3. Counting steps (Asymptotic analysis) [WiP]

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Algorithms (CC4010) 2023/2024

CISTER – U.Porto, Porto, Portugal

https://cister-labs.github.io/alg2324

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Overview

- Measuring precisely performance of algorithms
- Measuring asymptotically performance of algorithms
- Analysing recursive functions
- Next: beyond worst-/best-case scenarios
  - average time of a single operation
  - analysis of sequences of operations

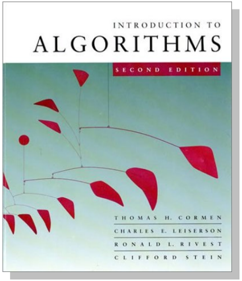
Motivation

slides by Charles E. Leiserson

pages 1-19

Introduction to Algorithms

6.046J/18.401J

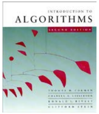


LECTURE 1

Analysis of Algorithms

- Insertion sort
- Asymptotic analysis
- Merge sort
- Recurrences

Prof. Charles E. Leiserson



Course information

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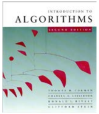
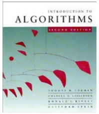


Analysis of algorithms

The theoretical study of computer-program performance and resource usage.

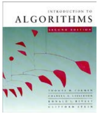
What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability



Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!



The problem of sorting

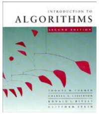
Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

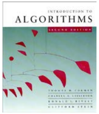
Output: 2 3 4 6 8 9



Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)        ⊳ A[1 . . n]
    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```



Insertion sort

“pseudocode”

INSERTION-SORT (A, n) $\rhd$ $A[1\,.\,.\,n]$

for $j \leftarrow 2$ **to** n

  do $key \leftarrow A[j]$

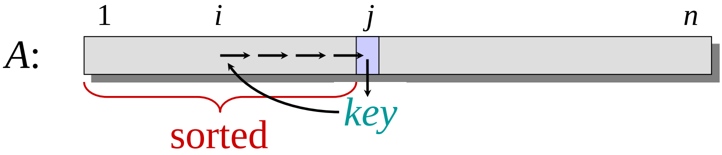
    $i \leftarrow j - 1$

    while $i > 0$ and $A[i] > key$

      do $A[i+1] \leftarrow A[i]$

        $i \leftarrow i - 1$

    $A[i+1] = key$



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

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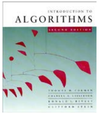

Example of insertion sort

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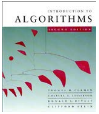

Example of insertion sort

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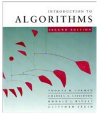
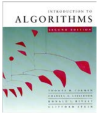


Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6

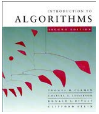


Example of insertion sort

8 2 4 9 3 6

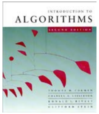
2 8 4 9 3 6

2 4 8 9 3 6



Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6



Example of insertion sort

8 2 4 9 3 6

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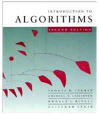
Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

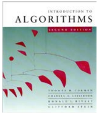


Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*



Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

slides by Pedro Ribeiro, slides 2

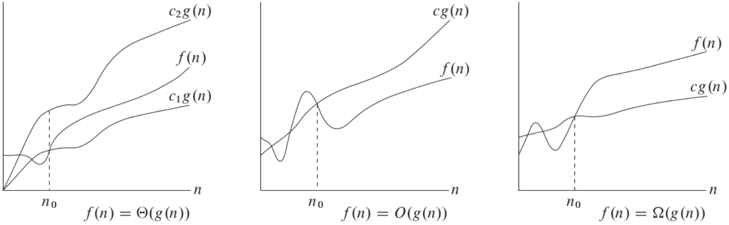
pages 1-2

Asymptotic Analysis

Pedro Ribeiro

DCC/FCUP

2018/2019

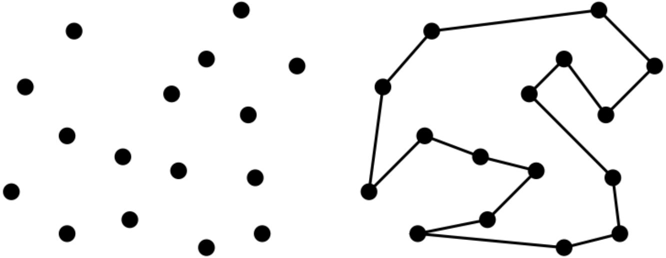


Motivational Example - TSP

Traveling Salesman Problem (Euclidean TSP version)

Input: a set S of n points in the plane
Output: the smallest possible path that starts on a point, visits all other points of S and then returns to the starting point.

