Natural Deduction Rules for Propositional Logic

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November 16, 2021

Conjunction

The rules:

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\land\mathbf{I} \qquad\qquad \frac{\varphi \land \psi}{\varphi}\land\mathbf{E}_l \qquad\qquad \frac{\varphi \land \psi}{\psi}\land\mathbf{E}_r$$

Representation in Fitch's style:

$$\begin{array}{r|ll} & \vdots & \\ n & \varphi & \\ & \vdots & \\ m & \psi & \\ & \vdots & \\ & \varphi \land \psi & \land\mathbf{I}(n,m) \end{array} \qquad \begin{array}{r|ll} & \vdots & \\ n & \varphi \land \psi & \\ & \vdots & \\ m & \varphi & \land\mathbf{E}_l(n) \end{array} \qquad \begin{array}{r|ll} & \vdots & \\ n & \varphi \land \psi & \\ & \vdots & \\ m & \psi & \land\mathbf{E}_r(n) \end{array}$$

Disjunction

The rules:

$$\frac{\varphi}{\varphi \lor \psi}\lor\mathbf{I}_l \qquad\qquad \frac{\psi}{\varphi \lor \psi}\lor\mathbf{I}_r \qquad\qquad \frac{\varphi \lor \psi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \theta\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \theta\end{array}}{\theta}\lor\mathbf{E}$$

Representation in Fitch's style:

$$
\begin{array}{r|ll}
 & \vdots & \\
n & \varphi & \\
 & \vdots & \\
 & \varphi \vee \psi & \vee\mathbf{I}_l(n)
\end{array}
\qquad
\begin{array}{r|ll}
 & \vdots & \\
n & \psi & \\
 & \vdots & \\
 & \varphi \vee \psi & \vee\mathbf{I}_r(n)
\end{array}
\qquad
\begin{array}{r|l|ll}
 & \multicolumn{2}{l}{\vdots} & \\
n & \multicolumn{2}{l}{\varphi \vee \psi} & \\
 & \multicolumn{2}{l}{\vdots} & \\
m_1 & & \varphi & \\ \cline{3-3}
 & & \vdots & \\
k_1 & & \theta & \\
 & \multicolumn{2}{l}{\vdots} & \\
m_2 & & \psi & \\ \cline{3-3}
 & & \vdots & \\
k_2 & & \theta & \\
 & \multicolumn{2}{l}{\vdots} & \\
 & \multicolumn{2}{l}{\theta} & \vee\mathbf{E}(n, m_1{-}k_1, m_2{-}k_2)
\end{array}
$$

Negation

The rules:

$$
\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\neg\varphi}\,\neg\mathbf{I}
\qquad\qquad
\dfrac{\neg\neg\varphi}{\varphi}\,\neg\mathbf{E}
$$

Representation in Fitch's style:

$$
\begin{array}{r|l}
 & \vdots \\
n & \quad \begin{array}{|l} \varphi \\ \hline \vdots \\ \bot \end{array} \\
m & \\
 & \neg\varphi \qquad \neg\mathbf{I}(n{-}m)
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & \vdots \\
n & \neg\neg\varphi \\
 & \vdots \\
 & \varphi \qquad \neg\mathbf{E}(n)
\end{array}
$$

False

The rules:

$$
\frac{\begin{array}{c} \varphi \\ \vdots \\ \neg\varphi \end{array}}{\bot}\ \bot\mathbf{I}
\qquad\qquad
\frac{\bot}{\varphi}\ \bot\mathbf{E}
$$

Representation in Fitch's style:

$$
\begin{array}{r|l}
 & \vdots \\
n & \varphi \\
 & \vdots \\
m & \neg\varphi \\
 & \vdots \\
 & \bot \qquad \bot\mathbf{I}(n,m)
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & \vdots \\
n & \bot \\
 & \vdots \\
 & \varphi \qquad \bot\mathbf{E}(n)
\end{array}
$$

Implication

The rules:

$$
\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \psi \end{array}}{\varphi \to \psi}\ {\to}\mathbf{I}
\qquad\qquad
\frac{\varphi \to \psi \qquad \varphi}{\psi}\ {\to}\mathbf{E}
$$

Representation in Fitch's style:

$$
\begin{array}{r|l l}
 & \vdots & \\
n & \quad \varphi & \\
 & \quad \vdots & \\
m & \quad \psi & \\
 & \varphi \to \psi & \to\mathbf{I}(n{-}m)
\end{array}
\qquad\qquad
\begin{array}{r|l l}
 & \vdots & \\
n & \varphi \to \psi & \\
 & \vdots & \\
m & \varphi & \\
 & \vdots & \\
 & \psi & \to\mathbf{E}(n,m)
\end{array}
$$