

## 2. RAMDE – Requirements and Model-driven Engineering

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Porto, Portugal

<https://cister-labs.github.io/ramde2122>

## Recalling Natural Deduction Rules in First Order Logic

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# Which are the new rules (on top of Propositional Logic)?

## Elimination rule for $\forall$

If we know that  $\forall x, \varphi$  holds, then we can conclude that  $\varphi$  holds for a specific term  $t$

$$\frac{\forall x \varphi}{\varphi[t/x]} \forall E$$

## Introduction rule for $\forall$

If we assume some term  $t$  and we are able to prove that  $\varphi[t/x]$  then we can conclude that  $\forall x, \varphi$ .

$$\frac{\begin{array}{c} [t] \\ \vdots \\ \varphi[t/x] \end{array}}{\forall x \varphi[t/x]} \forall I$$

# Which are the new rules (on top of Propositional Logic)?

## Elimination rule for $\exists$

If we know that  $\exists x, \varphi$  holds, and if assuming term  $t$  and  $\varphi[t/x]$  we can deduce  $\psi$ , then we can prove  $\psi$  overall.

$$\frac{\begin{array}{c} [t \ \varphi[t/x]] \\ \vdots \\ \psi \end{array}}{\exists x \varphi} \exists E$$

## Introduction rule for $\exists$

If we assume some term  $t$  and we are able to prove that  $\varphi[t/x]$  then we can conclude that  $\exists x, \varphi$ .

$$\frac{\varphi[t/x]}{\exists x, \varphi} \exists I$$

# Which are the new rules (on top of Propositional Logic)?

## Elimination rule for =

If we know that two terms  $t_1$  and  $t_2$  are equal and that  $\varphi[t_1/x]$  holds, then  $\varphi[t_2/x]$  must also hold.

$$\frac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]} = \mathbf{E}$$

## Introduction rule for =

If we assume some term  $t$  and we are able to prove that  $\varphi[t/x]$  then we can conclude that  $\forall x, \varphi$ .

$$\frac{}{t = t} = \mathbf{I}$$

# Practical Exercises

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**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

$$1 \quad \underbrace{\forall x(R(x) \wedge Q(x))}$$



**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

$$\begin{array}{l|l} 1 & \forall x(R(x) \wedge Q(x)) \\ \hline 2 & v \mid R(v) \wedge Q(v) \quad \forall E, 1 \end{array}$$

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

1		$\forall x(R(x) \wedge Q(x))$	
2		$R(v) \wedge Q(v)$	$\forall E, 1$
3		$R(v)$	$\wedge E, 2$

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

1		$\forall x(R(x) \wedge Q(x))$	
<hr/>			
2		$v$   $R(v) \wedge Q(v)$	$\forall E, 1$
3		$R(v)$	$\wedge E, 2$
4		$\forall xR(x)$	$\forall I, 2-3$

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

1		$\forall x(R(x) \wedge Q(x))$	
<hr/>			
2		$v \mid R(v) \wedge Q(v)$	$\forall E, 1$
3		$R(v)$	$\wedge E, 2$
4		$\forall xR(x)$	$\forall I, 2-3$
5		$v \mid R(v) \wedge Q(v)$	$\forall E, 1$

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

1		$\forall x(R(x) \wedge Q(x))$	
<hr/>			
2		$v \mid R(v) \wedge Q(v)$	$\forall E, 1$
3		$R(v)$	$\wedge E, 2$
4		$\forall xR(x)$	$\forall I, 2-3$
5		$v \mid R(v) \wedge Q(v)$	$\forall E, 1$
6		$Q(v)$	$\wedge E, 5$

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

1		$\forall x(R(x) \wedge Q(x))$	
2		$v \mid R(v) \wedge Q(v)$	$\forall E, 1$
3		$R(v)$	$\wedge E, 2$
4		$\forall xR(x)$	$\forall I, 2-3$
5		$v \mid R(v) \wedge Q(v)$	$\forall E, 1$
6		$Q(v)$	$\wedge E, 5$
7		$\forall xQ(x)$	$\forall I, 5-7$

**Ex1. Build a proof of  $\forall x(R(x) \wedge Q(x)) \vdash \forall xR(x) \wedge \forall xQ(x)$**

1	$\forall x(R(x) \wedge Q(x))$	
2	$v$   $R(v) \wedge Q(v)$	$\forall E, 1$
3	$R(v)$	$\wedge E, 2$
4	$\forall xR(x)$	$\forall I, 2-3$
5	$v$   $R(v) \wedge Q(v)$	$\forall E, 1$
6	$Q(v)$	$\wedge E, 5$
7	$\forall xQ(x)$	$\forall I, 5-7$
8	$\forall xR(x) \wedge \forall xQ(x)$	$\wedge I, 4, 8$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**



**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

$$1 \quad \underbrace{\forall x(R(x) \rightarrow Q(x))}$$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

$$\begin{array}{l|l} 1 & \forall x(R(x) \rightarrow Q(x)) \\ \hline 2 & \begin{array}{|l} \forall xR(x) \end{array} \end{array}$$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

1		$\forall x(R(x) \rightarrow Q(x))$	
2			
3			

---

2			$\forall xR(x)$	
3				
3				$R(v) \rightarrow Q(v) \quad \forall E, 1$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

1		$\forall x(R(x) \rightarrow Q(x))$	
2			
3			
4			

---

2		$\forall xR(x)$	
3			
3			
4			

---

3		$v$	$R(v) \rightarrow Q(v)$	$\forall E, 1$
4			$R(v)$	$\forall E, 2$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

1		$\forall x(R(x) \rightarrow Q(x))$	
<hr/>			
2		$\forall xR(x)$	
<hr/>			
3		$v$   $R(v) \rightarrow Q(v)$	$\forall E, 1$
4			$R(v)$ $\forall E, 2$
5			$Q(v)$ $\Rightarrow E, 3, 4$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

