

## 9. Modal Logic & Verification

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**Recall: What's in a logic?**

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# A logic

## A language

i.e. a collection of well-formed expressions to which meaning can be assigned.

## A semantics

describing how language expressions are interpreted as statements about something.

## A deductive system

i.e. a collection of rules to derive in a purely syntactic way facts and relationships among semantic objects described in the language.

## Note

- a purely syntactic approach (up to the 1940's; the **sacred form**)
- a model theoretic approach (A. Tarski legacy)

# Semantic reasoning: models

- sentences
- models & satisfaction:  $\mathfrak{M} \models \phi$
- validity:  $\models \phi$  ( $\phi$  is satisfied in every possible structure)
- logical consequence:  $\Phi \models \phi$  ( $\phi$  is satisfied in every model of  $\Phi$ )
- theory:  $Th \Phi$  (set of logical consequences of a set of sentences  $\Phi$ )

## Deductive systems $\vdash$

- sequents
  - Hilbert systems
  - natural deduction
  - tableaux systems
  - resolution
  - ...
- 
- derivation and proof
  - deductive consequence:  $\Phi \vdash \phi$
  - theorem:  $\vdash \phi$

# Soundness & completeness

- A deductive system  $\vdash$  is **sound** wrt a semantics  $\models$  if for all sentences  $\phi$

$$\vdash \phi \implies \models \phi$$

(every theorem is valid)

- ... **complete** ...

$$\models \phi \implies \vdash \phi$$

(every valid sentence is a theorem)

For logics with **negation** and a **conjunction** operator

- A sentence  $\phi$  is **refutable** if  $\neg\phi$  is a theorem (i.e.  $\vdash \neg\phi$ )
- A set of sentences  $\Phi$  is **refutable** if some finite conjunction of elements in  $\Phi$  is refutable
- $\phi$  or  $\Phi$  is **consistent** if it is not refutable.

$$\mathfrak{M} \models \phi$$

- Propositional logic (logic of **uninterpreted assertions**; models are **truth assignments**)
- Equational logic (formalises **equational** reasoning; models are **algebras**)
- First-order logic (logic of **predicates** and **quantification** over structures; models are **relational structures**)
- **Modal logics**
- ...



# Modal Logic

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## Modal logic (from P. Blackburn, 2007)

*Over the years modal logic has been applied in many different ways. It has been used as a tool for reasoning about **time**, **beliefs**, **computational systems**, **necessity** and **possibility**, and much else besides.*

*These applications, though diverse, have something important in common: the key ideas they employ (flows of time, relations between epistemic alternatives, transitions between computational states, networks of possible worlds) can all be represented as **simple graph-like structures**.*

Modal logics are

- **tools to talk about relational, or graph-like structures.**
- **fragments of classical ones**, with restricted forms of quantification ...
- ... which tend to be **decidable** and described in a pointfree notations.

## Syntax

$$\phi ::= p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle m \rangle \phi \mid [m] \phi$$

where  $p \in \text{PROP}$  and  $m \in \text{MOD}$

Disjunction ( $\vee$ ) and equivalence ( $\leftrightarrow$ ) are defined by abbreviation.

The *signature* of the basic modal language is determined by sets:

- **PROP** of **propositional** symbols (typically assumed to be denumerably infinite) and
- **MOD** of **modality** symbols.

## Notes

- if there is only one modality in the signature (i.e.,  $\text{MOD}$  is a singleton), write simply  $\Diamond\phi$  and  $\Box\phi$
- the language has some redundancy: in particular modal connectives are **dual** (as quantifiers are in first-order logic):  $[m]\phi$  is equivalent to  $\neg\langle m\rangle\neg\phi$

## Example

Models as LTSs over  $\text{Act}$ .

$\text{MOD} = \text{Act}$  (sets of actions)

$\langle a\rangle\phi$  can be read as “it **must** observe  $a$ , and  $\phi$  must hold after that.”

$[a]\phi$  can be read as “**if** it observes  $a$ , then  $\phi$  must hold after that.”

$\mathfrak{M}, s \models \phi$  – what does it mean?

## Model definition

A **model** for the language is a pair  $\mathfrak{M} = \langle \mathcal{L}, V \rangle$ , where

- $\mathcal{L} = \langle S, \text{MOD}, \longrightarrow \rangle$  is an **LTS**:
  - $S$  is a non-empty set of states (or points)
  - $\text{MOD}$  are the labels consisting of modality symbols
  - $\longrightarrow \subseteq S \times \text{MOD} \times S$  is the transition relation
- $V : \text{PROP} \longrightarrow \mathcal{P}(S)$  is a **valuation**.

