6. First Order Logic – Natural Deduction – Exercises

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Recalling Natural Deduction Rules in First Order Logic

Which are the new rules (on top of Propositional Logic)?

Elimination rule for $\forall$

If we know that $\forall x, \varphi$ holds, then we can conclude that φ holds for a specific term t

$$\frac{\forall x\, \varphi}{\varphi[t/x]}\ \forall\mathbf{E}$$

Introduction rule for $\forall$

If we assume some term t and we are able to prove that $\varphi[t/x]$ then we can conclude that $\forall x, \varphi$.

$$\frac{\begin{array}{c}[t] \\ \vdots \\ \varphi[t/x]\end{array}}{\forall x\, \varphi[t/x]}\ \forall\mathbf{I}$$

Which are the new rules (on top of Propositional Logic)?

Elimination rule for $\exists$

If we know that $\exists x, \varphi$ holds, and if assuming term t and $\varphi[t/x]$ we can deduce ψ, then we can prove ψ overall.

$$\dfrac{\exists x\, \varphi \qquad\qquad \begin{array}{c}[t \quad \varphi[t/x]] \\ \vdots \\ \psi\end{array}}{\psi}\ \exists\mathbf{E}$$

Introduction rule for $\exists$

If we assume some term t and we are able to prove that $\varphi[t/x]$ then we can conclude that $\forall x, \varphi$.

$$\dfrac{\varphi[t/x]}{\exists x, \varphi}\ \exists\mathbf{I}$$

Which are the new rules (on top of Propositional Logic)?

Elimination rule for $=$

If we know that two terms t_1 and t_2 are equal and that $\varphi[t_1/x]$ holds, then $\varphi[t_1/x]$ must also hold.

$$\frac{t_1 = t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]} =\mathbf{E}$$

Introduction rule for $=$

If we assume some term t and we are able to prove that $\varphi[t/x]$ then we can conclude that $\forall x, \varphi$.

$$\frac{}{t = t} =\mathbf{I}$$

Practical Exercises

Ex1. Build a proof of $\forall x(R(x) \land Q(x)) \vdash \forall x R(x) \land \forall x Q(x)$

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1 $\forall x(R(x) \land Q(x))$

Ex1. Build a proof of $\forall x(R(x) \land Q(x)) \vdash \forall x R(x) \land \forall x Q(x)$

1	$\forall x(R(x) \land Q(x))$	
2	$v \mid R(v) \land Q(v)$	$\forall$E, 1

Ex1. Build a proof of $\forall x(R(x) \land Q(x)) \vdash \forall x R(x) \land \forall x Q(x)$

1		$\forall x(R(x) \land Q(x))$	
2	v	$R(v) \land Q(v)$	$\forall$E, 1
3		$R(v)$	$\land$E, 2

Ex1. Build a proof of $\forall x(R(x) \wedge Q(x)) \vdash \forall x R(x) \wedge \forall x Q(x)$

1		$\forall x(R(x) \wedge Q(x))$	
2	v	$R(v) \wedge Q(v)$	$\forall$E, 1
3		$R(v)$	$\wedge$E, 2
4		$\forall x R(x)$	$\forall$I, 2–3

Ex1. Build a proof of $\forall x(R(x) \land Q(x)) \vdash \forall x R(x) \land \forall x Q(x)$

1		$\forall x(R(x) \land Q(x))$	
2	v	$R(v) \land Q(v)$	$\forall$E, 1
3		$R(v)$	$\land$E, 2
4		$\forall x R(x)$	$\forall$I, 2–3
5	v	$R(v) \land Q(v)$	$\forall$E, 1

Ex1. Build a proof of $\forall x(R(x) \land Q(x)) \vdash \forall x R(x) \land \forall x Q(x)$

1		$\forall x(R(x) \land Q(x))$	
2	v	$R(v) \land Q(v)$	$\forall$E, 1
3		$R(v)$	$\land$E, 2
4		$\forall x R(x)$	$\forall$I, 2–3
5	v	$R(v) \land Q(v)$	$\forall$E, 1
6		$Q(v)$	$\land$E, 5

Ex1. Build a proof of $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$

1		$\forall x(R(x) \wedge Q(x))$	
2	v	$R(v) \wedge Q(v)$	$\forall$E, 1
3		$R(v)$	$\wedge$E, 2
4		$\forall xR(x)$	$\forall$I, 2–3
5	v	$R(v) \wedge Q(v)$	$\forall$E, 1
6		$Q(v)$	$\wedge$E, 5
7		$\forall xQ(x)$	$\forall$I, 5–7

Ex1. Build a proof of $\forall x(R(x) \wedge Q(x)) \vdash \forall x R(x) \wedge \forall x Q(x)$

1		$\forall x(R(x) \wedge Q(x))$	
2	v	$R(v) \wedge Q(v)$	$\forall$E, 1
3		$R(v)$	$\wedge$E, 2
4		$\forall x R(x)$	$\forall$I, 2–3
5	v	$R(v) \wedge Q(v)$	$\forall$E, 1
6		$Q(v)$	$\wedge$E, 5
7		$\forall x Q(x)$	$\forall$I, 5–7
8		$\forall x R(x) \wedge \forall x Q(x)$	$\wedge$I, 4, 8

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

$$
\begin{array}{ll}
1 & \forall x(R(x) \to Q(x)) \\
\hline
\end{array}
$$

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

1 | $\forall x(R(x) \to Q(x))$

2 | | $\forall x R(x)$

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

1	$\forall x(R(x) \to Q(x))$	
2	$\quad \mid \ \forall x R(x)$	
3	$\quad \mid \ v \mid R(v) \to Q(v)$	$\forall$E, 1

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

1		$\forall x(R(x) \to Q(x))$	
2		$\forall x R(x)$	
3	v	$R(v) \to Q(v)$	$\forall$E, 1
4		$R(v)$	$\forall$E, 2

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

1		$\forall x(R(x) \to Q(x))$	
2		$\forall x R(x)$	
3	v	$R(v) \to Q(v)$	$\forall$E, 1
4		$R(v)$	$\forall$E, 2
5		$Q(v)$	$\Rightarrow$E, 3, 4

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

1	$\mid \forall x(R(x) \to Q(x))$	
	$\mid \underline{\qquad\qquad\qquad}$	
2	$\mid \quad \mid \forall x R(x)$	
	$\mid \quad \mid \underline{\qquad\qquad}$	
3	$\mid \quad \mid \; v \mid R(v) \to Q(v)$	$\forall$E, 1
4	$\mid \quad \mid \quad\; \mid R(v)$	$\forall$E, 2
5	$\mid \quad \mid \quad\; \mid Q(v)$	$\Rightarrow$E, 3, 4
6	$\mid \quad \mid \forall x Q(x)$	$\forall$I, 3–5