1		$\forall x(R(x) \rightarrow Q(x))$	
<hr/>			
2		$\forall xR(x)$	
<hr/>			
3		$v$   $R(v) \rightarrow Q(v)$	$\forall E, 1$
4		$R(v)$	$\forall E, 2$
5		$Q(v)$	$\Rightarrow E, 3, 4$
6		$\forall xQ(x)$	$\forall I, 3-5$

**Ex2. Build a proof of  $\forall x(R(x) \rightarrow Q(x)) \vdash \forall xR(x) \rightarrow \forall xQ(x)$**

1	$\forall x(R(x) \rightarrow Q(x))$	
2	$\forall xR(x)$	
3	$v \mid R(v) \rightarrow Q(v)$	$\forall E, 1$
4	$R(v)$	$\forall E, 2$
5	$Q(v)$	$\Rightarrow E, 3, 4$
6	$\forall xQ(x)$	$\forall I, 3-5$
7	$\forall xP(x) \rightarrow \forall xQ(x)$	$\Rightarrow I, 2-6$

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**



**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

$$1 \quad \underbrace{\exists x(R(x) \rightarrow Q(x))}$$

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

$$\begin{array}{l|l} 1 & \exists x(R(x) \rightarrow Q(x)) \\ \hline 2 & \begin{array}{|l} \exists xR(x) \end{array} \end{array}$$

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1		$\exists x(R(x) \rightarrow Q(x))$	
		<hr/>	
2			$\exists xR(x)$
			<hr/>
3			$v$   $R(v) \rightarrow Q(v)$
			<hr/>

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1		$\exists x(R(x) \rightarrow Q(x))$	
<hr/>			
2			$\exists xR(x)$
			<hr/>
3			
			$v$
			$R(v) \rightarrow Q(v)$
			<hr/>
4			
			$R(v)$
			<hr/>

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1		$\exists x(R(x) \rightarrow Q(x))$			
2			$\exists xR(x)$		
3				$R(v) \rightarrow Q(v)$	
4					$R(v)$
5					$Q(v)$
					$\Rightarrow E, 3, 4$

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1	$\exists x(R(x) \rightarrow Q(x))$				
2	$\exists xR(x)$				
3	$v$	$R(v) \rightarrow Q(v)$			
4		$R(v)$			
5		$Q(v)$	$\Rightarrow E, 3, 4$		
6		$\exists xQ(x)$	$\exists I, 5$		

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1		$\exists x(R(x) \rightarrow Q(x))$	
2		$\exists xR(x)$	
3		$v$   $R(v) \rightarrow Q(v)$	
4		$R(v)$	
5		$Q(v)$	$\Rightarrow E, 3, 4$
6		$\exists xQ(x)$	$\exists I, 5$
7		$\exists xQ(x)$	$\exists E, 2, 4-6$

**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1		$\exists x(R(x) \rightarrow Q(x))$	
2		$\exists xR(x)$	
3		$v$   $R(v) \rightarrow Q(v)$	
4		$R(v)$	
5		$Q(v)$	$\Rightarrow E, 3, 4$
6		$\exists xQ(x)$	$\exists I, 5$
7		$\exists xQ(x)$	$\exists E, 2, 4-6$
8		$\exists xQ(x)$	$\exists E, 1, 3-7$



**Ex3. Build a proof of  $\exists x(R(x) \rightarrow Q(x)) \vdash \exists xR(x) \rightarrow \exists xQ(x)$**

1	$\exists x(R(x) \rightarrow Q(x))$	
2	$\exists xR(x)$	
3	$v$   $R(v) \rightarrow Q(v)$	
4	$R(v)$	
5	$Q(v)$	$\Rightarrow E, 3, 4$
6	$\exists xQ(x)$	$\exists I, 5$
7	$\exists xQ(x)$	$\exists E, 2, 4-6$
8	$\exists xQ(x)$	$\exists E, 1, 3-7$
9	$\exists xP(x) \rightarrow \exists xQ(x)$	$\Rightarrow I, 2-8$

**Ex4. Build a proof of  $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$**

#### Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

$$1 \quad \boxed{\exists x \neg Q(x)}$$

#### Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

$$\begin{array}{l|l} 1 & \exists x \neg Q(x) \\ \hline 2 & \boxed{\forall x Q(x)} \end{array}$$

**Ex4. Build a proof of  $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$**

1		$\exists x \neg Q(x)$	
		<hr/>	
2			$\forall x Q(x)$
			<hr/>
3		$v$	$\neg Q(v)$

**Ex4. Build a proof of  $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$**

1		$\exists x \neg Q(x)$			
2			$\forall x Q(x)$		
3				$\neg Q(v)$	
4				$Q(v)$	$\forall E, 2$

#### Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1		$\exists x \neg Q(x)$			
2			$\forall x Q(x)$		
3				$\neg Q(v)$	
4				$Q(v)$	$\forall E, 2$
5				$\perp$	$\perp I, 3, 4$

#### Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1		$\exists x \neg Q(x)$		
2			$\forall x Q(x)$	
3				$\neg Q(v)$
4				$Q(v)$ $\forall E, 2$
5				$\perp$ $\perp I, 3, 4$
6			$\perp$	$\exists E, 1, 3-5$



#### Ex4. Build a proof of $\exists x \neg Q(x) \vdash \neg \forall x Q(x)$

1		$\exists x \neg Q(x)$			
2			$\forall x Q(x)$		
3				$\neg Q(v)$	
4				$Q(v)$	$\forall E, 2$
5				$\bot$	$\bot I, 3, 4$
6			$\bot$	$\exists E, 1, 3-5$	
7		$\neg \forall x Q(x)$	$\neg I, 2-6$		

**Ex5. Build a proof of  $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$**

## Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1    $\neg\exists x\neg Q(x)$

### Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

$$\begin{array}{l|l} 1 & \neg\exists x\neg Q(x) \\ \hline 2 & \boxed{\neg\forall xQ(x)} \end{array}$$

**Ex5. Build a proof of  $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$**

1	$\neg \exists x \neg Q(x)$		
2	$\neg \forall x Q(x)$		
3	<table style="border-collapse: collapse;"> <tr> <td style="border-right: 1px solid black; padding-right: 10px;"><math>v</math></td> <td style="padding-left: 10px;"><math>\neg Q(v)</math></td> </tr> </table>	$v$	$\neg Q(v)$
$v$	$\neg Q(v)$		

### Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1		$\neg\exists x\neg Q(x)$		
2			$\neg\forall xQ(x)$	
3				$\neg Q(v)$
4				$\exists x\neg Q(x)$ $\exists I, 3$

# Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1		$\neg\exists x\neg Q(x)$		
2			$\neg\forall xQ(x)$	
3				$\neg Q(v)$
4				$\exists x\neg Q(x)$ $\exists I, 3$
5				$\perp$ $\perp I, 1, 4$

# Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1				$\neg\exists x\neg Q(x)$			
2					$\neg\forall xQ(x)$		
3						$\neg Q(v)$	
4						$\exists x\neg Q(x)$	$\exists I, 3$
5						$\perp$	$\perp I, 1, 4$
6					$\neg\neg Q(v)$	$\neg I, 3-5$	



## Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1	$\neg\exists x\neg Q(x)$			
2	$\neg\forall xQ(x)$			
3	$v$	$\neg Q(v)$		
4		$\exists x\neg Q(x)$	$\exists I, 3$	
5		$\perp$	$\perp I, 1, 4$	
6		$\neg\neg Q(v)$	$\neg I, 3-5$	
7		$Q(v)$	$\neg E, 6$	

## Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1	$\neg\exists x\neg Q(x)$			
2	$\neg\forall xQ(x)$			
3	$v$	$\neg Q(v)$		
4		$\exists x\neg Q(x)$	$\exists I, 3$	
5		$\perp$	$\perp I, 1, 4$	
6		$\neg\neg Q(v)$	$\neg I, 3-5$	
7		$Q(v)$	$\neg E, 6$	
8	$\forall xQ(x)$		$\forall I, 3-7$	

## Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1	$\neg\exists x\neg Q(x)$			
2	$\neg\forall xQ(x)$			
3	$v$	$\neg Q(v)$		
4		$\exists x\neg Q(x)$	$\exists I, 3$	
5		$\perp$	$\perp I, 1, 4$	
6		$\neg\neg Q(v)$	$\neg I, 3-5$	
7		$Q(v)$	$\neg E, 6$	
8	$\forall xQ(x)$		$\forall I, 3-7$	
9	$\perp$		$\perp I, 2, 8$	

# Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1	$\neg\exists x\neg Q(x)$			
2	$\neg\forall xQ(x)$			
3	$v$	$\neg Q(v)$		
4		$\exists x\neg Q(x)$	$\exists I, 3$	
5		$\perp$	$\perp I, 1, 4$	
6		$\neg\neg Q(v)$	$\neg I, 3-5$	
7		$Q(v)$	$\neg E, 6$	
8		$\forall xQ(x)$	$\forall I, 3-7$	
9		$\perp$	$\perp I, 2, 8$	
10	$\neg\neg\forall xQ(x)$			$\neg I, 2-9$

# Ex5. Build a proof of $\neg\exists x\neg Q(x) \vdash \forall xQ(x)$

1	$\neg\exists x\neg Q(x)$			
2	$\neg\forall xQ(x)$			
3	$v$	$\neg Q(v)$		
4		$\exists x\neg Q(x)$	$\exists I, 3$	
5		$\perp$	$\perp I, 1, 4$	
6		$\neg\neg Q(v)$	$\neg I, 3-5$	
7		$Q(v)$	$\neg E, 6$	
8	$\forall xQ(x)$		$\forall I, 3-7$	
9	$\perp$		$\perp I, 2, 8$	
10	$\neg\neg\forall xQ(x)$			$\neg I, 2-9$
11	$\forall xQ(x)$			$\neg E, 10$

**Ex6. Build a proof of  $\forall x Q(x) \vdash \neg \exists x \neg Q(x)$**

**Ex6. Build a proof of  $\forall x Q(x) \vdash \neg \exists x \neg Q(x)$**

$$1 \quad \boxed{\forall x Q(x)}$$

**Ex6. Build a proof of  $\forall xQ(x) \vdash \neg\exists x\neg Q(x)$**

$$\begin{array}{l|l} 1 & \forall xQ(x) \\ \hline 2 & \begin{array}{l|l} & \exists x\neg Q(x) \\ \hline & \end{array} \end{array}$$



**Ex6. Build a proof of  $\forall xQ(x) \vdash \neg\exists x\neg Q(x)$**

1		$\forall xQ(x)$	
2			$\exists x\neg Q(x)$
3			$\neg\forall xQ(x)$ Ex4., 1

**Ex6. Build a proof of  $\forall xQ(x) \vdash \neg\exists x\neg Q(x)$**

1	$\forall xQ(x)$	
2	$\exists x\neg Q(x)$	
3	$\neg\forall xQ(x)$	Ex4., 1
4	$\perp$	$\perp$ I, 1, 3

**Ex6. Build a proof of  $\forall xQ(x) \vdash \neg\exists x\neg Q(x)$**

1	$\forall xQ(x)$	
2	$\exists x\neg Q(x)$	
3	$\neg\forall xQ(x)$	$\text{Ex4.}, 1$
4	$\perp$	$\perp\text{I}, 1, 3$
5	$\neg\exists xQ(x)$	$\neg\text{I}, 2-4$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

$$1 \quad | \quad \exists x(P(x) \wedge Q(x))$$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

