The Cantor-Lebesgue Function

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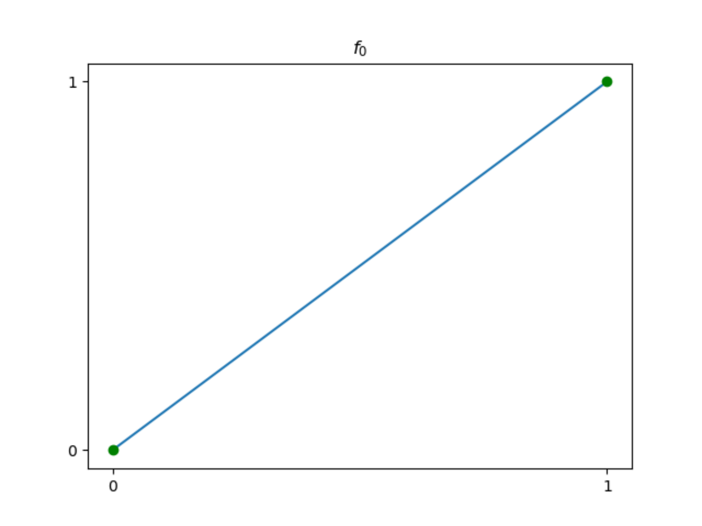
Define a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ recursively:

$$f_0(x) = x$$
$$f_n(x) = \begin{cases} \frac{1}{2} f_{n-1}(3x) & x \in [0, 1/3] \\ \frac{1}{2} & x \in (1/3, 2/3) \\ \frac{1}{2} + \frac{1}{2} f_{n-1}(3x - 2) & x \in [2/3, 1] \end{cases}$$

Let's see what the functions look like exactly $\downarrow$



$f_0 = x$
increasing on $C_0 := [0, 1]$
length of C_0: $|C_0| = 1$

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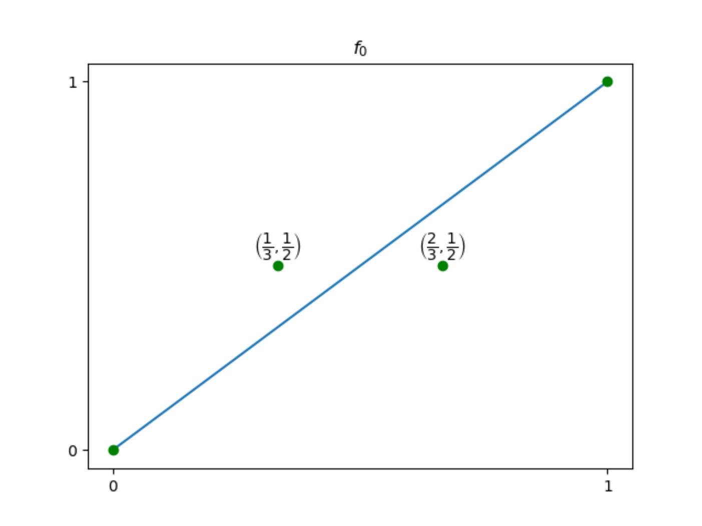

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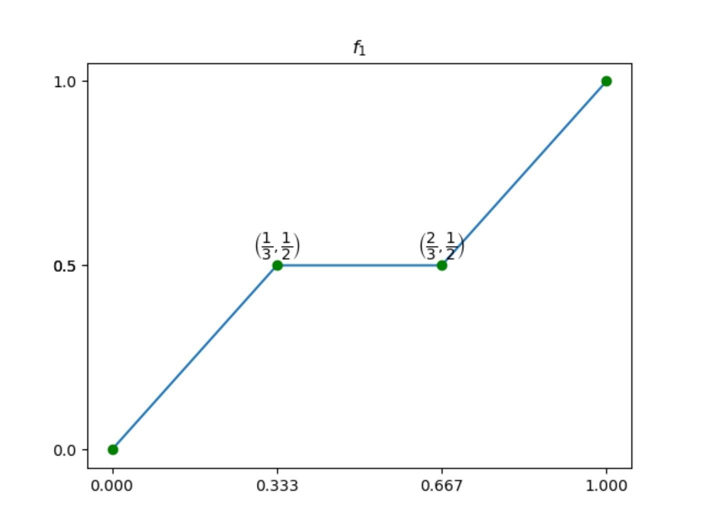
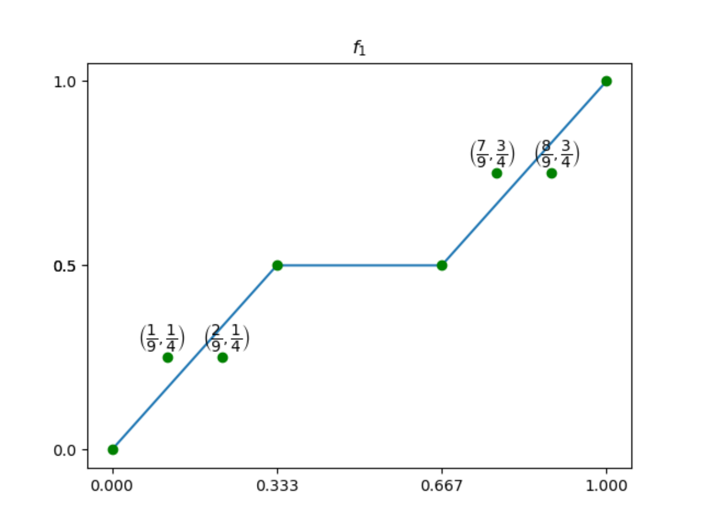


