3. Counting steps
(Asymptotic analysis) [WiP]

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Algorithms (CC4010) 2023/2024

CISTER – U.Porto, Porto, Portugal

https://cister-labs.github.io/alg2324

U.PORTO
FACULDADE DE CIÊNCIAS
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CISTER - Research Centre in
Real-Time & Embedded
Computing Systems

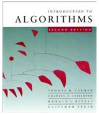
Overview

- Checking correctness of algorithms
- Measuring precisely performance of algorithms
- Measuring asymptotically performance of algorithms
- Analysing recursive functions
- Next: beyond worst-/best-case scenarios
    - average time of a single operation
    - analysis of sequences of operations (amortised analysis)

Motivation

slides by Charles E. Leiserson

pages 3-19

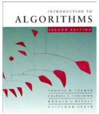


Analysis of algorithms

The theoretical study of computer-program performance and resource usage.

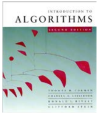
What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability



Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!



The problem of sorting

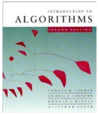
Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

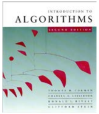
Output: 2 3 4 6 8 9



Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)        ⊳ A[1 . . n]
    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```



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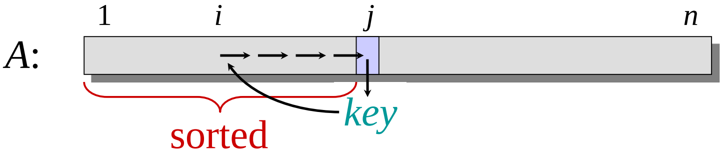
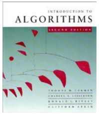
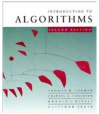




Example of insertion sort

8 2 4 9 3 6



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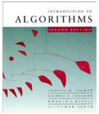

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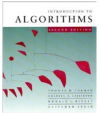

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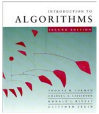
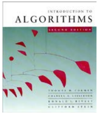


Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6



Example of insertion sort

8 2 4 9 3 6

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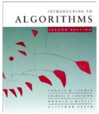
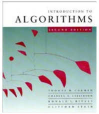


Example of insertion sort

8	**2**	4	9	3	6
2	8	**4**	9	3	6
2	4	8	**9**	3	6
2	4	8	9	3	6



Example of insertion sort

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Example of insertion sort

8	2	4	9	3	6
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2	4	8	9	3	6
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2	3	4	8	9	6

Example of insertion sort

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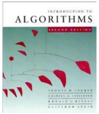
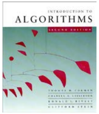


Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*



Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

slides by Pedro Ribeiro, slides 2

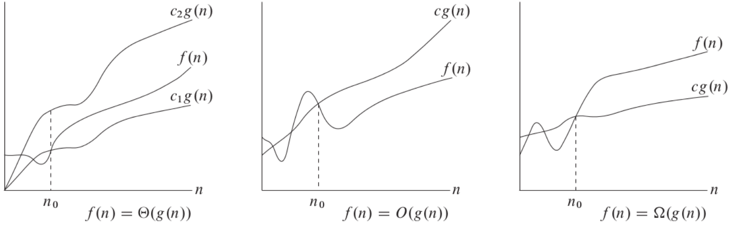
pages 1-2

Asymptotic Analysis

Pedro Ribeiro

DCC/FCUP

2018/2019

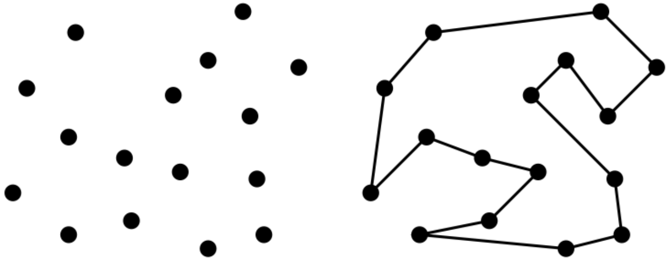


Motivational Example - TSP

Traveling Salesman Problem (Euclidean TSP version)

Input: a set S of n points in the plane
Output: the smallest possible path that starts on a point, visits all other points of S and then returns to the starting point.

An example:



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Motivational Example - TSP

How to solve the problem then?

A possible algorithm (exhaustive search a.k.a. "brute force")

$P_{min} \leftarrow$ any permutation of the points in S
For $P_i \leftarrow$ each of the permutations of points in S
    If $(cost(P_i) < cost(P_{min}))$ **then**
        $P_{min} \leftarrow P_i$
retorn Path formed by P_{min}

A correct algorithm, but extremely slow!

- $P(n) = n! = n \times (n-1) \times \ldots \times 1$
- For instance, $P(20) = 2,432,902,008,176,640,000$
- For a set of 20 points, even the fastest computer in the world would not solve it! (how long would it take?)

Motivational Example - TSP

- The present problem is a restricted version (euclidean) of one of the most well known "classic" hard problems, the **Travelling Salesman Problem** (**TSP**)
- This problem has **many possible applications**
  Ex: genomic analysis, industrial production, vehicle routing, ...
- There is no known **efficient solution** for this problem
  (with optimal results, not just approximated)
- The presented solution has $\mathcal{O}(n!)$ complexity
  The Held-Karp algorithm has $\mathcal{O}(2^n n^2)$ complexity
  (this notation will be the focus of this class)
- TSP belongs to the class of **NP-hard** problems
  The decision version belongs to the class of **NP-complete** problems
  (we will also talk about this at the end of the semester)

An experience - how many instructions

- How many instructions per second on a current computer?
  (just an approximation, an order of magnitude)

  On my notebook, about 10^9 instructions

- At this velocity, how much time for the following quantities of instructions?

Quant.	**100**	**1000**	**10000**
N	$< 0.01s$	$< 0.01s$	$< 0.01s$
N^2	$< 0.01s$	$< 0.01s$	$0.1s$
N^3	$< 0.01s$	$1.00s$	16 min
N^4	$0.1s$	16 min	115 days
2^N	10^{13} years	10^{284} years	10^{2993} years
$n!$	10^{141} years	10^{2551} years	10^{35642} years

An experience: - Permutations

- Let's go back to the idea of **permutations**

Exemple: the 6 permutations of $\{1, 2, 3\}$

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

- Recall that the number of permutations can be computed as: $P(n) = n! = n \times (n-1) \times \ldots \times 1$
  (do you understand the intuition on the formula?)

An experience: - Permutations

- What is the execution time of a program that goes through all permutations?
  (the following times are approximated, on my notebook)
  (what I want to show is **order of growth**)

$\mathbf{n} \leq \mathbf{7}$: $< 0.001s$
$\mathbf{n} = \mathbf{8}$: $0.001s$
$\mathbf{n} = \mathbf{9}$: $0.016s$
$\mathbf{n} = \mathbf{10}$: $0.185s$
$\mathbf{n} = \mathbf{11}$: $2.204s$
$\mathbf{n} = \mathbf{12}$: $28.460s$
...
$\mathbf{n} = \mathbf{20}$: 5000 years !

