

5. Behavioural equivalences

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Recall

1. Non-deterministic Finite Automata: 
2. Process algebra: $P = a.Q \quad Q = b.Q \quad P|Q$
3. Interaction between processes
4. Meaning of PA using NFA

Still missing

- When is a process P **equivalent** to a process Q ?
- When can a process P be **safely replaced** by a process Q ?
- When can a sequence of interactions be **safely implemented** as interacting components?

- High-level overview or requirements and associated processes
- Mathematical Preliminaries
 - Basic mathematical notations
 - Set theory
 - Propositional Logic
 - First Order Logic
 - The Z3 automatic theorem prover
- Behavioural modelling
 - Single component
 - Many components
 - Equivalences
 - Language Equivalence
 - (Bi)similarity
 - Realisability
 - Verification

Behavioural Equivalences – Intuition

Two automata (or LTS) should be **equivalent** if they cannot be distinguished by interacting with them.

Equality of functional behaviour

is not preserved by **parallel** composition: non **compositional** semantics, cf,

$$x:=4; \ x:=x+1 \quad \text{and} \quad x:=5$$

Graph isomorphism

is too strong (why?)

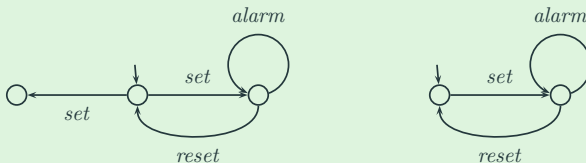
EQ1 – Language equivalence

Language equivalence

Definition

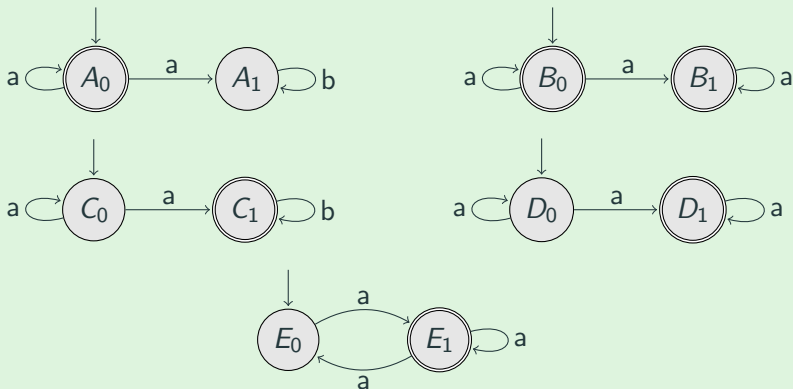
Two automata A, B are **language equivalent** iff $L_A = L_B$
(i.e. if they can perform the same finite sequences of transitions)

Example



Language equivalence applies when one can neither interact with a system, nor distinguish a slow system from one that has come to a stand still.

Ex. 5.1: Find pairs of automata with the same language



EQ2 – Similarity

the quest for a **behavioural equality**:
able to identify states that cannot be distinguished by any **realistic** form of observation

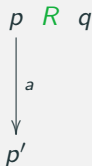
Simulation

A state q **simulates** another state p if
every transition from q is corresponded by a transition from p **and**
this capacity is kept along the whole life of the system to which state space q belongs
to.

Definition

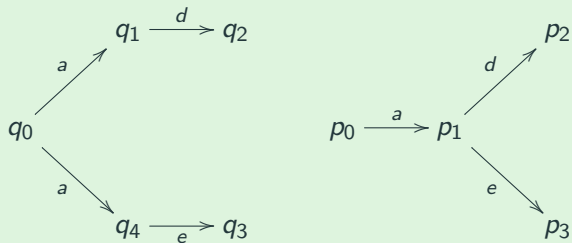
Given $\langle S_1, N, \longrightarrow_1 \rangle$ and $\langle S_2, N, \longrightarrow_2 \rangle$ over N (ignoring initial and final states) a relation $R \subseteq S_1 \times S_2$ is a **simulation** iff, for all $\langle p, q \rangle \in R$ and $a \in N$,

$$(1) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$



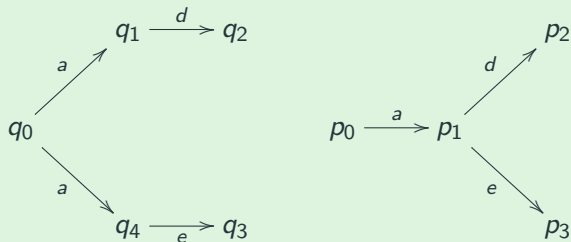
Example

Ex. 5.2: Find simulations



Example

Ex. 5.2: Find simulations



$$q_0 \lesssim p_0 \quad \text{cf.} \quad \{\langle q_0, p_0 \rangle, \langle q_1, p_1 \rangle, \langle q_4, p_1 \rangle, \dots\}$$

Definition

$$p \lesssim q \equiv \langle \exists R :: R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

We say *p is simulated by q*.

