Foundations of Elliptic Curves Cryptosystems

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ECC in a nutshell

- Mid-1980s
- GDLP in ECC
  - DHKE and DL-systems can be redefined in ECCs
- Same level of security of RSA and DL-system with considerably shorter operands
  - 160–256-bit vs 1024–3072 bit ➔ Performance advantages over RSA and DL-systems
  - However, RSA with short public parameter is faster than ECC

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Key Lenghts and Security Level

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- An algorithm has security level of *n* bit, if the best known algorithm requires 2^n steps
- Symmetric algorithms with security level of *n* have a key of length of *n* bits
- In asymmetric algorithms, the relationship between security level and cryptographic strengh is no as straightforward

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Key Lenghts and Security Level

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Algorithm Family	**Cryptosystem**	**Security Level**			
		80	**128**	**192**	**256**
Integer Factorization	RSA	1024 bit	3072 bit	7680 bit	15360 bit
Discrete Logarithm	DH, DSA, ElGamal	1024 bit	3072 bit	7680 bit	15360 bit
Elliptic curves	ECDH, ECDSA	160 bit	256 bit	384 bit	512 bit
Symmetric key	AES, 3DES	80 bit	128 bit	192 bit	256 bit

RULE OF THUMB - The computational complexity of the three public key algorithm families grows roughly with the cube of bit length

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Elliptic Curves Cryptosystem

HOW TO COMPUTE WITH ECC

How to Compute with ECC

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- ECC is based on GDLP, so we have to accomplish two tasks
  - **Task 1:** Define an elliptic-curve-based cyclic group
    - **Task 1.1**: Define a set of elements
    - **Task 1.2**: Define the group operations
  - **Task 2**: Show that DLP is hard in that group

Polynomials and curves

- We can form curves from polynomial equations
  - A curve is the set of points (x, y) which are the solutions of the equations
- Examples (in $\mathbb{R}$)
  - $x^2 + y^2 = r^2$ is a circle
  - $a \cdot x^2 + b \cdot y^2 = c$ is an ellipse

EC – definition

- We consider GF(p) = {0, 1, …. p – 1 }
  - Intuitively, GF is a finite set where you can add, subtract, multiply and invert
- Definition
  - The elliptic curve over $\mathbb{Z}_p$, p > 3, is the set of points $(x, y) \in \mathbb{Z}_p$ which fulfils

$$y^2 \equiv x^3 + a \cdot x + b \bmod p$$

  - together with an imaginary point of infinity $\mathcal{O}$,
  - where $a, b \in \mathbb{Z}_p$, and $4 \cdot a^3 + 27 \cdot b^2 \neq 0 \bmod p$
    - The curve is non-singular (no vertices, no self-intersections)

Group elements (Task 1.1)

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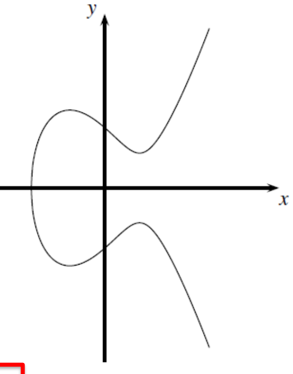
- Plotting in $\mathbb{R}$ for the sake of illustration
- Observations
  - 1, 3 intersections with x axis
  - Symmetric with respect to x axis
- Group elements are the points of the curve



$\boxed{y^2 = x^3 - 3x + 3 \text{ over } \mathbb{R}}$

Group operations (Task 1.2)

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- We call "addition" the group operation and denote it by "+" an operation that takes two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and produces a third point $R = (x_3, y_3)$ as a result

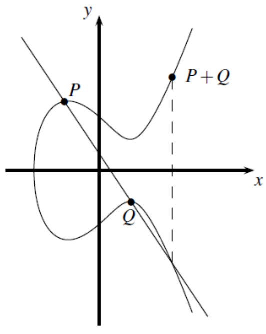
$$P + Q = R$$

- Geometrical interpretation of + in $\mathbb{R}$
  - Point Addition $P + Q$, $Q \neq P$
  - Point Doubling $P + P$