## When $\text{MOD} = 1$

- $\Diamond\phi$  and  $\Box\phi$  instead of  $\langle \cdot \rangle \phi$  and  $[\cdot] \phi$
- $\mathcal{L} = \langle S, \longrightarrow \rangle$  instead of  $\mathcal{L} = \langle S, \text{MOD}, \longrightarrow \rangle$
- $\longrightarrow \subseteq S \times S$  instead of  $\longrightarrow \subseteq S \times \text{MOD} \times S$

**Satisfaction:** for a model  $\mathfrak{M}$  and a point  $s$

$\mathfrak{M}, s \models \text{true}$

$\mathfrak{M}, s \not\models \text{false}$

$\mathfrak{M}, s \models p$

iff  $s \in V(p)$

$\mathfrak{M}, s \models \neg\phi$

iff  $\mathfrak{M}, s \not\models \phi$

$\mathfrak{M}, s \models \phi_1 \wedge \phi_2$

iff  $\mathfrak{M}, s \models \phi_1$  and  $\mathfrak{M}, s \models \phi_2$

$\mathfrak{M}, s \models \phi_1 \rightarrow \phi_2$

iff  $\mathfrak{M}, s \not\models \phi_1$  or  $\mathfrak{M}, s \models \phi_2$

$\mathfrak{M}, s \models \langle m \rangle \phi$

iff **there exists**  $v \in S$  st  $s \xrightarrow{m} v$  and  $\mathfrak{M}, v \models \phi$

$\mathfrak{M}, s \models [m] \phi$

iff **for all**  $v \in S$  st  $s \xrightarrow{m} v$  and  $\mathfrak{M}, v \models \phi$

## Satisfaction

A formula  $\phi$  is

- **satisfiable in a model**  $\mathfrak{M}$  if it is satisfied at some point of  $\mathfrak{M}$
- **globally satisfied** in  $\mathfrak{M}$  ( $\mathfrak{M} \models \phi$ ) if it is satisfied at all points in  $\mathfrak{M}$
- **valid** ( $\models \phi$ ) if it is globally satisfied in all models
- **a semantic consequence** of a set of formulas  $\Gamma$  ( $\Gamma \models \phi$ ) if for all models  $\mathfrak{M}$  and all points  $s$ , if  $\mathfrak{M}, s \models \Gamma$  then  $\mathfrak{M}, s \models \phi$

## Example: Hennessy-Milner logic

### Process logic (**Hennessy-Milner logic**)

- $\text{PROP} = \emptyset$  (hence  $V = \emptyset$ )
- $S = \mathcal{P}$  is a set states in a labelled transition system, typically process terms
- each subset  $K \subseteq \text{Act}$  of actions generates a modality corresponding to transitions labelled by an element of  $K$

Assuming the underlying LTS  $\mathcal{L} = \langle \mathcal{P}, \mathbb{P}(\text{Act}), \{ \langle p, K, p' \rangle \mid K \subseteq \text{Act} \} \rangle$  as the model's LTS, satisfaction is abbreviated as

$$\begin{array}{ll} p \models \langle K \rangle \phi & \text{iff } \exists_{q \in \{p' \mid p \xrightarrow{a} p' \wedge a \in K\}} . q \models \phi \\ p \models [K] \phi & \text{iff } \forall_{q \in \{p' \mid p \xrightarrow{a} p' \wedge a \in K\}} . q \models \phi \end{array}$$

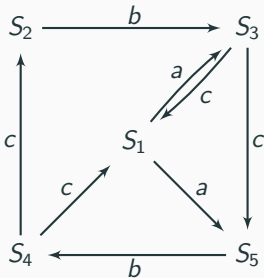


# Example: Hennessy-Milner logic

## Process Logic Syntax

$\phi ::= \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle K \rangle \phi \mid [K] \phi$

where  $K \subseteq \text{Act}$



## Ex. 9.1: Prove:

1.  $S_1 \models [a, b, c] (\langle b, c \rangle tt)$
2.  $S_2 \models [a] (\langle b \rangle tt \wedge \langle c \rangle tt)$
3.  $S_1 \not\models [a] (\langle b \rangle tt \wedge \langle c \rangle tt)$
4.  $S_2 \models [b][c] (\langle a \rangle tt \vee \langle b \rangle tt)$
5.  $S_1 \models [b][c] (\langle a \rangle tt \vee \langle b \rangle tt)$
6.  $S_1 \models [a, b] \langle b, c \rangle (\langle a \rangle tt)$

