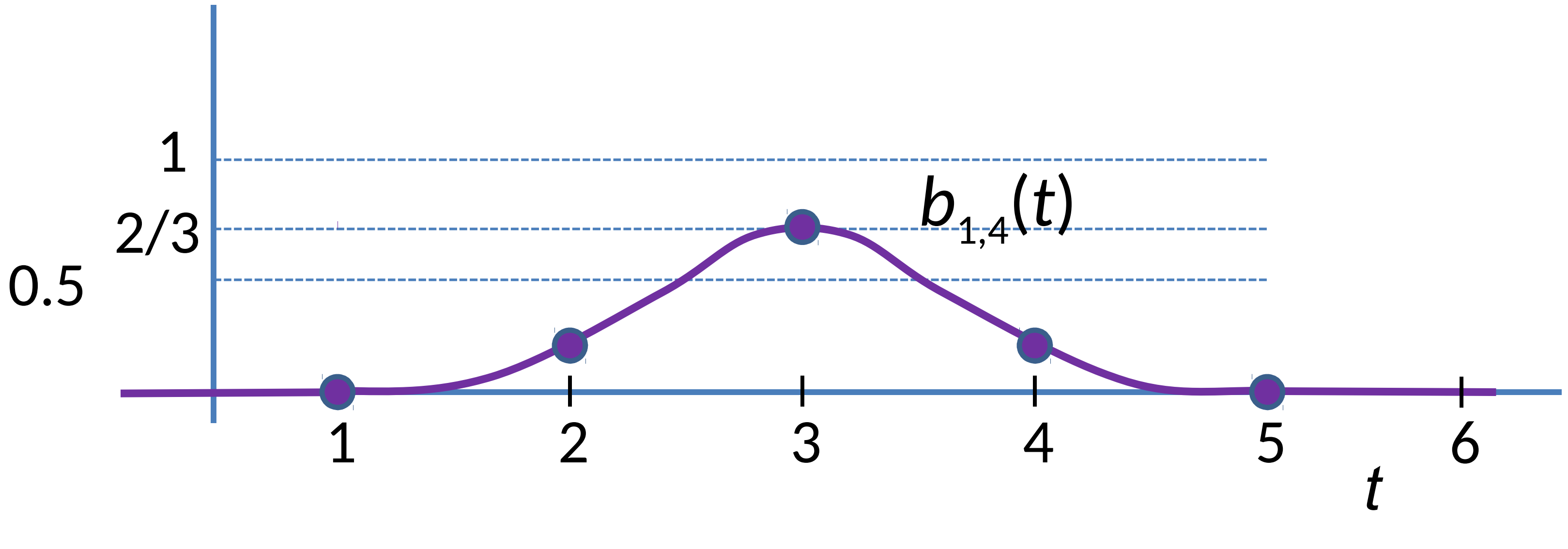
Uniform quadratic B-splines

- What is the starting point and what is the ending point of the uniform quadratic B-spline curve using control points

  $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$?

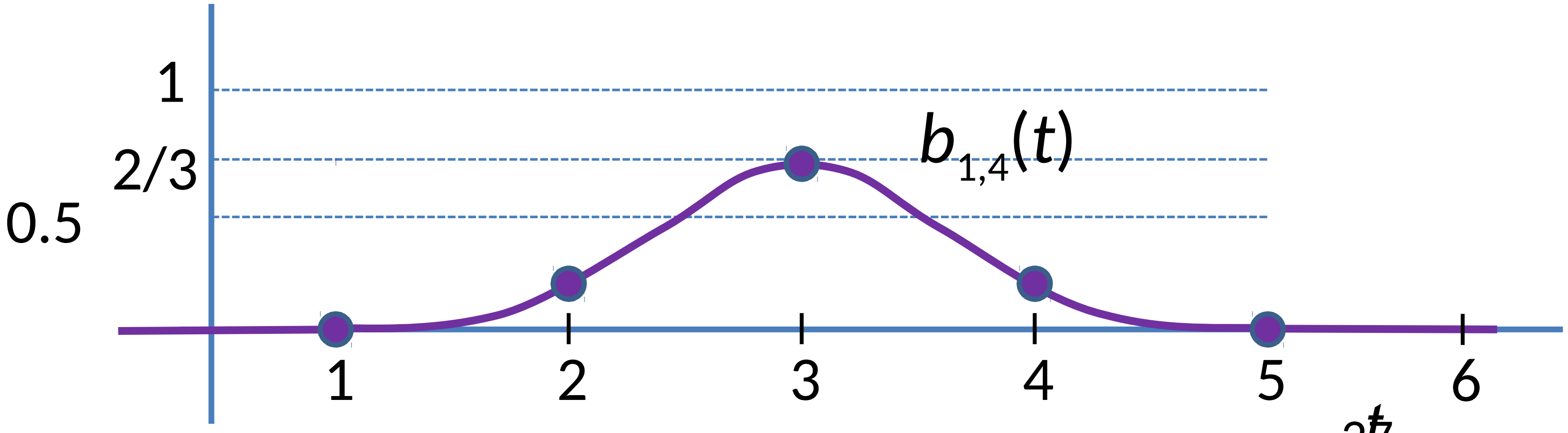
Uniform cubic B-splines

- Uniform cubic B-splines $b_{i,4}(t)$ have knots at i, i+1, i+2, i+3 and i+4
- They are shifted copies of each other



Uniform cubic B-splines

$$b_{i,4}(t) = \begin{cases} \frac{1}{6}u^3 & \text{if } t \in [i, i+1] \quad u = t - i \\ \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1) & \text{if } t \in [i+1, i+2] \quad u = t - (i+1) \\ \frac{1}{6}(3u^3 - 6u^2 + 4) & \text{if } t \in [i+2, i+3] \quad u = t - (i+2) \\ \frac{1}{6}(-u^3 + 3u^2 - 3u + 1) & \text{if } t \in [i+3, i+4] \quad u = t - (i+3) \\ 0 & \text{otherwise} \end{cases}$$



Uniform cubic B-splines

- Every location on the B-spline curve is determined by four control points and their B-splines
- Starting at the 4th knot, $t = 4$, the B-spline curve can be used, because the B-splines sum up to 1
- A cubic B-spline is C^2 continuous (the 1st and 2nd derivatives from the left and right are the same at the knots, and everywhere else too of course)
$\rightarrow$ a cubic B-spline *curve* is C^2 continuous

Cox-de Boor recurrence

- Cox-de Boor recurrence for defining B-splines with parameter k:

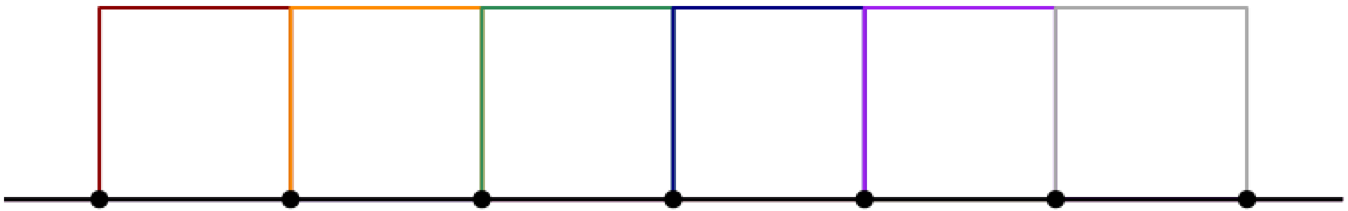
$$b_{i,1}(t) = \begin{cases} 1 & \text{if } i \le t < i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_{i,k}(t) = (t-i)/(k-1) \cdot b_{i,k-1}(t) + (i+k-1-t)/(k-1) \cdot b_{i+1,k-1}(t)$$

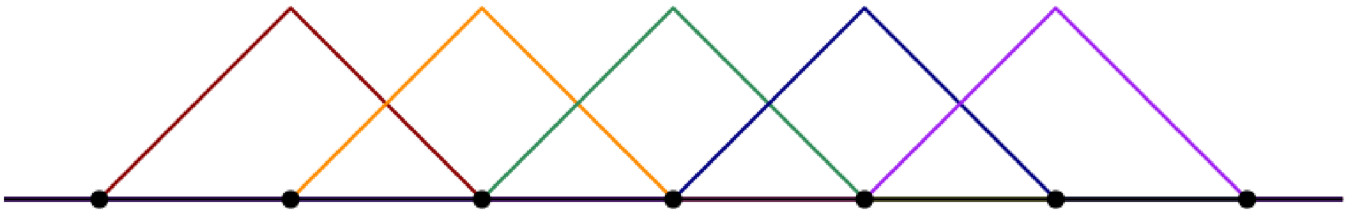
The recurrence shows that parameter k B-splines are a weighted interpolation of parameter $k-1$ B-splines, with weights linearly dependent on t

