

3. Propositional Logic – Exercises

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RAMDE 2021/2022

Requirements and Model-driven Engineering

CISTER – ISEP

Porto, Portugal

<https://cister-labs.github.io/ramde2122>

Propositional Logic - Practicing Natural Deduction

Natural Deduction Rules

Last RAMDE's class...

On the last class, you were introduced to Propositional Logic:

- its syntax and semantics
- normal forms: negative, disjunctive, and conjunctive
- rules for natural deduction

During this class...

You will be exposed to the practice of construction proofs about Propositional Logic's formulae using Natural Deduction

Warning: Becoming comfortable with this type of mathematics is not an easy task! Bare with me and be patient! Train a lot by doing the exercises at home once again, to start solidifying the types of proof patterns that naturally will appear...

Recalling the rules of introduction and elimination: conjunction

Introduction

If we know that both φ and ψ is hold, then so does their conjunction.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge \text{I}$$

Elimination

If know that $\varphi \wedge \psi$, then we can conclude that either of them also holds in isolation.

$$\frac{\varphi \wedge \psi}{\varphi} \wedge \text{E}_l$$

$$\frac{\varphi \wedge \psi}{\psi} \wedge \text{E}_r$$

Exercise

Prove that if $\varphi \wedge \psi$ holds, then $\psi \wedge \varphi$ also holds. That is $\varphi \wedge \psi \vdash \psi \wedge \varphi$

1	$\varphi \wedge \psi$	

2	φ	$\wedge E_l(1)$
3	ψ	$\wedge E_r(1)$
4	$\psi \wedge \varphi$	$\wedge I(2, 3)$

Recalling the rules of introduction and elimination: disjunction

Introduction

We can construct a new disjunction $\varphi \vee \psi$ if we know that either φ or ψ hold.

$$\frac{\varphi}{\varphi \vee \psi} \vee I_l$$

$$\frac{\psi}{\varphi \vee \psi} \vee I_r$$

Elimination

The elimination, in this case, assumes the form of introducing a new formula θ in case we can derive θ from both φ and ψ , and we know that $\varphi \vee \psi$ holds.

$$\frac{\varphi \vee \psi \quad \begin{array}{c} [\varphi] \\ \vdots \\ \theta \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \theta \end{array}}{\theta} \vee E$$

Quick exercise

Exercise

Prove that if $(\varphi \vee \psi) \wedge \theta$ holds, then $(\varphi \wedge \theta) \vee (\psi \wedge \theta)$ also holds.

1	$(\varphi \vee \psi) \wedge \theta$		
2	$\varphi \vee \psi$	$\wedge E_l(1)$	\vdots
3	θ	$\wedge E_r(1)$	7
4	φ		8
5	$\varphi \wedge \theta$	$\wedge I(4, 3)$	9
6	$(\varphi \wedge \theta) \vee (\psi \wedge \theta)$	$\vee I_l(5)$	10
	\vdots		
			ψ
			$\psi \wedge \theta$
			$(\varphi \wedge \theta) \vee (\psi \wedge \theta)$
			$(\varphi \wedge \theta) \vee (\psi \wedge \theta)$

Recalling the rules of introduction and elimination: Negation

Introduction

If we can derive false from φ , then we can conclude that φ does not hold, that is, its negation $\neg\varphi$ holds.

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \bot \end{array}}{\neg\varphi} \neg\text{I}$$

Elimination

If we know that $\neg\neg\varphi$ is false, then we can conclude that φ holds.

$$\frac{\neg\neg\varphi}{\varphi} \neg\text{E}$$

Recalling the rules of introduction and elimination: False

Introduction

If we assume φ and, still, we are able to derive $\neg\varphi$, then we can conclude false. In fact, we found a contradiction!

$$\frac{\begin{array}{c} \varphi \\ \vdots \\ \neg\varphi \end{array}}{\bot} \bot\text{I}$$

Elimination

From false, we can conclude anything!

$$\frac{\bot}{\varphi} \bot\text{E}$$

Quick exercise

Exercise

Prove that if $\neg(\varphi \vee \psi)$ holds, then $\neg\varphi \wedge \neg\psi$ also holds.