How many permutations per second?
About 10^7

On computer speed

- Will a **faster computer** be of any help? **No!**
  If $n = 20 \rightarrow 5000$ years, hypothetically:
  - 10x faster would still take 500 years
  - 5,000x would still take 1 year
  - 1,000,000x faster would still take two days, but
    $n = 21$ would take more than a month
    $n = 22$ would take more than a year!
- The **growth rate** of the execution time is what matters!

Algorithmic performance vs Computer speed

A better algorithm on a slower computer **will always win** against a worst algorithm on a faster computer, for sufficiently large instances

Why worry?

- What can we do with execution time/memory analysis?

Prediction

How much time/space does an algorithm need to solve a problem? How does it scale? Can we provide guarantees on its running time/memory?

Comparison

Is an algorithm A better than an algorithm B? Fundamentally, what is the best we can possibly do on a certain problem?

- We will study a **methodology** to answer these questions
- We will focus mainly on execution time analysis

Random Access Machine (RAM)

- We need a **model** that is **generic** and **independent** from the language and the machine.
- We will consider a Random Access Machine (RAM)
  - Each **simple operation** (ex: $+$, $-$, $\leftarrow$, **If**) takes **1 step**
  - Loops and procedures, for example, are not simple instructions!
  - Each **access to memory** takes also **1 step**
- We can measure execution time by... counting the number of steps as a function of the input size n: $T(n)$.
- Operations are **simplified**, but this is useful

  Ex: summing two integers does not cost the same as dividing two reals, but we will see that on a global vision, these specific values are not important

Random Access Machine (RAM)

A counting example

A simple program

```
int count = 0;
for (int i=0; i<n; i++)
    if (v[i] == 0) count++
```

Let's count the number of simple operations:

Variable declarations	2
Assignments:	2
"Less than" comparisons	$n + 1$
"Equality" comparisons:	n
Array access	n
Increment	between n and $2n$

Random Access Machine (RAM)

A counting example

A simple program

```
int count = 0;
for (int i=0; i<n; i++)
   if (v[i] == 0) count++
```

Total number of steps on the **worst** case:
$T(n) = 2 + 2 + (n+1) + n + n + 2n = 5 + 5n$

Total number of steps on the **best** case:
$T(n) = 2 + 2 + (n+1) + n + n + n = 5 + 4n$

Types of algorithm analysis

Worst Case analysis: **(the most common)**

- $T(n)$ = maximum amount of time for any input of size n

Average Case analysis: **(sometimes)**

- $T(n)$ = average time on all inputs of size n
- Implies knowing the statistical distribution of the inputs

Best Case analysis: **("deceiving")**

- It's almost like "cheating" with an algorithm that is fast just for **some** of the inputs

Next steps

1. Precise analysis: counting operations
2. Approximate analysis – Asymptotic notation$(O, \Theta, \Omega, o, \omega)$

Counting operations

Simpler counting

```
int count = 0;
for (int i=0; i<n; i++)
  if (v[i] == 0) count++
```

RAM

- worst-case: $T(n) = 5 + 5n$
- best-case: $T(n) = 5 + 4n$

#array-accesses + #count-increments

- worst-case: $T(n) = 2n$
- best-case: $T(n) = n$
- average-case: $\overline{T}(n) = n + \sum_{0 \le r < n} P(v[r] = 0)$

Exercises

```
void bubbleSort(int v[], int N){
  int i, j;
  for (i=N-1; i>0; i--)
    for (j=0; j<i; j++)
      if (v[j] > v[j+1])
          swap(v,j,j+1);
}
```

```
void iSort(int v[], int N){
  int i, j;
  for (i=1; i<N; i++)
    for (j=i; j>0 && v[j-1]>v[j];
          j--)
        swap(v,j,j-1);
}
```

Ex. 3.1: What is the best and worst case wrt comparisons between array elements?

Ex. 3.2: What is the best and worst case wrt swaps?

Ex. 3.3: How many of these operations are performed in both cases?

Exercises

```c
int mult1 (int x, int y){
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0){
    r = r+b;
    a = a-1;
  }
  return r;
}
```

```c
int mult2 (int x, int y){
  int a, b, r;
  a=x; b=y; r=0;
  while (a!=0) {
    if (a%2 == 1) r = r+b;
    a=a/2;
    b=b*2;
  return r;
}
```

Ex. 3.4: In each case, how many primitive operations (+ − *2 /2 %2) are performed?

Note: In `mult2`, consider the size N as the number of bits used to represent `x` and `y`; e.g., with 5 bits you can represent a positive integer until 31.

Exercises

```c
int maxgrow(int v[], int N) {
  int r = 1, i = 0, m;
  while (i<N-1) {
    m = grow(v+i, N-i);
    if (m>r) r = m;
    i++;
  }
  return r;
}
```

```c
int grow(int v[], int N) {
  int i;
  for (i=1; i<N; i++)
    if (v[i] < v[i-1]) break;
  return i;
}
```

Ex. 3.5: What is the best and worst case for `maxgrow` wrt comparisons of array elements?

Ex. 3.6: How many comparisons in each case?

Asymptotic Notation

slides by Pedro Ribeiro, slides 2

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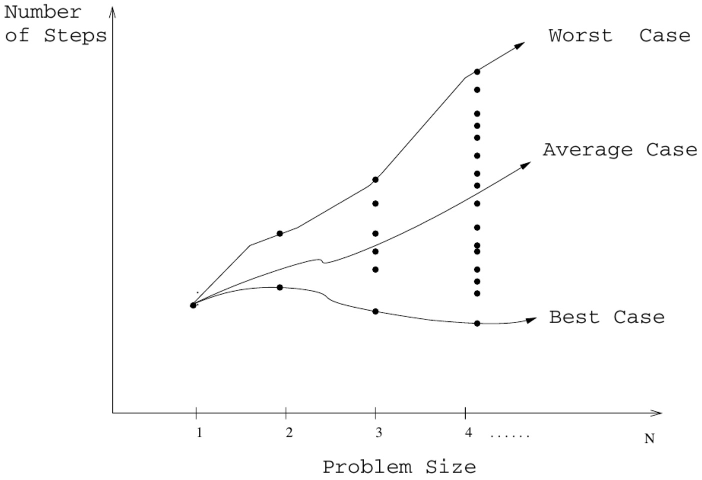
Types of algorithm analysis



Asymptotic Notation

We need a mathematical tool to **compare functions**

On algorithm analysis we use **Asymptotic Analysis**:

- "Mathematically": studying the behaviour of **limits** (as $n \to \infty$)
- Computer Science: studying the behaviour for arbitrary large input
  or
  "describing" **growth rate**

- A very specific **notation** is used: $O, \Omega, \Theta, o, \omega$
- It allows to focus on **orders of growth**