An example:



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Motivational Example - TSP

How to solve the problem then?

A possible algorithm (exhaustive search a.k.a. "brute force")

$P_{min} \leftarrow$ any permutation of the points in S
For $P_i \leftarrow$ each of the permutations of points in S
  If $(cost(P_i) < cost(P_{min}))$ **then**
    $P_{min} \leftarrow P_i$
retorn Path formed by P_{min}

A correct algorithm, but extremely slow!

- $P(n) = n! = n \times (n-1) \times \ldots \times 1$
- For instance, $P(20) = 2,432,902,008,176,640,000$
- For a set of 20 points, even the fastest computer in the world would not solve it! (how long would it take?)

Motivational Example - TSP

- The present problem is a restricted version (euclidean) of one of the most well known "classic" hard problems, the **Travelling Salesman Problem** (**TSP**)
- This problem has **many possible applications**
  Ex: genomic analysis, industrial production, vehicle routing, ...
- There is no known **efficient solution** for this problem
  (with optimal results, not just approximated)
- The presented solution has $\mathcal{O}(n!)$ complexity
  The Held-Karp algorithm has $\mathcal{O}(2^n n^2)$ complexity
  (this notation will be the focus of this class)
- TSP belongs to the class of **NP-hard** problems
  The decision version belongs to the class of **NP-complete** problems
  (we will also talk about this at the end of the semester)

An experience - how many instructions

- How many instructions per second on a current computer?
  (just an approximation, an order of magnitude)

  On my notebook, about 10^9 instructions

- At this velocity, how much time for the following quantities of instructions?

Quant.	**100**	**1000**	**10000**
N	$< 0.01s$	$< 0.01s$	$< 0.01s$
N^2	$< 0.01s$	$< 0.01s$	$0.1s$
N^3	$< 0.01s$	$1.00s$	16 min
N^4	$0.1s$	16 min	115 days
2^N	10^{13} years	10^{284} years	10^{2993} years
$n!$	10^{141} years	10^{2551} years	10^{35642} years

An experience: - Permutations

- Let's go back to the idea of **permutations**

Exemple: the 6 permutations of $\{1, 2, 3\}$

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

- Recall that the number of permutations can be computed as:
  $P(n) = n! = n \times (n-1) \times \ldots \times 1$
  (do you understand the intuition on the formula?)

An experience: - Permutations

- What is the execution time of a program that goes through all permutations?
  (the following times are approximated, on my notebook)
  (what I want to show is **order of growth**)

$\mathbf{n} \leq \mathbf{7}$: $< 0.001s$
$\mathbf{n} = \mathbf{8}$: $0.001s$
$\mathbf{n} = \mathbf{9}$: $0.016s$
$\mathbf{n} = \mathbf{10}$: $0.185s$
$\mathbf{n} = \mathbf{11}$: $2.204s$
$\mathbf{n} = \mathbf{12}$: $28.460s$
...
$\mathbf{n} = \mathbf{20}$: 5000 years !

How many permutations per second?
About 10^7

On computer speed

- Will a **faster computer** be of any help? **No!**
  If $n = 20 \rightarrow 5000$ years, hypothetically:
  - 10x faster would still take 500 years
  - 5,000x would still take 1 year
  - 1,000,000x faster would still take two days, but
    $n = 21$ would take more than a month
    $n = 22$ would take more than a year!
- The **growth rate** of the execution time is what matters!

Algorithmic performance vs Computer speed

A better algorithm on a slower computer **will always win** against a worst algorithm on a faster computer, for sufficiently large instances

Why worry?