Ex2. Build a proof of $\forall x(R(x) \to Q(x)) \vdash \forall x R(x) \to \forall x Q(x)$

1		$\forall x(R(x) \to Q(x))$	
2		$\forall x R(x)$	
3	v	$R(v) \to Q(v)$	∀E, 1
4		$R(v)$	∀E, 2
5		$Q(v)$	⇒E, 3, 4
6		$\forall x Q(x)$	∀I, 3–5
7		$\forall x P(x) \to \forall x Q(x)$	⇒I, 2–6

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

$$
\begin{array}{r|l}
1 & \exists x(R(x) \to Q(x)) \\ \hline
\end{array}
$$

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

1 $\exists x(R(x) \to Q(x))$

2 $\quad \exists x R(x)$

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

1		$\exists x(R(x) \to Q(x))$
2		$\exists x R(x)$
3	v	$R(v) \to Q(v)$

Ex3. Build a proof of $\exists x(R(x) \rightarrow Q(x)) \vdash \exists x R(x) \rightarrow \exists x Q(x)$

1 $\mid \exists x(R(x) \rightarrow Q(x))$

2 $\mid \quad \mid \exists x R(x)$

3 $\mid \quad \mid v \mid R(v) \rightarrow Q(v)$

4 $\mid \quad \mid \quad \mid \quad \mid R(v)$

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

1	$\mid\quad \exists x(R(x) \to Q(x))$	
2	$\mid\quad\mid\quad \exists x R(x)$	
3	$\mid\quad\mid\quad v \;\mid\quad R(v) \to Q(v)$	
4	$\mid\quad\mid\quad\mid\quad\mid\quad R(v)$	
5	$\mid\quad\mid\quad\mid\quad\mid\quad Q(v)$	⇒E, 3, 4

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

1	$\mid \underline{\exists x(R(x) \to Q(x))}$	
2	$\mid \quad \mid \underline{\exists x R(x)}$	
3	$\mid \quad \mid \; v \mid \underline{R(v) \to Q(v)}$	
4	$\mid \quad \mid \quad\; \mid \quad \mid \underline{R(v)}$	
5	$\mid \quad \mid \quad\; \mid \quad \mid Q(v)$	$\Rightarrow$E, 3, 4
6	$\mid \quad \mid \quad\; \mid \quad \mid \exists x Q(x)$	$\exists$I, 5

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

1	$\exists x(R(x) \to Q(x))$	
2	$\exists x R(x)$	
3	v $R(v) \to Q(v)$	
4	$R(v)$	
5	$Q(v)$	$\Rightarrow$E, 3, 4
6	$\exists x Q(x)$	$\exists$I, 5
7	$\exists x Q(x)$	$\exists$E, 2, 4–6

Ex3. Build a proof of $\exists x(R(x) \to Q(x)) \vdash \exists x R(x) \to \exists x Q(x)$

1		$\exists x(R(x) \to Q(x))$	
2		$\exists x R(x)$	
3	v	$R(v) \to Q(v)$	
4		$R(v)$	
5		$Q(v)$	$\Rightarrow$E, 3, 4
6		$\exists x Q(x)$	$\exists$I, 5
7		$\exists x Q(x)$	$\exists$E, 2, 4–6
8		$\exists x Q(x)$	$\exists$E, 1, 3–7

Ex3. Build a proof of $\exists x(R(x) \rightarrow Q(x)) \vdash \exists x R(x) \rightarrow \exists x Q(x)$

1		$\exists x(R(x) \rightarrow Q(x))$	
2		$\exists x R(x)$	
3	v	$R(v) \rightarrow Q(v)$	
4		$R(v)$	
5		$Q(v)$	$\Rightarrow$E, 3, 4
6		$\exists x Q(x)$	$\exists$I, 5
7		$\exists x Q(x)$	$\exists$E, 2, 4–6
8		$\exists x Q(x)$	$\exists$E, 1, 3–7
9		$\exists x P(x) \rightarrow \exists x Q(x)$	$\Rightarrow$I, 2–8

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1 $\exists x \neg Q(x)$

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1 | $\exists x \neg Q(x)$

2 | | $\forall x Q(x)$

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1	$\exists x \neg Q(x)$	
2	$\forall x Q(x)$	
3	v	$\neg Q(v)$

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1	$\exists x \neg Q(x)$			
2		$\forall x Q(x)$		
3		v	$\neg Q(v)$	
4			$Q(v)$	$\forall$E, 2

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1		$\exists x \neg Q(x)$	
2		$\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$Q(v)$	$\forall$E, 2
5		$\bot$	$\bot$I, 3, 4

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1		$\exists x \neg Q(x)$	
2		$\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$Q(v)$	∀E, 2
5		$\bot$	⊥I, 3, 4
6		$\bot$	∃E, 1, 3–5

Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1	$\exists x \neg Q(x)$			
2		$\forall x Q(x)$		
3		v	$\neg Q(v)$	
4			$Q(v)$	$\forall$E, 2
5			$\bot$	$\bot$I, 3, 4
6		$\bot$		$\exists$E, 1, 3–5
7	$\neg \forall x Q(x)$			$\neg$I, 2–6

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

$$\begin{array}{ll} 1 & \neg\exists x \neg Q(x) \end{array}$$

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1 $\neg\exists x \neg Q(x)$

2 $\neg\forall x Q(x)$

Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall x Q(x)$

1 $\neg\exists x\neg Q(x)$

2 $\neg\forall x Q(x)$

3 v $\neg Q(v)$

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1		$\neg\exists x \neg Q(x)$	
2		$\neg\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$\exists x \neg Q(x)$	∃I, 3

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1		$\neg\exists x \neg Q(x)$	
2		$\neg\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$\exists x \neg Q(x)$	$\exists$I, 3
5		$\bot$	$\bot$I, 1, 4

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1		$\neg\exists x \neg Q(x)$	
2		$\neg\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$\exists x \neg Q(x)$	$\exists$I, 3
5		$\bot$	$\bot$I, 1, 4
6		$\neg\neg Q(v)$	$\neg$I, 3–5

Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall x Q(x)$

1		$\neg\exists x\neg Q(x)$	
2		$\neg\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$\exists x\neg Q(x)$	$\exists$I, 3
5		$\bot$	$\bot$I, 1, 4
6		$\neg\neg Q(v)$	$\neg$I, 3–5
7		$Q(v)$	$\neg$E, 6