Lemma

The similarity relation is a preorder
(ie, reflexive and transitive)

EQ3 – Bisimilarity

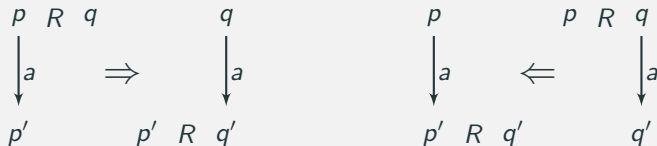
Definition

Given $\langle S_1, N, \longrightarrow_1 \rangle$ and $\langle S_2, N, \longrightarrow_2 \rangle$ over N , relation $R \subseteq S_1 \times S_2$ is a **bisimulation** iff both R and its converse R° are simulations.

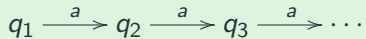
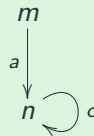
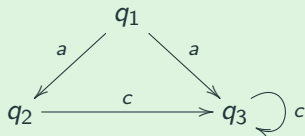
I.e., whenever $\langle p, q \rangle \in R$ and $a \in N$,

$$(1) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$(2) \quad q \xrightarrow{a}_2 q' \Rightarrow \langle \exists p' : p' \in S_1 : p \xrightarrow{a}_1 p' \wedge \langle p', q' \rangle \in R \rangle$$

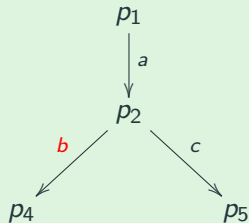
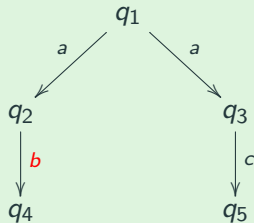
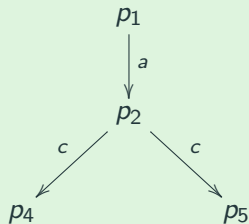
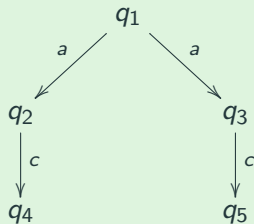


Ex. 5.3: Find bisimulations that include q_1



Examples

Ex. 5.4: Find bisimulations that include q_1



Definition

$$p \sim q \equiv \langle \exists R :: R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

We say *p is bisimilar to q*.

Lemma

Two processes P and Q are bisimilar if there is a bisimulation that includes $\langle P, Q \rangle$.

Warning

$\left[p \lesssim q \text{ and } q \lesssim p \right]$ does **not** imply $\left[p \sim q \right]$

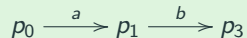
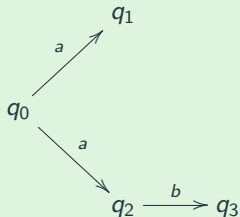
Properties

Warning

$[p \lesssim q \text{ and } q \lesssim p]$ does **not** imply $[p \sim q]$

Example

$q_0 \lesssim p_0, p_0 \lesssim q_0$ but $p_0 \not\sim q_0$



Similarity as the greatest simulation

$$\lesssim \triangleq \bigcup \{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

$$\sim \triangleq \bigcup \{S \mid S \text{ is a bisimulation}\}$$

Ex. 5.5: P,Q Bisimilar?

$$P = a.P_1$$

$$P_1 = b.P + c.P$$

$$Q = a.Q_1$$

$$Q_1 = b.Q_2 + c.Q$$

$$Q_2 = a.Q_3$$

$$Q_3 = b.Q + c.Q_2$$

Ex. 5.6: P,Q Bisimilar?