## Examples II

$(P, <)$  a strict partial order with infimum 0

i.e.,  $P = \{0, a, b, c, \dots\}$ ,

$a \rightarrow b$  means  $a < b$ ,

$a < b$  and  $b < c$  implies  $a < c$

$0 < x$ , for any  $x \neq 0$

there are no loops

some elements may not be comparable

- $P, x \models \Box \text{false}$  if  $x$  is a maximal element of  $P$
- $P, 0 \models \Diamond \Box \text{false}$  iff ...
- $P, 0 \models \Box \Diamond \Box \text{false}$  iff ...

### Temporal logic

- $\langle T, < \rangle$  where  $T$  is a set of time points (instants, execution states , ...) and  $<$  is the **earlier than** relation on  $T$ .
- Thus,  $\Box\varphi$  (respectively,  $\Diamond\varphi$ ) means that  $\varphi$  holds in all (respectively, some) time points.

### Epistemic logic (J. Hintikka, 1962)

- $W$  is a set of agents
- $\alpha \models [K_i] \phi$  means that agent  $i$  always knows that  $\phi$  is true.
- $\alpha \models \langle K_i \rangle \phi$  means that agent  $i$  can reach a state where he knows  $\phi$ .
- $\alpha \models (\neg[K_i] \phi) \wedge (\neg[K_i] \neg\phi)$  means that agent  $i$  does not know whether  $\phi$  is true or not.

Many variations exist, modelling knowledge and believes, knowledge of who knows what, distributed knowledge, etc.

### Deontic logic (G.H. von Wright, 1951)

- Obligations and permissions: **must** and **can** do.
- $\alpha \models \Box \phi$  means  $\phi$  is obligatory.
- $\alpha \models \Diamond \phi$  means  $\phi$  is a possibility.

Each logic accepts a different set of *principles* or *rules* (with variations), that makes their interpretation different.

## Ex. 9.2: Express the properties in Process Logic

- inevitability of  $a$ :
- progress:
- deadlock or termination:

## Ex. 9.3: What does this mean?

1.  $\langle - \rangle$  false
2.  $[-]$  true

“ $-$ ” stands for  $Act$ , and “ $-x$ ” abbreviates  $Act - \{x\}$

## Recall syntax

$$\begin{aligned} \phi &::= \text{true} \\ &| \text{false} \\ &| \neg \phi \\ &| \phi_1 \wedge \phi_2 \\ &| \phi_1 \rightarrow \phi_2 \\ &| \langle K \rangle \phi \\ &| [K] \phi \end{aligned}$$

where  $K \subseteq Act$

## Ex. 9.2: Express the properties in Process Logic

- inevitability of  $a$ :  $\langle - \rangle \text{ true} \wedge [-a] \text{ false}$
- progress:
- deadlock or termination:

## Ex. 9.3: What does this mean?

1.  $\langle - \rangle \text{ false}$
2.  $[-] \text{ true}$

“ $-$ ” stands for  $Act$ , and “ $-x$ ” abbreviates  $Act - \{x\}$

## Recall syntax

$$\begin{aligned} \phi &::= \text{true} \\ &| \text{false} \\ &| \neg \phi \\ &| \phi_1 \wedge \phi_2 \\ &| \phi_1 \rightarrow \phi_2 \\ &| \langle K \rangle \phi \\ &| [K] \phi \end{aligned}$$

where  $K \subseteq Act$

## Express the following using Process Logic

### Ex. 9.4: Coffee-machine

1. The user can have tea or coffee.
2. The user can have tea but not coffee.
3. The user can have tea after having 2 consecutive coffees.

### Ex. 9.5: *a*'s and *b*'s

1. It is possible to do *a* after 3 *b*'s, but not more than 1 *a*.
2. It must be possible to do *a* after [doing *a* and then *b*].
3. After doing *a* and then *b*, it is not possible to do *a*.



# Express the following using Process Logic

## Ex. 9.6: Taxi network

- $\phi_0 =$  In a taxi network, a car can *collect* a passenger or be *allocated* by the Central to a pending service
- $\phi_1 =$  This applies only to cars already *on-service*
- $\phi_2 =$  If a car is *allocated* to a service, it must first *collect* the passenger and then *plan* the route
- $\phi_3 =$  On detecting an *emergence* the taxi becomes inactive
- $\phi_4 =$  A car *on-service* is not inactive

## Process Logic Syntax

$$\phi ::= \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle E \rangle \phi \mid [E] \phi$$

where  $E$  is a **regular expression over  $Act$**

More expressive than Process Logic. Used by mCRL2.