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1		$\neg\exists x \neg Q(x)$	
2		$\neg\forall x Q(x)$	
3	v	$\neg Q(v)$	
4		$\exists x \neg Q(x)$	∃I, 3
5		$\bot$	⊥I, 1, 4
6		$\neg\neg Q(v)$	¬I, 3–5
7		$Q(v)$	¬E, 6
8		$\forall x Q(x)$	∀I, 3–7

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1	$\mid \underline{\neg\exists x \neg Q(x)}$	
2	$\mid \quad \mid \underline{\neg\forall x Q(x)}$	
3	$\mid \quad \mid v \mid \quad \mid \underline{\neg Q(v)}$	
4	$\mid \quad \mid \quad \mid \quad \mid \exists x \neg Q(x)$	∃I, 3
5	$\mid \quad \mid \quad \mid \quad \mid \bot$	⊥I, 1, 4
6	$\mid \quad \mid \quad \mid \neg\neg Q(v)$	¬I, 3–5
7	$\mid \quad \mid \quad \mid Q(v)$	¬E, 6
8	$\mid \quad \mid \forall x Q(x)$	∀I, 3–7
9	$\mid \quad \mid \bot$	⊥I, 2, 8

Ex5. Build a proof of $\neg\exists x \neg Q(x) \vdash \forall x Q(x)$

1	$\vert\ \neg\exists x\neg Q(x)$	
2	$\vert\ \vert\ \neg\forall x Q(x)$	
3	$\vert\ \vert\ v\ \vert\ \vert\ \neg Q(v)$	
4	$\vert\ \vert\ \ \vert\ \vert\ \exists x\neg Q(x)$	∃I, 3
5	$\vert\ \vert\ \ \vert\ \vert\ \bot$	⊥I, 1, 4
6	$\vert\ \vert\ \ \vert\ \neg\neg Q(v)$	¬I, 3–5
7	$\vert\ \vert\ \ \vert\ Q(v)$	¬E, 6
8	$\vert\ \vert\ \forall x Q(x)$	∀I, 3–7
9	$\vert\ \vert\ \bot$	⊥I, 2, 8
10	$\vert\ \neg\neg\forall x Q(x)$	¬I, 2–9

Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall x Q(x)$

1	$\vert\ \neg\exists x\neg Q(x)$	
2	$\vert\ \vert\ \neg\forall x Q(x)$	
3	$\vert\ \vert\ v\ \vert\ \vert\ \neg Q(v)$	
4	$\vert\ \vert\ \ \ \vert\ \vert\ \exists x\neg Q(x)$	$\exists$I, 3
5	$\vert\ \vert\ \ \ \vert\ \vert\ \bot$	$\bot$I, 1, 4
6	$\vert\ \vert\ \ \ \vert\ \neg\neg Q(v)$	$\neg$I, 3–5
7	$\vert\ \vert\ \ \ \vert\ Q(v)$	$\neg$E, 6
8	$\vert\ \vert\ \forall x Q(x)$	$\forall$I, 3–7
9	$\vert\ \vert\ \bot$	$\bot$I, 2, 8
10	$\vert\ \neg\neg\forall x Q(x)$	$\neg$I, 2–9
11	$\vert\ \forall x Q(x)$	$\neg$E, 10

Ex6. Build a proof of $\forall x Q(x) \vdash \neg \exists x \neg Q(x)$

Ex6. Build a proof of $\forall x Q(x) \vdash \neg \exists x \neg Q(x)$

$$\begin{array}{r|l} 1 & \forall x Q(x) \\ \hline \end{array}$$

Ex6. Build a proof of $\forall x Q(x) \vdash \neg\exists x \neg Q(x)$

1 $\forall x Q(x)$

2 $\quad \exists x \neg Q(x)$

Ex6. Build a proof of $\forall x Q(x) \vdash \neg \exists x \neg Q(x)$

1	$\forall x Q(x)$	
2	$\quad \exists x \neg Q(x)$	
3	$\quad \neg \forall x Q(x)$	Ex4., 1

Ex6. Build a proof of $\forall x Q(x) \vdash \neg\exists x \neg Q(x)$

1	$\forall x Q(x)$	
2	$\exists x \neg Q(x)$	
3	$\neg\forall x Q(x)$	Ex4., 1
4	$\bot$	$\bot$I, 1, 3

Ex6. Build a proof of $\forall x Q(x) \vdash \neg \exists x \neg Q(x)$

1	$\forall x Q(x)$	
2	$\quad \exists x \neg Q(x)$	
3	$\quad \neg \forall x Q(x)$	Ex4., 1
4	$\quad \bot$	$\bot$I, 1, 3
5	$\neg \exists x Q(x)$	$\neg$I, 2–4

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

$$
\begin{array}{l|l}
1 & \exists x(P(x) \land Q(x))
\end{array}
$$

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

$$
\begin{array}{r|l}
1 & \exists x(P(x) \land Q(x)) \\
2 & \forall x(P(x) \to R(x)) \\
\hline
\end{array}
$$

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

1		$\exists x(P(x) \land Q(x))$
2		$\forall x(P(x) \to R(x))$
3	v	$P(v) \land Q(v)$

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

1		$\exists x(P(x) \land Q(x))$	
2		$\forall x(P(x) \to R(x))$	
3	v	$P(v) \land Q(v)$	
4		$P(v)$	$\land$E, 3

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

1		$\exists x(P(x) \land Q(x))$	
2		$\forall x(P(x) \to R(x))$	
3	v	$P(v) \land Q(v)$	
4		$P(v)$	$\land$E, 3
5		$P(v) \to R(v)$	$\forall$E, 2

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

1		$\exists x(P(x) \land Q(x))$	
2		$\forall x(P(x) \to R(x))$	
3	v	$P(v) \land Q(v)$	
4		$P(v)$	$\land$E, 3
5		$P(v) \to R(v)$	$\forall$E, 2
6		$R(v)$	$\Rightarrow$E, 4, 5

Ex7. Build a proof of $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
3	v	$P(v) \wedge Q(v)$	
4		$P(v)$	$\wedge$E, 3
5		$P(v) \rightarrow R(v)$	$\forall$E, 2
6		$R(v)$	$\Rightarrow$E, 4, 5
7		$Q(v)$	$\wedge$E, 3

Ex7. Build a proof of $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \wedge P(x))$