- |   |                                    |
|---|------------------------------------|
| 1 | $\exists x(P(x) \wedge Q(x))$      |
| 2 | $\forall x(P(x) \rightarrow R(x))$ |
|   | <hr/>                              |

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$
2		$\forall x(P(x) \rightarrow R(x))$
<hr/>		
3		$v \mid \underline{P(v) \wedge Q(v)}$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3		$v$   $P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3		$v$   $P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$
5		$P(v) \rightarrow R(v)$	$\forall E, 2$



**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3		$v$   $P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$
5		$P(v) \rightarrow R(v)$	$\forall E, 2$
6		$R(v)$	$\Rightarrow E, 4, 5$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3		$v$   $P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$
5		$P(v) \rightarrow R(v)$	$\forall E, 2$
6		$R(v)$	$\Rightarrow E, 4, 5$
7		$Q(v)$	$\wedge E, 3$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3		$v$   $P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$
5		$P(v) \rightarrow R(v)$	$\forall E, 2$
6		$R(v)$	$\Rightarrow E, 4, 5$
7		$Q(v)$	$\wedge E, 3$
8		$R(v) \wedge Q(v)$	$\wedge I, 6, 7$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3		$v$   $P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$
5		$P(v) \rightarrow R(v)$	$\forall E, 2$
6		$R(v)$	$\Rightarrow E, 4, 5$
7		$Q(v)$	$\wedge E, 3$
8		$R(v) \wedge Q(v)$	$\wedge I, 6, 7$
9		$\exists x(R(x) \wedge Q(x))$	$\exists I, 8$

**Ex7. Build a proof of  $\exists x(P(x) \wedge Q(x)), \forall x(P(x) \rightarrow R(x)) \vdash \exists x(R(x) \wedge P(x))$**

1		$\exists x(P(x) \wedge Q(x))$	
2		$\forall x(P(x) \rightarrow R(x))$	
<hr/>			
3	v	$P(v) \wedge Q(v)$	
<hr/>			
4		$P(v)$	$\wedge E, 3$
5		$P(v) \rightarrow R(v)$	$\forall E, 2$
6		$R(v)$	$\Rightarrow E, 4, 5$
7		$Q(v)$	$\wedge E, 3$
8		$R(v) \wedge Q(v)$	$\wedge I, 6, 7$
9		$\exists x(R(x) \wedge Q(x))$	$\exists I, 8$
10		$\exists x(R(x) \wedge Q(x))$	$\exists E, 1, 3-9$

**Ex8. Build a proof of  $\forall x \forall y (x = y \rightarrow f(x) = f(y))$**

**Ex8. Build a proof of  $\forall x \forall y (x = y \rightarrow f(x) = f(y))$**

$$1 \quad \left| \begin{array}{c} u \\ v \end{array} \right| \quad \left| \begin{array}{c} u = v \end{array} \right|$$

**Ex8. Build a proof of  $\forall x \forall y (x = y \rightarrow f(x) = f(y))$**

$$\begin{array}{l|l|l|l} 1 & u & v & u = v \\ 2 & & & \neg(f(u) = f(v)) \end{array}$$



**Ex8. Build a proof of  $\forall x \forall y (x = y \rightarrow f(x) = f(y))$**

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	$=I$

# Ex8. Build a proof of $\forall x \forall y (x = y \rightarrow f(x) = f(y))$

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	=I
4						$f(u) = f(v)$	=E, 1, 3

**Ex8. Build a proof of  $\forall x \forall y (x = y \rightarrow f(x) = f(y))$**

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	=I
4						$f(u) = f(v)$	=E, 1, 3
5						$\perp$	$\perp$ I, 2-4

# Ex8. Build a proof of $\forall x \forall y (x = y \rightarrow f(x) = f(y))$

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	=I
4						$f(u) = f(v)$	=E, 1, 3
5						$\perp$	$\perp$ I, 2-4
6						$\neg\neg(f(u) = f(v))$	$\neg$ I, 2-5

# Ex8. Build a proof of $\forall x \forall y (x = y \rightarrow f(x) = f(y))$

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	=I
4						$f(u) = f(v)$	=E, 1, 3
5						$\perp$	$\perp$ I, 2-4
6						$\neg\neg(f(u) = f(v))$	$\neg$ I, 2-5
7						$f(u) = f(v)$	$\neg$ E, 6

# Ex8. Build a proof of $\forall x \forall y (x = y \rightarrow f(x) = f(y))$

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	=I
4						$f(u) = f(v)$	=E, 1, 3
5						$\perp$	$\perp$ I, 2-4
6						$\neg\neg(f(u) = f(v))$	$\neg$ I, 2-5
7						$f(u) = f(v)$	$\neg$ E, 6
8						$u = v \rightarrow f(u) = f(v)$	$\Rightarrow$ I, 1-7

# Ex8. Build a proof of $\forall x \forall y (x = y \rightarrow f(x) = f(y))$

1		$u$		$v$		$u = v$	
2						$\neg(f(u) = f(v))$	
3						$f(u) = f(u)$	=I
4						$f(u) = f(v)$	=E, 1, 3
5						$\perp$	$\perp$ I, 2-4
6						$\neg\neg(f(u) = f(v))$	$\neg$ I, 2-5
7						$f(u) = f(v)$	$\neg$ E, 6
8						$u = v \rightarrow f(u) = f(v)$	$\Rightarrow$ I, 1-7
9						$\forall y (u = y \rightarrow f(u) = f(y))$	$\forall$ I, 8