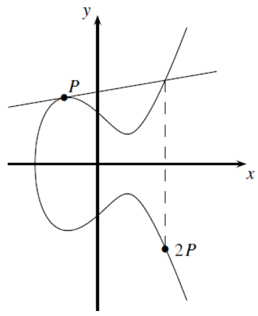
Group operations (task 1.2)

- Geometrical interpretation of “+” operation: the *tangent-and-chord* method

Point addition



Point doubling



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Group operations (task 1.2)

- Geometrical interpretation of +
  – The tangent-and-chord method only uses the four standard operations
- FACT
  – If addition + is defined this way, the group points fulfil most of necessary conditions of a group: closure, associativity, existence of an identity element and existence of an inverse

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Group operations (task 1.2)

- Analytic expressions of Point Addition and Point Doubling
  - $x_3 \equiv s^2 - x_1 - x_2 \bmod p$
  - $y_3 \equiv s \cdot (x_1 - x_3) - y_1 \bmod p$

  where
  - $s \equiv \frac{y_2 - y_1}{x_2 - x_1} \bmod p$ if P ≠ Q (point addition)
  - $s \equiv \frac{3 \cdot x_1^2 + a}{2 \cdot y_1} \bmod p$ if P = Q (point doubling)
  - with s the slope of chord/tangent

Point at infinity (task 1.2)

- An identity (neutral) element $\mathcal{O}$ is still missing
  - $\forall P \in E\colon\ P + \mathcal{O} = P$
- There exists not such a point on the curve
- Thus, we define $\mathcal{O}$ as the point at infinity
  - Located at "plus" infinity towards the y-axis or at "minus" infinity towards the y-axis
- Now, we also define –P (inverse): $P + (-P) = \mathcal{O}$

Group operations (task 1.2)

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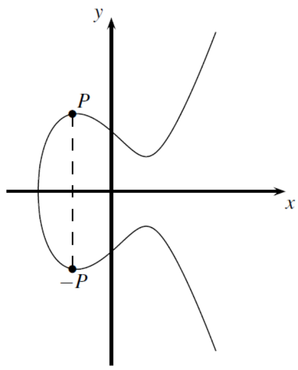
- Inverse of a point P on an elliptic curve
  - Apply the tangent-and-chord method
- In ECC over GF(p)
  - Given P = (x, y) then –P = (x, p – y)



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Elliptic Curve in GF(17) – an educational curve

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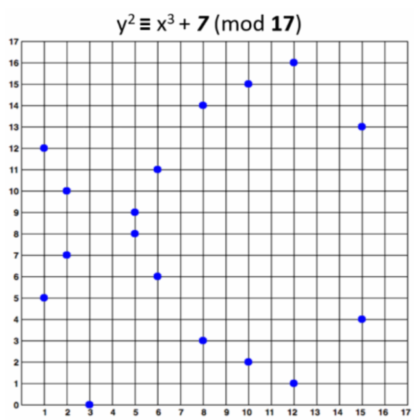

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Elliptic Curves Cryptosystem

BUILDING DLP ON EC

A useful theorem

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- THM
  - The points on an elliptic curve together with $\mathcal{O}$ have cyclic subgroups. Under certain conditions all points on an elliptic curve form a cyclic group
    - A primitive element must exist such that its powers generate the entire group

Example (1/3)

- E: $y^2 \equiv x^3 + 2 \cdot x + 2 \bmod 17$
  - #E (order of E) = 19
  - P = (5, 1) primitive element
  - "Powers" of P
    - 2P = (6, 3) – point doubling
    - 3P = (10, 6) – point addition 2P + P
    - 4P = (3, 1)
    - 5P = (9, 16)
    - 6P = (16, 13)
    - 7P = (0, 6)
    - 8P = (13, 7)
    - 9P = (7, 6)
    - 10P = (7, 11)
    - 11P = (13, 10)
    - 12P = (0, 11)
    - 13P = (16, 4)
    - 14P = (9, 1)
    - 15P = (3, 16)
    - 16P = (10, 11)
    - 17P = (6, 14)
    - 18P = (5, 16)
    - 19P = $\mathcal{O}$ = #E·P

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Example (2/3)