- What can we do with execution time/memory analysis?

Prediction

How much time/space does an algorithm need to solve a problem? How does it scale? Can we provide guarantees on its running time/memory?

Comparison

Is an algorithm A better than an algorithm B? Fundamentally, what is the best we can possibly do on a certain problem?

- We will study a **methodology** to answer these questions
- We will focus mainly on execution time analysis

Random Access Machine (RAM)

- We need a **model** that is **generic** and **independent** from the language and the machine.
- We will consider a Random Access Machine (RAM)
  - Each **simple operation** (ex: $+$, $-$, $\leftarrow$, **If**) takes **1 step**
  - Loops and procedures, for example, are not simple instructions!
  - Each **access to memory** takes also **1 step**
- We can measure execution time by... counting the number of steps as a function of the input size n: $T(n)$.
- Operations are **simplified**, but this is useful
  Ex: summing two integers does not cost the same as dividing two reals, but we will see that on a global vision, these specific values are not important

Random Access Machine (RAM)

A counting example

A simple program

```
int count = 0;
for (int i=0; i<n; i++)
   if (v[i] == 0) count++
```

Let's count the number of simple operations:

Variable declarations	2
Assignments:	2
"Less than" comparisons	$n+1$
"Equality" comparisons:	n
Array access	n
Increment	between n and $2n$

Random Access Machine (RAM)

A counting example

A simple program

```
int count = 0;
for (int i=0; i<n; i++)
   if (v[i] == 0) count++
```

Total number of steps on the **worst** case:
$T(n) = 2 + 2 + (n + 1) + n + n + 2n = 5 + 5n$

Total number of steps on the **best** case:
$T(n) = 2 + 2 + (n + 1) + n + n + n = 5 + 4n$

Types of algorithm analysis

Worst Case analysis: **(the most common)**

- $T(n)$ = maximum amount of time for any input of size n

Average Case analysis: **(sometimes)**

- $T(n)$ = average time on all inputs of size n
- Implies knowing the statistical distribution of the inputs

Best Case analysis: **("deceiving")**

- It's almost like "cheating" with an algorithm that is fast just for **some** of the inputs

Next steps

1. Precise analysis: counting operations
2. Approximate analysis – Asymptotic notation$(O, \Theta, \Omega, o, \omega)$

Counting operations

Exercises

[WiP: bubble sort, iSort, mult1, mult2]

[Recall arithmetic and geometric series, height of binary tree, ...]

[Proceed with Pedro's slides]

Asymptotic Notation

slides by Pedro Ribeiro, slides 2

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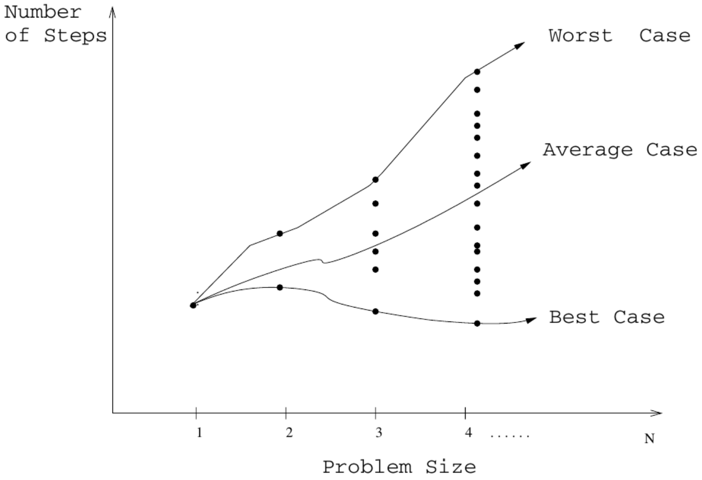
Types of algorithm analysis



Asymptotic Notation

We need a mathematical tool to **compare functions**

On algorithm analysis we use **Asymptotic Analysis**:

- "Mathematically": studying the behaviour of **limits** (as $n \to \infty$)
- Computer Science: studying the behaviour for arbitrary large input
  or
  "describing" **growth rate**

- A very specific **notation** is used: $O, \Omega, \Theta, o, \omega$
- It allows to focus on **orders of growth**