$$P = a.(b.0 + 0)$$

$$Q = a.b.0$$

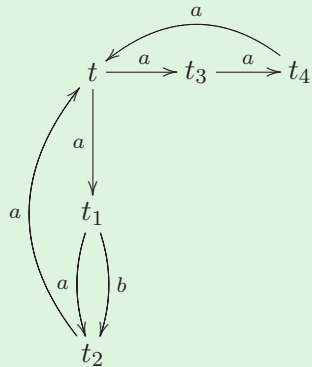
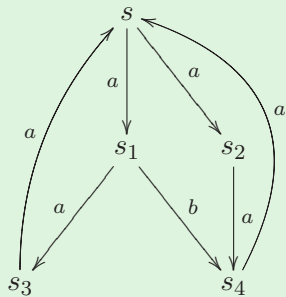
Ex. 5.7: P,Q Bisimilar?

$$P = a.(b.0 + c.0)$$

$$Q = a.b.0 + a.c.0$$

Draw their LTS. If bisimilar, find the bisimulation.

Ex. 5.8: Find a bisimulation



Weak bisimilarity

Considering τ -transitions

Weak transition

$$\begin{aligned} p \xRightarrow{\alpha} q & \text{ iff } p (\xrightarrow{\tau})^* q_1 \xrightarrow{a} q_2 (\xrightarrow{\tau})^* q \\ p \xRightarrow{\tau} q & \text{ iff } p (\xrightarrow{\tau})^* q \end{aligned}$$

where $\alpha \neq \tau$ and $(\xrightarrow{\tau})^*$ is the reflexive and transitive closure of $\xrightarrow{\tau}$.

Weak bisimulation (vs. strong)

Given $\langle S_1, N, \xrightarrow{}_1 \rangle$ and $\langle S_2, N, \xrightarrow{}_2 \rangle$ over N , relation $R \subseteq S_1 \times S_2$ is a **bisimulation** iff for all $\langle p, q \rangle \in R$ and $a \in N \cup \{\tau\}$,

$$(1) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xRightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$(2) \quad q \xrightarrow{a}_2 q' \Rightarrow \langle \exists p' : p' \in S_1 : p \xRightarrow{a}_1 p' \wedge \langle p', q' \rangle \in R \rangle$$

Considering τ -transitions

Branching bisimulation

Given $\langle S_1, N, \longrightarrow_1 \rangle$ and $\langle S_2, N, \longrightarrow_2 \rangle$ over N , relation $R \subseteq S_1 \times S_2$ is a **bisimulation** iff for all $\langle p, q \rangle \in R$ and $a \in N \cup \{\tau\}$,

(1) if $p \xrightarrow{a}_1 p'$ then either

(1.1) $a = \tau$ and $\langle p', q \rangle \in R$ or

(1.2) $\langle \exists q', q'' \in S_2 :: q (\xrightarrow{\tau}_2)^* q' \xrightarrow{a}_2 q'' \wedge \langle p, q' \rangle \in R \wedge \langle p', q'' \rangle \in R \rangle$

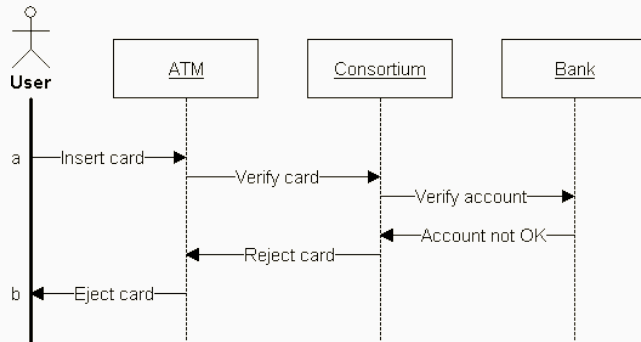
(2) if $q \xrightarrow{a}_2 q'$ then either

(2.1) $a = \tau$ and $\langle p', q' \rangle \in R$ or

(2.2) $\langle \exists p', p'' \in S_1 :: p (\xrightarrow{\tau}_1)^* p' \xrightarrow{a}_1 p'' \wedge \langle p', q \rangle \in R \wedge \langle p'', q' \rangle \in R \rangle$

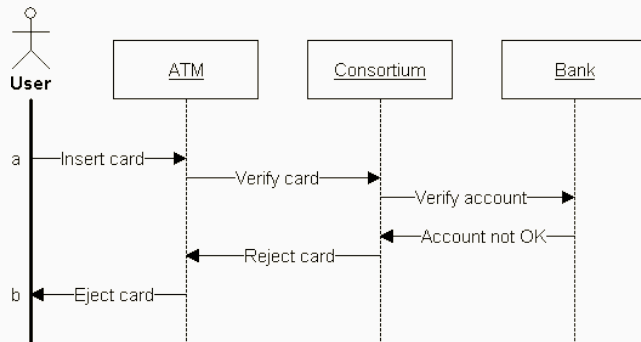
Realisability of Sequence Diagrams

Recall: Sequence Diagrams as Interactive Processes



- **Objects** as **Processes**
(e.g., processes U , A , C , B)
- **Send** actions (e.g., *insertCard*)
- **Receive** actions (e.g., $\overline{\text{insertCard}}$)
- Unique action for each object pair
- Do not write $(\dots + \mathbf{0})$

