5. First Order Logic – Natural Deduction

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Natural Deduction in First Order Logic

Recalling FOL...

FOL language

A language of FOL considers the following sets of symbols:

logical symbols of one of the following forms: a **set of variables** $S = \{x, y, \ldots, x_0, y_0, \ldots\}$; **logical connectives** $\wedge, \vee, \neg$, and $\rightarrow$; **quantifiers** $\forall$ (for all) and $\exists$ (exists); parenthesis (and); possibly, the **equality** symbol $=$

Non-logical symbols of one of the following forms: a (possibly empty) set of **functional symbols** for each n-arity, represented as $\mathcal{F}_n$ (when referring to constants, we are actually talking about functional symbols with arity 0). Typically, f, g, h, $\ldots$; a (possibly empty) set of **relation symbols** for each n-arity, represented as $\mathcal{R}_n$. Typically, P, Q, R, $\ldots$

First Order Logic - Syntax

FOL Terms and Atoms

Let $\mathcal{L}$ be a FOL language. A term is inductively/recursively defined as follows:

- a variable $x \in \mathcal{V}$ is a term;
- a constant i.e., a symbol $c \in \mathcal{F}_0$ is also a term;
- if $t_0, \ldots, t_n$ are terms and $f \in \mathcal{F}_n$ is a functional symbol, then $f(t_0, \ldots, t_n)$ is a term.

A FOL term is said to be **closed** if no variables occur in such term.

Let $\mathcal{L}$ be a FOL language. An **atom** (from the term atomic formula) is inductively/recursively defined as follows:

- if $t_0, \ldots, t_n$ are terms and $R \in \mathcal{R}_n$ is a relational symbol, then $R(t_0, \ldots, t_n)$ is an atom;
- if $\mathcal{L}$ include the equality symbol $=$ and if t_1 and t_2 are terms, then $t_1 = t_2$ is an atom.

First Order Logic - Syntax

FOL Formulae

Let $\mathcal{L}$ be a FOL language. The set of **formulae** is inductively/recursively defined as follows:

- an atom is a formula;
- if φ is a formula, then so is $\neg\varphi$;
- if φ and ψ are formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$;
- if φ is a formula and x is a variable, then $\forall x, \varphi$ and $\exists x, \varphi$ are also formulas.

Bound and free variables

Bound Variable

A variable x is said to be **bound** to a formula φ if φ has a subformula ψ whose schema is $\forall x, \theta$ or $\exists x, \theta$ and x occurs in θ.

Free Variable

A variable x is said to be **free** if it is not bound.

Proposition

A formula is said to be a proposition if it does not contain free variables.

Variable Substitution

Substitution

Let $\mathcal{L}$ be a FOL language, φ a formula, t a term, and $x \in \mathcal{L}$ a variable. The substitution of the variable x by the term t in φ is denoted by $\varphi[t/x]$ and corresponds to replacing all the free occurrences of x in φ by the term t.

Natural Deduction Rules

Which are the new rules (on top of Propositional Logic)?

Elimination rule for $\forall$

If we know that $\forall x, \varphi$ holds, then we can conclude that φ holds for a specific term t

$$\frac{\forall x\, \varphi}{\varphi[t/x]}\ \forall\mathbf{E}$$

Introduction rule for $\forall$

If we assume some term t and we are able to prove that $\varphi[t/x]$ then we can conclude that $\forall x, \varphi$.

$$\frac{\begin{array}{l}[t] \\ \quad\quad\vdots \\ \quad\quad\quad\quad\varphi[t/x]\end{array}}{\forall x\, \varphi[t/x]}\ \forall\mathbf{I}$$

Examples of reasoning about "for all"

Lets prove that $\forall x, (P(x) \to Q(x)), \forall x, P(x) \vdash \forall x, Q(x)$.

1		$\forall x, (P(x) \to Q(x))$	
2		$\forall x, P(x)$	
3	t	$P(t) \to Q(t)$	$\forall\mathbf{E}(1)$
4		$P(t)$	$\forall\mathbf{E}(2)$
5		$Q(t)$	$\to\mathbf{E}(3, 4)$
6		$\forall x, Q(x)$	$\forall\mathbf{I}(3\text{-}5)$

Examples of reasoning about "for all"

Lets prove that $P(t), \forall x(P(x) \to Q(x)) \vdash \neg Q(t)$.

1	$P(t)$	
2	$\forall x, (P(x) \to Q(x))$	
3	$P(t) \to Q(t)$	$\forall\mathbf{E}(2)$
4	$\neg Q(t)$	$\to\mathbf{E}(3,1)$

Examples of reasoning about "for all"

Lets prove that $\vdash \forall x(P(x) \rightarrow Q(x)) \rightarrow (\forall x, P(x) \rightarrow \forall x, Q(x))$.

1	$\forall x(P(x) \rightarrow Q(x))$	
2	$\forall x P(x)$	
3	$t \quad P(t)$	$\forall \mathbf{E}(2)$
4	$P(t) \rightarrow Q(t)$	$\forall \mathbf{E}(1)$
5	$Q(t)$	$\rightarrow \mathbf{E}(3,4)$
6	$\forall x Q(x)$	$\forall \mathbf{I}$(3-5)
7	$\forall x P(x) \rightarrow \forall x Q(x)$	$\rightarrow \mathbf{I}(2-6)$
8	$\forall x(P(x) \rightarrow Q(x)) \rightarrow (\forall x, P(x) \rightarrow \forall x, Q(x))$	$\rightarrow \mathbf{I}(1-7)$

Which are the new rules (on top of Propositional Logic)?