- The cyclic structure becomes visible
  - 20 P = 19P + P = $\mathcal{O}$ + P = P
  - 21P = 19P + 2P = 2P
  - …
- Furthermore
  - 19P = $\mathcal{O}$, thus 18P + P = $\mathcal{O}$, then P^{-1} = 18P and vice versa
  - Verification
    - P = (5, 1), 18P = (5, 16)
    - $x_p = x_{18P} = 5$
    - $y_p + y_{18p} \equiv 0 \bmod 17$

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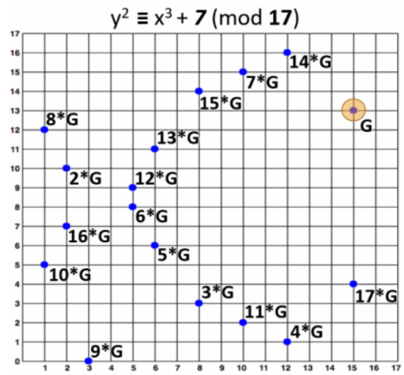
Example (3/3)

- G = (15, 13)
  - #E = 18
- G' = (5, 9)
  - #E' = 3
    - 1G' = (5, 9)
    - 2G' = (5, 8)
    - 3G' = $\mathcal{O}$



Hasse's Theorem

- Hasse's theorem
  - Given an elliptic curve E modulo p, the number of points on the curve is denoted by #E and is bounded by:
    $$p + 1 - 2\sqrt{p} \leq \#E \leq p + 1 + \sqrt{p}$$
  - The number of points is roughly in the range of p (Hasse's bound)
- Example
  - If you need an EC with 2^{160} points, you have to use a prime p of about 160 bit

ECDLP – point multiplication

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- Elliptic Curve Discrete Logarithm Problem (ECDLP)
  - Given an elliptic curve E. We consider a primitive element P and another element T. The DL problem is finding the integer d, where $1 \le d \le \#E$, such that:

$$\underbrace{P + P + \ldots + P}_{\text{d times}} = d \cdot P = T$$

  - d is the private key, T is the public key
  - Point multiplication $\stackrel{\text{def}}{=}$ T = d·P

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Square-and-multiply

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- Point multiplication is analogue to exponentiation in multiplicative groups $(\mathbb{Z}_p^*, \times)$ ➔ we can adopt the square-and-multiply algorithm
- Example
  - $26P = (11010)_2 P = (d_4 d_3 d_2 d_1 d_0)_2\ P$
  - Step
    - #0 P = **1**P — init setting, bit processed: $d_4 = 1$
    - #1a P+P = 2P = **10**P — DOUBLE, bit processed: d_3
    - #1b 2P+P = 3P = 10P+1P = **11**P — ADD, since $d_3 = 1$
    - #2a 3P+3P = 6P = 2(11P) = **110**P — DOUBLE, bit processed: d_2
    - #2b — *no ADD*, since $d_2 = 0$
    - #3a 6P+6P = 12P = 2(110P) = **1100**P — DOUBLE, bit processed: d_1
    - #3b 12P+P = 13P = 1100P+1P = **1101**P — ADD, since $d_1 = 1$
    - #4a 13P+13P = 26P = 2(1101P) = **11010**P — DOUBLE, bit processed: d_0
    - #4b — *no ADD*, since $d_0 = 0$

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EC Cryptosystem

- Private key: d
  - Randomly generated integer
- Public key: T
- Geometrical interpretation of ECDLP
  - Given P, we compute 2P, 3P,…, d·P = T, we actually jump back and forth on the EC
  - Given the starting point P and the final point T (public key), the adversary has to figure out how often we “jumped” on the EC

Standard curves

- Elliptic Curve Cryptography, NIST
  - Standards for digital signatures and key establishment schemes
- RFC
  - Elliptic Curve Cryptography (ECC) Brainpool Standard Curves and Curve Generation (RFC 5639)
  - Fundamental Elliptic Curve Cryptography Algorithms (RFC 6090)
  - Elliptic Curve Cryptography (ECC) Cipher Suites for Transport Layer Security (TLS) Versions 1.2 and Earlier (RFC 8422)