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \to R(x))$	
3	v	$P(v) \wedge Q(v)$	
4		$P(v)$	$\wedge$E, 3
5		$P(v) \to R(v)$	$\forall$E, 2
6		$R(v)$	$\Rightarrow$E, 4, 5
7		$Q(v)$	$\wedge$E, 3
8		$R(v) \wedge Q(v)$	$\wedge$I, 6, 7

Ex7. Build a proof of $\exists x(P(x) \land Q(x)), \forall x(P(x) \to R(x)) \vdash \exists x(R(x) \land P(x))$

1	$\exists x(P(x) \land Q(x))$	
2	$\forall x(P(x) \to R(x))$	
3	v $P(v) \land Q(v)$	
4	$P(v)$	$\land$E, 3
5	$P(v) \to R(v)$	$\forall$E, 2
6	$R(v)$	$\Rightarrow$E, 4, 5
7	$Q(v)$	$\land$E, 3
8	$R(v) \land Q(v)$	$\land$I, 6, 7
9	$\exists x(R(x) \land Q(x))$	$\exists$I, 8

Ex7. Build a proof of $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
3	v	$P(v) \wedge Q(v)$	
4		$P(v)$	$\wedge$E, 3
5		$P(v) \rightarrow R(v)$	$\forall$E, 2
6		$R(v)$	$\Rightarrow$E, 4, 5
7		$Q(v)$	$\wedge$E, 3
8		$R(v) \wedge Q(v)$	$\wedge$I, 6, 7
9		$\exists x(R(x) \wedge Q(x))$	$\exists$I, 8
10		$\exists x(R(x) \wedge Q(x))$	$\exists$E, 1, 3–9

Ex8. Build a proof of $\forall x \forall y (x = y \to f(x) = f(y))$

Ex8. Build a proof of $\forall x \forall y (x = y \to f(x) = f(y))$

$$1 \quad \Big| \; u \; \Big| \; v \; \Big| \quad \underline{\Big| \; u = v \;}$$

Ex8. Build a proof of $\forall x \forall y(x = y \to f(x) = f(y))$

$$\begin{array}{l|l|l|l|ll}
1 & u & v & & \underline{u = v} & \\
2 & & & & & \multicolumn{1}{|l}{\underline{\neg(f(u) = f(v))}}
\end{array}$$

Ex8. Build a proof of $\forall x \forall y (x = y \to f(x) = f(y))$

$$
\begin{array}{l|l|l|l|l|ll}
1 & u & v & & \underline{u = v} & & \\
2 & & & & & \vert\ \underline{\neg(f(u) = f(v))} & \\
3 & & & & & \vert\ f(u) = f(u) & =\text{I}
\end{array}
$$

Ex8. Build a proof of $\forall x \forall y(x = y \rightarrow f(x) = f(y))$

1	u	v	$u = v$		
2				$\neg(f(u) = f(v))$	
3				$f(u) = f(u)$	=I
4				$f(u) = f(v)$	=E, 1, 3

Ex8. Build a proof of $\forall x \forall y(x = y \to f(x) = f(y))$

1	u	v	$u = v$	
2			$\neg(f(u) = f(v))$	
3			$f(u) = f(u)$	$=$I
4			$f(u) = f(v)$	$=$E, 1, 3
5			$\bot$	$\bot$I, 2–4

Ex8. Build a proof of $\forall x \forall y (x = y \to f(x) = f(y))$

1	u	v	$u = v$	
2			$\mid\ \neg(f(u) = f(v))$	
3			$\mid\ f(u) = f(u)$	=I
4			$\mid\ f(u) = f(v)$	=E, 1, 3
5			$\mid\ \bot$	⊥I, 2–4
6			$\neg\neg(f(u) = f(v))$	¬I, 2–5

Ex8. Build a proof of $\forall x \forall y (x = y \to f(x) = f(y))$

1	u	v	$u = v$	
2			$\neg(f(u) = f(v))$	
3			$f(u) = f(u)$	$=$I
4			$f(u) = f(v)$	$=$E, 1, 3
5			$\bot$	$\bot$I, 2–4
6			$\neg\neg(f(u) = f(v))$	$\neg$I, 2–5
7			$f(u) = f(v)$	$\neg$E, 6

Ex8. Build a proof of $\forall x \forall y(x = y \to f(x) = f(y))$

1	u	v	$u = v$	
2			$\neg(f(u) = f(v))$	
3			$f(u) = f(u)$	=I
4			$f(u) = f(v)$	=E, 1, 3
5			$\bot$	⊥I, 2–4
6			$\neg\neg(f(u) = f(v))$	¬I, 2–5
7			$f(u) = f(v)$	¬E, 6
8			$u = v \to f(u) = f(v)$	⇒I, 1–7

Ex8. Build a proof of $\forall x \forall y(x = y \to f(x) = f(y))$

1	u	v	$u = v$	
2			$\neg(f(u) = f(v))$	
3			$f(u) = f(u)$	$=$I
4			$f(u) = f(v)$	$=$E, 1, 3
5			$\bot$	$\bot$I, 2–4
6			$\neg\neg(f(u) = f(v))$	$\neg$I, 2–5
7			$f(u) = f(v)$	$\neg$E, 6
8			$u = v \to f(u) = f(v)$	$\Rightarrow$I, 1–7
9			$\forall y(u = y \to f(u) = f(y))$	$\forall$I, 8

Ex8. Build a proof of $\forall x \forall y(x = y \to f(x) = f(y))$

1	$\vert\; u \;\vert\; v \;\vert\; \quad \vert\; \underline{u = v}$	
2	$\vert \quad \vert \quad \vert \quad \vert \quad \vert\; \underline{\neg(f(u) = f(v))}$	
3	$\vert \quad \vert \quad \vert \quad \vert \quad \vert\; f(u) = f(u)$	=I
4	$\vert \quad \vert \quad \vert \quad \vert \quad \vert\; f(u) = f(v)$	=E, 1, 3
5	$\vert \quad \vert \quad \vert \quad \vert \quad \vert\; \bot$	⊥I, 2–4
6	$\vert \quad \vert \quad \vert \quad \vert\; \neg\neg(f(u) = f(v))$	¬I, 2–5
7	$\vert \quad \vert \quad \vert \quad \vert\; f(u) = f(v)$	¬E, 6
8	$\vert \quad \vert \quad \vert\; u = v \to f(u) = f(v)$	⇒I, 1–7
9	$\vert \quad \vert\; \forall y(u = y \to f(u) = f(y))$	∀I, 8
10	$\vert\; \forall x \forall y(x = y \to f(x) = f(y))$	∀I, 9