## Examples

- “ $\langle a.b.c \rangle$  true” means “ $\langle a \rangle \langle b \rangle \langle c \rangle$  true”
- “ $[a.b.c]$  false” means “ $[a][b][c]$  false”
- “ $\langle a^*.b \rangle$  true” means that  $b$  can be taken after some number of  $a$ 's.
- “ $\langle -^*.a \rangle$  true” means that  $a$  can **eventually** be taken
- “ $[-^*] \langle a + b \rangle$  true” means it is **always** possible to do  $a$  or  $b$

## Ex. 9.7: What does this mean?

1.  $\langle - \rangle$  true
2.  $[-^*]\langle - \rangle$  true
3.  $[-^*.a]\langle b \rangle$  true
4.  $\langle -^*.send \rangle$   
 $\langle (-send)^*.recv \rangle$  true

## Ex. 9.8: Express using logic

1. The user can only have coffee after the coffee button is pressed.
2. The user must have coffee after the coffee button is pressed.
3. It is always possible to turn off the coffee machine.
4. It is always possible to reach a state where the coffee machine can be turned off.
5. It is never possible to add chocolate right after pressing the *latte button*.

## Bisimulation and modal equivalence

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## Definition

Given two models  $\mathfrak{M} = \langle \mathcal{L}, V \rangle$  and  $\mathfrak{M}' = \langle \mathcal{L}', V' \rangle$ , a **bisimulation of  $\mathcal{L}$  and  $\mathcal{L}'$**  is also a **bisimulation of  $\mathfrak{M}$  and  $\mathfrak{M}'$**  if,

whenever  $s R s'$ , then  $V(s) = V'(s')$

## Lemma (invariance: bisimulation implies modal equivalence)

Given two models  $\mathfrak{M}$  and  $\mathfrak{M}'$ , and a bisimulation  $R$  between their states:

if two states  $s, s'$  are related by  $R$  (i.e.  $sRs'$ ),

then  $s, s'$  satisfy the same basic modal formulas.

(i.e., for all  $\phi$ :  $\mathfrak{M}, s \models \phi \Leftrightarrow \mathfrak{M}', s' \models \phi$ )

## Hence

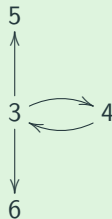
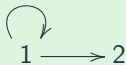
Given 2 models  $\mathfrak{M}$  and  $\mathfrak{M}'$ , if you can find  $\phi$  such that

$$\mathfrak{M} \models \phi \text{ and } \mathfrak{M}' \not\models \phi$$

then they are NOT bisimilar.

## Ex. 9.9: Bisimilarity and modal equivalence

Consider the following transition systems:



Give a modal formula that can be satisfied at point 1 but not at 3.

## Richer modal logics

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can be obtained in different ways, e.g.

- axiomatic extensions
- introducing more complex satisfaction relations
- support novel semantic capabilities
- ...

Examples

- richer temporal logics
- hybrid logic
- modal  $\mu$ -calculus

## Until and Since

$\mathfrak{M}, w \models \phi \mathcal{U} \psi$       iff there **exists**  $v$  st  $w \leq v$  and  $\mathfrak{M}, v \models \psi$ , and  
for **all**  $u$  st  $w \leq u < v$ , one has  $\mathfrak{M}, u \models \phi$

$\mathfrak{M}, w \models \phi \mathcal{S} \psi$       iff there **exists**  $v$  st  $v \leq w$  and  $\mathfrak{M}, v \models \psi$ , and  
for **all**  $u$  st  $v < u \leq w$ , one has  $\mathfrak{M}, u \models \phi$

- Defined for temporal frames  $\langle T, < \rangle$  (transitive, asymmetric).
- note the  $\exists \forall$  qualification pattern: these operators are neither diamonds nor boxes.
- More general definition for other frames – it becomes more expressive than modal logics.