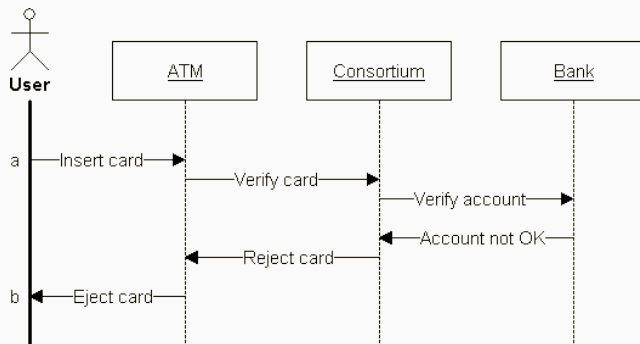
Recall: Language of Sequence Diagrams, Informally



This example has only 1 word and its prefixes

$$L_{sd} = \text{insertCard} \cdot \text{verifyCard} \cdot \text{verifyAccount} \cdot \text{accountNotOK} \cdot \\ \text{rejectedCard} \cdot \text{ejectCard}$$

Recall: Sequence Diagrams as Interactive Processes



We can specify a SD as interactive processes

$$\begin{aligned} \text{Sys} &= (U|A|C|E) \backslash \dots \\ U &= \text{insertCard}.\overline{\text{ejectCard}}.0 \\ A &= \dots \\ C &= \dots \\ E &= \dots \end{aligned}$$

Sequence Diagrams covered by Interactive Processes

- **Sequence diagrams** depict **scenarios**
(possible sequence of actions)
- **Processes** abstract **implementations**
(simplified view of concrete implementations)

Processes can do more

E.g., an ATM that also *accepts* cards can (and should) still support the *rejection* scenario.

Observing the interactions

We want to **observe** interactions in such processes

Modified CCS semantics

$$\begin{array}{c} \text{(com1)} \\ \frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q} \\ \text{(com2)} \\ \frac{Q \xrightarrow{\alpha} Q'}{P|Q \xrightarrow{\alpha} P|Q'} \\ \text{(com3)} \\ \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\bar{a}} Q'}{P|Q \xrightarrow{\tau_a} P'|Q'} \end{array}$$

$\alpha \in N \cup \bar{N} \cup \{\tau_a \mid a \in N\}$ is an action

Recall Sys from Slide 24 and its diagram sd .

$$L_{sd} = \{iC \cdot vC \cdot cA \cdot aN \cdot rC \cdot eC\}$$

$$L_{Sys} = \{\tau_{iC} \cdot \tau_{vC} \cdot \tau_{cA} \cdot \tau_{aN} \cdot \tau_{rC} \cdot \tau_{eC}\}$$

Language inclusion

P includes sd

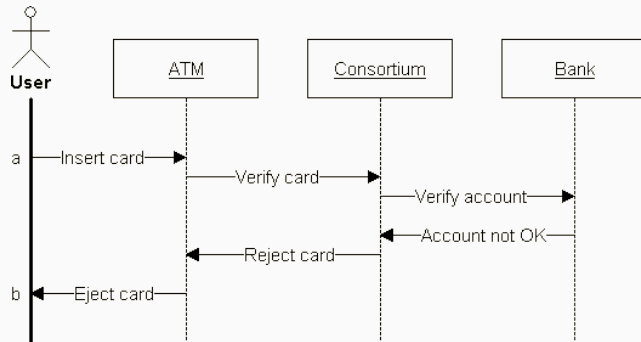
iff

$$L_{sd} \subseteq L_{P^\dagger}$$

P^\dagger modifies P 's LTS by:

filtering actions of sd and replacing τ_a by a

Are words enough?



Ex. 5.9: Let sd be the diagram above and recall Slide 24

Does Sys still includes sd if U is instead defined as below?

1. $U = insertCard.\overline{ejectCard}.0 + insertCard.0$
2. $U = (insertCard.\overline{ejectCard}.0) + goAway.0$

Is language coverage enough?