# Ex8. Build a proof of $\forall x \forall y (x = y \rightarrow f(x) = f(y))$

1	$u$	$v$	$u = v$	
2			$\neg(f(u) = f(v))$	
3			$f(u) = f(u)$	=I
4			$f(u) = f(v)$	=E, 1, 3
5			$\perp$	$\perp$ I, 2-4
6			$\neg\neg(f(u) = f(v))$	$\neg$ I, 2-5
7			$f(u) = f(v)$	$\neg$ E, 6
8			$u = v \rightarrow f(u) = f(v)$	$\Rightarrow$ I, 1-7
9		$\forall y (u = y \rightarrow f(u) = f(y))$		$\forall$ I, 8
10	$\forall x \forall y (x = y \rightarrow f(x) = f(y))$			$\forall$ I, 9



**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

---

**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1    $\mid \forall x P(a, x, x)$

**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1  $\forall x P(a, x, x)$

2  $\forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z)))$

**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1	$\forall x P(a, x, x)$	
2	$\forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z)))$	
3	$P(a, a, a)$	$\forall E, 1$

**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1	$\forall x P(a, x, x)$	
2	$\forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z)))$	
<hr/>		
3	$P(a, a, a)$	$\forall E, 1$
4	$\forall y \forall z (P(a, y, z) \rightarrow P(f(a), y, f(z)))$	$\forall E, 2$

**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1	$\forall x P(a, x, x)$	
2	$\forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z)))$	
<hr/>		
3	$P(a, a, a)$	$\forall E, 1$
4	$\forall y \forall z (P(a, y, z) \rightarrow P(f(a), y, f(z)))$	$\forall E, 2$
5	$\forall z (P(a, a, z) \rightarrow P(f(a), a, f(z)))$	$\forall E, 4$

**Ex9. Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$**

1	$\forall x P(a, x, x)$	
2	$\forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z)))$	
<hr/>		
3	$P(a, a, a)$	$\forall E, 1$
4	$\forall y \forall z (P(a, y, z) \rightarrow P(f(a), y, f(z)))$	$\forall E, 2$
5	$\forall z (P(a, a, z) \rightarrow P(f(a), a, f(z)))$	$\forall E, 4$
6	$P(a, a, a) \rightarrow P(f(a), a, f(a))$	$\forall E, 5$

**Ex9.** Build a proof of  $\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1	$\forall x P(a, x, x)$	
2	$\forall x \forall y \forall z (P(x, y, z) \rightarrow P(f(x), y, f(z)))$	
<hr/>		
3	$P(a, a, a)$	$\forall E, 1$
4	$\forall y \forall z (P(a, y, z) \rightarrow P(f(a), y, f(z)))$	$\forall E, 2$
5	$\forall z (P(a, a, z) \rightarrow P(f(a), a, f(z)))$	$\forall E, 4$
6	$P(a, a, a) \rightarrow P(f(a), a, f(a))$	$\forall E, 5$
7	$P(f(a), a, f(a))$	$\Rightarrow E, 3, 6$



**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

$$1 \quad \mid \quad \exists x \exists y (H(x, y) \vee H(y, x))$$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

- |   |  |  |
|---|--|--|
| 1 |  | $\exists x \exists y (H(x, y) \vee H(y, x))$ |
| 2 |  | $\neg \exists x H(x, x)$                     |
|   |  | _____  |

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

1		$\exists x \exists y (H(x, y) \vee H(y, x))$		
2		$\neg \exists x H(x, x)$		
<hr/>				
3		$u$	$v$	$u = v$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

1	$\exists x \exists y (H(x, y) \vee H(y, x))$				
2	$\neg \exists x H(x, x)$				
3	$u$	$v$		$u = v$	
4				$H(u, v) \vee H(v, u)$	
5				$H(u, v)$	

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

1	$\exists x \exists y (H(x, y) \vee H(y, x))$				
2	$\neg \exists x H(x, x)$				
<hr/>					
3	$u$	$v$		$u = v$	
				<hr/>	
4				$H(u, v) \vee H(v, u)$	
				<hr/>	
5				$H(u, v)$	
				<hr/>	
6				$H(u, u)$	$=E, 3, 5$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

1	$\exists x \exists y (H(x, y) \vee H(y, x))$			
2	$\neg \exists x H(x, x)$			
3	$u$	$v$	$u = v$	
4			$H(u, v) \vee H(v, u)$	
5			$H(u, v)$	
6			$H(u, u)$	$=E, 3, 5$
7			$\exists x H(x, x)$	$\exists I, 5$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

1	$\exists x \exists y (H(x, y) \vee H(y, x))$			
2	$\neg \exists x H(x, x)$			
3	$u$	$v$	$u = v$	
4			$H(u, v) \vee H(v, u)$	
5			$H(u, v)$	
6			$H(u, u)$	$=E, 3, 5$
7			$\exists x H(x, x)$	$\exists I, 5$
8			$\perp$	$\perp I, 2, 7$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**



**Ex10. Build a proof of**  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$

8 | | | | | :

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						⋮	
9							$H(v, u)$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						⋮	
9							$H(v, u)$
10							$H(v, v)$
							$=E, 3, 9$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						⋮	
9							$H(v, u)$
10							$H(v, v) \quad =E, 3, 9$
11							$\exists x H(x, x) \quad \exists I, 10$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						⋮	
9							$H(v, u)$
10							$H(v, v) \quad =E, 3, 9$
11							$\exists x H(x, x) \quad \exists I, 10$
12							$\perp \quad \perp I, 2, 7$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						$\vdots$	
9						$H(v, u)$	
10						$H(v, v)$	$=E, 3, 9$
11						$\exists x H(x, x)$	$\exists I, 10$
12						$\perp$	$\perp I, 2, 7$
13						$\perp$	$\vee E, 4, 5-8, 9-12$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						⋮	
9						$H(v, u)$	
10						$H(v, v)$	=E, 3, 9
11						$\exists x H(x, x)$	∃I, 10
12						$\perp$	⊥I, 2, 7
13						$\perp$	∀E, 4, 5–8, 9–12
14						$\perp$	∃E, 1, 4–13