$f(n) = \mathcal{O}(g(n))$

It means that $c \times g(n)$ is an **upper bound** of $f(n)$

$f(n) = \Omega(g(n))$

It means that $c \times g(n)$ is a **lower bound** of $f(n)$

$f(n) = \Theta(g(n))$

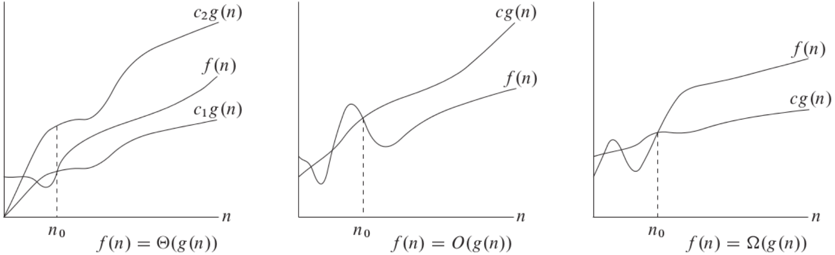
It means that $c_1 \times g(n)$ is a **lower bound** of $f(n)$ and $c_2 \times g(n)$ is an **upper bound** of $f(n)$

Note: $\in$ could be used instead of $=$

Asymptotic Notation

A graphical depiction

Θ $\mathcal{O}$ Ω



The definitions imply an n from which the function is bounded. The small values of n do not "matter".

Asymptotic Notation

Formalization

- $\mathbf{f(n)} = \mathcal{O}(\mathbf{g(n)})$ if there exist positive constants n_0 and c such that $f(n) \leq c \times g(n)$ for all $n \geq n_0$
- $\mathbf{f(n)} = \mathbf{\Omega}(\mathbf{g(n)})$ if there exist positive constants n_0 and c such that $f(n) \geq c \times g(n)$ for all $n \geq n_0$
- $\mathbf{f(n)} = \mathbf{\Theta}(\mathbf{g(n)})$ if there exist positive constants n_0, c_1 and $c2$ such that $c_1 \times g(n) \leq f(n) \leq c_2 \times g(n)$ for all $n \geq n_0$

- $\mathbf{f(n)} = \mathbf{o}(\mathbf{g(n)})$ if for any positive constant c there exists n_0 such that $f(n) < c \times g(n)$ for all $n \geq n_0$
- $\mathbf{f(n)} = \omega(\mathbf{g(n)})$ if for any positive constant c there exists n_0 such that $f(n) > c \times g(n)$ for all $n \geq n_0$

Examples

Big Oh (O)

$$3n^2 - 100n + 6 =? \; O(n^2)$$
$$3n^2 - 100n + 6 =? \; O(n^3)$$
$$3n^2 - 100n + 6 =? \; O(n)$$

Big Omega (Ω)

$$3n^2 - 100n + 6 =? \; \Omega(n^2)$$
$$3n^2 - 100n + 6 =? \; \Omega(n^3)$$
$$3n^2 - 100n + 6 =? \; \Omega(n)$$

Big Theta (Θ)

$$3n^2 - 100n + 6 =? \; \Theta(n^2)$$
$$3n^2 - 100n + 6 =? \; \Theta(n^3)$$
$$3n^2 - 100n + 6 =? \; \Theta(n)$$

Examples

Big Oh (O)

$3n^2 - 100n + 6 = O(n^2)$ because $3n^2 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = O(n^3)$ because $0.01n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq O(n)$ because $c \cdot n < 3n^2$ when $n > c$

Big Omega (Ω)

$3n^2 - 100n + 6 =? \ \Omega(n^2)$

$3n^2 - 100n + 6 =? \ \Omega(n^3)$

$3n^2 - 100n + 6 =? \ \Omega(n)$

Big Theta (Θ)

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Examples

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$3n^2 - 100n + 6 = O(n^3)$ because $0.01n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq O(n)$ because $c \cdot n < 3n^2$ when $n > c$

Big Omega (Ω)

$3n^2 - 100n + 6 = \Omega(n^2)$ because $2.99n^2 < 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq \Omega(n^3)$ because $n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = \Omega(n)$ because $10^{10^{10}} n < 3n^2 - 100 + 6$

Big Theta (Θ)

$3n^2 - 100n + 6 =?\ \Theta(n^2)$

$3n^2 - 100n + 6 =?\ \Theta(n^3)$

$3n^2 - 100n + 6 =?\ \Theta(n)$

Examples

Big Oh (O)

$3n^2 - 100n + 6 = O(n^2)$ because $3n^2 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = O(n^3)$ because $0.01n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq O(n)$ because $c \cdot n < 3n^2$ when $n > c$

Big Omega (Ω)

$3n^2 - 100n + 6 = \Omega(n^2)$ because $2.99n^2 < 3n^2 - 100n + 6$

$3n^2 - 100n + 6 \neq \Omega(n^3)$ because $n^3 > 3n^2 - 100n + 6$

$3n^2 - 100n + 6 = \Omega(n)$ because $10^{10^{10}} n < 3n^2 - 100 + 6$

Big Theta (Θ)

$3n^2 - 100n + 6 = \Theta(n^2)$ because O and Ω

$3n^2 - 100n + 6 \neq \Theta(n^3)$ because O only

$3n^2 - 100n + 6 \neq \Theta(n)$ because Ω only

slides by Pedro Ribeiro, slides 2

pages 24-31

Comparison between two functions f and g and two numbers a and b:

$f(n) = \mathcal{O}(g(n))$	is like	$a \leq b$	upper bound	at least as good as
$f(n) = \mathbf{\Omega}(g(n))$	is like	$a \geq b$	lower bound	at least as bad as
$f(n) = \mathbf{\Theta}(g(n))$	is like	$a = b$	tight bound	as good as
$f(n) = \mathbf{o}(g(n))$	is like	$a < b$	strict upper b.	strictly better than
$f(n) = \omega(g(n))$	is like	$a > b$	strict lower b.	strictly worst than

Asymptotic Notation

A few consequences

- $f(n) = \Theta(g(n)) \rightarrow f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$
- $f(n) = \mathcal{O}(g(n)) \rightarrow f(n) \neq \omega(g(n))$
- $f(n) = \Omega(g(n)) \rightarrow f(n) \neq \mathbf{o}(g(n))$
- $f(n) = \mathbf{o}(g(n)) \rightarrow f(n) \neq \Omega(g(n))$
- $f(n) = \omega(g(n)) \rightarrow f(n) \neq \mathcal{O}(g(n))$

- $f(n) = \Theta(g(n)) \rightarrow g(n) = \Theta(f(n))$
- $f(n) = \mathcal{O}(g(n)) \rightarrow g(n) = \Omega(f(n))$
- $f(n) = \Omega(g(n)) \rightarrow g(n) = \mathcal{O}(f(n))$
- $f(n) = \mathbf{o}(g(n)) \rightarrow g(n) = \omega(f(n))$
- $f(n) = \omega(g(n)) \rightarrow g(n) = \mathbf{o}(f(n))$

Asymptotic Notation

A few practical rules

- **Multiplying by a constant** does not affect:
  $\Theta(c \times f(n)) = \Theta(f(n))$
  $99 \times n^2 = \Theta(n^2)$

- On a polynomial of the form $a_x n^x + a_{x-1} n^{x-1} + \ldots + a_2 n^2 + a_1 n + a_0$ we can focus on the term with the **largest exponent**:
  $3\mathbf{n^3} - 5n^2 + 100 = \Theta(n^3)$
  $6\mathbf{n^4} - 20^2 = \Theta(n^4)$
  $0.8\mathbf{n} + 224 = \Theta(n)$

- More than that, on a sum we can focus on the **dominant** term:
  $\mathbf{2^n} + 6n^3 = \Theta(2^n)$
  $\mathbf{n}! - 3n^2 = \Theta(n!)$
  $n \log n + 3\mathbf{n^2} = \Theta(n^2)$

When is a function **better** than another?