Implementations can have:

- extra undesirable behaviour
- less behaviour

Alternative: change the inclusion/equivalence

Let $SD = \{sd_1, sd_2, \dots\}$ be a set of sequence diagrams.

Language inclusion: $L_{SD} \subseteq L_{P^\dagger}$

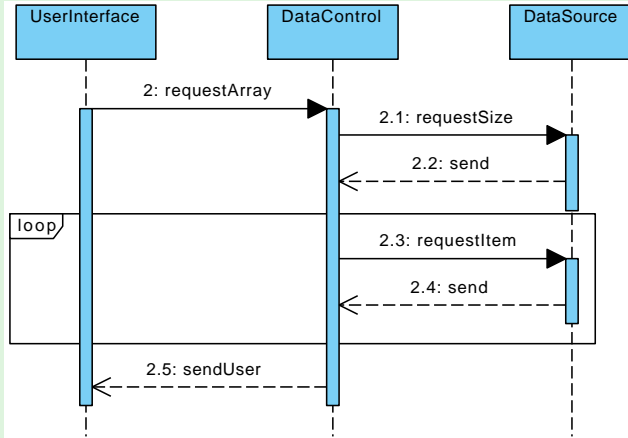
Language equivalence: $L_{SD} = L_{P^\dagger}$

Similarity: $NFA(SD) \lesssim P^\dagger$

Bisimilarity: $NFA(SD) \sim P^\dagger$

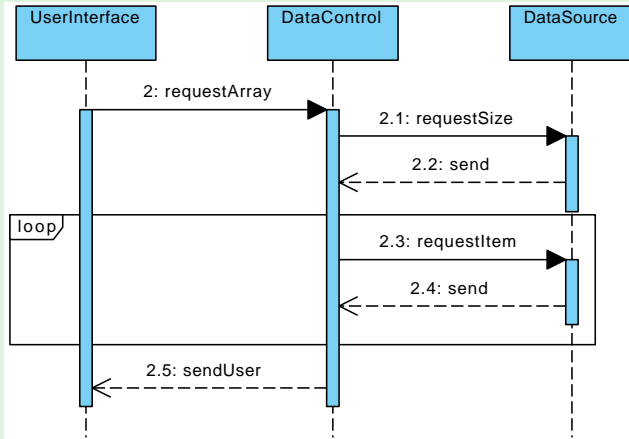
Exercise

Ex. 5.10: Draw an NFA that captures the following diagram



Exercise

Ex. 5.11: Write a process for each object of the diagram



Realisability

Question: after encoding SD into processes:

Can we recover the behaviour of the original sequence diagram
by composing
the encoded processes?

Realisability

A set SD of sequence diagrams is **realisable**

iff

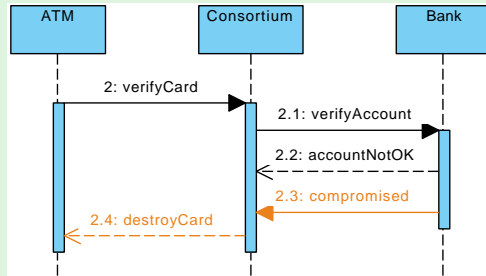
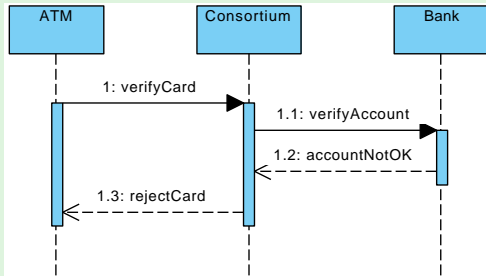
$$\text{NFA}(SD) \sim \text{Comp}(\text{Proc}(SD))^{\dagger}$$

$\text{Proc}(SD)$ returns the set of encoded processes for each $sd \in SD$

$\text{Comp}(P_1, P_2, \dots) = (P_1 | P_2 | \dots) \setminus \{\text{actions of } SD\}$

Exercise

Ex. 5.12: Are the diagrams below realisable?



1. draw $NFA(SD)$
2. calculate $Proc(SD)$
Hint: $B = \overline{vA}.(aN.0 + aN.c.0)$
3. draw $Comp(\cdot)$
4. search for a bisimulation

Ex. 5.13: Verify if the diagram in Slide 31 is realisable.

Exercise

Ex. 5.14: Verify if the diagram is realisable.