Cox-de Boor recurrence

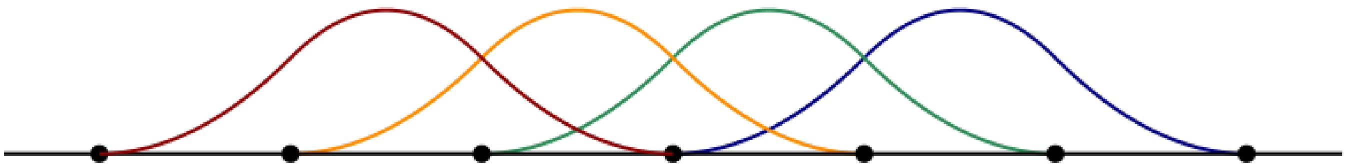
$b_{i,1}(t)$



$b_{i,2}(t)$

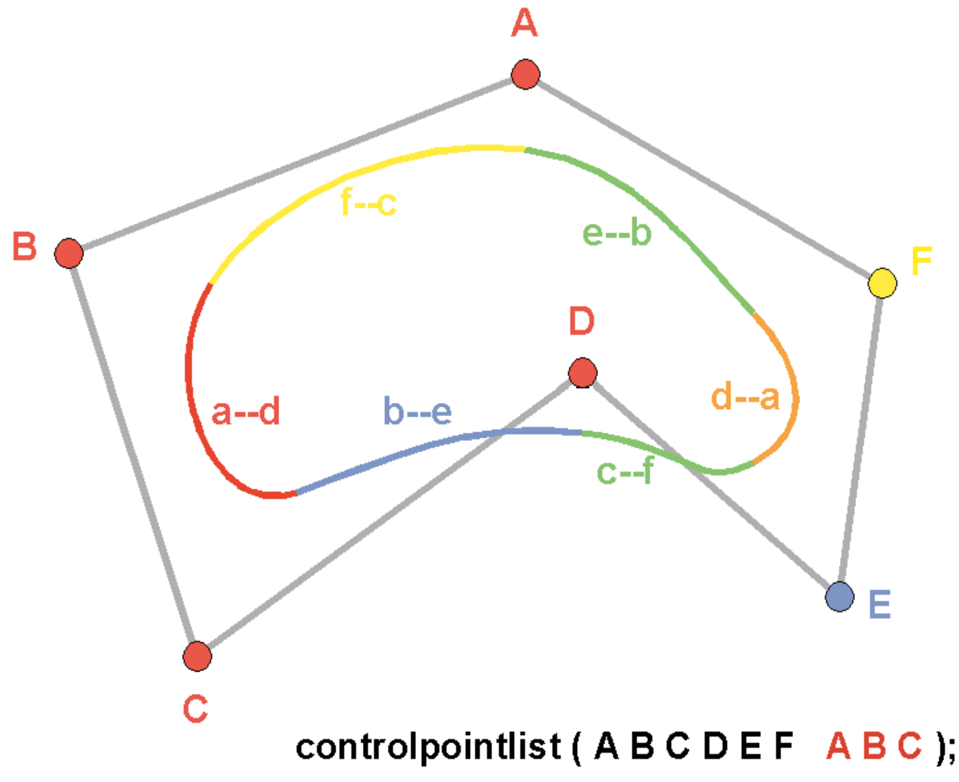


$b_{i,3}(t)$



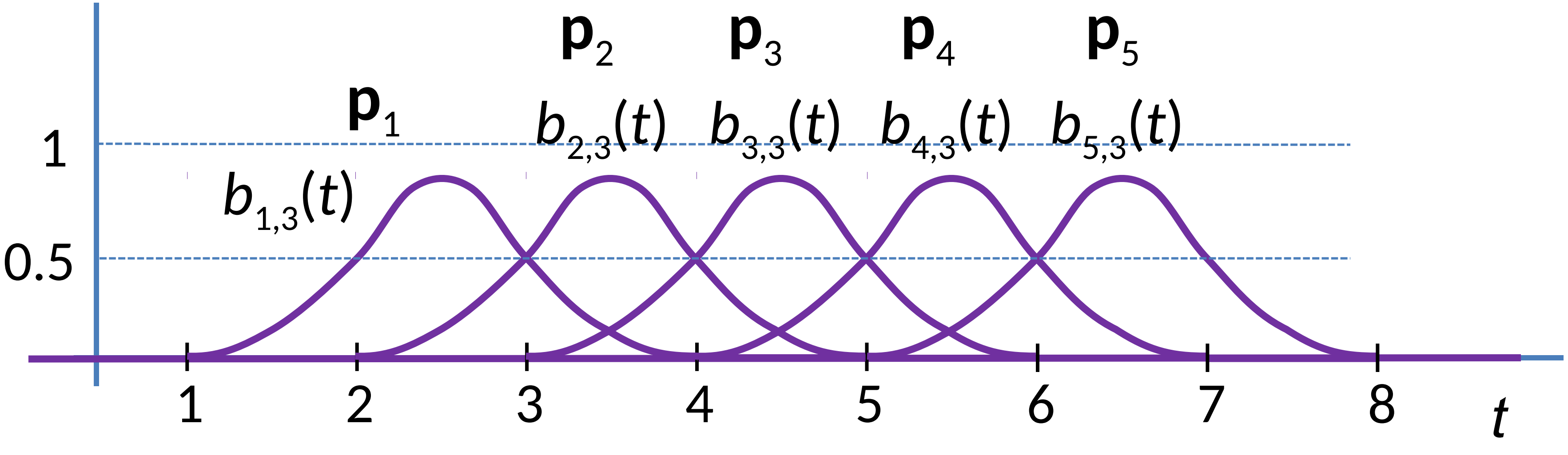
Closed uniform B-spline curves

- Simply repeat the first k – 1 points as the last points
  - quadratic: $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \rightarrow \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n, \mathbf{p}_1, \mathbf{p}_2$
  - cubic: $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_n \rightarrow \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$



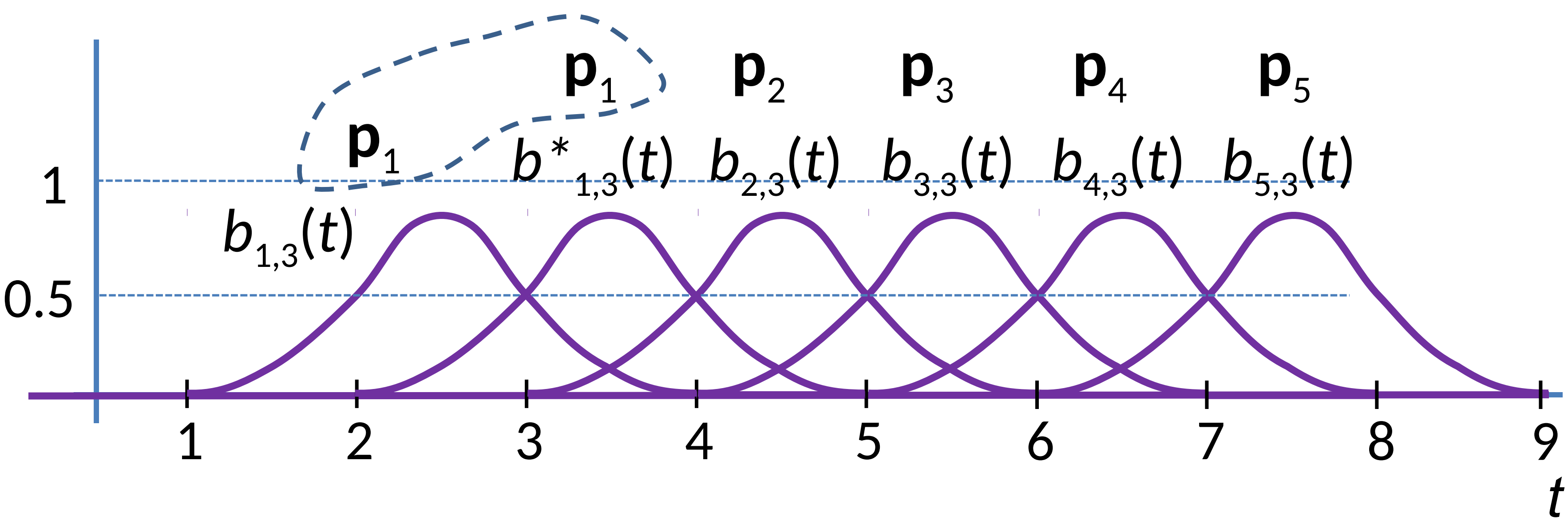
Repeating control points in B-spline curves

- Consider quadratic B-spline curves, in the figure:
  - Five B-splines of control points
  - 5+3 = 8 knots (t = 1, 2, ..., 8), useful in interval [3, 6]



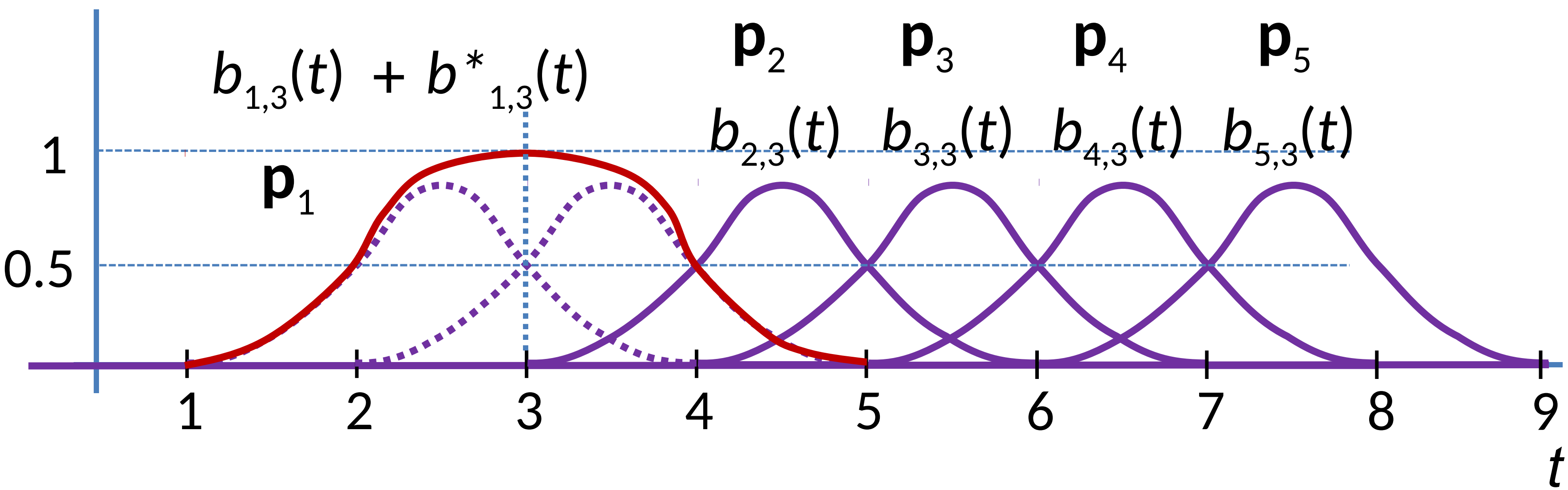
Repeating control points in B-spline curves