- If we want to minimize time, **"smaller" functions are better**
- A function dominates another if as n grows it keeps getting larger
- Mathematically: $f(n) \gg g(n)$ if $\lim_{n \to \infty} g(n)/f(n) = 0$

Dominance Relations

$$n! \gg 2^n \gg n^3 \gg n^2 \gg n \log n \gg n \gg \log n \gg 1$$

If an operation takes 10^{-9} seconds...

	$\log n$	n	$n \log n$	n^2	n^3	2^n	$n!$
10	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$
20	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	77 years
30	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$1.07s$	
40	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	18.3 min	
50	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	13 days	
100	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	10^{13} *years*	
10^3	$< 0.01s$	$< 0.01s$	$< 0.01s$	$< 0.01s$	$1s$		
10^4	$< 0.01s$	$< 0.01s$	$< 0.01s$	$0.1s$	16.7 min		
10^5	$< 0.01s$	$< 0.01s$	$< 0.01s$	$10s$	11 days		
10^6	$< 0.01s$	$< 0.01s$	$0.02s$	16.7 min	31 years		
10^7	$< 0.01s$	$0.01s$	$0.23s$	1.16 days			
10^8	$< 0.01s$	$0.1s$	$2.66s$	115 days			
10^9	$< 0.01s$	$1s$	$29.9s$	31 years			

Function	Name	Examples
1	constant	summing two numbers
$\log n$	logarithmic	binary search, inserting in a heap
n	linear	1 loop to find maximum value
$n \log n$	linearithmic	sorting (ex: mergesort, heapsort)
n^2	quadratic	2 loops (ex: verifying, bubblesort)
n^3	cubic	3 loops (ex: Floyd-Warshall)
2^n	exponential	exhaustive search (ex: subsets)
$n!$	factorial	all permutations

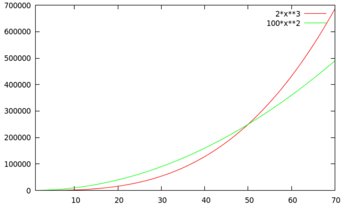
Asymptotic Growth

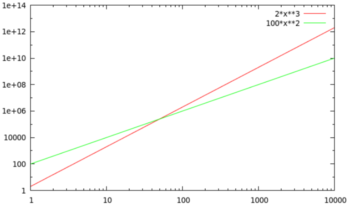
Drawing functions

An useful program for drawing functions is gnuplot.

(comparing $2n^3$ with $100n^2$ for $1 \leq n \leq 100$)

```
gnuplot> plot [1:70] 2*x**3, 100*x**2
gnuplot> set logscale xy 10
gnuplot> plot [1:10000] 2*x**3, 100*x**2
```





(which grows faster: $\sqrt{n}$ or $\log_2 n$?)

```
gnuplot> plot [1:1000000] sqrt(x), log(x)/log(2)
```

Asymptotic Analysis

A few more examples

- A program has two pieces of code A and B, executed one after the other, with A running in $\Theta(n \log n)$ and B in $\Theta(n^2)$.
  The program runs in $\Theta(n^2)$, because $n^2 \gg n \log n$
- A program calls n times a function $\Theta(\log n)$, and then it calls again n times another function $\Theta(\log n)$
  The program runs in $\Theta(n \log n)$
- A program has 5 loops, all called sequentially, each one of them running in $\Theta(n)$
  The program runs in $\Theta(n)$
- A program P_1 has execution time proportional to $100 \times n \log n$.
  Another program P_2 runs in $2 \times n^2$.
  Which one is more efficient?
  P_1 is more efficient because $n^2 \gg n \log n$. However, for a small n, P_2 is quicker and it might make sense to have a program that calls P_1 or P_2 depending on n.

slides by Pedro Ribeiro, exercises 2

pages 1-2

Exercises #2
Asymptotic Analysis

Theoretical Background

Remember the asymptotic notation:

- $\mathbf{f(n) = O(g(n))}$ if there exist positive constants n_0 and c such that $f(n) \le cg(n)$ for all $n \ge n_0$.
- $\mathbf{f(n) = \Omega(g(n))}$ if there exist positive constants n_0 and c such that $f(n) \ge cg(n)$ for all $n \ge n_0$.
- $\mathbf{f(n) = \Theta(g(n))}$ if there exist positive constants n_0, c_1 and c_2 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$.
- $\mathbf{f(n) = o(g(n))}$ if for any positive constant c there exists n_0 such that $f(n) < cg(n)$ for all $n \ge n_0$.
- $\mathbf{f(n)} = \omega(\mathbf{g(n)})$ if for any positive constant c there exists n_0 such that $f(n) > cg(n)$ for all $n \ge n_0$.

Asymptotic Notation

1. Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$. Justify your answer with brief proofs.

2. For each pair of functions $f(n)$ and $g(n)$, indicate whether $f(n)$ is O, o, Ω, ω, or Θ of $g(n)$. Your answer should be in the form of a "yes" or "no" for each cell of the table.

	$f(n)$	$g(n)$	O	o	Ω	ω	Θ
(a)	$2n^3 - 10n^2$	$25n^2 + 37n$					
(b)	56	$\log_2 30$					
(c)	$log_3 n$	$\log_2 n$					
(d)	n^3	3^n					
(e)	$n!$	2^n					
(f)	$n!$	n^n					
(g)	$n \log_2 n + n^2$	n^2					
(h)	$\sqrt{n}$	$\log_2 n$					
(i)	$\log_3(\log_3 n)$	$\log_3 n$					
(j)	$\log_2 n$	$\log_2 n^2$					

3. For each of the following conjectures, indicate if they are true or false, explaining why.

   You can assume that functions $f(n)$ and $g(n)$ are asymptotically positive, i.e., they are positive from some point on ($\exists n_0 : f(n) > 0$ for all $n \geq n_0$)

   (a) $f(n) = O(g(n))$ implies that $g(n) = O(f(n))$
   (b) $f(n) = O(g(n))$ implies that $g(n) = \Omega(f(n))$
   (c) $f(n) + g(n) = \Theta(min(f(n), g(n)))$
   (d) $f(n) + g(n) = \Theta(max(f(n), g(n)))$
   (e) $(n + c)^k = \Theta(n^k)$, where c and k are positive integer constants
   (f) $f(n) + o(f(n)) = \Theta(f(n))$
   (g) $n^2 = \Theta(16^{\log_4 n})$

Growth Ratio

4. Imagine a program A running with time complexity $\Theta(f(n))$, taking t seconds for an input of size k. What would your estimation be for the execution time for an input of size $2k$ for the following functions: n, n^2 , n^3, 2^n, $\log_2 n$. Is this growth ratio constant for any k or is it changing?