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

$$\begin{array}{r|l} 1 & \forall x P(a,x,x) \end{array}$$

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

$$
\begin{array}{r|l}
1 & \forall x P(a,x,x) \\
2 & \forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z))) \\
\hline
\end{array}
$$

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

1	$\forall x P(a,x,x)$	
2	$\forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z)))$	
3	$P(a,a,a)$	$\forall$E, 1

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

1	$\forall x P(a,x,x)$	
2	$\forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z)))$	
3	$P(a,a,a)$	$\forall$E, 1
4	$\forall y \forall z (P(a,y,z) \rightarrow P(f(a),y,f(z)))$	$\forall$E, 2

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

1	$\forall x P(a,x,x)$	
2	$\forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z)))$	
3	$P(a,a,a)$	$\forall$E, 1
4	$\forall y \forall z (P(a,y,z) \rightarrow P(f(a),y,f(z)))$	$\forall$E, 2
5	$\forall z (P(a,a,z) \rightarrow P(f(a),a,f(z)))$	$\forall$E, 4

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z(P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

1	$\forall x P(a,x,x)$	
2	$\forall x \forall y \forall z(P(x,y,z) \rightarrow P(f(x),y,f(z)))$	
3	$P(a,a,a)$	∀E, 1
4	$\forall y \forall z(P(a,y,z) \rightarrow P(f(a),y,f(z)))$	∀E, 2
5	$\forall z(P(a,a,z) \rightarrow P(f(a),a,f(z)))$	∀E, 4
6	$P(a,a,a) \rightarrow P(f(a),a,f(a))$	∀E, 5

Ex9. Build a proof of $\forall x P(a,x,x), \forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash P(f(a),a,f(a))$

1	$\forall x P(a,x,x)$	
2	$\forall x \forall y \forall z (P(x,y,z) \rightarrow P(f(x),y,f(z)))$	
3	$P(a,a,a)$	∀E, 1
4	$\forall y \forall z (P(a,y,z) \rightarrow P(f(a),y,f(z)))$	∀E, 2
5	$\forall z (P(a,a,z) \rightarrow P(f(a),a,f(z)))$	∀E, 4
6	$P(a,a,a) \rightarrow P(f(a),a,f(a))$	∀E, 5
7	$P(f(a),a,f(a))$	⇒E, 3, 6

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

1 | $\exists x \exists y (H(x,y) \lor H(y,x))$

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

1 $\exists x \exists y (H(x,y) \lor H(y,x))$

2 $\neg \exists x H(x,x)$

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

1 $\mid \exists x \exists y (H(x,y) \lor H(y,x))$

2 $\mid \underline{\neg \exists x H(x,x)}$

3 $\mid u \mid v \mid \quad \mid \underline{u = v}$

Ex10. Build a proof of $\exists x \exists y(H(x,y) \lor H(y,x)), \neg\exists x H(x,x) \vdash \exists x \exists y \neg(x = y)$

1	$\exists x \exists y(H(x,y) \lor H(y,x))$					
2	$\neg\exists x H(x,x)$					
3	u	v		$u = v$		
4					$H(u,v) \lor H(v,u)$	
5						$H(u,v)$

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

1	$\exists x \exists y (H(x,y) \vee H(y,x))$	
2	$\neg \exists x H(x,x)$	
3	u $\mid$ v $\mid$ $\mid$ $u = v$	
4	$H(u,v) \vee H(v,u)$	
5	$H(u,v)$	
6	$H(u,u)$	=E, 3, 5

Ex10. Build a proof of $\exists x \exists y(H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg(x = y)$

1		$\exists x \exists y(H(x,y) \lor H(y,x))$	
2		$\neg \exists x H(x,x)$	
3	u v	$u = v$	
4		$H(u,v) \lor H(v,u)$	
5		$H(u,v)$	
6		$H(u,u)$	$=$E, 3, 5
7		$\exists x H(x,x)$	$\exists$I, 5

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

1	$\exists x \exists y (H(x,y) \lor H(y,x))$	
2	$\neg \exists x H(x,x)$	
3	u v $u = v$	
4	$H(u,v) \lor H(v,u)$	
5	$H(u,v)$	
6	$H(u,u)$	=E, 3, 5
7	$\exists x H(x,x)$	∃I, 5
8	$\bot$	⊥I, 2, 7

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8 | | | | | $\vdots$

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8 | | | | | $\vdots$

9 | | | | | | $\underline{H(v,u)}$

Ex10. Build a proof of $\exists x \exists y (H(x, y) \lor H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$

8	$\vdots$	
9	$H(v, u)$	
10	$H(v, v)$	$=$E, 3, 9

Ex10. Build a proof of $\exists x \exists y (H(x,y) \lor H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8	$\vdots$	
9	$H(v,u)$	
10	$H(v,v)$	$=$E, 3, 9
11	$\exists x H(x,x)$	$\exists$I, 10

Ex10. Build a proof of $\exists x \exists y (H(x, y) \lor H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$

8	$\vdots$	
9	$H(v, u)$	
10	$H(v, v)$	$=$E, 3, 9
11	$\exists x H(x, x)$	$\exists$I, 10
12	$\bot$	$\bot$I, 2, 7

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8	$\vdots$	
9	$H(v,u)$	
10	$H(v,v)$	=E, 3, 9
11	$\exists x H(x,x)$	∃I, 10
12	$\bot$	⊥I, 2, 7
13	$\bot$	∨E, 4, 5–8, 9–12

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8	$\vdots$	
9	$H(v,u)$	
10	$H(v,v)$	=E, 3, 9
11	$\exists x H(x,x)$	∃I, 10
12	$\bot$	⊥I, 2, 7
13	$\bot$	∨E, 4, 5–8, 9–12
14	$\bot$	∃E, 1, 4–13

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8	$\vert\ \vert\ \vert\ \vert\ \vert\ \vdots$	
9	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ \underline{H(v,u)}$	
10	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ H(v,v)$	$=$E, 3, 9
11	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ \exists x H(x,x)$	$\exists$I, 10
12	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ \bot$	$\bot$I, 2, 7
13	$\vert\ \vert\ \vert\ \vert\ \vert\ \bot$	$\vee$E, 4, 5–8, 9–12
14	$\vert\ \vert\ \vert\ \vert\ \bot$	$\exists$E, 1, 4–13
15	$\vert\ \vert\ \vert\ \neg(u = v)$	$\neg$I, 3–14