1		$\neg(\varphi \vee \psi)$	
		—	
2			
2		φ	$\wedge E_I(1)$
3		$\varphi \vee \psi$	$\vee I_I(2)$
4		\perp	$\perp I(1, 3)$
5		$\neg\varphi$	$\neg I(2-4)$

6		ψ	
		—	
7			
7		$\varphi \vee \psi$	$\vee I_r(6)$
8		\perp	$\perp I(1, 7)$
9		$\neg\psi$	$\neg I(6-8)$
10		$\neg\varphi \wedge \neg\psi$	$\wedge I(5, 8)$

Natural Deduction Rules - Implication

Introduction of implication

If we assume φ and we can derive ψ from it, then we can conclude that $\varphi \rightarrow \psi$.

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow \mathbf{I}$$

Elimination of implementation

From false, we can conclude whatever we want.

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow \mathbf{E}$$

Quick exercise

Exercise

Prove that if $(\varphi \vee \psi) \rightarrow \theta$ and φ hold, then $\psi \rightarrow \theta$ also holds.

1		$(\varphi \vee \psi) \rightarrow \theta$	
2		φ	
		—	
3			
4			
5			
6			

		ψ	
		—	
		$\varphi \vee \psi$	$\vee\text{I}_1(2)$
		θ	$\rightarrow\text{E}(1, 4)$
		$\psi \rightarrow \theta$	$\rightarrow\text{I}(3-5)$

Natural Deduction Rules - Derived rules

$$\frac{\varphi \rightarrow \psi \quad \neg\psi}{\neg\varphi} \text{ MT}$$

$[\neg\varphi]$

$$\frac{\vdots}{\bot} \text{ RA}$$

$$\frac{\varphi}{\neg\neg\varphi} \neg\neg\text{I}$$

$$\frac{}{\varphi \vee \neg\varphi} \text{ ET}$$

Lets prove the derived rules?

Exercise

Prove that $\varphi \rightarrow \psi, \neg\psi \vdash \neg\varphi$

1	$\varphi \rightarrow \psi$	
2	$\neg\psi$	
<hr/>		
3	φ	
4	ψ	$\rightarrow\mathbf{E}(1, 3)$
5	\perp	$\perp\mathbf{I}(3, 4)$
6	$\neg\varphi$	$\neg\mathbf{I}(3-5)$

Lets prove the derived rules?

Exercise

Prove that $\varphi \vdash \neg\neg\varphi$

1	φ	
2	$\neg\varphi$	
3	F	$\perp I(1, 2)$
4	$\neg\neg\varphi$	$\neg I(2-3)$

Lets prove the derived rules?

Exercise

Prove that $\neg\varphi \rightarrow F \vdash \varphi$

1	$\neg\varphi \rightarrow F$	

2	$\neg\varphi$	

3	F	$\perp\text{I}(1,2)$
4	$\neg\neg\varphi$	$\neg\text{I}(2-3)$
5	φ	$\neg\text{E}(4)$

Lets prove the derived rules?

Exercise

Whatever φ we have $\vdash \varphi \vee \neg\varphi$

1	$\neg(\varphi \vee \neg\varphi)$	
2	φ	
3	$\varphi \vee \neg\varphi$	$\vee I_r(2)$
4	\perp	$\perp I(1,3)$
5	$\neg\varphi$	$\neg I(2-4)$
6	$\varphi \vee \neg\varphi$	$\vee I_l(5)$

	\vdots	
7	\perp	$\perp I(1,6)$
8	$\neg\neg(\varphi \vee \neg\varphi)$	$\neg I(1-7)$
9	$\varphi \vee \neg\varphi$	$\neg E(8)$

Lets continue with more exercises

Exercise

Build the derivations for each of the statements below:

- $\vdash (\varphi \wedge \psi) \rightarrow \psi$
- $\vdash \varphi \rightarrow (\varphi \vee \psi)$
- $\vdash (\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$
- $\theta \rightarrow (\varphi \rightarrow \psi), \neg\psi, \theta \vdash \neg\varphi$
- $\theta, \neg\varphi \vdash \neg(\theta \rightarrow \varphi)$
- $(\psi \wedge \theta) \rightarrow \neg\delta, \varphi \rightarrow \delta, \theta, \varphi \vdash \neg\psi$
- $(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$

Solutions for each of the statements are given in the slides that follow...