$\mathbf{f(n) = \mathcal{O}(g(n))}$

It means that $c \times g(n)$ is an **upper bound** of $f(n)$

$\mathbf{f(n) = \Omega(g(n))}$

It means that $c \times g(n)$ is a **lower bound** of $f(n)$

$\mathbf{f(n) = \Theta(g(n))}$

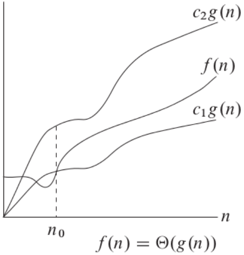
It means that $c_1 \times g(n)$ is a **lower bound** of $f(n)$ and $c_2 \times g(n)$ is an **upper bound** of $f(n)$

Note: $\in$ could be used instead of $=$

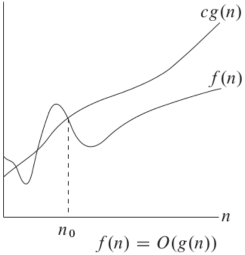
Asymptotic Notation

A graphical depiction

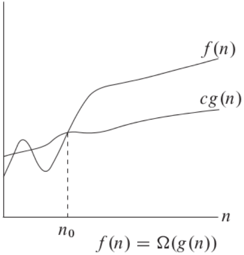
Θ



$\mathcal{O}$



Ω



The definitions imply an n from which the function is bounded. The small values of n do not "matter".

Asymptotic Notation
Formalization

- $\mathbf{f(n)} = \mathcal{O}(\mathbf{g(n)})$ if there exist positive constants n_0 and c such that $f(n) \leq c \times g(n)$ for all $n \geq n_0$
- $\mathbf{f(n)} = \mathbf{\Omega}(\mathbf{g(n)})$ if there exist positive constants n_0 and c such that $f(n) \geq c \times g(n)$ for all $n \geq n_0$
- $\mathbf{f(n)} = \mathbf{\Theta}(\mathbf{g(n)})$ if there exist positive constants n_0, c_1 and $c2$ such that $c_1 \times g(n) \leq f(n) \leq c_2 \times g(n)$ for all $n \geq n_0$

- $\mathbf{f(n)} = \mathbf{o}(\mathbf{g(n)})$ if for any positive constant c there exists n_0 such that $f(n) < c \times g(n)$ for all $n \geq n_0$
- $\mathbf{f(n)} = \omega(\mathbf{g(n)})$ if for any positive constant c there exists n_0 such that $f(n) > c \times g(n)$ for all $n \geq n_0$

Examples

Big Oh (O)

$3n^2 - 100n + 6 =? \; O(n^2)$

$3n^2 - 100n + 6 =? \; O(n^3)$

$3n^2 - 100n + 6 =? \; O(n)$

Big Omega (Ω)

$3n^2 - 100n + 6 =? \; \Omega(n^2)$

$3n^2 - 100n + 6 =? \; \Omega(n^3)$

$3n^2 - 100n + 6 =? \; \Omega(n)$

Big Theta (Θ)

$3n^2 - 100n + 6 =? \; \Theta(n^2)$

$3n^2 - 100n + 6 =? \; \Theta(n^3)$

$3n^2 - 100n + 6 =? \; \Theta(n)$

Examples

Big Oh (O)

$3n^2 - 100n + 6 = O(n^2)$ because $3n^2 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = O(n^3)$ because $0.01n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq O(n)$ because $c \cdot n < 3n^2 when n > c$

Big Omega (Ω)

$3n^2 - 100n + 6 =? \Omega(n^2)$

$3n^2 - 100n + 6 =? \Omega(n^3)$

$3n^2 - 100n + 6 =? \Omega(n)$

Big Theta (Θ)

$3n^2 - 100n + 6 =? \Theta(n^2)$

$3n^2 - 100n + 6 =? \Theta(n^3)$

$3n^2 - 100n + 6 =? \Theta(n)$

Examples

Big Oh (O)

$3n^2 - 100n + 6 = O(n^2)$ because $3n^2 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = O(n^3)$ because $0.01n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq O(n)$ because $c \cdot n < 3n^2 when n > c$