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8	$\mid\ \mid\ \mid\ \mid\ \mid\ \vdots$	
9	$\mid\ \mid\ \mid\ \mid\ \mid\ \mid\ \underline{H(v,u)}$	
10	$\mid\ \mid\ \mid\ \mid\ \mid\ \mid\ H(v,v)$	$=$E, 3, 9
11	$\mid\ \mid\ \mid\ \mid\ \mid\ \mid\ \exists x H(x,x)$	$\exists$I, 10
12	$\mid\ \mid\ \mid\ \mid\ \mid\ \mid\ \bot$	$\bot$I, 2, 7
13	$\mid\ \mid\ \mid\ \mid\ \mid\ \bot$	$\vee$E, 4, 5–8, 9–12
14	$\mid\ \mid\ \mid\ \mid\ \bot$	$\exists$E, 1, 4–13
15	$\mid\ \mid\ \mid\ \neg(u = v)$	$\neg$I, 3–14
16	$\mid\ \mid\ \exists y \neg(u = y)$	$\exists$I, 15

Ex10. Build a proof of $\exists x \exists y (H(x,y) \vee H(y,x)), \neg \exists x H(x,x) \vdash \exists x \exists y \neg (x = y)$

8	$\vdots$	
9	$H(v,u)$	
10	$H(v,v)$	=E, 3, 9
11	$\exists x H(x,x)$	∃I, 10
12	$\bot$	⊥I, 2, 7
13	$\bot$	∨E, 4, 5–8, 9–12
14	$\bot$	∃E, 1, 4–13
15	$\neg(u = v)$	¬I, 3–14
16	$\exists y \neg(u = y)$	∃I, 15
17	$\exists x \exists y \neg(x = y)$	∃I, 16

Ex11. Build a proof of $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$

Ex11. Build a proof of $\exists y \exists x Q(y,x) \vdash \exists x \exists y Q(y,x)$

1 | $\underline{\exists y \exists x Q(y,x)}$

Ex11. Build a proof of $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$

1	$\exists y \exists x Q(y, x)$	
2	u	$\exists x Q(u, x)$

Ex11. Build a proof of $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$

1	$\exists y \exists x Q(y, x)$		
2	u	$\exists x Q(u, x)$	
3		v	$Q(u, v)$

Ex11. Build a proof of $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$

1	$\exists y \exists x Q(y, x)$			
2	u	$\exists x Q(u, x)$		
3		v	$Q(u, v)$	
4			$\exists y Q(y, v)$	∃I, 3

Ex11. Build a proof of $\exists y \exists x Q(y,x) \vdash \exists x \exists y Q(y,x)$

1	$\exists y \exists x Q(y,x)$			
2	u	$\exists x Q(u,x)$		
3		v	$Q(u,v)$	
4			$\exists y Q(y,v)$	∃I, 3
5		$\exists y Q(y,v)$		∃E, 2, 3–4

Ex11. Build a proof of $\exists y \exists x Q(y,x) \vdash \exists x \exists y Q(y,x)$

1			$\exists y \exists x Q(y,x)$	
2	u		$\exists x Q(u,x)$	
3		v	$Q(u,v)$	
4			$\exists y Q(y,v)$	∃I, 3
5			$\exists y Q(y,v)$	∃E, 2, 3–4
6			$\exists x \exists y Q(y,x)$	∃I, 5

Ex11. Build a proof of $\exists y \exists x Q(y,x) \vdash \exists x \exists y Q(y,x)$

1	$\exists y \exists x Q(y,x)$	
2	$u \mid \exists x Q(u,x)$	
3	$\mid v \mid Q(u,v)$	
4	$\mid \quad \mid \exists y Q(y,v)$	∃I, 3
5	$\mid \exists y Q(y,v)$	∃E, 2, 3–4
6	$\mid \exists x \exists y Q(y,x)$	∃I, 5
7	$\exists x \exists y Q(y,x)$	∃E, 1, 2–6

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

1 $\exists x P(x) \to \forall x R(x)$

2 u $P(u)$

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

1		$\exists x P(x) \to \forall x R(x)$	
2	u	$P(u)$	
3		$\exists x P(x)$	∃I, 2

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

1		$\exists x P(x) \to \forall x R(x)$	
2	u	$P(u)$	
3		$\exists x P(x)$	∃I, 2
4		$\forall x R(x)$	⇒E, 1, 3

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

1		$\exists x P(x) \to \forall x R(x)$	
2	u	$P(u)$	
3		$\exists x P(x)$	∃I, 2
4		$\forall x R(x)$	⇒E, 1, 3
5		$R(u)$	∀E, 4

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

1		$\exists x P(x) \to \forall x R(x)$	
2	u	$P(u)$	
3		$\exists x P(x)$	∃I, 2
4		$\forall x R(x)$	⇒E, 1, 3
5		$R(u)$	∀E, 4
6		$P(u) \to Q(u)$	⇒I, 2–5

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x(P(x) \to Q(x)))$

1		$\exists x P(x) \to \forall x R(x)$	
2	u	$P(u)$	
3		$\exists x P(x)$	$\exists$I, 2
4		$\forall x R(x)$	$\Rightarrow$E, 1, 3
5		$R(u)$	$\forall$E, 4
6		$P(u) \to Q(u)$	$\Rightarrow$I, 2–5
7		$\forall x(P(x) \to Q(x))$	$\forall$I, 2–6

Ex12. Build a proof of $\vdash (\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$

1		$\exists x P(x) \to \forall x R(x)$	
2	u	$P(u)$	
3		$\exists x P(x)$	∃I, 2
4		$\forall x R(x)$	⇒E, 1, 3
5		$R(u)$	∀E, 4
6		$P(u) \to Q(u)$	⇒I, 2–5
7		$\forall x (P(x) \to Q(x))$	∀I, 2–6
8		$(\exists x P(x) \to \forall x R(x)) \to (\forall x (P(x) \to Q(x)))$	⇒I, 1–7

Ex13. Build a proof of $\vdash x = f(y) \to \forall z(P(x,z) \to P(f(y),z))$

Ex13. Build a proof of $\vdash x = f(y) \to \forall z(P(x,z) \to P(f(y),z))$

1 $\quad | \quad | \; x = f(y)$

Ex13. Build a proof of $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$

1	$x = f(y)$	
2	u	$P(x, u)$

Ex13. Build a proof of $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$

1		$x = f(y)$	
2	u	$P(x, u)$	
3		$P(f(y), u)$	$=$E, 1, 2

Ex13. Build a proof of $\vdash x = f(y) \to \forall z(P(x,z) \to P(f(y),z))$

1		$x = f(y)$	
2	u	$P(x,u)$	
3		$P(f(y),u)$	=E, 1, 2
4		$P(x,u) \to P(f(y),u)$	⇒I, 2–3

