Perfect Cipher

Gianluca Dini
Dept. of Ingegneria dell'Informazione
University of Pisa
gianluca.dini@unipi.it
Version: 2024-02-26

1

Towards a secure cipher

- Attacker's ability: (one) cipher-text only attack
- Security requirements
  - Attacker cannot recover the secret key
  - Attacker cannot recover the plaintext
- Intuition of perfectly secure cipher
  - Regardless of *any prior information* the attacker has about the plaintext, the cyphertext should leak *no additional information* about the plaintext

2

A probabilistic approach

- Message M is a random variable
  - Plaintext distribution
  - Example
    - Pr[M = "attack today"] = 0.7
    - Pr[M = "don't attack] = 0.3
  - Prior knowledge of the attacker
- Gen() defines a probability distribution over **K**
  - Pr[K = k] = Pr[k ← Gen()]
- Random variables M and K are independent

3

A probabilistic approach

- Ciphertext generation process
  - Choose a message m
  - Generate a key k, k ← Gen()
  - Compute c ← E_k(m)
- The ciphertext is a random variable C
- Encryption defines a distribution over the ciphertext **C**

4

Perfect secrecy (informal)

- We formalize «information about the plaintext» in terms of probability distribution
- The adversary's *a-priori* knowledge of the plaintext distribution, i.e. before observing a ciphertext, and the adversary's *a-posteriori* knowledge of the plaintex distribution, i.e. after observing the ciphertext, must be equal

feb-24 Perfect cipher 5

5

Perfect secrecy (Shannon, 1949)

- Definition of Perfect secrecy – For every every *m* in M, every *c* in C, with Pr[C = c] > 0, it holds Pr[M = m | C = c] = Pr[M = m]
- An equivalent formulation
  - $\forall\ m, m' \in M, \forall c \in C, \Pr[E_k(m) = c] = \Pr[E_k(m') = c]$
  - The distribution of the ciphertext does not depend on the plaintex

feb-24 Perfect cipher 6

6

Shannon's Theorem

- Shannon's Theorem – In a perfect cipher, $|\mathbf{K}| \geq |\mathbf{M}|$
  - i.e., the number of keys cannot be smaller than the number of messages
  - Proof. By contradiction.
    a) Let $|\mathbf{K}|<|\mathbf{M}|$
    b) It must be $|\mathbf{C}| \geq |\mathbf{M}|$ or, otherwise, the cipher is not invertible
    c) Therefore, $|\mathbf{C}| > |\mathbf{K}|$
    d) Select m in **M**, s.t., $Pr[M = m] \neq 0$; $c_i \leftarrow E(k_i, m)$ for all k_i in **K**
    e) Because of c), there exists at least one c s.t. $c \neq c_i$, for all i
    f) Therefore $Pr[M = m|C = c] = 0$, that is different of $Pr[M = m]$

feb-24 Perfect cipher 7

7

Shannon's Theorem

- **FACT**. Let Π = (Gen, Enc, Dec) an encryption scheme with $|\mathcal{M}| = |\mathcal{K}| = |\mathcal{C}|$. The scheme is perfectly secret iff
  1. Every $k \in \mathcal{K}$ is chosen with equal probaility $1/|\mathcal{K}|$ by Gen
  2. For every $m \in \mathcal{M}$ and every $c \in \mathcal{C}$ there exists a unique key $k \in \mathcal{K}$ such that $E_k(m) = c$
- Useful for deciding whether a given scheme if perfectly secure
  - Condition 1 is easy to check
  - Condition 2 does not require computing any probabilities

feb-24 Perfect cipher 8

8

Unconditional security

- Perfect secrecy is equivalent to unconditional security
  - An adversary is assumed to have infinite computing resources
  - Observation of the CT provides the adversary no information whatsoever
- Necessary conditions
  - Key bits are truly randomly chosen
  - Key len $\geq$ msg len (Shannon theorem)