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8					⋮	
9					$H(v, u)$	
10					$H(v, v)$	=E, 3, 9
11					$\exists x H(x, x)$	$\exists I, 10$
12					$\perp$	$\perp I, 2, 7$
13					$\perp$	$\vee E, 4, 5-8, 9-12$
14					$\perp$	$\exists E, 1, 4-13$
15					$\neg(u = v)$	$\neg I, 3-14$



**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						$\vdots$	
9						$H(v, u)$	
10						$H(v, v)$	$=E, 3, 9$
11						$\exists x H(x, x)$	$\exists I, 10$
12						$\perp$	$\perp I, 2, 7$
13						$\perp$	$\vee E, 4, 5-8, 9-12$
14						$\perp$	$\exists E, 1, 4-13$
15						$\neg(u = v)$	$\neg I, 3-14$
16						$\exists y \neg(u = y)$	$\exists I, 15$

**Ex10. Build a proof of  $\exists x \exists y (H(x, y) \vee H(y, x)), \neg \exists x H(x, x) \vdash \exists x \exists y \neg (x = y)$**

8						$\vdots$	
9						$H(v, u)$	
10						$H(v, v)$	$=E, 3, 9$
11						$\exists x H(x, x)$	$\exists I, 10$
12						$\perp$	$\perp I, 2, 7$
13						$\perp$	$\vee E, 4, 5-8, 9-12$
14						$\perp$	$\exists E, 1, 4-13$
15						$\neg(u = v)$	$\neg I, 3-14$
16						$\exists y \neg(u = y)$	$\exists I, 15$
17						$\exists x \exists y \neg(x = y)$	$\exists I, 16$

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

1    $\underbrace{\exists y \exists x Q(y, x)}$

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

$$\begin{array}{l|l} 1 & \exists y \exists x Q(y, x) \\ \hline 2 & u \mid \exists x Q(u, x) \\ & \hline \end{array}$$

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

1		$\exists y \exists x Q(y, x)$	
2		$u$	$\exists x Q(u, x)$
3		$v$	$Q(u, v)$

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

1		$\exists y \exists x Q(y, x)$	
2		$u$   $\exists x Q(u, x)$	
3		$v$   $Q(u, v)$	
4		$\exists y Q(y, v)$	$\exists I, 3$

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

1		$\exists y \exists x Q(y, x)$	
<hr/>			
2		$u$   $\exists x Q(u, x)$	
<hr/>			
3		$v$   $Q(u, v)$	
<hr/>			
4		$\exists y Q(y, v)$	$\exists I, 3$
<hr/>			
5		$\exists y Q(y, v)$	$\exists E, 2, 3-4$



**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

1		$\exists y \exists x Q(y, x)$	
2		$u$   $\exists x Q(u, x)$	
3		$v$   $Q(u, v)$	
4		$\exists y Q(y, v)$	$\exists I, 3$
5		$\exists y Q(y, v)$	$\exists E, 2, 3-4$
6		$\exists x \exists y Q(y, x)$	$\exists I, 5$

**Ex11. Build a proof of  $\exists y \exists x Q(y, x) \vdash \exists x \exists y Q(y, x)$**

1		$\exists y \exists x Q(y, x)$	
2		$u$   $\exists x Q(u, x)$	
3			$v$   $Q(u, v)$
4			$\exists y Q(y, v)$ $\exists I, 3$
5		$\exists y Q(y, v)$	$\exists E, 2, 3-4$
6		$\exists x \exists y Q(y, x)$	$\exists I, 5$
7		$\exists x \exists y Q(y, x)$	$\exists E, 1, 2-6$

**Ex12. Build a proof of  $\vdash (\exists x P(x) \rightarrow \forall x R(x)) \rightarrow (\forall x (P(x) \rightarrow Q(x)))$**

**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

$$\begin{array}{l|l|l} 1 & & \exists xP(x) \rightarrow \forall xR(x) \\ 2 & u & P(u) \end{array}$$

**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

1			$\exists xP(x) \rightarrow \forall xR(x)$	
2		$u$	$P(u)$	
3			$\exists xP(x)$	$\exists I, 2$

**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

1			$\exists xP(x) \rightarrow \forall xR(x)$	
2		$u$	$P(u)$	
3			$\exists xP(x)$	$\exists I, 2$
4			$\forall xR(x)$	$\Rightarrow E, 1, 3$

**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

1			$\exists xP(x) \rightarrow \forall xR(x)$	
2		$u$	$P(u)$	
3			$\exists xP(x)$	$\exists I, 2$
4			$\forall xR(x)$	$\Rightarrow E, 1, 3$
5			$R(u)$	$\forall E, 4$

**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

1			$\exists xP(x) \rightarrow \forall xR(x)$	
2		$u$	$P(u)$	
3			$\exists xP(x)$	$\exists I, 2$
4			$\forall xR(x)$	$\Rightarrow E, 1, 3$
5			$R(u)$	$\forall E, 4$
6			$P(u) \rightarrow Q(u)$	$\Rightarrow I, 2-5$



**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

1			$\exists xP(x) \rightarrow \forall xR(x)$	
2		$u$	$P(u)$	
3			$\exists xP(x)$	$\exists I, 2$
4			$\forall xR(x)$	$\Rightarrow E, 1, 3$
5			$R(u)$	$\forall E, 4$
6			$P(u) \rightarrow Q(u)$	$\Rightarrow I, 2-5$
7			$\forall x(P(x) \rightarrow Q(x))$	$\forall I, 2-6$

**Ex12. Build a proof of  $\vdash (\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$**

1			$\exists xP(x) \rightarrow \forall xR(x)$	
2			$u$   $P(u)$	
3			$\exists xP(x)$	$\exists I, 2$
4			$\forall xR(x)$	$\Rightarrow E, 1, 3$
5			$R(u)$	$\forall E, 4$
6			$P(u) \rightarrow Q(u)$	$\Rightarrow I, 2-5$
7			$\forall x(P(x) \rightarrow Q(x))$	$\forall I, 2-6$
8			$(\exists xP(x) \rightarrow \forall xR(x)) \rightarrow (\forall x(P(x) \rightarrow Q(x)))$	$\Rightarrow I, 1-7$