Big Omega (Ω)

$3n^2 - 100n + 6 = \Omega(n^2)$ because $2.99n^2 < 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq \Omega(n^3)$ because $n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = \Omega(n)$ because $10^{10^{10}} n < 3n^2 - 100 + 6$

Big Theta (Θ)

$3n^2 - 100n + 6 =? \Theta(n^2)$

$3n^2 - 100n + 6 =? \Theta(n^3)$

$3n^2 - 100n + 6 =? \Theta(n)$

Examples

Big Oh (O)

$$3n^2 - 100n + 6 = O(n^2) \quad \text{because} \quad 3n^2 > 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 = O(n^3) \quad \text{because} \quad 0.01n^3 > 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 \neq O(n) \quad \text{because} \quad c \cdot n < 3n^2 when n > c$$

Big Omega (Ω)

$$3n^2 - 100n + 6 = \Omega(n^2) \quad \text{because} \quad 2.99n^2 < 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 \neq \Omega(n^3) \quad \text{because} \quad n^3 > 3n^2 - 100n + 6$$
$$3n^2 - 100n + 6 = \Omega(n) \quad \text{because} \quad 10^{10^{10}} n < 3n^2 - 100 + 6$$

Big Theta (Θ)

$$3n^2 - 100n + 6 = \Theta(n^2) \quad \text{because} \quad O \text{ and } \Omega$$
$$3n^2 - 100n + 6 \neq \Theta(n^3) \quad \text{because} \quad O \text{ only}$$
$$3n^2 - 100n + 6 \neq \Theta(n) \quad \text{because} \quad \Omega \text{ only}$$

slides by Pedro Ribeiro, slides 2

pages 24-31

Asymptotic Notation
Analogy

Comparison between two functions f and g and two numbers a and b:

$f(n) = \mathcal{O}(g(n))$	is like	$a \leq b$	upper bound	at least as good as
$f(n) = \mathbf{\Omega}(g(n))$	is like	$a \geq b$	lower bound	at least as bad as
$f(n) = \mathbf{\Theta}(g(n))$	is like	$a = b$	tight bound	as good as
$f(n) = \mathbf{o}(g(n))$	is like	$a < b$	strict upper b.	strictly better than
$f(n) = \omega(g(n))$	is like	$a > b$	strict lower b.	strictly worst than

Asymptotic Notation

A few consequences

- $f(n) = \mathbf{\Theta}(g(n)) \rightarrow f(n) = \mathcal{O}(g(n))$ and $f(n) = \mathbf{\Omega}(g(n))$
- $f(n) = \mathcal{O}(g(n)) \rightarrow f(n) \neq \omega(g(n))$
- $f(n) = \mathbf{\Omega}(g(n)) \rightarrow f(n) \neq \mathbf{o}(g(n))$
- $f(n) = \mathbf{o}(g(n)) \rightarrow f(n) \neq \mathbf{\Omega}(g(n))$
- $f(n) = \omega(g(n)) \rightarrow f(n) \neq \mathcal{O}(g(n))$

- $f(n) = \mathbf{\Theta}(g(n)) \rightarrow g(n) = \mathbf{\Theta}(f(n))$
- $f(n) = \mathcal{O}(g(n)) \rightarrow g(n) = \mathbf{\Omega}(f(n))$
- $f(n) = \mathbf{\Omega}(g(n)) \rightarrow g(n) = \mathcal{O}(f(n))$
- $f(n) = \mathbf{o}(g(n)) \rightarrow g(n) = \omega(f(n))$
- $f(n) = \omega(g(n)) \rightarrow g(n) = \mathbf{o}(f(n))$

Asymptotic Notation

A few practical rules

- **Multiplying by a constant** does not affect:
  $\Theta(c \times f(n)) = \Theta(f(n))$
  $99 \times n^2 = \Theta(n^2)$
- On a polynomial of the form $a_x n^x + a_{x-1} n^{x-1} + \ldots + a_2 n^2 + a_1 n + a_0$ we can focus on the term with the **largest exponent**:
  $3\mathbf{n^3} - 5n^2 + 100 = \Theta(n^3)$
  $6\mathbf{n^4} - 20^2 = \Theta(n^4)$
  $0.8\mathbf{n} + 224 = \Theta(n)$
- More than that, on a sum we can focus on the **dominant** term:
  $\mathbf{2^n} + 6n^3 = \Theta(2^n)$
  $\mathbf{n}! - 3n^2 = \Theta(n!)$
  $n \log n + 3\mathbf{n^2} = \Theta(n^2)$

When is a function **better** than another?