- Consider quadratic B-spline curves, in the figure:
  - Suppose $\mathbf{p}_1$ is repeated
  - 9 knots, useful interval [3, 7]



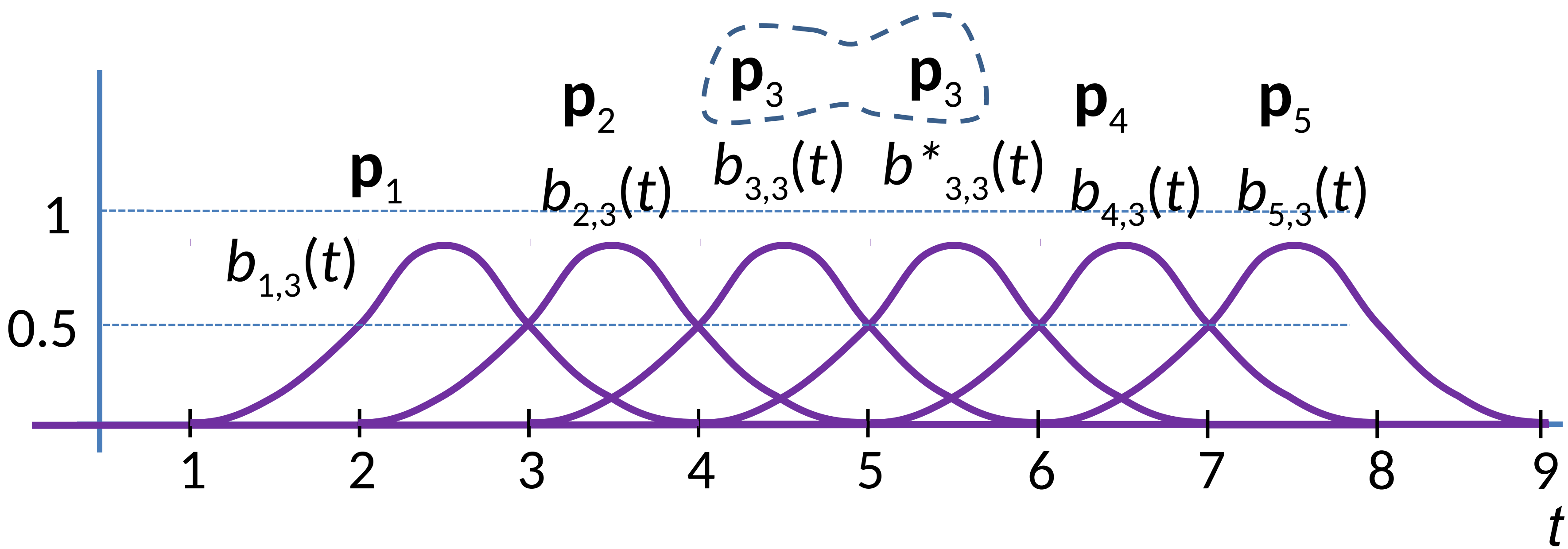
Repeating control points in B-spline curves

- Consider quadratic B-spline curves, in the figure:
  - Suppose $\mathbf{p}_1$ is repeated
  - 9 knots, useful interval [3, 7]
  - The B-spline curve starts at $\mathbf{p}_1$ at knot 3



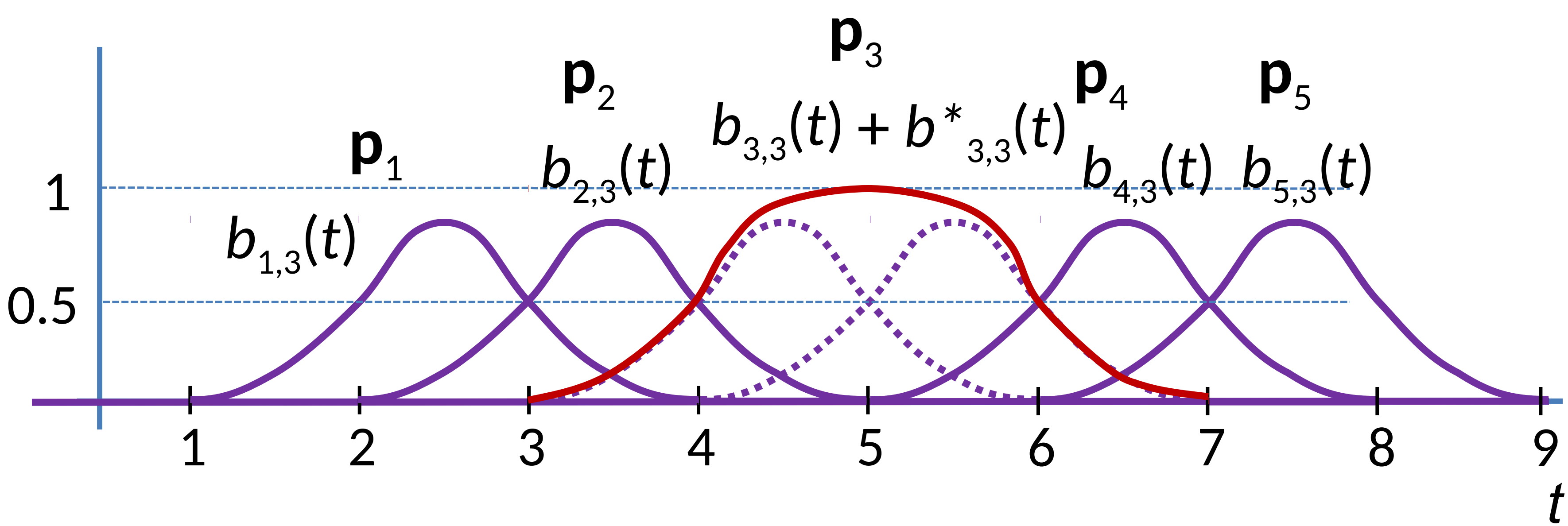
Repeating control points in B-spline curves

- Consider quadratic B-spline curves, in the figure:
  - Suppose $\mathbf{p}_3$ is repeated
  - 9 knots, useful interval [3, 7]



Repeating control points in B-spline curves

- Consider quadratic B-spline curves, in the figure:
  - Suppose $\mathbf{p}_3$ is repeated
  - 9 knots, useful interval [3, 7]
  - The B-spline curve passes through point $\mathbf{p}_3$