feb-24 Perfect cipher 9

9

Perfect indistinguishability

- Yet another definition of perfect secrecy
- **Definition** – An encryption scheme Π = (G, E, D) over ($\mathcal{K}$, $\mathcal{M}$, $\mathcal{C}$) has *perfect indistinguishability* iff
  - For all m_1, $m_2 \in \mathcal{M}$,, $|m_1| = |m_2|$
  - with k ← Gen() (uniform)
  - For all $c \in \mathcal{C}$, $Pr[E(k, m_1) = c] = Pr[E(k, m_2) = c]$
- **Fact** – Π has perfectly indistinguishability iff it is perfectly secure

feb-24 Perfect cipher 10

10

Perfect Cipher

ONE-TIME PAD

One Time Pad

- Patented in 1917 by Vernam
  - Known 35 years earlier
- Proven perfect by Shannon in 1949
- Moscow-Washington “red telephone”
  - In reality a secure direct communication link
    - Teletype, fax machine, secure computer link (email)
  - Never a telephone (not even red)

Preliminary

- Or-exclusive (xor)
  - Truth table

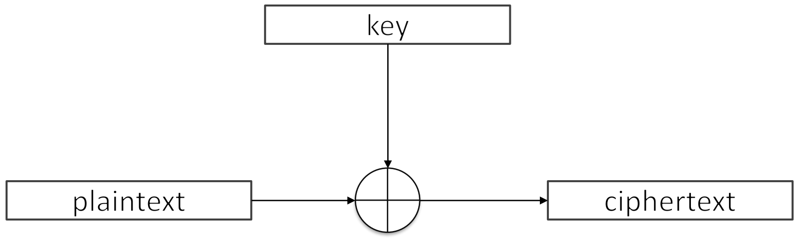
x	y	$z = x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

  - Matematically
    - $z = x \oplus y = (x + y) \bmod 2$

One Time Pad

- Assumptions
  - Let x be a t-bit message, i.e., $x \in \{0,1\}^t$
  - Let k be a t-bit key stream, $k \in \{0, 1\}^t$, where each bit is truly random chosen
- Encryption
  - For all i in [1,…,t], $y_i = m_i \oplus k_i$ i.e., $y_i = m_i + k_i \bmod 2$
- Decryption
  - For all i in [1,…, t], $x_i = c_i \oplus k_i$, i.e., $x_i = y_i + k_i \bmod 2$
- Consistency property can be easily proven

One-Time Pad



Xor is a good encryption function

- Theorem – Let X be a random variable over $\{0, 1\}^n$, and K an independent uniform variable over $\{0,1\}^n$. Then, $Y = X \oplus K$ is uniform over $\{0,1\}^n$.
  - Proof (for n = 1).
    - Let $Pr[X = 0] = X0$, $Pr[X = 1] = X1$, $X0 + X1 = 1$
    - $Pr[Y = 0] =$
      $= Pr[(X = 0)\wedge(K = 0)] + Pr[(X = 1)\wedge(K = 1)] =$
      $= Pr[X = 0]\times Pr[K = 0] + Pr[X = 1] \times Pr[K = 1] =$
      $= X0 \times 0.5 + X1 \times 0.5 = 0.5 \times (X0 + X1) =$
      $= 0.5$

OTP has perfect secrecy

- Theorem – OTP has perfect secrecy
  - Proof
    - a) $\Pr[M = m | C = c]$ = (Bayes law)
      $= \Pr[C = c \mid M = m] \times \Pr[M = m]/\Pr[C = c]$
    - b) $\Pr[C = c]$ = (Total probability law)
      $= \Sigma_i \Pr[C = c | M = m_i] \times \Pr[M = m_i] =$
      $= \Sigma_i \Pr[K = c \oplus m_i] \times \Pr[M = m_i] =$
      $= \Sigma i\ 2^{-k} \times \Pr[M = m_i] = 2^{-k}$
    - c) Put b) into a)
      $\Pr[M = m | C = c] =$
      $= \Pr[K = c \oplus m] \times \Pr[M = m]/2^{-k}$
      $= 2^{-k} \times \Pr[M = m]/2^{-k} =$
      $\Pr[M = m]$