f_1 :
increasing on each interval of
$C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$,
$|C_1| = 2/3 \approx 0.66667$



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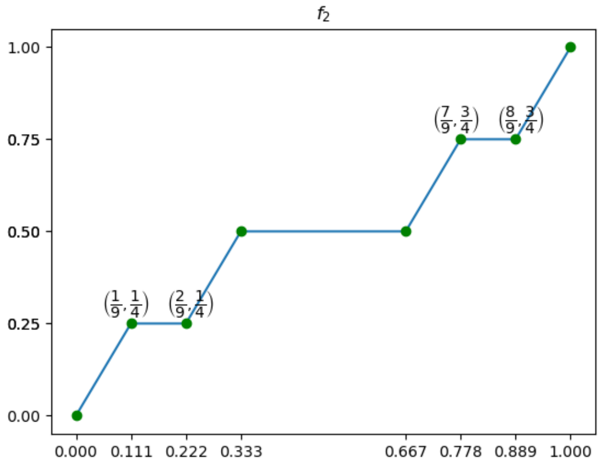

f_2:
increasing on each interval of

$$C_2 := \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right],$$

$$|C_2| = 4/9 = (2/3)^2 \approx 0.44444$$

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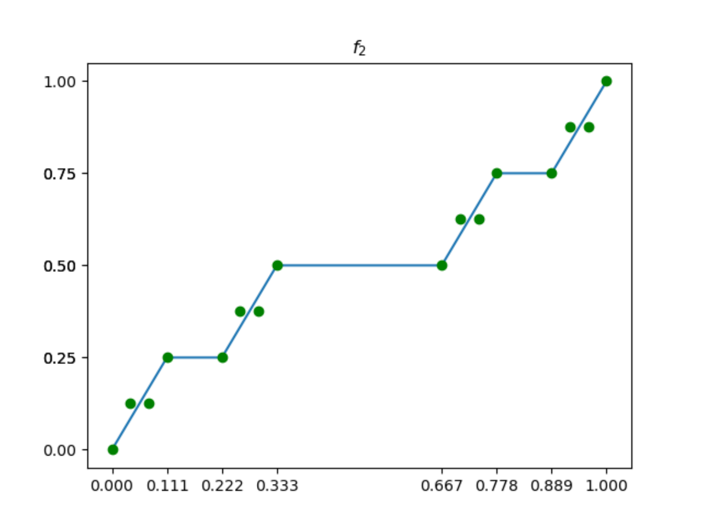

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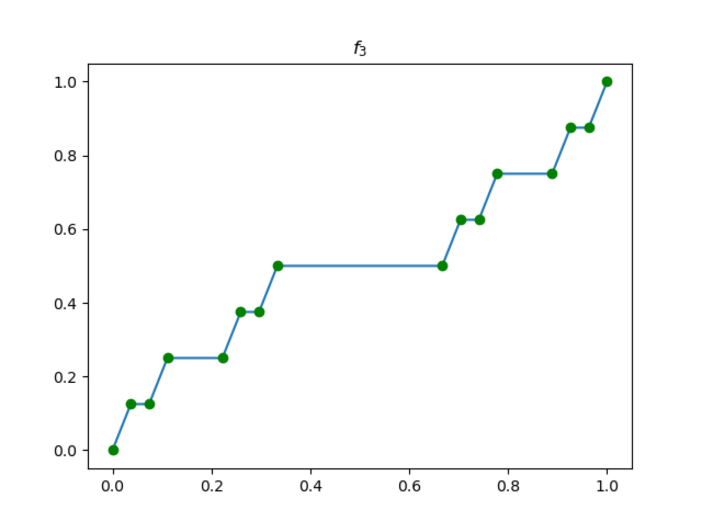