## Temporal logics - rewrite using $\mathcal{U}$

- $\Diamond\psi =$
- $\Box\psi =$

## Temporal logics - rewrite using $\mathcal{U}$

- $\Diamond\psi = tt\mathcal{U}\psi$
- $\Box\psi =$

## Temporal logics - rewrite using $\mathcal{U}$

- $\Diamond\psi = tt\mathcal{U}\psi$
- $\Box\psi = \neg(\Diamond\neg\psi) = \neg(tt\mathcal{U}\neg\psi)$

# Linear temporal logic (LTL)

$$\phi := \text{true} \mid p \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \bigcirc\phi \mid \phi_1 \mathcal{U} \phi_2$$

mutual exclusion	$\Box(\neg c_1 \vee \neg c_2)$
liveness	$\Box\Diamond c_1 \wedge \Box\Diamond c_2$
starvation freedom	$(\Box\Diamond w_1 \rightarrow \Box\Diamond c_1) \wedge (\Box\Diamond w_2 \rightarrow \Box\Diamond c_2)$
progress	$\Box(w_1 \rightarrow \Diamond c_1)$
weak fairness	$\Diamond\Box w_1 \rightarrow \Box\Diamond c_1$
eventually forever	$\Diamond\Box w_1$

- First temporal logic to reason about reactive systems [Pnueli, 1977]
- Formulas are interpreted over **execution paths**
- Express **linear-time properties**

# Computational tree logic (CTL, CTL\*)

**state** formulas to express properties of a state:

$$\Phi := \text{true} \mid \Phi \wedge \Phi \mid \neg \Phi \mid \exists \psi \mid \forall \psi$$

**path** formulas to express properties of a path:

$$\psi := \bigcirc \Phi \mid \Phi \mathcal{U} \Psi$$

mutual exclusion	$\forall \square (\neg c_1 \vee \neg c_2)$
liveness	$\forall \square \forall \Diamond c_1 \wedge \forall \square \forall \Diamond c_2$
order	$\forall \square (c_1 \vee \forall \bigcirc c_2)$

- Branching time structure encode transitive, irreflexive but not necessarily linear flows of time
- flows are **trees**: past linear; branching future

## Motivation

Add the possibility of **naming** points and reason about their **identity**

Compare:

$$\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q)$$

with

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

for  $i \in \text{NOM}$  (a **nominal**)

## Syntax

$$\phi ::= \dots \mid p \mid \langle m \rangle \phi \mid [m] \phi \mid i \mid @_i \phi$$

where  $p \in \text{PROP}$  and  $m \in \text{MOD}$  and  $i \in \text{NOM}$



## Nominals $i$

- Are special propositional symbols that hold exactly on one state (the state they **name**)
- In a model the **valuation**  $V$  is extended from

$$V : \text{PROP} \longrightarrow \mathcal{P}(W)$$

to

$$V : \text{PROP} \longrightarrow \mathcal{P}(W) \quad \text{and} \quad V : \text{NOM} \longrightarrow W$$

where NOM is the set of nominals in the model

- Satisfaction:

$$\mathfrak{M}, w \models i \quad \text{iff } w = V(i)$$

## The @<sub>i</sub> operator

$\mathfrak{M}, s \models \text{true}$

$\mathfrak{M}, s \not\models \text{false}$

$\mathfrak{M}, s \models p$

iff  $s \in V(p)$

$\mathfrak{M}, s \models \neg\phi$

iff  $\mathfrak{M}, s \not\models \phi$

$\mathfrak{M}, s \models \phi_1 \wedge \phi_2$

iff  $\mathfrak{M}, s \models \phi_1$  and  $\mathfrak{M}, s \models \phi_2$

$\mathfrak{M}, s \models \phi_1 \rightarrow \phi_2$

iff  $\mathfrak{M}, s \not\models \phi_1$  or  $\mathfrak{M}, s \models \phi_2$

$\mathfrak{M}, s \models \langle m \rangle \phi$

iff **there exists**  $v \in S$  st  $s \xrightarrow{m} v$  and  $\mathfrak{M}, v \models \phi$

$\mathfrak{M}, s \models [m] \phi$

iff **for all**  $v \in S$  st  $s \xrightarrow{m} v$  and  $\mathfrak{M}, v \models \phi$

$\mathfrak{M}, s \models @_i \phi$

iff  $\mathfrak{M}, u \models \phi$  and  $u = V(i)$

[ $u$  is the state denoted by  $i$ ]

## Summing up

- basic hybrid logic is a simple notation for capturing the **bisimulation-invariant fragment of first-order logic with constants and equality**, i.e., a mechanism for equality reasoning in propositional modal logic.
- comes **cheap**: up to a polynomial, the complexity of the resulting decision problem is no worse than for the basic modal language