Ex13. Build a proof of $\vdash x = f(y) \rightarrow \forall z(P(x,z) \rightarrow P(f(y),z))$

1		$x = f(y)$	
2	u	$P(x,u)$	
3		$P(f(y),u)$	$=$E, 1, 2
4		$P(x,u) \rightarrow P(f(y),u)$	$\Rightarrow$I, 2–3
5		$\forall z(P(x,z) \rightarrow P(f(y),z))$	$\forall$I, 2–4

Ex13. Build a proof of $\vdash x = f(y) \to \forall z(P(x,z) \to P(f(y),z))$

$$\begin{array}{lll}
1 & \vert\quad \vert\ \underline{x = f(y)} & \\
2 & \vert\quad \vert\ u\ \vert\quad \vert\ \underline{P(x,u)} & \\
3 & \vert\quad \vert\ \ \vert\quad \vert\ P(f(y),u) & =\text{E, 1, 2} \\
4 & \vert\quad \vert\ \ \vert\ P(x,u) \to P(f(y),u) & \Rightarrow\text{I, 2–3} \\
5 & \vert\quad \vert\ \forall z(P(x,z) \to P(f(y),z)) & \forall\text{I, 2–4} \\
6 & \vert\ x = f(y) \to \forall z(P(x,z) \to P(f(y),z)) & \Rightarrow\text{I, 1–5}
\end{array}$$

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \to (y = v \to f(x, y) = f(u, v)))$

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1 | a | b | c | d | $\underline{a = b}$

Ex14. Build a proof of $\forall x \forall y \forall u \forall v(x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1 $\quad | \; a \; | \; b \; | \; c \; | \; d \; | \quad | \; \underline{a = b}$

2 $\quad | \quad | \quad | \quad | \quad | \quad | \quad | \; \underline{c = d}$

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \to (y = v \to f(x, y) = f(u, v)))$

1	a	b	c	d	$a = b$		
2						$c = d$	
3						$f(a, c) = f(a, c)$	=I

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1	a	b	c	d	$a = b$		
2						$c = d$	
3						$f(a, c) = f(a, c)$	=I
4						$f(a, c) = f(b, c)$	=E, 1, 3

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1	a	b	c	d	$a = b$	
2					$c = d$	
3					$f(a, c) = f(a, c)$	=I
4					$f(a, c) = f(b, c)$	=E, 1, 3
5					$f(a, c) = f(b, d)$	=E, 2, 4

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \to (y = v \to f(x, y) = f(u, v)))$

1	a	b	c	d	$a = b$	
2					$\quad c = d$	
3					$\quad f(a, c) = f(a, c)$	=I
4					$\quad f(a, c) = f(b, c)$	=E, 1, 3
5					$\quad f(a, c) = f(b, d)$	=E, 2, 4
6					$c = d \to f(a, c) = f(b, d)$	⇒I, 2–5

Ex14. Build a proof of $\forall x \forall y \forall u \forall v(x = u \to (y = v \to f(x, y) = f(u, v)))$

1	a	b	c	d	$a = b$	
2					$c = d$	
3					$f(a, c) = f(a, c)$	=I
4					$f(a, c) = f(b, c)$	=E, 1, 3
5					$f(a, c) = f(b, d)$	=E, 2, 4
6					$c = d \to f(a, c) = f(b, d)$	⇒I, 2–5
7					$a = b \to (c = d \to f(a, c) = f(b, d))$	⇒I, 1–6

Ex14. Build a proof of $\forall x \forall y \forall u \forall v(x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1	a b c d	$a = b$	
2		$c = d$	
3		$f(a, c) = f(a, c)$	=I
4		$f(a, c) = f(b, c)$	=E, 1, 3
5		$f(a, c) = f(b, d)$	=E, 2, 4
6		$c = d \rightarrow f(a, c) = f(b, d)$	⇒I, 2–5
7		$a = b \rightarrow (c = d \rightarrow f(a, c) = f(b, d))$	⇒I, 1–6
8		$\forall v(a = b \rightarrow (c = v \rightarrow f(a, c) = f(b, v)))$	∀I, 7

Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \to (y = v \to f(x, y) = f(u, v)))$

1	a b c d	$a = b$	
2		$c = d$	
3		$f(a, c) = f(a, c)$	=I
4		$f(a, c) = f(b, c)$	=E, 1, 3
5		$f(a, c) = f(b, d)$	=E, 2, 4
6		$c = d \to f(a, c) = f(b, d)$	⇒I, 2–5
7		$a = b \to (c = d \to f(a, c) = f(b, d))$	⇒I, 1–6
8		$\forall v (a = b \to (c = v \to f(a, c) = f(b, v)))$	∀I, 7
9		$\forall u \forall v (a = u \to (c = v \to f(a, c) = f(u, v)))$	∀I, 8

Ex14. Build a proof of $\forall x \forall y \forall u \forall v(x = u \to (y = v \to f(x,y) = f(u,v)))$

1	a b c d $a = b$	
2	$c = d$	
3	$f(a,c) = f(a,c)$	=I
4	$f(a,c) = f(b,c)$	=E, 1, 3
5	$f(a,c) = f(b,d)$	=E, 2, 4
6	$c = d \to f(a,c) = f(b,d)$	⇒I, 2–5
7	$a = b \to (c = d \to f(a,c) = f(b,d))$	⇒I, 1–6
8	$\forall v(a = b \to (c = v \to f(a,c) = f(b,v)))$	∀I, 7
9	$\forall u \forall v(a = u \to (c = v \to f(a,c) = f(u,v)))$	∀I, 8
10	$\forall y \forall u \forall v(a = u \to (y = v \to f(a,y) = f(u,v)))$	∀I, 9

Ex14. Build a proof of $\forall x\forall y\forall u\forall v(x = u \to (y = v \to f(x, y) = f(u, v)))$

