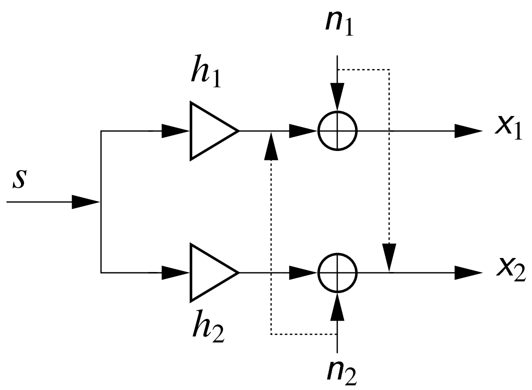
2. Maximum Likelihood Estimation

2.1 Maximum Likelihood Estimation

A known signal (respectively a signal sequence) s is sent over two transmission lines with constant channel gains h_1 and h_2. The transmission gains h_1 and h_2 shall be estimated from the observed



receive signals x_1 and x_2. The observation of the i-th transmission line is disturbed by additive Gaussian noise n_i, i.e.,

$$\boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{C}) \quad \text{with} \quad \boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

and $c_{i,j} = \mathrm{Cov}\left[n_i, n_j\right]$.

a) What is the underlying statistical model? State the likelihood function and determine the ML estimator $\hat{\boldsymbol{h}}_{\mathrm{ML}}$ for $\boldsymbol{h} = [h_1, h_2]^{\mathrm{T}}$ if only one observation is available.

In the following N observations are available. The i-th signal s of the known sequence is denoted by $s^{(i)}$ and we assume $\boldsymbol{n}^{(1)}, \ldots, \boldsymbol{n}^{(N)}$ to be jointly Gaussian.

b) If the random variables $\boldsymbol{n}^{(1)}, \ldots, \boldsymbol{n}^{(N)}$ are uncorrelated, what is the ML estimator $\hat{\boldsymbol{h}}_{\mathrm{ML}}$ for $\boldsymbol{h}$?

c) Although N observations are available, only the first observation is used to design an estimator $\hat{\boldsymbol{h}} = \boldsymbol{x}^{(1)}/s^{(1)}$. Is the estimator unbiased? Is it consistent?

d) If the random variables $\boldsymbol{n}^{(1)}, \ldots, \boldsymbol{n}^{(N)}$ are correlated, what is the ML estimator $\hat{\boldsymbol{h}}_{\mathrm{ML}}$ for $\boldsymbol{h}$?

In the following, the channel gains h_1 and h_2 are known and the random variables $\boldsymbol{n}^{(1)}, \ldots, \boldsymbol{n}^{(N)}$ are uncorrelated. The unknown covariance matrix $\boldsymbol{C}$ shall be estimated.

e) What is the ML estimator $\hat{\boldsymbol{C}}_{\mathrm{ML}}$ for the noise covariance matrix $\boldsymbol{C}$?

Hint:

$$
\begin{aligned}
\frac{\partial \det(\boldsymbol{A})}{\partial \boldsymbol{A}} &= \det(\boldsymbol{A})\,(\boldsymbol{A}^{-1})^{\mathrm{T}} \\
\frac{\partial \mathrm{tr}\left\{\boldsymbol{A}\boldsymbol{X}^{-1}\boldsymbol{B}\right\}}{\partial \boldsymbol{X}} &= -(\boldsymbol{X}^{-1}\boldsymbol{B}\boldsymbol{A}\boldsymbol{X}^{-1})^{\mathrm{T}} \\
\frac{\partial \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{B}\right\}}{\partial \boldsymbol{X}} &= \boldsymbol{B}^{\mathrm{T}}
\end{aligned}
$$

f) The ML estimator $\hat{\boldsymbol{C}}_{\mathrm{ML}}$ is consistent, but is it unbiased as well?