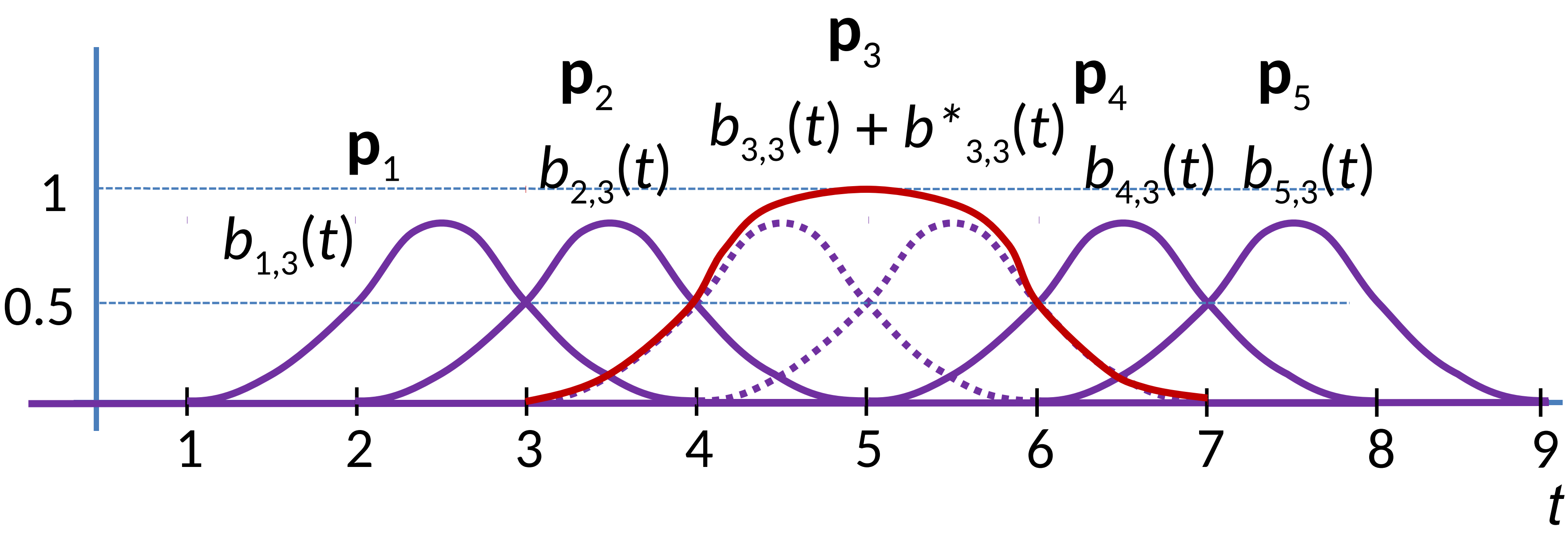


Repeating control points in B-spline curves

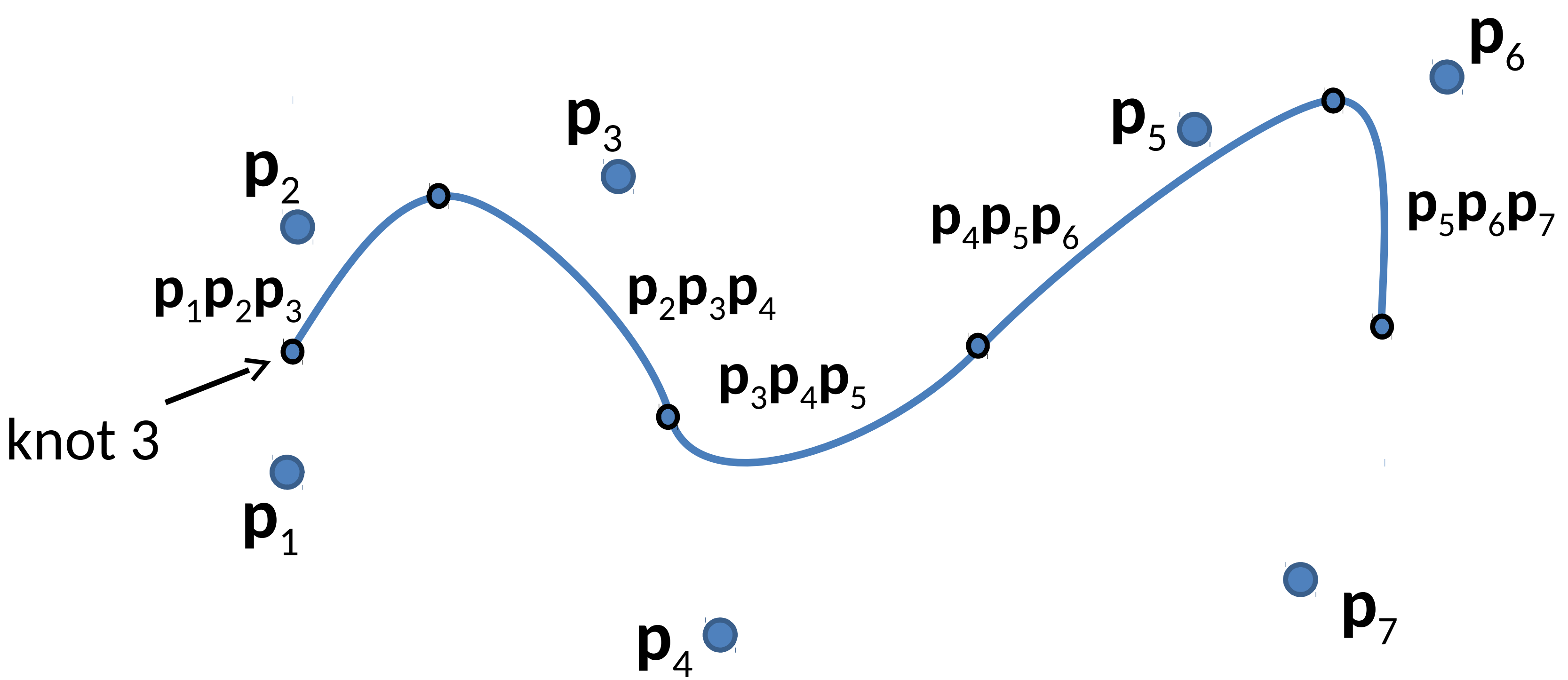
- A degree $k - 1$ B-spline curve will pass through the first and last control points if each occurs $k - 1$ times
- Similarly, we can make it pass through an intermediate control point by having $k - 1$ copies of that control point
- The level of continuity at an intermediate repeated control point decreases

Repeating control points in B-spline curves

- On the parameter interval [4,5], only $\mathbf{p}_2$ and $\mathbf{p}_3$ have a non-zero B-spline
  $\rightarrow$ the curve is a line segment!
- Also on parameter interval [5,6]