**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

$$1 \quad | \quad \underline{x = f(y)}$$

**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

$$\begin{array}{l|l|l} 1 & & x = f(y) \\ & \hline 2 & u & \boxed{P(x, u)} \end{array}$$

**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

1			$x = f(y)$	
2			$u$	$P(x, u)$
3				$P(f(y), u)$

=E, 1, 2

**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

1			$x = f(y)$	
2			$u$   $P(x, u)$	
3			$P(f(y), u)$	$=E, 1, 2$
4			$P(x, u) \rightarrow P(f(y), u)$	$\Rightarrow I, 2-3$

**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

1			$x = f(y)$	
2			$u$   $P(x, u)$	
3			$P(f(y), u)$	$=E, 1, 2$
4			$P(x, u) \rightarrow P(f(y), u)$	$\Rightarrow I, 2-3$
5			$\forall z(P(x, z) \rightarrow P(f(y), z))$	$\forall I, 2-4$



**Ex13. Build a proof of  $\vdash x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$**

1			$x = f(y)$	
2			$u$   $P(x, u)$	
3			$P(f(y), u)$	$=E, 1, 2$
4			$P(x, u) \rightarrow P(f(y), u)$	$\Rightarrow I, 2-3$
5			$\forall z(P(x, z) \rightarrow P(f(y), z))$	$\forall I, 2-4$
6			$x = f(y) \rightarrow \forall z(P(x, z) \rightarrow P(f(y), z))$	$\Rightarrow I, 1-5$

**Ex14. Build a proof of  $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$**

**Ex14. Build a proof of**  $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1     $| \ a \ | \ b \ | \ c \ | \ d \ | \ \underline{\hspace{1cm}} \ a = b$

**Ex14. Build a proof of  $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$**

1		$a$		$b$		$c$		$d$			$a = b$
2											$c = d$

**Ex14. Build a proof of  $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$**

1		$a$		$b$		$c$		$d$		$a = b$	
2											$c = d$
3											$f(a, c) = f(a, c)$

=I

**Ex14. Build a proof of  $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$**

1	$a$	$b$	$c$	$d$	$a = b$	
2					$c = d$	
3					$f(a, c) = f(a, c)$	=I
4					$f(a, c) = f(b, c)$	=E, 1, 3

**Ex14. Build a proof of  $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$**

1	$a$	$b$	$c$	$d$	$a = b$	
2					$c = d$	
3					$f(a, c) = f(a, c)$	=I
4					$f(a, c) = f(b, c)$	=E, 1, 3
5					$f(a, c) = f(b, d)$	=E, 2, 4

# Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1		<i>a</i>		<i>b</i>		<i>c</i>		<i>d</i>		<i>a</i> = <i>b</i>	
2										<i>c</i> = <i>d</i>	
3										<i>f</i> ( <i>a</i> , <i>c</i> ) = <i>f</i> ( <i>a</i> , <i>c</i> )	=I
4										<i>f</i> ( <i>a</i> , <i>c</i> ) = <i>f</i> ( <i>b</i> , <i>c</i> )	=E, 1, 3
5										<i>f</i> ( <i>a</i> , <i>c</i> ) = <i>f</i> ( <i>b</i> , <i>d</i> )	=E, 2, 4
6										<i>c</i> = <i>d</i> → <i>f</i> ( <i>a</i> , <i>c</i> ) = <i>f</i> ( <i>b</i> , <i>d</i> )	⇒I, 2-5



# Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1		<i>a</i>		<i>b</i>		<i>c</i>		<i>d</i>		<i>a = b</i>	
2										<i>c = d</i>	
3										$f(a, c) = f(a, c)$	=I
4										$f(a, c) = f(b, c)$	=E, 1, 3
5										$f(a, c) = f(b, d)$	=E, 2, 4
6										$c = d \rightarrow f(a, c) = f(b, d)$	$\Rightarrow$ I, 2-5
7										$a = b \rightarrow (c = d \rightarrow f(a, c) = f(b, d))$	$\Rightarrow$ I, 1-6

# Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1		<i>a</i>		<i>b</i>		<i>c</i>		<i>d</i>		<i>a</i> = <i>b</i>	
2										<i>c</i> = <i>d</i>	
3										$f(a, c) = f(a, c)$	=I
4										$f(a, c) = f(b, c)$	=E, 1, 3
5										$f(a, c) = f(b, d)$	=E, 2, 4
6										$c = d \rightarrow f(a, c) = f(b, d)$	$\Rightarrow$ I, 2-5
7										$a = b \rightarrow (c = d \rightarrow f(a, c) = f(b, d))$	$\Rightarrow$ I, 1-6
8										$\forall v (a = b \rightarrow (c = v \rightarrow f(a, c) = f(b, v)))$	$\forall$ I, 7

# Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1						$a = b$	
2							$c = d$
3							$f(a, c) = f(a, c)$ =I
4							$f(a, c) = f(b, c)$ =E, 1, 3
5							$f(a, c) = f(b, d)$ =E, 2, 4
6							$c = d \rightarrow f(a, c) = f(b, d)$ $\Rightarrow$ I, 2-5
7							$a = b \rightarrow (c = d \rightarrow f(a, c) = f(b, d))$ $\Rightarrow$ I, 1-6
8							$\forall v (a = b \rightarrow (c = v \rightarrow f(a, c) = f(b, v)))$ $\forall$ I, 7
9							$\forall u \forall v (a = u \rightarrow (c = v \rightarrow f(a, c) = f(u, v)))$ $\forall$ I, 8

# Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1		a		b		c		d		a = b	
2										c = d	
3										f(a, c) = f(a, c)	=I
4										f(a, c) = f(b, c)	=E, 1, 3
5										f(a, c) = f(b, d)	=E, 2, 4
6										c = d → f(a, c) = f(b, d)	⇒I, 2-5
7										a = b → (c = d → f(a, c) = f(b, d))	⇒I, 1-6
8										∀v(a = b → (c = v → f(a, c) = f(b, v)))	∀I, 7
9										∀u∀v(a = u → (c = v → f(a, c) = f(u, v)))	∀I, 8
10										∀y∀u∀v(a = u → (y = v → f(a, y) = f(u, v)))	∀I, 9

# Ex14. Build a proof of $\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$

1	$a$	$b$	$c$	$d$	$a = b$	
2					$c = d$	
3					$f(a, c) = f(a, c)$	=I
4					$f(a, c) = f(b, c)$	=E, 1, 3
5					$f(a, c) = f(b, d)$	=E, 2, 4
6					$c = d \rightarrow f(a, c) = f(b, d)$	$\Rightarrow$ I, 2-5
7					$a = b \rightarrow (c = d \rightarrow f(a, c) = f(b, d))$	$\Rightarrow$ I, 1-6
8					$\forall v (a = b \rightarrow (c = v \rightarrow f(a, c) = f(b, v)))$	$\forall$ I, 7
9					$\forall u \forall v (a = u \rightarrow (c = v \rightarrow f(a, c) = f(u, v)))$	$\forall$ I, 8
10					$\forall y \forall u \forall v (a = u \rightarrow (y = v \rightarrow f(a, y) = f(u, v)))$	$\forall$ I, 9
11					$\forall x \forall y \forall u \forall v (x = u \rightarrow (y = v \rightarrow f(x, y) = f(u, v)))$	$\forall$ I, 10

**Ex15. Build a proof of**  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

**Ex15. Build a proof of**  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

1     $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$

**Ex15. Build a proof of**  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$

$$\begin{array}{l|l|l} 1 & \exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \\ 2 & a \quad b & \boxed{P(a) \wedge P(b)} \end{array}$$



**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	

**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$

**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$
5			$P(b) \rightarrow b = a$	$\wedge E, 3$

**Ex15. Build a proof of  $\exists x\forall y(P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x\forall y((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x\forall y(P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$
5			$P(b) \rightarrow b = a$	$\wedge E, 3$
6			$b = a$	$\Rightarrow E, 4, 5$

**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$
5			$P(b) \rightarrow b = a$	$\wedge E, 3$
6			$b = a$	$\Rightarrow E, 4, 5$
7			$b = b$	$=I$

**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$
5			$P(b) \rightarrow b = a$	$\wedge E, 3$
6			$b = a$	$\Rightarrow E, 4, 5$
7			$b = b$	$=I$
8			$a = b$	$=E, 6, 7$

**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$
5			$P(b) \rightarrow b = a$	$\wedge E, 3$
6			$b = a$	$\Rightarrow E, 4, 5$
7			$b = b$	$=I$
8			$a = b$	$=E, 6, 7$
9			$a = b$	$\exists E, 1, 3-8$

**Ex15. Build a proof of  $\exists x\forall y(P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x\forall y((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x\forall y(P(x) \wedge (P(y) \rightarrow y = x))$			
2	$a$	$b$	$P(a) \wedge P(b)$	
3			$P(a) \wedge (P(b) \rightarrow b = a)$	
4			$P(b)$	$\wedge E, 2$
5			$P(b) \rightarrow b = a$	$\wedge E, 3$
6			$b = a$	$\Rightarrow E, 4, 5$
7			$b = b$	$=I$
8			$a = b$	$=E, 6, 7$
9			$a = b$	$\exists E, 1, 3-8$
10			$(P(a) \wedge P(b)) \rightarrow a = b$	$\Rightarrow I, 2-9$



**Ex15. Build a proof of  $\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$**

1	$\exists x \forall y (P(x) \wedge (P(y) \rightarrow y = x))$		
2	$a$	$b$	$P(a) \wedge P(b)$
3			$P(a) \wedge (P(b) \rightarrow b = a)$
4			$P(b)$ $\wedge E, 2$
5			$P(b) \rightarrow b = a$ $\wedge E, 3$
6			$b = a$ $\Rightarrow E, 4, 5$
7			$b = b$ $=I$
8			$a = b$ $=E, 6, 7$
9		$a = b$	$\exists E, 1, 3-8$
10		$(P(a) \wedge P(b)) \rightarrow a = b$	$\Rightarrow I, 2-9$
11	$\forall y ((P(a) \wedge P(y)) \rightarrow a = y)$		$\forall I, 10$

# Ex15. Build a proof of $\exists x\forall y(P(x) \wedge (P(y) \rightarrow y = x)) \vdash \forall x\forall y((P(x) \wedge P(y)) \rightarrow x = y)$

1	$\exists x\forall y(P(x) \wedge (P(y) \rightarrow y = x))$		
2	$a$	$b$	$P(a) \wedge P(b)$
3			$P(a) \wedge (P(b) \rightarrow b = a)$
4			$P(b)$ $\wedge E, 2$
5			$P(b) \rightarrow b = a$ $\wedge E, 3$
6			$b = a$ $\Rightarrow E, 4, 5$
7			$b = b$ $=I$
8			$a = b$ $=E, 6, 7$
9			$a = b$ $\exists E, 1, 3-8$
10			$(P(a) \wedge P(b)) \rightarrow a = b$ $\Rightarrow I, 2-9$
11			$\forall y((P(a) \wedge P(y)) \rightarrow a = y)$ $\forall I, 10$
12			$\forall x\forall y((P(x) \wedge P(y)) \rightarrow x = y)$ $\forall I, 11$