Solution for $\vdash (\varphi \wedge \psi) \rightarrow \psi$

1			$\varphi \wedge \psi$	
2			ψ	$\wedge\mathbf{E}_I(1)$
3			$(\varphi \wedge \psi) \rightarrow \psi$	$\rightarrow\mathbf{I}(1-2)$

Solution for $\vdash \varphi \rightarrow (\varphi \vee \psi)$

1			φ	
2			$\varphi \vee \psi$	$\vee\text{I}_r(1)$
3			$\varphi \rightarrow (\varphi \vee \psi)$	$\rightarrow\text{I}(1-2)$

Solution for $\vdash (\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$

1			$\varphi \vee \psi$	
			—	
2			φ	
			—	
3			$\psi \vee \varphi$	$\vee I_r(2)$
			—	
4			ψ	
			—	
5			$\psi \vee \varphi$	$\vee I_l(4)$

			\vdots	
6			$\psi \vee \varphi$	$\vee E(1, 2-3, 4-5)$
7			$(\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$	$\rightarrow I(1-6)$

Solution for $\theta \rightarrow (\varphi \rightarrow \psi), \neg\psi, \theta \vdash \neg\varphi$

1	$\varphi \rightarrow \psi$	
2	$\neg\psi$	
3	θ	
<hr/>		
4	φ	
<hr/>		
5	ψ	$\rightarrow\mathbf{E}(1, 4)$
6	\perp	$\perp\mathbf{I}(2, 5)$
7	$\neg\varphi$	$\neg\mathbf{I}(4-6)$

Solution for $\theta, \neg\varphi \vdash \neg(\theta \rightarrow \varphi)$

1		θ	
2		$\neg\varphi$	
		—	
3			
3			$\theta \rightarrow \varphi$
			—
4			φ $\rightarrow\mathbf{E}(1, 3)$
5			\perp $\perp\mathbf{I}(2, 4)$
6			—
6		$\neg(\theta \rightarrow \varphi)$ $\neg\mathbf{I}(3-5)$	

Solution for $(\psi \wedge \theta) \rightarrow \neg\delta, \varphi \rightarrow \delta, \theta, \varphi \vdash \neg\psi$

1 $(\psi \wedge \theta) \rightarrow \neg\delta$

2 $\varphi \rightarrow \delta$

3 θ

4 φ

5 ψ

6 $\psi \wedge \theta$ $\wedge\mathbf{I}(5, 3)$

7 $\neg\delta$ $\rightarrow\mathbf{E}(1, 6)$

\vdots

8 δ $\rightarrow\mathbf{E}(2, 4)$

9 \perp $\perp\mathbf{I}(7, 8)$

10 $\neg\psi$ $\neg\mathbf{I}(5-9)$

Solution for $(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$ (I)

1	$(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi)$	
2	$\psi \rightarrow \varphi$	$\wedge\mathbf{E}_l(1)$
3	$\varphi \rightarrow \psi$	$\wedge\mathbf{E}_r(1)$
4	$\neg((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$	
5	ψ	
6	φ	$\rightarrow\mathbf{E}(2, 5)$
7	$\varphi \wedge \psi$	$\wedge\mathbf{I}(6, 5)$
8	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\vee\mathbf{I}_l(7)$
9	\perp	$\perp\mathbf{I}(4, 8)$

Solution for $(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$ (II)

	\vdots	
10	$\neg\psi$	$\neg\text{I}(5-9)$
11	φ	
12	ψ	$\rightarrow\text{E}(3, 11)$
13	$\varphi \wedge \psi$	$\wedge\text{I}(11, 12)$
14	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\vee\text{I}_1(13)$
15	\perp	$\perp\text{I}(4, 14)$
16	$\neg\varphi$	$\neg\text{I}(11-15)$

Solution for $(\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi) \vdash (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$ (III)

	\vdots	
17	$\neg\varphi \wedge \neg\psi$	$\wedge\text{I}(10, 16)$
18	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\vee\text{I}_r(17)$
19	\perp	$\perp\text{I}(4, 18)$
20	$\neg\neg((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$	$\neg\text{I}(4-19)$
21	$(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$	$\neg\text{E}(20)$