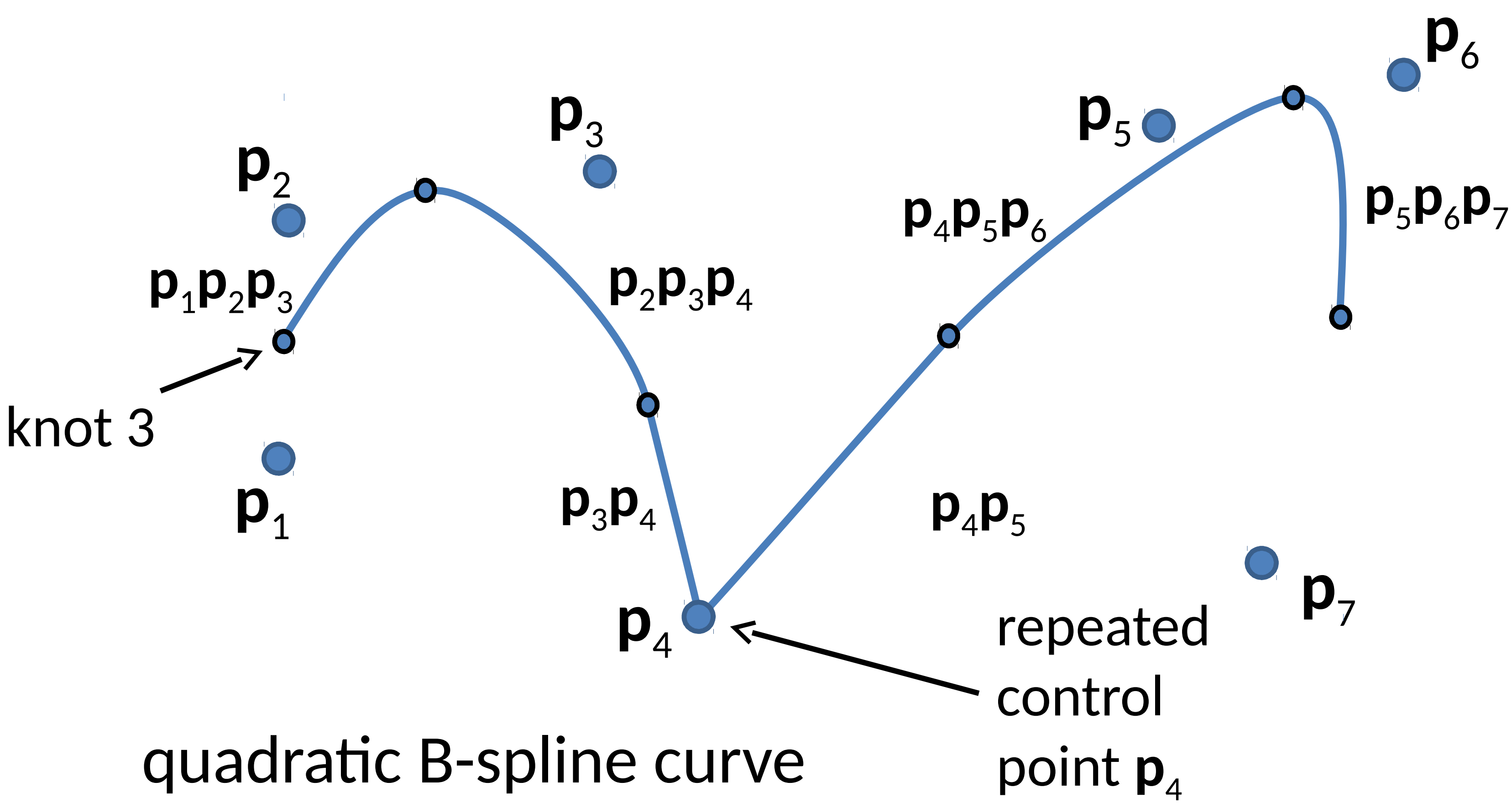


Repeating control points in B-spline curves



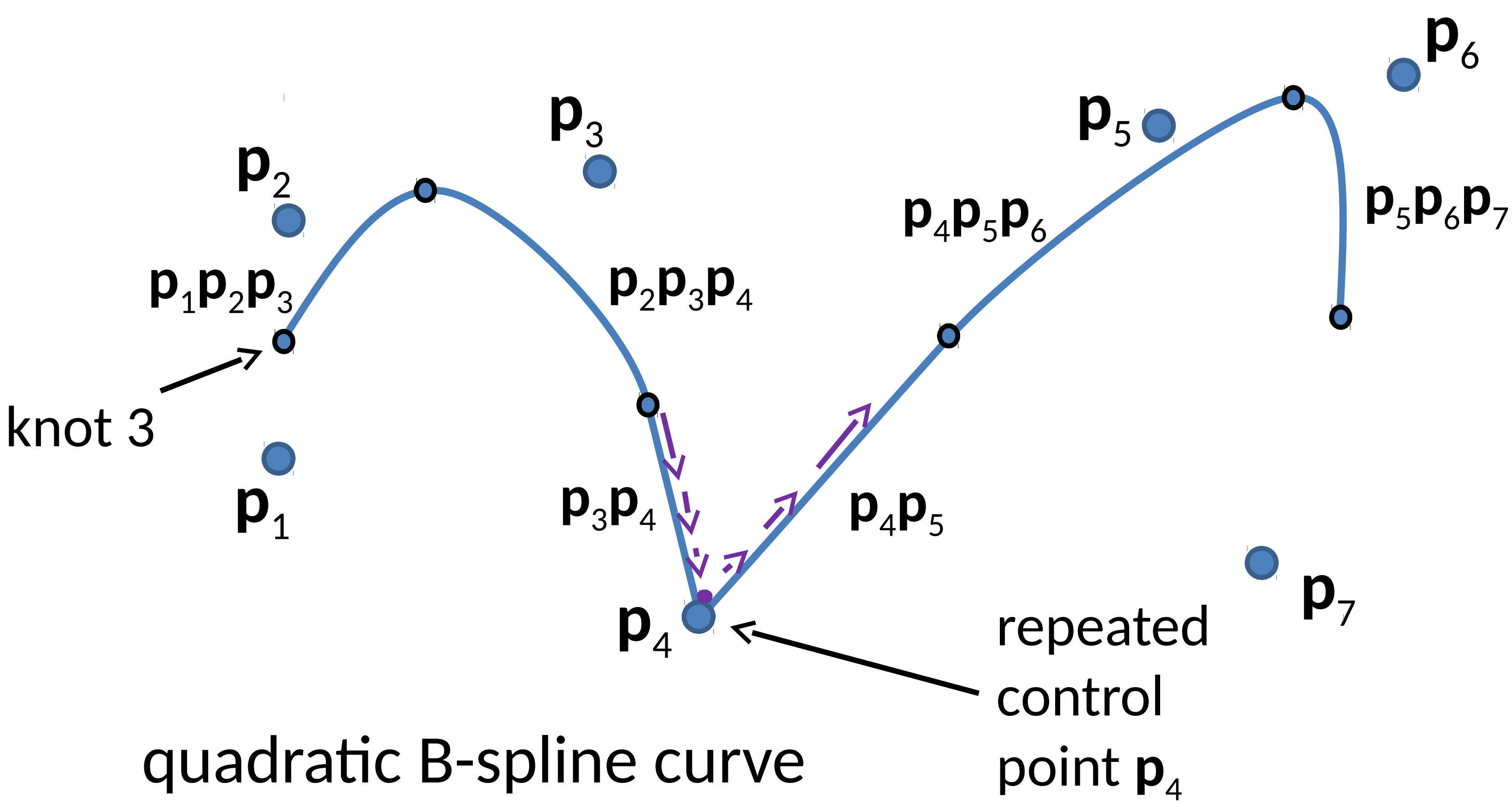
quadratic B-spline curve

Repeating control points in B-spline curves



quadratic B-spline curve with repeated control point

Repeating control points in B-spline curves



quadratic B-spline curve
with repeated control point

Non-uniform B-splines

- Uniform B-splines have knots at 1, 2, 3, …, but generally knots can have any parameter value → *non-uniform* B-splines
- The knot vector $\mathbf{t} = [t_1, \ldots, t_{n+k}]$ is a sequence of non-decreasing values specifying where the knots for the parameter t occur

Cox-de Boor recurrence

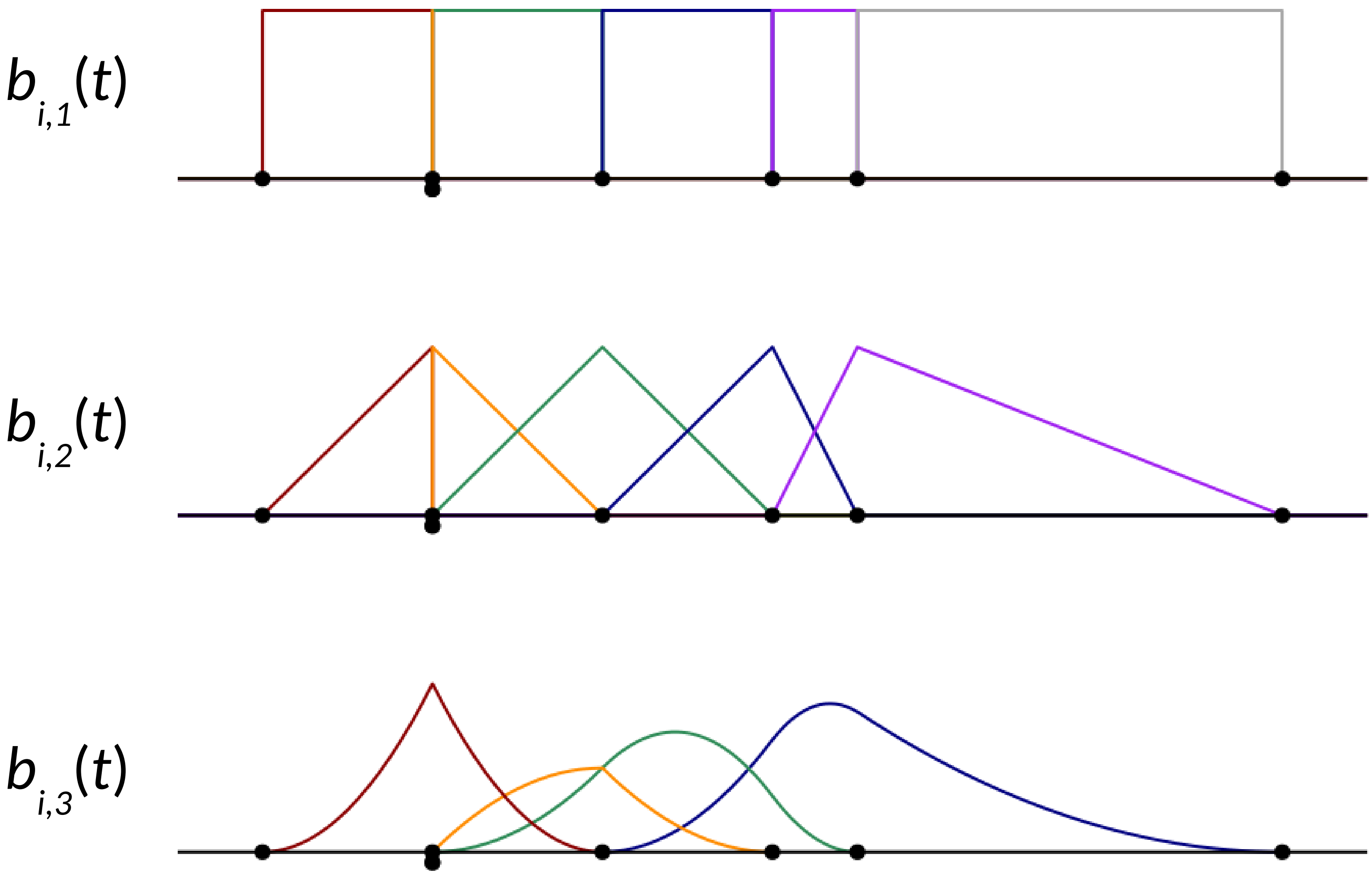
- Cox-de Boor recurrence for defining B-splines with parameter k and knot vector $\mathbf{t} = [t_1, \ldots, t_{n+k}]$:

$$b_{i,1}(t) = \begin{cases} 1 & \text{if } t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$b_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} b_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} b_{i+1,k-1}(t)$$

The recurrence shows that parameter k B-splines are a weighted interpolation of parameter $k - 1$ B-splines, with weights linearly dependent on t

Cox-de Boor recurrence



Repeating knots in B-spline curves

- Repeating a knot means that one sequence of control points no longer occurs
  - For example, for quadratic B-spline curves, the part between knots 4 and 5 (was based on points $\mathbf{p}_2$ $\mathbf{p}_3$ $\mathbf{p}_4$)
  - The curve part before (based on points $\mathbf{p}_1$ $\mathbf{p}_2$ $\mathbf{p}_3$) directly connects to the curve part after (based on points $\mathbf{p}_3$ $\mathbf{p}_4$ $\mathbf{p}_5$)
  - The B-splines get different shapes to accommodate the missing part (because the B-splines must still be continuous and sum up to 1)

Repeating knots in B-spline curves

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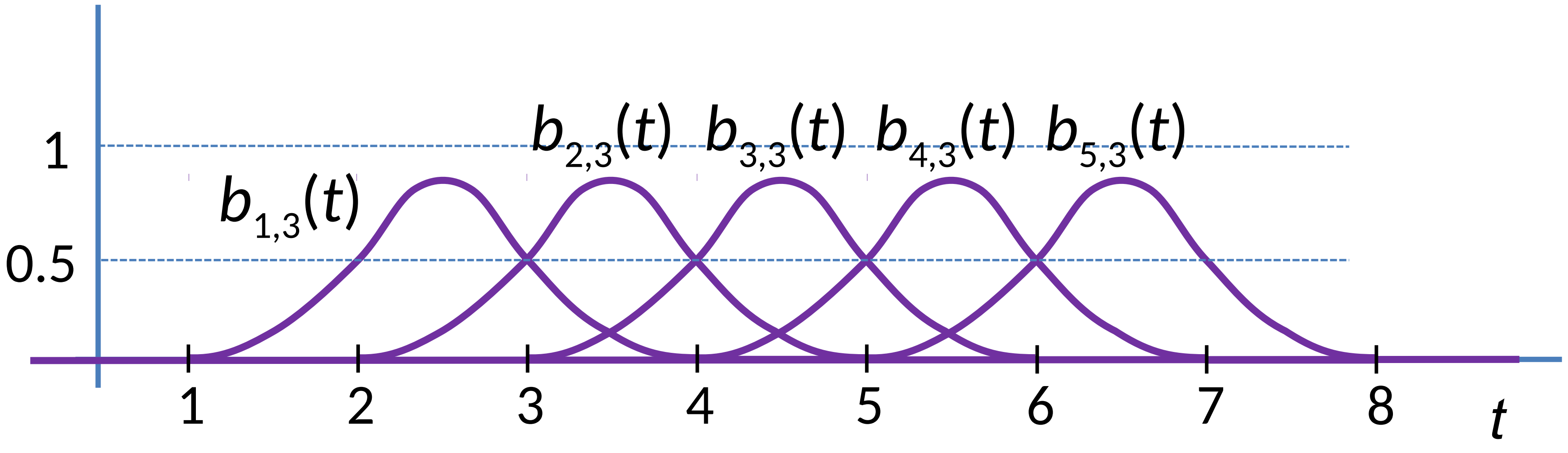