5. Consider two programs implementing algorithms A and B, both trying to solve the same problem for an input of size n. They measured the execution times for test cases of different sizes and got the following table:

Algorithm	$n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$
A	0.003s	0.024s	0.081s	0.192s	0.375s
B	0.040s	0.160s	0.360s	0.640s	1.000s

   (a) Which program is more efficient? Why?
   (b) Could you produce a program that uses both algorithms in order to produce an algorithm C that would be at least as good as A and B for any test case?

Analysis of recursive functions

Binary search

To analyse the complexity of a recursive function, we typically define the time T using *recurring equations*.

```c
int bsearch(int x, int v[], int N){
  int i;
  if (N<=0) i = -1;
  else {
    m = N/2;
    if (v[m]==x) i = m;
    else if (v[m] > x)
      i = bsearch(x, v, m);
    else {
      i = bsearch(x, v+m+1, N-m-1);
      if (i!=-1) i = i+m+1
    }
  return i ;
}
```

Counting the number of comparisons with array elements:

$$T(N) = \begin{cases} 0 & \text{if } N = 0 \\ T(N/2) + 2 & \text{if } N > 0 \end{cases}$$

Binary search

To analyse the complexity of a recursive function, we typically define the time T using *recurring equations*.

```
int bsearch(int x, int v[], int N){
  int i;
  if (N<=0) i = -1;
  else {
    m = N/2;
    if (v[m]==x) i = m;
    else if (v[m] > x)
      i = bsearch(x, v, m);
    else {
      i = bsearch(x, v+m+1, N-m-1);
      if (i!=-1) i = i+m+1
    }
  return i ;
}
```

$$(\mathrm{T(N) = T(N/2)+2 \;\; if \;\; N>0})$$

$$T(0) = 0$$
$$T(1) = T(2^0) = 2$$
$$T(2) = T(2^1) = 2 + T(2/2) = 2 + 2 = 4$$
$$T(4) = T(2^2) = 2 + T(4/2) = 2 + 2 + 2 = 6$$
$$\dots$$
$$T(2^i) = \underbrace{2 + 2 + \cdots + 2}_{i\text{-times}} + 2 = 2i + 2$$

Binary search

To analyse the complexity of a recursive function, we typically define the time T using *recurring equations*.

```
int bsearch(int x, int v[], int N){
  int i;
  if (N<=0) i = -1;
  else {
    m = N/2;
    if (v[m]==x) i = m;
    else if (v[m] > x)
      i = bsearch(x, v, m);
    else {
      i = bsearch(x, v+m+1, N-m-1);
      if (i!=-1) i = i+m+1
    }
  return i ;
}
```

(T(N) = T(N/2)+2 if N>0)

$$
\begin{aligned}
T(0) &= 0 \\
T(1) &= T(2^0) = 2 \\
T(2) &= T(2^1) = 2 + T(2/2) = 2 + 2 = 4 \\
T(4) &= T(2^2) = 2 + T(4/2) = 2 + 2 + 2 = 6 \\
&\dots \\
T(2^i) &= \underbrace{2 + 2 + \cdots + 2}_{i\text{-times}} + 2 = 2i + 2 \\
T(N) &= T(2^{log_2(N)}) \\
&= 2 * log_2(N) + 2 \quad (= \Theta(log(N)))
\end{aligned}
$$

slides by Pedro Ribeiro, slides 2

pages 38-61

Divide and Conquer

We are often interested in algorithms that are expressed in a recursive way

Many of these algorithms follow the divide and conquer strategy:

Divide and Conquer

Divide the problem in a set of subproblems which are smaller instances of the same problem

Conquer the subproblems solving them recursively. If the problem is small enough, solve it directly.

Combine the solutions of the smaller subproblems on a solution for the original problem

We now describe the MergeSort algorithm for sorting an array of size n

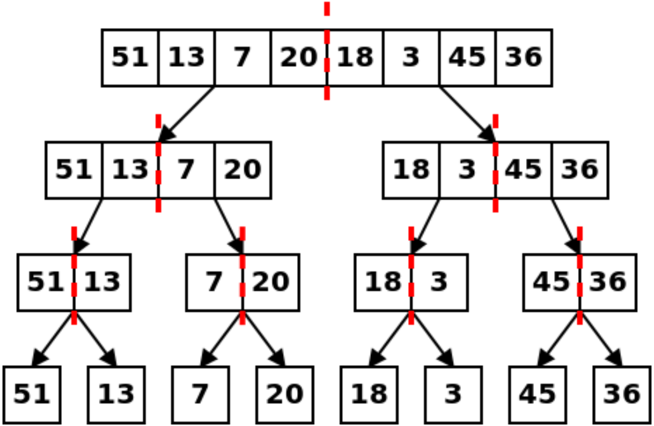
MergeSort

Divide: partition the initial array in two halves

Conquer: recursively sort each half. If we only have one number, it is sorted.

Combine: merge the two sorted halves in a final sorted array

Divide:


51
13
7
20
18
3
45
36
51
13
7
20
18
3
45
36
51
13
7
20
18
3
45
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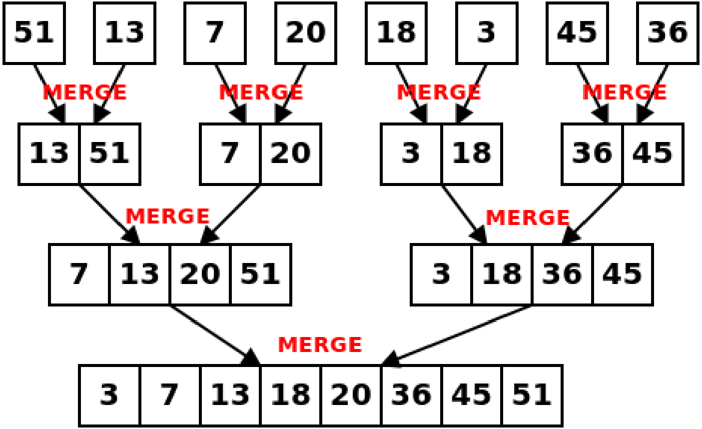
51
13
7
20
18
3
45
36

Divide and Conquer

MergeSort

Conquer:



What is the **execution time** of this algorithm?