f_3:
increasing on C_3 with
$|C_3| = 8/27 = (2/3)^3 \approx 0.29630$

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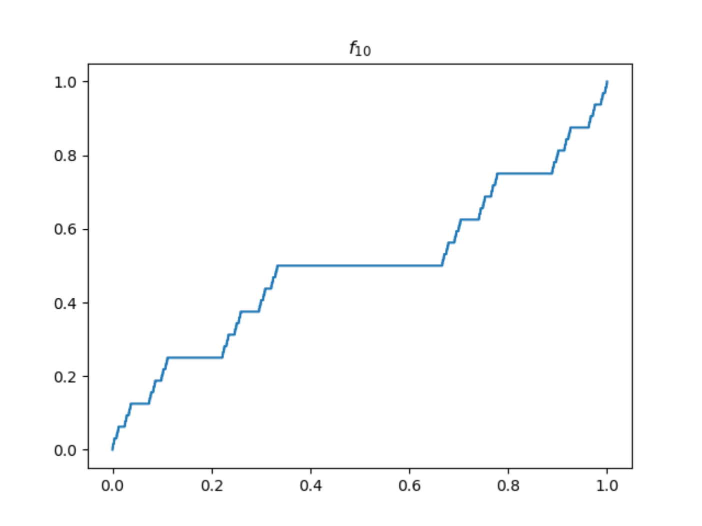
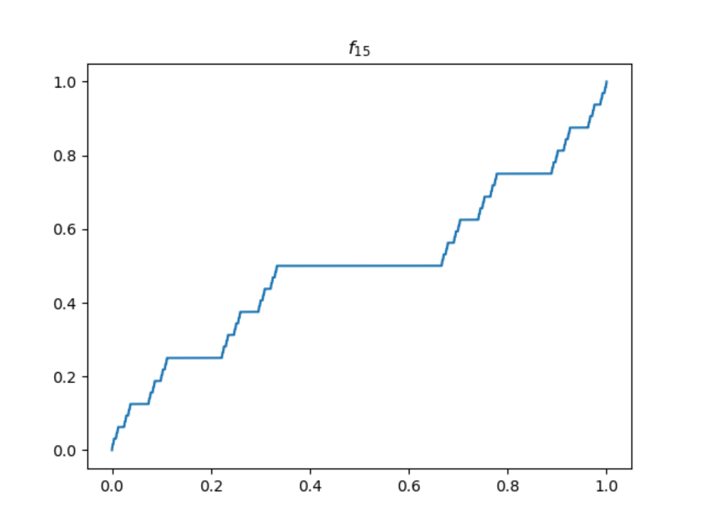


$$|C_{10}| = (2/3)^{10} \approx 0.01734$$



$$|C_{15}| = (2/3)^{15} \approx 0.00228$$

Properties of this sequence of functions

- Each f_n is continuous and non-decreasing.
- $f_n(0) = 0, f_n(1) = 1$ for each n.
- $|f_{n+1}(x) - f_n(x)| \leq 2^{-n}$ for all $x \in [0, 1]$
- f_n converges uniformly to a function $f : [0, 1] \to \mathbb{R}$.

The Cantor-Legesgue function

The Cantor-Lebesgue function is the uniform limit of that sequence of functions.

It is also called the Devil's staircase.

Properties of the Cantor-Lebesgue function:

- f is continuous and non-decreasing on $[0, 1]$.
- $f(0) = 0, f(1) = 1$.
- For all $x \notin \bigcap C_n$, f is constant in a neighborhood of x, and thus $f'(x) = 0$.
- However, f is not differentiable in $\bigcap C_n$, and $\bigcap C_n$ has measure (or "length") 0 .
- But f' is still (Lebesgue) integrable on $[0, 1]$.
- Nevertheless, $\int_{[0,1]} f' \neq f(1) - f(0)$, which does not agree with the fundamental theorem of calculus. Recall that the FTC requires f' to be defined everywhere.

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