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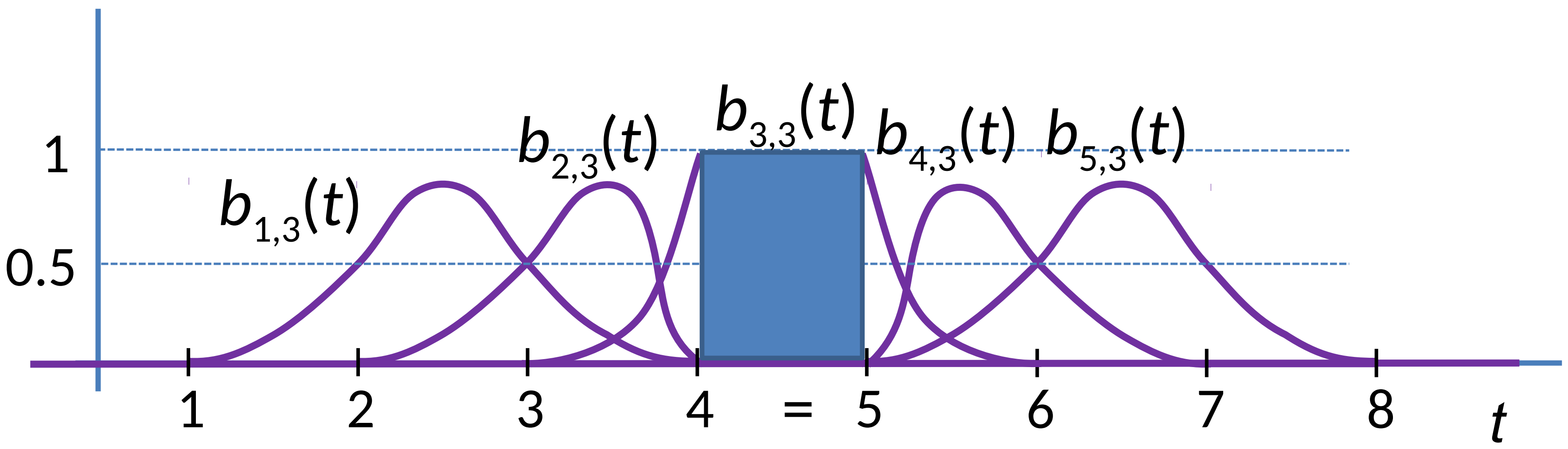

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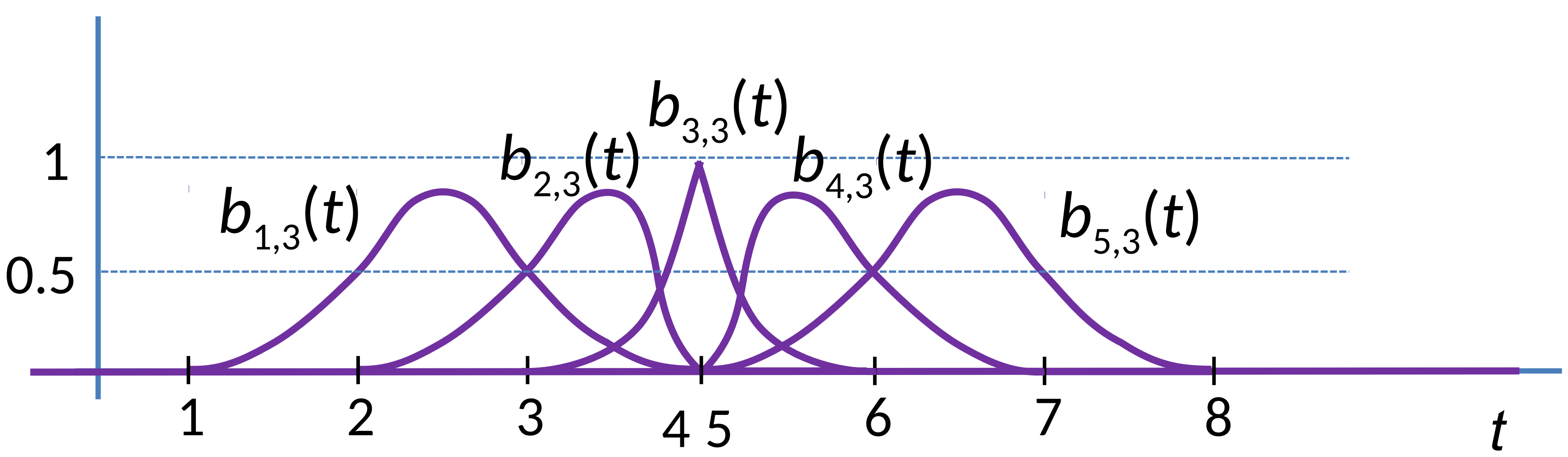
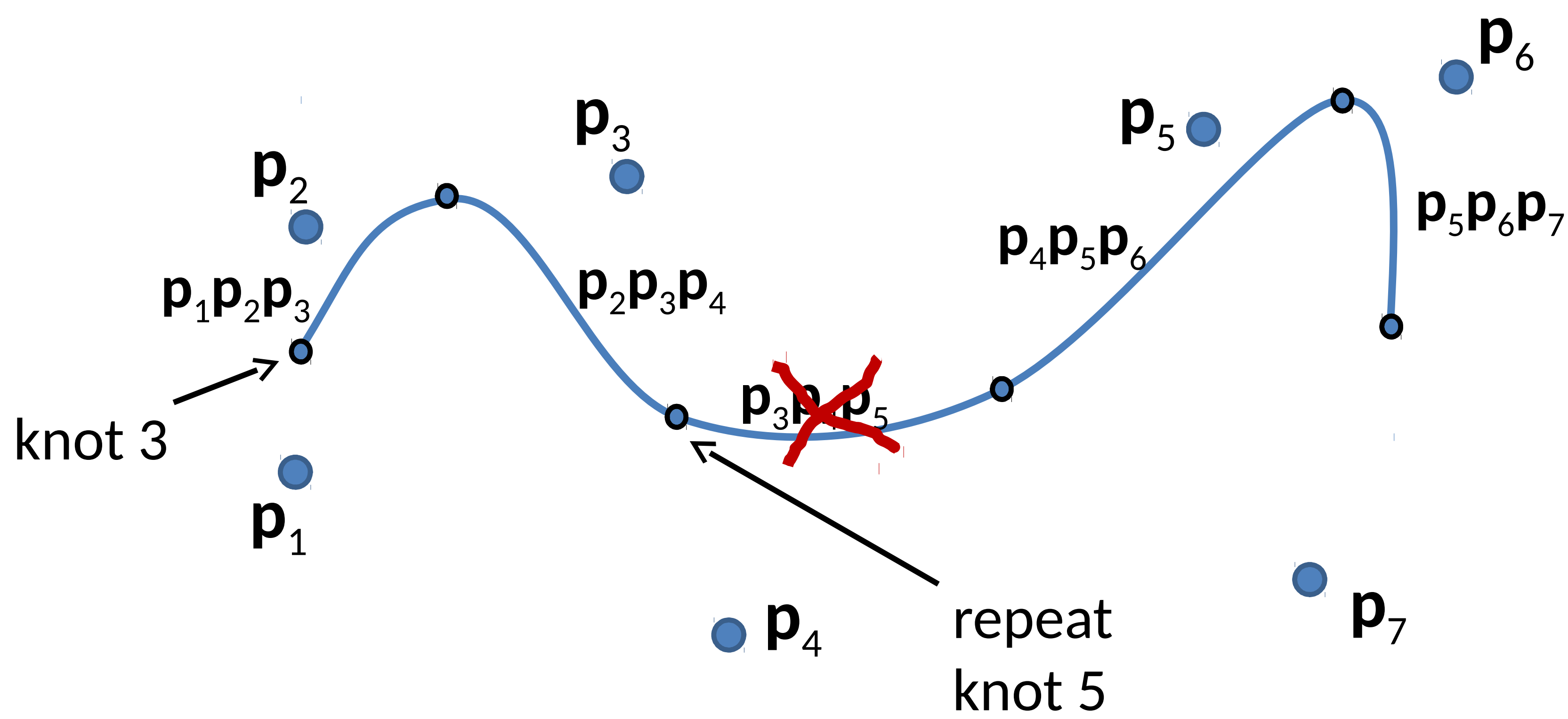
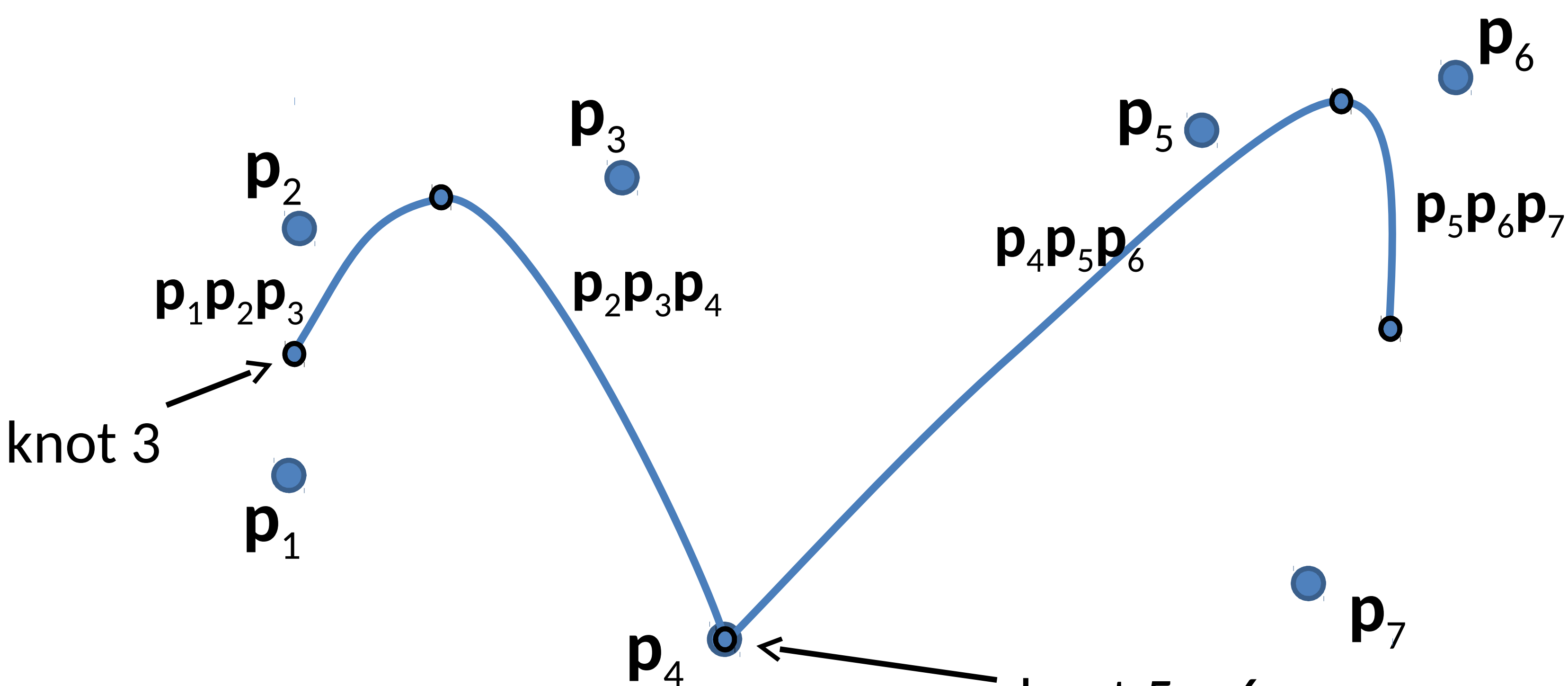