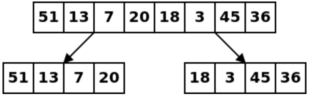
- **D(n)** - Time to partition an array of size n in two halves
- **M(n)** - Time to merge two sorted arrays of size n
- **T(n)** - Time for a MergeSort on an array of size n

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ D(n) + 2T(n/2) + M(n) & \text{if } n > 1 \end{cases}$$

In practice, we are ignoring certain details, but it suffices
(ex: when n is odd, the size of subproblem is not exactly $n/2$)

Divide and Conquer
MergeSort

D(**n**) - Time to partition an array of size n in two halves



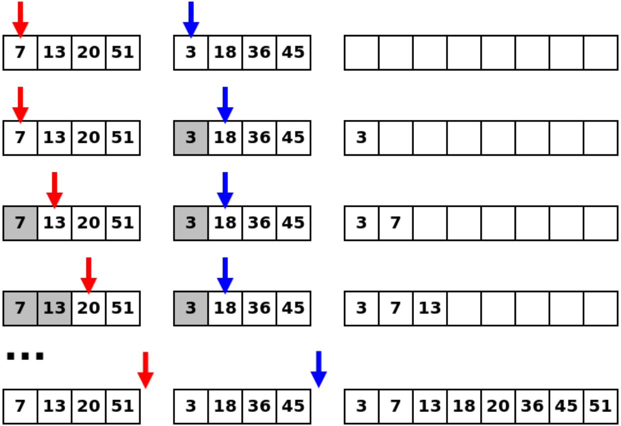
We can do it in constant time! $\boldsymbol{\Theta(1)}$

`mergesort(a,b)`: (sort from position a to b)

In the beginning, call `mergesort(0,n-1)`

Let $m = \lfloor (a+b)/2 \rfloor$ (middle position)
Call `mergesort(a,m)` and `mergesort(m+1,b)`

M(n) - Time to merge two sorted arrays of size n



We can do it in linear time! $\Theta(\mathbf{n})$ ($2n$ comparisons)

Back to the mergesort recurrence:

- **D(n)** - Time to partition an array of size n in two halves
- **M(n)** - Time to merge two sorted arrays of size n
- **T(n)** - Time for a MergeSort on an array of size n

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ D(n) + 2T(n/2) + M(n) & \text{if } n > 1 \end{cases}$$

becomes

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

For sufficiently small inputs, an algorithm generally takes constant time.
This means that for a small n, we have $T(n) = \Theta(1)$

For convenience, we can can generally **omit the boundary condition of the recurrence**.

Examples:

- Mergesort: $T(n) = 2T(n/2) + \Theta(n)$
- Binary Search: $T(n) = T(n/2) + \Theta(1)$
- Finding Maximum with tail recursion: $T(n) = T(n-1) + \Theta(1)$

How to solve recurrences like this?

We are going to talk about 4 methods:

- **Unrolling:** unroll the recurrence to obtain an expression (ex: summation) you can work with
- **Substitution:** guess the answer and prove by induction
- **Recursion Tree:** draw a tree representing the recursion and sum all the work done in the nodes
- **Master Theorem:** If the recurrence is of the form $\mathbf{aT(n/b) + cn^k}$, the answer follows a certain pattern

Some recurrences can be solved by **unrolling** them to get a summation:

$$T(n) = T(n-1) + \Theta(n) = \Theta(n) + \Theta(n-1) + \Theta(n-2) + \ldots + \Theta(1)$$

$$T(n) = T(n-1) + cn = cn + c(n-1) + c(n-2) + \ldots + c$$

There are n terms and each one is at most cn, so the summation is **at most** cn^2.

Similarly, since the first $n/2$ terms are each **at least** $cn/2$, this summation is at least $(n/2)(cn/2) = cn^2/4$.

Given this, the recurrence is $\Theta(n^2)$.

We could have also used arithmetic progressions:
$T(n) = c[n + (n-1) + \ldots + c] = c\frac{(n+c)n}{2} = cn^2 + c^2n/2$

Another possible method is to make a **guess and then prove** the guess correctness using **induction**

- "Strong" vs "Weak" induction
  - With **weak induction** we assume it is valid for n and then we prove $n+1$
  - With **strong induction** we assume it is valid for all $n_0 < n$ and we prove it for n.
- There are two "main" ways to use the substitution method:
  - We have an **exact guess**, with no "unknowns" (ex: $3n^2 - n$)
  - We only have **an idea of the class it belongs to** (ex: cn^2)
- How to prove that some $f(n)$ is $\Theta(g(n))$?
  - If we have an exact formula, just use it
  - Else, it may be "easier" to separately prove O and Ω
    - Ex: to prove O we can show it is less than $c.g(n)$
    - Ex: to prove Ω we can show it is more than $c.g(n)$

"Prove that $\mathbf{T(n) = T(n-1) + n}$ is $\mathbf{\Theta(n^2)}$"

Can we have an **exact guess**?

Let's assume $T(1) = 1$

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&= T(n-2) + (n-1) + n \\
&= T(n-3) + (n-2) + (n-1) + n \\
&= 1 + 2 + 3 + \ldots + (n-1) + n \\
&= \frac{(n+1)n}{2} \text{ (An arithmetic progression)}
\end{aligned}
$$

"Prove that $\mathbf{T(n) = T(n-1) + n}$ is $\mathbf{\Theta(n^2)}$"

Our (exact) guess is $\mathbf{\frac{(n+1)n}{2}}$

Now, let's try to prove by substituting.

Assuming it is true for $n-1$:

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&= \frac{n(n-1)}{2} + n \\
&= \frac{n^2-n}{2} + n \\
&= \frac{n^2-n+2n}{2} \\
&= \frac{n^2+n}{2} \\
&= \frac{(n+1)n}{2} \qquad \square \text{ (An we have proved our guess!)}
\end{aligned}
$$

"Prove that $\mathbf{T(n) = T(n/2) + 1}$ is $\mathbf{\Theta(\log_2 n)}$"
And if we don't have an exact guess?

Let's try to prove that $\mathbf{T(n) = \mathcal{O}(\log_2 n)}$

We basically need to prove that $T(n) \leq c \log_2 n$, with $n \geq n_0$, for a correct choice of c and n_0.

Let's assume $T(1) = 0$ and $T(2) = 1$. For these base cases:

- $T(1) \leq c \log_2 1$ for any c, because $\log_2 1 = 0$
- $T(2) \leq c \log_2 2$ is true as long as $c \geq 1$.

Now, assuming it is true for all $n' < n$:

$$\begin{aligned} T(n) &\leq c \log_2(n/2) + 1 \\ &= c(\log_2 n - \log_2 2) + 1 \\ &= c \log_2 n - c + 1 \\ &\leq c \log_2 n, \text{ as long as } c \geq 1 \quad \square \text{ (We proved } \mathbf{T(n) = \mathcal{O}(\log_2 n)}) \end{aligned}$$

"Prove that $\mathbf{T(n) = T(n/2) + 1}$ is $\mathbf{\Theta(\log_2 n)}$"
Let's try to prove that $\mathbf{T(n) = \Omega(\log_2 n)}$

We basically need to prove that $T(n) \geq c \log_2 n$, with $n \geq n_0$, for a correct choice of c and n_0.