feb-24 Perfect cipher 17

17

OTP has perfect secrecy: intuition

- **c[i] = m[i] + k[i] mod 26**
- **m = “SUPPORT JAMES BOND”**

m =	S	U	P	P	O	R	T	J	A	M	E	S	B	O	N	D
k =	W	C	L	N	B	T	D	E	F	J	A	Z	G	U	I	R
c =	O	W	A	C	P	K	W	N	F	V	E	R	H	I	V	U

c =	O	W	A	C	P	K	W	N	F	V	E	R	H	I	V	U
k' =	M	W	L	J	V	T	S	E	F	J	A	Z	G	U	I	R
m =	C	A	P	T	U	R	E	J	A	M	E	S	B	O	N	D

feb-24 Perfect cipher 19

19

Pros and Cons

- Pros
  - Unconditionally secure
    - A cryptosystem is unconditionally or information-theoretically secure if it cannot be broken even with infinite computational resources
  - OTP is optimal
    - Only one key maps m into c
    - $|\mathcal{M}| = |\mathcal{K}| = |\mathcal{C}|$
- Very fast enc/dec

Pros and Cons

- Cons
  - Long keys: unpractical!
    - Key len == msg len
    - In general, $|\mathcal{K}| \geq |\mathcal{M}|$
  - Keys must be used once: avoid two-time pad!
    - Let C1 = M1 xor K and C2 = M2 xor K ➔
    - C1 xor C2 = M1 xor M2

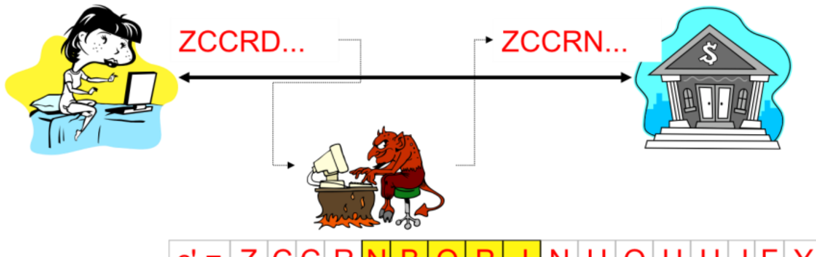
Pros and Cons

- Cons
  - A Known-PlainText attack breaks OTP
    - Given (m, c) => k = m xor c
  - OTP is malleable
    - Modifications to cipher-text are undetected and have predictable impact on plain-text

22

OTP is malleable

m =	D	A	R	E	C	E	N	T	O	E	U	R	O	A	B	O	B
k =	W	C	L	N	B	T	D	E	F	J	A	Z	G	U	I	R	X
c =	Z	C	C	R	D	X	Q	X	T	N	U	Q	U	U	J	F	Y



c' =	Z	C	C	R	N	B	O	P	J	N	U	Q	U	U	J	F	Y
k =	W	C	L	N	B	T	D	E	F	J	A	Z	G	U	I	R	X
m =	D	A	R	E	M	I	L	L	E	E	U	R	O	A	B	O	B

23

Malleability

- Malleability
  - A crypto scheme is said to be *malleable* if the attacker is capable of transforming the ciphertext into another ciphertext which leads to a *known* transformation of the plaintext
    - The attacker does not decrypt the ciphertext, but (s)he is able to manipulate the plaintext in a predictable manner

feb-24 Perfect cipher 24

24

On OTP malleability

- Attack against integrity
  - Alice sends Bob: $c = p \oplus k$
  - The adversary
    - intercepts c and
    - transmits Bob $c' = c \oplus r$, with r called *perturbation*
  - Bob
    - receives c'
    - Computes $p' = c' \oplus k = c \oplus r \oplus k = p \oplus k \oplus r \oplus k$ so obtaining $p' = p \oplus r$
    - The perturbation goes undetected and
    - The perturbation has a predictable impact on the plaintext