Repeating knots in B-spline curves



quadratic B-spline curve

Repeating knots in B-spline curves

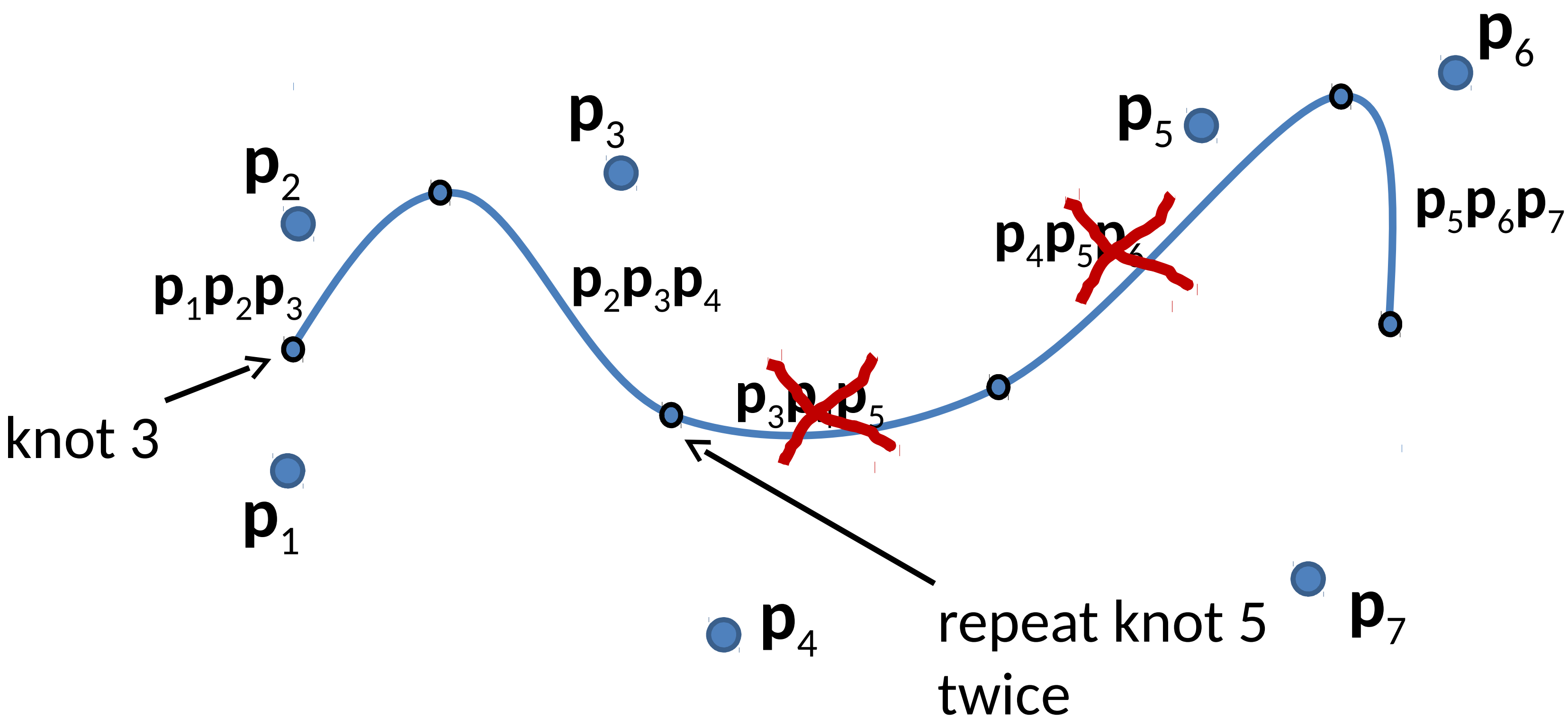


quadratic B-spline curve
with repeated knot

Repeating knots in B-spline curves

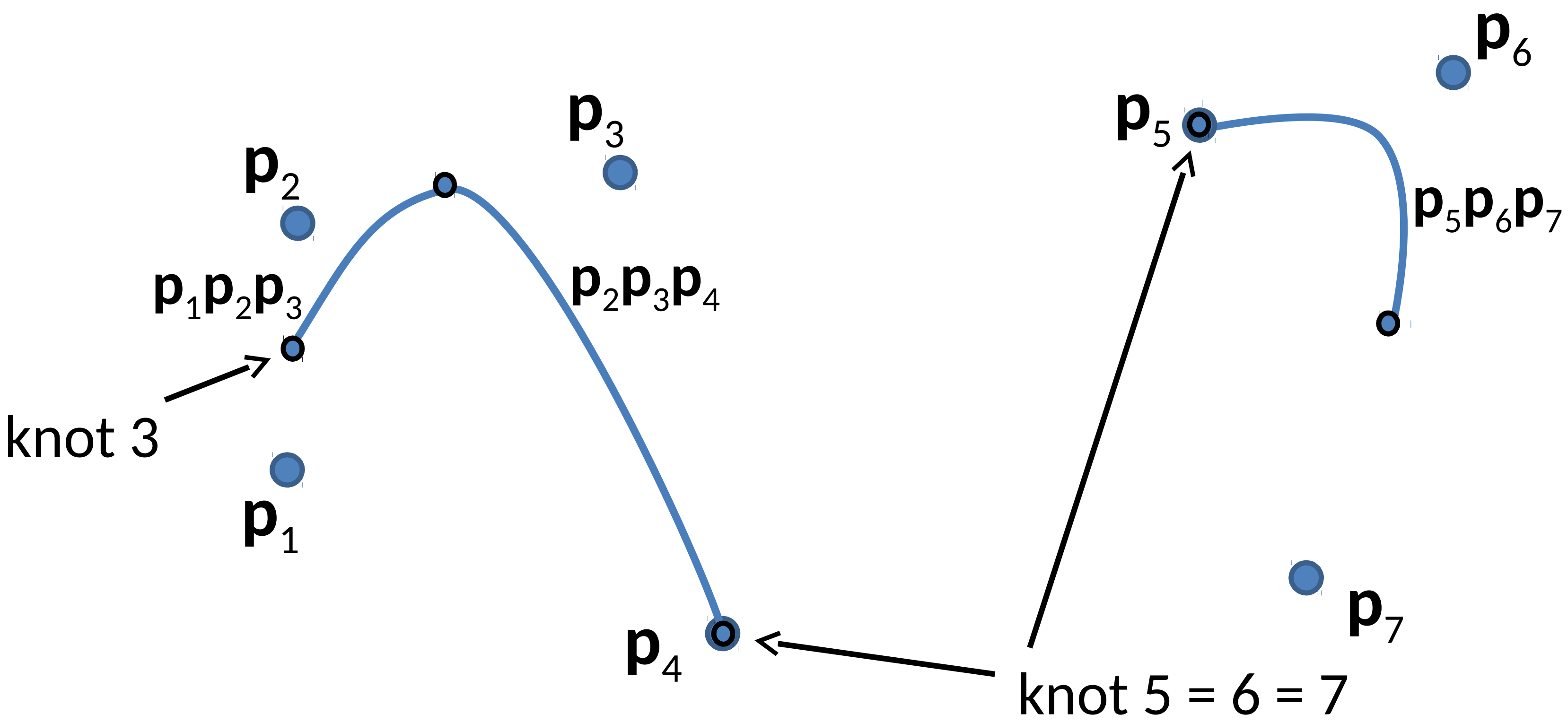
- Repeating a *control point* creates line segments on either side, but *repeating a knot* does not have this effect on quadratic B-spline curves
- Repeating a knot *twice* (three same knots in a sequence) creates a discontinuity in a quadratic
B-spline curve
  - Two consecutive curve parts have no point in common
  - The curve part before the two disappearing parts (e.g., based on points $\mathbf{p}_1$ $\mathbf{p}_2$ $\mathbf{p}_3$) is disconnected from the curve part after those parts (based on points $\mathbf{p}_4$ $\mathbf{p}_5$ $\mathbf{p}_6$)