Let's assume $T(1) = 0$ and $T(2) = 1$. For these base cases:

- $T(1) \geq c \log_2 1$ for any c, because $\log_2 1 = 0$
- $T(2) \geq c \log_2 2$ is true as long as $c \leq 1$.

Now, assuming it is true for all $n' < n$:

$$\begin{aligned} T(n) &\geq c \log_2(n/2) + 1 \\ &= c(\log_2 n - \log_2 2) + 1 \\ &= c \log_2 n - c + 1 \\ &\geq c \log_2 n, \text{ as long as } c \leq 1 \end{aligned}$$

$\square$ (We proved $\mathbf{T(n) = \Omega(\log_2 n)}$)

$T(n) = \mathcal{O}(\log_2 n))$ and $T(n) = \Omega(\log_2 n) \rightarrow \mathbf{T(n) = \Theta(\log_2 n)})$

If the guess is wrong, often we will gain clues for a better guess.

Recurrence to solve: $T(n) = 4T(n/4) + n$

Guess #1: $T(n) \leq cn$ (which would mean $T(n) = \mathcal{O}(n)$)

Attempt to prove Guess #1:
If $T(1) = c$, then the base case is true. For the rest of the induction, assuming it is true for $n' < n$, we can substitute using $n' = n/4$:

$$\begin{aligned} T(n) &\leq 4(cn/4) + n \\ &= cn + n \\ &= (c+1)n \qquad \text{but } (c+1)n \text{ is never } \leq cn \text{ for a positive } c \text{ (the guess is wrong!)} \end{aligned}$$

We guess that we night need an higher function than simply $\mathcal{O}(n)$

Recurrence to solve: $T(n) = 4T(n/4) + n$

Guess #2: $T(n) \leq n \log_4 n$
(I'm proving a more tight bound than simply $cn \log_4 n$)

Attempt to prove Guess #2:
If $T(1) = 1$, then the base case is true. For the rest of the induction, assuming it is true for $n' < n$, we can substitute using $n' = n/4$:

$$\begin{aligned} T(n) &\leq 4[(n/4)\log_4(n/4)] + n \\ &= n\log_4(n/4) + n \\ &= n\log_4(n) - n + n \\ &= n\log_4(n) \quad \square \text{ [correct guess! In fact, } T(n) = \Theta(n\log_4 n)\text{]} \end{aligned}$$

Sometimes you might correctly guess an asymptotic bound on the solution of a recurrence, but somehow the *math fails to work out in the induction*.

The problem frequently turns out to be that the **inductive assumption is not strong enough** to prove the detailed bound. If you **revise the guess by subtracting a lower-order term** when you hit such a snag, the math often goes through.

Let's observe an example of this:
Recurrence to solve: $T(n) = 4T(n/2) + n$

As you will see later, $T(n) = \Theta(n^2)$

Let's try to prove that directly.

Recurrence to solve: $T(n) = 4T(n/2) + n$

Guess #1: $T(n) \leq cn^2$

Attempt to prove Guess #1:

If $T(1) = 1$, then the base case is true as long as $c \leq 1$.

Now, assuming it is true for $n' < n$

$$\begin{aligned} T(n) &\leq 4[c(n/2)^2] + n \\ &= cn^2 + n \quad \text{[which is not } \leq cn^2 \text{ for any positive } n\text{]} \end{aligned}$$

Although the bound is correct, the math does not work out...

We need a tighter bound to form a **stronger induction hypothesis**.

Let's **subtract a lower order-term** and try $T(n) \leq c_1 n^2 - c_2 n$

Recurrence to solve: $T(n) = 4T(n/2) + n$

Guess #2: $T(n) \leq c_1 n^2 - c_2 n$

Attempt to prove Guess #2:
If $T(1) = 1$, then the base case is true as long as $c_1 - c_2 \leq 1$

Now, assuming it is true for $n' < n$

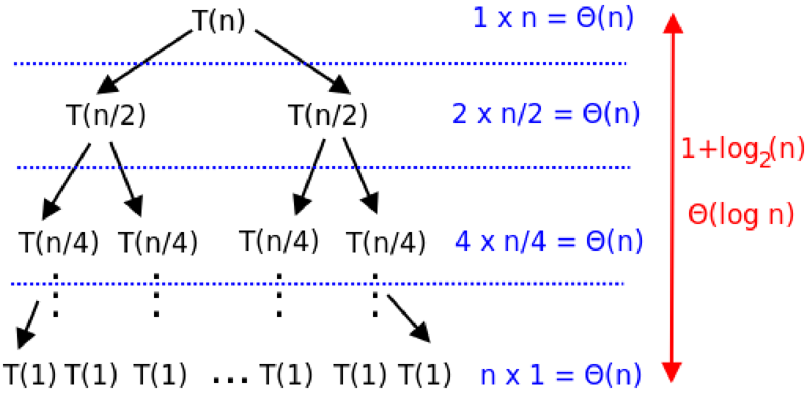
$$\begin{aligned} T(n) &\leq 4[c_1(n/2)^2 - c_2(n/2)] + n \\ &= c_1 n^2 - 2c_2 n + n \\ &= c_1 n^2 - c_2 n \quad \text{[correct guess!]} \end{aligned}$$

Another method is to **draw a recursion tree** and analyse it, by summing all the work in the tree nodes.

This method could be also used to get a good guess which we could then prove by induction.

Let us try it out with MergeSort: $T(n) = 2(n/2) + n$

(for a cleaner explanation we will assume $n = 2^k$,
but the results holds for any n)

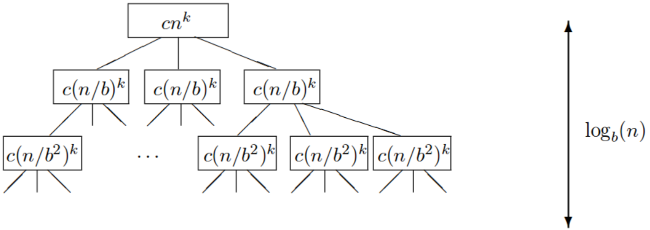


Summing everything we get that **MergeSort** is $\mathbf{\Theta(n \log_2 n)}$

We can use the **master theorem** for recurrences of the following form:

$$\mathbf{T(n) = aT(n/b) + cn^k}$$

This is well suited for divide and conquer recurrences and corresponds to an algorithm that divides the problem into **a** pieces of size **n/b** and takes $\mathbf{cn^k}$ time for partitioning+combining.



In the mergesort case, $a = 2, b = 2, k = 1$.