1	$a \quad b \quad c \quad d \quad a = b$	
2	$c = d$	
3	$f(a, c) = f(a, c)$	=I
4	$f(a, c) = f(b, c)$	=E, 1, 3
5	$f(a, c) = f(b, d)$	=E, 2, 4
6	$c = d \to f(a, c) = f(b, d)$	⇒I, 2–5
7	$a = b \to (c = d \to f(a, c) = f(b, d))$	⇒I, 1–6
8	$\forall v(a = b \to (c = v \to f(a, c) = f(b, v)))$	∀I, 7
9	$\forall u\forall v(a = u \to (c = v \to f(a, c) = f(u, v)))$	∀I, 8
10	$\forall y\forall u\forall v(a = u \to (y = v \to f(a, y) = f(u, v)))$	∀I, 9
11	$\forall x\forall y\forall u\forall v(x = u \to (y = v \to f(x, y) = f(u, v)))$	∀I, 10

Ex15. Build a proof of $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

Ex15. Build a proof of $\exists x \forall y (P(x) \land (P(y) \to y = x)) \vdash \forall x \forall y ((P(x) \land P(y)) \to x = y)$

1 $\quad \exists x \forall y (P(x) \land (P(y) \to y = x))$

Ex15. Build a proof of $\exists x \forall y (P(x) \land (P(y) \to y = x)) \vdash \forall x \forall y ((P(x) \land P(y)) \to x = y)$

1 $\quad \exists x \forall y (P(x) \land (P(y) \to y = x))$

2 $\quad a \mid b \mid \quad P(a) \land P(b)$

Ex15. Build a proof of $\exists x \forall y(P(x) \land (P(y) \to y = x)) \vdash \forall x \forall y((P(x) \land P(y)) \to x = y)$

1 $\exists x \forall y(P(x) \land (P(y) \to y = x))$

2 a b $P(a) \land P(b)$

3 $P(a) \land (P(b) \to b = a)$

Ex15. Build a proof of $\exists x \forall y(P(x) \wedge (P(y) \to y = x)) \vdash \forall x \forall y((P(x) \wedge P(y)) \to x = y)$

1	$\exists x \forall y(P(x) \wedge (P(y) \to y = x))$	
2	a | b | $P(a) \wedge P(b)$	
3	$P(a) \wedge (P(b) \to b = a)$	
4	$P(b)$	$\wedge$E, 2

Ex15. Build a proof of $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$					
2		a	b	$P(a) \wedge P(b)$		
3					$P(a) \wedge (P(b) \rightarrow b = a)$	
4					$P(b)$	$\wedge$E, 2
5					$P(b) \rightarrow b = a$	$\wedge$E, 3

Ex15. Build a proof of $\exists x \forall y (P(x) \land (P(y) \to y = x)) \vdash \forall x \forall y ((P(x) \land P(y)) \to x = y)$

1			$\exists x \forall y (P(x) \land (P(y) \to y = x))$	
2	a	b	$P(a) \land P(b)$	
3			$P(a) \land (P(b) \to b = a)$	
4			$P(b)$	$\land$E, 2
5			$P(b) \to b = a$	$\land$E, 3
6			$b = a$	$\Rightarrow$E, 4, 5

Ex15. Build a proof of $\exists x \forall y (P(x) \land (P(y) \to y = x)) \vdash \forall x \forall y ((P(x) \land P(y)) \to x = y)$

1	$\exists x \forall y (P(x) \land (P(y) \to y = x))$	
2	$a \quad b \quad P(a) \land P(b)$	
3	$P(a) \land (P(b) \to b = a)$	
4	$P(b)$	$\land$E, 2
5	$P(b) \to b = a$	$\land$E, 3
6	$b = a$	$\Rightarrow$E, 4, 5
7	$b = b$	=I

Ex15. Build a proof of $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

1			$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$	
2	a	b	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge$E, 2
5			$P(b) \rightarrow b = a$	$\wedge$E, 3
6			$b = a$	$\Rightarrow$E, 4, 5
7			$b = b$	$=$I
8			$a = b$	$=$E, 6, 7

Ex15. Build a proof of $\exists x \forall y (P(x) \land (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \land P(y)) \rightarrow x = y)$

1	$\exists x \forall y (P(x) \land (P(y) \rightarrow y = x))$	
2	a b $P(a) \land P(b)$	
3	$P(a) \land (P(b) \rightarrow b = a)$	
4	$P(b)$	$\land$E, 2
5	$P(b) \rightarrow b = a$	$\land$E, 3
6	$b = a$	$\Rightarrow$E, 4, 5
7	$b = b$	=I
8	$a = b$	=E, 6, 7
9	$a = b$	$\exists$E, 1, 3–8

Ex15. Build a proof of $\exists x \forall y (P(x) \land (P(y) \to y = x)) \vdash \forall x \forall y ((P(x) \land P(y)) \to x = y)$

1		$\exists x \forall y (P(x) \land (P(y) \to y = x))$	
2	a b	$P(a) \land P(b)$	
3		$P(a) \land (P(b) \to b = a)$	
4		$P(b)$	$\land$E, 2
5		$P(b) \to b = a$	$\land$E, 3
6		$b = a$	$\Rightarrow$E, 4, 5
7		$b = b$	$=$I
8		$a = b$	$=$E, 6, 7
9		$a = b$	$\exists$E, 1, 3–8
10		$(P(a) \land P(b)) \to a = b$	$\Rightarrow$I, 2–9

Ex15. Build a proof of $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$	
2	$a \quad b \quad P(a) \wedge P(b)$	
3	$P(a) \wedge (P(b) \rightarrow b = a)$	
4	$P(b)$	$\wedge$E, 2
5	$P(b) \rightarrow b = a$	$\wedge$E, 3
6	$b = a$	$\Rightarrow$E, 4, 5
7	$b = b$	$=$I
8	$a = b$	$=$E, 6, 7
9	$a = b$	$\exists$E, 1, 3–8
10	$(P(a) \wedge P(b)) \rightarrow a = b$	$\Rightarrow$I, 2–9
11	$\forall y ((P(a) \wedge P(y)) \rightarrow a = y)$	$\forall$I, 10

Ex15. Build a proof of $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$	
2	a b $P(a) \wedge P(b)$	
3	$P(a) \wedge (P(b) \rightarrow b = a)$	
4	$P(b)$	$\wedge$E, 2
5	$P(b) \rightarrow b = a$	$\wedge$E, 3
6	$b = a$	$\Rightarrow$E, 4, 5
7	$b = b$	=I
8	$a = b$	=E, 6, 7
9	$a = b$	$\exists$E, 1, 3–8
10	$(P(a) \wedge P(b)) \rightarrow a = b$	$\Rightarrow$I, 2–9
11	$\forall y ((P(a) \wedge P(y)) \rightarrow a = y)$	$\forall$I, 10
12	$\forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$	$\forall$I, 11