- If we want to minimize time, **"smaller" functions are better**
- A function dominates another if as n grows it keeps getting larger
- Mathematically: $f(n) \gg g(n)$ if $\lim_{n \to \infty} g(n)/f(n) = 0$

Dominance Relations

$$n! \gg 2^n \gg n^3 \gg n^2 \gg n \log n \gg n \gg \log n \gg 1$$

If an operation takes 10^{-9} seconds...

	$\log n$	n	$n \log n$	n^2	n^3	2^n	$n!$
10	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$
20	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	77 years
30	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$1.07s$	
40	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	18.3 min	
50	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	13 days	
100	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	10^{13} *years*	
10^3	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$1s$		
10^4	$< 0.01s$	$< 0.01s$	$< 0.01s$	$0.1s$	16.7 min		
10^5	$< 0.01s$	$< 0.01s$	$< 0.01s$	$10s$	11 days		
10^6	$< 0.01s$	$< 0.01s$	$0.02s$	16.7 min	31 years		
10^7	$< 0.01s$	$0.01s$	$0.23s$	1.16 days			
10^8	$< 0.01s$	$0.1s$	$2.66s$	115 days			
10^9	$< 0.01s$	$1s$	$29.9s$	31 years			

Asymptotic Notation

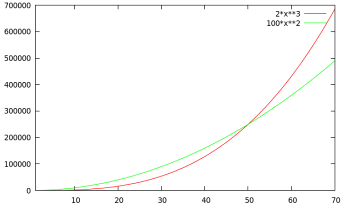
Common Functions

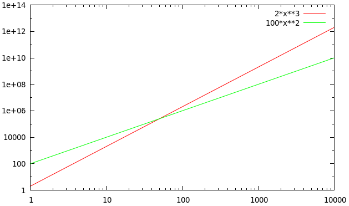
Function	Name	Examples
1	constant	summing two numbers
$\log n$	logarithmic	binary search, inserting in a heap
n	linear	1 loop to find maximum value
$n \log n$	linearithmic	sorting (ex: mergesort, heapsort)
n^2	quadratic	2 loops (ex: verifying, bubblesort)
n^3	cubic	3 loops (ex: Floyd-Warshall)
2^n	exponential	exhaustive search (ex: subsets)
$n!$	factorial	all permutations

An useful program for drawing functions is gnuplot.

(comparing $2n^3$ with $100n^2$ for $1 \leq n \leq 100$)

```
gnuplot> plot [1:70] 2*x**3, 100*x**2
gnuplot> set logscale xy 10
gnuplot> plot [1:10000] 2*x**3, 100*x**2
```





(which grows faster: $\sqrt{n}$ or $\log_2 n$?)

```
gnuplot> plot [1:1000000] sqrt(x), log(x)/log(2)
```

- A program has two pieces of code A and B, executed one after the other, with A running in $\Theta(n \log n)$ and B in $\Theta(n^2)$.
  The program runs in $\Theta(n^2)$, because $n^2 \gg n \log n$
- A program calls n times a function $\Theta(\log n)$, and then it calls again n times another function $\Theta(\log n)$
  The program runs in $\Theta(n \log n)$
- A program has 5 loops, all called sequentially, each one of them running in $\Theta(n)$
  The program runs in $\Theta(n)$
- A program P_1 has execution time proportional to $100 \times n \log n$.
  Another program P_2 runs in $2 \times n^2$.
  Which one is more efficient?
  P_1 is more efficient because $n^2 \gg n \log n$. However, for a small n, P_2 is quicker and it might make sense to have a program that calls P_1 or P_2 depending on n.

Recursive functions

...