Ex. 3.7: Solve using recursion trees (assume $T(0)$ is a constant)

1. $T(n) = k + T(n-1)$ where k is a constant
2. $T(n) = k + T(n/2)$ where k is a constant
3. $T(n) = k + 2 * T(n/2)$ where k is a constant
4. $T(n) = n + T(n-1)$
5. $T(n) = n + T(n/2)$
6. $T(n) = n + 2 * T(n/2)$

Exercises

[more exercises: maxSumR; hanoi; heightBT] [maybe use PR's exercises too.]

slides by Pedro Ribeiro, slides 2

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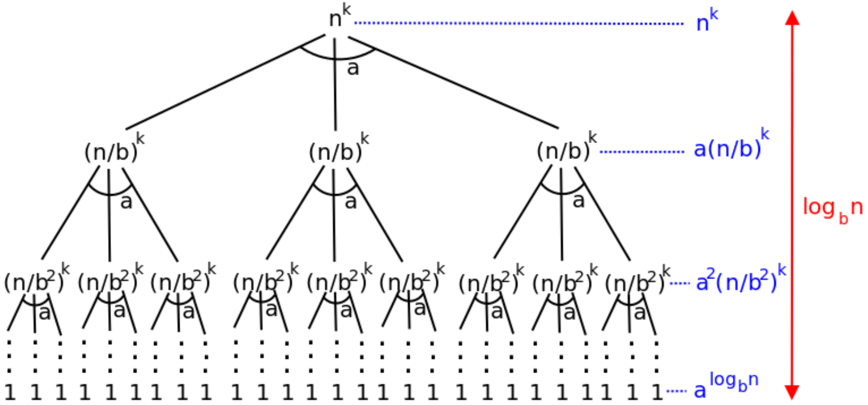
$\mathbf{aT(n/b) + n^k}$ (I assume $c = 1$ for a cleaner explanation)



Master Theorem

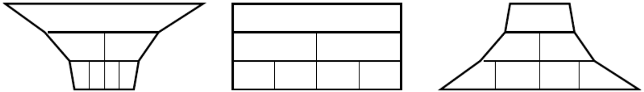
Intuition behind it

- **Root (first level):** n^k
- **Depth i (intermediate):** $a^i(n/b^i)^k = a^i/b^{ik}n^k = (a/b^k)^i n^k$
- **Leafs (last level):** $a^{\log_b n} = n^{\log_b a}$

So the weight of depth i is: $\mathbf{(a/b^k)^i n^k}$

(1) $a < b^k$	implies that	a/b^k is lower than 1	(weight is shrinking)
(2) $a = b^k$	implies that	a/b^k is equal to 1	(weight is constant)
(3) $a > b^k$	implies that	a/b^k is higher than 1	(weight is growing)

- (1) The time is dominated by the **top level**
- (2) The time is (uniformly) **distributed** along the recursion tree
- (3) The time is dominated by the **last level**



Master Theorem

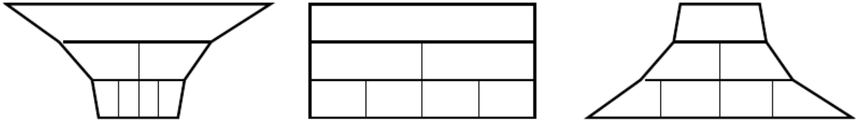
Master Theorem - A practical version

A recurrence $\mathbf{aT(n/b) + cn^k}$ $(a \geq 1, b > 1, c$ and k are constants$)$ solves to:

$$
\begin{array}{lll}
(1) & T(n) = \Theta(n^k) & \text{if } a < b^k \\
(2) & T(n) = \Theta(n^k \log n) & \text{if } a = b^k \\
(3) & T(n) = \Theta(n^{\log_b a}) & \text{if } a > b^k
\end{array}
$$

If you think on the recursion tree, intuitively, these 3 cases correspond to:

- (1) The time is dominated by the **top level**
- (2) The time is (uniformly) **distributed** along the recursion tree
- (3) The time is dominated by the **last level**



Master Theorem

Master Theorem - A practical version

A recurrence $\mathbf{aT(n/b) + cn^k}$ ($a \geq 1, b > 1, c$ and k are constants) solves to:

$$\begin{array}{lll} (1) & T(n) = \Theta(n^k) & \text{if } a < b^k \\ (2) & T(n) = \Theta(n^k \log n) & \text{if } a = b^k \\ (3) & T(n) = \Theta(n^{\log_b a}) & \text{if } a > b^k \end{array}$$

Example of Case (1):

$T(n) = 2T(n/2) + n^2$

$a = 2, b = 2, k = 2,\ a < b^k$ since $2 < 4$.

The recurrence solves to $\mathbf{\Theta(n^2)}$

Master Theorem

Master Theorem - A practical version

A recurrence $\mathbf{aT(n/b) + cn^k}$ ($a \geq 1, b > 1, c$ and k are constants) solves to:

$$\begin{array}{lll} (1) & T(n) = \Theta(n^k) & \text{if } a < b^k \\ (2) & T(n) = \Theta(n^k \log n) & \text{if } a = b^k \\ (3) & T(n) = \Theta(n^{\log_b a}) & \text{if } a > b^k \end{array}$$

Example of Case (2):

$T(n) = 2T(n/2) + n$ *(ex: mergesort)*

$a = 2, b = 2, k = 1,\ a = b^k$ since $2 = 2$.

The recurrence solves to $\mathbf{\Theta(n \log n)}$ (as we already knew).

Master Theorem

Master Theorem - A practical version

A recurrence $\mathbf{aT(n/b) + cn^k}$ ($a \geq 1, b > 1, c$ and k are constants) solves to:

$$\begin{array}{lll} (1) & T(n) = \Theta(n^k) & \text{if } a < b^k \\ (2) & T(n) = \Theta(n^k \log n) & \text{if } a = b^k \\ (3) & T(n) = \Theta(n^{\log_b a}) & \text{if } a > b^k \end{array}$$

Example of Case (3):

$T(n) = 2T(n/2) + 1$

$a = 2, b = 2, k = 0,\ a > b^k$ since $2 > 1$.

The recurrence solves to $\mathbf{\Theta(n)}$

Examples:

(1) $T(n) = 2T(n/2) + n^2 = \Theta(n^2)$
$n^2 + n^2/2 + n^2/4 + \ldots + n \leftarrow$ (n^2 dominates, i.e., the root)

(2) $T(n) = 2T(n/2) + n = \Theta(n \log n)$
$n + n + \ldots + n \leftarrow$ (distributed among all levels)

(3) $T(n) = 2T(n/2) + 1 = \Theta(n)$
$1 + 2 + 4 + \ldots + n \leftarrow$ (n dominates, i.e., the leaf)

Master Theorem

For the sake of completeness, here is the master theorem version presented in the book **"Introduction to Algorithms"**.

Master Theorem

A more general version A recurrence $\mathbf{aT(n/b) + f(n)}$ ($a \geq 1, b > 1$ are constants) solves to:

(1) If $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
(2) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
(3) If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

(cases 1 and 3 are inverted in relation to the practical version I've shown)

Exercises

Ex. 3.8: Calculate the asymptotic complexity of the 3 exercises above (`maxSumR`, `hanoi`, and `heightBT`), indicating which case you used.