feb-24 Perfect cipher 25

25

OTP Malleabililty: example

- Assume the adversary intercepts an encrypted email. The adversary does not know anything about the email, but Bob is the sender. Furthermore, since the message comes from Bob, then the adversary knows that the first line of the message is “from: Bob”. The adversary wants to make the message to appear as coming from Eve.
- The adversary has only to apply a change to bytes 7-9 and transform the from ‘B’ ‘o’ ‘b’ to ‘E’ ‘v’ ‘e’. This is quite simple:
- X = [‘B’ ‘o’ ‘b’] xor [‘E’ ‘v’ ‘e’] (byte-wise xor)
- If we consider the Ascii codes
  - B o b → 42 6F 62, E v e → 45 76 65
- X = Bob xor Eve (byte-wise xor) = 07 19 07

Remainder of probability theory

- Random variable, probability distribution
- Conditional probability
  - $Pr[A|B] = Pr[A \wedge B]/Pr[B]$
- Bayes’ Theorem
  - $Pr[A|B] = Pr[B|A] \times Pr[A]/Pr[B]$
- Law of total probability
  - $\{E_i\}$ are a *partition* of all possible events
    - For all i, j, $i \neq j$, E_i and E_j are pairwise impossible ($E_i \cap E_j = \emptyset$)
    - At least some E_i occurs
  - For any event A, $Pr[A] = \Sigma_i Pr[A \wedge E_i] = \Sigma_i Pr[A|E_i] \times Pr[E_i]$

Example 1

- Shift cipher
  - K = {0, …, 26}, Pr[K = k] = 1/26 (random)
  - Pr[M = ‘a’] = 0.7; Pr[M = ‘z’] = 0.3 (a-priori distribution)
  - Compute Pr[C = ‘b’]
    - Result = 1/26

28

Example 2

- Shift cipher
  - K = {0, …, 26}, Pr[K = k] = 1/26 (random)
  - m1 = «ONE», m2 = «TEN»
  - Pr[M = m1] = Pr[M = m2] = 0.5 (a-priori distribution)
  - Compute Pr[C = «RQH»]
    - Result = 1/52

29

Example 3

- Shift cipher
  - K = {0, …, 26}, Pr[K = k] = 1/26 (random)
  - m1 = «ONE», m2 = «TEN»
  - Pr[M = m1] = Pr[M = m2] = 0.5 (a-priori distribution)
  - Compute Pr[M=«TEN»|C = «RQH»]
    - Result = 0 that is different of Pr[M = «TEN»] ➔
  - Shift cipher is not perfect

feb-24 Perfect cipher 30

30

Example 4

- Shift cipher
- Message distribution
  - Pr[M = «HI»] = 0.3
  - Pr[M = «NO»] = 0.2
  - Pr[M = «IN»] = 0.5
- Compute Pr[M = «HI»|C = «XY»]
  - Pr[M=«HI»|C=«XY»] = (Bayes' law) =
    = Pr[C = «XY»|M=«HI»]·Pr[M=«HI»]/Pr[C=«XY»]
  - Pr[C = «XY»|M=«HI»] = Pr[K = 16] = 1/26 (continue)

feb-24 Perfect cipher 31

31

Example 4 continued

- Compute Pr[M = «HI»|C = «XY»]
  - Pr[C = «XY»] = (law of total probability)
    Pr[C=«XY»|M=«HI»]·Pr[M=«HI»]+
    Pr[C=«XY»|M=«NO»]·Pr[M=«NO»]+
    Pr[C=«XY»|M=«IN»]·Pr[M=«IN»] =
    = (1/26)·0.3 + (1/26)·0.2 + 0·0.5 =
    = 1/52
  - Pr[M = «HI»|C = «XY»] = (1/26)·0.3/(1/52) = 0.6
    ≠ Pr[M = «HI»] ➔
- Shift cipher is not perfect