Repeating knots in B-spline curves



quadratic B-spline curve

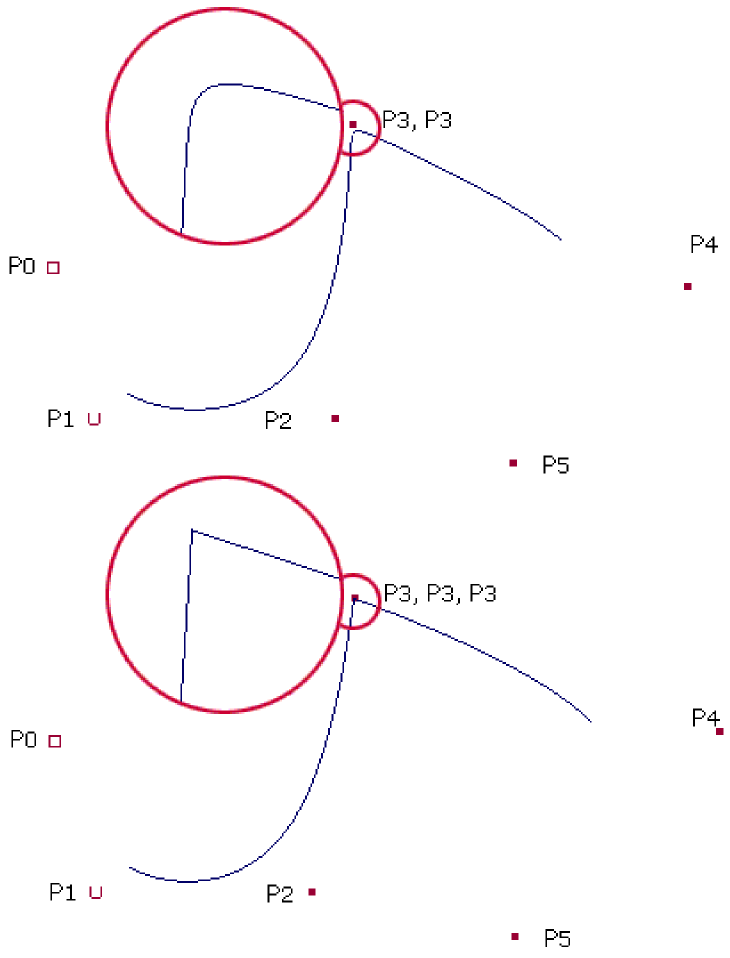
Repeating knots in B-spline curves



quadratic B-spline curve

Repeating knots in B-spline curves

- In cubic B-spline curves
  - a double knot gives C^1 / G^1 continuity
  - a triple knot gives C^0 / G^0 continuity
  - a quadruple knot disconnects the curve
  - a triple knot at the start lets $\mathbf{p}_1$ be the start of the curve
  - a triple knot at the end lets $\mathbf{p}_n$ be the end of the curve



Repeating knots in B-spline curves

- Cubic B-splines for a B-spline curve passing through its endpoints, knot vector [1, 1, 1, 2, 3, 4, 5, 6, 7, 8, 8, 8]
- The first three and last three B-splines are different from the standard cubic B-spline shape

Non-uniform rational B-splines

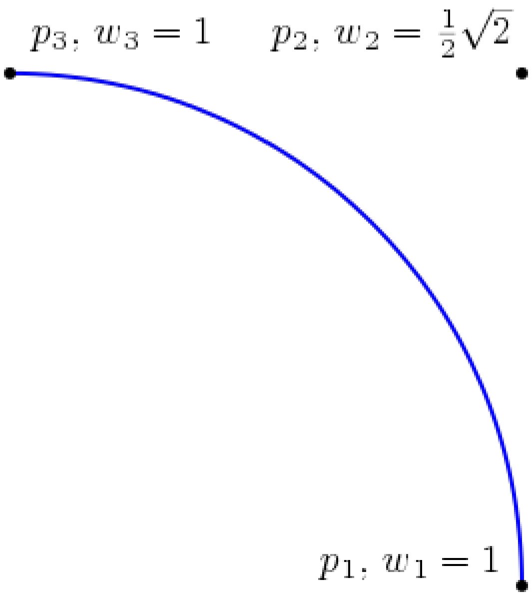
- Standard B-splines cannot have weights
- Rational B-splines *can* have weight

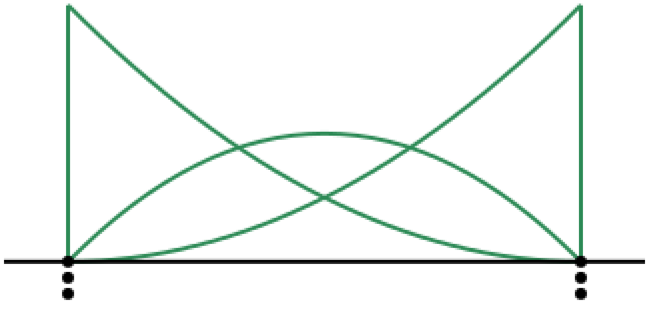
$$f(u) = \frac{\sum_{i=1}^{k} b_i(u) w_i p_i}{\sum_{i=1}^{k} b_i(u) w_i}$$

- Can be combined with knot vectors
  → non-uniform *rational* B-splines (NURBS)
- More flexible
- More complicated

Example: Circle as NURBS

- Knot vector: [0,0,0,1,1,1]
- Quadratic blending functions
- Weights are 1 or ½√2



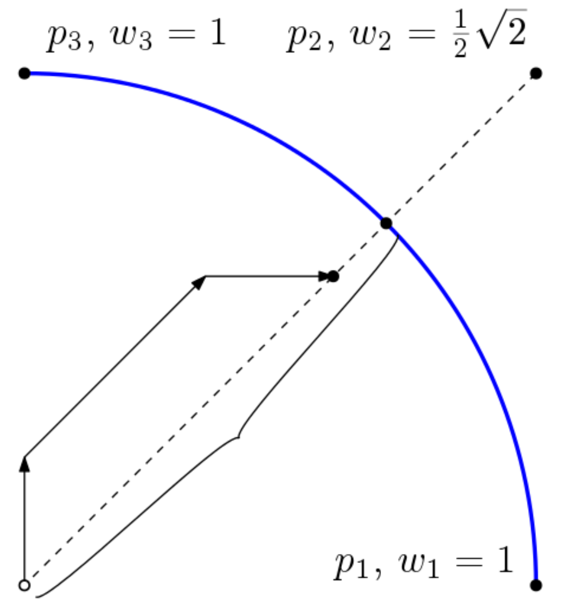


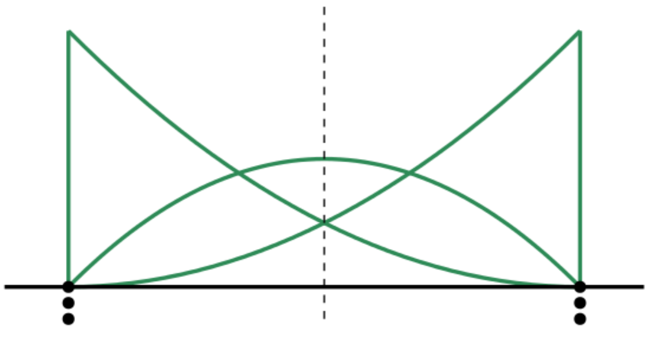
Example: Circle as NURBS

- Knot vector: [0,0,0,1,1,1]
- Quadratic blending functions
- Weights are 1 or ½√2

- f(0) = (1,0)
- f(1) = (0,1)
- f(½) = (½√2,½√2)

- Try other values yourself!





Summary

- B-spline curves are defined by B-splines and provide great flexibility in curve design
- They exist of any order (degree) and continuity
- Repeating control points or knots allows interpolation instead of approximation
- B-spline surfaces are a direct generalization
- Even more general are NURBS:
  Non-Uniform Rational B-Splines
  - defined using ratios of two polynomials
  - allow conic sections (circles, ellipses, hyperbolas)