Elimination rule for $\exists$

If we know that $\exists x, \varphi$ holds, and if assuming term t and $\varphi[t/x]$ we can deduce ψ, then we can prove ψ overall.

$$\dfrac{\exists x\, \varphi \qquad \begin{array}{c}[t \quad \varphi[t/x]] \\ \vdots \\ \psi\end{array}}{\psi}\ \exists\mathbf{E}$$

Introduction rule for $\exists$

If we assume some term t and we are able to prove that $\varphi[t/x]$ then we can conclude that $\forall x, \varphi$.

$$\dfrac{\varphi[t/x]}{\exists x, \varphi}\ \exists\mathbf{I}$$

Examples of reasoning about "for all"

Lets prove that $\forall x, \varphi \vdash \exists x, \varphi$.

$$
\begin{array}{r|ll}
1 & \forall x \varphi & \\
\hline
2 & \varphi[t/x] & \forall \mathbf{E}(1) \\
3 & \exists x, \varphi & \exists \mathbf{I}(2)
\end{array}
$$

Examples of reasoning about "for all"

Lets prove that $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$.

1		$\forall x P(x) \to Q(x)$	
2		$\exists x Q(x)$	
3	t	$P(t)$	
4		$P(t) \to Q(t)$	$\forall\mathbf{E}(1)$
5		$Q(t)$	$\to\mathbf{E}(3,4)$
6		$\exists x Q(x)$	$\exists\mathbf{I}(5)$
7		$\exists x Q(x)$	$\exists\mathbf{E}(3\text{–}6)$

Examples of reasoning about "for all"

Lets prove that $\exists x P(x), \forall x \forall y (P(x) \to Q(y)) \vdash \forall y Q(y)$.

1	$\exists x Q(x)$	
2	$\forall x \forall y (P(x) \to Q(y))$	
3	$t \quad u \quad P(t)$	
4	$\forall y (P(t) \to Q(y))$	$\forall \mathbf{E}(2)$
5	$P(t) \to Q(u)$	$\forall \mathbf{E}(4)$
6	$Q(u)$	$\to \mathbf{E}(3, 5)$
7	$Q(u)$	$\exists \mathbf{E}(1{-}3-6)$
8	$\forall y Q(y)$	$\forall \mathbf{I}$(3-7)

Which are the new rules (on top of Propositional Logic)?

Elimination rule for $=$

If we know that two terms t_1 and t_2 are equal and that $\varphi[t_1/x]$ holds, then $\varphi[t_1/x]$ must also hold.

$$\frac{t_1 = t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]} =\mathbf{E}$$

Introduction rule for $=$

If we assume some term t and we are able to prove that $\varphi[t/x]$ then we can conclude that $\forall x, \varphi$.

$$\frac{}{t = t} =\mathbf{I}$$

Examples of reasoning about equality

Lets prove that if $t_1 = t_2$ then $t_2 = t_1$.

1	$t_1 = t_2$	
2	$t_1 = t_1$	$=\mathbf{I}$
3	$t_2 = t_1$	$=\mathbf{E}(\varphi \text{ is } x = t_1, 1, 2)$

Lets prove that if $t_1 = t_2$ and $t_2 = t_3$, then $t_1 = t_3$.

1	$t_1 = t_2$	
2	$t_2 = t_3$	$=\mathbf{I}$
3	$t_1 = t_3$	$=\mathbf{E}(\varphi \text{ is } t_1 = x, 2, 2)$

Exercises on FOL Natural Deduction

Proving that $\exists x \neg \varphi \vdash \neg \forall x \varphi$

1	$\exists x \neg \varphi$	
2	$\quad \forall x \varphi$	
3	$\quad u \quad \neg \varphi[u/x]$	
4	$\qquad \varphi[u/x]$	$\forall\mathbf{E}(2)$
5	$\qquad \bot$	$\bot\mathbf{I}(3,4)$
6	$\quad \bot$	$\exists\mathbf{E}(1\text{–}3-5)$
7	$\neg \forall x \varphi$	$\neg\mathbf{I}(2\text{–}6)$

Proving that $\forall x \varphi \wedge \psi \vdash \forall x(\varphi \wedge \psi)$ and x is not free in ψ

1	$\forall x \varphi \wedge \psi$	
2	$\forall x \varphi$	$\wedge \mathbf{E}_l(1)$
3	ψ	$\wedge \mathbf{E}_r(1)$
4	u | $\varphi[u/x]$	
5	| $\varphi[u/x] \wedge \psi$	$\wedge \mathbf{I}(4, 3)$
6	| $(\varphi \wedge \psi)[u/x]$	x free in ψ
7	$\forall x(\varphi \wedge \psi)$	$\forall \mathbf{I}$(4-6)

Proving that $\forall x(\varphi \wedge \psi) \vdash \forall x\varphi \wedge \psi$ and x is not free in ψ

1		$\forall x(\varphi \wedge \psi)$	
2	u	$(\varphi \wedge \psi)[u/x]$	$\forall\mathbf{E}(1)$
3		$\varphi[u/x] \wedge \psi$	x not free in ψ
4		ψ	$\wedge\mathbf{E}_r(3)$
5		$\varphi[u/x]$	$\wedge\mathbf{E}_l(3)$
6		$\forall x\varphi$	$\forall\mathbf{I}(2\text{-}5)$
7		$\forall x\varphi \wedge \psi$	$\wedge\mathbf{I}(6, 4)$